An introduction to fractional calculus

Fundamental ideas and numerics

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The original idea

The concept of differentiation and integration to noninteger order goes as far back as the concept we are used to work with. Leibniz mentions it in a letter to L'Hôspital in 1695:

> *"John Bernoulli seems to have told you of my having mentioned to him a marvelous analogy which makes it possible to say in a way that successive differentials are in geometric progression.* ***One can ask what would be a differential having as its exponent a fraction****. You see that the result can be expressed by an infinite series. Although this seems removed from Geometry, which does not yet know of such fractional exponents,* ***it appears that one day these paradoxes will yield useful consequences, since there is hardly a paradox without utility****. Thoughts that mattered little in themselves may give occasion to more beautiful ones."*

(Leibniz, 1646-1716)

Who cares?

Derivatives of non integer order help

- modeling of *viscoelastic* phenomena, *e.g.*, (Bagley and Torvik 1986; Müller et al. 2011)
- restate fundamental model from physics [gravity (Giusti, Garrappa, and Vachon 2020), Schrödinger (Laskin 2002), waves (Luchko 2013), ...],
- modeling of heterogeneous cardiac tissues (Cusimano et al. 2015),
- describing phenomena with *memory* and *non locality* aspects, *e.g.*, (Benzi et al. 2020; Riascos and Mateos 2014)

⋮

This is a **booming topic**, and many new applications frequently arise.

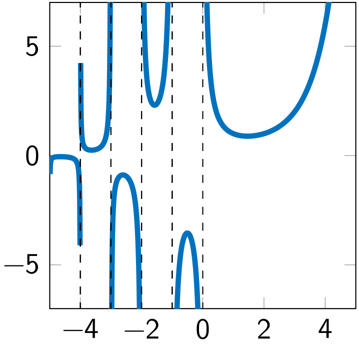
Fractional integrals

Euler Γ-function

The Γ **function** $\Gamma(z)$ is defined for complex numbers with a positive real part via the convergent improper integral:

$$\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x}\,\mathrm{d}x, \qquad \Re(z) > 0,$$

and then extended by **analytic continuation** to a *meromorphic* function that is holomorphic in the whole complex plane except zero and the negative integers, where the function has simple poles.



$$\Gamma(z+1) = z\,\Gamma(z)$$

Bounded in:

$$S = \{z \in \mathbb{C} : \Re z \in [1,2)\}$$

A formula for repeated integration

Swapping Integrals

If $G(x,t)$ is jointly continuous on $[c,b] \times [c,b]$:

$$\int_c^x \mathrm{d}x_1 \int_c^{x_1} G(x_1,x_2)\mathrm{d}x_2 = \int_c^x \mathrm{d}x_2 \int_{x_2}^x G(x_1,x_2)\mathrm{d}x_1.$$

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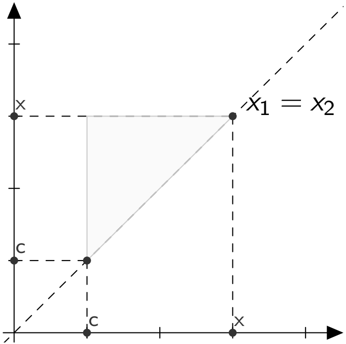
Fubini's Theorem

Given $(X, \mathfrak{S}_X, \mu_x)$, $(Y, \mathfrak{S}_Y, \mu_y)$ measure spaces with σ-finite complete measures μ_x, μ_y on the σ-algebras $\mathfrak{S}_X$, and $\mathfrak{S}_Y$. If the function $f(x,y)$ is integrable on the product $X \times Y$ w.r.t. the product measure $\mu = \mu_x \times \mu_y$, then the following equality holds true

$$\int_{X\times Y} f(x,y)\,\mathrm{d}\mu = \int_Y \mathrm{d}\mu_y \int_X f(x,y)\mathrm{d}\mu_x.$$



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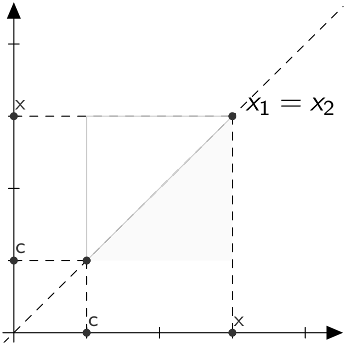

