

From FEM Discretizations to Saddle-Point Matrices

Iterative Methods for Large-Scale Saddle-Point Problems

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George Pólya (1887–1985)

*“In order **to solve** this differential equation you look at it till a solution occurs to you.”*

How to Solve It (Princeton 1945)



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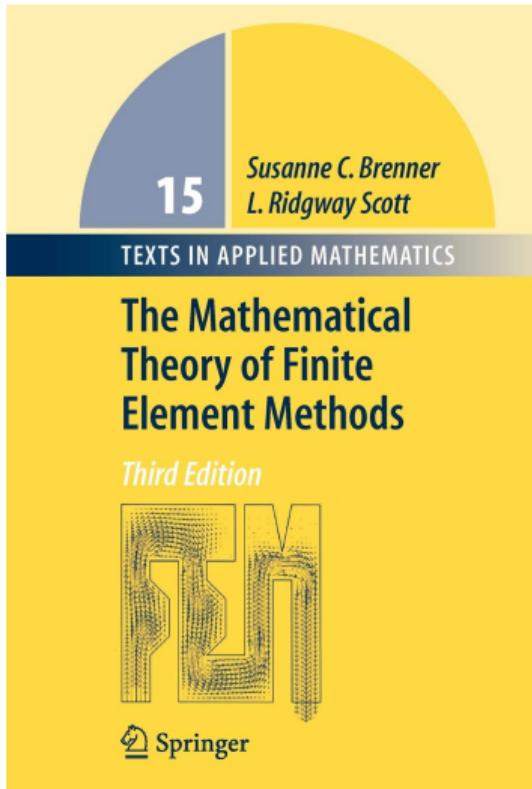
How to Solve It (Princeton 1945)

We are gonna settle for **approximating its solution**.

Overview

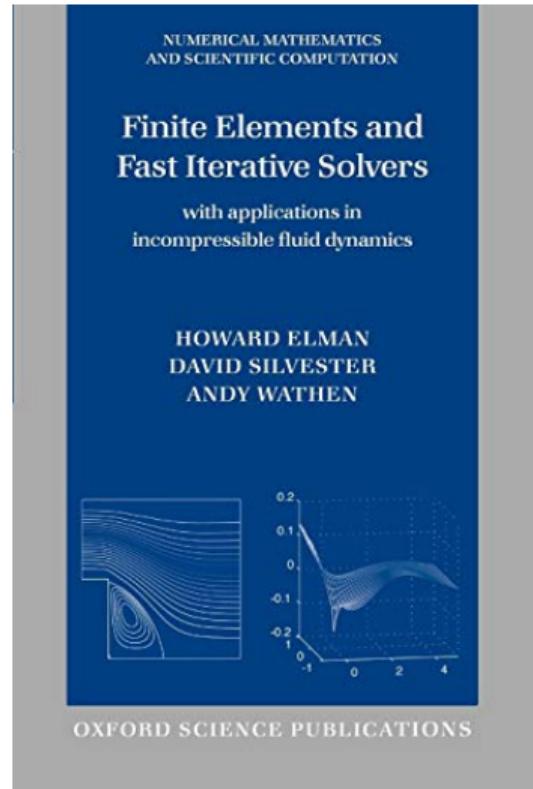
- 1. Basic Concepts**
- 2. Finite Element Spaces**
- 3. Variational crimes**
- 4. Mixed methods**
 - 4.1 The Poisson Equation
 - 4.2 The Stokes Equation
 - Stable discretizations
 - Stabilized discretizations
 - 4.3 The Navier-Stokes Equation

The main sources



S. C. Brenner and L. R. Scott (2008). *The mathematical theory of finite element methods*. Third. Vol. 15. Texts in Applied Mathematics. Springer, New York, pp. xviii+397. ISBN: 978-0-387-75933-3

H. C. Elman, D. J. Silvester, and A. J. Wathen (2014). *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*. Second. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, pp. xiv+479. ISBN: 978-0-19-967880-8



Basic Concepts

Consider the **two-point boundary value problem** (BVP):

$$\begin{cases} -\frac{d^2u}{dx^2} = f, & \text{in } (0, 1), \\ u(0) = 0, & u'(1) = 0. \end{cases}$$

If u is the solution and $v \in V$ is a sufficiently regular function for which $v(0) = 0$, then **integration by parts** yields:

$$\begin{aligned} (f, v) &= \int_0^1 f(x)v(x) dx = - \int_0^1 u''(x)v(x) dx \\ &= \int_0^1 u'(x)v'(x) dx = a(u, v). \end{aligned}$$

Then **the solution** u to our BVP is characterized by

$$\text{find } u \in V \text{ such that } a(u, v) = (f, v) \quad \forall v \in V.$$

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Sobolev Spaces: multi-index notation

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First some notation

Given a **multi-index** $\alpha \in \mathbb{N}^n$ we denote with

$$|\alpha| = \sum_{i=1}^n \alpha_i,$$

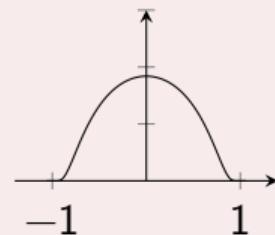
the length of the multi-index. For a function $\varphi \in \mathcal{C}^\infty$, we denote the usual **pointwise partial derivative** by

$$D^\alpha \varphi = D_x^\alpha \varphi = \left(\frac{\partial}{\partial x} \right)^\alpha \varphi = \varphi^{(\alpha)} = \partial_x^\alpha \varphi = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} \varphi.$$

Sobolev Spaces: building blocks

Definition: compact support functions

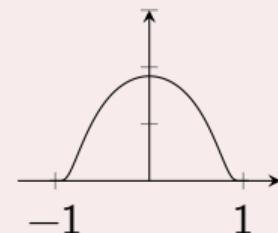
Let $\Omega \subseteq \mathbb{R}^n$ a domain. We denote by $\mathcal{D}(\Omega)$ or $\mathcal{C}_0^\infty(\Omega)$ the set of $\mathcal{C}^\infty(\Omega)$ **functions with compact support** in Ω , i.e., the the $\mathcal{C}^\infty(\Omega)$ functions for which the closure of the set of the points in which they are not zero is compact in Ω .



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Definition: locally integrable functions

Given a domain Ω we define the set of **locally integrable functions** as

$$\mathbb{L}_{\text{loc}}^1(\Omega) = \{f : f \in \mathbb{L}^1(K) \forall K \subset \overset{\circ}{\Omega} \text{ } K \text{ compact}\}.$$

Sobolev Spaces: **weak derivatives**

Definition: weak derivative

We say that a function $f \in \mathbb{L}_{\text{loc}}^1(\Omega)$ has a **weak derivative**, $D_w^\alpha f$ provided that there exists a function $g \in \mathbb{L}_{\text{loc}}^1(\Omega)$ such that

$$\int_{\Omega} g(x)\varphi(x)dx = (-1)^{|\alpha|} \int_{\Omega} f(x)\varphi^{(\alpha)}(x)dx, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega).$$

If such g exists then we define $D_w^\alpha f = g$.

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A couple of **examples**:

- $f(x) = 1 - |x|$ admits as first weak derivative $D_w^1 f = g = \chi_{x<0} + \chi_{x>0}$,
- If $f \in \mathcal{C}^{|\alpha|}(\Omega)$ for an arbitrary α , then $D_w^\alpha f = D^\alpha f$.

Sobolev space

Definition: Sobolev norms and spaces

Let $k \in \mathbb{N}$, $f \in \mathbb{L}_{\text{loc}}^1(\Omega)$, suppose that the weak derivative $D_w^\alpha f$ exists for all $|\alpha| \leq k$. We define the **Sobolev norm**

$$\|f\|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{\alpha: |\alpha| \leq k} \|D_w^\alpha f\|_{\mathbb{L}^p(\Omega)}^p \right)^{\frac{1}{p}}, & 1 \leq p < +\infty \\ \max_{\alpha: |\alpha| \leq k} \|D_w^\alpha f\|_{\mathbb{L}^\infty(\Omega)}, & p = \infty. \end{cases}$$

We define the **Sobolev space** $W_p^k(\Omega)$ as

$$W_p^k(\Omega) = \left\{ f \in \mathbb{L}_{\text{loc}}^1(\Omega) : \|f\|_{W_p^k(\Omega)} < \infty \right\}.$$

Sobolev space: a collection of results

Theorem(s)

- (i) The Sobolev space $W_p^k(\Omega)$ is a Banach space,
- (ii) Let Ω be any open set, then $C^\infty(\Omega) \cap W_p^k(\Omega)$ is dense in $W_p^k(\Omega)$ for $p < \infty$,
- (iii) $k, m \in \mathbb{N}$, $k \leq m$, $1 \leq p \leq \infty \Rightarrow W_p^m(\Omega) \subset W_p^k(\Omega)$,
- (iv) Ω bounded, $k \in \mathbb{N}$, $1 \leq p \leq q \leq \infty \Rightarrow W_q^k(\Omega) \subset W_p^k(\Omega)$,

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Definition: Lipschitz boundary

Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$. Ω is a **Lipschitz domain** if $\forall p \in \partial\Omega$ exists a hyperplane H of dimension $n - 1$ through p , a Lipschitz-continuous function $g : H \rightarrow \mathbb{R}$ over that hyperplane, and reals $r > 0$ and $h > 0$ such that

- $\Omega \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < g(x)\}$,
- $(\partial\Omega) \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, g(x) = y\}$,

where \vec{n} is a unit vector that is normal to H , $B_r(p) := \{x \in \mathbb{R}^n \mid \|x - p\| < r\}$ is the open ball of radius r , $C := \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < h\}$.

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- (iv) Ω bounded, $k \in \mathbb{N}$, $1 \leq p \leq q \leq \infty \Rightarrow W_q^k(\Omega) \subset W_p^k(\Omega)$,
- (v) If $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary, $\forall k \in \mathbb{N}$, $1 \leq p \leq \infty$, there exist $E : W_p^k(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$ satisfying $Ev|_\Omega = v \quad \forall v \in W_p^k(\Omega)$, and $\|Ev\|_{W_p^k(\mathbb{R}^n)} \leq C\|v\|_{W_p^k(\Omega)}$ with C independent of v ,
- (vi) If $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary, $\forall k \in \mathbb{N}$, $1 \leq p < \infty$, and $m < k$, then

$$\exists C > 0 : \forall u \in W_p^k(\Omega) \quad \|u\|_{W_\infty^m(\Omega)} \leq C\|u\|_{W_p^k(\Omega)} \quad \begin{cases} k - m \geq n, & p = 1, \\ k - m > \frac{n}{p}, & p > 1. \end{cases}$$

And there exist a function in C^m in the \mathbb{L}^p equivalence class of u .

Sobolev space: finally we have got an answer!

If you have forgotten the question, we were trying to understand for what V the solution u characterized by

$$\text{find } u \in V \text{ such that } a(u, v) = (f, v) \quad \forall v \in V.$$

was a meaningful solution to our initial BVP.

The space

$$V = \{v \in W_2^1(\Omega) : v(0) = 0\}.$$

By the **extension property** and the **Sobolev inequality** we now know that **pointwise values are well defined** for functions $W_2^1(\Omega)$.

But all this machinery was needed just to **validate the formulation**, how do we go to a **discrete solution**?

Building a discrete space

To move to a discrete setting, we need to select a **finite subspace** $S \subset V$. With this, we can impose the **Ritz-Galerkin** conditions:

$$\text{find } u_S \in S \text{ such that } a(u_S, v) = (f, v) \quad \forall v \in S.$$

- Since S is finite-dimensional, there exists a basis ϕ_1, \dots, ϕ_n of S ,
- Thus, $u_S = \sum_{i=1}^n U_i \phi_i \in S$, $U_i \in \mathbb{R}$ for $i = 1, \dots, n$,
- Ritz-Galerkin conditions are now a **system of linear equations** for the unknown coefficients U_i :

$$K\mathbf{U} = \mathbf{F},$$

with

- $\mathbf{U} = (U_1, \dots, U_n)^T \in \mathbb{R}^n$,
- $\mathbf{F} = (F_1, \dots, F_n)^T \in \mathbb{R}^n$, for $F_i = (f, \phi_i)$,
- $\mathbf{K} = (K_{ij}) \in \mathbb{R}^{n \times n}$, for $K_{ij} = a(\phi_i, \phi_j)$.

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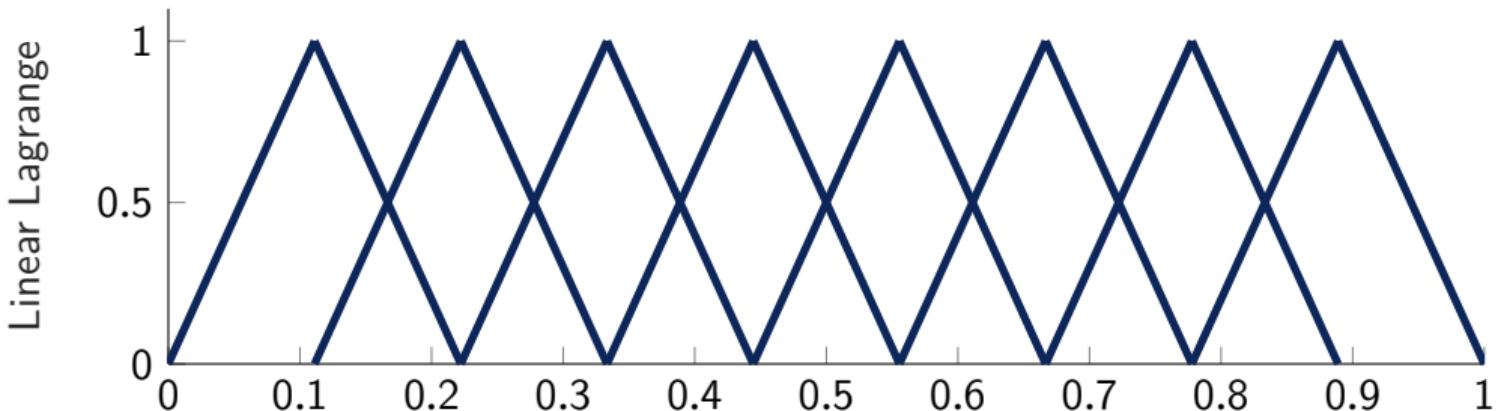
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What are examples of such S ?

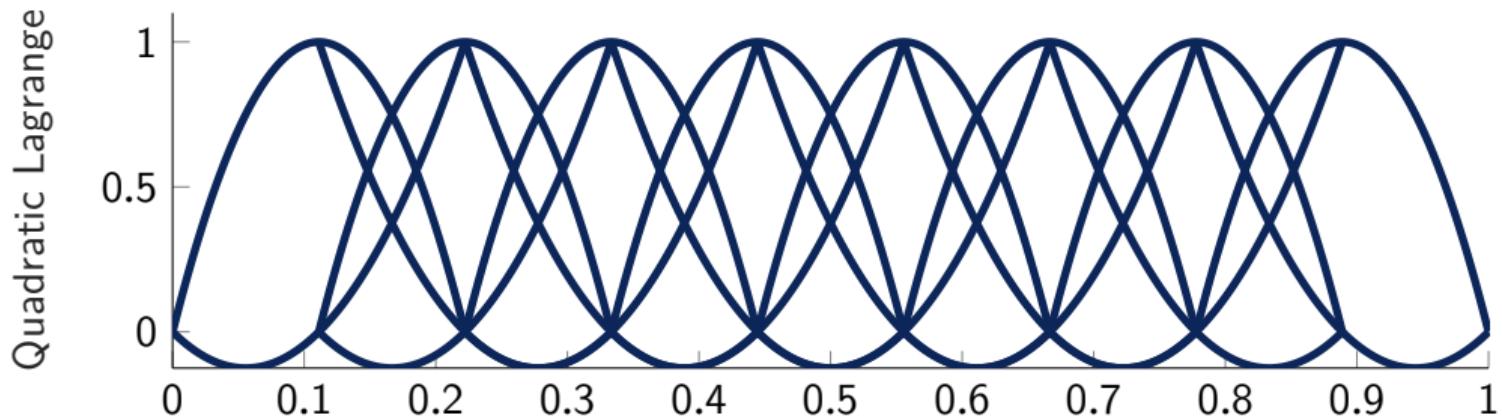
Lagrange basis



Let $0 = x_0 < x_1 < x_2 < \dots < x_n = 1$, we consider the linear space of functions $v \in S$ s.t.

- (i) $v \in C^0([0, 1])$,
- (ii) $v|_{[x_{i-1}, x_i]}$ is a linear polynomial, $i = 1, \dots, n$, and
- (iii) $v(0) = 0$.

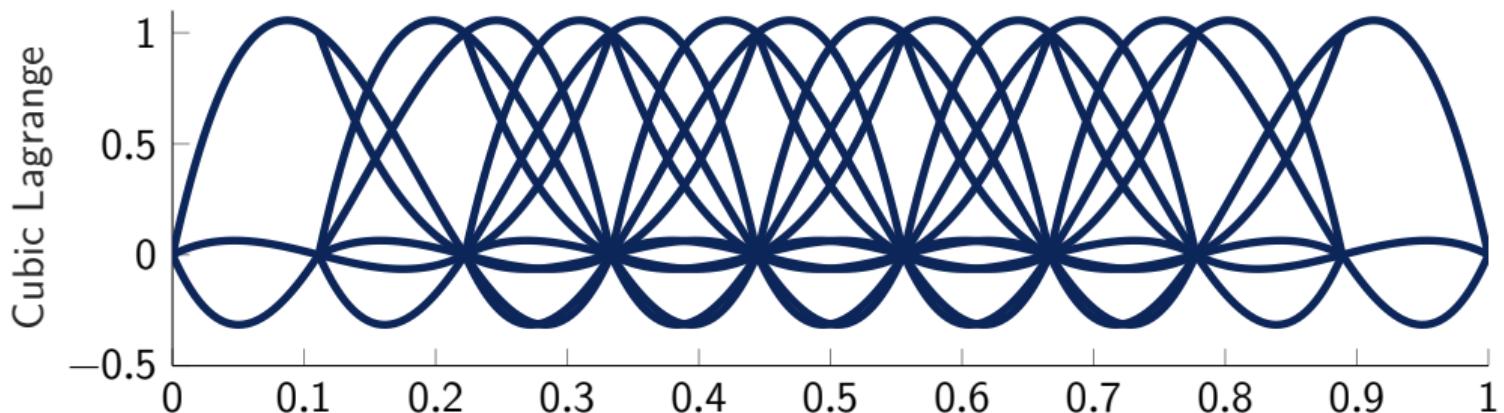
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- (ii) $v|_{[x_{i-2}, x_i]}$ is a quadratic polynomial, $i = 2, \dots, n$, and
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Lagrange basis



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- (i) $v \in C^0([0, 1])$,
- (ii) $v|_{[x_{i-3}, x_i]}$ is a cubic polynomial, $i = 3, \dots, n$, and
- (iii) $v(0) = 0$.

Convergence and approximation properties

We have a **theoretical framework for solutions**, examples of **discrete spaces**, but **what about convergence?**

Sobolev meets Hilbert

W_p^k is a Hilbert space for $p = 2$, with inner product

$$\langle f, g \rangle_{W_2^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g).$$

We write: $H^k(\Omega) \equiv W_2^k(\Omega)$, and $H_0^k(\Omega) = \{v \in W_2^k(\Omega) : v \equiv 0 \text{ on } \partial\Omega\}$.

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Our V space

$$V = H_0^1([0, 1]).$$

Convergence and approximation properties

Variational problem

For a given Hilbert space V , a bilinear form $a : V \times V \rightarrow \mathbb{R}$ and a linear functional $F : V \rightarrow \mathbb{R}$, find $u \in V$ such that:

$$a(u, v) = F(v), \quad \text{for all } v \in V. \tag{VP}$$

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Theorem (Lax-Milgram).

Let V be a Hilbert space, $a : V \times V \rightarrow \mathbb{R}$ a bilinear form, and $F : V \rightarrow \mathbb{R}$ a linear functional s.t.

Coercivity $\exists c_1 > 0$ s.t. $a(v, v) \geq c_1 \|v\|_V^2$ for all $v \in V$.

Continuity $\exists c_2, c_3 > 0$ s.t. $a(v, w) \leq c_2 \|v\|_V \|w\|_V$, and $|F(v)| \leq c_3 \|v\|_V$ for all $v, w \in V$.

$\Rightarrow \exists! u \in V$ satisfying (VP), and $\|u\|_V \leq \frac{1}{c_1} \|F\|_{V^*}$.

Convergence and approximation properties

In the **conforming Galerkin approach** we chose a (finite-dimensional) closed subspace $V_h \subset V$ and look for $u_h \in V_h$ satisfying

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Theorem

Under the assumptions of the Lax-Milgram, for any closed subspace $V_h \subset V$, there exists a unique solution $u_h \in V_h$ of (VP_h) satisfying

$$\|u_h\|_V \leq \frac{1}{c_1} \|F\|_{V^*}.$$

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Céa's lemma

Let u_h be the solution of (VP_h) for given $V_h \subset V$ and u be the solution of variational problem (VP) . Then,

$$\|u - u_h\|_V \leq \frac{c_2}{c_1} \inf_{v_h \in V_h} \|u - v_h\|_V,$$

where c_1 and c_2 are the constants from the **coercivity** and **continuity** assumptions.

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The conforming idea

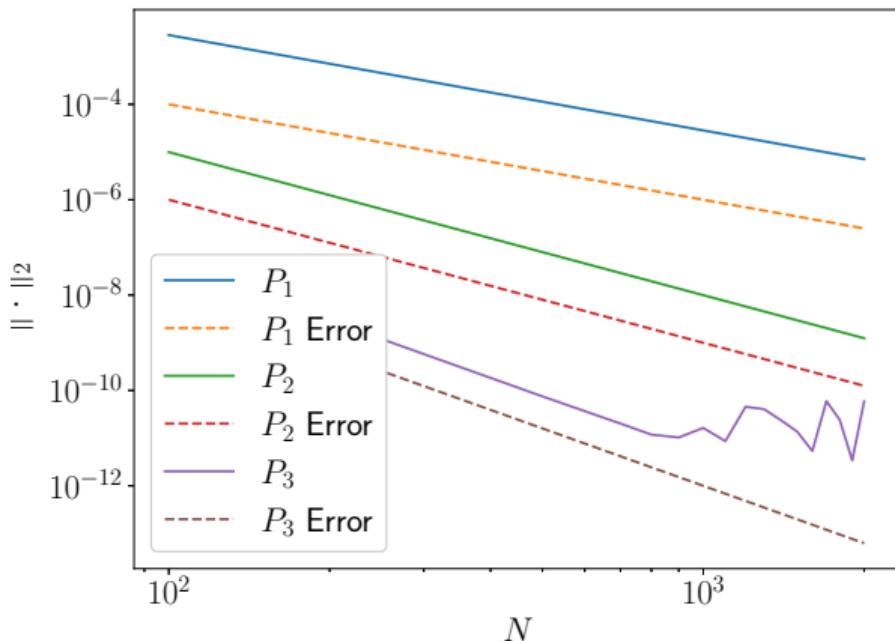
The **error** of the **conforming Galerkin** approach is **determined** by the **approximation error** of the exact solution **in** V_h .

Error estimate on 1D problem

The **test problem** we consider is

$$\begin{cases} -u_{xx} = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega \end{cases}$$

for $f(x) = 2 \cos(x)/e^x$ and $g(x) = \sin(x)/e^x$ on $\Omega = (0, 10)$. We discretize it with Lagrangian 1, 2 and 3 elements and report the computed error: $\|u - u_{\text{ex}}\|_{L^2(\Omega_h)}$ on the uniform grid with N points.



FEniCSx Code Example

We can implement this simple case in the FEniCSx Library in few lines of code

1. First we need to load some packages

```
from mpi4py import MPI # Needed for the MPI environment
import numpy as np # The numpy package support
from dolfinx import mesh # Handler for the meshes
from dolfinx import fem # FEM building blocks
from dolfinx.fem import FunctionSpace # FEM Function Spaces
import ufl # Language for building up variational formulations
```

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```
nx = 500
Omegah = mesh.create_interval(comm=MPI.COMM_WORLD, nx=nx,
    ↪ points=(0,10))
V = FunctionSpace(Omegah, ("CG", 1))
```

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3. Then we need a bit of work to impose **essential boundary conditions**

```
g = fem.Function(V)
g.interpolate(lambda x: np.sin(x[0])/np.exp(x[0]))
tdim = Omegah.topology.dim
fdim = tdim - 1
Omegah.topology.create_connectivity(fdim, tdim)
boundary_facets =
    np.flatnonzero(mesh.compute_boundary_facets(Omegah.topology))
boundary_dofs = fem.locate_dofs_topological(V, fdim,
    boundary_facets)
bc = fem.dirichletbc(g, boundary_dofs)
```

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```
u = ufl.TrialFunction(V)
v = ufl.TestFunction(V)
```

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5. We build the source and the **variational formulation**

```
f = fem.Function(V)
f.interpolate(lambda x: 2.0*np.cos(x[0])/np.exp(x[0]))
a = ufl.dot(ufl.grad(u), ufl.grad(v)) * ufl.dx
F = f * v * ufl.dx
```

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6. Finally we **solve** the **linear system** (directly...it's 1D!)

```
problem = fem.petsc.LinearProblem(a, F, bcs=[bc],  
    ↳ petsc_options={"ksp_type": "preonly", "pc_type": "lu"})  
uh = problem.solve()
```

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7. and **compute the error**: Error_L2 : 1.14e-04

```
V2 = fem.FunctionSpace(Omegah, ("CG", 2))
uex = fem.Function(V2)
uex.interpolate(lambda x: np.sin(x[0])/np.exp(x[0]))
L2_error = fem.form(ufl.inner(uh - uex, uh - uex) * ufl.dx)
error_local = fem.assemble_scalar(L2_error)
error_L2 = np.sqrt(Omegah.comm.allreduce(error_local, op=MPI.SUM))
```

FEniCSx Code Example

We can implement this simple case in the FEniCSx Library in few lines of code

1. First we need to load some packages
2. Then we **build** the **mesh** and the **function space**
3. Then we need a bit of work to impose **essential boundary conditions**
4. We create **test** and **trial functions**
5. We build the source and the **variational formulation**
6. Finally we **solve** the **linear system** (directly...it's 1D!)
7. and **compute the error**: Error_L2 : 1.14e-04

To **run** the example there is a  Python notebook using FEniCSx shared through
 bit.ly/3tTEBfl (and executed on **Google Colab**).

