Generalized Kac's moment formula for positive continuous additive functionals of symmetric Markov processes

Naotaka Kajino* Ryoichiro Noda[†]

Abstract

We establish a formula for moments of certain random variables involving positive continuous additive functionals of symmetric Hunt processes whose Dirichlet forms are regular, generalizing the classical Kac's moment formula.

Keywords: Symmetric Hunt process, positive continuous additive functional (PCAF) in the strict sense, smooth measure in the strict sense, Revuz correspondence, generalized Kac's moment formula.

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1 Introduction

The aim of this paper is to extend Kac's moment formula [8] (see (1.1) below) to general positive continuous additive functionals (PCAFs) of symmetric Markov processes and to certain more general random variables involving PCAFs. Let us first recall Kac's moment formula for the Brownian motion. Let $X = (X_t)_{t\geq 0}$ be the one-dimensional Brownian motion, $f: \mathbb{R} \to [0, \infty)$ bounded and Borel measurable, and $(p_t(x,y))_{t>0, x,y\in\mathbb{R}}$ the transition density of the Brownian motion, i.e., $p_t(x,y) = (2\pi)^{-1/2} \exp(-(x-y)^2/2t)$. Consider the additive functional $A = (A_t)_{t\geq 0}$ of X given by setting $A_t \coloneqq \int_0^t f(X_s) \, ds$. Then, Kac's moment formula reads as follows: for any $t \in (0,\infty), x \in \mathbb{R}$, and positive integer k,

$$E_{x}[A_{t}^{k}] = k! \int_{0}^{t} dt_{1} \int_{t_{1}}^{t} dt_{2} \cdots \int_{t_{k-1}}^{t} dt_{k} \int_{\mathbb{R}} f(y_{1}) dy_{1} \int_{\mathbb{R}} f(y_{2}) dy_{2}$$
$$\cdots \int_{\mathbb{R}} f(y_{k}) dy_{k} p_{t_{1}}(x, y_{1}) p_{t_{2}-t_{1}}(y_{1}, y_{2}) \cdots p_{t_{k}-t_{k-1}}(y_{k-1}, y_{k}).$$
(1.1)

Kac's moment formula plays fundamental roles in studying occupation times and Feynman–Kac semigroups (see, e.g., [4, 14]). To obtain (1.1), one does not need to use any specific properties of Brownian motion, except for the Markov property. Therefore, it is not difficult to extend (1.1) to general Markov processes with transition densities. On the other hand, the condition that the additive functional A is of the form $A_t = \int_0^t f(X_s) ds$ is crucial for the applicability of Fubini's theorem, and it is not clear how (1.1) can be extended to more general additive functionals.

In our framework, we consider symmetric Hunt processes that are associated with regular Dirichlet forms and satisfy the absolute continuity condition, and their positive continuous additive functionals (PCAFs) in the strict sense. (The precise definitions of these objects are given in Section 2 below.) Since some preparation is required to state the main theorems of this paper (Theorems 2.7 and 2.9), here we explain briefly what our version of Kac's moment formula (1.1) looks like in the present general setting. Fix a locally compact separable metrizable topological space S, a Radon measure m on S with full support, and a regular symmetric

^{*}Research Institute for Mathematical Sciences, Kyoto University, nkajino@kurims.kyoto-u.ac.jp

 $^{^\}dagger Research\ Institute\ for\ Mathematical\ Sciences,\ Kyoto\ University,\ sgrndr@kurims.kyoto-u.ac.jp$

Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(S, m)$. Let $X = (X_t)_{t \geq 0}$ be an m-symmetric Hunt process whose Dirichlet form is $(\mathcal{E}, \mathcal{F})$ and assume that X admits a jointly Borel measurable transition density $(p_t(x,y))_{t>0, x,y\in S}$ with respect to m. For each $\alpha \in (0,\infty)$, we define the α -potential density $(r_{\alpha}(x,y))_{x,y\in S}$ of X by setting $r_{\alpha}(x,y) \coloneqq \int_0^{\infty} e^{-\alpha t} p_t(x,y) dt$. Let $A = (A_t)_{t\geq 0}$ be a PCAF in the strict sense of X. By the Revuz correspondence (see Theorem 2.4), there exists a unique Borel measure μ on S, called the Revuz measure of A, satisfying

$$E_x \left[\int_0^\infty e^{-\alpha t} f(X_s) dA_s \right] = \int_S r_\alpha(x, y) f(y) \,\mu(dy) \tag{1.2}$$

for any $\alpha \in (0, \infty)$, $x \in S$, and Borel measurable function $f: S \to [0, \infty]$. In this setting, our second main theorem (Theorem 2.9) implies that, for any $t \in (0, \infty)$, $x \in S$, and positive integer k,

$$E_x[A_t^k] = k! \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k \int_S \mu(dy_1) \int_S \mu(dy_2) \cdots \int_S \mu(dy_k)$$

$$p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k)$$
(1.3)

(see Corollary 2.10 below). This is a generalization of (1.1). Indeed, if we consider a PCAF $A = (A_t)_{t \geq 0}$ of X given by $A_t := \int_0^t f(X_s) \, ds$ for a bounded Borel measurable function $f : S \to [0, \infty)$, then the measure μ satisfying (1.2) is $\mu(dx) = f(x) \, m(dx)$. Thus, (1.1) is recovered from (1.3).

The special case of (1.3) where X is the α -dimensional Bessel process on $[0, \infty)$ for $\alpha \in (0, 2)$ and A is its local time at 0 was obtained by Molchanov and Ostrovskii in [11, p. 129], by applying (1.1) to absolutely continuous PCAFs $A^{(n)}$ of X converging to A and then by letting $n \to \infty$ on the basis of explicit quantitative estimates on X. Unlike this argument in [11], our proof of the main results of this paper including (1.3) is based only on the strong Markov property of X and a change-of-variable formula for Lebesgue–Stieltjes integrals and does not require any quantitative estimates on X or A.