A formula for repeated integration

Cauchy's formula

The indefinite integral of order $n \in \mathbb{N}$ of function $f(t)$ is given by

$$I^n_{c,t} f(t) = \int_c^t \cdots \int_c^t f(t)\,\mathrm{d}t \cdots \mathrm{d}t = \frac{1}{(n-1)!} \int_c^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau,$$
$$I^n_{t,c} f(t) = \int_t^c \cdots \int_t^c f(t)\,\mathrm{d}t \cdots \mathrm{d}t = \frac{1}{(n-1)!} \int_c^t (\tau-t)^{n-1} f(\tau)\,\mathrm{d}\tau.$$

- Can be proved **by induction** using Fubini's Theorem/the previous formula,

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- Can be proved **by induction** using Fubini's Theorem/the previous formula,
- We have introduced the Γ function so let's use it,
- Now we use it to move from the integer case to the **real one**.

Riemann–Liouville Fractional Integrals

Riemann–Liouville Fractional Integral

Let $\Re\alpha > 0$, and let $f \in \mathbb{L}^1([a,b])$. Then for $t \in [a,b]$ we call

$$I^{\alpha}_{[a,t]} f(t) = {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau,$$
$$I^{\alpha}_{[t,b]} f(t) = {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau.$$

the **Riemann–Liouville** fractional integrals of f of order α, we set it to be the identity operator whenever $\alpha = 0$.

- **the idea** is that we have substituted the integer number n of repetition of the integral with the real order α,
- but does this makes sense?

RL Fractional Integrals: properties - I

Theorem (Existence).

Lef $f \in \mathbb{L}^1[a, b]$, and $\alpha > 0$. Then, the integral $I^{\alpha}_{[a,t]} f(t)$ exists for almost every $t \in [a, b]$. Moreover, the function $I^{\alpha}_{[a,t]} f$ itself is also an element of $\mathbb{L}^1[a, b]$.

Proof. It is sufficient to recognize that we can write the integral in question as a convolution on $\mathbb{R}$, indeed:

$$\int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau = \int_{-\infty}^{+\infty} \Phi_1(t-\tau)\Phi_2(\tau)\, \mathrm{d}\tau,$$

where

$$\Phi_1(u) = \begin{cases} u^{\alpha-1}, & \text{for } 0 < t \leq b-a, \\ 0, \text{ otherwise}, \end{cases} \quad \text{and } \Phi_2(u) = \begin{cases} f(u), & \text{for } u \in [a, b], \\ 0, & \text{otherwise}. \end{cases}$$

By construction both the Φ_j, $j = 1, 2$, are in $\mathbb{L}^1(\mathbb{R})$, and thus the integral exists and is a member of $\mathbb{L}^1$ as a convolution of $\mathbb{L}^1$ functions (We are using again *Fubini's Theorem*).

RL Fractional Integrals: properties - II

Theorem (Semigroup property).

The RL fractional integral operators $\{I_c^\alpha : \mathbb{L}^1[a,b] \to \mathbb{L}^1[a,b],\ \alpha \geq 0\}$ form a commutative semigroup with respect to the concatenation operation, that is

$$I_c^\alpha(I_c^\beta f(t)) = I_c^{\alpha+\beta} f(t)), \text{ and } I_c^\beta(I_c^\alpha f(t)) = I_c^{\alpha+\beta} f(t)).$$

The neutral element of this semigroup is the I_c^0 operator.

Proof. We prove it for one side, the other is analogous.

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Proof. We have just proved that the integral exists, then by using *Fubini's theorem* we can interchange the order of integration:

$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_\tau^x (x-t)^{\alpha-1}(t-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

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We now use the substitution $t = \tau + s(x-\tau)$, $\mathrm{d}t = (x-\tau)\mathrm{d}s$.

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$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) \int_0^1 [(x-\tau)(1-s)]^{\alpha-1}[s(x-\tau)]^{\beta-1}(x-\tau)\,\mathrm{d}s\,\mathrm{d}\tau$$

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$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau)(x-\tau)^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1}\, \mathrm{d}s\, \mathrm{d}\tau.$$

RL Fractional Integrals: properties - II

Euler's β-function

The Euler's β-function is defined as:

$$\beta(x,y) \triangleq \int_0^1 u^{x-1}(1-u)^{y-1}\,du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \Re x > 0, \Re y > 0,$$

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$$I^{\alpha}_{[a,t]} I^{\beta}_{[a,t]} f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau)(x-\tau)^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1}\,\mathrm{d}s\,\mathrm{d}\tau.$$

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$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-\tau)^{\alpha+\beta-1} f(\tau)\,\mathrm{d}\tau = I_{[a,t]}^{\alpha+\beta} f(x), \quad \text{a.e. on } [a,b].$$

The same works also if we exchange α and β, while we have the 0th order operator being the neutral element by definition.