FEM Spaces

We can build many different types of Finite Elements.

FE Definition (Ciarlet, 1978)

A *finite element* is a triple $(K, \mathcal{P}, \mathcal{N})$ where

- (i) $K \subset \mathbb{R}^n$ is a simply connected bounded open set with piecewise smooth boundary (*element domain*);
- (ii) \mathcal{P} is a finite-dimensional space of functions defined on K (*space of shape functions*);
- (iii) $\mathcal{N} = \{N_1, \dots, N_d\}$ is a basis of \mathcal{P}^* (*degrees of freedom*).

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Dual basis definition

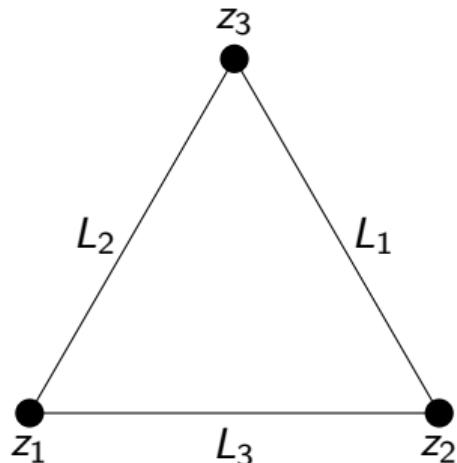
Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. A basis $\{\psi_1, \dots, \psi_d\}$ of \mathcal{P} is called *dual basis* or *nodal basis* to \mathcal{N} if $N_i(\psi_j) = \delta_{ij}$.

A lineup of some usual (and unusual) suspects

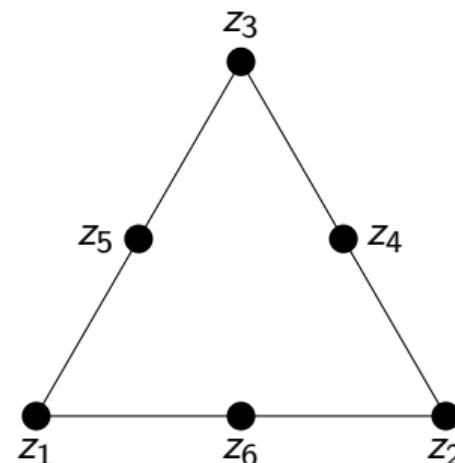


FEM Spaces: triangular finite elements

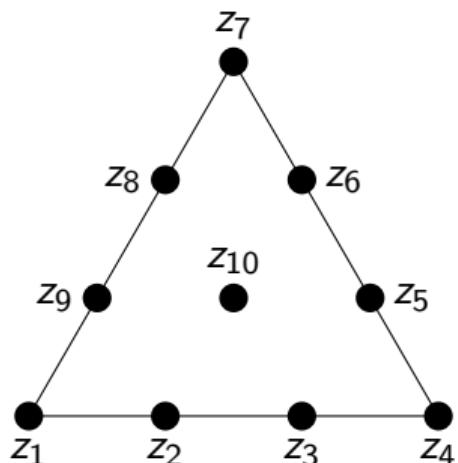
K any triangle, space \mathcal{P}_k of bivariate polynomials of degree $\leq k$,



Linear Lagrange element



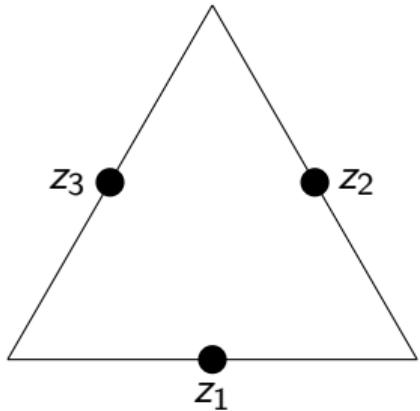
Quadratic Lagrange element



Cubic Lagrange element

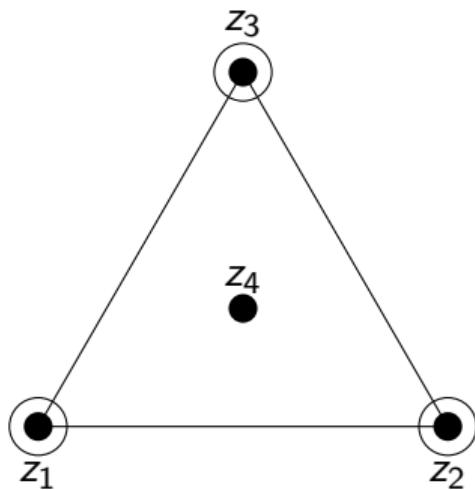
“●” Point evaluations determining the $\mathcal{N} = \{\mathcal{N}_1, \dots, \mathcal{N}_{\frac{1}{2}(k+1)(k+2)}\}$.

FEM Spaces: triangular finite elements

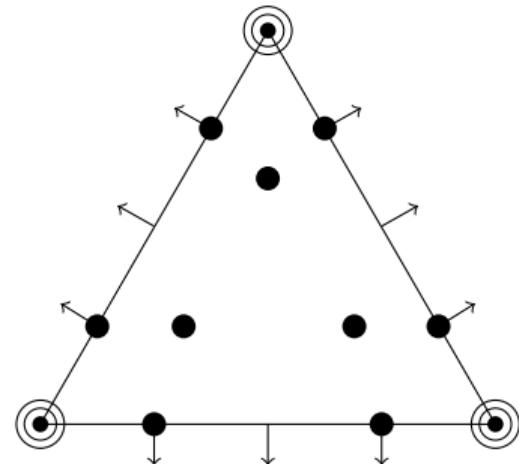


Linear nonconforming
Crouzeix-Raviart element

- Point evaluations, ○ Gradient evaluations, ⊙ Three second derivative, ↑ Normal derivative.



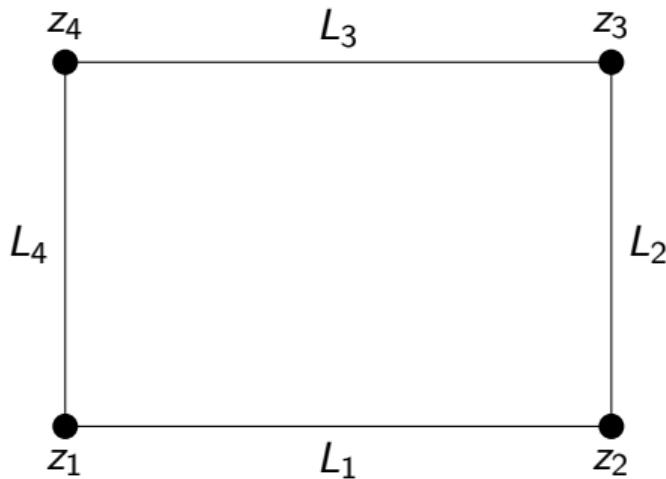
Cubic Hermite element



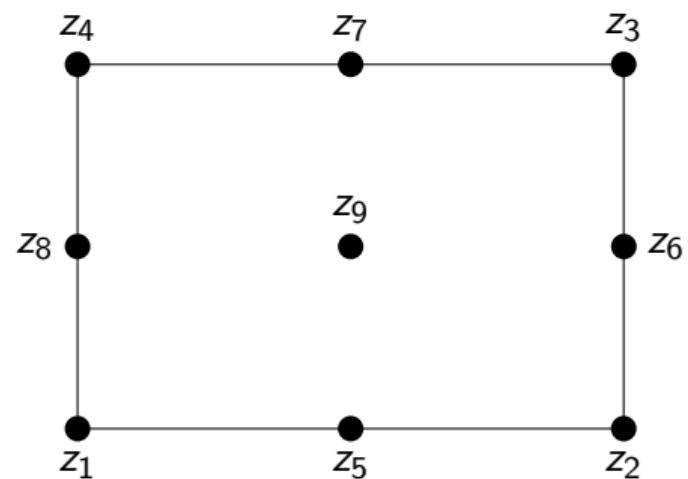
Quintic Argyris element

FEM Spaces: rectangular finite elements

K any rectangle, space $\mathcal{Q}_k = \left\{ \sum_j c_j p_j(x) q_j(x), \ p_j, q_j \in \mathbb{P}_{\leq k}[x] \right\}$,



Bilinear Lagrange element



Biquadratic Lagrange element

- Point evaluations for $\mathcal{N} = \{N_1, \dots, N_d\}$, $d = \dim \mathcal{Q}_k = (\dim \mathbb{P}_{\leq k}[x])^2$.

Periodic Table of the Finite Elements



$\text{d}=0$	$\text{d}=1$	$\text{d}=2$	$\text{d}=3$
\mathcal{P}_1	$\mathcal{T}_1, \mathcal{P}_1, \mathcal{A}_1$	$\mathcal{P}_1, \mathcal{P}_1, \mathcal{A}_1$	$\mathcal{P}_1, \mathcal{P}_1, \mathcal{A}_1$
\mathcal{P}_2	$\mathcal{T}_2, \mathcal{P}_2, \mathcal{A}_2$	$\mathcal{P}_2, \mathcal{P}_2, \mathcal{A}_2$	$\mathcal{P}_2, \mathcal{P}_2, \mathcal{A}_2$
\mathcal{P}_3	$\mathcal{T}_3, \mathcal{P}_3, \mathcal{A}_3$	$\mathcal{P}_3, \mathcal{P}_3, \mathcal{A}_3$	$\mathcal{P}_3, \mathcal{P}_3, \mathcal{A}_3$
$\mathcal{Q}_r \Lambda^k$	$\mathcal{T}_r, \mathcal{P}_r, \mathcal{A}_r$	$\mathcal{P}_r, \mathcal{P}_r, \mathcal{A}_r$	$\mathcal{P}_r, \mathcal{P}_r, \mathcal{A}_r$
$\mathcal{S}_r \Lambda^k$	$\mathcal{T}_r, \mathcal{P}_r, \mathcal{A}_r$	$\mathcal{P}_r, \mathcal{P}_r, \mathcal{A}_r$	$\mathcal{P}_r, \mathcal{P}_r, \mathcal{A}_r$

Plane elements

Plane elements are defined on quadrilaterals and triangles. They are divided into three categories: **Shape functions**, **Degrees of freedom**, and **Integration points**.

Volume elements

Volume elements are defined on hexahedra and tetrahedra. They are divided into three categories: **Shape functions**, **Degrees of freedom**, and **Integration points**.

References

1. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
2. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
3. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
4. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
5. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
6. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
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8. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
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10. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.
11. S. R. Vollmer, "A Periodic Table of Finite Elements," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3370–3388, 2010.

FEM Spaces: it's a vast world

Much of what we discussed and of what we are going to discuss in the next slides can be applied to *FEM-adjacent* methods, a (obviously not exhaustive) list of ideas:

DG: Discontinuous Galerkin, (Cockburn, Karniadakis, and Shu 2000) for a general overview, linear solvers (Ayuso de Dios et al. 2014; Dobrev et al. 2006)...

IgA: Isogeometric Analysis, (Cottrell, Hughes, and Bazilevs 2009) for a general overview, adaptive meshes (Giannelli, Jüttler, and Speleers 2012; Patrizi and Dokken 2020), linear solvers (Donatelli et al. 2015; Horníková, Vuik, and Egermaier 2021; Sangalli and Tani 2016)...

VEM: Virtual Elements, (Beirão da Veiga et al. 2014, 2016) for a general overview, linear solvers (Antonietti, Mascotto, and Verani 2018; Dassi and Scacchi 2020)...

Another nice source of information is: defelement.com.

Variational crimes



“The crime is now logical and reasonable.”

Murder for Christmas, A. Christie

The Penal Code

- 💣 *Petrov–Galerkin* approaches, where the function u satisfying $a(u, v)$ for all $v \in V$ is an element of $U \neq V$;
- 💣 *non-conforming* approaches, where the discrete spaces U_h and V_h are not subspaces of U and V , respectively; and
- 💣 *non-consistent* approaches, where the discrete problem involves a bilinear form $a_h \neq a$ (and a_h might not be well-defined for all $u \in U$).

We thus need a **more general framework** that covers these cases as well.

- U, V be Banach spaces, with V reflexive, U^*, V^* denote their topological duals
- Given $a : U \times V \rightarrow \mathbb{R}$ bilinear, $F \in V^*$ continuous we look for $u \in U$ satisfying

$$a(u, v) = F(v) \quad \text{for all } v \in V. \tag{W}$$

Existence and uniqueness in a world full of crimes

Theorem Banach–Nečas–Babuška

Let U and V be Banach spaces and V be reflexive. If $a : U \times V \rightarrow \mathbb{R}$ and $F : V \rightarrow \mathbb{R}$ satisfy:

- (i) *Inf-sup condition*: there exists a $c_1 > 0$ such that

$$\inf_{u \in U} \sup_{v \in V} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq c_1.$$

- (ii) *Continuity*: there exist c_2, c_3 such that

$$|a(u, v)| \leq c_2 \|u\|_U \|v\|_V, \quad |F(v)| \leq c_3 \|v\|_V, \quad \forall u \in U, \forall v \in V$$

- (iii) *Injectivity*: for any $v \in V$ $a(u, v) = 0$ for all $u \in U$ implies $v = 0$.

Then there exists a unique solution $u \in U$ to (\mathcal{W}) , which satisfies

$$\|u\|_U \leq \frac{1}{c_1} \|F\|_{V^*}.$$

Mixed Methods - The Poisson equation

Let us start again from the **Poisson equation** with homogeneous Dirichlet conditions

$$\begin{cases} -\Delta u = -\nabla \cdot \nabla u = -\nabla^2 u = -\operatorname{div} \operatorname{grad} u = f, & \mathbf{x} \in \Omega \subset \mathbb{R}^n \\ u = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

We introduce an **auxiliary variable** $\sigma = \nabla u \in \mathbb{L}^2(\Omega)^n$ and rewrite it as

$$\begin{cases} \nabla u - \sigma = 0, \\ -\nabla \cdot \sigma = f. \end{cases}$$

This system can be formulated in variational form in two different ways:

1. we *formally* integrate by parts in the second equation \Rightarrow *primal* approach,
2. we *formally* integrate by parts in the first equation \Rightarrow *dual* approach.

Mixed Methods - The Poisson equation - Primal

We look for $(\sigma, u) \in \mathbb{L}^2(\Omega)^n \times \mathbb{H}_0^1(\Omega)$ satisfying

$$\begin{cases} (\sigma, \tau) - (\tau, \nabla u) = 0 & \text{for all } \tau \in \mathbb{L}^2(\Omega)^n, \\ -(\sigma, \nabla v) = -(f, v) & \text{for all } v \in \mathbb{H}_0^1(\Omega). \end{cases}$$

that we can restate in **abstract form** as

$$a(\sigma, \tau) = (\sigma, \tau) : V \times V \rightarrow \mathbb{R}, \quad b(v, \mu) = -(v, \nabla \mu) : V \times M \rightarrow \mathbb{R},$$

on the two (reflexive) Banach spaces $V = \mathbb{L}^2(\Omega)^n$ and $M = \mathbb{H}_0^1(\Omega)$ for the problem

Find u, λ s.t. $\begin{cases} a(u, v) + b(v, \lambda) = \langle f, v \rangle_{V^*, V} & \text{for all } v \in V, \\ b(u, \mu) = \langle g, \mu \rangle_{M^*, M} & \text{for all } \mu \in M. \end{cases}$

Mixed Methods - Abstract Saddle-Point formulation

To uncover the connection with the discrete case we are aiming at, let us reformulate the previous in operator form by introducing

$$\begin{aligned} A : V &\rightarrow V^*, \quad \langle Au, v \rangle_{V^*, V} = a(u, v) \quad \text{for all } v \in V, \\ B : V &\rightarrow M^*, \quad \langle Bu, \mu \rangle_{M^*, M} = b(u, \mu) \quad \text{for all } \mu \in M, \\ B^* : M &\rightarrow V^*, \quad \langle B^*\lambda, v \rangle_{V^*, V} = b(v, \lambda) \quad \text{for all } v \in V. \end{aligned}$$

From which we rewrite our problem as

$$\text{Find } u, \lambda \text{ s.t. } \begin{cases} Au + B^*\lambda = f & \text{in } V^*, \\ Bu = g & \text{in } M^*. \end{cases}$$

At this stage, this should be very familiar!

Mixed Methods - Abstract Saddle-Point formulation

Abstract Saddle-Point

$$\text{Find } u, \lambda \text{ s.t. } \begin{cases} Au + B^* \lambda = f & \text{in } V^*, \\ Bu = g & \text{in } M^*. \end{cases}$$

- If B is *invertible* \Rightarrow **existence** and **uniqueness** first of u and then of λ follow immediately,

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- If B is *invertible* \Rightarrow **existence** and **uniqueness** first of u and then of λ follow immediately,
- Usually, we **are not this lucky**, remember our **starting example**:

$$\langle B\sigma, \mu \rangle_{(\mathbb{H}_0^1)^*, \mathbb{H}_0^1} = b(\sigma, \mu) = -(\sigma, \nabla \mu),$$

that is we have to **require** that A is **injective** and **coercive** on $\ker B$ to obtain a **unique** u ...

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that is we have to **require** that A is **injective** and **coercive** on $\ker B$ to obtain a **unique** u ...

- ...for the **existence** of λ we then need B^* to be **surjective**.

Mixed Methods - Banach–Nečas–Babuška

Theorem (Continuous Brezzi)

We assume that

- (i) $a : V \times V \rightarrow \mathbb{R}$ satisfies the conditions of the Banach–Nečas–Babuška Theorem for $U = V = \ker B$
- (ii) $b : V \times M \rightarrow \mathbb{R}$ is such that the Ladyzhenskaya–Babuška–Brezzi condition holds

$$\exists \beta > 0 : \inf_{\mu \in M} \sup_{v \in V} \frac{b(v, \mu)}{\|v\|_V \|\mu\|_M} \geq \beta.$$

$\Rightarrow \exists! (u, \lambda) \in V \times M$ solving the mixed saddle-point system and satisfying

$$\|u\|_V + \|\lambda\|_M \leq C(\|f\|_{V^*} + \|g\|_{M^*}).$$

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- ☛ $a(u, v)$ has to satisfy the BNB condition only on $\ker B$, not on all of V !
- ☛ LBB condition couples V and M spaces, this is going to have repercussions in a moment!

Mixed Methods - Back to Poisson

Mixed Continuous Primal Poisson Problem

Find $(\sigma, u) \in \mathbb{L}^2(\Omega)^n \times \mathbb{H}_0^1(\Omega)$ s.t.

$$\begin{cases} (\sigma, \tau) - (\tau, \nabla u) = 0 & \forall \tau \in \mathbb{L}^2(\Omega)^n, \\ -(\sigma, \nabla v) = -(f, v) & \forall v \in \mathbb{H}_0^1(\Omega). \end{cases}$$

Coercivity: a is coercive on V with constant $\alpha = 1$,

LBB: chose $v \in \mathbb{H}_0^1(\Omega) = M$ and take $\tau = -\nabla v \in \mathbb{L}^2(\Omega)^n = V$, then

$$\sup_{\tau \in V} \frac{b(\tau, v)}{\|\tau\|_V} = \sup_{\tau \in V} \frac{-(\tau, \nabla v)}{\|\tau\|_{\mathbb{L}^2(\Omega)^n}} \geq \frac{(\nabla v, \nabla v)}{\|\nabla v\|_{\mathbb{L}^2(\Omega)^n}} = |v|_{\mathbb{H}^1} \geq c_\Omega^{-1} \|v\|_M$$

❶ to get C_Ω^{-1} we use **Poincaré inequality**: for $1 \leq p < \infty$, Ω an open bounded set \Rightarrow

$\exists c_\Omega : \|f\|_{W_p^1(\Omega)} \leq c_\Omega |f|_{W_p^1(\Omega)}$ depending only on p and Ω .

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Mixed Methods - Galerkin Approach

Abstract Saddle-Point

Find u, λ s.t. $\begin{cases} Au + B^* \lambda = f & \text{in } V^*, \\ Bu = g & \text{in } M^*. \end{cases}$

- $V_h \subset V, M_h \subset M,$
- 👑 V_h and M_h **cannot be** selected independently!

Mixed Methods - Galerkin Approach

Abstract Discrete Saddle-Point

Find u_h, λ_h such that

$$\begin{cases} a(u_h, v_h) + b(v_h, \lambda_h) = \langle f, v_h \rangle_{V^*, V} \quad \forall v_h \in V_h, \\ b(u_h, \mu_h) = \langle g, \mu_h \rangle_{M^*, M} \quad \forall \mu_h \in M_h. \end{cases}$$

- $V_h \subset V, M_h \subset M,$
- V_h and M_h **cannot be** selected independently!

Theorem (Discrete Brezzi)

$$\text{If } \exists \alpha_h > 0 : \inf_{u_h \in \ker B_h} \sup_{v_h \in \ker B_h} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} \geq \alpha_h,$$

$$\text{If } \exists \beta_h > 0 : \inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \mu_h)}{\|v_h\|_V \|\mu_h\|_M} \geq \beta_h.$$

$\Rightarrow \exists! (u_h, \lambda_h) \in V_h \times M_h$ solving the discrete saddle-point and satisfying

$$\|u_h\|_V + \|\lambda_h\|_M \leq C_h(\|f\|_{V^*} + \|g\|_{M^*}).$$

Mixed Methods - Dual approach

We integrate by parts the **first equation**:

$$\int_{\Omega} (\operatorname{div} \tau) w \, dx + \int_{\Omega} \tau \cdot \nabla w \, dx = \int_{\partial\Omega} (\tau \cdot \nu) w \, dx$$

We need to define the **proper Sobolev space**.

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$\mathbb{H}(\operatorname{div})$

We define the space

$$\mathbb{H}(\operatorname{div}) = \{\tau \in \mathbb{L}^2(\Omega)^n : \operatorname{div} \tau \in \mathbb{L}^2(\Omega)\},$$

with the norm

$$\|\tau\|_{\mathbb{H}(\operatorname{div})}^2 := \|\tau\|_{\mathbb{L}^2(\Omega)^n}^2 + \|\operatorname{div} \tau\|^2.$$

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Well posedness of the *normal trace*.

$\mathcal{C}^\infty(\overline{\Omega})^n$ is dense in $\mathbb{L}^2(\Omega)^n \supset \mathbb{H}(\operatorname{div}) \Rightarrow \tau \in \mathbb{H}(\operatorname{div})$ has $(\tau|_{\partial\Omega} \cdot \nu) \in \mathbb{H}^{-1/2}(\partial\Omega)$.

Mixed Methods - Dual approach

The **dual problem** is then

$$\text{find } (\sigma, u) \in \mathbb{H}(\text{div}) \times \mathbb{L}^2 \text{ s.t. } \begin{cases} (\sigma, \tau) + (\text{div } \tau, u) = 0 & \forall \tau \in \mathbb{H}(\text{div}), \\ (\text{div } \sigma, v) = -(f, v) & \forall v \in \mathbb{L}^2. \end{cases}$$

- we have used that $u|_{\partial\Omega} = 0$,

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- we have used that $u|_{\partial\Omega} = 0$,
- This is a **general saddle problem** with $V = \mathbb{H}(\text{div})$, and $M = \mathbb{L}^2$:

$$a(\sigma, \tau) = (\sigma, \tau), \quad b(\sigma, v) = (\text{div } \sigma, v).$$

and a and b bounded by Cauchy–Schwarz inequality.

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$$a(\sigma, \tau) = (\sigma, \tau), \quad b(\sigma, v) = (\text{div } \sigma, v).$$

and a and b bounded by Cauchy–Schwarz inequality.

- For the **existence of the solution** we need to prove **coercivity** for a .

$$\ker B = \{\tau \in \mathbb{H}(\text{div}) : (\text{div } \tau, v) = 0 \forall v \in \mathbb{L}^2\}$$

Since $\text{div } \tau \in \mathbb{L}^2$ we have $\|\text{div } \tau\|_{\mathbb{L}^2} = 0$ whenever $\tau \in \ker B \subset \mathbb{H}(\text{div})$, and therefore

$$a(\tau, \tau) = \|\tau\|_{\mathbb{L}^2(\Omega)^n}^2 = \|\tau\|_{\mathbb{H}(\text{div})}^2 \quad \forall \tau \in \ker B,$$

indeed we have just proved **coercivity with $\alpha = 1$** .

Mixed Methods - Dual approach

The **dual problem** is then

$$\text{find } (\sigma, u) \in \mathbb{H}(\text{div}) \times \mathbb{L}^2 \text{ s.t. } \begin{cases} (\sigma, \tau) + (\text{div } \tau, u) = 0 & \forall \tau \in \mathbb{H}(\text{div}), \\ (\text{div } \sigma, v) = -(f, v) & \forall v \in \mathbb{L}^2. \end{cases}$$

- we have used that $u|_{\partial\Omega} = 0$,
- This is a **general saddle problem** with $V = \mathbb{H}(\text{div})$, and $M = \mathbb{L}^2$:

$$a(\sigma, \tau) = (\sigma, \tau), \quad b(\sigma, v) = (\text{div } \sigma, v).$$

and a and b bounded by Cauchy–Schwarz inequality.

- The form a is coercive on $\ker B$ with $\alpha = 1$,
- We now need to verify the **LBB condition**. This requires some work.

Mixed Methods - Dual approach - LBB

Assumption:

We make the simplifying assumption of having $\partial\Omega$ represented by a C^1 function or, analogously, having Ω convex.

Lemma (Surjectivity)

For any $f \in \mathbb{L}^2$, there exists a function $\tau \in \mathbb{H}(\text{div})$ with $\text{div } \tau = f$ and $\|\tau\|_{\mathbb{H}(\text{div})} \leq C\|f\|_{\mathbb{L}^2}$.

