

An introduction to fractional calculus

Fundamental ideas and numerics

Fabio Durastante

Università di Pisa

✉ fabio.durastante@unipi.it

🌐 fdurastante.github.io



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Sylvester with quasiseparable matrices

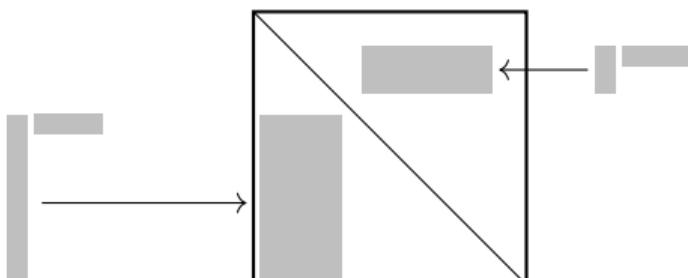
Let's start again from the problem we wanted to solve

$$AX + XB^T = C, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{m \times m}, \quad X, C \in \mathbb{R}^{n \times m},$$

with A , B , and C **quasiseparable**

Quasiseparable matrix

A matrix A is *quasiseparable* of order k if the maximum of the ranks of all its submatrices contained in the strictly upper or lower part is less or equal than k .



💡 We have seen that A , B , and C quasiseparable \Rightarrow

X with **decay of the singular values** of off-diagonal blocks of C .

Sylvester with quasiseparable matrices

Theorem (Massei, Palitta, and Robol 2018, Theorem 2.12)

Let A, B be matrices of quasiseparable rank k_A and k_B respectively and such that $W(A) \subseteq E$ and $W(-B) \subseteq F$. Consider the Sylvester equation $AX + XB = C$, with C of quasiseparable rank k_C . Then a generic off-diagonal block Y of the solution X satisfies

$$\frac{\sigma_{1+k\ell}(Y)}{\sigma_1(Y)} \leq \mathcal{C}^2 \cdot Z_\ell(E, F), \quad k := k_A + k_B + k_C.$$

Where $Z_\ell(E, F)$ is the solution of the **Zolotarev problem**

$$Z_\ell(E, F) \triangleq \inf_{r(x) \in \mathcal{R}_{\ell,\ell}} \frac{\max_{x \in E} |r(x)|}{\min_{y \in F} |r(y)|}, \quad \ell \geq 1,$$

for $\mathcal{R}_{\ell,\ell}$ is the set of rational functions of degree at most (ℓ, ℓ) , and \mathcal{C} is the Crouzeix universal constant.

Sylvester with quasiseparable matrices

- ③ Do the *decaying singular values* in the blocks implies the existence of a **quasiseparable approximant**?

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ϵ -quasiseparable matrices of rank k (ϵ -qsrank k)

We say that A has **ϵ -quasiseparable rank k** if, for every off-diagonal block Y , $\sigma_{k+1}(Y) \leq \epsilon$. If the property holds for the lower (respectively upper) offdiagonal blocks, we say that A has lower (respectively upper) ϵ -quasiseparable rank k .

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Submatrices and off-diagonal blocks

If a matrix A has ϵ -quasiseparable rank k , then any of its principal submatrix A' has ϵ -quasiseparable rank k .

Any off-diagonal block Y of A' is also an off-diagonal block of $A \Rightarrow \sigma_{k+1}(Y) \leq \epsilon$.

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For \oplus the direct sum

Technical lemma

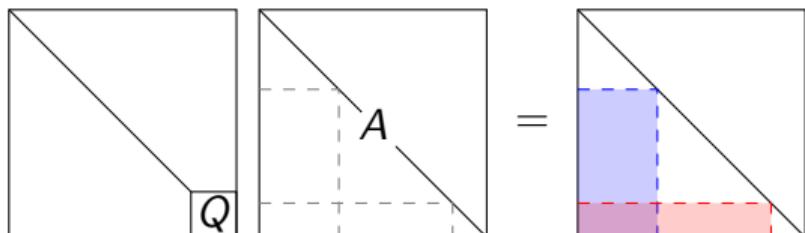
Let A be a matrix with ϵ -quasiseparable rank k , Q any $(k + 1) \times (k + 1)$ unitary matrix. Then, $(I_{n-k-1} \oplus Q)A$ also has ϵ -quasiseparable rank k .

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Q acts on the **tall block of A** without changing its singular values, while **the small one** has small rank thanks to the small number of rows.

Sylvester with quasiseparable matrices

Theorem (Massei, Palitta, and Robol 2018, Theorem 2.16)

Let A be of ϵ -quasiseparable rank k , for $\epsilon > 0$. Then, there exists a matrix δA of norm bounded by $\|\delta A\|_2 \leq 2\sqrt{n} \cdot \epsilon$ so that $A + \delta A$ is k -quasiseparable.

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- ➋ If the **spectra** of A and $-B$ are **well-separated** in the Zolotarev sense, we can **preserve structure**!

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💡 Hierarchical matrix formats!

Hierarchical matrix formats

There exist **many hierarchical matrix formats**:

- 🔧 \mathcal{H} -Matrices,
- 🔧 \mathcal{H}^2 -Matrices,
- 🔧 Hierarchical **O**ff-**D**iagonal **L**ow-**R**ank (HODLR),
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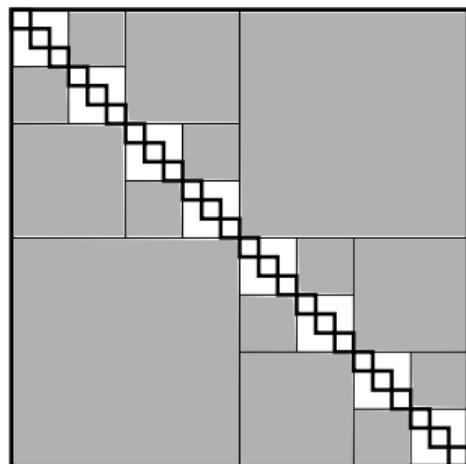
- 🔧 \mathcal{H} -Matrices,
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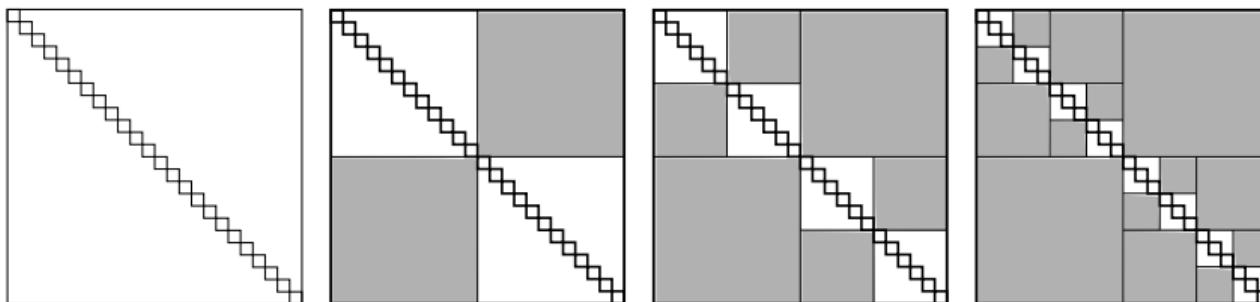
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HODLR-matrices

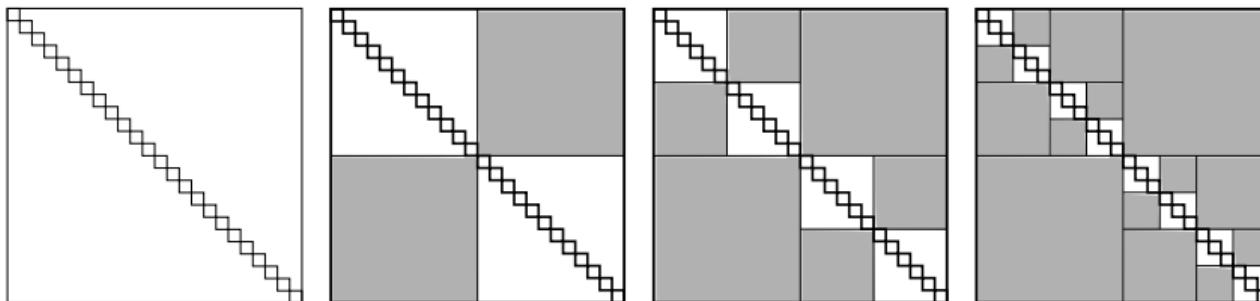
The general idea:



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- 📦 The **grey blocks** are **low rank matrices** represented **in** a **compressed form**,
- 📦 the *diagonal blocks* in the last step are *stored as dense matrices*.
- 📦 We need now a **formal definition** and a way to **define operations**.

HODLR-matrices: trees

Cluster tree

Given $n \in \mathbb{N}$, let \mathcal{T}_p be a completely balanced binary tree of depth p whose nodes are subsets of $\{1, \dots, n\}$. We say that \mathcal{T}_p is a *cluster tree* if it satisfies:

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-  The root is $I_1^0 := I = \{1, \dots, n\}$.
-  The nodes at level ℓ , denoted by $I_1^\ell, \dots, I_{2^\ell}^\ell$, form a partitioning of $\{1, \dots, n\}$ into consecutive indices:

$$I_i^\ell = \{n_{i-1}^{(\ell)} + 1, \dots, n_i^{(\ell)} - 1, n_i^{(\ell)}\}$$

for some integers $0 = n_0^{(\ell)} \leq n_1^{(\ell)} \leq \dots \leq n_{2^\ell}^{(\ell)} = n$, $\ell = 0, \dots, p$. In particular, if $n_{i-1}^{(\ell)} = n_i^{(\ell)}$ then $I_i^\ell = \emptyset$.

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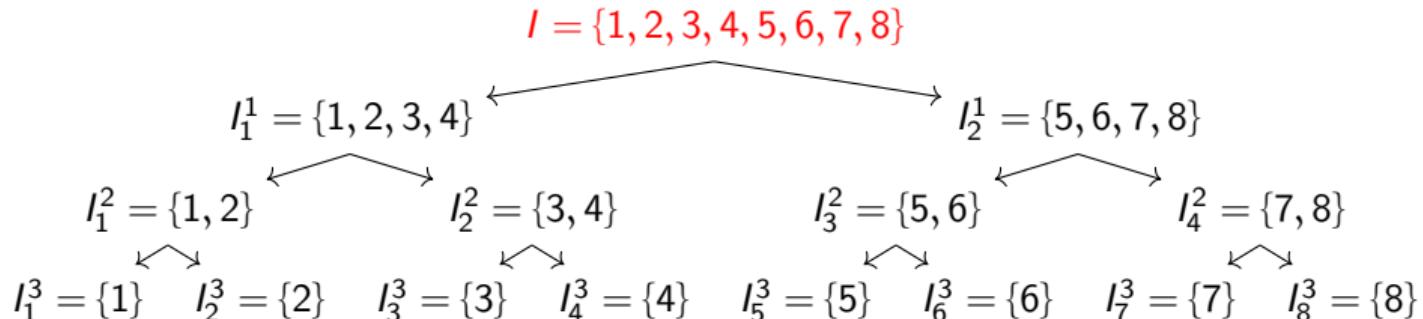
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- Nodes at a level ℓ partition A into a $2^\ell \times 2^\ell$ block matrix with blocks $\{A(I_i^\ell, I_j^\ell)\}_{i,j=1}^{2^\ell}$.