RL Fractional Integrals: properties - III

A note on regularity.

Observe that in the proof we could say something more on the regularity of the resulting functions. Indeed if f is a continuous function on $[a, b]$, then also $I^{\alpha}_{[a,t]}f$ is continuous.

Therefore we have that also the concatenation $I^{\alpha}_{[a,t]}I^{\beta}_{[a,t]}$ and $I^{\alpha+\beta}_{[a,t]}$ are continuous. Then what we have proved is that we have two continuous function that are **almost everywhere equal**, and therefore they most coincide everywhere. Furthermore, if $f \in \mathbb{L}^1[a, b]$ and $\alpha + \beta \geq 1$ we can use Semigroup property to write

$$I^{\alpha}_{[a,t]}I^{\beta}_{[a,t]}f = I^{\alpha+\beta}_{[a,t]}f = I^{\alpha+\beta-1}_{[a,t]}I^{1}_{[a,t]}f, \text{ a.e.}$$

Now, since $I^{1}_{[a,t]}f$ is continuos, we also get that the other two way of writing it are continuous, and thus we can conclude the equality everywhere by the same argument as before.

Computing a Riemann–Liouville fractional integral.

$$I^{\alpha}_{[0,t]} t^{\mu} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\mu} \, d\tau,$$

This should be the simplest possible example, and indeed it is as simple as using again the **Euler** β **Function**:

$$\beta(x,y) \triangleq \int_0^1 u^{x-1}(1-u)^{y-1} \, du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \Re x > 0, \Re y > 0.$$

To obtain it, we do the substitution for $u = \frac{\tau}{t}$, then

$$\begin{aligned} I^{\alpha}_{[0,t]} t^{\mu} =& \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \int_0^1 u^{\mu}(1-u)^{\alpha-1} \, du \\ =& \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \frac{\Gamma(\mu+1)\Gamma(\alpha)}{\Gamma(\alpha+\mu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}. \end{aligned}$$

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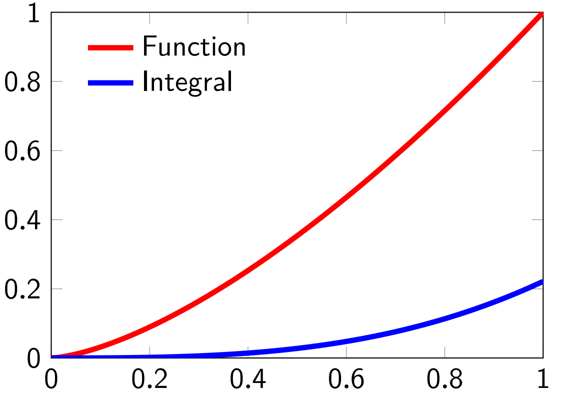
```
t = linspace(0,1,100);
I = @(alpha,mu,t)
 ↪  gamma(mu+1)*t.^(alpha+mu)/
 ↪  gamma(alpha+mu+1);
mu = 1.5;
alpha = 1.5;
plot(t,t.^mu,'r-',t,I(alpha,mu,t),
 ↪  'b-','Linewidth',2);
legend('Function','Integral');
```



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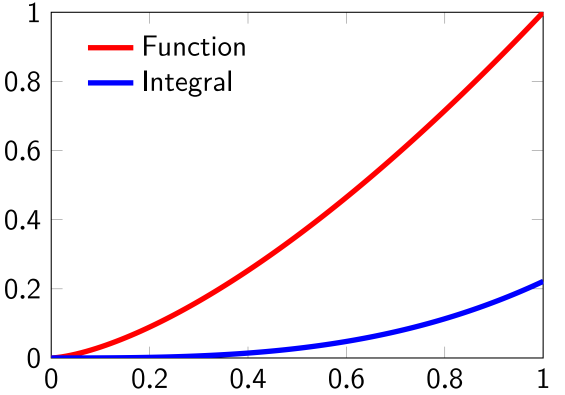

☹ They are hard to compute!

