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On the representation of rational numbers via Euler's totient function

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Abstract Let $b > 1$ be an odd positive integer and $k, l \in \mathbb{N}$. In this paper, we show that every positive rational number can be written as $\varphi(m^2)/(\varphi(n^2))^b$ and $\varphi(k(m^2-1))/\varphi(ln^2)$, where $m, n \in \mathbb{N}$ and φ is the Euler's totient function. At the end, some further results are discussed.

Keywords representation of rational numbers, Euler's totient function, Diophantine equation, Dirichlet's theorem

MR(2020) Subject Classification 11A25, 11D85

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, the Euler's totient function $\varphi(n)$ is defined as the number of positive integers up to n that are relatively prime to n. Suppose that

$$n = \prod_{i=1}^{s} p_i^{\alpha_i}, \text{ where } p_1 < p_2 < \cdots < p_s \text{ are primes and } \alpha_i \in \mathbb{N},$$

is the standard factorization of n. It is well-known that (see, e.g., [1, p. 20])

$$\varphi(n) = \varphi\left(\prod_{i=1}^{s} p_i^{\alpha_i}\right) = \prod_{i=1}^{s} p_i^{\alpha_i - 1}(p_i - 1).$$

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Throughout this paper, we always suppose that $\mathrm{rad}(n) = \prod_{i=1}^{s} p_i$, and let $v_{p_i}(n) = \alpha_i$ denote the p_i-valuation of n. For $q = m/n \in \mathbb{Q}^+$, where $m, n \in \mathbb{N}$, the notation $v_{p_i}(q)$ means that $v_{p_i}(q) = v_{p_i}(m) - v_{p_i}(n)$. As an easy exercise of the above formula, we have

$$\frac{\varphi(m)}{\varphi(n)} = \frac{m}{n}$$

when $\mathrm{rad}(n) = \mathrm{rad}(m)$.

In 2020, Krachun and Sun [2] proved that every positive rational number can be written in the form $\varphi(m^2)/\varphi(n^2)$, where $m, n \in \mathbb{N}$. Li, Yuan and Bai [4] proved that such representation is unique with some natural restrictions on $\gcd(m, n)$. Recently, Krachun and Sun's result has been generalized in the following two forms

$$\frac{\varphi(km^r)}{\varphi(ln^s)} \quad \text{and} \quad \frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}, \quad \text{where} \quad k, l, r, s, a, b \in \mathbb{N}.$$

Of the two, the former has been clearly studied by Li, Yuan and Bai [3]. Let $k, l, r, s \in \mathbb{N}$ with $\max\{r, s\} \geqslant 2$. They proved that every positive rational number can be written in the form $\varphi(km^r)/\varphi(ln^s)$, where $m, n \in \mathbb{N}$, if and only if $\gcd(r, s) = 1$ or $(k, l, r, s) = (1, 1, 2, 2)$. Moreover, if $\gcd(r, s) > 1$, then the proper representation of such representation is unique.

In 2022, T. T. Vu [5] modified Krachun and Sun's proof and showed that if $\gcd(ar, bs) = 1$, then every positive rational number can be written in the form $(\varphi(m^r))^a/(\varphi(n^s))^b$, where $m, n \in \mathbb{N}$. In fact, Vu's proof shows that there exist infinitely many such positive integer pairs (m, n). At the end of [5], Vu proposed the following open problem.

Problem 1.1. *Besides $(a, b, r, s) = (1, 1, 2, 2)$ and (a, b, r, s) with $\gcd(ar, bs) = 1$, are there any other positive integer quadruples (a, b, r, s) such that every positive rational number can be written in the form $(\varphi(m^r))^a/(\varphi(n^s))^b$, where $m, n \in \mathbb{N}$?*

In this paper, we answer the above problem by establishing the following theorem.