HODLR-matrices: trees

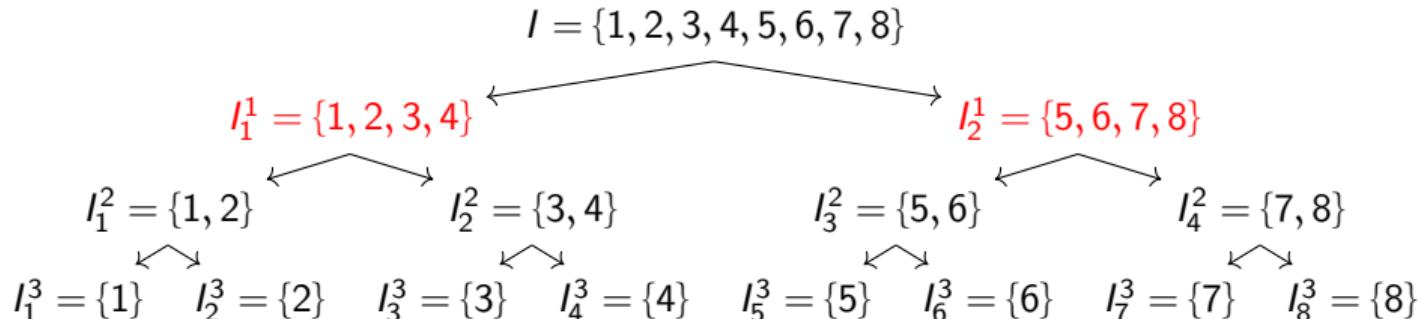


$$\ell = 0$$

👉 The root $I = \{1, \dots, 8\}$,

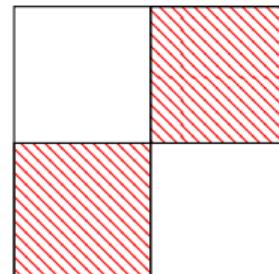


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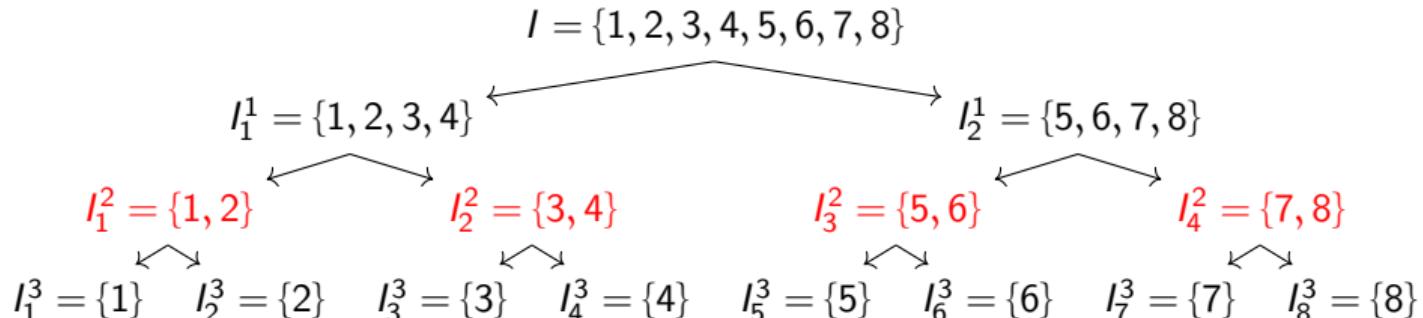


- leaf The root $I = \{1, \dots, 8\}$,
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HODLR-matrices: trees



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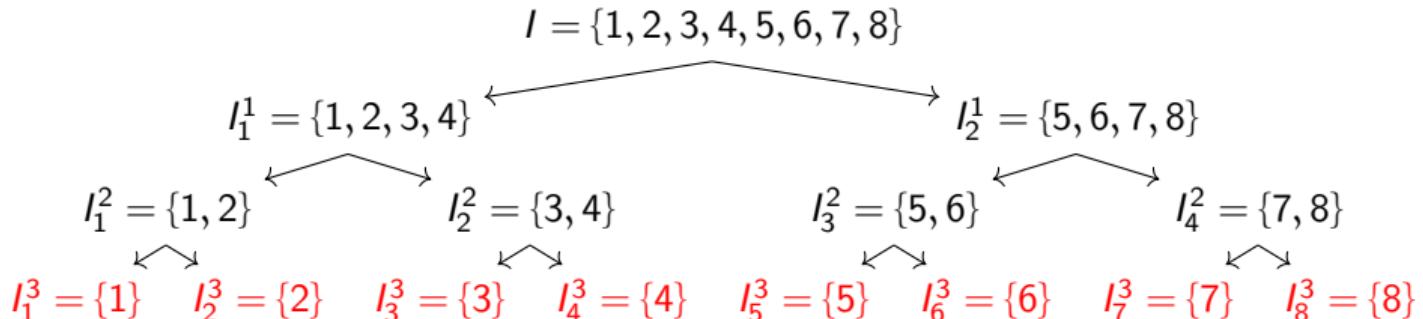
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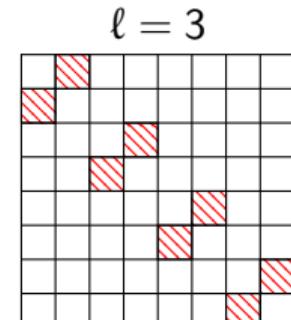
Nodes at level 2: $\mathcal{B}(I_1^1) = \{I_1^2, I_2^2\}$, $\mathcal{B}(I_2^1) = \{I_3^2, I_4^2\}$,

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- Nodes at level 3: $\mathcal{P}(I_1^2) = \{I_1^3, I_2^3\}$, ..., $\mathcal{P}(I_4^2) = \{I_7^3, I_8^3\}$.



HODLR-matrices: definition

HODLR matrix

Let $A \in \mathbb{R}^{n \times n}$ and consider a cluster tree \mathcal{T}_p .

- Given $k \in \mathbb{N}$, A is said to be a (\mathcal{T}_p, k) -HODLR matrix if every off-diagonal block

$$A(I_i^\ell, I_j^\ell) \quad \text{such that } I_i^\ell \text{ and } I_j^\ell \text{ are siblings in } \mathcal{T}_p, \quad \ell = 1, \dots, p,$$

has rank at most k .

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- ⚙️ The classical choice is to have a **binary tree**, i.e., $n = 2^p n_{\min}$.

HODLR-matrices: occupied space

If we assume **identical ranks** k and a **balanced partitioning** then

- Storage for off-diagonal blocks $A(I_i^\ell, I_j^\ell) = U_i^{(\ell)}(V_j^{(\ell)})^T$, $U_i^{(\ell)}, V_j^{(\ell)} \in \mathbb{R}^{m_\ell \times k}$.
On level $\ell > 0$ there are 2^ℓ off-diagonal blocks

$$2k \sum_{\ell=1}^p 2^\ell m_\ell = 2kn_0 \sum_{\ell=1}^p 2^\ell 2^{p-\ell} 2kn_0 p 2^p = 2knp = 2kn \log_2(n/n_0),$$

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- Both **requirements** on ranks and partitioning can be **relaxed to obtain similar results**.

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If A is **sparse**:

- 💻 Use a **two sided Lanczos method** only requiring matrix-vector multiplications with an off-diagonal block and its transpose, combined with recompression to each off-diagonal block.

HODLR-matrices: building the representation

⚠ Is **non trivial** to construct structured representations efficiently, especially if you want to avoid computing the whole n^2 coefficients!

👉 Build a **cluster tree** \mathcal{T}_p for the given index set,

If A is **dense**:

- 💻 Use Householder **QR decomposition** with column pivoting or **SVD** on off-diagonal blocks,
- 🔧 The **rank** of each off-diagonal block $A(I_i^P, I_j^P)$ is chosen such that the spectral **norm of the approximation error** is bounded by ϵ times $\|A(I_i^P, I_j^P)\|_2$.

If A is **sparse**:

- 💻 Use a **two sided Lanczos method** only requiring matrix-vector multiplications with an off-diagonal block and its transpose, combined with recompression to each off-diagonal block.

If A is **structured** use an *ad-hoc* constructor!

HODLR of Grünwald–Letnikov

Theorem (Fiedler 2010, Theorem A)

Let \mathbf{x}, \mathbf{y} two real vectors of length N , with ascending and descending ordered entries, respectively. Moreover, we denote with $C(\mathbf{x}, \mathbf{y})$ the Cauchy matrix defined by

$$C_{ij} = \frac{1}{x_i - y_j}, \quad i, j = 1, \dots, N.$$

If $C(\mathbf{x}, \mathbf{y}) = C(\mathbf{x}, \mathbf{y})^T$, $x_i \in [a, b]$, $y_j \in [c, d]$ with $a > d$, then $C(\mathbf{x}, \mathbf{y})$ is positive definite.

HODLR of Grünwald–Letnikov

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Theorem (Beckermann and Townsend 2019, Theorem 5.5)

Let H be a positive semidefinite Hankel matrix of size N . Then, the ϵ -rank of H is bounded by

$$\text{rank}_\epsilon(H) \leq 2 + 2 \left\lceil \frac{2}{\pi^2} \log \left(\frac{4}{\pi} N \right) \log \left(\frac{16}{\epsilon} \right) \right\rceil \triangleq \mathfrak{B}(N, \epsilon).$$

HODLR of Grünwald–Letnikov

We need to work with $G_N \in \mathbb{R}^{N \times N}$

$$G_N = - \begin{bmatrix} g_1^{(\alpha)} & g_0^{(\alpha)} & 0 & \cdots & 0 & 0 \\ g_2^{(\alpha)} & g_1^{(\alpha)} & g_0^{(\alpha)} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ g_{N-1}^{(\alpha)} & \ddots & \ddots & \ddots & g_1^{(\alpha)} & g_0^{(\alpha)} \\ g_N^{(\alpha)} & g_{N-1}^{(\alpha)} & \cdots & \cdots & g_2^{(\alpha)} & g_1^{(\alpha)} \end{bmatrix}$$

Lemma (Massei, Mazza, and Robol 2019)

Consider the Hankel matrix H defined as

$$H = (h_{ij}), \quad h_{ij} = g_{i+j}^{(\alpha)},$$

for $1 \leq \alpha \leq 2$. Then, H is positive semidefinite.

- ↗ Show that H is obtained as the sum of a positive definite Cauchy matrix and a positive semidefinite matrix.
- ↗ Use the result by Beckermann and Townsend 2019.

HODLR of Grünwald–Letnikov

Proof. For $k \geq 2$ we rewrite $g_k^{(\alpha)}$ as

$$\begin{aligned} g_k^{(\alpha)} &= \frac{(-1)^k}{k!} \alpha(\alpha - 1) \dots (\alpha - k + 1) \\ &= \frac{\alpha(\alpha - 1)}{k!} (k - \alpha - 1)(k - \alpha - 2) \dots (2 - \alpha) \\ &= \alpha(\alpha - 1) \frac{\Gamma(k - \alpha)}{\Gamma(k + 1)\Gamma(2 - \alpha)}. \end{aligned}$$

HODLR of Grünwald–Letnikov

Proof. For $k \geq 2$ we rewrite $g_k^{(\alpha)}$ as

$$g_k^{(\alpha)} = \alpha(\alpha - 1) \frac{\Gamma(k - \alpha)}{\Gamma(k + 1)\Gamma(2 - \alpha)}.$$

Use the Gauss representation of the Euler Γ

$$\Gamma(z) = \lim_{m \rightarrow \infty} \frac{m! m^z}{z(z+1)(z+2)\dots(z+m)}, \quad z \neq \{0, -1, -2, \dots\},$$

we rewrite

$$g_k^{(\alpha)} = \alpha(\alpha - 1) \lim_{m \rightarrow \infty} \frac{1}{m! m^3} \prod_{p=0}^m \frac{k + 1 + p}{k - \alpha + p} (2 - \alpha + p).$$

HODLR of Grünwald–Letnikov

Proof. For $k \geq 2$ we rewrite $g_k^{(\alpha)}$ as

$$g_k^{(\alpha)} = \alpha(\alpha - 1) \frac{\Gamma(k - \alpha)}{\Gamma(k + 1)\Gamma(2 - \alpha)}.$$

We rewrite

$$H = \lim_{m \rightarrow +\infty} H_0 \circ \dots \circ H_m, \quad (H_p)_{ij} = \frac{i+j+1+p}{i+j-\alpha+p}$$

for \circ the Hadamard product, $\{H_j\}_{j=0}^m$ Hankel matrices.