Quadratures for Fractional Integrals

Quadrature idea

Let us assume that $f(t)$ is suitably smooth on an interval (a, b). Let

$$h = \frac{b-a}{N}, \quad t_k = a + kh, \quad \text{with } k = 0, 1, 2, \ldots, N, \quad N \in \mathbb{N}$$

then we can approximate for $t = t_N$ the fractional integral as

$$ {}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} = \frac{1}{\Gamma(\alpha)} \int_a^{t_N} (t_N - \tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_k - \tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau.$$

We approximate $f(x)$ with a polynomial $p(x)$ such that we can compute exactly the involved integrals, this yields quadratures by the usual look

$$ {}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx \sum_{k=0}^{N-1} \omega_k f(t_k).$$

Piecewise constant approximation

We approximate $f(t)$ on the intervals $[t_k, t_k+1)$, $k=0,\dots,N-1$, selecting

$$f(t) \approx p(t) \equiv p(t_k), \quad t \in [t_k, t_k+1),\ k=0,1,\dots,N-1,$$

from which we get the formula

$$\begin{aligned}
{}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} f(t_k) \int_{t_k}^{t_k+1} (t_N-\tau)^{\alpha-1}\,\mathrm{d}\tau = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} f(t_k) \left[-\frac{1}{\alpha}(t_N-\tau)^{\alpha}\right]_{t_k}^{t_{k+1}} \\
&= \sum_{k=0}^{N-1} f(t_k) \frac{1}{\alpha\Gamma(\alpha)} \left[(t_N-t_k)^{\alpha} - (t_N-t_{k+1})^{\alpha}\right] \\
&= \sum_{k=0}^{N-1} f(t_k) \frac{1}{\alpha\Gamma(\alpha)} \left[(a+hn-a-kh)^{\alpha} - (a+hn-a-(k+1)h)^{\alpha}\right] \\
&= \sum_{k=0}^{N-1} f(t_k) \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left[(n-k)^{\alpha} - (N-k-1)^{\alpha}\right] = \sum_{k=0}^{N-1} b_{N-k-1} f(t_k),
\end{aligned}$$

Piecewise constant approximation

We approximate $f(t)$ on the intervals $[t_k, t_k + 1)$, $k = 0, \ldots, N - 1$, selecting

$$f(t) \approx p(t) \equiv p(t_k), \quad t \in [t_k, t_k + 1),\ k = 0, 1, \ldots, N - 1,$$

from which we get the formula

$${}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx \sum_{k=0}^{N-1} b_{N-k-1} f(t_k),$$

where we have defined

$$b_k = \frac{h^\alpha}{\Gamma(\alpha + 1)}[(k + 1)^\alpha - k^\alpha], \qquad 0 \leq k \leq N - 1.$$

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Analogously we get the case in which we select the **right approximation**

$$f(t) \approx p(t) \equiv p(t_{k+1}), \quad t \in [t_k, t_k+1),\ k = 0, 1, \ldots, N-1,$$

and, more generally, for the **weighted formula** in which we select

$$f(t) \approx p(t) \equiv \lambda p(t_k) + (1-\lambda)p(t_{k+1}), \quad t \in [t_k, t_k+1),\ k = 0, 1, \ldots, N-1,\ \lambda \in [0, 1].$$

Piecewise constant approximation

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from which we get the formula

$${}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx \sum_{k=0}^{N-1} b_{N-k-1} f(t_k), \qquad b_k = \frac{h^\alpha}{\Gamma(\alpha+1)}[(k+1)^\alpha - k^\alpha].$$

The general **weighted formula** is then given by

$${}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx \sum_{k=0}^{N-1} b_{N-k-1} \left[\lambda p(t_k) + (1-\lambda) p(t_{k+1})\right], \quad \lambda \in [0,1].$$

Implementation

This is a simple procedure to implement

```
function I = constfracint(f,a,t,alpha,N,lambda)
%CONSTFRACINT computes the fractional integral with the weighted piecewise
%constant approximation of the function f between a and t, over N uniformly
%distributed intervals.
h  = (t-a)/N;
tk = (a:h:t)';
b  = zeros(N,1);
for k=0:N-1
   b(k+1) = (k+1)^alpha - k^alpha;
end
b = h^alpha*b/gamma(alpha+1);
p = f(tk);
I = flipud(b)'*(lambda*p(1:N) + (1-lambda)*p(2:N+1));
end
```

Implementation - II

And we can **test the results** using the fractional integral we have computed by hand

```
f     = @(t,mu) t.^mu;
Itrue = @(alpha,mu,t) gamma(mu+1)*t.^(alpha+mu)/ gamma(alpha+mu+1);
mu    = 1;
alpha = 1.5;

N = 100;
lambda = 1;
I = constfracint(@(t) f(t,mu),0,1,alpha,N,1);
fprintf('Relative error is: %e\n',abs(I-Itrue(alpha,mu,1))./abs(Itrue(alpha,mu,1)));
```

That returns us

```
Relative error is: 1.246939e-02
```

But what about convergence?