- The space $\mathbb{H}^1(\Omega)^n \subset \mathbb{H}(\text{div})$, thus if we take a $v \in M = \mathbb{L}^2$ and the corresponding $\tau_v \in \mathbb{H}(\text{div})$ given by the surjectivity lemma (i.e., $\text{div } \tau_v = v$) we find

$$\sup_{\tau \in V} \frac{b(\tau, v)}{\|\tau\|_V} = \sup_{\tau \in V} \frac{(\text{div } \tau, v)}{\|\tau\|_{\mathbb{H}(\text{div})}} \geq \frac{(\tau_v, v)}{\|\tau_v\|_{\mathbb{H}(\text{div})}} \geq \frac{(v, v)}{C\|v\|_{\mathbb{L}^2(\Omega)}} = \frac{1}{c}\|v\|_{\mathbb{L}^2(\Omega)}.$$

We have then proved the LBB condition for $\beta = \frac{1}{C}$.

Mixed Methods - Dual approach - LBB

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- We have then proved the LBB condition for $\beta = \frac{1}{C}$.
- By **Continuous Brezzi** we have $\exists! (\sigma, u) \in V \times M$ solving the **saddle problem** and such that

$$\|\sigma\|_{\mathbb{H}(\text{div})} + \|u\|_{\mathbb{L}^2(\Omega)} \leq C\|f\|_{\mathbb{L}^2(\Omega)}.$$

Mixed Methods - Dual approach - LBB

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- The solution u we have obtained seem to be only in \mathbb{L}^2 ...but u satisfies

$$(\sigma, \tau) + (\text{div } \tau, u) = 0,$$

thus an **integration by parts** shows that u has a weak derivative and satisfies boundary conditions, that is, u is where it should be $u \in \mathbb{H}_0^1(\Omega)$.

Mixed Methods - Galerkin Approach for Poisson

Let us build the **discrete problem**.

A property of the form

We can (but won't) show that for any partition Ω_h of Ω

$$\{\tau \in \mathbb{L}^2(\Omega)^n : \tau|_{\Omega_j} \in \mathbb{H}^1(\Omega_j) \text{ and } \tau|_{\Omega_j} \cdot \hat{\mathbf{n}} = \tau_{\Omega_i} \cdot \hat{\mathbf{n}} \forall \overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset\} \subset \mathbb{H}^1(\Omega).$$

In layman terms, piecewise differentiable functions with continuous normal traces across elements are in $\mathbb{H}^1(\Omega)$.

This observation is crucial for building **conformal FEM spaces** for this problem.

- We could consider are the *Raviart-Thomas elements* (Raviart and Thomas 1977),
- The other usual option are the **Brezzi-Douglas-Marini elements** (Brezzi, Douglas, and Marini 1985),

Mixed Methods - Galerkin Approach for Poisson

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- We could consider are the *Raviart-Thomas elements* (Raviart and Thomas 1977),
- The other usual option are the **Brezzi-Douglas-Marini elements** (Brezzi, Douglas, and Marini 1985),
- To build the matrices we will use the code from (Zhang 2015).

Mixed Methods - Test problem

We consider a **more general formulation** of the Poisson problem

$\Omega \subset \mathbb{R}^2$ a polygonal domain, with boundary

$$\partial\Omega = \Gamma_D \cap \Gamma_N \quad (\Gamma_D \cap \Gamma_N = \emptyset, \mu(\Gamma_D) \neq 0)$$

$$\begin{cases} -\nabla \cdot (\alpha(x)\nabla u) = f, & \text{in } \Omega, \\ -\alpha\nabla u \cdot \hat{\mathbf{n}} = g_N, & \text{on } \Gamma_N, \\ u = g_D, & \text{on } \Gamma_D. \end{cases}$$

With

- $f \in \mathbb{L}^2(\Omega)$,
- $g_D \in \mathbb{H}^{1/2}(\Gamma_D)$ and $g_N \in \mathbb{L}^2(\Gamma_N)$,
- $\alpha(x)$ positive **piecewise constant**.

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The weak form is then for
 $(\tau, v) \in \mathbb{H}_N(\text{div}) \times \mathbb{L}^2(\Omega)$

$$\begin{cases} (\alpha^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\text{div } \boldsymbol{\tau}, u) = -(\boldsymbol{\tau} \cdot \hat{\mathbf{n}}, g_D)_{\Gamma_D} \\ (\text{div } \boldsymbol{\sigma}, v) = (f, v) \end{cases}$$

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where

- $\boldsymbol{\sigma} = -\alpha(x)\nabla u$ is the **flux**,
- $\mathbb{H}_N(\text{div}) = \{\boldsymbol{\tau} \in \mathbb{H}(\text{div}) : \boldsymbol{\tau} \cdot \hat{\mathbf{n}} = 0 \text{ on } \Gamma_N\}$.

Mixed Methods - Test problem

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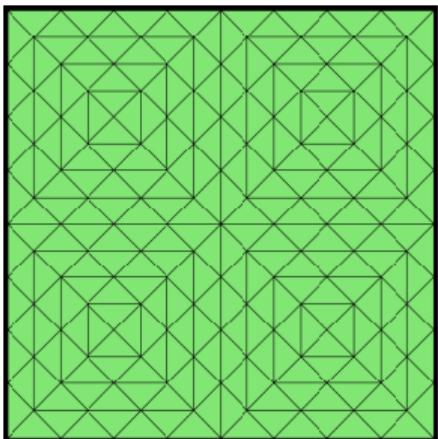
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- $\boldsymbol{\sigma} = -\alpha(x)\nabla u$ is the **flux**,
- $\mathbb{H}_N(\text{div}) = \{\boldsymbol{\tau} \in \mathbb{H}(\text{div}) : \boldsymbol{\tau} \cdot \hat{\mathbf{n}} = 0 \text{ on } \Gamma_N\}$.

Existence theory is not substantially different, just longer to write, see (Boffi, Brezzi, and Fortin 2013, Chapter 7).

Mixed Methods - Galerkin Approach for Poisson

To apply the **discrete version of Brezzi's Theorem**, for which we select the Brezzi-Douglas-Marini and piecewise constant elements to build our mixed space.



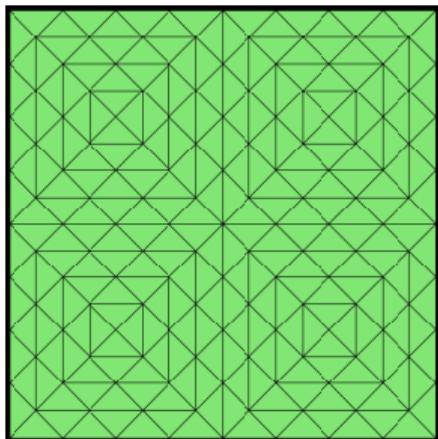
Mesh: shape-regular affine triangulation Ω_h

- **Mesh:**

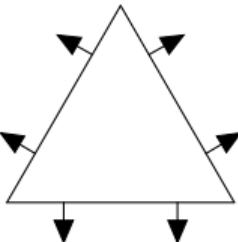
```
RefinementLevels = 2;
node = [-1 1; 0 1; 1 1; -0.5 0.5; 0.5 0.5; -1 0; 0 0; 1 0;
         ↪ -0.5 -0.5; 0.5 -0.5;-1.0 -1.0; 0.0 -1.0;1.0 -1.0];
elem =[4 2 1;4 1 6;4 6 7;4 7 2;5 3 2;5 2 7;5 7 8;5 8 3;9
         ↪ 7 6; 9 6 11;9 11 12;9 12 7;10 8 7;10 7 12;10 12 13;10
         ↪ 13 8];
bdEdge = [2 0 0;1 0 0; 0 0 0;0 0 0;2 0 0;0 0 0;0 0 0;0 0 0;1 0
         ↪ 0; 0 0 0;1 0 0;1 0 0;0 0 0;0 0 0;0 0 0;1 0 0;1 0 0];
for i=1:RefinementLevels
    [node,elem,bdEdge] = uniformrefine(node,elem,bdEdge);
end
```

Mixed Methods - Galerkin Approach for Poisson

To apply the **discrete version of Brezzi's Theorem**, for which we select the Brezzi-Douglas-Marini and piecewise constant elements to build our mixed space.



Mesh: shape-regular affine triangulation Ω_h



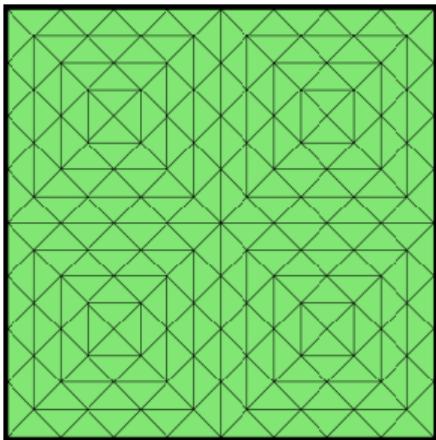
- **Mesh:** Ω_h
- The BDM_1 elements

In general, $\mathbf{q} \in BDM_k = (\mathbb{P}_k)^2$, thus $\operatorname{div} \mathbf{q} \in \mathbb{P}_{k-1}$, and to complete the definition we impose the values on the normal trace $\phi = \mathbf{q} \cdot \hat{\mathbf{n}}$ on ∂K belonging to

$$\{\phi \mid \phi \in \mathbb{L}^2(K), \phi|_{\partial\Omega} \in \mathbb{P}_k\}.$$

Mixed Methods - Galerkin Approach for Poisson

To apply the **discrete version of Brezzi's Theorem**, for which we select the Brezzi-Douglas-Marini and piecewise constant elements to build our mixed space.



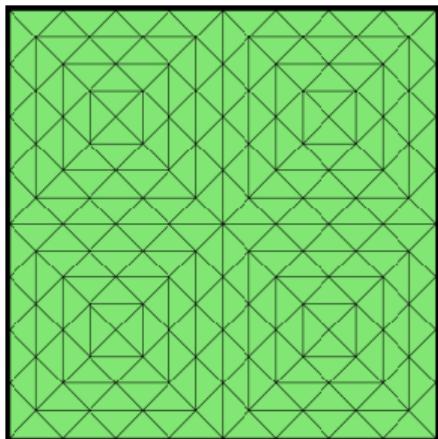
- **Mesh:** Ω_h
- The BDM_1 elements
- The P_0 elements

$$P_0 = \{v : v|_K \in \mathbb{P}_0(K), \quad \forall K \in \Omega_h\}.$$

Mesh: shape-regular affine triangulation Ω_h

Mixed Methods - Galerkin Approach for Poisson

To apply the **discrete version of Brezzi's Theorem**, for which we select the Brezzi-Douglas-Marini and piecewise constant elements to build our mixed space.

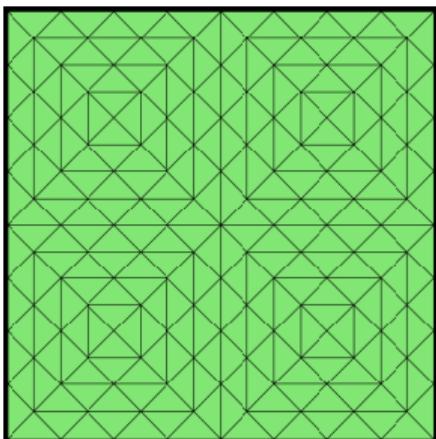


Mesh: shape-regular affine triangulation Ω_h

- **Mesh:** Ω_h
- The BDM_1 elements $\rightsquigarrow V_h$
- The P_0 elements $\rightsquigarrow M_h$
- For the convergence analysis see (Brezzi, Douglas, and Marini 1985, Section 3 and 4).

Mixed Methods - Galerkin Approach for Poisson

To apply the **discrete version of Brezzi's Theorem**, for which we select the Brezzi-Douglas-Marini and piecewise constant elements to build our mixed space.



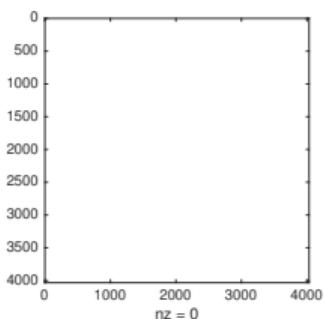
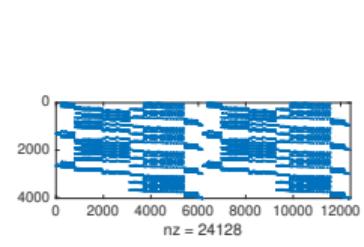
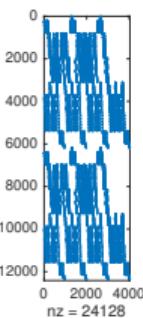
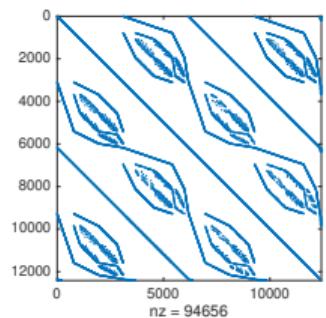
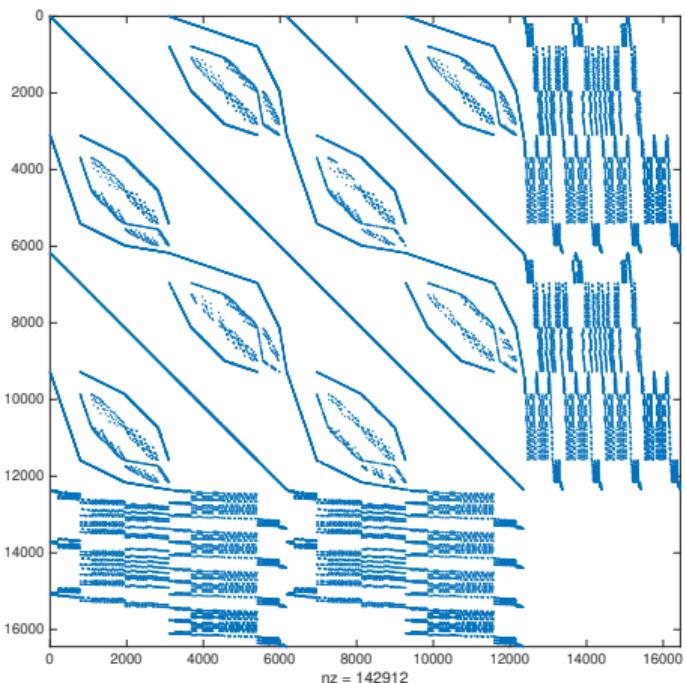
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- **Mesh:** Ω_h
- The BDM_1 elements
- The P_0 elements
- For the convergence analysis see (Brezzi, Douglas, and Marini 1985, Section 3 and 4).
- And look at the code for assembling the matrix

```
NT = size(elem,1); % Number of triangles  
NE = size(edge,1); % Number of edges  
sol = zeros(2*NE+NT,1); % Space to store the solution  
inva = 1./exactalpha((node(elem(:,1)) + node(elem(:,2)) +  
    ↳ node(elem(:,3)))/3);  
[a,b,area] = gradlambda(node,elem);  
M = assemblebdm(NT,NE,a,b,area,elem2edge,signedge,inva);
```

Mixed Methods - The Saddle-Point Matrix

We can finally look at our first **saddle-point** matrix for the Poisson problem.



Mixed Methods - Eigenvalue Bounds

One of the results you have seen in the **morning lectures** concerns eigenvalue bounds for these matrices. Let us look at it numerically.

Theorem (Rusten and Winther 1992)

Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$ be the eigenvalues of A , $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m > 0$ the singular values of B . If we denote as $\sigma(\mathcal{A})$ the spectrum of \mathcal{A} , then

$$\sigma(\mathcal{A}) \subset I = I^- \cup I^+,$$

where

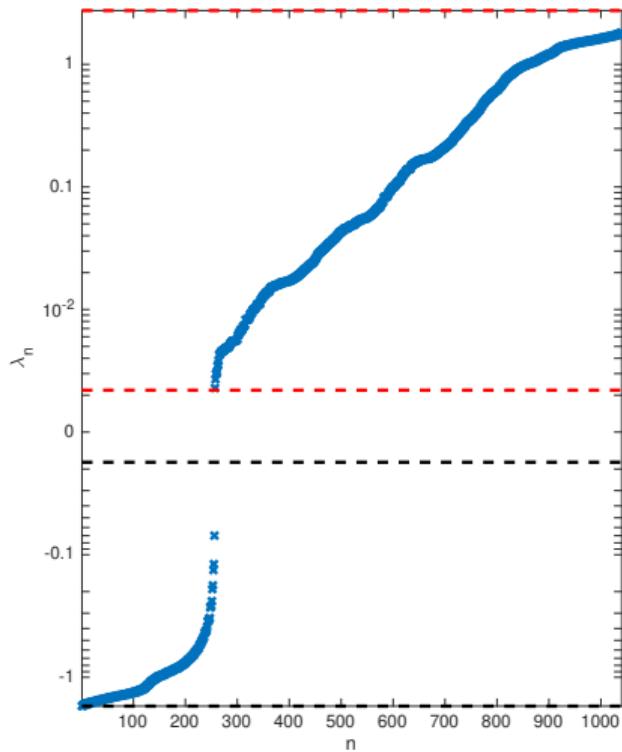
$$I^- = \left[\frac{1}{2} \left(\mu_n - \sqrt{\mu_n^2 + 4\sigma_1^2} \right), \frac{1}{2} \left(\mu_1 - \sqrt{\mu_1^2 + 4\sigma_m^2} \right) \right],$$

$$I^+ = \left[\mu_n, \frac{1}{2} \left(\mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right) \right].$$

Mixed Methods - Eigenvalue Bounds

We can compute the bounds with few lines of code:

```
lambda = eig(M(freeDof,freeDof));  
mun = eigs(A,1,'smallestabs');  
mu1 = eigs(A,1,'largestabs');  
sigma1 = svds(BT,1,'largest');  
sigmam = svds(BT,1,'smallest');  
  
Iminus(1) = 0.5*(mun - sqrt(mun^2+4*sigma1^2));  
Iminus(2) = 0.5*(mu1 - sqrt(mu1^2+4*sigmam^2));  
Iplus(1) = mun;  
Iplus(2) = 0.5*(mu1 + sqrt(mu1^2 +  
    4*sigma1^2));
```



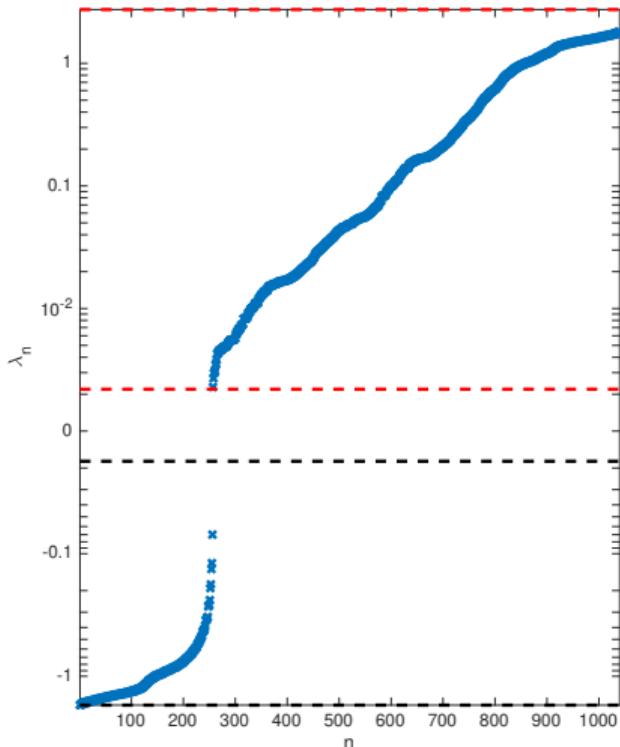
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Iplus(1) = mun;
Iplus(2) = 0.5*(mu1 + sqrt(mu1^2 +
    4*sigma1^2));
```

 **Next week**, after you become familiar with iterative methods, we will focus on **preconditioning**.



Mixed Methods - The Stokes equation

Let us consider the **Stokes equations** for the *steady flow* of a **very viscous fluid**

$$\begin{cases} -\nabla^2 \mathbf{u} + \nabla p = \mathbf{0}, & \text{Momentum equation,} \\ \nabla \cdot \mathbf{u} = 0, & \text{Incompressibility constraint.} \end{cases}$$

- \mathbf{u} is a *vector-valued function* representing the velocity of the fluid,
- p is a *scalar* function representing the pressure.

Modeling assumption

The crucial **modeling assumption** is that the flow is “low speed” we **neglect** effects due to **convection**.

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Modeling assumption

The crucial **modeling assumption** is that the flow is “low speed” we **neglect** effects due to **convection**.

Why do we care?

Stokes equations represent a limiting case of the more general Navier–Stokes equations

The Stokes equation: weak formulation

Let us build the **weak formulation**

$$\begin{aligned}-\nabla^2 \mathbf{u} + \nabla p &= \mathbf{0}, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in V \times M$

$$\int_{\Omega} \mathbf{v} \cdot (-\nabla^2 \mathbf{u} + \nabla p) = \mathbf{0},$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u} = 0,$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in V \times M$

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \left(\frac{\partial \mathbf{u}}{\partial n} - p \hat{\mathbf{n}} \right) \cdot \mathbf{v} = 0,$$
$$\int_{\Omega} q \nabla \cdot \mathbf{u} = 0,$$

- Here $\nabla \mathbf{u} : \nabla \mathbf{v}$ is the **componentwise** scalar product, e.g., in dimension 2, this is $\nabla u_x \cdot \nabla v_x + \nabla u_y \cdot \nabla v_y$

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- Here $\nabla \mathbf{u} : \nabla \mathbf{v}$ is the **componentwise** scalar product
- We select boundary conditions $\partial\Omega = \Gamma_N \cup \Gamma_D$ $\Gamma_D \cap \Gamma_N = \emptyset$, $\mu(\Gamma_D) \neq 0$:

$$\mathbf{u} = \mathbf{w} \text{ on } \Gamma_D, \quad \frac{\partial \mathbf{u}}{\partial n} - p \hat{\mathbf{n}} = \mathbf{s} \text{ on } \Gamma_N$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in \mathbb{H}_{E_0}^1 \times \mathbb{L}^2$

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \left(\frac{\partial \mathbf{u}}{\partial n} - p \hat{\mathbf{n}} \right) \cdot \mathbf{v} = 0,$$
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- We define the spaces

$$\mathbb{H}_E^1 = \{ \mathbf{u} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{u} = \mathbf{w} \text{ on } \Gamma_D \}, \quad \mathbb{H}_{E_0}^1 = \{ \mathbf{v} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_D \}.$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in \mathbb{H}_{E_0}^1 \times \mathbb{L}^2$

$$\begin{aligned}\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \mathbf{s} \cdot \mathbf{v} &= \mathbf{0}, \\ \int_{\Omega} q \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

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The Stokes equation: weak formulation

Let us build the **weak formulation**

Find $(\mathbf{u}, p) \in \mathbb{H}_E^1 \times \mathbb{L}^2(\Omega)$ s.t. $\begin{cases} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \mathbf{s} \cdot \mathbf{v} = \mathbf{0}, & \forall \mathbf{v} \in \mathbb{H}_{E_0}^1 \\ \int_{\Omega} q \nabla \cdot \mathbf{u} = 0, & \forall q \in \mathbb{L}^2. \end{cases}$

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$$\mathbb{H}_E^1 = \{\mathbf{u} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{u} = \mathbf{w} \text{ on } \Gamma_D\}, \quad \mathbb{H}_{E_0}^1 = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

The Stokes equation: issues with BCs

$$\text{Find } (\mathbf{u}, p) \in \mathbb{H}_E^1 \times \mathbb{L}^2(\Omega) \text{ s.t. } \begin{cases} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \mathbf{s} \cdot \mathbf{v} = \mathbf{0}, & \forall \mathbf{v} \in \mathbb{H}_{E_0}^1, \\ \int_{\Omega} q \nabla \cdot \mathbf{u} = 0, & \forall q \in \mathbb{L}^2. \end{cases}$$

Words of caution

1. For a **unique velocity solution** the Dirichlet part of the boundary has to be nontrivial,
2. If the velocity is fixed everywhere on the boundary ($\Gamma_D \equiv \partial\Omega$) the pressure solution is only unique up to a constant (*hydrostatic pressure level*) and \mathbf{w} has to satisfy

$$0 = \int_{\Omega} \nabla \cdot \mathbf{u} = \int_{\partial\Omega} \mathbf{u} \cdot \hat{\mathbf{n}} = \int_{\partial\Omega} \mathbf{w} \cdot \hat{\mathbf{n}},$$

i.e., the **volume of fluid entering** the domain must be **matched** by the **volume of fluid flowing out** of it.

The Stokes equation: Mixed Elements

As we have done for Poisson, we need to select $V_h \subset V = \mathbb{H}_{E_0}^1$ and $M_h \subset M = \mathbb{L}^2(\Omega)$:

Find $(\mathbf{u}_h, p_h) \in V_h \times M_h$ s.t.
$$\begin{cases} \int_{\Omega} \nabla \mathbf{u}_h : \nabla \mathbf{v}_h - \int_{\Omega} p_h \nabla \cdot \mathbf{v}_h - \int_{\partial\Omega} \mathbf{s} \cdot \mathbf{v}_h = \mathbf{0}, & \forall \mathbf{v}_h \in V_h, \\ \int_{\Omega} q_h \nabla \cdot \mathbf{u}_h = 0, & \forall q_h \in M_h. \end{cases}$$

To determine the subspaces V_h and M_h we want to apply the **Theorem (Discrete Brezzi)**

$$\min_{q_h \neq \text{const.}} \max_{\mathbf{v}_h \neq \mathbf{0}} \frac{|(q_h, \nabla \cdot \mathbf{v}_h)|}{\|\mathbf{v}_h\|_V \|q_h\|_M} \geq \beta.$$

where

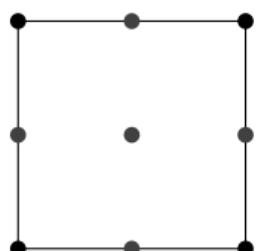
- $\|\mathbf{v}\|_V = (\int_{\Omega} \mathbf{v} \cdot \mathbf{v} + \nabla \mathbf{v} : \nabla \mathbf{v})^{\frac{1}{2}}$,
- $\|q\|_M = \|q - \mu(\Omega)^{-1} \int_{\Omega} q\|$.

The Stokes equation: Mixed Elements

Idea for finding inf-sup stable elements

The idea is to consider “local enclosed flow Stokes problems” posed on a subdomain $\mathcal{M} \subset \Omega$ ($\mathbf{w} \cdot \hat{\mathbf{n}} = 0$ on $\partial\mathcal{M}$) called a *macroelement* that has a topology that is regular and simple enough (so that we can actually do estimates and computations).

Q_2 - Q_1 Elements



Two velocity components

We approximate the 2 components of velocity with a single Q_2 FEM space

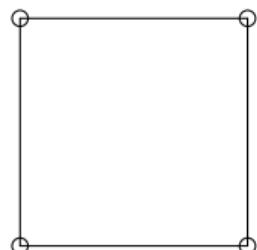
$$\{\Phi_1, \dots, \Phi_{2n}\} = \{(\phi_1, 0)^T, \dots, (\phi_n, 0)^T, (0, \phi_1)^T, \dots, (0, \phi_n)^T\}$$

The Stokes equation: Mixed Elements

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Q_2 - Q_1 Elements



Pressure

We approximate the 2 components of velocity with a single Q_2 FEM space $\{\Phi_j\}_{j=1}^{n_u}$. And the scalar pressure component with Q_1 FEM space $\{\psi_j\}_{j=1}^{n_p}$ giving:

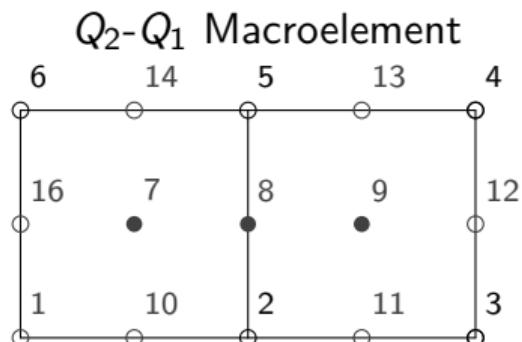
$$\mathcal{A} = \begin{bmatrix} A & O & B_x^T \\ O & A & B_y^T \\ B_x & B_y & O \end{bmatrix} \quad \begin{aligned} a_{i,j} &= \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j, \\ b_{x,ki} &= - \int_{\Omega} \psi_k \partial_x \phi_i, \\ b_{y,kj} &= - \int_{\Omega} \psi_k \partial_y \phi_j. \end{aligned}$$

Since we have an **enclosed flow** $\ker B^T = \{1\}$.

The Stokes equation: Mixed Elements

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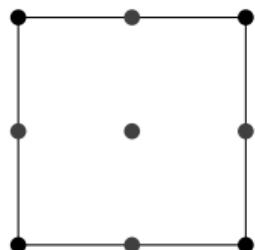
Three interior velocity nodes
and six pressure nodes.

B^T is a 6×6 matrix, with some effort we can compute all the entries and verify that $\ker B^T = \{\mathbf{1}\}$ (part of the computations are done in (Elman, Silvester, and Wathen 2014, Section 3.3.1)).

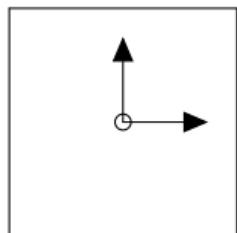
- Then stability holds for all patches of elements with the same topology,
- Any grid made of an even number of cells can be decomposed this way.

The Stokes equation: other stable elements

Q_2-P_{-1} Elements

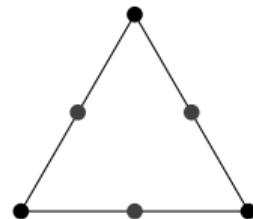


Velocity

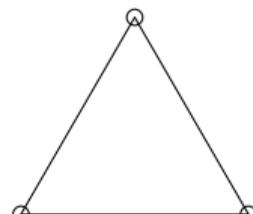


Pressure, and pressure derivatives

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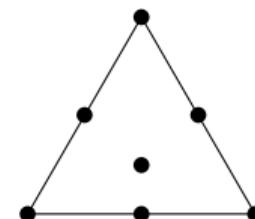


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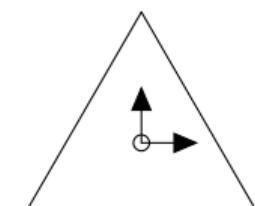


Pressure

$P_{2*}-P_{-1}$ Element



Velocity

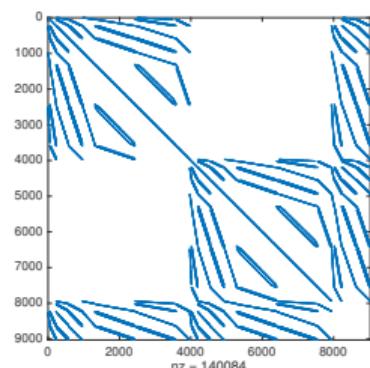


Pressure, and pressure derivatives

The Stokes equation: the associated saddles

The P_2-P_1 (Taylor-Hood) case for the **colliding flow** test problem.

- $\Omega = [-1, 1] \times [-1, 1]$
- $u_x = 20xy^3, u_y = 5x^4 - 5y^4,$
 $p = 60x^2y - 20y^3 + c,$
- Dirichlet boundary condition on all the square
 $\psi(x, y) = 5xy^4 - x^5.$

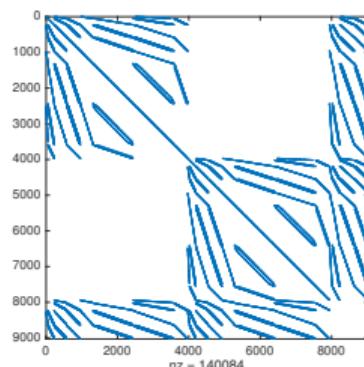