HODLR of Grünwald–Letnikov

Proof. For $k \geq 2$ we rewrite $g_k^{(\alpha)}$ as

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for \circ the Hadamard product, $\{H_j\}_{j=0}^m$ Hankel matrices. **Schur Product Theorem** tells us that
“*the Hadamard product of two positive definite matrices is also a positive definite matrix*”
 \Rightarrow If $H_0 \circ \dots \circ H_m$ is positive semidefinite for every m then H is also positive semidefinite.

HODLR of Grünwald–Letnikov

Proof. For $k \geq 2$ we rewrite $g_k^{(\alpha)}$ as

$$g_k^{(\alpha)} = \alpha(\alpha - 1) \frac{\Gamma(k - \alpha)}{\Gamma(k + 1)\Gamma(2 - \alpha)}.$$

We rewrite

$$H = \lim_{m \rightarrow +\infty} H_0 \circ \dots \circ H_m, \quad (H_p)_{ij} = \frac{i+j+1+p}{i+j-\alpha+p}$$

for \circ the Hadamard product, $\{H_j\}_{j=0}^m$ Hankel matrices. Rewrite

$$(H_p)_{ij} = \frac{i+j+1+p}{i+j-\alpha+p} = 1 + \frac{\alpha + 1}{i+j-\alpha+p}$$

HODLR of Grünwald–Letnikov

Proof. For $k \geq 2$ we rewrite $g_k^{(\alpha)}$ as

$$g_k^{(\alpha)} = \alpha(\alpha - 1) \frac{\Gamma(k - \alpha)}{\Gamma(k + 1)\Gamma(2 - \alpha)}.$$

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for \circ the Hadamard product, $\{H_j\}_{j=0}^m$ Hankel matrices. Rewrite

$$(H_p)_{ij} = 1 + \frac{\alpha + 1}{i + j - \alpha + p}, \quad H_p = \mathbf{1}\mathbf{1}^T + (\alpha + 1) \cdot C(\mathbf{x}, -\mathbf{x}), \quad \mathbf{x} = \begin{bmatrix} 1 \\ \vdots \\ N \end{bmatrix} + \frac{p - \alpha}{2} \mathbf{1},$$

$\mathbf{x} \geq 0$ for $\alpha < 2$, thus $C(\mathbf{x}, -\mathbf{x})$ is PD. Then H_p is positive semidefinite as the sum of a PD and positive semidefinite matrix. □

HODLR of Grünwald–Letnikov

Proposition (Massei, Mazza, and Robol 2019, Lemma 3.15)

For every $\epsilon > 0$, the ϵ -qsrank of G_N is bounded by

$$\text{qsrank}_\epsilon(G_N) \leq \mathfrak{B}\left(N, \frac{\epsilon}{2}\right) = 2 + 2 \left\lceil \frac{2}{\pi^2} \log\left(\frac{4}{\pi}N\right) \log\left(\frac{32}{\epsilon}\right) \right\rceil.$$

Proof.

HODLR of Grünwald–Letnikov

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Proof. We just need to work on the lower triangle, for the upper the rank is at most 1 (Hessenberg).

HODLR of Grünwald–Letnikov

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Proof. Let $Y \in \mathbb{R}^{s \times t}$ be any lower off-diagonal block of G_N . *Without loss of generality* we assume that Y is maximal, i.e. $s + t = N$.

HODLR of Grünwald–Letnikov

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Proof. Let $Y \in \mathbb{R}^{s \times t}$ be any lower off-diagonal block of G_N . *Without loss of generality* we assume that Y is maximal, i.e. $s + t = N$. (If $\text{rank}(Y + \delta Y) = k$ and $\|\delta Y\|_2 \leq \epsilon \|G_N\|_2$ then the submatrices of δY verify the analogous claim for the corresponding ones of Y .)

HODLR of Grünwald–Letnikov

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Proof. Let $Y \in \mathbb{R}^{s \times t}$ be any lower off-diagonal block of G_N . *Without loss of generality* we assume that Y is maximal, i.e. $s + t = N$.

Entries Y are given by $Y_{ij} = -g_{1+i-j+t}^{(\alpha)}$. Call $h = \max\{s, t\}$, and A the $h \times h$ matrix defined by $A_{ij} = -g_{1+i-j+h}^{(\alpha)}$.

HODLR of Grünwald–Letnikov

$$A = \begin{array}{|c|c|} \hline & Y \\ \hline \end{array}$$

For every $1 \leq i \leq s$ and $1 \leq j \leq t$ one have
 $Y_{ij} = -g_{1+i-j+t}^{(\alpha)} = -g_{1+i-(j-t+h)+h}^{(\alpha)} = A_{i,j-t+h}.$

Proof. Let $Y \in \mathbb{R}^{s \times t}$ be any lower off-diagonal block of G_N . *Without loss of generality* we assume that Y is maximal, i.e. $s + t = N$.

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HODLR of Grünwald–Letnikov

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Entries Y are given by $Y_{ij} = -g_{1+i-j+t}^{(\alpha)}$. Call $h = \max\{s, t\}$, and A the $h \times h$ matrix defined by $A_{ij} = -g_{1+i-j+h}^{(\alpha)}$. Y coincides with either the last t columns or the first s rows of A . In particular, Y is a submatrix of A and therefore $\|Y\|_2 \leq \|A\|_2$.  We need now to estimate $\|A\|_2$ in terms of $\|G_N\|_2$, thus we partition

$$A = \begin{bmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{bmatrix}, \quad A^{(ij)} \in \mathbb{C}^{m_{ij} \times n_{ij}}, \quad \begin{cases} m_{1j} = n_{i1} = \lceil \frac{h}{2} \rceil \\ m_{2j} = n_{i2} = \lfloor \frac{h}{2} \rfloor \end{cases}, \quad \begin{cases} h \leq N-1, \\ m_{i,j} + n_{i,j} \leq N, \end{cases}$$

HODLR of Grünwald–Letnikov

Proof. and consider the subdiagonal block $T^{(ij)}$ of G_N defined by

$$T^{(ij)} = G_N(N - m_{ij} + 1 : N, N - m_{ij} - n_{ij} + 1 : N - m_{ij}), \quad i, j = 1, 2, \quad T^{(ij)} \in \mathbb{R}^{m_{ij} \times n_{ij}}, \\ m_{ij} + n_{ij} \leq N.$$

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- ⦿ Since $g_j^{(\alpha)} > g_{j+1}^{(\alpha)} > 0$, $|T^{(ij)}| \geq |A^{(ij)}|$ for every $i, j = 1, 2$,

HODLR of Grünwald–Letnikov

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- Being $T^{(ij)}$ and $A^{(ij)}$ nonpositive and the 2 norm monotonous, $\|A^{(ij)}\|_2 \leq \|T^{(ij)}\|_2$.

HODLR of Grünwald–Letnikov

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- 💡 By exploiting

$$\begin{aligned} \|A\|_2 &\leq \left\| \begin{bmatrix} A^{(11)} & \\ & A^{(22)} \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} & A^{(12)} \\ A^{(21)} & \end{bmatrix} \right\|_2 \Rightarrow \|A\|_2 \leq 2\|G_N\|_2. \\ &= \max\{\|A^{(11)}\|_2, \|A^{(22)}\|_2\} + \max\{\|A^{(12)}\|_2, \|A^{(21)}\|_2\} \end{aligned}$$

HODLR of Grünwald–Letnikov

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- ✳ Conclude by the result on Hankel matrices!

HODLR of Grünwald–Letnikov

Proposition (Massei, Mazza, and Robol 2019, Lemma 3.15)

For every $\epsilon > 0$, the ϵ -qsrank of G_N is bounded by

$$\text{qsrank}_\epsilon(G_N) \leq \mathfrak{B}\left(N, \frac{\epsilon}{2}\right) = 2 + 2 \left\lceil \frac{2}{\pi^2} \log\left(\frac{4}{\pi}N\right) \log\left(\frac{32}{\epsilon}\right) \right\rceil.$$

Proof. We call J the $h \times h$ flip matrix, so that $-AJ$ is Hankel and positive semidefinite:

$$\text{rank}_{\frac{\epsilon}{2}}(A) = \text{rank}_{\frac{\epsilon}{2}}(AJ) \leq \mathfrak{B}\left(N, \frac{\epsilon}{2}\right).$$

HODLR of Grünwald–Letnikov

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Y is a submatrix of A , thus there exists δY such that

$$\|\delta Y\|_2 \leq \varepsilon \|G_N\|_2 \text{ and } \text{rank}(Y + \delta Y) \leq \mathfrak{B}\left(N, \frac{\epsilon}{2}\right).$$

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$$\Rightarrow \text{qsrank}_\epsilon(G_N) \leq \mathfrak{B}\left(N, \frac{\epsilon}{2}\right).$$

□

HODLR of Grünwald–Letnikov

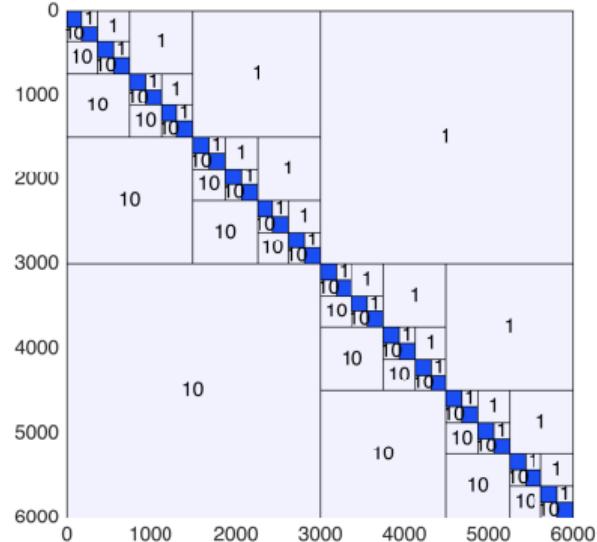
Let's do some experiments with the  hm-toolbox (Massei, Robol, and Kressner 2020).

```
function G = glhodlrmatrix(N,alpha,tol)
%%GLMATRIX produces the GL discretization of
% the Riemann-Liouville derivative in HODLR
% format
g = gl(N,alpha);
c = zeros(N,1);
r = zeros(1,N);
r(1:2) = g(2:-1:1);
c(1:N) = g(2:end);
hodlroption( 'threshold', tol);
G = hodlr('toeplitz',c,r);
end
```