There is also a work in a similar research direction by Fitzsimmons and Pitman [5], where they considered more general Markov processes and their moment formula is described in terms of certain potential operators associated with PCAFs rather than transition densities and corresponding Revuz measures as in (1.3). A striking difference between their results and ours is that they considered moments of additive functionals at Markov killing times of X, whereas we consider moments of them at deterministic times. Here, a Markov killing time of X refers to a random time with the property that the process X killed at the time is again Markov, and a finite deterministic time is usually not a Markov killing time of X. For further discussions, see Remarks 2.8 and 2.11.

We expect that our generalized moment formula will be useful in the study of various objects involving PCAFs such as Feynman–Kac semigroups, just as the classical Kac's moment formula is. In fact, in [12], on the basis of our results, the convergence of PCAFs is shown to be implied by the convergence of the potentials of the corresponding Revuz measures.

The remainder of this article is organized as follows. In Section 2, we introduce the framework, state the main theorems of this article, Theorems 2.7 and 2.9, and give some corollaries of them. Then, in Section 3, we prove Theorems 2.7 and 2.9.

2 Setting and main results

This section is divided into two subsections. In Subsection 2.1, we set out the framework for the discussions of this article and introduce PCAFs and smooth measures in the strict sense. Then, in Subsection 2.2, we present our main theorems, Theorems 2.7 and 2.9, and two corollaries of them (Corollaries 2.10 and 2.12). Throughout this paper, we fix a locally compact separable

metrizable topological space S and a Radon measure m on S with full support. We write $S_{\Delta} = S \cup \{\Delta\}$ for the one-point compactification of S. (NB. If S is compact, then we add Δ to S as an isolated point.) Any $[-\infty, \infty]$ -valued function f defined on S is regarded as a function on S_{Δ} by setting $f(\Delta) := 0$, and we define $||f||_{\infty} := \sup\{|f(x)| \mid x \in S\}$ for each such f. The symbol $\mathbb N$ denotes the set of positive integers. For a non-empty set E and $E_0 \subseteq E$, we write $1_{E_0} : E \to \mathbb R$ for the function defined by setting $1_{E_0}(x) := 1$ for $x \in E_0$ and $1_{E_0}(x) := 0$ for $x \in E \setminus E_0$. Given a topological space E, we write $\mathcal{B}(E)$ for the Borel σ -algebra of E.

2.1 Setting, PCAFs and smooth measures in the strict sense

In this subsection, we clarify the setting for our discussions and introduce the main objects: PCAFs and smooth measures in the strict sense. Our results are established within the theory of regular symmetric Dirichlet forms and symmetric Hunt processes. For details of this theory, the reader is referred to [3, 6].

We first fix the setting that is assumed throughout the rest of this paper. We let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L^2(S, m)$ (see [6, Section 1.1] for the definition of the notion of regular symmetric Dirichlet form), and let $X = (\Omega, \mathcal{M}, (X_t)_{t \in [0,\infty]}, (P_x)_{x \in S_{\Delta}}, (\theta_t)_{t \in [0,\infty]})$ be an m-symmetric Hunt process on S whose Dirichlet form is $(\mathcal{E}, \mathcal{F})$ in the sense of [6, Theorem 7.2.1]. Here, θ_t denotes the shift operator, i.e., a map $\theta_t \colon \Omega \to \Omega$ satisfying $X_s \circ \theta_t = X_{s+t}$ for any $s \in [0,\infty]$. We let ζ denote the life time of X, i.e., a $[0,\infty]$ -valued function on Ω satisfying $\{X_t = \Delta\} = \{\zeta \leq t\}$ for each $t \in [0,\infty]$, and write $\mathcal{F}_* = (\mathcal{F}_t)_{t \in [0,\infty]}$ for the minimum augmented admissible filtration of X in Ω (see [3, p. 397]). A Borel subset N of S is said to be properly exceptional for X if it satisfies m(N) = 0 and $P_x(\{X_t, X_{t-} \mid t \in (0,\infty)\} \subseteq S_{\Delta} \setminus N) = 1$ for all $x \in S \setminus N$, where $X_{t-}(\omega) := \lim_{s \uparrow t} X_s(\omega) \in S_{\Delta}$ for $(t,\omega) \in (0,\infty) \times \Omega$. We assume that X satisfies the following absolute continuity condition (AC).

(AC) For all $x \in S$ and $t \in (0, \infty)$, the Borel measure $P_x(X_t \in dy)$ on S is absolutely continuous with respect to m(dy).

Then, by [1, Proof of Theorem 3.8], there exists a unique Borel measurable function $p: (0, \infty) \times S \times S \to [0, \infty]$ satisfying, for any $s, t \in (0, \infty)$ and $x, y \in S$, $P_x(X_t \in dz) = p_t(x, z) m(dz)$ (as Borel measures on S), $p_t(x, y) = p_t(y, x)$, and $p_{t+s}(x, y) = \int_S p_t(x, z) p_s(z, y) m(dz)$. The function p is called the *transition density* (or *heat kernel*) of X (with respect to m). For convenience, we extend the domain of p to $(0, \infty] \times S \times S_{\Delta}$ by setting

$$p_t(x,\Delta) := 1 - \int_S p_t(x,y) \, m(dy) = P_x(X_t = \Delta)$$
 and $p_\infty(x,y) := 1_{\{\Delta\}}(y)$

for $t \in (0, \infty)$, $x \in S$, and $y \in S_{\Delta}$, and extend the measure m to a Borel measure m_{Δ} on S_{Δ} by setting $m_{\Delta} := m(\cdot \cap S) + \delta_{S_{\Delta},\Delta}$, where $\delta_{S_{\Delta},\Delta}$ denotes the Dirac measure on S_{Δ} putting mass 1 at Δ . It is then easy to check that, for any $t \in (0, \infty]$, $x \in S$, and Borel measurable function $f: S_{\Delta} \to [0, \infty]$ (that is not necessarily 0 at Δ),