```
% Building the mesh
RefinementLevels = 2;
square = [0,1,0,1];
h = 0.25;
[node,elem] = squaremesh(square,h);
for i=1:RefinementLevels
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end
% Building the test problem: colliding flows
bdFlag = setboundary(node,elem,'Dirichlet');
pde = Stokesdata1;
options.solver='none'; % We just perform the build
[soln,eqn,info] =
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Question:

Can we apply (Rosten and Winther 1992)?

The Stokes equation: properties of the matrix

Theorem (Elman, Silvester, and Wathen 2014, Theorem 3.2.1)

With P_1 , P_2 , Q_1 or Q_2 approximation on a *shape-regular, quasi-uniform* subdivision of \mathbb{R}^2 , the matrix A for the *discrete vector Laplacian* satisfies

$$ch^2 \leq \frac{\langle A\mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \leq C \quad \forall \mathbf{v} \in \mathbb{R}^{n_u},$$

where h is the length of the longest edge in the mesh, and c and C are **constants independent** of h .

- This gives us information the behavior of the **smallest and largest eigenvalue** of A (*Rayleigh Principle*)!

h^2	$\lambda_{\min}(A)$	$h^2/\lambda_{\min}(A)$	$\lambda_{\max}(A)$
0.0156	0.0768	0.2035	10.5391
0.0039	0.0193	0.2028	10.6346
0.0010	0.0048	0.2027	10.6586
0.0002	0.0012	0.2027	10.6647

The Stokes equation: properties of the matrix

To uncover information on the B matrices, we need to introduce a **discrete representation of the norm of $M_h \subset \mathbb{L}^2$** :

$$p_h \in M_h : \|p_h\| = \langle Qp_h, p_h \rangle^{1/2}, \quad Q = (q_{kl}), q_{k,l} = \int_{\Omega} \psi_k \psi_l, \quad k, l = 1, \dots, n_p.$$

- ! The matrix Q is called **mass matrix** for the pressure space, *in general*, we call mass-matrices all the matrices obtained in this way for the basis of a given FEM space.

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Generalized singular values

We call **generalized singular values** the **real** numbers σ associated with the following generalized eigenvalue problem

$$\begin{bmatrix} O & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix} = \sigma \begin{bmatrix} A & O \\ O & Q \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix}.$$

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$\sigma \neq 0$ we select vector $(\mathbf{v}^T, -\mathbf{q}^T)^T$ and obtain

$$\begin{aligned} <\mathbf{v}, B^T \mathbf{q}> - <\mathbf{q}, B\mathbf{v}> &= 0 = \sigma (<\mathbf{v}, A\mathbf{v}> - <\mathbf{q}, Q\mathbf{q}>) \\ \Rightarrow < A\mathbf{v}, \mathbf{v}> &= < Q\mathbf{q}, \mathbf{q}>. \end{aligned}$$

That is

$$\frac{< BA^{-1}B^T \mathbf{q}, \mathbf{q}>}{< Q\mathbf{q}, \mathbf{q}>} = \sigma^2 = \frac{< B^T Q^{-1} B\mathbf{v}, \mathbf{v}>}{< A\mathbf{v}, \mathbf{v}>}.$$

The Stokes equation: properties of the matrix

$$\frac{\langle BA^{-1}B^T \mathbf{q}, \mathbf{q} \rangle}{\langle Q\mathbf{q}, \mathbf{q} \rangle} = \sigma^2 = \frac{\langle B^T Q^{-1} B \mathbf{v}, \mathbf{v} \rangle}{\langle A\mathbf{v}, \mathbf{v} \rangle}.$$

- If $\ker B^T = \mathbf{0}$ then B has n_p positive singular values.
- This is linked to the inf-sup condition:

$$\begin{aligned}\beta &\leq \min_{q_h \neq \text{const}} \max_{\mathbf{v}_h \neq \mathbf{0}} \frac{|(q_h, \nabla \cdot \mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\| \|q_h\|} = \min_{\mathbf{q} \neq \mathbf{1}} \max_{\mathbf{v} \neq \mathbf{0}} \frac{|\langle \mathbf{q}, B\mathbf{v} \rangle|}{\langle A\mathbf{v}, \mathbf{v} \rangle^{1/2} \langle Q\mathbf{q}, \mathbf{q} \rangle^{1/2}} \\ &= \min_{\mathbf{q} \neq \mathbf{1}} \frac{1}{\langle Q\mathbf{q}, \mathbf{q} \rangle^{1/2}} \max_{\mathbf{w} = A^{1/2}\mathbf{v} \neq \mathbf{0}} \frac{|\langle \mathbf{q}, BA^{-1/2}\mathbf{w} \rangle|}{\langle \mathbf{w}, \mathbf{w} \rangle^{1/2}} \\ &= \min_{\mathbf{q} \neq \mathbf{1}} \frac{\langle A^{-1/2}B^T \mathbf{q}, A^{-1/2}B^T \mathbf{q} \rangle^{1/2}}{\langle Q\mathbf{q}, \mathbf{q} \rangle^{1/2}} = \min_{\mathbf{q} \neq \mathbf{1}} \frac{\langle BA^{-1}B^T \mathbf{q}, \mathbf{q} \rangle^{1/2}}{\langle Q\mathbf{q}, \mathbf{q} \rangle^{1/2}} = \sigma_{\min}\end{aligned}$$

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The Stokes equation: properties of the matrix

Theorem (Elman, Silvester, and Wathen 2014, Theorem 3.22)

Let $\partial\Omega \equiv \Gamma_D$, the Stokes problem discretized with a uniformly stable mixed approximation on a shape-regular, quasi-uniform subdivision of \mathbb{R}^2 , has a **Schur complement matrix** $BA^{-1}B^T$ that is spectrally equivalent to the pressure mass matrix Q :

$$\beta^2 \leq \frac{\langle BA^{-1}B^T \mathbf{q}, \mathbf{q} \rangle}{\langle Q \mathbf{q}, \mathbf{q} \rangle} \leq 1, \quad \forall \mathbf{q} \in \mathbb{R}^{n_p} : \mathbf{q} \neq \mathbf{1}.$$

The inf-sup constant β is bounded away from zero independently of h and the condition number (discarding the zero eigenvalue) $\kappa^e(BA^{-1}B^T) \leq C/(c\beta)^2$ for c and C given by

$$ch^2 \leq \frac{\langle Q \mathbf{q}, \mathbf{q} \rangle}{\langle \mathbf{q}, \mathbf{q} \rangle} \leq Ch^2, \quad \forall \mathbf{q} \in \mathbb{R}^{n_p}.$$

The Stokes equation: properties of the matrix

We can **run this test** on the usual test problem
by running the code in the folder

`</> E3-Stokes/stokesmatrixproperties.m`

This tests both:

1. The bound on the vector Laplacian,
2. The bounds on the Schur complement.

for the P_2-P_1 elements.

h	λ_2	λ_{n_p}
0.2500	0.1352	0.9932
0.1250	0.1341	0.9996
0.0625	0.1336	1.0000
0.0312	0.1334	1.0000

Generalized eigenvalues for:

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Why do we care?

As you will see in the following, these information are **useful** for the **design of iterative solvers**.

The Stokes equation: stabilized discretizations

We have seen that the matter of obtain a **stable discretization** depends on the null-space of B^T .

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Idea behind stabilization

If the discretization is not stable $\exists \mathbf{p} \neq \mathbf{0}$ such that $B^T \mathbf{p} = \mathbf{0}$, that is $(\mathbf{0}^T, \mathbf{p}^T)^T$ is a null vector for the homogeneous saddle-point system. The **idea** behind stabilization is **relaxing the incompressibility constraint** so that this vector is no longer in the kernel **and** we still obtain a reasonable error bound for the convergence of the method.

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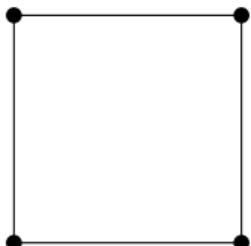
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Technique

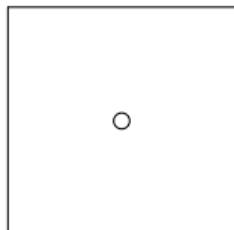
The **technique** to devise stabilization is again using *macroelements*.

The Stokes equation: stabilized Q_1-P_0

- This is the simplest *unstable* element,



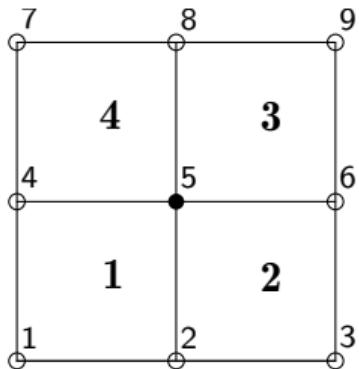
Velocity



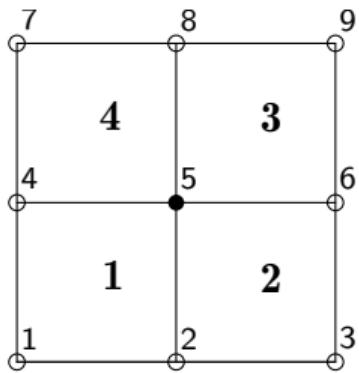
Pressure

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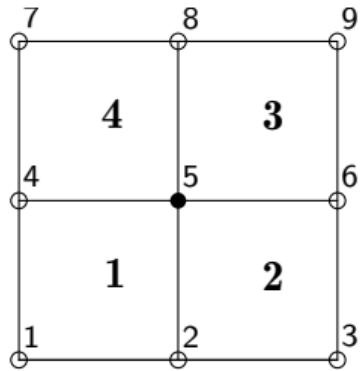


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- The pressure coefficient \mathbf{p} solves

$$BA^{-1}B^T \mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

the Schur complement $BA^{-1}B^T$ has **two eigenvectors** for the 0 eigenvalue: $\mathbf{1}$, and $\mathbf{q}_2 = (1, -1, 1, -1)^T$.

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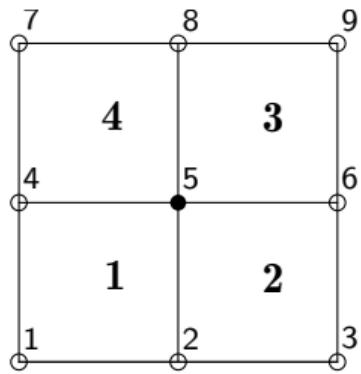
- We relax the incompressibility constraint

$$\begin{bmatrix} A & B^T \\ B & -\gamma C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

$$S_\gamma \mathbf{p} \equiv (BA^{-1}B^T + \gamma C)\mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

selecting C such that $C^T = C$, $C \geq 0$, $S_\gamma \geq 0$ and $\ker S_\gamma = \{\mathbf{1}\}$.

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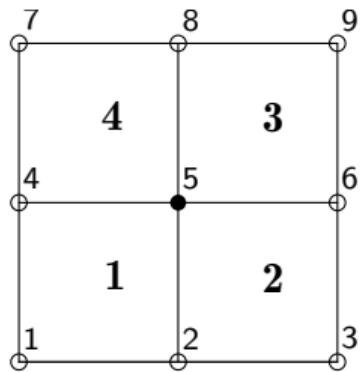
$$\begin{bmatrix} A & B^T \\ B & -\gamma C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

$$C_* = \mathbf{q}_2 \mathbf{q}_2^T$$

$$S_\gamma \mathbf{p} \equiv (BA^{-1}B^T + \gamma C)\mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

selecting C such that $C^T = C$, $C \geq 0$, $S_\gamma \geq 0$ and $\ker S_\gamma = \{\mathbf{1}\}$.

The Stokes equation: stabilized Q_1-P_0



- This is the simplest *unstable* element,
- To devise a stabilization we need a 2×2 *macroelement*
- The pressure coefficient \mathbf{p} solves

$$BA^{-1}B^T \mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

the Schur complement $BA^{-1}B^T$ has **two eigenvectors** for the 0 eigenvalue: $\mathbf{1}$, and $\mathbf{q}_2 = (1, -1, 1, -1)^T$.

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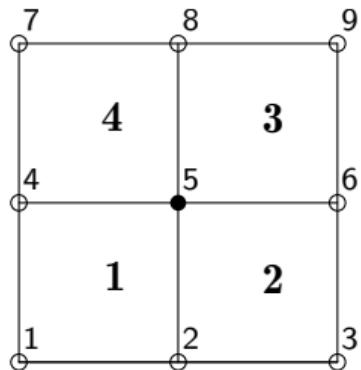
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- What about γ ?

The Stokes equation: stabilized Q_1-P_0

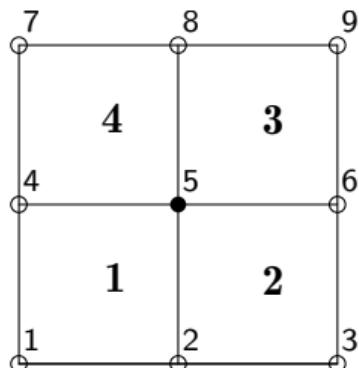


- We have moved the 0 eigenvalue with eigenvector \mathbf{q}_2 to 4γ , if we take $\gamma = 1/4$ we get a spectral orthogonal projector for C ,

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$$C_* = \frac{1}{4} \mathbf{q}_2 \mathbf{q}_2^T$$

The Stokes equation: stabilized Q_1-P_0



- We have moved the 0 eigenvalue with eigenvector \mathbf{q}_2 to 4γ , if we take $\gamma = 1/4$ we get a spectral orthogonal projector for C ,
- But γ has to be selected to balance both **stability** and **accuracy**, a natural choice would be selecting

$$\gamma_* = \frac{1}{4} h_x h_y, \quad h_x h_y = \text{"area of the element"}$$

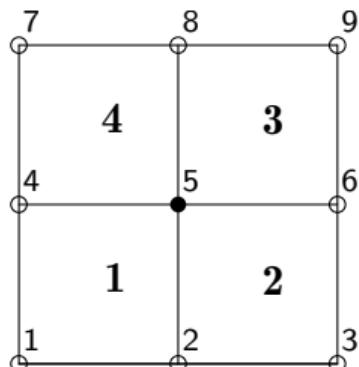
If Q is the **mass matrix** associate with the P_0 elements ($Q = h_x h_y \times I$), then γ_* is the largest value for which

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$$C_* = \mathbf{q}_2 \mathbf{q}_2^T$$

$$\gamma \frac{\mathbf{p}^T C_* \mathbf{p}}{\mathbf{p}^T Q \mathbf{p}} \leq 1 \quad \forall \mathbf{p} \in \mathbb{R}^{n_p}.$$

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- Complete stabilization matrix: $C = \text{blockdiag}(C_*, \dots, C_*)$.

The Stokes equation: stabilized Q_1-P_0

Other stabilization are possible

This is not the only possible stabilization matrix, other choices are possible, consider, e.g.,

$$C^* = h_x h_y \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

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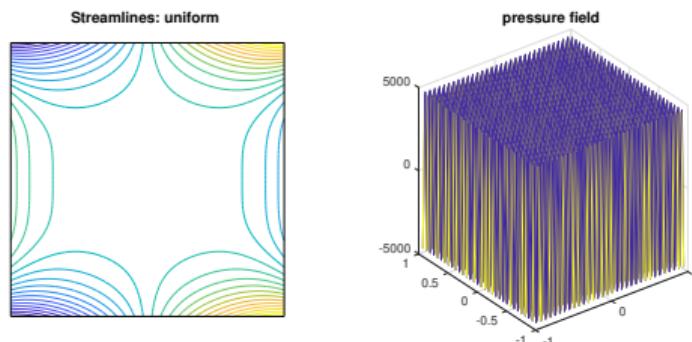
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We can visually see the effect of the stabilization on the same **colliding flow** problem.

$$\gamma = 0$$



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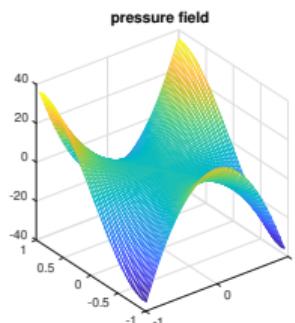
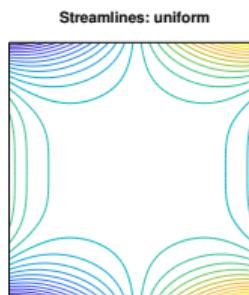
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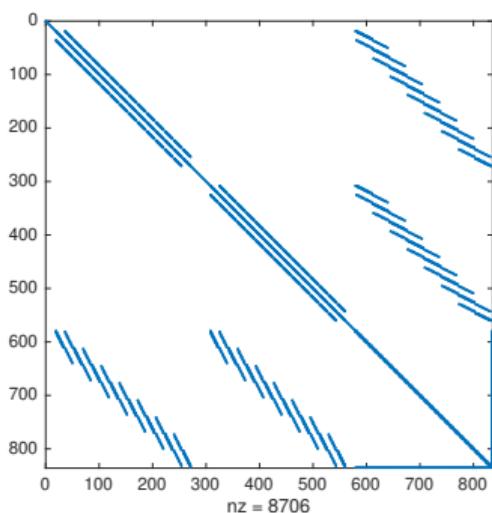
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$$\gamma = -1/4$$



The Stokes equation: stabilized matrix properties

What can we say about the **spectral properties** of the **stabilized matrices**?



- To substitute the inf-sup condition we introduce the operator:

$$s(q_h) : M_h \rightarrow \mathbb{R}$$

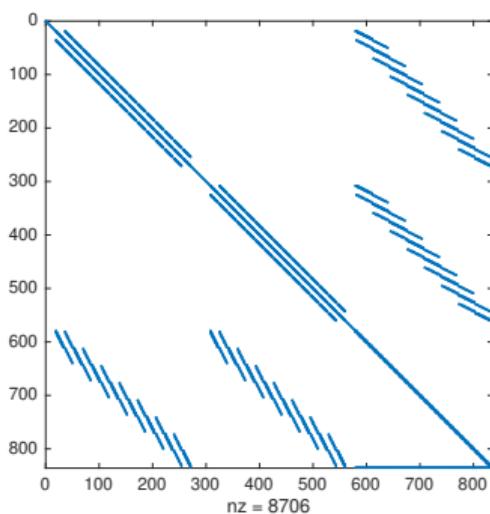
$$q_h \mapsto s(q_h) = \max_{\mathbf{v}_h \neq 0} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|} + c(q_h, q_h)^{1/2},$$

where $c(\cdot, \cdot)$ is the stabilization operator that generates the matrix C .