Theorem 1.1. *Let $b > 1$ be an odd integer. Then every positive rational number can be written as $\varphi(m^2)/(\varphi(n^2))^b$, where $m, n \in \mathbb{N}$.*

Note that $q \in \mathbb{Q}^+$ if and only if $1/q \in \mathbb{Q}^+$. Let $a > 1$ and $b > 1$ be odd integers. Then Theorem 1.1 shows that $(a, 1, 2, 2)$ and $(1, b, 2, 2)$ are other positive integer quadruples that satisfy the conditions in the above problem. Further discussion of the above problem will occur in Section 3.

Another main purpose of this paper is to prove the following result.

Theorem 1.2. *Let k and l be positive integers. Then every positive rational number can be written as $\varphi(k(m^2 - 1))/\varphi(ln^2)$, where $m, n \in \mathbb{N}$.*

Let (k, l, t) be a positive integer triple. If we generalize Krachun and Sun's result in the form $\varphi(k(m^2 - t))/\varphi(ln^2)$, where $m, n \in \mathbb{N}$, then Krachun and Sun's result and Theorem 1.2 show that this form can express all positive rational number for the triples $(1, 1, 0)$ and $(k, l, 1)$. For a further study of this form, we will mention it at the end of this paper.

2 Proofs and Examples

In this section, we give our proofs in more detail and present some examples for Theorems 1.1–1.2.

Proof of Theorem 1.1. Let $2 = p_1 < p_2 < p_3 < \dots$ be the sequence of all primes. Let $Q_0 = \{1\}$ and for every $l \in \mathbb{N}$, let $Q_l = \left\{\prod_{i=1}^{l} p_i^{\alpha_i} : \alpha_i \in \mathbb{Z}, 1 \leqslant i \leqslant l\right\}$ be the set of all positive rational numbers that can be expressed as the product of integer powers of the first l primes. Let q be a positive rational number. If $q = 1$, then the statement is true for $m = n = 1$. For the case $q \neq 1$, we assume that $q \in Q_s \setminus Q_{s-1}$ for some $s \in \mathbb{N}$. We now give a construction for m and n as follows.

Step 1: By Dirichlet's theorem on primes in an arithmetic progression, we can choose a prime q_1 such that $q_1 \equiv 1 \left(\bmod \ (p_s!)^t\right)$, where $t \in \mathbb{N}$ such that

$$t(b-1) > \max_{1 \leqslant j \leqslant s}\left\{v_{p_j}\left(\frac{1}{q}\prod_{i=1}^{s}(p_i - 1)\right)\right\}.$$

Since b is an odd integer, there are positive integers x_1 and y_1 such that

$$(2x_1 - 1) - (2y_1 - 1)b = \alpha_1(b-1), \text{ where } \alpha_1 = 0.$$

Let $m_1 = q_1^{x_1}$ and $n_1 = q_1^{y_1}$. Then we have

$$\frac{\varphi(m_1^2)}{(\varphi(n_1^2))^b} = \frac{\varphi(q_1^{2x_1})}{(\varphi(q_1^{2y_1}))^b} = \frac{q_1^{2x_1-1}(q_1-1)}{q_1^{(2y_1-1)b}(q_1-1)^b} = \frac{q_1^{\alpha_1(b-1)}}{(q_1-1)^{b-1}}.$$

Step 2: Let q_2 be the maximal prime factor of $(q_1 - 1)/q_1^{\alpha_1}$. Since b is an odd integer, there are positive integers x_2 and y_2 such that

$$(2x_2 - 1) - (2y_2 - 1)b = \alpha_2(b-1), \text{ where } \alpha_2 = v_{q_2}(q_1 - 1).$$

Let $m_2 = q_2^{x_2}$ and $n_2 = q_2^{y_2}$. Then we have

$$\frac{\varphi(m_2^2)}{(\varphi(n_2^2))^b} = \frac{\varphi(q_2^{2x_2})}{(\varphi(q_2^{2y_2}))^b} = \frac{q_2^{2x_2-1}(q_2-1)}{q_2^{(2y_2-1)b}(q_2-1)^b} = \frac{q_2^{\alpha_2(b-1)}}{(q_2-1)^{b-1}}.$$