$$E_x[f(X_t)] = \int_{S_\Delta} p_t(x, y) f(y) m_\Delta(dy). \tag{2.1}$$

For each $\alpha \in (0, \infty)$, the α -potential density $r_{\alpha} \colon S \times S \to [0, \infty]$ of X is defined by setting

$$r_{\alpha}(x,y) := \int_{0}^{\infty} e^{-\alpha t} p_{t}(x,y) dt.$$

Given a Borel measure ν on S and $\alpha \in (0, \infty)$, we define $R_{\alpha}\nu \colon S \to [0, \infty]$ by setting

$$R_{\alpha}\nu(x) \coloneqq \int_{S} r_{\alpha}(x,y)\,\nu(dy).$$

Now, we introduce PCAFs and smooth measures in the strict sense below.

Definition 2.1 (PCAF, [6, p. 222]). Let $A = (A_t)_{t\geq 0}$ be an \mathcal{F}_* -adapted $[0, \infty]$ -valued stochastic process defined on Ω . It is called a *positive continuous additive functional (PCAF)* of X if there exist a set $\Lambda \in \mathcal{F}_{\infty}$, called a *defining set* of A, and a properly exceptional set $N \in \mathcal{B}(S)$ for X, called an *exceptional set* of A, satisfying the following.

- (i) It holds that $P_x(\Lambda) = 1$ for all $x \in S \setminus N$ and $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \in [0, \infty)$.
- (ii) For every $\omega \in \Lambda$, $[0, \infty) \ni t \mapsto A_t(\omega)$ is a $[0, \infty]$ -valued continuous function with $A_0(\omega) = 0$ such that for any $s, t \in [0, \infty)$, $A_t(\omega) < \infty$ if $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ if $t \ge \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$.

We set $A_{\infty} := \lim_{t \to \infty} A_t$ on Λ and $A_{\infty} := 0$ on $\Omega \setminus \Lambda$.

Definition 2.2 (PCAF in the strict sense, [6, pp. 235–236]). A PCAF of X is called a PCAF in the strict sense of X if it admits a defining set Λ with $P_x(\Lambda) = 1$ for all $x \in S$. In other words, it is a PCAF of X such that the empty set \emptyset can be taken as an exceptional set of it. We say that two PCAFs $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ in the strict sense of X are equivalent if there exists a set $\Lambda \in \mathcal{F}_{\infty}$ which is a defining set of both A and B and satisfies $P_x(\Lambda) = 1$ for all $x \in S$ and $A_t(\omega) = B_t(\omega)$ for all $(t, \omega) \in [0, \infty) \times \Lambda$. We write $\mathbf{A}_{c,1}^+$ for the collection of all PCAFs in the strict sense of X.

The next definition requires one more piece of notation. For each $E \subseteq S_{\Delta}$, we define the first exit time $\tau_E \colon \Omega \to [0,\infty]$ of X from E by setting $\tau_E(\omega) := \inf\{t \in [0,\infty) \mid X_t(\omega) \notin E\}$ (inf $\emptyset := \infty$), so that τ_E is an \mathcal{F}_* -stopping time for any $E \in \mathcal{B}(S_{\Delta})$ by [6, Theorem A.2.3] or [3, Theorem A.1.19].

Definition 2.3 (Smooth measure in the strict sense, [6, p. 238]). We define S_{00} to be the collection of finite Borel measures μ on S satisfying $||R_1\mu||_{\infty} < \infty$. A Borel measure μ on S is called a *smooth measure in the strict sense* if there exists an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of Borel subsets of S such that $1_{S_n} \cdot \mu \in S_{00}$ for every $n \in \mathbb{N}$ and $P_x(\lim_{n\to\infty} \tau_{S_n} \ge \zeta) = 1$ for each $x \in S$. We write S_1 for the collection of smooth measures in the strict sense.

It is known that there is a one-to-one correspondence between the set of equivalence classes of PCAFs of X and a certain class of Borel measures on S called smooth measures. This correspondence is due to Revuz [13] and referred to as the Revuz correspondence, and the smooth measure corresponding under it to a given PCAF A of X is called the Revuz measure of A; for details see also [6, Theorems 5.1.3 and 5.1.4], which we follow for the terminology. By [6, Theorem 5.1.7], restricting the Revuz correspondence gives a one-to-one correspondence between the equivalence classes of PCAFs in the strict sense of X and the smooth measures in the strict sense.

Theorem 2.4 (Revuz correspondence). Let $A = (A_t)_{t \geq 0} \in \mathbf{A}_{c,1}^+$, $\mu \in \mathcal{S}_1$, and consider the following condition (RC).

(RC) For any $\alpha \in (0, \infty)$, $x \in S$, and Borel measurable function $f: S \to [0, \infty]$,

$$E_x \left[\int_0^\infty e^{-\alpha t} f(X_s) \, dA_s \right] = \int_S r_\alpha(x, y) f(y) \, \mu(dy).$$

Then, μ is the Revuz measure of A if and only if (RC) holds.

Proof. If (RC) holds, then by the symmetry of r_{α} and Fubini's theorem we immediately obtain [6, Theorem 5.1.3(iv)], and hence μ is the Revuz measure of A by [6, Theorems 5.1.7(ii) and 5.1.3]. Conversely, if μ is the Revuz measure of A, then one can verify (RC) by following [9, Proof of Proposition 2.32].