HODLR of Grünwald–Letnikov

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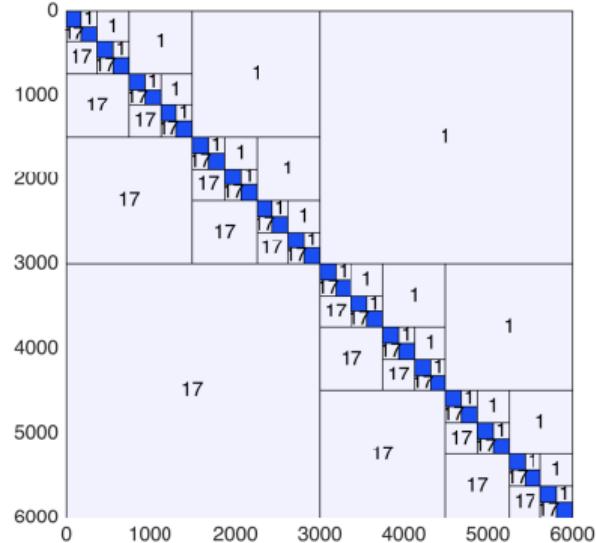


```
G = glhodlrmatrix(6000,1.5,1e-6);
```

HODLR of Grünwald–Letnikov

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```

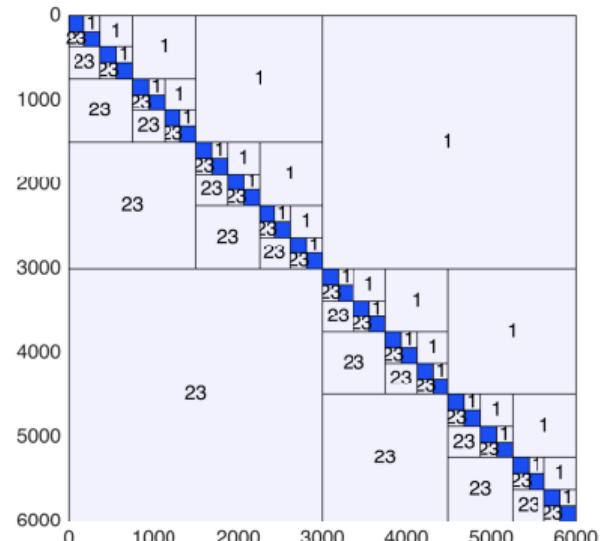


```
G = glhodlrmatrix(6000,1.5,1e-9);
```

HODLR of Grünwald–Letnikov

Let's do some experiments with the  hm-toolbox (Massei, Robol, and Kressner 2020).

```
function G = glhodlrmatrix(N,alpha,tol)
%%GLMATRIX produces the GL discretization of
% the Riemann-Liouville derivative in HODLR
% format
g = gl(N,alpha);
c = zeros(N,1);
r = zeros(1,N);
r(1:2) = g(2:-1:1);
c(1:N) = g(2:end);
hodlroption( 'threshold', tol);
G = hodlr('toeplitz',c,r);
end
```



```
G = glhodlrmatrix(6000,1.5,1e-12);
```

HODLR Matrix: the whole discretization

Matrix G_N was only a piece of the whole discretization matrix

$$A_N = I_N + \frac{\Delta t}{h^\alpha} \left(D_{(m)}^+ G_N + D_{(m)}^- G_N^T \right),$$

does it share the same structure?

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does it share the same structure?

Corollary (Massei, Mazza, and Robol 2019, Corollary 3.16)

$$\text{qsrank}_\epsilon(A_N) \leq 3 + 2 \left\lceil \frac{2}{\pi^2} \log \left(\frac{4}{\pi} N \right) \log \left(\frac{32}{\hat{\epsilon}} \right) \right\rceil, \quad \hat{\epsilon} \triangleq \frac{\|A_N\|}{\|G_N\| \cdot \max\{\|D_{(m)}^+\|, \|D_{(m)}^-\|\}} \epsilon.$$

HODLR Matrix: the whole discretization

Matrix G_N was only a piece of the whole discretization matrix

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Proof. Result is invariant under scaling, so assume wlog that $\frac{\Delta t}{h^\alpha} = 1$. A generic off-diagonal block Y , wlog in the lower triangular part, If Y does not intersect the first subdiagonal, is a subblock of $D_{(m)}^+ G_N$, so there exists a perturbation δY with norm bounded by $\|\delta Y\| \leq \|D_{(m)}^+\| \|G_N\| \cdot \hat{\epsilon}$ such that $Y + \delta Y$ has rank at most $\mathfrak{B}(N, \hat{\epsilon}/2)$. In particular, δY satisfies $\|\delta Y\| \leq \|A_N\| \cdot \epsilon$.

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Proof. Result is invariant under scaling, so assume wlog that $\frac{\Delta t}{h^\alpha} = 1$. Since we have excluded one subdiagonal, a generic off-diagonal block Y we can find a perturbation with norm bounded by $\|A_N\| \cdot \epsilon$ such that $Y + \delta Y$ has rank $1 + \mathfrak{B}(N, \hat{\epsilon}/2)$. \square

A HODLR right-hand side

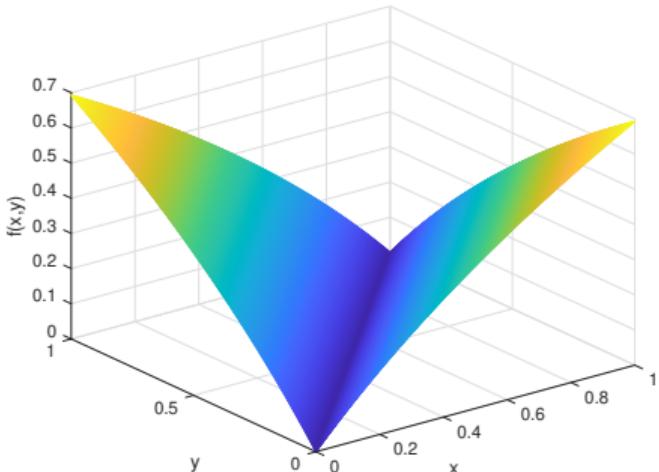
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Consider the function

$$f(x, y) = \log(\tau + |x - y|), \quad \tau > 0.$$



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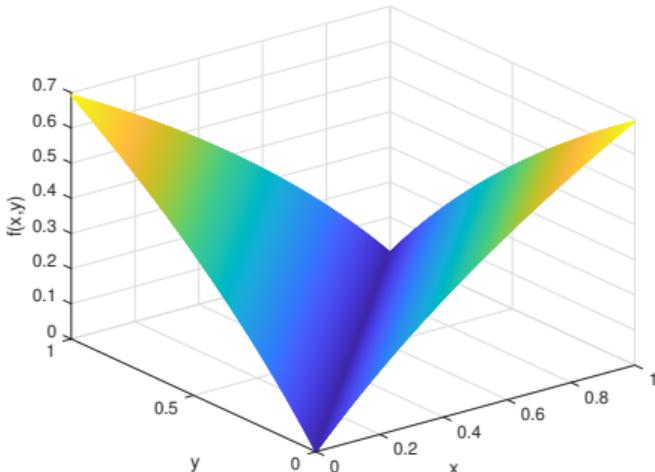
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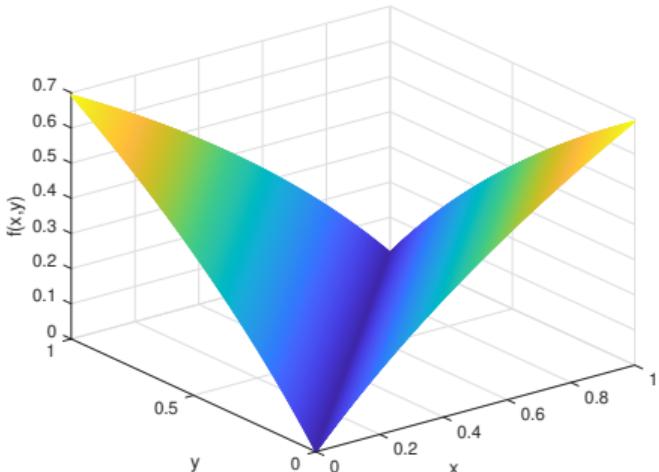
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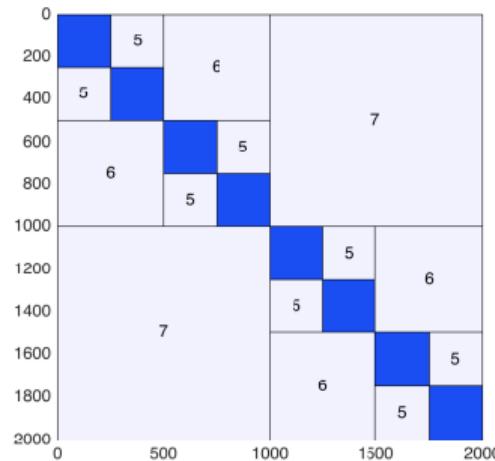
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- 🔧 We can use again Chebyshev basis to approximate it in a separable fashion.



```
x = linspace(0,1,N); y = linspace(0,1,N);
[X,Y] = meshgrid(x,y); tau = 1;
C = log(tau + abs(X-Y)); hC = hodlr(C);
```

Separability (a bit more formally)

Separable expansion (Hackbusch 2015, Definition 4.4)

Take a function $\chi(x, y) : X \times Y \rightarrow \mathbb{R}$, we call

$$\chi(x, y) = \sum_{v=1}^r \phi_v^{(r)}(x) \psi_v^{(r)}(y) + R_r(x, y), \quad \text{for } x \in X, y \in Y,$$

a *separable expansion* of χ with r terms in $X \times Y$ with remainder R_r .

🔧 To have an idea of the **goodness** of the *separable expansion*, we would like to have $\{\|R_r\|_\infty, \|R_r\|_{\mathbb{L}^p}\} \xrightarrow{r \rightarrow 0} 0$ **as fast as possible**, e.g., **exponentially**.

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- 👉 We can use Taylor expansions, Chebyshev expansion, Hermite/Lagrange interpolation, cross approximation... In all the cases, the behavior of R_r is tied to the regularity of $\chi(x, y)$; see (Hackbusch 2015, Chapter 4).

BLAS with HODLR format

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$\mathbf{y} = \mathbf{A}\mathbf{x}$: Matrix-vector products, *recursively*:

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Master theorem (divide and conquer): $c_{\mathbf{A}\cdot\mathbf{x}}(n) = (4k + 1) \log_2(n)n$.

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$C = A + B$: Adding two equally partitioned HODLR matrices **increases the ranks** of off-diagonal blocks by a factor 2.

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- ⚙️ We need truncation $\mathfrak{T}_k(A(I_1^\ell, I_j^\ell) + B(I_1^\ell, I_j^\ell))$, costs

$$c_{\text{LR+LR}} = c_{\text{SVD}} \times (nk^2 + k^3),$$

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Total cost is then:

$$\begin{aligned} \sum_{\ell=1}^p 2^\ell c_{\text{LR+LR}}(m_\ell) &= c_{\text{SVD}} \sum_{\ell=1}^p 2^\ell (k^3 + m_\ell k^2) \\ &\leq c_{\text{SVD}} \left(2^{p+1} k^3 + \sum_{\ell=1}^p 2^\ell 2^{p-\ell} n_0 k^2 \right) \\ &\leq c_{\text{SVD}} (2nk^3 + n \log_2(n) k^2). \end{aligned}$$

BLAS with HODLR format

$C = AB$: Matrix-matrix multiplication can also be done recursively

$$\begin{array}{c} \text{Diagram of two 4x4 HODLR matrices} \\ \cdot \\ \text{Diagram of two 4x4 HODLR matrices} \end{array} = \left[\begin{array}{cc} \text{Diagram of 2x2 HODLR matrix} \cdot \text{Diagram of 2x2 HODLR matrix} + \text{Diagram of low-rank block} \cdot \text{Diagram of low-rank block} & \text{Diagram of 2x2 HODLR matrix} \cdot \text{Diagram of low-rank block} + \text{Diagram of low-rank block} \cdot \text{Diagram of 2x2 HODLR matrix} \\ \text{Diagram of low-rank block} \cdot \text{Diagram of 2x2 HODLR matrix} + \text{Diagram of 2x2 HODLR matrix} \cdot \text{Diagram of low-rank block} & \text{Diagram of low-rank block} \cdot \text{Diagram of low-rank block} + \text{Diagram of 2x2 HODLR matrix} \cdot \text{Diagram of 2x2 HODLR matrix} \end{array} \right]$$

where is a $n/2 \times n/2$ HODLR matrix and is a low-rank block.