Step 3: Let q_3 be the maximal prime factor of $(q_1 - 1)(q_2 - 1)/(q_1^{\alpha_1} q_2^{\alpha_2})$. Since b is an odd integer, there are positive integers x_3 and y_3 such that

$$(2x_3 - 1) - (2y_3 - 1)b = \alpha_3(b-1), \text{ where } \alpha_3 = v_{q_3}\left((q_1-1)(q_2-1)\right).$$

Let $m_3 = q_3^{x_3}$ and $n_3 = q_3^{y_3}$. Then we have

$$\frac{\varphi(m_3^2)}{(\varphi(n_3^2))^b} = \frac{\varphi(q_3^{2x_3})}{(\varphi(q_3^{2y_3}))^b} = \frac{q_3^{2x_3-1}(q_3-1)}{q_3^{(2y_3-1)b}(q_3-1)^b} = \frac{q_3^{\alpha_3(b-1)}}{(q_3-1)^{b-1}}.$$

Continuing this procedure, we assume that k is the maximal positive integer such that $q_k > p_s$. If $k > 1$, then we will proceed to the following step k.

Step k**:** Let q_k be the maximal prime factor of $\prod_{i=1}^{k-1}(q_i-1)/\prod_{i=1}^{k-1}q_i^{\alpha_i}$. Since b is an odd integer, there are positive integers x_k and y_k such that

$$(2x_k-1)-(2y_k-1)b=\alpha_k(b-1),\ \text{where } \alpha_k=v_{q_k}\left(\prod_{i=1}^{k-1}(q_i-1)\right).$$

Let $m_k=q_k^{x_k}$ and $n_k=q_k^{y_k}$. Then we have

$$\frac{\varphi(m_k^2)}{(\varphi(n_k^2))^b}=\frac{\varphi(q_k^{2x_k})}{(\varphi(q_k^{2y_k}))^b}=\frac{q_k^{2x_k-1}(q_k-1)}{q_k^{(2y_k-1)b}(q_k-1)^b}=\frac{q_k^{\alpha_k(b-1)}}{(q_k-1)^{b-1}}.$$

Therefore, if $k>1$, then we have

$$\frac{\varphi\left(\left(\prod_{i=1}^k m_i\right)^2\right)}{\left(\varphi\left(\left(\prod_{i=1}^k n_i\right)^2\right)\right)^b}=\prod_{i=1}^k\frac{\varphi(m_i^2)}{(\varphi(n_i^2))^b}=\frac{\prod_{i=1}^k q_i^{\alpha_i(b-1)}}{\prod_{i=1}^k(q_i-1)^{b-1}}=\frac{1}{A_0},$$

where $A_0\in Q_s$ is a positive integer such that $v_{p_i}(A_0)\geqslant t(b-1)$ for every $i=1,2,\ldots,s$. Note that this conclusion also holds for $k=1$.

Next, we will continue with the following s steps.

Step $k+1$**:** Let $q_{k+1}=p_s$ and $\alpha_{k+1}=v_{q_{k+1}}(qA_0)$. Since

$$v_{q_{k+1}}(A_0)\geqslant t(b-1)>\max_{1\leqslant j\leqslant s}\left\{v_{p_j}\left(\frac{1}{q}\prod_{i=1}^s(p_i-1)\right)\right\}>-v_{q_{k+1}}(q),$$

it follows that

$$\alpha_{k+1}=v_{q_{k+1}}(qA_0)=v_{q_{k+1}}(q)+v_{q_{k+1}}(A_0)>0.$$

Note that b is an odd integer. If $2\mid\alpha_{k+1}$, then there are positive integers x_{k+1} and y_{k+1} such that