Remark 2.5. Let $A = (A_t)_{t\geq 0}$ be a PCAF in the strict sense of X with defining set Λ . Then, noting that $\{\zeta = 0\} = \{X_0 = \Delta\}$ satisfies either $\{\zeta = 0\} \subset \Omega_0$ or $\{\zeta = 0\} \cap \Omega_0 = \emptyset$ for any $\Omega_0 \in \mathcal{F}_{\infty}$, we easily see that $\Lambda \in \mathcal{F}_0$, and thus that $(A_t 1_{\Lambda})_{t\geq 0}$ is a PCAF in the strict sense of X equivalent to A with defining set $\Lambda \cup \{\zeta = 0\} \in \mathcal{F}_0$. In view of this observation and the Revuz correspondence described just before Theorem 2.4, we may and do assume without loss of generality that every $PCAF A = (A_t)_{t\geq 0}$ in the strict sense of X with defining set Λ considered in the rest of this paper satisfies $A_t(\omega) = 0$ for any $(t, \omega) \in [0, \infty) \times (\Omega \setminus \Lambda)$ and $\{\zeta = 0\} \subset \Lambda$.

The following fact will be used in the proof of Proposition 3.2 below.

Lemma 2.6 ([3, Exercise A.3.3 and Theorem A.3.5(iii)]). Let $A = (A_t)_{t\geq 0}$ be a PCAF in the strict sense of X, let μ be the Revuz measure of A, and let $E \in \mathcal{B}(S)$. Then, the process $B = (B_t)_{t\geq 0}$ defined by setting $B_t := \int_0^t 1_E(X_s) dA_s$ is a PCAF in the strict sense of X with Revuz measure $1_E \cdot \mu$.

2.2 Main results

In this subsection, we state our main theorems, Theorems 2.7 and 2.9, and provide two corollaries of them, Corollaries 2.10 and 2.12. The proofs of Theorems 2.7 and 2.9 are given later in Section 3. We continue to assume the setting specified in Subsection 2.1. For measurable spaces (Y, A) and (Z, B), we write $A \otimes B$ for the product σ -algebra of A and B in $Y \times Z$.

Theorem 2.7. Let $A = (A_t)_{t\geq 0}$ be a PCAF in the strict sense of X and let μ be the Revuz measure of A. Let $F: (0,\infty) \times \Omega \to [0,\infty]$ be $\mathcal{B}((0,\infty)) \otimes \mathcal{F}_{\infty}$ -measurable, set $F_t(\omega) := F(t,\omega)$ for $(t,\omega) \in (0,\infty) \times \Omega$, and assume that the following conditions are satisfied:

- (i) there exists $\Lambda \in \mathcal{F}_{\infty}$ with $P_x(\Lambda) = 1$ for all $x \in S$ such that the function $(0, \infty) \times \Omega \ni (t, \omega) \mapsto 1_{\Lambda}(\omega) F_t \circ \theta_t(\omega) = 1_{\Lambda}(\omega) F_t(\theta_t \omega)$ is $\mathcal{B}((0, \infty)) \otimes \mathcal{F}_{\infty}$ -measurable;
- (ii) the function $(0, \infty) \times S \ni (t, x) \mapsto E_x[F_t]$ is Borel measurable.

Then, for any $t \in (0, \infty]$ and $x \in S$,

$$E_x \left[\int_0^t F_s \circ \theta_s \, dA_s \right] = \int_0^t \int_S p_s(x, y) E_y[F_s] \, \mu(dy) \, ds. \tag{2.2}$$

Remark 2.8. As mentioned in Section 1, a formula similar to (2.2) is stated in [5, Equation (21)], which deals with expectations of Lebesgue–Stieltjes integrals with respect to PCAFs up to Markov killing times of X rather than up to deterministic times t as in (2.2). As stated in Remark 2.11 below, it is possible to extend our results to the case where t is replaced by the first exit time t0 from a non-empty open subset t0 of t3, one of the most typical examples of Markov killing times of t4. We emphasize here that our proof of Theorem 2.7 is quite elementary and based only on the strong Markov property of t5 and a change-of-variable formula for Lebesgue–Stieltjes integrals, whereas it is indicated in [5] that the (omitted) proof of [5, Equation (21)] uses the Ray–Knight compactification and is therefore technically demanding.

By using Theorem 2.7, we further obtain the following generalizations of Kac's moment formula (1.1) to various random variables involving general PCAFs in the strict sense of X.

Theorem 2.9. Let $k \in \mathbb{N}$, and for each $i \in \{1, 2, ..., k\}$, let $A^{(i)} = (A_t^{(i)})_{t \geq 0}$ be a PCAF in the strict sense of X and let μ_i be the Revuz measure of $A^{(i)}$. Then, for any $t \in (0, \infty]$, $x \in S$, and

Borel measurable function $f: S_{\Delta} \to [0, \infty]$,

$$E_{x}\left[f(X_{t})\prod_{i=1}^{k}A_{t}^{(i)}\right]$$

$$=\sum_{\pi\in\mathfrak{S}_{k}}\int_{0}^{t}dt_{1}\int_{t_{1}}^{t}dt_{2}\cdots\int_{t_{k-1}}^{t}dt_{k}\int_{S}\mu_{\pi_{1}}(dy_{1})\int_{S}\mu_{\pi_{2}}(dy_{2})\cdots\int_{S}\mu_{\pi_{k}}(dy_{k})\int_{S_{\Delta}}m_{\Delta}(dz)$$

$$p_{t_{1}}(x,y_{1})p_{t_{2}-t_{1}}(y_{1},y_{2})\cdots p_{t_{k}-t_{k-1}}(y_{k-1},y_{k})p_{t-t_{k}}(y_{k},z)f(z),$$
(2.3)

where \mathfrak{S}_k denotes the set of all the bijections $\pi = (\pi_i)_{i=1}^k$ from $\{1, 2, \ldots, k\}$ to itself.

Corollary 2.10. Let $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ be PCAFs in the strict sense of X, and let μ and ν be the Revuz measures of A and B, respectively. Then, the following statements hold.