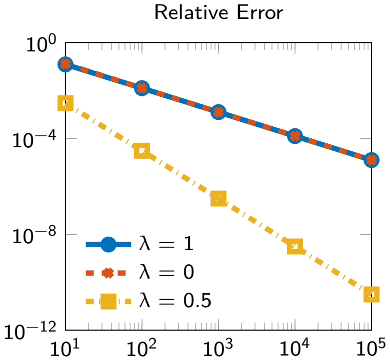
Convergence

Fractional Newton-Cotes formula

Lef $f(t)$ be approximated by a polynomial $p_{k,r}(t)$ of degree r on the grid points $\{t_k = t_0^{(k)}, \dots, t_r^{(k)} = t_{k+1}\}$. Then the error estimate for an $f \in \mathcal{C}^{r+1}([a,b])$ on each sub-interval $[t_k, t_{k+1}]$ is given by

$$f(t) - p_{k,r}(t) = \frac{f^{(r+1)}(\tau_k)}{(r+1)!} \prod_{j=0}^{r} (t - t_j^{(k)}),$$

for $r \in \mathbb{N}$, $t, \tau_k \in [t_k, t_{k+1}]$, i.e., the formula is of order $O(h^{r+1})$.



N/λ	0	1	$1/2$
1e+01	1.3e-01	1.2e-01	2.9e-03
1e+02	1.3e-02	1.2e-02	3.1e-05
1e+03	1.3e-03	1.2e-03	3.1e-07
1e+04	1.3e-04	1.2e-04	3.1e-09
1e+05	1.3e-05	1.2e-05	3.1e-11

Convergence

Proof. The interpolating polynomial can be expressed in the **Lagrange basis**

$$p_{k,r}(t) = \sum_{i=0}^{r} l_{k,i}(t) f(t_i^{(k)}), \quad l_{k,i}(t) = \prod_{\substack{j=0 \\ j\neq i}}^{r} \frac{t - t_j^{(k)}}{t_i^{(k)} - t_j^{(k)}}, \quad 0 \leq i \leq r,\ t \in [t_k, t_{k+1}].$$

Then the fractional Newton-Coates formula si given by

$${}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx {}_aD_b^{-\alpha} p_{k,r}(t)\Big|_{t=t_N} = \sum_{k=0}^{N-1} \sum_{i=0}^{r} C_{i,n}^{(k)} f(t_i^{(k)}),$$

for

$$C_{i,n}^{(k)} = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha-1} l_{k,i}(\tau)\, \mathrm{d}\tau.$$

Convergence

Proof. Then the fractional Newton-Coates formula si given by

$$
{}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx {}_aD_b^{-\alpha} p_{k,r}(t)\Big|_{t=t_N} = \sum_{k=0}^{N-1}\sum_{i=0}^{r} C_{i,n}^{(k)} f(t_i^{(k)}),
$$

from which we obtain the error estimate as

$$
\begin{aligned}
\left|{}_aD_b^{-\alpha} f(t) - {}_aD_b^{-\alpha} p_{k,r}(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_k-\tau)^{\alpha-1} \left|f(\tau)-p_{k,r}(\tau)\right| \mathrm{d}\tau \\
&\leq \max_{t\in[a,t_N]} \frac{|f^{(r+1)}(t)|}{(r+1)!\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_n-\tau)^{\alpha-1} \prod_{j=0}^{r} |\tau - t_j^{(k)}| \,\mathrm{d}\tau \\
&\leq \max_{t\in[a,t_N]} \left|f^{(r+1)}(t)\right| \frac{h^{r+1}}{(r+1)!\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_N-\tau)^{\alpha-1} \,\mathrm{d}\tau.
\end{aligned}
$$