$$(2x_{k+1}-1)-(2y_{k+1}-1)b=\alpha_{k+1}.$$

Let $m_{k+1}=q_{k+1}^{x_{k+1}}$ and $n_{k+1}=q_{k+1}^{y_{k+1}}$. Then we have

$$\frac{\varphi\left(m_{k+1}^2\right)}{\left(\varphi\left(n_{k+1}^2\right)\right)^b}=\frac{\varphi\left(q_{k+1}^{2x_{k+1}}\right)}{\left(\varphi\left(q_{k+1}^{2y_{k+1}}\right)\right)^b}=\frac{q_{k+1}^{2x_{k+1}-1}(q_{k+1}-1)}{q_{k+1}^{(2y_{k+1}-1)b}(q_{k+1}-1)^b}=\frac{q_{k+1}^{\alpha_{k+1}}}{A_1},$$

where $A_1=(q_{k+1}-1)^{b-1}$. If $2\nmid\alpha_{k+1}$, then $x_{k+1}=(\alpha_{k+1}+1)/2$ is a positive integer. Let $m_{k+1}=q_{k+1}^{x_{k+1}}$ and $n_{k+1}=1$. Then we have

$$\frac{\varphi\left(m_{k+1}^2\right)}{\left(\varphi\left(n_{k+1}^2\right)\right)^b}=\frac{\varphi\left(q_{k+1}^{2x_{k+1}}\right)}{\left(\varphi\left(1^2\right)\right)^b}=q_{k+1}^{2x_{k+1}-1}(q_{k+1}-1)=\frac{q_{k+1}^{\alpha_{k+1}}}{A_1},$$

where $A_1=1/(q_{k+1}-1)$.

Step $k+2$**:** Let $q_{k+2}=p_{s-1}$ and $\alpha_{k+2}=v_{q_{k+2}}(qA_0A_1)$. Since

$$v_{q_{k+2}}(A_0)\geqslant t(b-1)>\max_{1\leqslant j\leqslant s}\left\{v_{p_j}\left(\frac{1}{q}\prod_{i=1}^{s}(p_i-1)\right)\right\}>v_{q_{k+2}}(q_{k+1}-1)-v_{q_{k+2}}(q),$$

it follows that

$$\alpha_{k+2}=v_{q_{k+2}}(qA_0A_1)\geqslant v_{q_{k+2}}(q)+v_{q_{k+2}}(A_0)-v_{q_{k+2}}(q_{k+1}-1)>0.$$

Note that b is an odd integer. If $2\mid\alpha_{k+2}$, then there are positive integers x_{k+2} and y_{k+2} such that

$$(2x_{k+2}-1)-(2y_{k+2}-1)b=\alpha_{k+2}.$$

Let $m_{k+2}=q_{k+2}^{x_{k+2}}$ and $n_{k+2}=q_{k+2}^{y_{k+2}}$. Then we have

$$\frac{\varphi\left(m_{k+2}^2\right)}{\left(\varphi\left(n_{k+2}^2\right)\right)^b}=\frac{\varphi\left(q_{k+2}^{2x_{k+2}}\right)}{\left(\varphi\left(q_{k+2}^{2y_{k+2}}\right)\right)^b}=\frac{q_{k+2}^{2x_{k+2}-1}(q_{k+2}-1)}{q_{k+2}^{(2y_{k+2}-1)b}(q_{k+2}-1)^b}=\frac{q_{k+2}^{\alpha_{k+2}}}{A_2},$$

where $A_2=(q_{k+2}-1)^{b-1}$. If $2\nmid\alpha_{k+2}$, then $x_{k+2}=(\alpha_{k+2}+1)/2$ is a positive integer. Let $m_{k+2}=q_{k+2}^{x_{k+2}}$ and $n_{k+2}=1$. Then we have

$$\frac{\varphi\left(m_{k+2}^2\right)}{\left(\varphi\left(n_{k+2}^2\right)\right)^b}=\frac{\varphi\left(q_{k+2}^{2x_{k+2}}\right)}{\left(\varphi\left(1^2\right)\right)^b}=q_{k+2}^{2x_{k+2}-1}(q_{k+2}-1)=\frac{q_{k+2}^{\alpha_{k+2}}}{A_2},$$

where $A_2=1/(q_{k+2}-1)$.