(i) For any $t \in (0, \infty]$, $x \in S$, and Borel measurable function $f: S_{\Delta} \to [0, \infty]$,

$$E_x[f(X_t)A_t] = \int_0^t \int_S p_s(x,y) \int_{S_\Delta} p_{t-s}(y,z)f(z) \, m_\Delta(dz) \, \mu(dy) \, ds. \tag{2.4}$$

(ii) For any $t \in (0, \infty]$, $x \in S$, and $k \in \mathbb{N}$,

$$E_{x}[A_{t}^{k}] = k! \int_{0}^{t} dt_{1} \int_{t_{1}}^{t} dt_{2} \cdots \int_{t_{k-1}}^{t} dt_{k} \int_{S} \mu(dy_{1}) \int_{S} \mu(dy_{2}) \cdots \int_{S} \mu(dy_{k})$$

$$p_{t_{1}}(x, y_{1}) p_{t_{2}-t_{1}}(y_{1}, y_{2}) \cdots p_{t_{k}-t_{k-1}}(y_{k-1}, y_{k}). \tag{2.5}$$

(iii) For any $t \in (0, \infty]$ and $x \in S$,

$$E_{x}[A_{t}B_{t}] = \int_{0}^{t} dt_{1} \int_{t_{1}}^{t} dt_{2} \int_{S} \mu(dy_{1}) \int_{S} \nu(dy_{2}) p_{t_{1}}(x, y_{1}) p_{t_{2}-t_{1}}(y_{1}, y_{2})$$

$$+ \int_{0}^{t} dt_{1} \int_{t_{1}}^{t} dt_{2} \int_{S} \nu(dy_{1}) \int_{S} \mu(dy_{2}) p_{t_{1}}(x, y_{1}) p_{t_{2}-t_{1}}(y_{1}, y_{2}). \tag{2.6}$$

Proof. These are immediate consequences of Theorem 2.9.

Remark 2.11. More generally, we can extend each of (2.2), (2.3), (2.4), (2.5) and (2.6) to a similar formula for the random variable given by replacing t with $t \wedge \tau_D$ (and requiring $f|_{S_\Delta \setminus D}$ to be constant) for any non-empty open subset D of S. Namely, fix any such D, let $D_\Delta = D \cup \{\Delta_D\}$ be the one-point compactification of D, set $m|_D \coloneqq m|_{\mathcal{B}(D)}$, $P_{\Delta_D} \coloneqq P_{\Delta}$, and for each $t \in [0, \infty]$ define $X_t^D \colon \Omega \to D_\Delta$ by setting $X_t^D(\omega) \coloneqq X_t(\omega)$ for $\omega \in \{t < \tau_D\}$ and $X_t^D(\omega) \coloneqq \Delta_D$ for $\omega \in \{t \ge \tau_D\}$. Then $X^D \coloneqq (\Omega, \mathcal{F}_\infty, (X_t^D)_{t \in [0,\infty]}, (P_x)_{x \in D_\Delta})$, called the part process of X on D (killed upon exiting D), is an $m|_D$ -symmetric Hunt process on D by [6, Theorem A.2.10 and Lemma 4.1.3] (see also [3, Exercise 3.3.7(ii), (3.3.4) and Exercise 4.1.9(i)]), the Dirichlet form of X^D is regular by [6, Theorems 4.4.2 and 4.4.3], and X^D satisfies (AC) by (AC) of X and hence has a unique transition density p^D with respect to $m|_D$. We further set $p_t^D(x, \Delta_D) \coloneqq 1 - \int_D p_t^D(x,y) m(dy) = P_x(t \ge \tau_D)$ and $p_\infty^D(x,y) \coloneqq 1_{\{\Delta_D\}}(y)$ for $t \in (0,\infty), x \in D$, and $y \in D_\Delta$, and extend $m|_D$ to a Borel measure $(m|_D)_\Delta$ on D_Δ by setting $(m|_D)_\Delta \coloneqq m|_D(\cdot \cap D) + \delta_{D_\Delta,\Delta_D}$. Then we have the following.

Theorems 2.7, 2.9 and Corollary 2.10, with $S, S_{\Delta}, p, m_{\Delta}$ replaced by $D, D_{\Delta}, p^{D}, (m|_{D})_{\Delta}$ and t, X in the left-hand sides of (2.2), (2.3), (2.4), (2.5) and (2.6) replaced by $t \wedge \tau_{D}, X^{D}$, hold.

Indeed, Theorem 2.9 and Corollary 2.10 (and Lemma 3.1 and Proposition 3.2 below) are shown to extend in this form to general D by applying them to X^D on the basis of [3, Exercise 4.1.9 and Proposition 4.1.10], and the proof of Theorem 2.7 given in Section 3 below is easily seen to extend to the present case of general D by replacing $t, 1_S$ with $t \wedge \tau_D, 1_D$ in (3.2). (Note that this extension of Theorem 2.7 is *not* implied by a mere application of Theorem 2.7 to X^D , since the minimum augmented admissible filtration of X^D usually does not coincide with \mathcal{F}_* .)

Following [10], we define $\mathcal{S}_{EK} := \{ \mu \in \mathcal{S}_1 \mid \lim_{\alpha \to \infty} \|R_{\alpha}\mu\|_{\infty} < 1 \}$, and call \mathcal{S}_{EK} the extended Kato class. This class of smooth measures plays an important role in the study of Feynman–Kac semigroups (see, e.g., [7, 15]). As a consequence of (2.5), we can prove that any PCAF in the strict sense of X with Revuz measure in \mathcal{S}_{EK} has uniformly bounded exponential moments at any finite time, as follows. Essentially the same result was proved in [7, 15], but we give a detailed proof here since our setting and class \mathcal{S}_{EK} of smooth measures are different from those in [7, 15].