Convergence

Proof. Then the fractional Newton-Coates formula si given by

$$
{}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx {}_aD_b^{-\alpha} p_{k,r}(t)\Big|_{t=t_N} = \sum_{k=0}^{N-1} \sum_{i=0}^{r} C_{i,n}^{(k)} f(t_i^{(k)}),
$$

from which we obtain the error estimate as

$$
\begin{aligned}
\left| {}_aD_b^{-\alpha} f(t) - {}_aD_b^{-\alpha} p_{k,r}(t) \right| &\leq \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_k - \tau)^{\alpha-1} \left| f(\tau) - p_{k,r}(\tau) \right| \mathrm{d}\tau \\
&\leq \max_{t \in [a, t_N]} \frac{|f^{(r+1)}(t)|}{(r+1)!\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha-1} \prod_{j=0}^{r} |\tau - t_j^{(k)}| \, \mathrm{d}\tau \\
&\leq \max_{t \in [a, t_N]} \left| f^{(r+1)}(t) \right| \frac{h^{r+1}}{(r+1)!\Gamma(\alpha+1)} (t_n - t_0)^{\alpha}.
\end{aligned}
$$

Convergence

Proof. Then the fractional Newton-Coates formula si given by

$$ {}_aD_b^{-\alpha} f(t)\Big|_{t=t_N} \approx {}_aD_b^{-\alpha} p_{k,r}(t)\Big|_{t=t_N} = \sum_{k=0}^{N-1} \sum_{i=0}^{r} C_{i,n}^{(k)} f(t_i^{(k)}), $$

from which we obtain the error estimate as

$$ \left| {}_aD_b^{-\alpha} f(t) - {}_aD_b^{-\alpha} p_{k,r}(t) \right| \in O(h^{r+1}). $$

Remark

The error estimate does not coincide completely with the classical one for Newton-Coates formulas, this is due to the nonsymmetry of the integral kernel $(t_n - t)^{\alpha-1}$.

Suggested exercises, and some extensions

(i) Rewrite (and implement) the fractional weighted constant approximation for the *other-sided* Riemann-Liuoville fractional integral,

(ii) Denote with $t_{k+1/2} = {}^{t_k+t_{k+1}}/_2$ on each sub-interval $[t_k, t_{k+1}]$, approximate $f(t)$ with a *piecewise quadratic polynomial*, derive and implement the fractional Simpson's formula
   - ⚠ The closed form of the coefficients for this case is cumbersome...

Extensions

By mimicking the usual procedure for deriving collocation/spectral type quadrature formulas, we could approximate $f(t)$ by using, e.g., Jacobi polynomials to obtain the related quadrature formulas (when you have obtained formulas for Jacobi, then *Chebyshev* and *Legendre* follow with relative "ease").

Riemann–Liouville Fractional Derivatives

Now that we've gotten a little bit of familiarity with Riemann–Liouville integral operators, we can finally **introduce the corresponding differential operators**.

The key idea

Let f be a function having a continuous nth derivative on the interval $[a, b]$, and let $m \in \mathbb{N}$ be such that $m > n$, then

$$\frac{d^n}{dt^n} f(t) = \frac{1}{(m-n-1)!} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-n-1} f(\tau)\,\mathrm{d}\tau = \frac{d^m}{dt^m} I_a^{m-n} f,$$

simply by employing the Fundamental Theorem of (Classical) Calculus

$$f = \frac{d^{m-n}}{dt^{m-n}} I_a^{m-n} f,$$

and applying the operator $\frac{d^n}{dt^n}$ to both side of it.

Riemann–Liouville Fractional Derivatives

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The key idea now we go from integers to real numbers!

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Riemann–Liouville Fractional Derivatives

Substitute the integer n with a real positive number α and select an $m \in \mathbb{N}$ s.t. $m > \alpha$.

RL Derivative

Let $\alpha \in \mathbb{R}_+$ and $m = \lceil \alpha \rceil$, we define the Riemann-Liouville operator ${}_{\mathrm{RL}}D_a^\alpha$ as

$$ {}_{\mathrm{RL}}D_a^\alpha f(t) \triangleq \frac{d^m}{dt^m} I_a^{m-\alpha} f(t), $$

and we set ${}_{\mathrm{RL}}D_a^0$ to the identity operator.

⚠ The right-hand side of our definition remains valid, **but** now the resulting operator depends on the choice of the point a.

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- ⚠ The right-hand side of our definition remains valid, **but** now the resulting operator depends on the choice of the point a.
- ? for what functions f does this definition make sense?