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

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Uniform stabilization

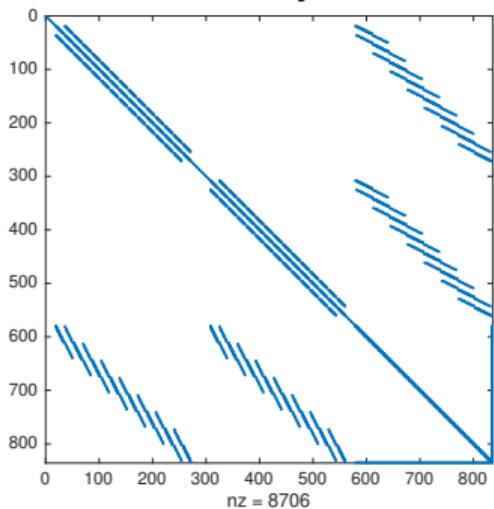
The Stokes problem is said to be **uniformly stabilized** if there exists β **independent of h** such that

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

$$s(q_h) \geq \beta^2 \|q_h\|, \quad \forall q_h \in M_h.$$

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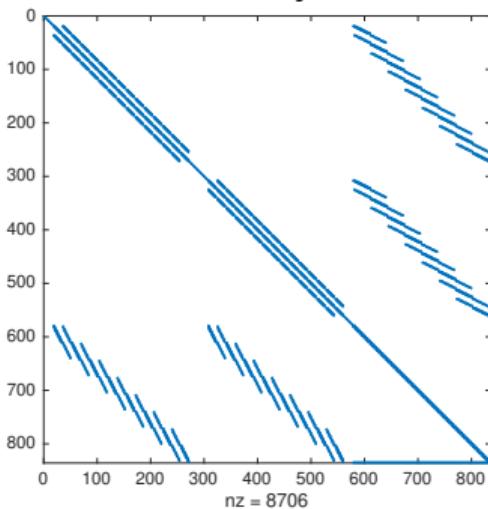
- As we have done for the stable case, we can express everything in terms of matrices $\forall \mathbf{q} \in \mathbb{R}^{n_p}$

$$\langle BA^{-1}B^T \mathbf{q}, \mathbf{q} \rangle^{1/2} + \langle C\mathbf{q}, \mathbf{q} \rangle^{1/2} \geq \frac{1}{2}\beta^2 \langle Q\mathbf{q}, \mathbf{q} \rangle^{1/2} .$$

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- Then the generalized inf-sup conditions is

$$\beta^2 = 2 \min_{\mathbf{q} \neq 0} \frac{\langle (BA^{-1}B^T + C)\mathbf{q}, \mathbf{q} \rangle}{\langle Q\mathbf{q}, \mathbf{q} \rangle}.$$

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The Stokes equation: stabilized matrix properties

Theorem (Elman, Silvester, and Wathen 2014, Theorem 3.29)

Let $\partial\Omega \equiv \Gamma_D$, the Stokes problem discretized with an *ideally* stabilized mixed approximation on a shape-regular, quasi-uniform subdivision of \mathbb{R}^2 , has a **Schur complement matrix** $BA^{-1}B^T + C$ that is spectrally equivalent to the pressure mass matrix Q :

$$\beta^2 \leq \frac{\langle (BA^{-1}B^T + C)\mathbf{q}, \mathbf{q} \rangle}{\langle Q\mathbf{q}, \mathbf{q} \rangle} \leq 2, \quad \forall \mathbf{q} \in \mathbb{R}^{n_p} : \mathbf{q} \neq \mathbf{1}.$$

The *generalized* inf-sup constant β is bounded away from zero independently of h .

$\ell(h = 2^{-\ell})$	β^2	λ_{n_p}
3	0.280929	1.7238
4	0.252201	1.74406
5	0.233876	1.74859
6	0.221837	1.74965

The **colliding flow** problem can be tested with

«/» E3-Stokes/stokesmatrixpropertiesstab.m

The Navier-Stokes Equation

We add to the Stokes problem a **forcing term** and a **convection term** obtaining

$$\begin{aligned}-\nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

where

- $\nu > 0$ is the *kinematic viscosity*,
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- The equation is nonlinear!
- We need boundary conditions on $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$:

$$\mathbf{u} = \mathbf{w} \text{ on } \Gamma_D, \quad \nu \frac{\partial \mathbf{u}}{\partial n} - \hat{\mathbf{n}} p = \mathbf{0} \text{ on } \partial\Gamma_N$$

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$$\partial\Omega \equiv \Gamma_D$$

If the velocity is specified everywhere on the boundary, then the pressure solution to the Navier-Stokes problem is only unique up to a *hydrostatic constant*.

The Navier-Stokes Equation: normalization

We normalize the system

$$\begin{aligned}-\nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

to better highlight if the system is **diffusion dominated** or **advection dominated**.

- Let L denote a *characteristic length scale* for the domain Ω ,
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$$\begin{aligned} -\frac{1}{\mathcal{R}} \nabla^2 \mathbf{u}_* + \mathbf{u}_* \cdot \nabla \mathbf{u}_* + \nabla p_* &= \frac{L}{U^2} \mathbf{f}, \quad \mathcal{R} = UL/\nu \\ \nabla \cdot \mathbf{u}_* &= 0, \end{aligned}$$

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Reynolds number

We call $\mathcal{R} = UL/\nu$ the Reynolds number. If $\mathcal{R} \leq 1$ then the problem is diffusion dominated, for increasing values of \mathcal{R} we get instead convection dominated problems.

The Navier-Stokes Equation: weak formulation

Can be written *similarly to the Stokes problem*

$$\text{Find } (\mathbf{u}, p) \in V \times M : \begin{cases} \nu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} - \int_{\Omega} p(\nabla \cdot \mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \\ \int_{\Omega} q(\nabla \cdot \mathbf{u}) = 0. \end{cases}$$

We need again the **suitable spaces**

- $\mathbf{u} \in \mathbb{H}_E^1 = \{\mathbf{u} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{u} = \mathbf{w} \text{ on } \Gamma_D\} \equiv V, p \in \mathbb{L}^2(\Omega) \equiv M,$
- $\mathbf{v} \in \mathbb{H}_{E_0}^1 = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_D\},$
- New addition is a **trilinear form** for the **velocity term**:

$$c : \mathbb{H}_{E_0}^1 \times \mathbb{H}_{E_0}^1 \times \mathbb{H}_{E_0}^1 \rightarrow \mathbb{R}$$
$$(\mathbf{z}, \mathbf{u}, \mathbf{v}) \mapsto c(\mathbf{z}, \mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{z} \cdot \nabla \mathbf{u}) \cdot \mathbf{v}.$$

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- We restate the problem

$$\text{Find } (u, p) \in V \times M : \begin{cases} a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in V, \\ b(\mathbf{u}, q) = 0, & \forall q \in M, \end{cases} \quad (\text{NS})$$

with

$$a(\mathbf{w}, \mathbf{v}) = \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) : V \times V \rightarrow \mathbb{R}, \quad b : V \times Q \rightarrow \mathbb{R} = -(q, \nabla \cdot \mathbf{v}),$$

$$c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^d \left(u_j \frac{\partial u_i}{\partial x_j}, v_i \right) : V \times V \times V \rightarrow \mathbb{R}$$

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The existence proof then follows in few steps.

1. We consider the problem on the space $V_{\text{div}} = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d : \operatorname{div} \mathbf{v} = 0\}$, then a solution of (NS) is a solution also of the problem on this space

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2. Then we prove that to a solution on the reduced space corresponds a solution of the full problem.

Lemma (Quarteroni and Valli 1994, Lemma 10.1.1)

Let \mathbf{u} be a solution of (NS_{div}) . Then there exists a unique $p \in M$ such that (\mathbf{u}, p) is a solution of problem (NS).

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3. Prove that (NS_{div}) has a *unique* solution.

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Theorem (Quarteroni and Valli 1994, Theorem 10.1.1)

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$$\|\mathbf{f}\| < \frac{\nu^2}{\hat{C} C_\Omega^{1/2}},$$

where $\hat{C} > 0$ is the *continuity constant* for the trilinear form c , i.e.,

$$|c(\mathbf{w}, \mathbf{z}, \mathbf{v})| \leq \hat{C} |\mathbf{w}|_1 |\mathbf{z}|_1 |\mathbf{v}|_1 \quad \forall \mathbf{w}, \mathbf{z}, \mathbf{v} \in \mathbb{H}_0^1(\Omega)^d,$$

and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\text{div}}$ to (NS_{div}) .

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Idea of the proof.

1. Use Lax-Milgram for problem $\mathcal{A}_{\mathbf{w}}(\mathbf{z}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$, $\forall, \mathbf{v} \in V_{\text{div}}$ and $\mathcal{A}_{\mathbf{w}}(\mathbf{z}, \mathbf{v}) = a(\mathbf{z}, \mathbf{v}) + c(\mathbf{w}, \mathbf{z}, \mathbf{v})$ to prove existence for every \mathbf{w} .

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and C_{Ω} is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\text{div}}$ to (NS_{div}) .

Idea of the proof.

2. The solution we look for is then a fixed point of the map $\Phi : \mathbf{w} \rightarrow \mathbf{z}$. First we prove that such solution is in a ball in V_{div} .

The Navier-Stokes Equation: existence

Theorem (Quarteroni and Valli 1994, Theorem 10.1.1)

Let $\mathbf{f} \in \mathbb{H}_{\text{div}} = \{\mathbf{v} \in \mathbb{L}^2(\Omega)^d \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$, with

$$\|\mathbf{f}\| < \frac{v^2}{\hat{C} C_{\Omega}^{1/2}},$$

where $\hat{C} > 0$ is the *continuity constant* for the trilinear form c , i.e.,

$$|c(\mathbf{w}, \mathbf{z}, \mathbf{v})| \leq \hat{C} |\mathbf{w}|_1 |\mathbf{z}|_1 |\mathbf{v}|_1 \quad \forall \mathbf{w}, \mathbf{z}, \mathbf{v} \in \mathbb{H}_0^1(\Omega)^d,$$

and C_{Ω} is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\text{div}}$ to (NS_{div}) .

Idea of the proof.

3. Finally, we apply *Banach contraction Theorem* (using the hypothesis on \mathbf{f}) to prove that there exist a unique fixed point for the problem.

The Navier-Stokes Equation: existence

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Let $\mathbf{f} \in \mathbb{H}_{\text{div}} = \{\mathbf{v} \in \mathbb{L}^2(\Omega)^d \mid \text{div } \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$, with

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where $\hat{C} > 0$ is the *continuity constant* for the trilinear form c and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\text{div}}$ to (NS_{div}) .

Conditions - 1

It is not restrictive to assume $\mathbf{f} \in \mathbb{H}_{\text{div}}$, any $\mathbf{f} \in \mathbb{L}^2(\Omega)^d$ can be decomposed as the sum of a function in \mathbb{H}_{div} and a function that is a gradient of an $\mathbb{H}^1(\Omega)$ function. The gradient component of the external force field \mathbf{f} doesn't play a role, $(\mathbf{v}, \nabla q) = 0 \forall q \in \mathbb{H}^1$ and $\mathbf{v} \in \mathbb{H}_{\text{div}}$.

The Navier-Stokes Equation: existence

Theorem (Quarteroni and Valli 1994, Theorem 10.1.1)

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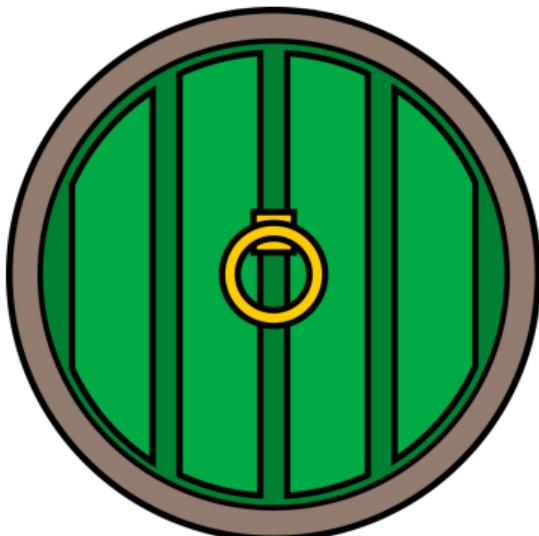
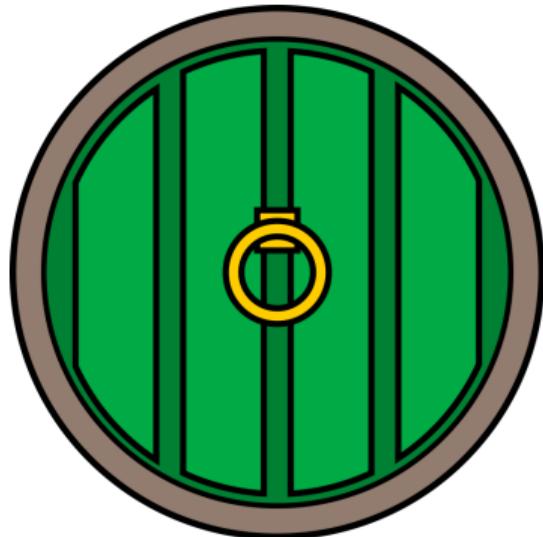
where $\hat{C} > 0$ is the *continuity constant* for the trilinear form c and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\text{div}}$ to (NS_{div}) .

Conditions - 2

The smallness condition on the viscosity ν is necessary for proving uniqueness, and is restrictive. The solution may not be unique when ν is small w.r.t. \mathbf{f} , even for reasonable \mathbf{f} .

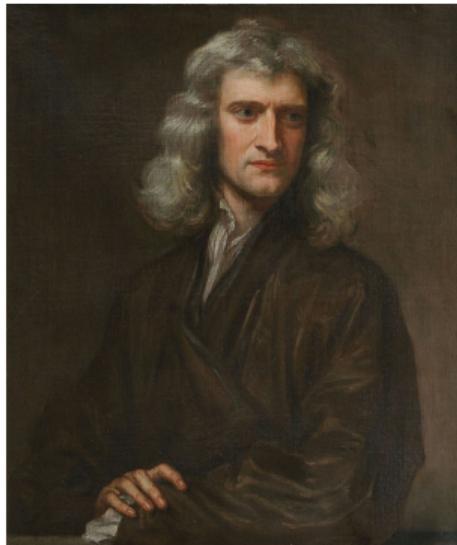
The Navier-Stokes Equation: linearizations

Since we only now how to solve linear problems, to face (NS) we discuss two types of **nonlinear iteration** with a **linearized problem being solved at every step**.



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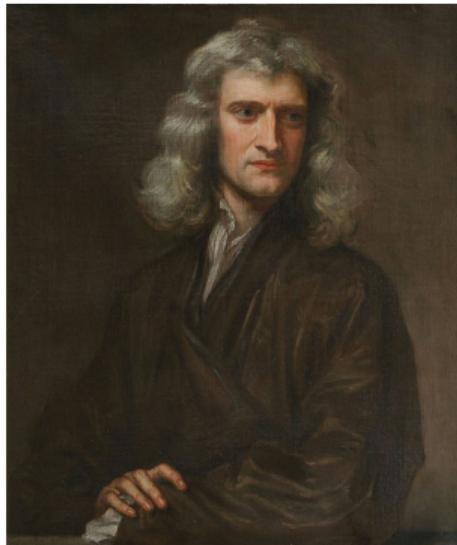
Newton Method



Picard's Iteration

The Navier-Stokes Equation: linearizations

Since we only now how to solve linear problems, to face (NS) we discuss two types of **nonlinear iteration** with a **linearized problem being solved at every step**.



Newton Method

We introduce both method first in the continuous context.



Picard's Iteration

The Navier-Stokes Equation: Newton method

1. We have a guess $\{\mathbf{u}_k, p_k\}$ for the solution,
2. We compute the residual pairs

$$\begin{bmatrix} R_k \\ r_k \end{bmatrix} = \begin{bmatrix} \int_{\Omega} \mathbf{f} \cdot \mathbf{v} - c(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) - \nu \int_{\Omega} \nabla \mathbf{u}_k : \nabla \mathbf{v} + \int_{\Omega} p_k (\nabla \cdot \mathbf{v}) \\ - \int_{\Omega} q (\nabla \cdot \mathbf{u}_k) \end{bmatrix} \quad \begin{aligned} \mathbf{v} &\in \mathbb{H}_{E_0}^1, \\ q &\in \mathbb{L}^2(\Omega). \end{aligned}$$

3. Then update the solution as

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \delta \mathbf{u}_k, \quad p_{k+1} = p_k + \delta p_k,$$

for $\delta \mathbf{u}_k \in \mathbb{H}_{E_0}^1$ and $\delta p_k \in \mathbb{L}^2(\Omega)$ the solution of

$$\begin{cases} c(\delta \mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) + c(\mathbf{u}_k, \delta \mathbf{u}_k, \mathbf{v}) + \nu \int_{\Omega} \nabla \delta \mathbf{u}_k : \nabla \mathbf{v} - \int_{\Omega} \delta p_k (\nabla \cdot \mathbf{v}) = R_k, & \forall \mathbf{v} \in \mathbb{H}_{E_0}^1, \\ \int_{\Omega} q (\nabla \cdot \delta \mathbf{u}_k) = r_k, & \forall q \in \mathbb{L}^2 \end{cases}$$

The Navier-Stokes Equation: Discrete Newton

As we have done for the **Stokes problem** we select $V_h \subset \mathbb{H}_{E_0}^1$ and $M_h \subset \mathbb{L}^2(\Omega)$,

- The Newton **updates** are then computed by solving $\forall \mathbf{v} \in V_h, \forall q_h \in M_h$

$$\begin{cases} c(\delta \mathbf{u}_h^{(k)}, \mathbf{u}_h^{(k)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(k)}, \delta \mathbf{u}_h^{(k)}, \mathbf{v}) + \nu \int_{\Omega} \nabla \delta \mathbf{u}_h^{(k)} : \nabla \mathbf{v}_h - \int_{\Omega} \delta p_h^{(k)} (\nabla \cdot \mathbf{v}_h) = R_h^{(k)} \\ \int_{\Omega} q_h (\nabla \cdot \delta \mathbf{u}_h^{(k)}) = r_h^{(k)} \end{cases}$$

where $R_k(\mathbf{v}_h)$, and $r_k(q_h)$ are the nonlinear residuals w.r.t. discrete formulation.

- Selecting basis $V_h = \text{Span}\{\phi_j\}$, $M_h = \text{Span}\{\psi_j\}$ and representing (dropping the k)

$$\mathbf{u}_h = \sum_{j=1}^{n_u} \mathbf{u}_j \phi_j + \sum_{n_u+1}^{n_u+n_\partial} \mathbf{u}_j \phi_j, \quad p_h = \sum_{k=1}^{n_p} \mathbf{p}_k \psi_k,$$

and

$$\delta \mathbf{u}_h \sum_{j=1}^{n_u} + \delta \mathbf{u}_j \phi_j, \quad \delta p_h = \sum_{k=1}^{n_p} \delta \mathbf{p}_k \psi_k,$$

we get the corresponding discrete system

The Navier-Stokes Equation: Discrete Newton

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where $R_k(\mathbf{v}_h)$, and $r_k(q_h)$ are the nonlinear residuals w.r.t. discrete formulation.

- we get the corresponding discrete system

$$\mathcal{A}\delta = \begin{bmatrix} \nu \mathbf{A} + \mathbf{N} + \mathbf{W} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

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$$\mathcal{A}\delta = \begin{bmatrix} \nu \mathbf{A} + \mathbf{N} + \mathbf{W} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{bmatrix} \quad a_{i,j} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j, \\ \mathbf{B} = \begin{bmatrix} B_x & B_y \end{bmatrix} \quad b_{x,ki} = - \int_{\Omega} \psi_k \partial_x \phi_i, \\ b_{y,kj} = - \int_{\Omega} \psi_k \partial_y \phi_j.$$

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- If we use *unstable* elements, we need a stabilization matrix.

The Navier-Stokes Equation: Picard's Iteration

The second approach for linearization is **Picard's iteration**, we start again from

1. We have a guess $\{\mathbf{u}_k, p_k\}$ for the solution,
2. We compute the residual pairs

$$\begin{bmatrix} R_k \\ r_k \end{bmatrix} = \begin{bmatrix} \int_{\Omega} \mathbf{f} \cdot \mathbf{v} - c(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) - \nu \int_{\Omega} \nabla \mathbf{u}_k : \nabla \mathbf{v} + \int_{\Omega} p_k (\nabla \cdot \mathbf{v}) \\ - \int_{\Omega} q (\nabla \cdot \mathbf{u}_k) \end{bmatrix} \quad \begin{array}{l} \mathbf{v} \in \mathbb{H}_{E_0}^1, \\ q \in \mathbb{L}^2(\Omega). \end{array}$$

3. Then update the solution as

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \delta \mathbf{u}_k, \quad p_{k+1} = p_k + \delta p_k,$$

for $\delta \mathbf{u}_k \in \mathbb{H}_{E_0}^1$ and $\delta p_k \in \mathbb{L}^2(\Omega)$ the solution of

$$\begin{cases} c(\delta \mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) + c(\mathbf{u}_k, \delta \mathbf{u}_k, \mathbf{v}) + \nu \int_{\Omega} \nabla \delta \mathbf{u}_k : \nabla \mathbf{v} - \int_{\Omega} \delta p_k (\nabla \cdot \mathbf{v}) = R_k, & \forall \mathbf{v} \in \mathbb{H}_{E_0}^1, \\ \int_{\Omega} q (\nabla \cdot \delta \mathbf{u}_k) = r_k, & \forall q \in \mathbb{L}^2 \end{cases}$$

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for $\delta \mathbf{u}_k \in \mathbb{H}_{E_0}^1$ and $\delta p_k \in \mathbb{L}^2(\Omega)$ the solution of the **Oseen system**

$$\begin{cases} c(\mathbf{u}_k, \delta \mathbf{u}_k, \mathbf{v}) + \nu \int_{\Omega} \nabla \delta \mathbf{u}_k : \nabla \mathbf{v} - \int_{\Omega} \delta p_k (\nabla \cdot \mathbf{v}) = R_k, & \forall \mathbf{v} \in \mathbb{H}_{E_0}^1, \\ \int_{\Omega} q (\nabla \cdot \delta \mathbf{u}_k) = r_k, & \forall q \in \mathbb{L}^2 \end{cases}$$

The Navier-Stokes Equation: Discrete Picard

The **discrete system** is the same of the Newton method **without** the Newton matrix \mathbf{W} :

$$\mathcal{A}\delta = \begin{bmatrix} \nu\mathbf{A} + \mathbf{N} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

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Theorem

Consider the generic saddle-point system

$$\mathcal{A} = \begin{bmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & -\mathbf{C} \end{bmatrix},$$

where \mathbf{C} is symmetric and positive-semidefinite matrix. If $\langle \mathbf{F}\mathbf{u}, \mathbf{u} \rangle > 0 \ \forall \mathbf{u} \neq \mathbf{0}$, then

$$\ker \mathcal{A} = \left\{ \begin{bmatrix} \mathbf{0} \\ \mathbf{p} \end{bmatrix} \mid \mathbf{p} \in \ker(B\mathbf{F}^{-1}\mathbf{B}^T + \mathbf{C}) \right\}.$$

The Navier-Stokes Equation: Newton and Picard

Newton

$$\mathcal{A} = \begin{bmatrix} \nu A + N + W_{xx} & W_{xy} & B_x^T \\ W_{yx} & \nu A + N + W_{yy} & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

Picard

$$\mathcal{A} = \begin{bmatrix} \nu A + N & O & B_x^T \\ O & \nu A + N & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

- Coupled $M_{1,1}$ block,
- Quadratic convergence,
- Locally convergent for “large enough” ν , and “close enough” initial guess.

- Decoupled $M_{1,1}$ block,
- Linear convergence,
- Converges under the existence condition: $\|\mathbf{f}\| < \nu^2 / \hat{\mathcal{C}} C_\Omega^{1/2}$.

The Navier-Stokes Equation: Newton and Picard

Newton

$$\mathcal{A} = \begin{bmatrix} \nu A + N + W_{xx} & W_{xy} & B_x^T \\ W_{yx} & \nu A + N + W_{yy} & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

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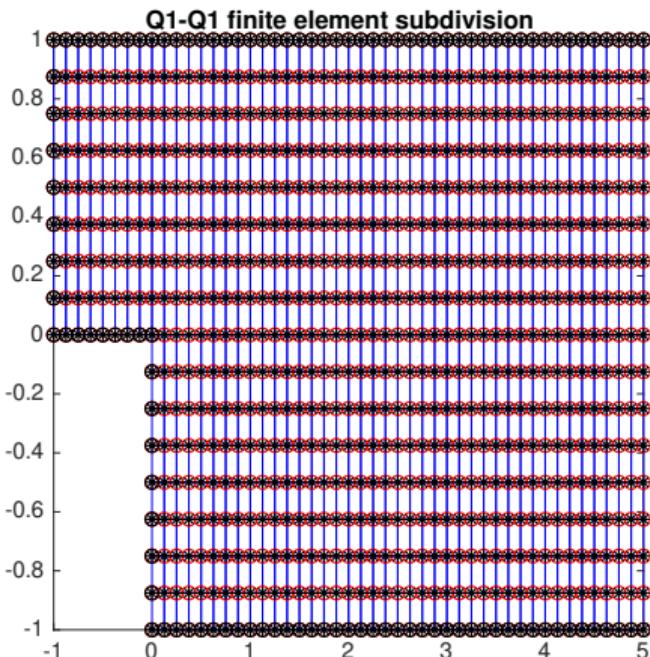
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Next week we will delve into some **numerical experiments**, and try several **preconditioners** discussed in the morning lectures.

Navier-Stokes: backward facing step

Test problem:

- L -shaped domain Ω , parabolic inflow boundary condition, natural outflow boundary condition,



You can run the example as `E4-NavierStokes/navierstokes_solution.m`.