Corollary 2.12. Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X whose Revuz measure belongs to S_{EK} . Then, there exist $c_1, c_2 \in (0, \infty)$ such that for any $t \in [0, \infty)$,

$$\sup_{x \in S} E_x \left[e^{A_t} \right] \le c_1 e^{c_2 t}.$$

Proof. Let μ be the Revuz measure of A. By $\mu \in \mathcal{S}_{EK}$, we can take $\alpha \in (0, \infty)$ satisfying $\|R_{\alpha}\mu\|_{\infty} < 1$. Choose $s_{\alpha} \in (0, \infty)$ so that $c := e^{s_{\alpha}} \|R_{\alpha}\mu\|_{\infty} < 1$, set $t_{\alpha} := s_{\alpha}/\alpha$, and let $x \in S$. Then,

$$\int_{0}^{t_{\alpha}} \int_{S} p_{t}(x, y) \, \mu(dy) \, dt \leq e^{\alpha t_{\alpha}} \int_{S} \int_{0}^{t_{\alpha}} e^{-\alpha t} p_{t}(x, y) \, dt \, \mu(dy) \leq e^{\alpha t_{\alpha}} \|R_{\alpha}\mu\|_{\infty} = c < 1, \qquad (2.7)$$

which together with Corollary 2.10(ii) shows that $E_x[A_{t_\alpha}^k] \leq k!c^k$ for each $k \in \mathbb{N}$. This then yields

$$E_x[e^{A_{t_{\alpha}}}] = \sum_{k>0} \frac{1}{k!} E_x[A_{t_{\alpha}}^k] \le \frac{1}{1-c} =: c_1 \in [1, \infty).$$
 (2.8)

Now for each $t \in [0, \infty)$, taking $n = n_t \in \mathbb{N}$ such that $(n-1)t_{\alpha} \leq t < nt_{\alpha}$, and noting that $E_{\Delta}[e^{A_{t_{\alpha}}}] = 1 \leq c_1$ by Remark 2.5, we see from the Markov property of X and (2.8) that

$$E_x[e^{A_t}] \le E_x[e^{A_{nt_{\alpha}}}] = E_x[e^{A_{(n-1)t_{\alpha}}} E_{X_{(n-1)t_{\alpha}}}[e^{A_{t_{\alpha}}}]] \le c_1 E_x[e^{A_{(n-1)t_{\alpha}}}] \le c_1^n \le c_1^{1+t/t_{\alpha}},$$

which completes the proof.

Remark 2.13. It is not always the case that a PCAF $A = (A_t)_{t\geq 0}$ in the strict sense of X with Revuz measure in S_{EK} has uniformly bounded exponential moments at the life time ζ , i.e., $\sup_{x\in S} E_x[e^{A_\zeta}] < \infty$. If this is the case, then (X,A) is said to be gaugeable. The gaugeability is important in the study of Feynman–Kac semigroups and Schrödinger operators and there has been plenty of research on it. An analytic characterization of the gaugeability can be found in [10], for example.

3 Proofs of the main theorems

In this section, we prove our main theorems, Theorems 2.7 and 2.9. We continue to assume the setting specified in Subsection 2.1. The main ingredient of the proofs is the following fact.

Lemma 3.1 ([9, Proposition 2.32]). Let $A = (A_t)_{t\geq 0}$ be a PCAF in the strict sense of X, let μ be the Revuz measure of A, and let $f: S \to [0, \infty]$ be Borel measurable. Then, for any $t \in (0, \infty]$ and $x \in S$,

$$E_x \left[\int_0^t f(X_s) \, dA_s \right] = \int_0^t \int_S p_s(x, y) f(y) \, \mu(dy) \, ds.$$

A simple application of the monotone class theorem allows us to extend Lemma 3.1 as follows.

Proposition 3.2. Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X, let μ be the Revuz measure of A, and let $f: (0, \infty) \times S \to [0, \infty]$ be Borel measurable. Then, with the convention that $f(t, \Delta) := 0$ for any $t \in (0, \infty)$, it holds that for any $t \in (0, \infty)$ and $x \in S$,

$$E_x \left[\int_0^t f(s, X_s) \, dA_s \right] = \int_0^t \int_S p_s(x, y) f(s, y) \, \mu(dy) \, ds. \tag{3.1}$$

Proof. By the monotone convergence theorem, it suffices to show (3.1) for $t \in (0, \infty)$. Let $t \in (0, \infty)$, $x \in S$, and assume that (3.1) holds when $\mu \in S_{00}$. To see (3.1) in the general case, recalling that $\mu \in S_1$ by [6, Theorem 5.1.7], let $(S_n)_{n \in \mathbb{N}}$ be as in Definition 2.3. By Lemma 2.6, for each $n \in \mathbb{N}$, the process $A^{(n)} = (A_t^{(n)})_{t \geq 0}$ defined by setting $A_t^{(n)} \coloneqq \int_0^t 1_{S_n}(X_s) \, dA_s$ is a PCAF in the strict sense of X with Revuz measure $1_{S_n} \cdot \mu_n$. Recall that $(S_n)_{n \in \mathbb{N}}$ is increasing, and note also that $S = \bigcup_{n \in \mathbb{N}} S_n$ since, for any $y \in S$, $P_y(\lim_{n \to \infty} \tau_{S_n} > 0) \ge P_y(\lim_{n \to \infty} \tau_{S_n} \ge \zeta) = 1$ and hence $y \in \bigcup_{n \in \mathbb{N}} S_n$. Now we conclude (3.1) by using the monotone convergence theorem to let $n \to \infty$ in the equality

$$E_x \left[\int_0^t f(s, X_s) 1_{S_n}(X_s) dA_s \right] = E_x \left[\int_0^t f(s, X_s) dA_s^{(n)} \right] = \int_0^t \int_S p_s(x, y) f(s, y) 1_{S_n}(y) \, \mu(dy) \, ds$$

implied for any $n \in \mathbb{N}$ by the assumed validity of (3.1) for $A^{(n)}$ and $1_{S_n} \cdot \mu \in \mathcal{S}_{00}$.