BLAS with HODLR format

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$$\begin{array}{c} \text{Diagram of two 4x4 matrices with red and grey blocks} \\ \cdot \\ = \end{array} \left[\begin{array}{cc} \text{Diagram of 2x2 HODLR blocks} & \cdot \text{Diagram of 2x2 HODLR blocks} + \text{Diagram of low-rank block} \cdot \text{Diagram of low-rank block} & \text{Diagram of 2x2 HODLR blocks} \cdot \text{Diagram of low-rank block} + \text{Diagram of low-rank block} \cdot \text{Diagram of 2x2 HODLR blocks} \\ \text{Diagram of low-rank block} \cdot \text{Diagram of 2x2 HODLR blocks} + \text{Diagram of 2x2 HODLR blocks} \cdot \text{Diagram of low-rank block} & \text{Diagram of low-rank block} \cdot \text{Diagram of low-rank block} + \text{Diagram of 2x2 HODLR blocks} \cdot \text{Diagram of 2x2 HODLR blocks} & \text{Diagram of 2x2 HODLR blocks} \cdot \text{Diagram of 2x2 HODLR blocks} \end{array} \right]$$

where $\boxed{\textcolor{red}{\square}}$ is a $n/2 \times n/2$ HODLR matrix and \square is a low-rank block.

1. $\boxed{\textcolor{red}{\square}} \cdot \boxed{\textcolor{red}{\square}} \cdot$ of 2 HODLR $n/2$ matrices,

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BLAS with HODLR format

$C = AB$: Matrix-matrix multiplication can also be done recursively

$$\begin{array}{c} \text{Diagram showing two 4x4 matrices with red and grey blocks being multiplied.} \\ \cdot = \left[\begin{array}{cc} \text{Diagram of } 2 \times 2 \text{ HODLR blocks} & \text{Diagram of } 2 \times 2 \text{ low-rank blocks} \\ \text{Diagram of } 2 \times 2 \text{ low-rank blocks} & \text{Diagram of } 2 \times 2 \text{ HODLR blocks} \end{array} \right] + \left[\begin{array}{cc} \text{Diagram of } 2 \times 2 \text{ HODLR blocks} & \text{Diagram of } 2 \times 2 \text{ low-rank blocks} \\ \text{Diagram of } 2 \times 2 \text{ low-rank blocks} & \text{Diagram of } 2 \times 2 \text{ HODLR blocks} \end{array} \right] \end{array}$$

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$$\begin{array}{c} \text{Diagram showing two 4x4 matrices with red and grey blocks multiplied together.} \\ \cdot = \left[\begin{array}{cc} \text{Diagram of } 2 \times 2 \text{ HODLR blocks} & \text{Diagram of } 2 \times 2 \text{ low-rank blocks} \\ \text{Diagram of } 2 \times 2 \text{ low-rank blocks} & \text{Diagram of } 2 \times 2 \text{ HODLR blocks} \end{array} \right] + \left[\begin{array}{cc} \text{Diagram of } 2 \times 2 \text{ HODLR blocks} & \text{Diagram of } 2 \times 2 \text{ low-rank blocks} \\ \text{Diagram of } 2 \times 2 \text{ low-rank blocks} & \text{Diagram of } 2 \times 2 \text{ HODLR blocks} \end{array} \right] \end{array}$$

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BLAS with HODLR format

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$$\begin{array}{c} \text{Diagram showing two 4x4 matrices being multiplied, where each matrix is partitioned into four 2x2 blocks. The result is shown as a sum of eight terms, each involving a 2x2 HODLR matrix and a 2x2 low-rank block.} \\ \text{Diagram: } \begin{matrix} \text{Red} & \text{Red} & \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Red} & \text{Red} & \text{Grey} \\ \text{Red} & \text{Grey} & \text{Red} & \text{Red} \\ \text{Grey} & \text{Grey} & \text{Red} & \text{Red} \end{matrix} \cdot \begin{matrix} \text{Red} & \text{Red} & \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Red} & \text{Red} & \text{Grey} \\ \text{Red} & \text{Grey} & \text{Red} & \text{Red} \\ \text{Grey} & \text{Grey} & \text{Red} & \text{Red} \end{matrix} = \left[\begin{matrix} \text{Red} & \text{Red} & \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Red} & \text{Red} & \text{Grey} \\ \text{Red} & \text{Grey} & \text{Red} & \text{Red} \\ \text{Grey} & \text{Grey} & \text{Red} & \text{Red} \end{matrix} \cdot \begin{matrix} \text{Red} & \text{Red} \\ \text{Grey} & \text{Red} \end{matrix} + \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \cdot \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \quad \begin{matrix} \text{Red} & \text{Red} & \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Red} & \text{Red} & \text{Grey} \\ \text{Red} & \text{Grey} & \text{Red} & \text{Red} \\ \text{Grey} & \text{Grey} & \text{Red} & \text{Red} \end{matrix} \cdot \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} + \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \cdot \begin{matrix} \text{Red} & \text{Red} \\ \text{Grey} & \text{Red} \end{matrix} \right] \\ \left[\begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \cdot \begin{matrix} \text{Red} & \text{Red} \\ \text{Grey} & \text{Red} \end{matrix} + \begin{matrix} \text{Red} & \text{Red} \\ \text{Grey} & \text{Red} \end{matrix} \cdot \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \quad \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \cdot \begin{matrix} \text{Red} & \text{Red} \\ \text{Grey} & \text{Red} \end{matrix} + \begin{matrix} \text{Red} & \text{Red} \\ \text{Grey} & \text{Red} \end{matrix} \cdot \begin{matrix} \text{Grey} & \text{Grey} \\ \text{Grey} & \text{Grey} \end{matrix} \right] \end{array}$$

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1. $\boxed{\text{Red}} \cdot \boxed{\text{Red}} \cdot$ of 2 HODLR $n/2$ matrices,
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3. $\boxed{\text{Red}} \cdot \square \cdot$ of HODLR times low-rank,
4. $\square \cdot \boxed{\text{Red}} \cdot$ of low-rank times HODLR.

$$\begin{aligned} c_{H \cdot H}(n) = & 2(c_{H \cdot H}(n/2) + c_{LR \cdot LR}(n/2) + c_{H \cdot LR}(n/2) + c_{LR \cdot H}(n/2) \\ & + c_{H+LR}(n/2) + c_{LR+LR}(n/2)) \end{aligned}$$

BLAS with HODLR format

$C = AB$: Matrix-matrix multiplication can also be done recursively

$$\begin{array}{c} \text{Diagram showing two 4x4 matrices with red and grey blocks being multiplied, resulting in a sum of eight terms involving HODLR and low-rank blocks.} \\ \text{The result is:} \\ \left[\begin{array}{c} \text{Diagram of term 1: } \text{HODLR} \cdot \text{HODLR} \\ \text{Diagram of term 2: } \text{LowRank} \cdot \text{HODLR} \\ \text{Diagram of term 3: } \text{HODLR} \cdot \text{LowRank} \\ \text{Diagram of term 4: } \text{LowRank} \cdot \text{LowRank} \end{array} + \dots \right] \end{array}$$

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3. $\boxed{\textcolor{red}{\square}} \cdot \square \cdot$ of HODLR times low-rank,
4. $\square \cdot \boxed{\textcolor{red}{\square}} \cdot$ of low-rank times HODLR.

$$c_{H \cdot H}(n) = 2(c_{H \cdot H}(n/2) + c_{LR \cdot LR}(n/2) + c_{H \cdot LR}(n/2) + c_{LR \cdot H}(n/2) + c_{H+LR}(n/2) + c_{LR+LR}(n/2))$$

$$c_{LR \cdot LR}(n) = 4nk^2$$

BLAS with HODLR format

$C = AB$: Matrix-matrix multiplication can also be done recursively

$$\begin{array}{c} \text{Diagram showing two 4x4 matrices with red and grey blocks being multiplied, resulting in a sum of eight terms where each term is a product of a 2x2 HODLR matrix and a 2x2 low-rank block.} \\ \text{Diagram: Two 4x4 matrices are shown. The first has red blocks at (1,1), (1,2), (2,1), and (3,4). The second has red blocks at (1,1), (1,3), (2,2), and (3,4). They are multiplied to produce a sum of eight terms, each consisting of a 2x2 HODLR matrix (red) multiplied by a 2x2 low-rank block (grey).} \\ = \left[\begin{array}{l} \text{Term 1: } \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Grey} & \text{Grey} \\ \hline \text{Grey} & \text{Grey} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Grey} & \text{Grey} \\ \hline \text{Grey} & \text{Grey} \\ \hline \end{array} \\ \text{Term 2: } \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Grey} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Grey} & \text{Grey} \\ \hline \text{Grey} & \text{Grey} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} \\ \text{Term 3: } \begin{array}{|c|c|} \hline \text{Grey} & \text{Grey} \\ \hline \text{Grey} & \text{Grey} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Grey} & \text{Grey} \\ \hline \text{Grey} & \text{Grey} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} \\ \text{Term 4: } \begin{array}{|c|c|} \hline \text{Grey} & \text{Grey} \\ \hline \text{Grey} & \text{Grey} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \text{Red} & \text{Red} \\ \hline \text{Grey} & \text{Red} \\ \hline \end{array} \end{array} \right] \end{array}$$

where $\boxed{\text{Red}}$ is a $n/2 \times n/2$ HODLR matrix and $\boxed{\text{Grey}}$ is a low-rank block.

1. $\boxed{\text{Red}} \cdot \boxed{\text{Red}} \cdot$ of 2 HODLR $n/2$ matrices,
2. $\boxed{\text{Grey}} \cdot \boxed{\text{Grey}} \cdot$ of 2 low-rank blocks,
3. $\boxed{\text{Red}} \cdot \boxed{\text{Grey}} \cdot$ of HODLR times low-rank,
4. $\boxed{\text{Grey}} \cdot \boxed{\text{Red}} \cdot$ of low-rank times HODLR.

$$\begin{aligned} c_{H \cdot H}(n) = & 2(c_{H \cdot H}(n/2) + c_{LR \cdot LR}(n/2) + \cancel{c_{H \cdot LR}(n/2)} + \cancel{c_{LR \cdot H}(n/2)} \\ & + \cancel{c_{H+LR}(n/2)} + c_{LR+LR}(n/2)) \end{aligned}$$

$$c_{H \cdot LR}(n) = c_{LR \cdot H} = kc_{Hv}(n) = k(4k+1)\log_2(n)n$$

BLAS with HODLR format

$C = AB$: Matrix-matrix multiplication can also be done recursively

$$\begin{array}{c} \text{Diagram showing two 4x4 matrices being multiplied, where each matrix is a block-diagonal sum of HODLR and low-rank blocks.} \\ \cdot \\ = \end{array} \left[\begin{array}{c} \text{Diagram showing the recursive decomposition of the product into four terms, each involving a 2x2 HODLR matrix and a 2x2 low-rank matrix.} \\ + \\ \text{Diagram showing the recursive decomposition of the product into four terms, each involving a 2x2 low-rank matrix and a 2x2 HODLR matrix.} \\ + \\ \text{Diagram showing the recursive decomposition of the product into four terms, each involving a 2x2 low-rank matrix and a 2x2 low-rank matrix.} \\ + \\ \text{Diagram showing the recursive decomposition of the product into four terms, each involving a 2x2 HODLR matrix and a 2x2 low-rank matrix.} \end{array} \right]$$

where is a $n/2 \times n/2$ HODLR matrix and is a low-rank block.