Riemann–Liouville Fractional Derivatives Existence

The $\mathbb{A}^n$ functions

We call $\mathbb{A}^n[a,b]$, or simply $\mathbb{A}^n$ when the interval is clear from the context, the space of function with an **absolutely continuous** $(n-1)$st derivative, i.e., the functions f for which there exists almost everywhere a (generalized) nth derivative function $g \in \mathbb{L}^1[a,b]$ for which holds

$$f^{(n-1)}(t) = f^{(n-1)}(a) + \int_a^t g(\tau)\,\mathrm{d}\tau.$$

Remind: For a compact interval:

continuously differentiable $\subseteq$ Lipschitz continuous $\subseteq$ absolutely continuous $\subseteq$ bounded variation $\subseteq$ differentiable almost everywhere

Example: $f(t) = \sqrt[3]{t}$ is absolutely continuous on any bounded interval I **but** not Lipschitz continuous on any interval I such that $0 \in I$.

Riemann–Liouville Fractional Derivatives Existence

Theorem (Existence)

Lef $f \in \mathbb{A}^1[a,b]$, and $0 < \alpha < 1$. Then ${}_{\mathrm{RL}}D_a^\alpha f(t)$ exists almost everywhere in $[a,b]$. Moreover, ${}_{\mathrm{RL}}D_a^\alpha f(t) \in \mathbb{L}^p$ for $1 \le p < \alpha^{-1}$ and

$$
{}_{\mathrm{RL}}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \int_a^t f'(\tau)(t-\tau)\,\mathrm{d}\tau \right).
$$

Proof. We use directly the two definitions

$$
\begin{aligned}
{}_{\mathrm{RL}}D_a^\alpha f(t) =& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)\,\mathrm{d}\tau \\
=& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(f(a) + \int_a^\tau f'(s)\,\mathrm{d}s \right) (t-\tau)\,\mathrm{d}\tau
\end{aligned}
$$

Riemann–Liouville Fractional Derivatives Existence

Theorem (Existence)

Lef $f \in \mathbb{A}^1[a,b]$, and $0 < \alpha < 1$. Then ${}_{\mathrm{RL}}D_a^{\alpha} f(t)$ exists almost everywhere in $[a,b]$. Moreover, ${}_{\mathrm{RL}}D_a^{\alpha} f(t) \in \mathbb{L}^p$ for $1 \le p < \alpha^{-1}$ and

$$ {}_{\mathrm{RL}}D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^{\alpha}} + \int_a^t f'(\tau)(t-\tau)\,\mathrm{d}\tau \right). $$

Proof. We use directly the two definitions

$$ \begin{aligned} {}_{\mathrm{RL}}D_a^{\alpha} f(t) =& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(f(a) + \int_a^{\tau} f'(s)\,\mathrm{d}s \right) (t-\tau)\,\mathrm{d}\tau \\ =& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(f(a) \int_a^t \frac{\mathrm{d}t}{(x-t)^{\alpha}} + \int_a^{\tau} \int_a^{s} f'(s)(t-\tau)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}\tau \right) \end{aligned} $$

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$$

Proof. We use directly the two definitions, and apply again *Fubini's Theorem*

$$
\begin{aligned}
{}_{\mathrm{RL}}D_a^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(f(a)\int_a^t \frac{\mathrm{d}t}{(x-t)^\alpha} + \int_a^\tau\int_a^s f'(s)(t-\tau)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}\tau\right)\\
&= \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(a)}{(t-a)^\alpha} + \frac{d}{dt}\int_a^t\int_a^s f'(s)(t-\tau)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}\tau\right)\\
\text{(Fubini)}\quad &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(a)}{(t-a)^\alpha} + \frac{d}{dt}\int_a^t f'(s)\frac{(t-s)^{1-\alpha}}{1-\alpha}\,\mathrm{d}s\right),
\end{aligned}
$$

Riemann–Liouville Fractional Derivatives Existence

Theorem (Existence)

Lef $f \in \mathbb{A}^1[a,b]$, and $0 < \alpha < 1$. Then ${}_{\mathrm{RL}}D_a^{\alpha} f(t)$ exists almost everywhere in $[a,b]$. Moreover, ${}_{\mathrm{RL}}D_a^{\alpha} f(t) \in \mathbb{L}^p$ for $1 \leq p < \alpha^{-1}$ and