It thus remains to prove (3.1) when $\mu \in \mathcal{S}_{00}$. Define \mathcal{H} to be the collection of $[0, \infty]$ -valued Borel measurable functions on $(0, \infty) \times S$ satisfying (3.1) for any $t \in (0, \infty)$ and $x \in S$. Then \mathcal{H} is closed under multiplication by any element of $[0, \infty]$ and under sum, and $1_{(0,\infty)\times S} \in \mathcal{H}$ by Lemma 3.1. Moreover, for any $t \in (0,\infty)$, $x \in S$, and $E \in \mathcal{B}(S)$, the same argument as (2.7) yields $\int_0^t \int_S p_s(x,y) 1_E(y) \, \mu(dy) \, ds \leq e^t \|R_1\mu\|_{\infty}$, which is finite since $\mu \in \mathcal{S}_{00}$. It therefore follows from Lemma 3.1 that $1_{(s_1,s_2]\times E} \in \mathcal{H}$ for any $s_1,s_2 \in [0,\infty)$ with $s_1 < s_2$ and any $E \in \mathcal{B}(S)$, and hence from the monotone class theorem (see [2, Chapter 0, Proof of Theorem 2.3]) that \mathcal{H} is equal to the set of all $[0,\infty]$ -valued Borel measurable functions on $(0,\infty)\times S$, completing the proof.

Now, we are ready to prove Theorems 2.7 and 2.9.

Proof of Theorem 2.7. Set $F_{\infty}(\omega) := 0 =: F_0(\omega)$ for $\omega \in \Omega$ and, for each $t \in [0, \infty)$, define $\tau_t \colon \Omega \to [0, \infty]$ by setting $\tau_t(\omega) := \inf\{s \in [0, \infty) \mid A_s(\omega) > t\}$ (inf $\emptyset := \infty$), so that τ_t is an \mathcal{F}_* -stopping time by [3, Proposition A.3.8(i)] and the function $[0, \infty) \ni s \mapsto \tau_s(\omega) \in [0, \infty]$ is non-decreasing and right-continuous for any $\omega \in \Omega$. In particular, by [6, Proof of Lemma A.2.4], as maps in $(s, \omega) \in [0, \infty) \times \Omega$, the functions $\tau_s(\omega)$ and $1_{[0,t \wedge \zeta(\omega))}(\tau_s(\omega))$ for $t \in (0, \infty]$ are $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_{\infty}$ -measurable, the S_{Δ} -valued map $X_{\tau_s}(\omega) := X_{\tau_s(\omega)}(\omega)$ is $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_{\infty}/\mathcal{B}(S_{\Delta})$ -measurable, and hence the functions $1_{\Lambda}(\omega)F_{\tau_s}\circ\theta_{\tau_s}(\omega)$ and $E_{X_{\tau_s}(\omega)}[F_u]|_{u=\tau_s(\omega)}$ are $\mathcal{B}([0, \infty))\otimes\mathcal{F}_{\infty}$ -measurable, where Λ is as in condition (i) and $F_{\tau_s}\circ\theta_{\tau_s}(\omega) := F_{\tau_s(\omega)}(\theta_{\tau_s(\omega)}\omega)$. Letting $t \in (0, \infty]$ and $x \in S$, and noting $P_x(\Lambda) = 1$, the $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_{\infty}$ -measurability of the functions in the previous sentence, and that $1_{[0,t \wedge \zeta)}(\tau_s)$ is \mathcal{F}_{τ_s} -measurable for each $s \in [0, \infty)$ (see, e.g., [2, Chapter I, Proposition 6.8]), now we obtain

$$\begin{split} E_x \bigg[\int_0^t F_s \circ \theta_s \, dA_s \bigg] &= E_x \bigg[\int_0^\infty \mathbf{1}_{[0, t \wedge \zeta)}(\tau_s) \mathbf{1}_\Lambda \cdot F_{\tau_s} \circ \theta_{\tau_s} \, ds \bigg] \quad \text{(by [3, Lemma A.3.7(i)])} \\ &= \int_0^\infty E_x \big[\mathbf{1}_{[0, t \wedge \zeta)}(\tau_s) \mathbf{1}_\Lambda \cdot F_{\tau_s} \circ \theta_{\tau_s} \big] \, ds \quad \text{(by Fubini's theorem)} \\ &= \int_0^\infty E_x \big[\mathbf{1}_{[0, t \wedge \zeta)}(\tau_s) E_x [F_{\tau_s} \circ \theta_{\tau_s} \mid \mathcal{F}_{\tau_s}] \big] \, ds \\ &= \int_0^\infty E_x \big[\mathbf{1}_{[0, t \wedge \zeta)}(\tau_s) E_{X_{\tau_s}} [F_u]|_{u = \tau_s} \big] \, ds \end{split}$$

$$= E_x \left[\int_0^\infty 1_{[0,t \wedge \zeta)}(\tau_s) E_{X_{\tau_s}}[F_u]|_{u=\tau_s} ds \right] \quad \text{(by Fubini's theorem)}$$

$$= E_x \left[\int_0^t 1_S(X_s) E_{X_s}[F_s] dA_s \right] \quad \text{(by [3, Lemma A.3.7(i)]);} \tag{3.2}$$

here the fourth equality follows by applying [2, Chapter I, Exercise 8.16], which holds by the strong Markov property of X (see, e.g., [3, Theorem A.1.21]), to the \mathcal{F}_* -stopping time τ_s and the $\mathcal{F}_{\tau_s} \otimes \mathcal{F}_{\infty}$ -measurable function $\Omega \times \Omega \ni (\omega, \omega') \mapsto F_{\tau_s(\omega)}(\omega')$. Combining (3.2) with Proposition 3.2, we get (2.2).