1. · · of 2 HODLR $n/2$ matrices,
2. · · of 2 low-rank blocks,
3. · · of HODLR times low-rank,
4. · · of low-rank times HODLR.

$$c_{H \cdot H}(n) = 2(c_{H \cdot H}(n/2) + c_{LR \cdot LR}(n/2) + c_{H \cdot LR}(n/2) + c_{LR \cdot H}(n/2) + c_{H+LR}(n/2) + c_{LR+LR}(n/2))$$

$$c_{H+LR}(n) = c_{H+H}(n) = c_{SVD}(nk^3 + n \log(n)k^2)$$

BLAS with HODLR format

$C = AB$: Matrix-matrix multiplication can also be done recursively

$$\begin{array}{c} \text{Diagram showing two } n/2 \times n/2 \text{ HODLR matrices being multiplied. The result is a sum of eight terms, each involving a low-rank block and a HODLR matrix.} \\ \text{Diagram: Two } n/2 \times n/2 \text{ HODLR matrices are multiplied. The result is a sum of eight terms, each involving a low-rank block and a HODLR matrix.} \end{array}$$

where is a $n/2 \times n/2$ HODLR matrix and is a low-rank block.

1. · · of 2 HODLR $n/2$ matrices,
2. · · of 2 low-rank blocks,
3. · · of HODLR times low-rank,
4. · · of low-rank times HODLR.

Total cost $c_{H \cdot H}(n) \in O(k^3 n \log n + k^2 n \log^2 n)$.

BLAS with HODLR format

Approximate solution of a linear system $Ax = b$ with HODLR matrix A :

BLAS with HODLR format

Approximate solution of a linear system $Ax = b$ with HODLR matrix A :

$A \approx LU$ Approximate LU-factorization $A \approx LU$ in HODLR format:

$$\begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & \\ \textcolor{lightgray}{\square} & \textcolor{red}{\square} & \textcolor{red}{\square} & \textcolor{white}{\square} & \\ \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \\ \hline \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \\ \textcolor{white}{\square} & \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \textcolor{red}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \end{array} \approx \begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \\ \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \textcolor{white}{\square} & \\ \textcolor{white}{\square} & \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \hline \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \textcolor{white}{\square} & \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \textcolor{red}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \end{array} \cdot \begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \\ \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \textcolor{white}{\square} & \\ \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \hline \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \textcolor{white}{\square} & \textcolor{lightgray}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \\ \textcolor{red}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{red}{\square} & \end{array}$$

BLAS with HODLR format

Approximate solution of a linear system $Ax = b$ with HODLR matrix A :

$A \approx LU$ Approximate LU-factorization $A \approx LU$ in HODLR format:

$$\begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & \\ \textcolor{red}{\square} & & \textcolor{red}{\square} & & \\ \textcolor{lightgray}{\square} & & & \textcolor{red}{\square} & \\ \hline \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{red}{\square} & & & & \end{array} \approx \begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & & \\ \textcolor{lightgray}{\square} & & \textcolor{red}{\square} & & \\ \textcolor{lightgray}{\square} & & & \textcolor{red}{\square} & \\ \hline \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{red}{\square} & & & & \end{array} \cdot \begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & & \\ \textcolor{lightgray}{\square} & & \textcolor{red}{\square} & & \\ \textcolor{lightgray}{\square} & & & \textcolor{red}{\square} & \\ \hline \textcolor{white}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{white}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{red}{\square} & & & & \end{array}$$

Forward substitution to solve $Ly = b$,

Backward substitution to solve $Ux = y$.

BLAS with HODLR format

Approximate solution of a linear system $Ax = b$ with HODLR matrix A :

$A \approx LU$ Approximate LU-factorization $A \approx LU$ in HODLR format:

$$\begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & \\ \textcolor{red}{\square} & & \textcolor{red}{\square} & & \\ \textcolor{lightgray}{\square} & & & \textcolor{red}{\square} & \\ \hline \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{red}{\square} & & & & \end{array} \approx \begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & & \\ \textcolor{lightgray}{\square} & & \textcolor{red}{\square} & & \\ \textcolor{lightgray}{\square} & & & \textcolor{red}{\square} & \\ \hline \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{lightgray}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{red}{\square} & & & & \end{array} \cdot \begin{array}{c|cc|cc} \textcolor{red}{\square} & \textcolor{lightgray}{\square} & \textcolor{lightgray}{\square} & & \\ \textcolor{lightgray}{\square} & & \textcolor{red}{\square} & & \\ \textcolor{lightgray}{\square} & & & \textcolor{red}{\square} & \\ \hline \textcolor{white}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{white}{\square} & & & & \textcolor{red}{\square} \\ \textcolor{red}{\square} & & & & \end{array}$$

Forward substitution to solve $Ly = b$,

Backward substitution to solve $Ux = y$.

We need to analyze the two steps separately.

BLAS with HODLR format

Approximate LU factorization, on level $\ell = 1$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$$

It is done in four steps

BLAS with HODLR format

Approximate LU factorization, on level $\ell = 1$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$$

It is done in four steps

1. Compute LU factors L_{11}, U_{11} of A_{11} ,

BLAS with HODLR format

Approximate LU factorization, on level $\ell = 1$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$$

It is done in four steps

1. Compute LU factors L_{11} , U_{11} of A_{11} ,
2. Compute $U_{12} = L_{11}^{-1}A_{12}$ by **forward substitution**,

BLAS with HODLR format

Approximate LU factorization, on level $\ell = 1$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$$

It is done in four steps

1. Compute LU factors L_{11} , U_{11} of A_{11} ,
2. Compute $U_{12} = L_{11}^{-1}A_{12}$ by **forward substitution**,
3. Compute $L_{21} = A_{21}U_{11}^{-1}$ by **backward substitution**,

BLAS with HODLR format

Approximate LU factorization, on level $\ell = 1$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$$

It is done in four steps

1. Compute LU factors L_{11}, U_{11} of A_{11} ,
2. Compute $U_{12} = L_{11}^{-1} A_{12}$ by **forward substitution**,
3. Compute $L_{21} = A_{21} U_{11}^{-1}$ by **backward substitution**,
4. Compute LU factors L_{22}, U_{22} of $A_{22} - L_{21} U_{12}$.

BLAS with HODLR format

Approximate LU factorization, on level $\ell = 1$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$$

It is done in four steps

1. Compute LU factors L_{11}, U_{11} of A_{11} ,
2. Compute $U_{12} = L_{11}^{-1} A_{12}$ by **forward substitution**,
3. Compute $L_{21} = A_{21} U_{11}^{-1}$ by **backward substitution**,
4. Compute LU factors L_{22}, U_{22} of $A_{22} - L_{21} U_{12}$.

The analysis of the cost is *analogous to the matrix-matrix multiplication case*, **but** we **need** to know how to do and how-much does forward/backward substitution costs.

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,
2. Compute $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 - L_{21}\mathbf{y}_1$,

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,
2. Compute $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 - L_{21}\mathbf{y}_1$,
3. Solve $L_{22}\mathbf{y}_2 = \tilde{\mathbf{b}}_2$.

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,
2. Compute $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 - L_{21}\mathbf{y}_1$,
3. Solve $L_{22}\mathbf{y}_2 = \tilde{\mathbf{b}}_2$.

Cost recursively:

$$c_{\text{forw}} = 2c_{\text{forw}}(n/2) + (2k + 1)n.$$

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,
2. Compute $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 - L_{21}\mathbf{y}_1$,
3. Solve $L_{22}\mathbf{y}_2 = \tilde{\mathbf{b}}_2$.

Cost recursively:

$$c_{\text{forw}} = 2c_{\text{forw}}(n/2) + (2k + 1)n.$$

On level $\ell = p$, we have the direct solution of $2^p = n/n_0$ linear systems of size $n_0 \times n_0$.

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,
2. Compute $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 - L_{21}\mathbf{y}_1$,
3. Solve $L_{22}\mathbf{y}_2 = \tilde{\mathbf{b}}_2$.

Cost recursively:

$$c_{\text{forw}} = 2c_{\text{forw}}(n/2) + (2k+1)n.$$

On level $\ell = p$, we have the direct solution of $2^p = n/n_0$ linear systems of size $n_0 \times n_0$.

■ Total cost $c_{\text{forw}} \in O(kn \log(n))$, and **analogously for backward substitution**.

BLAS with HODLR format

Forward substitution with lower triangular L in HODLR format: $\mathbf{y} = L^{-1}\mathbf{b}$

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with L_{21} low-rank, and L_{11} , L_{22} HODLR.

1. Solve $L_{11}\mathbf{y}_1 = \mathbf{b}_1$,
2. Compute $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 - L_{21}\mathbf{y}_1$,
3. Solve $L_{22}\mathbf{y}_2 = \tilde{\mathbf{b}}_2$.

Cost recursively:

$$c_{\text{forw}} = 2c_{\text{forw}}(n/2) + (2k+1)n.$$

On level $\ell = p$, we have the direct solution of $2^p = n/n_0$ linear systems of size $n_0 \times n_0$.

- Total cost $c_{\text{forw}} \in O(kn \log(n))$, and **analogously for backward substitution**.
- Total cost $c_{\text{LU}}(n) \lesssim c_{H.H}(n) \in O(k^3 n \log n + k^2 n \log^2 n)$.

BLAS with HODLR format

The  hm-toolbox (Massei, Robol, and Kressner 2020) contains all the routines.

- ↳ They *overload* the standard MATLAB operation by the same name, i.e., if you have variables in the right class you operate directly in this format.
- 👉 One can use different **cluster tree** \mathcal{T}_p to get smaller ranks. They are determined by the partitioning of the index set on the leaf level and represented as the vector $\mathbf{c} = [n_1^{(p)}, \dots, n_{2^p}^{(p)}]$, change it to change the HODLR matrix.

Operation	HODLR complexity
$\mathbf{A} * \mathbf{v}$	$\mathcal{O}(kn \log n)$
$\mathbf{A} \backslash \mathbf{v}$	$\mathcal{O}(k^2 n \log^2 n)$
$\mathbf{A} + \mathbf{B}$	$\mathcal{O}(k^2 n \log n)$
$\mathbf{A} * \mathbf{B}$	$\mathcal{O}(k^2 n \log^2 n)$
$\mathbf{A} \backslash \mathbf{B}$	$\mathcal{O}(k^2 n \log^2 n)$
<code>inv(A)</code>	$\mathcal{O}(k^2 n \log^2 n)$
$\mathbf{A} . * \mathbf{B}^2$	$\mathcal{O}(k^4 n \log n)$
<code>lu(A)</code> , <code>chol(A)</code>	$\mathcal{O}(k^2 n \log^2 n)$
<code>qr(A)</code>	$\mathcal{O}(k^2 n \log^2 n)$
compression	$\mathcal{O}(k^2 n \log(n))$

²The complexity of the Hadamard product is dominated by the recompression stage due to the k^2 HODLR rank of $A \circ B$. Without recompression the cost is $\mathcal{O}(k^2 n \log n)$.