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Proof. We use directly the two definitions, and finally Leibniz rule for the derivative of integral functions,

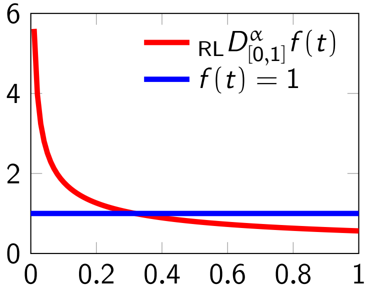
$$ \begin{aligned} {}_{\mathrm{RL}}D_a^{\alpha} f(t) =& \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^{\alpha}} + \frac{d}{dt} \int_a^t f'(s) \frac{(t-s)^{1-\alpha}}{1-\alpha}\,\mathrm{d}s \right), \\ =& \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^{\alpha}} + \int_a^t f'(\tau)(t-\tau)\,\mathrm{d}\tau \right). \end{aligned} $$

Computing our first RL derivatives

To keep things simple we can compute, first of all, the fractional derivative of order $\alpha \in (0,1)$ of the **constant function** $f(t) = 1$ in $[0,t]$:

We simply apply the previous representation theorem, and thus:

$$
\begin{aligned}
{}_{\mathrm{RL}}D^{\alpha}_{[0,1]} f(t) =& \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(0)}{(t-0)^{\alpha}} + \int_0^t f'(\tau)(t-\tau)\,\mathrm{d}\tau \right) = \\
=& \frac{1}{\Gamma(1-\alpha)} \frac{1}{(t-0)^{\alpha}} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}
\end{aligned}
$$



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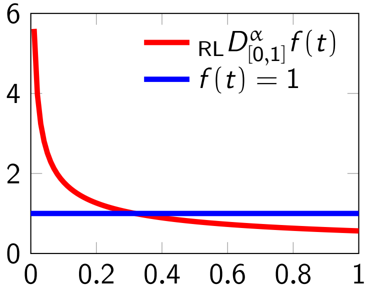

The RL derivative of a constant is not zero!

Computing our first RL derivatives

Let $f(t) = (t-a)^\beta$ for some $\beta > -1$ and compute its RL derivative of order $\alpha > 0$ on an interval $[a, b]$.
First we compute the **fractional integral** part of the definition:

$$
\begin{aligned}
I^{\alpha}_{[a,t]} f(t) =& \frac{1}{\Gamma(\alpha)} \int_a^t (\tau - a)^\beta (t-\tau)^{\alpha-1}\,\mathrm{d}\tau = \\
=& \frac{1}{\Gamma(\alpha)} \int_0^{t-a} s^\beta (t-a-s)^{\alpha-1}\,\mathrm{d}s = \leftarrow \left(\int_0^x s^{\beta-1}(x-s)^{\alpha-1}\mathrm{d}s = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} \right) \\
=& \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta},
\end{aligned}
$$

Computing our first RL derivatives

Let $f(t) = (t-a)^{\beta}$ for some $\beta > -1$ and compute its RL derivative of order $\alpha > 0$ on an interval $[a, b]$.

First we compute the **fractional integral** part of the definition:

$$I^{\alpha}_{[a,t]} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta},$$

Then we just have to compute the derivative with the correct indexes

$${}_{\mathrm{RL}}D^{\alpha}_{[0,1]} f(t) = \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} I^{\lceil\alpha\rceil-\alpha}_{[a,t]} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\lceil\alpha\rceil-\alpha+\beta+1)} \left.\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}(\cdot - a)^{\lceil\alpha\rceil-\alpha+\beta}\right|_{t},$$

now, if $\alpha - \beta \in \mathbb{N}$ the right-hand side vanishes ($\lceil\alpha\rceil$-derivative of a polynomial of lower degree), if $\alpha - \beta \notin \mathbb{N}$, we find

$${}_{\mathrm{RL}}D^{\alpha}_{[0,1]} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha}.$$

Summary and anticipations

We did

- Definition and properties of Riemann–Liouville Integrals,
- Some examples of Fractional Newton-Cotes formulas for RL integral computations,
- Definition and existence of Riemann–Liouville Derivatives,
- A couple of by-hand computations of RL derivatives of simple functions.

Next up

- Properties and interactions between Riemann–Liouville Integrals and Derivatives,
- The Caputo fractional derivative,
- An introduction to Fractional Differential Equations.

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