Navier-Stokes: backward facing step

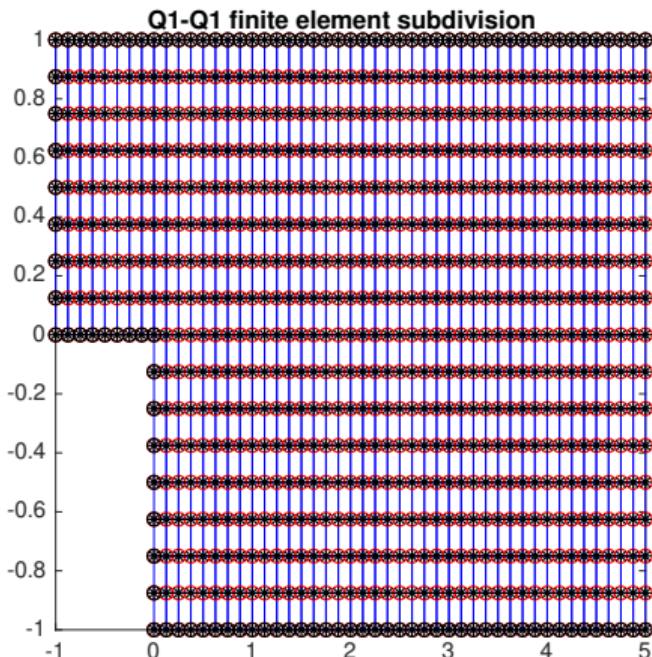
Test problem:

- L-shaped domain Ω , **parabolic inflow boundary condition**, natural outflow boundary condition,

Poiseuille flow

It is a steady horizontal flow in a channel driven by a pressure difference between the two ends

$$u_x = 1 - y^2, \quad u_y = 0, \quad p = -2\gamma x + \text{constant}.$$

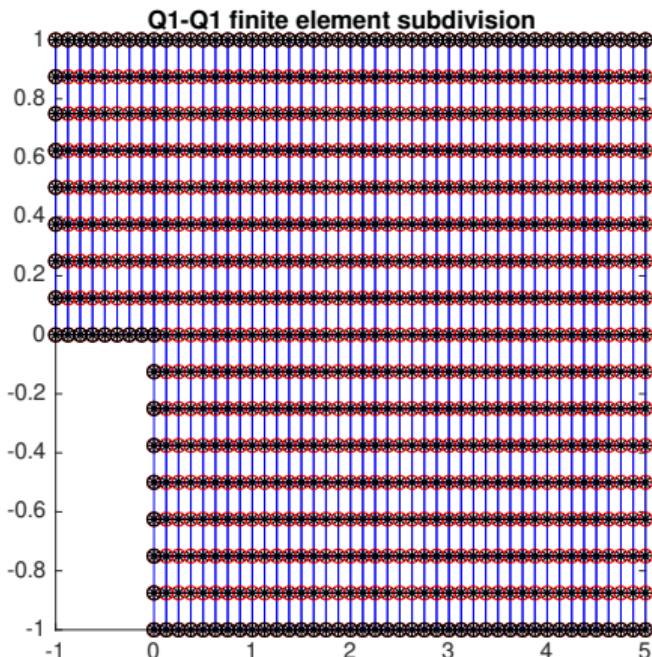


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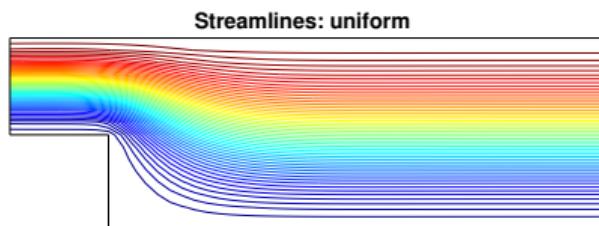
- L -shaped domain Ω , parabolic inflow boundary condition, natural outflow boundary condition,
- **Inflow** $x = -1, 0 \leq y \leq 1$,
No flow on the boundary,
Neumann condition at the outflow $x = L$,
 $-1 < y < 1$.
- Discretized with (unstable) Q1-Q1 elements.



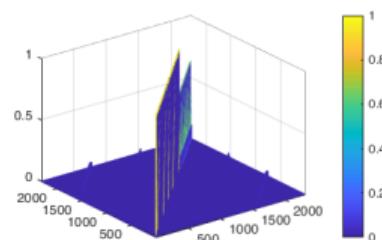
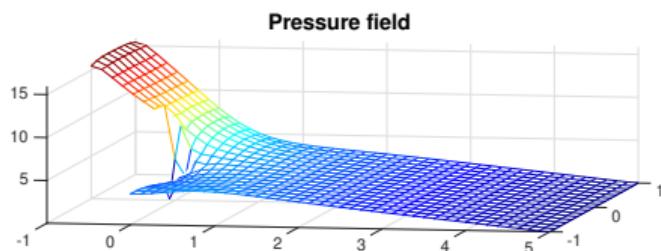
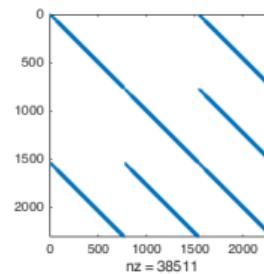
You can run the example as `E4-NavierStokes/navierstokes_solution.m`.

Navier-Stokes: backward facing step

Initial guess:



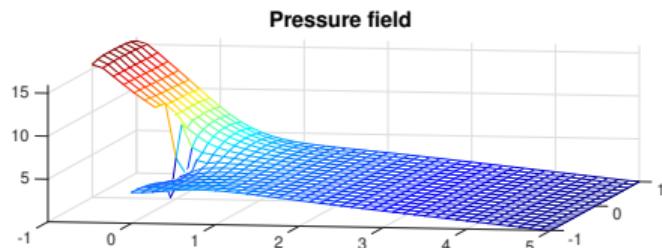
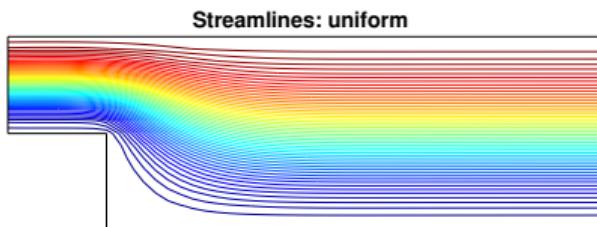
Picard's Iteration



Solution of the associated Stokes problem.

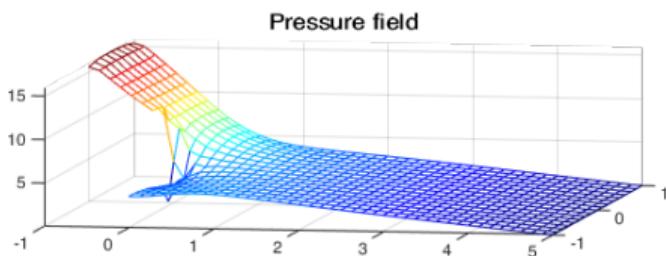
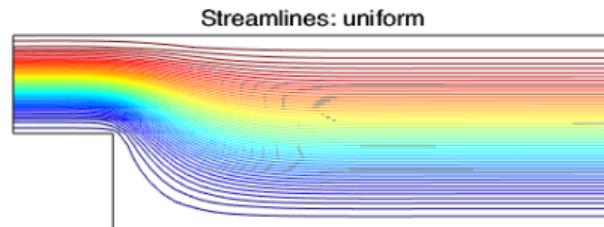
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

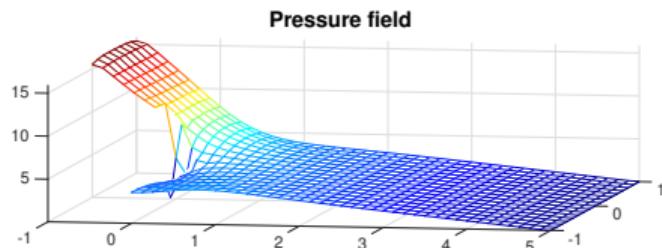
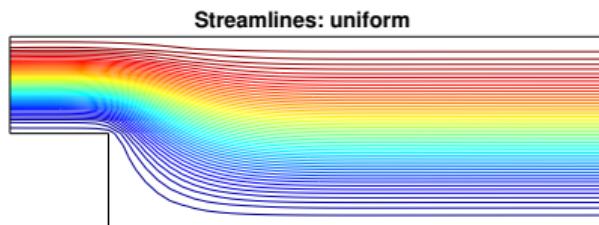
Picard's Iteration



Iteration 1

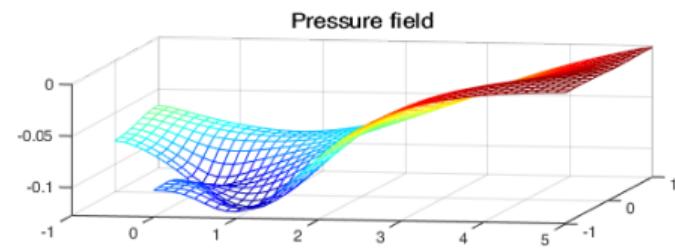
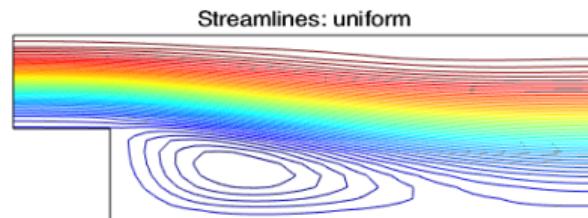
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

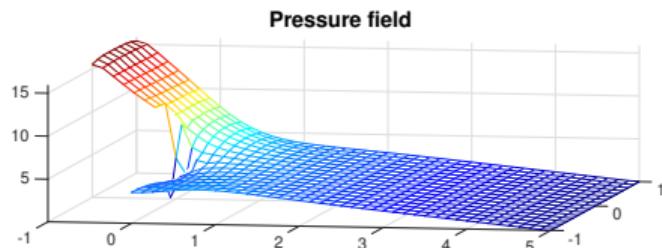
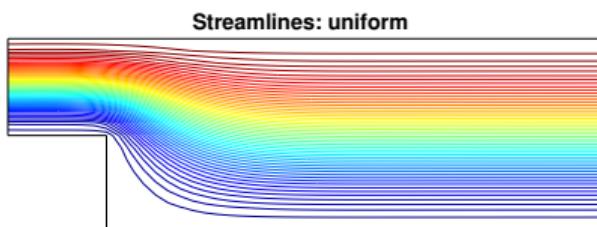
Picard's Iteration



Iteration 2

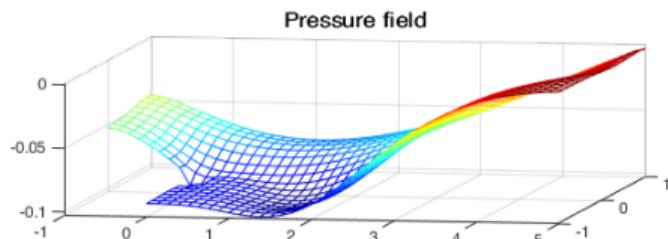
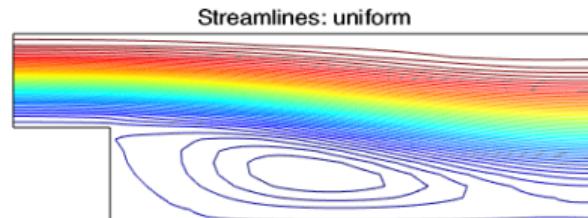
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

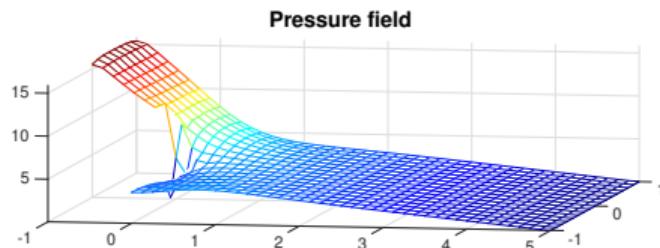
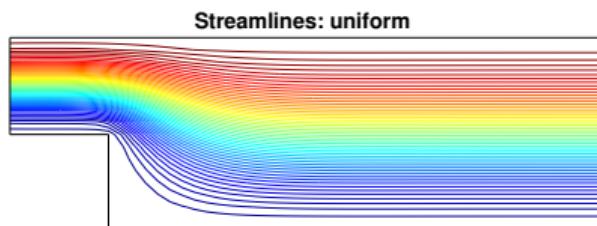
Picard's Iteration



Iteration 3

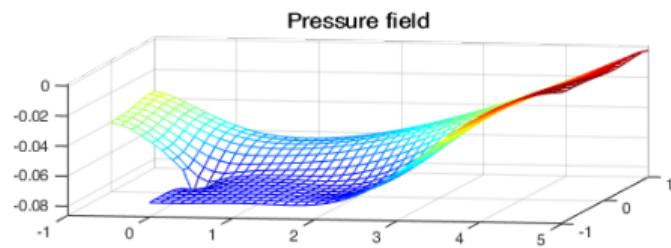
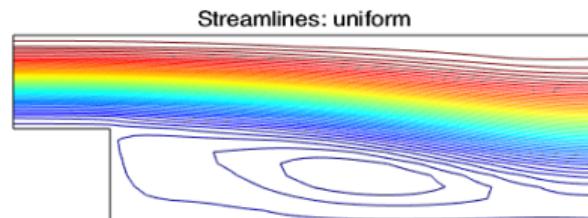
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

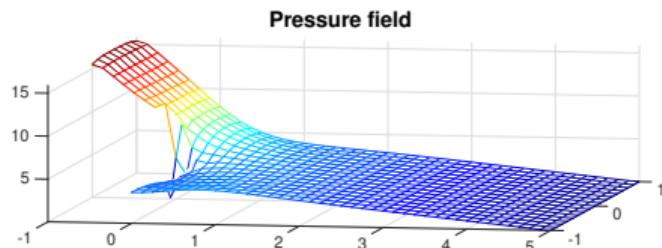
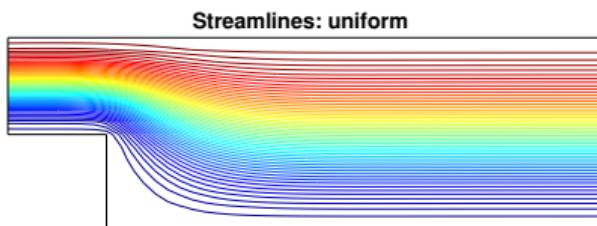
Picard's Iteration



Iteration 4

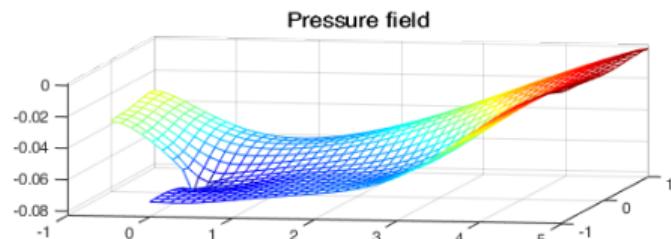
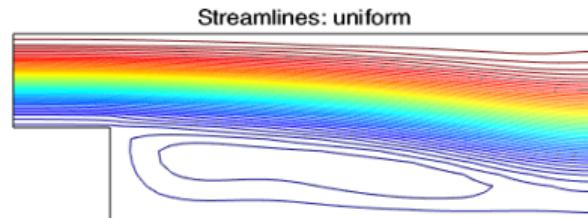
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

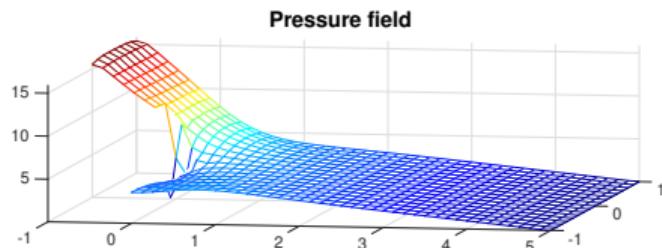
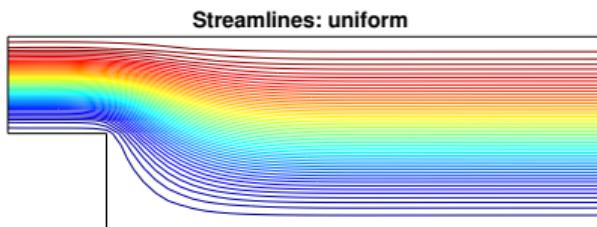
Picard's Iteration



Iteration 5

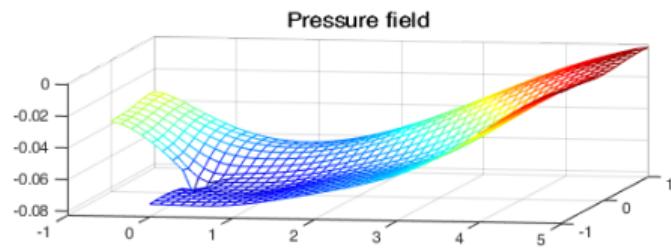
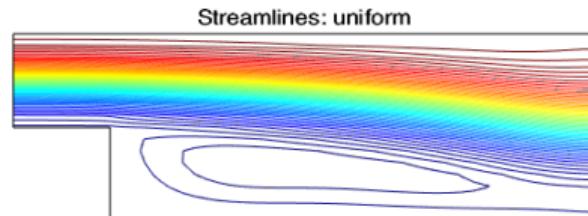
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

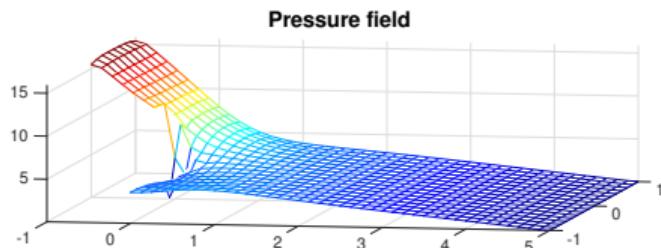
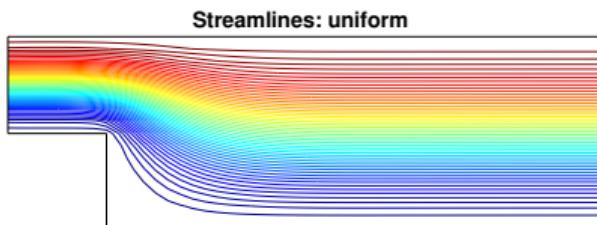
Picard's Iteration



Iteration 6

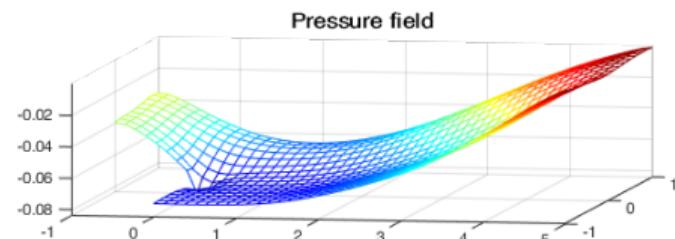
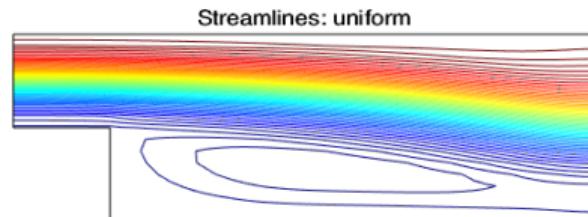
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

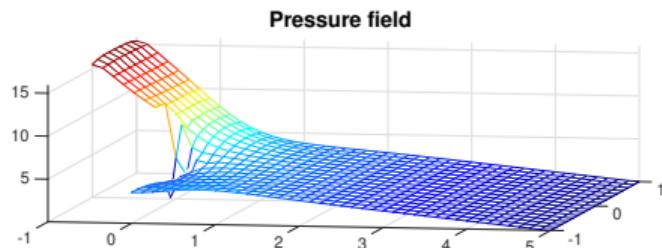
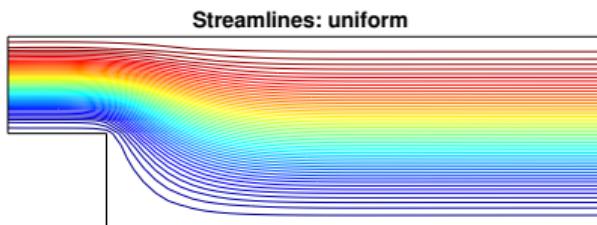
Picard's Iteration



Iteration 7

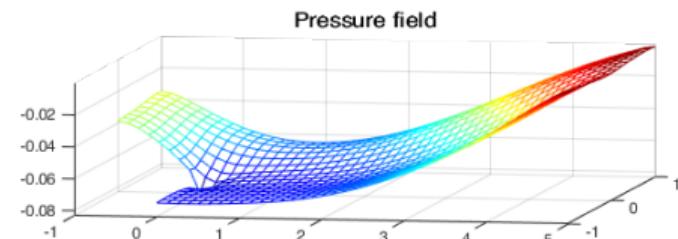
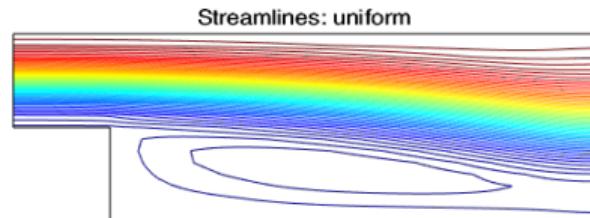
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

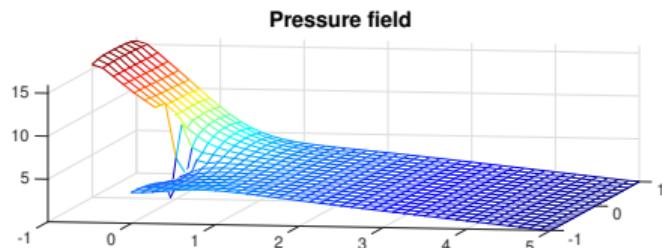
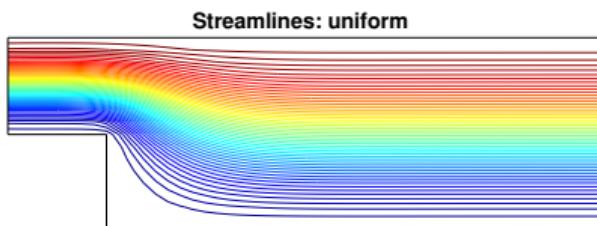
Picard's Iteration



Iteration 8

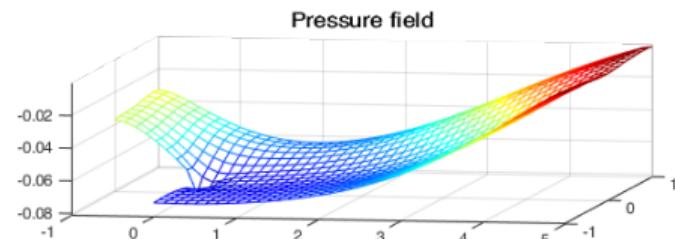
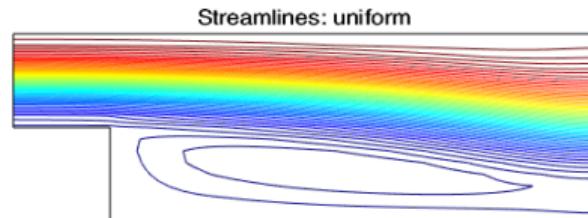
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

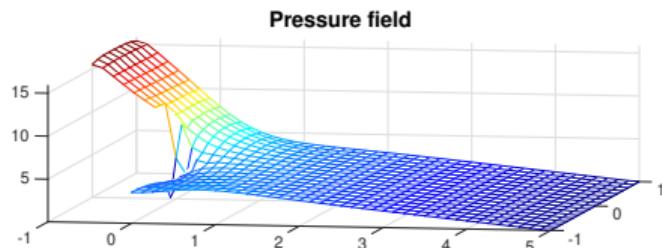
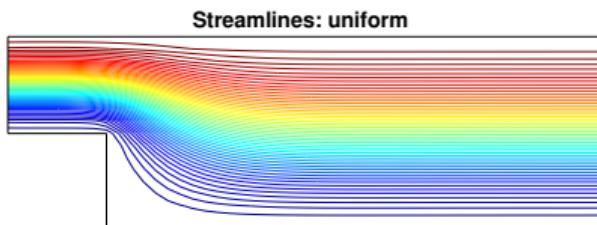
Picard's Iteration



Iteration 9

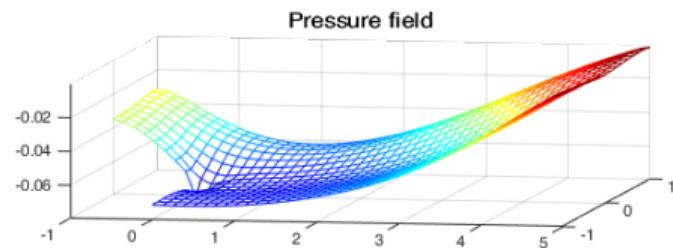
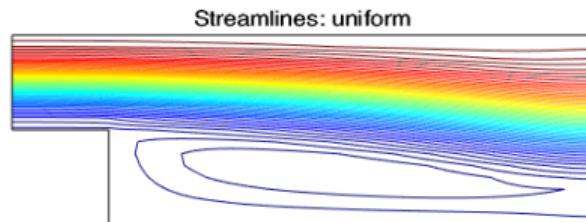
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

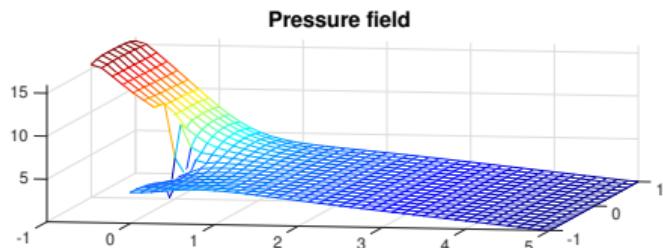
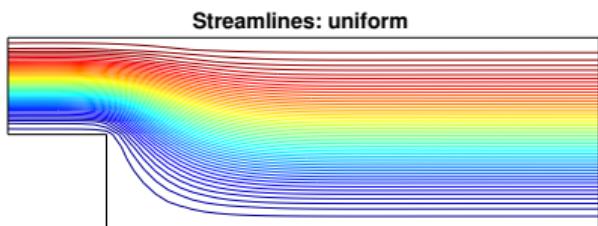
Picard's Iteration



Iteration 10

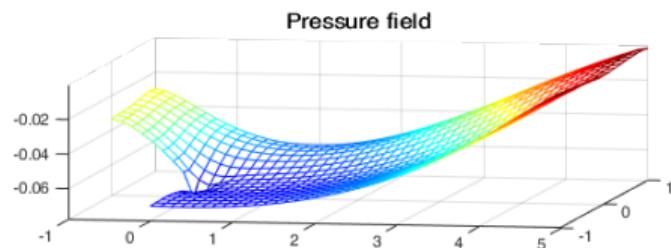
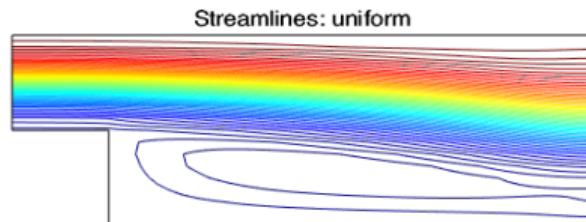
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