Step $k+3$**:** Let $q_{k+3}=p_{s-2}$ and $\alpha_{k+3}=v_{q_{k+3}}(qA_0A_1A_2)$. Since

$$\begin{aligned}v_{q_{k+3}}(A)\geqslant t(b-1)&>\max_{1\leqslant j\leqslant s}\left\{v_{p_j}\left(\frac{1}{q}\prod_{i=1}^{s}(p_i-1)\right)\right\}\\&>v_{q_{k+3}}\left((q_{k+1}-1)(q_{k+2}-1)\right)-v_{q_{k+3}}(q),\end{aligned}$$

it follows that

$$\alpha_{k+3}=v_{q_{k+3}}(qA_0A_1A_2)\geqslant v_{q_{k+3}}(q)+v_{q_{k+3}}(A)-v_{q_{k+3}}\left((q_{k+1}-1)(q_{k+2}-1)\right)>0.$$

Note that b is an odd integer. If $2\mid\alpha_{k+3}$, then there are positive integers x_{k+3} and y_{k+3} such that

$$(2x_{k+3}-1)-(2y_{k+3}-1)b=\alpha_{k+3}.$$

Let $m_{k+3}=q_{k+3}^{x_{k+3}}$ and $n_{k+3}=q_{k+3}^{y_{k+3}}$. Then we have

$$\frac{\varphi\left(m_{k+3}^2\right)}{\left(\varphi\left(n_{k+3}^2\right)\right)^b}=\frac{\varphi\left(q_{k+3}^{2x_{k+3}}\right)}{\left(\varphi\left(q_{k+3}^{2y_{k+3}}\right)\right)^b}=\frac{q_{k+3}^{2x_{k+3}-1}(q_{k+3}-1)}{q_{k+3}^{(2y_{k+3}-1)b}(q_{k+3}-1)^b}=\frac{q_{k+3}^{\alpha_{k+3}}}{A_3},$$

where $A_3 = (q_{k+3} - 1)^{b-1}$. If $2 \nmid \alpha_{k+3}$, then $x_{k+3} = (\alpha_{k+3} + 1)/2$ is a positive integer. Let $m_{k+3} = q_{k+3}^{x_{k+3}}$ and $n_{k+3} = 1$. Then we have

$$\frac{\varphi\left(m_{k+3}^2\right)}{\left(\varphi\left(n_{k+3}^2\right)\right)^b} = \frac{\varphi\left(q_{k+3}^{2x_{k+3}}\right)}{\left(\varphi\left(1^2\right)\right)^b} = q_{k+3}^{2x_{k+3}-1}(q_{k+3} - 1) = \frac{q_{k+3}^{\alpha_{k+3}}}{A_3},$$

where $A_3 = 1/(q_{k+3} - 1)$.

Continuing this procedure, we may end at step $k + s$.

Step $k + s$**:** Let $q_{k+s} = p_1 = 2$ and $\alpha_{k+s} = v_2\left(q \prod_{i=0}^{s-1} A_i\right)$. Since

$$v_2(A_0) \geqslant t(b-1) > \max_{1 \leqslant j \leqslant s} \left\{ v_{p_j}\left(\frac{1}{q}\prod_{i=1}^{s}(p_i - 1)\right)\right\} \geqslant v_2\left(\prod_{i=1}^{s-1}(q_{k+i} - 1)\right) - v_2(q),$$

it follows that

$$\alpha_{k+s} = v_2\left(q\prod_{i=0}^{s-1} A_i\right) \geqslant v_2(q) + v_2(A_0) - v_2\left(\prod_{i=1}^{s-1}(q_{k+i} - 1)\right) > 0.$$

Note that b is an odd integer. If $2 \mid \alpha_{k+s}$, then there are positive integers x_{k+s} and y_{k+s} such that

$$(2x_{k+s} - 1) - (2y_{k+s} - 1)b = \alpha_{k+s}.$$

Let $m_{k+s} = 2^{x_{k+s}}$ and $n_{k+s} = 2^{y_{k+s}}$. Then we have

$$\frac{\varphi\left(m_{k+s}^2\right)}{\left(\varphi\left(n_{k+s}^2\right)\right)^b} = \frac{\varphi\left(2^{2x_{k+s}}\right)}{\left(\varphi\left(2^{2y_{k+s}}\right)\right)^b} = \frac{2^{2x_{k+s}-1}(2-1)}{2^{(2y_{k+s}-1)b}(2-1)^b} = 2^{\alpha_{k+s}}.$$