HODLR solver for the 1D case

We can modify our first example to get a solution for the 1D problem in the new format.

```
% Discretization
N = 2^7; hN = 1/(N-1); x = 0:hN:1; dt = hN;
alpha = 1.5; % Coefficients
dplus=@(x)gamma(3-alpha).*x.^alpha;
dminus=@(x)gamma(3-alpha).*(1-x).^alpha;
w = @(x) 5*x.* (1-x);
tol = 1e-9; % HODLR building
tic;
G = glhodlrmatrix(N,alpha,tol);
Dplus = hodlr('diagonal',dplus(x));
Dminus = hodlr('diagonal',dminus(x));
I = hodlr('eye', N);
nu = hN^alpha/dt;
A = nu*I -(Dplus*G + Dminus*G');
buildtime = toc;
```

```
% Solving
[L,U] = lu(A);
flu = @() lu(A);
timelu = timeit(flu,2);
w = w(x).';
solvetime = 0;
for i=1:N
    tic;
    w = U\ (L\ (nu*w));
    solvetime = solvetime + toc;
end
solvetime = solvetime/N;
```

HODLR solver for the 1D case

We can modify our first example to get a solution for the 1D problem in the new format.

```
% Discretization
N = 2^7; hN = 1/(N-1); x = 0:hN:1; dt = hN;
alpha = 1.5; % Coefficients
dplus=@(x)gamma(3-alpha).*x.^alpha;
dminus=@(x)gamma(3-alpha).*(1-x).^alpha;
w = @(x) 5*x.* (1-x);
tol = 1e-9; % HODLR building
tic;
G = glhodlrmatrix(N,alpha,tol);
Dplus = hodlr('diagonal',dplus(x));
Dminus = hodlr('diagonal',dminus(x));
I = hodlr('eye', N);
nu = hN^alpha/dt;
A = nu*I -(Dplus*G + Dminus*G');
buildtime = toc;
```

```
% Solving
[L,U] = lu(A);
flu = @() lu(A);
timelu = timeit(flu,2);
w = w(x).';
solvetime = 0;
for i=1:N
    tic;
    w = U\ (L\ (nu*w));
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```

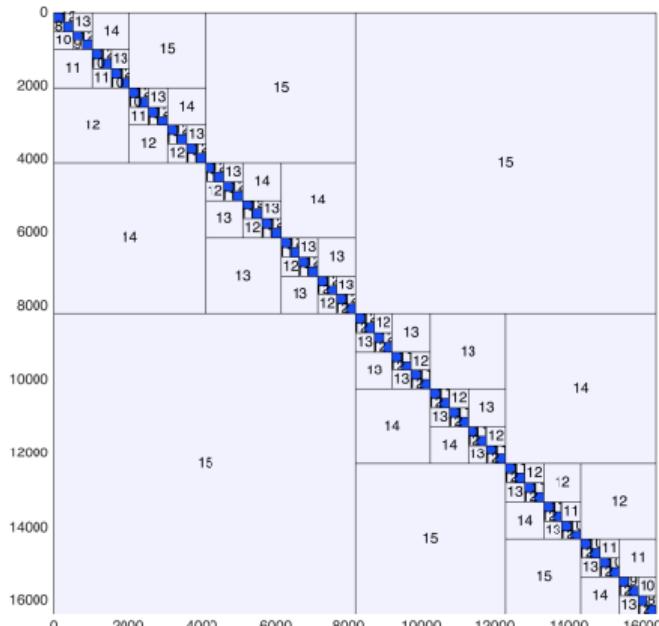


Let us try looking at the timings.

HODLR solver for the 1D case

We take $\alpha = 1.5$, and $\varepsilon = 10^{-9}$

N	Build (s)	LU (s)	Avg. Solve (s)
2^7	8.96e-03	1.44e-04	2.93e-04
2^8	1.35e-02	4.63e-04	3.33e-04
2^9	3.14e-02	2.05e-03	5.41e-04
2^{10}	7.28e-02	6.21e-03	9.35e-04
2^{11}	1.59e-01	1.63e-02	1.75e-03
2^{12}	3.85e-01	4.33e-02	3.68e-03
2^{13}	8.81e-01	1.27e-01	7.99e-03
2^{14}	2.19e+00	3.73e-01	1.55e-02

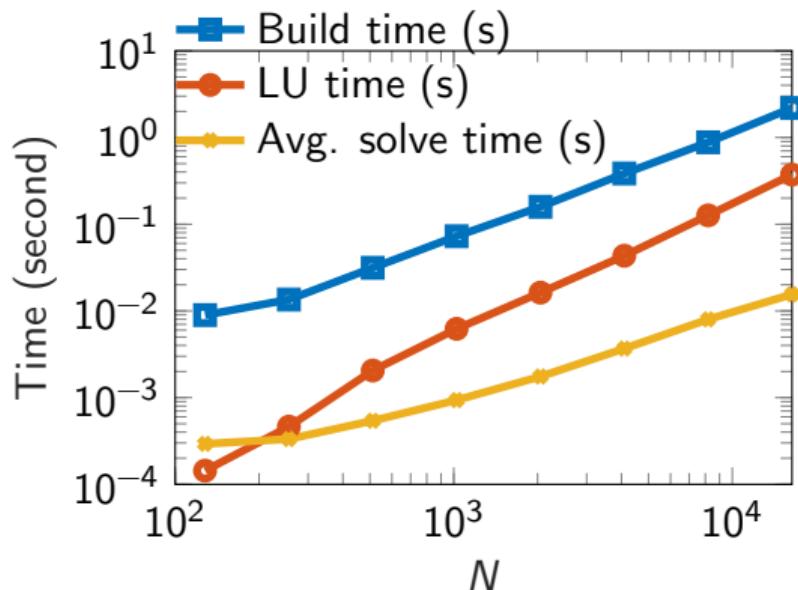


■ Largest matrix occupies 46.25 Mb, against the 2 Gb of the dense storage and the 0.87 Mb of storing three diagonals and $2 \times (2N - 1)$ for the Toeplitz storage.

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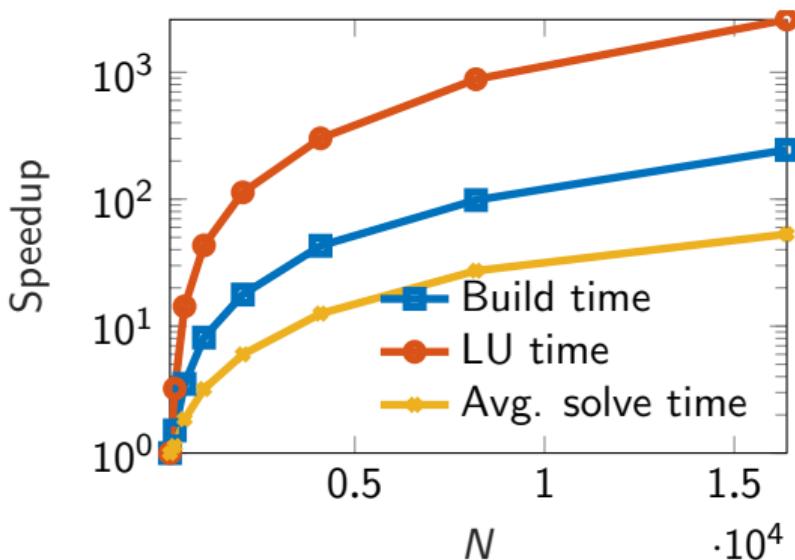


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Back to Sylvester (Massei, Palitta, and Robol 2018)

To solve the Sylvester equation with HODLR coefficients

$$AX + XB^T = C, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{m \times m}, \quad X, C \in \mathbb{R}^{n \times m},$$

we can use the integral formulation

$$X = \int_0^{+\infty} e^{-At} Ce^{-B^T t} dt.$$

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We perform the *change of variables*: $t = f(\theta) \triangleq L \cdot \cot\left(\frac{\theta}{2}\right)^2$, rewriting the integral as

$$X = 2L \int_0^\pi \frac{\sin(\theta)}{(1 - \cos(\theta))^2} e^{-Af(\theta)} Ce^{-B^T f(\theta)} d\theta,$$

with L a parameter to be optimized for convergence.

Back to Sylvester (Massei, Palitta, and Robol 2018)

We now have an integral on a finite domain \Rightarrow **Gauss-Legendre quadrature**

$$X \approx \sum_{j=1}^m \omega_j \cdot e^{-Af(\theta_j)} Ce^{-B^T f(\theta_j)},$$

for $\{\theta_j, w_j\}_{j=1}^m$ are the Legendre points and weights, and $\omega_j = 2Lw_j \cdot \frac{\sin(\theta_j)}{(1-\cos(\theta_j))^2}$.

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1 (d, d)-Padé with *scaling and squaring* $e^A = (e^{2^{-k}A})^{2^k}$ and $k = \lceil \log_2 \|A\|_2 \rceil$.

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¶ (d, d)-Padé with *scaling and squaring* $e^A = (e^{2^{-k}A})^{2^k}$ and $k = \lceil \log_2 \|A\|_2 \rceil$.

¶ Rational Chebyshev function (Popolizio and Simoncini 2008):

$$e^x \approx \frac{r_1}{x - s_1} + \dots + \frac{r_d}{x - s_d}.$$

requiring d inversions and additions that is uniformly accurate for every positive value of t , and thus is better in the case in which $\|A\|_2$ is large.

Back to Sylvester (Massei, Palitta, and Robol 2018)

Input: lyap_integral

$A, B, C, m;$

/ Solves $AX + XB^T = C$ with m integration points */*

$L \leftarrow 100$; */* Should be tuned for accuracy! */*

$[w, \theta] \leftarrow \text{GaussLegendrePts } m$;

/ Integration points and weights on $[0, \pi]$ */*

$X \leftarrow 0_{n \times n}$;

for $i = 1, \dots, m$ **do**

$f \leftarrow L \cdot \cot(\frac{\theta_i}{2})^2$;

$X \leftarrow X + w_i \frac{\sin(\theta_i)}{(1 - \cos \theta_i)^2} \cdot \expm(-f \cdot A) \cdot C \cdot \expm(-f \cdot B^T)$;

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Mixed structures

If the right-hand side C is low-rank, and the structure in the matrices A and B is HODLR, thus permitting to perform fast matrix vector multiplications and system solutions; then we can apply the *extended Krylov subspace method* we had already seen.

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Build