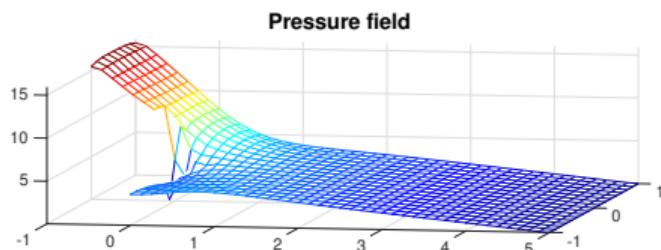
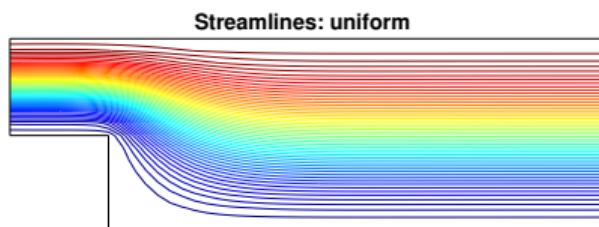
Picard's Iteration



Iteration 11

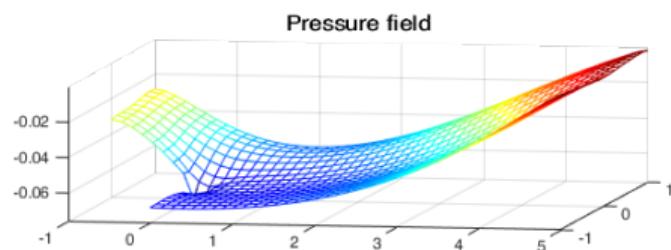
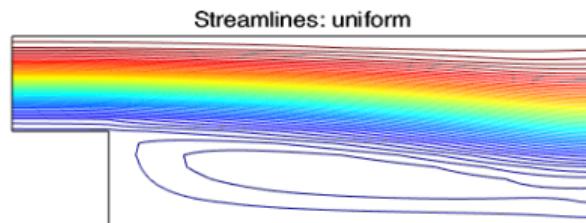
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

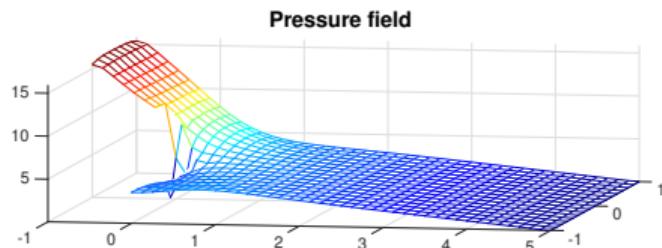
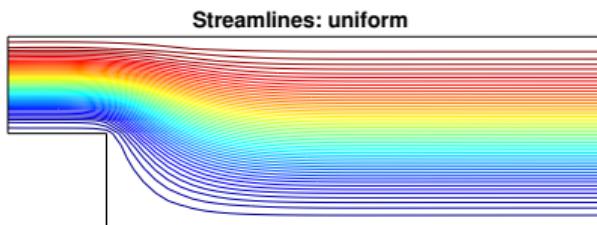
Picard's Iteration



Iteration 12

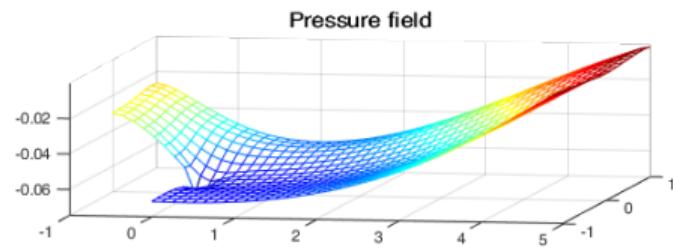
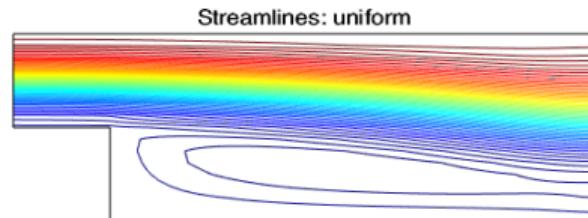
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

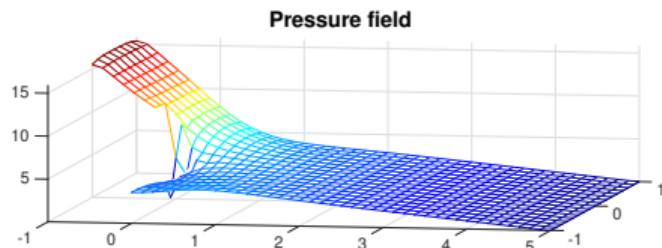
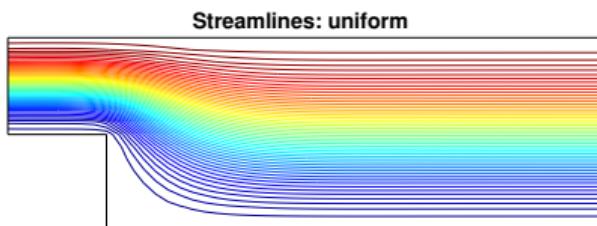
Picard's Iteration



Iteration 13

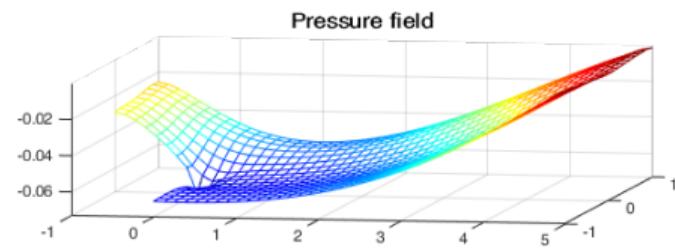
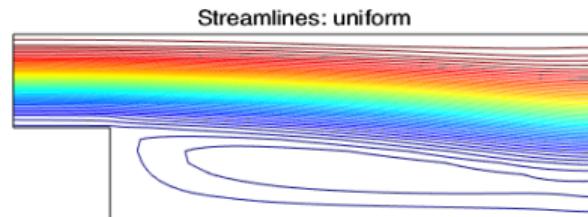
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

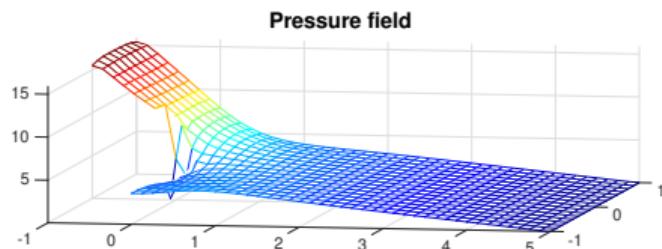
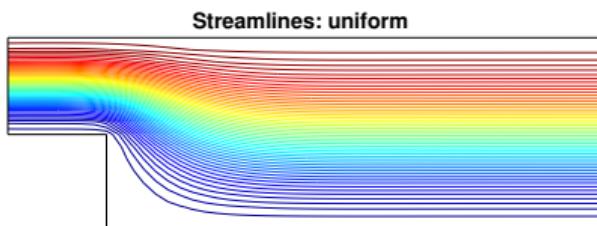
Picard's Iteration



Iteration 14

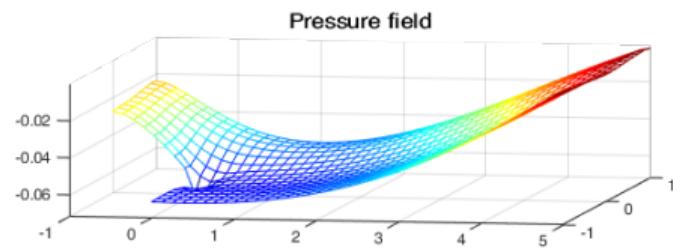
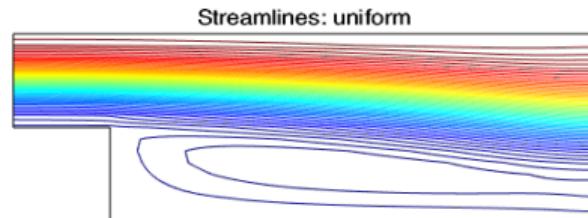
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

Picard's Iteration

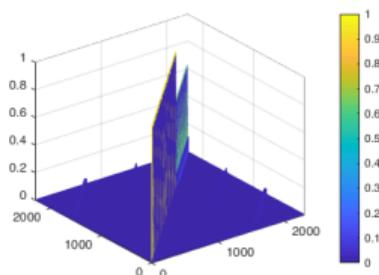
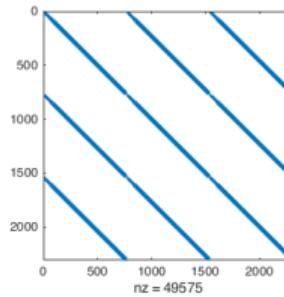
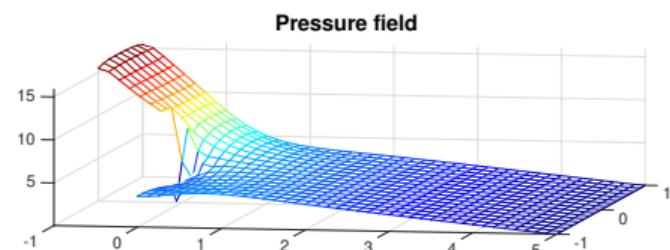
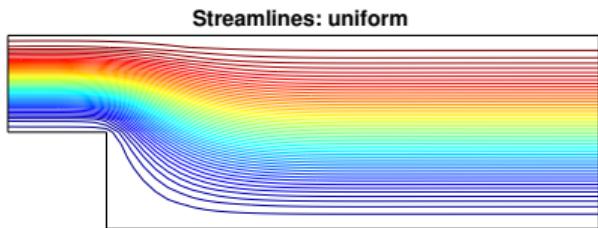


Iteration 15

Navier-Stokes: backward facing step

Newton Method

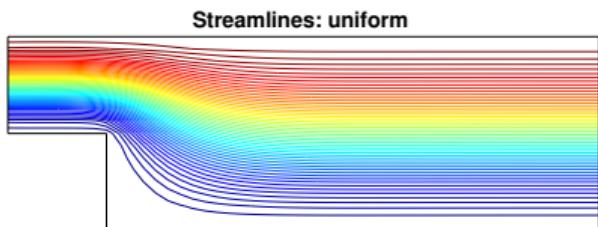
Initial guess:



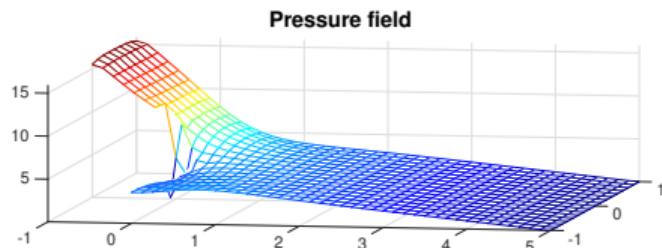
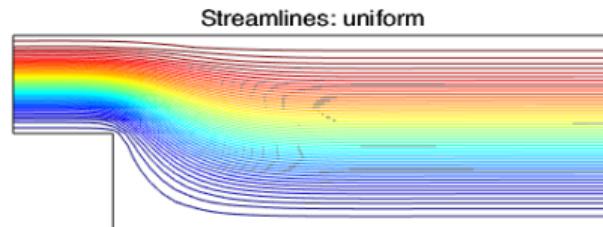
Solution of the associated Stokes problem.

Navier-Stokes: backward facing step

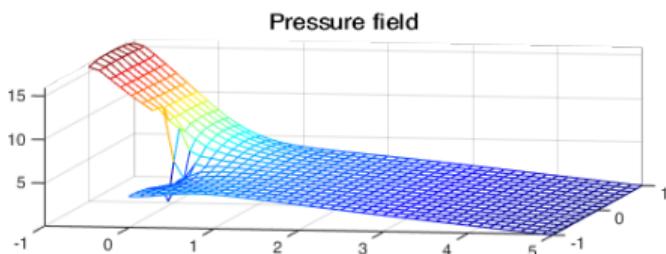
Initial guess:



Newton Method



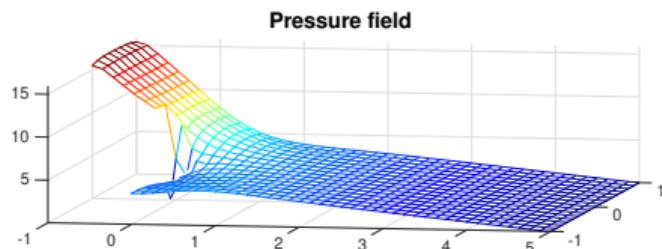
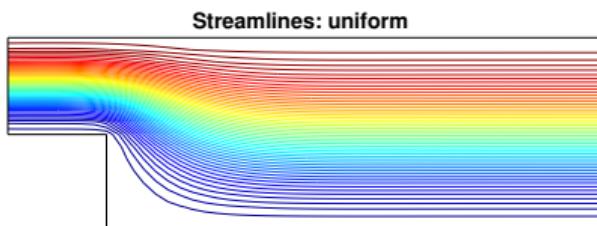
Solution of the associated Stokes problem.



Iteration 1

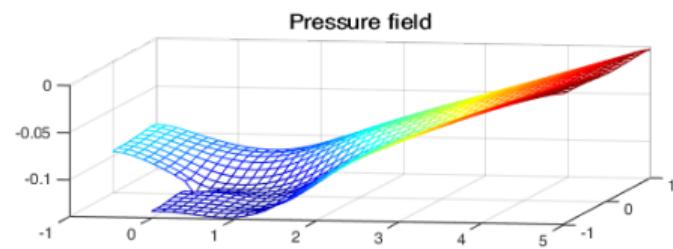
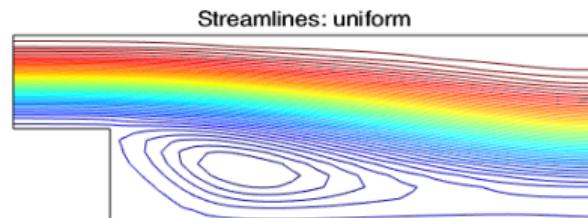
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

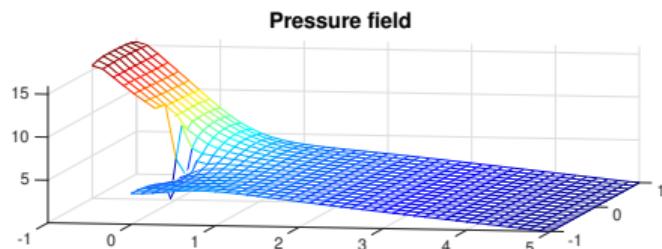
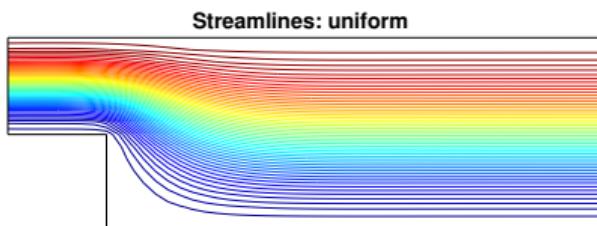
Newton Method



Iteration 2

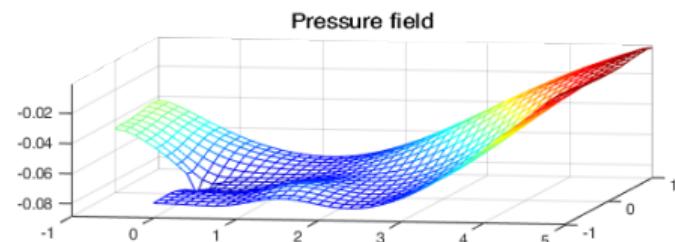
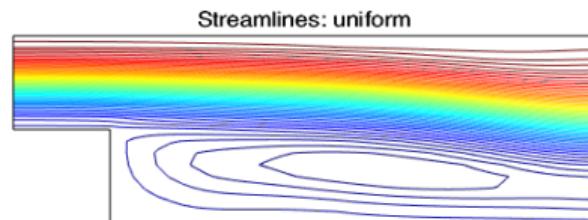
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

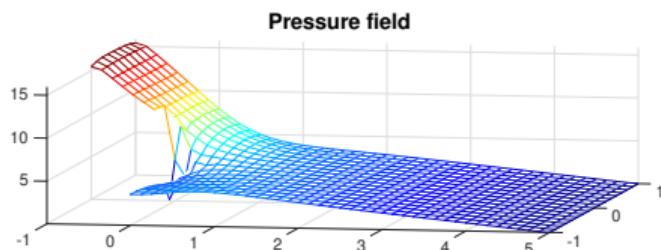
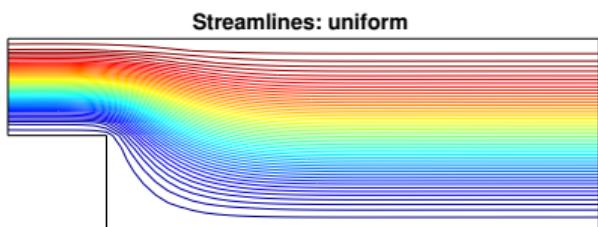
Newton Method



Iteration 3

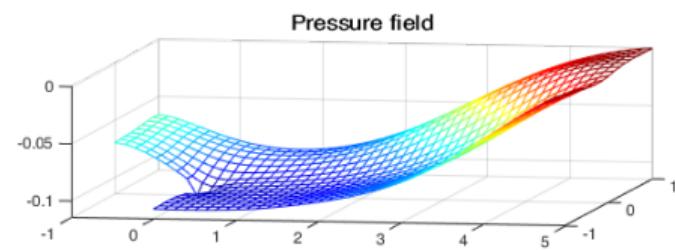
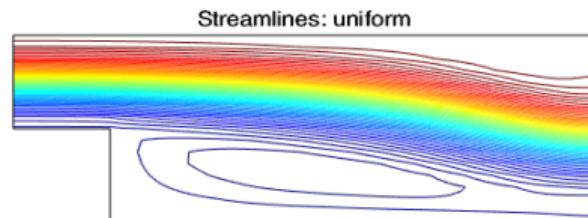
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

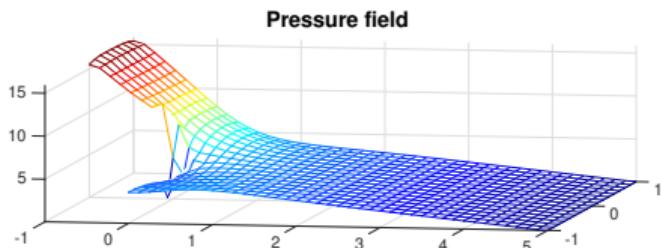
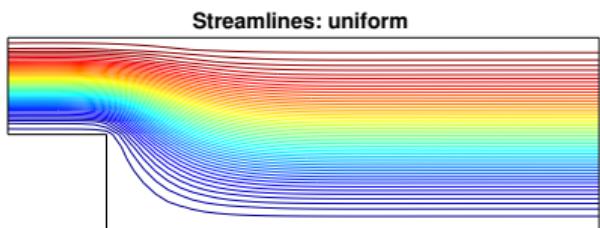
Newton Method



Iteration 4

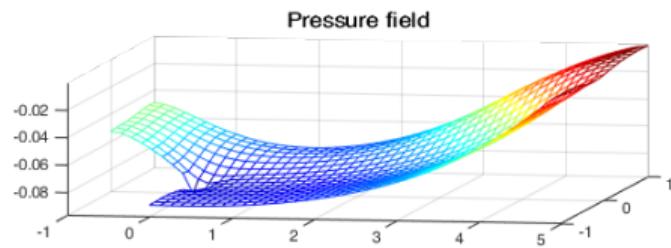
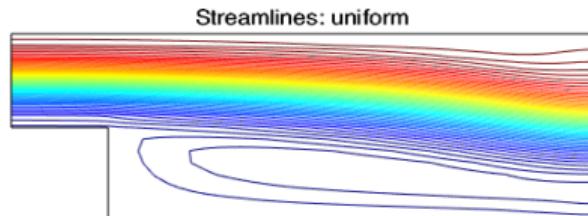
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

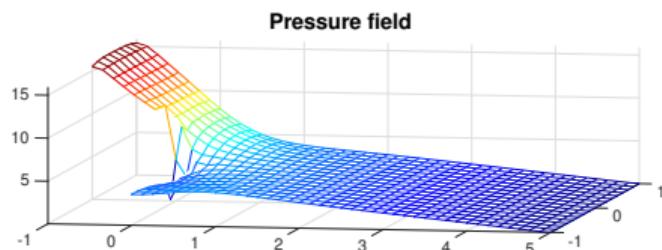
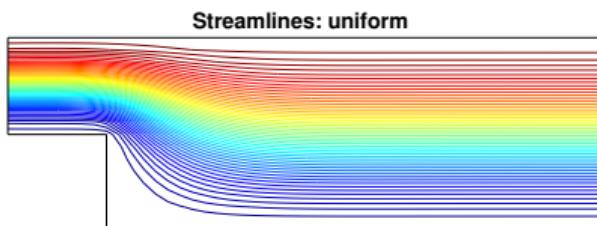
Newton Method



Iteration 5

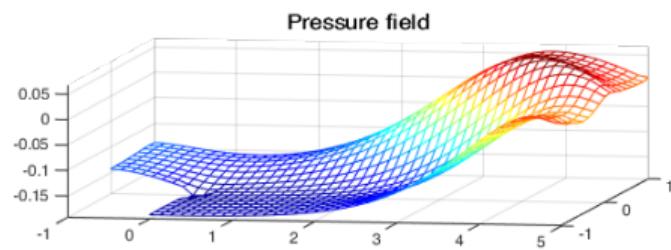
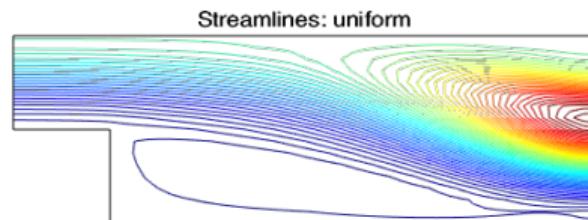
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

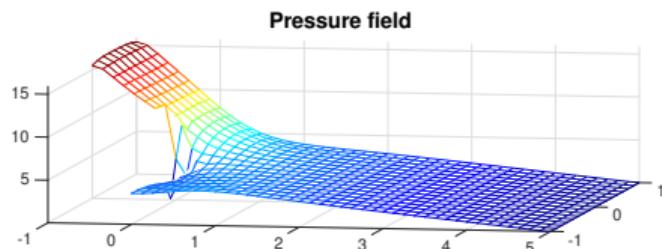
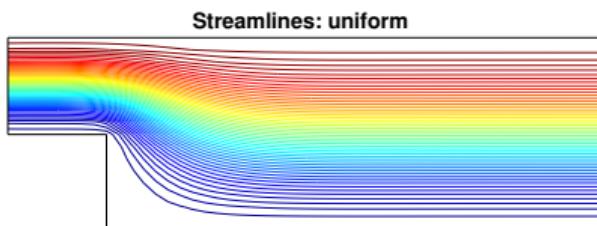
Newton Method



Iteration 6

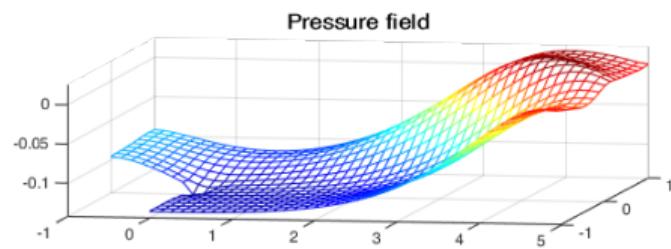
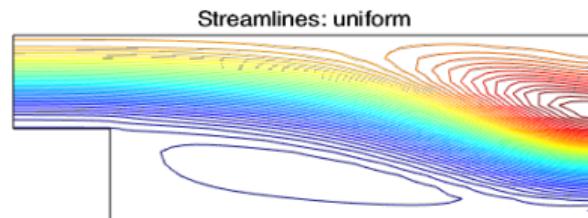
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

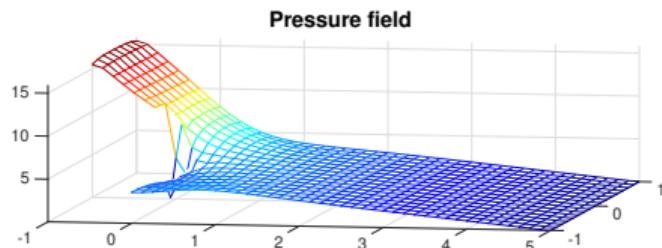
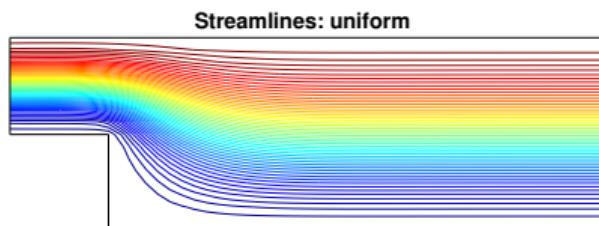
Newton Method



Iteration 7

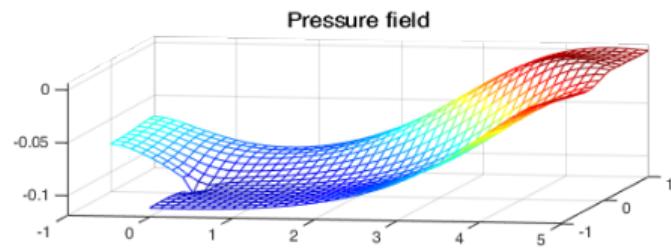
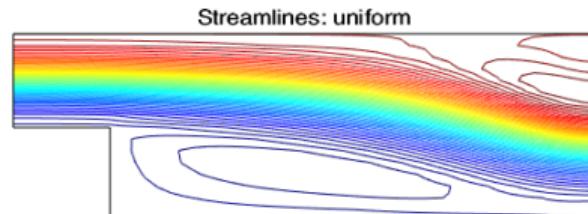
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