Proof of Theorem 2.9. Let $f: S_{\Delta} \to [0, \infty]$ be Borel measurable. First, for any $t \in (0, \infty]$ and $x \in S$,

$$E_x \left[f(X_t) \prod_{i=1}^k A_t^{(i)} \right] = \sum_{\pi \in \mathfrak{S}_k} E_x \left[\int_0^t dA_{t_1}^{(\pi_1)} \int_{t_1}^t dA_{t_2}^{(\pi_2)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(\pi_k)} f(X_t) \right].$$

It thus suffices to show by induction on $k \in \mathbb{N}$ that, for any $A^{(1)}, \ldots, A^{(k)}$ with their respective Revuz measures μ_1, \ldots, μ_k as in the statement, $t \in (0, \infty]$, and $x \in S$,

$$E_{x} \left[\int_{0}^{t} dA_{t_{1}}^{(1)} \int_{t_{1}}^{t} dA_{t_{2}}^{(2)} \cdots \int_{t_{k-1}}^{t} dA_{t_{k}}^{(k)} f(X_{t}) \right]$$

$$= \int_{0}^{t} dt_{1} \int_{t_{1}}^{t} dt_{2} \cdots \int_{t_{k-1}}^{t} dt_{k} \int_{S} \mu_{1}(dy_{1}) \int_{S} \mu_{2}(dy_{2}) \cdots \int_{S} \mu_{k}(dy_{k}) \int_{S_{\Delta}} m_{\Delta}(dz)$$

$$p_{t_{1}}(x, y_{1}) p_{t_{2}-t_{1}}(y_{1}, y_{2}) \cdots p_{t_{k}-t_{k-1}}(y_{k-1}, y_{k}) p_{t-t_{k}}(y_{k}, z) f(z). \tag{3.3}$$

This claim holds for k=1; indeed, applying Theorem 2.7 with $F_s=f(X_{(t-s)\vee 0})$, we have

$$\begin{split} E_x \bigg[\int_0^t dA_s^{(1)} f(X_t) \bigg] &= E_x \bigg[\int_0^t f(X_{t-s}) \circ \theta_s \, dA_s^{(1)} \bigg] \\ &= \int_0^t \int_S p_s(x,y) E_y [f(X_{t-s})] \, \mu_1(dy) \, ds \quad \text{(by Theorem 2.7)} \\ &= \int_0^t \int_S \int_{S_\Delta} p_s(x,y) p_{t-s}(y,z) f(z) \, m_\Delta(dz) \, \mu_1(dy) \, ds \quad \text{(by (2.1))}. \end{split}$$

Next, let $k \geq 2$ and suppose that the above claim with k-1 in place of k holds. Let $t \in (0, \infty]$ and, for each $s \in (0, \infty)$, define $F_s^{(1)}, F_s^{(2)}, F_s \colon \Omega \to [0, \infty]$ by setting

$$F_s^{(1)} \coloneqq \int_0^{(t-s)\vee 0} dA_{t_2}^{(2)} \int_{t_2}^{(t-s)\vee 0} dA_{t_3}^{(3)} \cdots \int_{t_{k-1}}^{(t-s)\vee 0} dA_{t_k}^{(k)}, \quad F_s^{(2)} \coloneqq f(X_{(t-s)\vee 0}), \quad F_s \coloneqq F_s^{(1)} F_s^{(2)},$$

so that, taking a defining set $\Lambda_i \in \mathcal{F}_{\infty}$ of $A^{(i)}$ for each $i \in \{2, ..., k\}$ and setting $\Lambda := \bigcap_{i=2}^k \Lambda_i$, we have $P_x(\Lambda) = 1$ for all $x \in S$ and

$$1_{\Lambda} \cdot F_s^{(1)} \circ \theta_s = 1_{\Lambda} \int_{s \wedge t}^t dA_{t_2}^{(2)} \int_{t_2}^t dA_{t_3}^{(3)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(k)}. \tag{3.4}$$

In particular, $F_s^{(1)}(\omega)$ and $1_{\Lambda}(\omega)F_s^{(1)}\circ\theta_s(\omega)$ are right-continuous as $[0,\infty]$ -valued maps in $s\in(0,\infty)$ for each $\omega\in\Omega$ by the monotone convergence theorem, and hence are $\mathcal{B}((0,\infty))\otimes\mathcal{F}_{\infty}$ -measurable as functions in $(s,\omega)\in(0,\infty)\times\Omega$ by [6, Proof of Lemma A.2.4]. Moreover, for any $s\in(0,\infty)$ and $x\in S$, $E_x[F_s]=0$ if $s\geq t$, otherwise by the induction hypothesis $E_x[F_s]$ is equal to the right-hand side of (3.3) with $k,A^{(i)},\mu_i,t$ replaced by $k-1,A^{(i+1)},\mu_{i+1},t-s$, and

therefore $(0, \infty) \times S \ni (s, x) \mapsto E_x[F_s]$ is Borel measurable. Theorem 2.7 is thus applicable to $F_s = F_s^{(1)} F_s^{(2)}$ and, combined with (3.4), shows that for any $x \in S$,

$$E_x \left[\int_0^t dA_{t_1}^{(1)} \int_{t_1}^t dA_{t_2}^{(2)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(k)} f(X_t) \right] = E_x \left[\int_0^t F_{t_1} \circ \theta_{t_1} dA_{t_1}^{(1)} \right]$$

$$= \int_0^t \int_S p_{t_1}(x, y_1) E_{y_1}[F_{t_1}] \, \mu_1(dy_1) \, dt_1,$$

which, together with the above-mentioned expression of $E_{y_1}[F_{t_1}]$ implied by the induction hypothesis, yields (3.3) and thereby completes the proof.

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