$$\mathbb{EK}_s(A, U) = \text{span}\{U, A^{-1}U, AU, \dots\}$$

$$\mathbb{EK}_s(B^T, V) = \text{span}\{V, B^{-T}V, B^TV, \dots\},$$

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```
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A numerical test (Massei, Mazza, and Robol 2019)

We use the usual square $[0, 1]^2$, and the source f

$$f(x, y, t) = 100 \cdot (\sin(10\pi x) \cos(\pi y) + \sin(10t) \sin(\pi x) \cdot y(1 - y)).$$

for both **constant coefficient** $d^+ = d^- = 1$, and variable coefficients

$$\begin{aligned}d_1^+(x) &= \Gamma(1.2)(1+x)^{\alpha_1}, & d_1^-(x) &= \Gamma(1.2)(2-x)^{\alpha_1}, \\d_2^+(y) &= \Gamma(1.2)(1+y)^{\alpha_2}, & d_2^-(y) &= \Gamma(1.2)(2-y)^{\alpha_2}.\end{aligned}$$

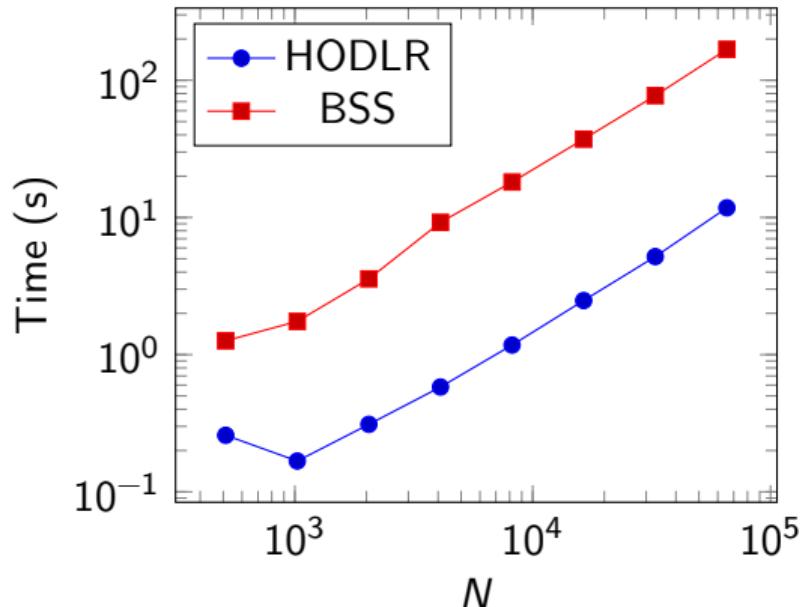
The fractional orders are $\alpha_1 = 1.3$, $\alpha_2 = 1.7$, and $\alpha_1 = 1.7$, $\alpha_2 = 1.9$. Methods are

- ☞ Sylvester by Extended-Krylov with stopping $\epsilon = 10^{-6}$ (HODLR),
 - ☞ HODLR arithmetic is set to work with a truncation of 10^{-8} .
- ☞ Sylvester by Extended-Krylov with stopping $\epsilon = 10^{-6}$ (Breiten, Simoncini, and Stoll 2016),
 - ☞ Inner solve with: GMRES with tolerance 10^{-7} and *structured preconditioners*,

A numerical test (Massei, Mazza, and Robol 2019)

Constant coefficient with $\alpha_1 = 1.3$ and $\alpha_2 = 1.7$.

N	t_{HODLR}	t_{BSS}	rank_e	qsrank_e
512	0.26	1.26	14	11
1,024	0.17	1.75	15	11
2,048	0.31	3.57	15	12
4,096	0.58	9.21	16	12
8,192	1.17	18.14	16	13
16,384	2.48	37.24	16	13
32,768	5.18	77.28	16	14
65,536	11.76	168.29	15	14

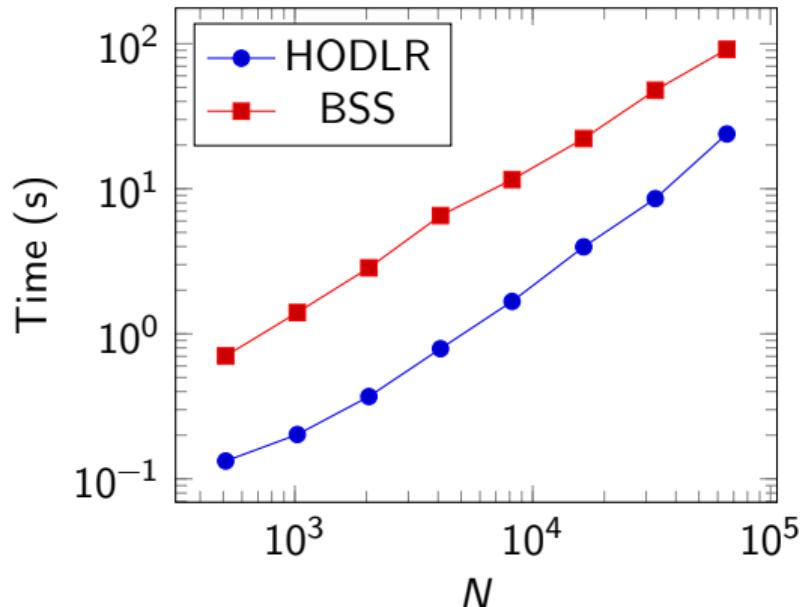


FD_Example.m from github.com/numpi/fme

A numerical test (Massei, Mazza, and Robol 2019)

Constant coefficient with $\alpha_1 = 1.7$ and $\alpha_2 = 1.9$.

N	t_{HODLR}	t_{BSS}	rank_ϵ	qsrank_ϵ
512	0.13	0.7	17	10
1,024	0.2	1.4	18	10
2,048	0.37	2.85	19	11
4,096	0.79	6.53	20	11
8,192	1.67	11.57	20	11
16,384	3.98	22.2	21	11
32,768	8.56	47.75	22	11
65,536	23.86	91.53	23	11

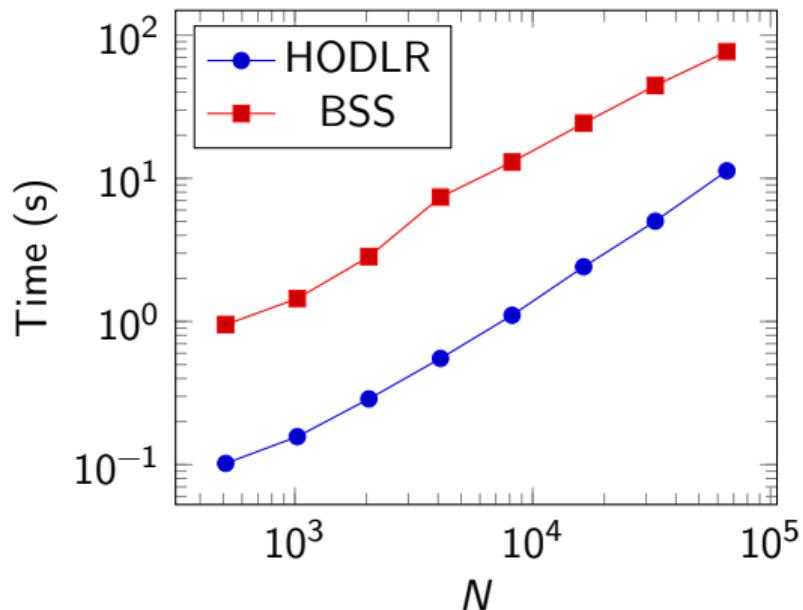


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A numerical test (Massei, Mazza, and Robol 2019)

Non-constant coefficient case with $\alpha_1 = 1.3$ and $\alpha_2 = 1.7$.

N	t_{HODLR}	t_{BSS}	rank_ϵ	qsrank_ϵ
512	0.1	0.95	14	10
1,024	0.16	1.45	14	11
2,048	0.29	2.83	15	12
4,096	0.55	7.39	16	12
8,192	1.11	13.02	16	13
16,384	2.41	24.27	16	13
32,768	5.02	44.5	16	14
65,536	11.28	76.78	16	14

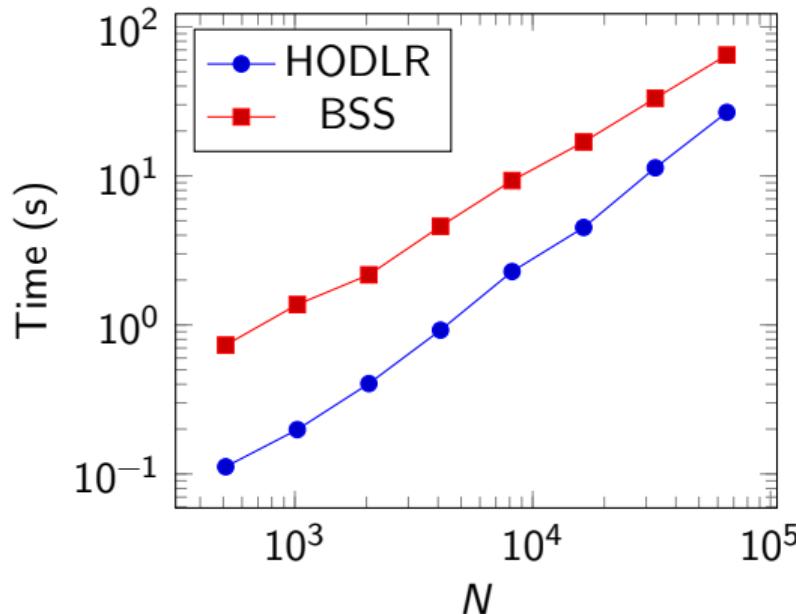


FD_Example_vc.m from github.com/numpi/fme

A numerical test (Massei, Mazza, and Robol 2019)

Non-constant coefficient case with $\alpha_1 = 1.7$ and $\alpha_2 = 1.9$.

N	t_{HODLR}	t_{BSS}	rank_ϵ	qsrank_ϵ
512	0.11	0.73	18	10
1,024	0.2	1.37	19	10
2,048	0.4	2.17	20	11
4,096	0.92	4.59	21	11
8,192	2.28	9.31	22	11
16,384	4.51	16.89	22	11
32,768	11.33	33.19	23	12
65,536	26.71	64.73	24	12



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$$\left(\frac{1}{2} I_{N_x} - \Delta t \tilde{G}_{N_x} \right) \tilde{W}^{(m+1)} + \tilde{W}^{(m+1)} \left(\frac{1}{2} I_{N_y} - \Delta t \tilde{G}_{N_y} \right)^T = \tilde{W}^{(m)} + \Delta t F^{(m+1)}, \quad m = 0, \dots, M-1.$$

here the spectrum is *fictitiously independent from the discretization*, i.e., all matrix-equation solvers perform a number of iteration independent from the system size: the cost is reduced to the extended Krylov subspace cost! **But** we still have time-stepping to do.

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$$\left(\frac{1}{2} I_{N_x} - \Delta t \tilde{G}_{N_x} \right) \tilde{W}^{(m+1)} + \tilde{W}^{(m+1)} \left(\frac{1}{2} I_{N_y} - \Delta t \tilde{G}_{N_y} \right)^T = \tilde{W}^{(m)} + \Delta t F^{(m+1)}, \quad m = 0, \dots, M-1.$$

here the spectrum is *fictitiously independent from the discretization*, i.e., all matrix-equation solvers perform a number of iteration independent from the system size: the cost is reduced to the extended Krylov subspace cost! **But** we still have time-stepping to do.

- ⌚ The case in which the matrix equation solver has a number of iterations dependent on the problem size is not yet resolved:
 - 😊 Low-rank *but* 🚫 no preconditioner – VS – 🚫 Full memory *but* 😊 preconditioners

The tale of the matrix equation: the moral of the story.

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 - ❗ Still looking for a way to solve **everything** all-at-once compactly.

Conclusion and summary

- ✓ We have seen how to work with matrices in HODLR format,
- ✓ We have discussed a couple of strategy to solve Sylvester equations with HODLR coefficients,
- ✓ We have applied all the machinery to solve a time step of a 2D equation FDE.

Next up

- 📋 Back to *all-at-once* solution with respect to both space and time,
- 📋 Linear multistep formulas in boundary value form,
- 📋 Structured preconditioner for LMFs,
- 📋 Tensor-Train reformulation of the problem.

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