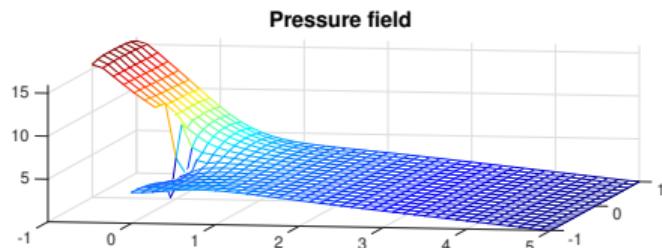
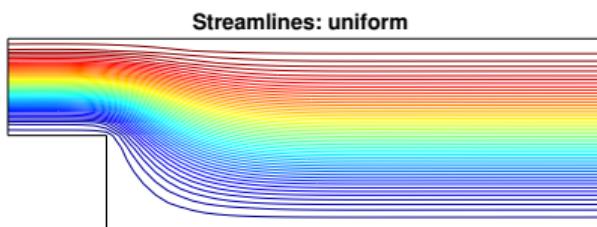
Newton Method



Iteration 8

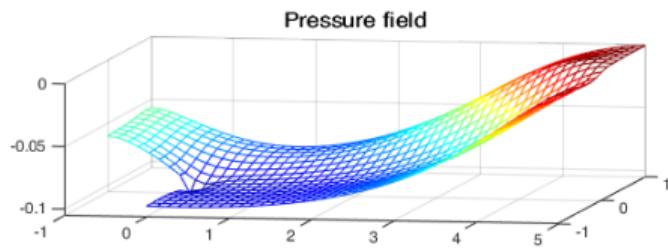
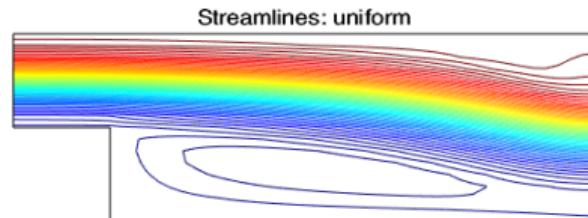
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

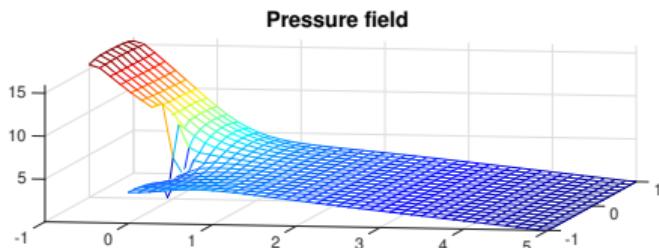
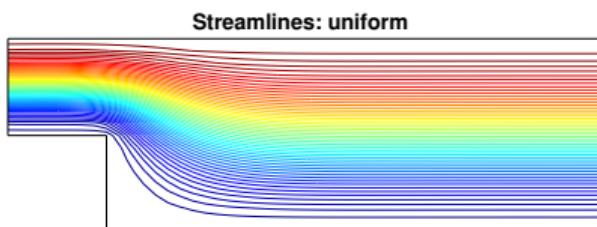
Newton Method



Iteration 9

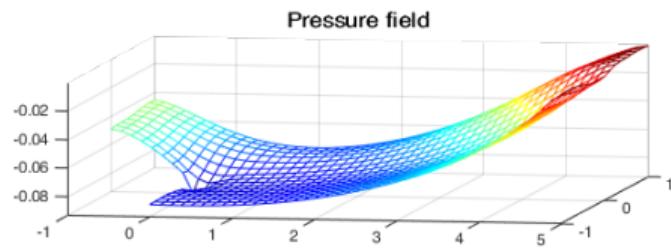
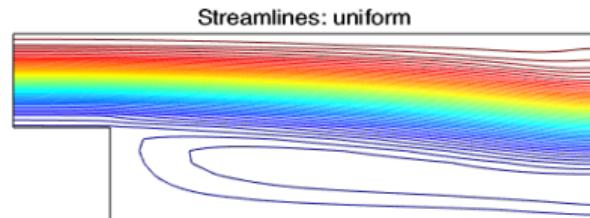
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

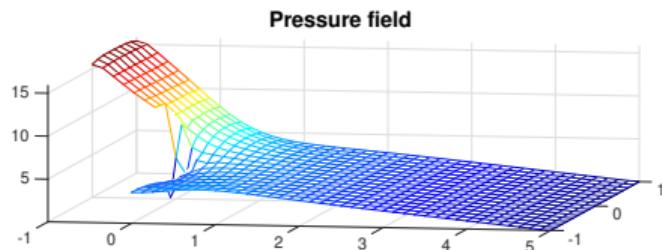
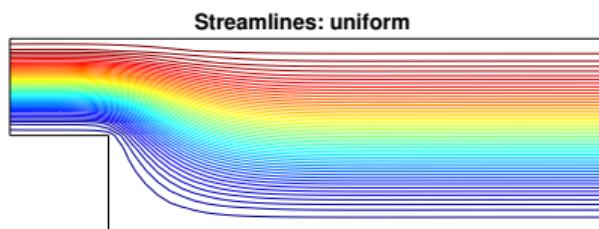
Newton Method



Iteration 10

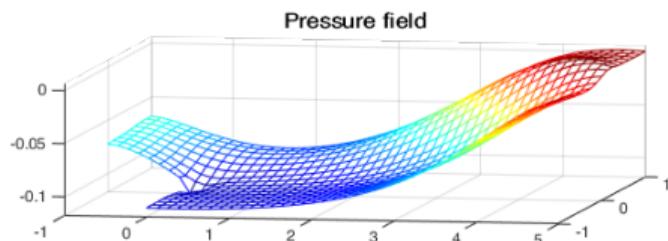
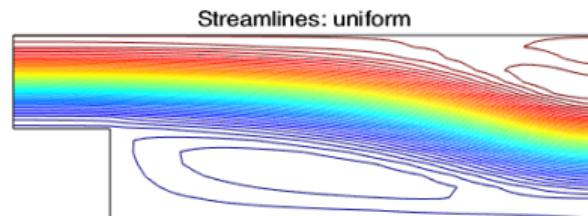
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

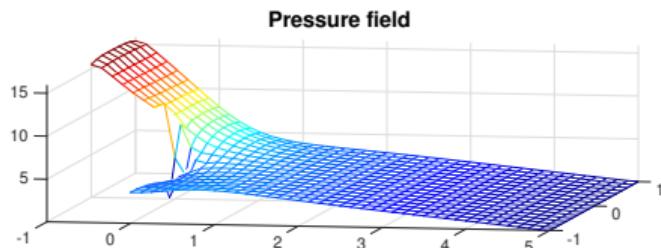
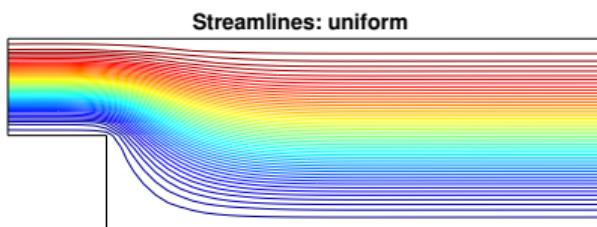
Newton Method



Iteration 11

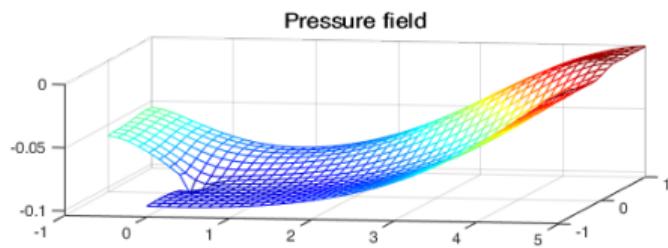
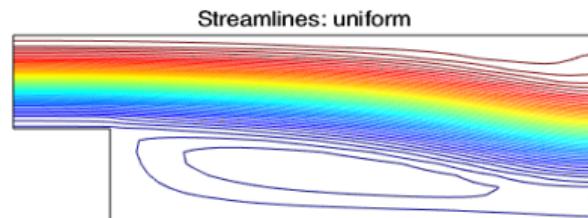
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

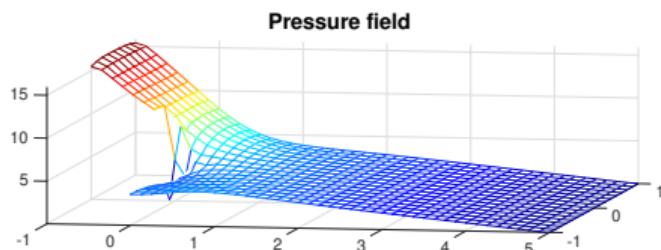
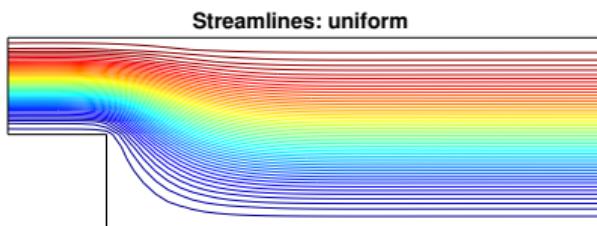
Newton Method



Iteration 12

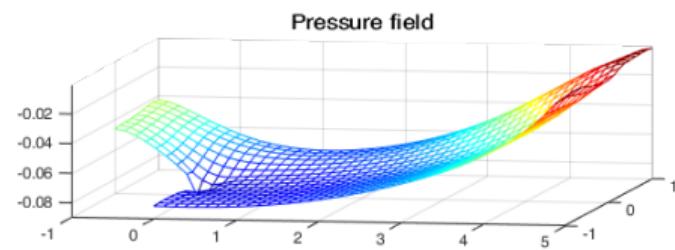
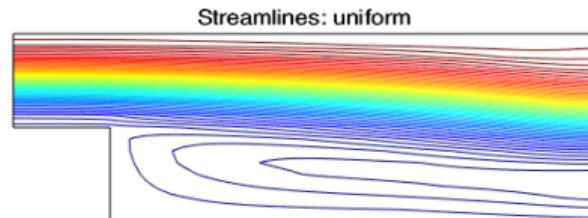
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

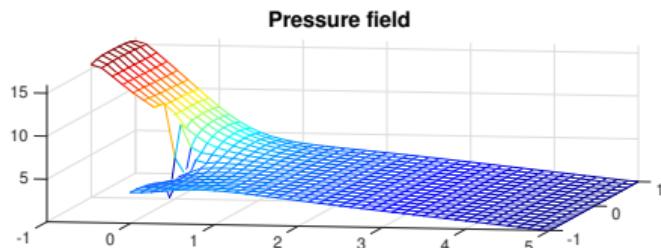
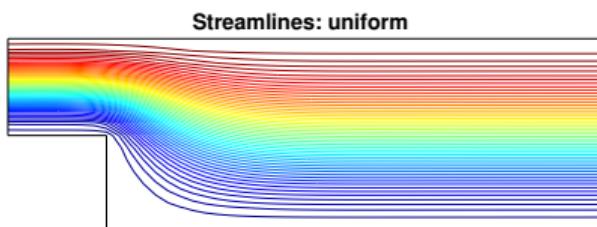
Newton Method



Iteration 13

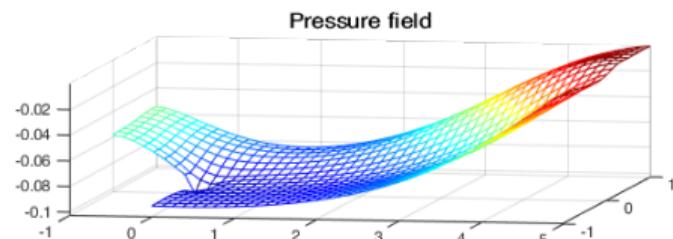
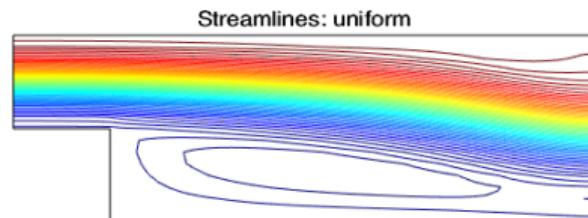
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

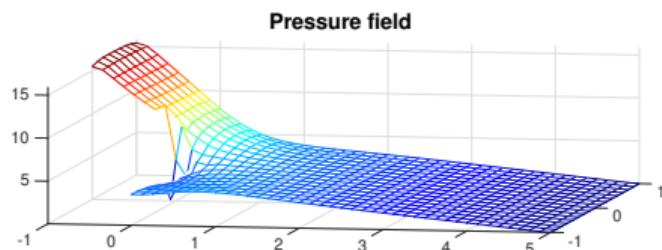
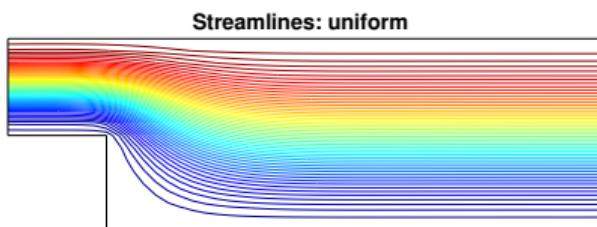
Newton Method



Iteration 14

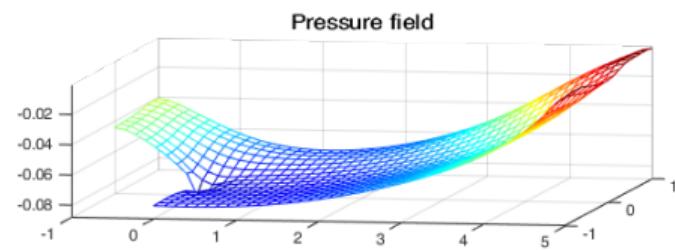
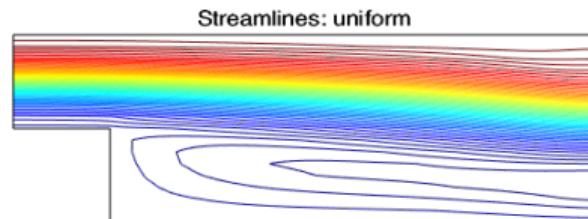
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

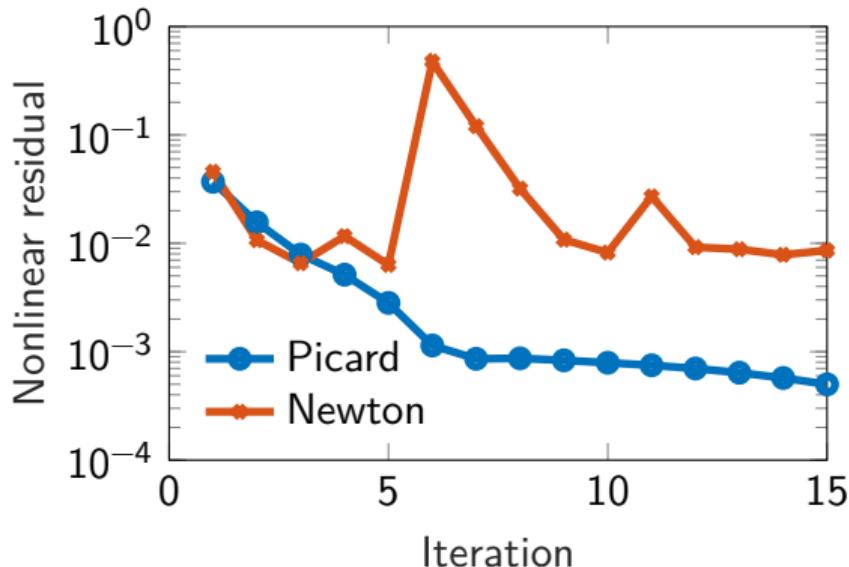
Newton Method



Iteration 15

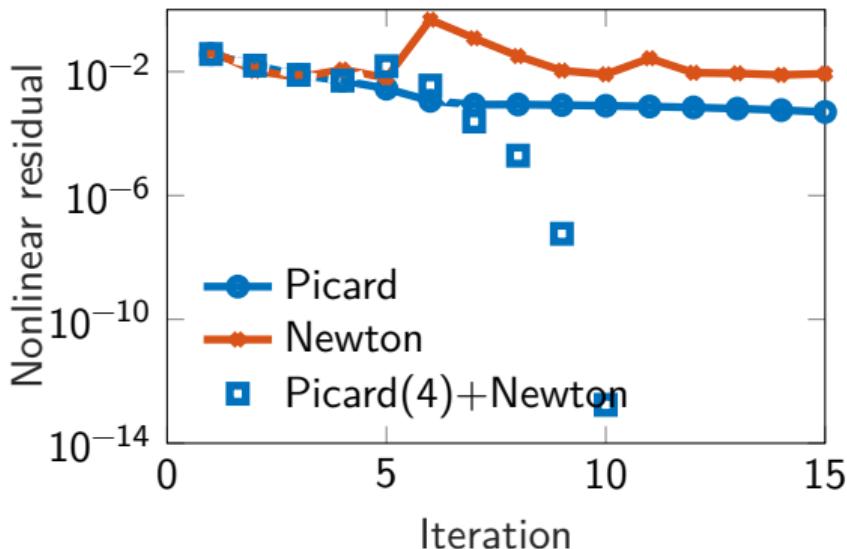
Navier-Stokes: backward facing step

- For this test problem convergence of the Newton method from the Stokes initial data is quite poor, what can we do?



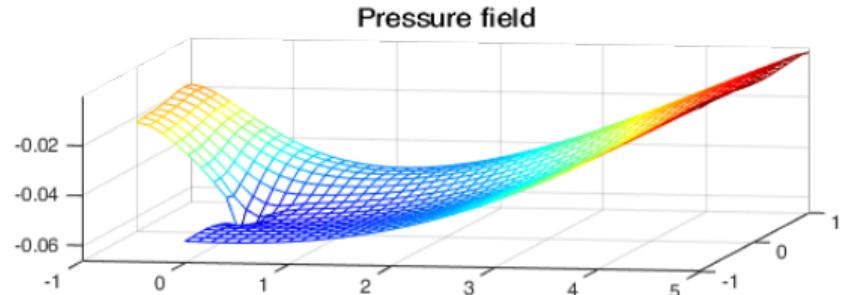
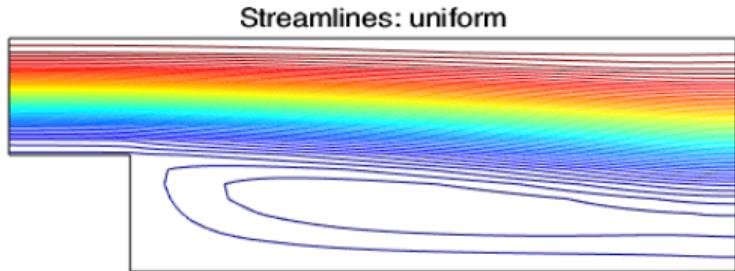
Navier-Stokes: backward facing step

- For this test problem convergence of the Newton method from the Stokes initial data is quite poor, what can we do?
- We start from Stokes, then perform few steps of Picard's iteration, and finally accelerate with Newton.



Navier-Stokes: backward facing step

- For this test problem convergence of the Newton method from the Stokes initial data is quite poor, what can we do?
- We start from Stokes, then perform few steps of Picard's iteration, and finally accelerate with Newton.
- The “mess” doesn't end here – unfortunately or fortunately, I'm not yet sure...*boundary layers, bifurcations, absence of stable flows,...*



Bibliography I

Main references references on the theory of FEM and their connection with iterative solvers

-  Brenner, S. C. and L. R. Scott (2008). *The mathematical theory of finite element methods*. Third. Vol. 15. Texts in Applied Mathematics. Springer, New York, pp. xviii+397. ISBN: 978-0-387-75933-3.
-  Elman, H. C., D. J. Silvester, and A. J. Wathen (2014). *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*. Second. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, pp. xiv+479. ISBN: 978-0-19-967880-8.

Exotic elements literature on exotic elements and approaches

-  Antonietti, P. F., L. Mascotto, and M. Verani (2018). "A multigrid algorithm for the p -version of the virtual element method". In: *ESAIM Math. Model. Numer. Anal.* 52.1, pp. 337–364. ISSN: 0764-583X. DOI: [10.1051/m2an/2018007](https://doi.org/10.1051/m2an/2018007).

Bibliography II

-  Ayuso de Dios, B. et al. (2014). "A simple preconditioner for a discontinuous Galerkin method for the Stokes problem". In: *J. Sci. Comput.* 58.3, pp. 517–547. ISSN: 0885-7474. DOI: [10.1007/s10915-013-9758-0](https://doi.org/10.1007/s10915-013-9758-0).
-  Beirão da Veiga, L. et al. (2014). "The hitchhiker's guide to the virtual element method". In: *Math. Models Methods Appl. Sci.* 24.8, pp. 1541–1573. ISSN: 0218-2025. DOI: [10.1142/S021820251440003X](https://doi.org/10.1142/S021820251440003X).
-  — (2016). "Virtual element method for general second-order elliptic problems on polygonal meshes". In: *Math. Models Methods Appl. Sci.* 26.4, pp. 729–750. ISSN: 0218-2025. DOI: [10.1142/S0218202516500160](https://doi.org/10.1142/S0218202516500160).
-  Cockburn, B., G. E. Karniadakis, and C.-W. Shu, eds. (2000). *Discontinuous Galerkin methods*. Vol. 11. Lecture Notes in Computational Science and Engineering. Theory, computation and applications, Papers from the 1st International Symposium held in Newport, RI, May 24–26, 1999. Springer-Verlag, Berlin, pp. xii+470. ISBN: 3-540-66787-3. DOI: [10.1007/978-3-642-59721-3](https://doi.org/10.1007/978-3-642-59721-3).

Bibliography III

-  Cottrell, J. A., T. J. R. Hughes, and Y. Bazilevs (2009). *Isogeometric analysis. Toward integration of CAD and FEA*. John Wiley & Sons, Ltd., Chichester, pp. xvi+335. ISBN: 978-0-470-74873-2. DOI: [10.1002/9780470749081](https://doi.org/10.1002/9780470749081).
-  Dassi, F. and S. Scacchi (2020). "Parallel block preconditioners for three-dimensional virtual element discretizations of saddle-point problems". In: *Comput. Methods Appl. Mech. Engrg.* 372, pp. 113424, 17. ISSN: 0045-7825. DOI: [10.1016/j.cma.2020.113424](https://doi.org/10.1016/j.cma.2020.113424).
-  Dobrev, V. A. et al. (2006). "Two-level preconditioning of discontinuous Galerkin approximations of second-order elliptic equations". In: *Numer. Linear Algebra Appl.* 13.9, pp. 753–770. ISSN: 1070-5325. DOI: [10.1002/nla.504](https://doi.org/10.1002/nla.504).
-  Donatelli, M. et al. (2015). "Robust and optimal multi-iterative techniques for IgA Galerkin linear systems". In: *Comput. Methods Appl. Mech. Engrg.* 284, pp. 230–264. ISSN: 0045-7825. DOI: [10.1016/j.cma.2014.06.001](https://doi.org/10.1016/j.cma.2014.06.001).

Bibliography IV

-  Giannelli, C., B. Jüttler, and H. Speleers (2012). "THB-splines: the truncated basis for hierarchical splines". In: *Comput. Aided Geom. Design* 29.7, pp. 485–498. ISSN: 0167-8396. DOI: [10.1016/j.cagd.2012.03.025](https://doi.org/10.1016/j.cagd.2012.03.025).
-  Horníková, H., C. Vuik, and J. Egermaier (2021). "A comparison of block preconditioners for isogeometric analysis discretizations of the incompressible Navier-Stokes equations". In: *Internat. J. Numer. Methods Fluids* 93.6, pp. 1788–1815. ISSN: 0271-2091. DOI: [10.1002/fld.4952](https://doi.org/10.1002/fld.4952).
-  Patrizi, F. and T. Dokken (2020). "Linear dependence of bivariate minimal support and locally refined B-splines over LR-meshes". In: *Comput. Aided Geom. Design* 77, pp. 101803, 22. ISSN: 0167-8396. DOI: [10.1016/j.cagd.2019.101803](https://doi.org/10.1016/j.cagd.2019.101803).
-  Sangalli, G. and M. Tani (2016). "Isogeometric preconditioners based on fast solvers for the Sylvester equation". In: *SIAM J. Sci. Comput.* 38.6, A3644–A3671. ISSN: 1064-8275. DOI: [10.1137/16M1062788](https://doi.org/10.1137/16M1062788).

Bibliography V

Poisson problem references for the mixed form of the Poisson problem

-  Boffi, D., F. Brezzi, and M. Fortin (2013). *Mixed finite element methods and applications*. Vol. 44. Springer Series in Computational Mathematics. Springer, Heidelberg, pp. xiv+685. ISBN: 978-3-642-36518-8; 978-3-642-36519-5. DOI: [10.1007/978-3-642-36519-5](https://doi.org/10.1007/978-3-642-36519-5). URL: <https://doi.org/10.1007/978-3-642-36519-5>.
-  Brezzi, F., J. Douglas Jr., and L. D. Marini (1985). “Two families of mixed finite elements for second order elliptic problems”. In: *Numer. Math.* 47.2, pp. 217–235. ISSN: 0029-599X. DOI: [10.1007/BF01389710](https://doi.org/10.1007/BF01389710). URL: <https://doi.org/10.1007/BF01389710>.
-  Raviart, P.-A. and J. M. Thomas (1977). “A mixed finite element method for 2nd order elliptic problems”. In: *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, 292–315. Lecture Notes in Math., Vol. 606.

Bibliography VI

-  Rusten, T. and R. Winther (1992). "A preconditioned iterative method for saddlepoint problems". In: vol. 13. 3. Iterative methods in numerical linear algebra (Copper Mountain, CO, 1990), pp. 887–904. DOI: [10.1137/0613054](https://doi.org/10.1137/0613054). URL: <https://doi.org/10.1137/0613054>.
-  Zhang, S. (2015). "An efficient implementation of Brezzi-Douglas-Marini (BDM) mixed finite element method in MATLAB". In: *arXiv preprint arXiv:1508.06445*.

Stokes and Navier-Stokes references for the solution of the Stokes and Navier-Stokes problems

-  Quarteroni, A. and A. Valli (1994). *Numerical approximation of partial differential equations*. Vol. 23. Springer Series in Computational Mathematics. Springer-Verlag, Berlin, pp. xvi+543. ISBN: 3-540-57111-6.