If $2 \nmid \alpha_{k+s}$, then $x_{k+s} = (\alpha_{k+s} + 1)/2$ is a positive integer. Let $m_{k+s} = 2^{x_{k+s}}$ and $n_{k+s} = 1$. Then we have

$$\frac{\varphi\left(m_{k+s}^2\right)}{\left(\varphi\left(n_{k+s}^2\right)\right)^b} = \frac{\varphi\left(2^{2x_{k+s}}\right)}{\left(\varphi\left(1^2\right)\right)^b} = 2^{2x_{k+s}-1}(2-1) = 2^{\alpha_{k+s}}.$$

Let $m = \prod_{i=1}^{k+s} m_i$ and $n = \prod_{i=1}^{k+s} n_i$. Then we have

$$\frac{\varphi(m^2)}{\left(\varphi(n^2)\right)^b} = \prod_{i=1}^{k+s}\frac{\varphi\left(m_i^2\right)}{\left(\varphi\left(n_i^2\right)\right)^b} = \frac{1}{A}\cdot\prod_{i=k+1}^{k+s}\frac{\varphi\left(m_i^2\right)}{\left(\varphi\left(n_i^2\right)\right)^b} = \frac{\prod_{i=1}^{s} p_{k+i}^{\alpha_{k+i}}}{\prod_{i=0}^{s-1} A_i} = q.$$

This completes the proof.

Proof of Theorem 1.2. Let $q = u/v$ be a positive rational number, where $u, v \in \mathbb{N}$ and $\gcd(u, v) = 1$. Suppose that

$$kluv = \prod_{i=1}^{s} p_i^{\alpha_i}, \text{ where } p_1 < p_2 < \cdots < p_s \text{ are primes and } \alpha_i \in \mathbb{N}, \tag{1}$$

be the standard factorization of $kluv$. Let $d = \prod_{i=1}^{s} p_i^{\delta_i}$, where

$$\delta_i = \begin{cases} 1, & \text{if } 2 \nmid \alpha_i, \\ 0, & \text{if } 2 \mid \alpha_i. \end{cases}$$

We distinguish two cases as follows.

Case 1: $d = 1$

In this case, $\delta_i = 0$, that is, $2 \mid \alpha_i$ for every $i = 1, 2, \ldots, s$. This implies that $kluv = c^2$ for some $c \in \mathbb{N}$. Thus, we have $q = u/v = c^2/(klv^2)$. By Dirichlet's theorem on primes in an arithmetic progression, there is a positive integer t such that $1+v^2c^2lt$ is a prime greater than $\max\{2, k\}$. Let $m = 2v^2c^2lt+1$ and $n = 2v^3cltk$. Note that $\mathrm{rad}(4v^2c^2ltk) = \mathrm{rad}(4v^6c^2l^3t^2k^2)$. Then we have

$$\begin{aligned}\frac{\varphi(k(m^2-1))}{\varphi(ln^2)} &= \frac{\varphi(4v^2c^2ltk(1+v^2c^2lt))}{\varphi(4v^6c^2l^3t^2k^2)} \\ &= \frac{\varphi(4v^2c^2ltk)}{\varphi(4v^6c^2l^3t^2k^2)} \cdot v^2c^2lt = \frac{4v^2c^2ltk}{4v^6c^2l^3t^2k^2} \cdot v^2c^2lt = \frac{c^2}{klv^2} = q.\end{aligned}$$

Case 2: $d > 1$

In this case, we have $\alpha_i \equiv \delta_i \pmod 2$, that is, $v_{p_i}(kluv) \equiv v_{p_i}(d) \pmod 2$ for every $i = 1, 2, \ldots, s$. This implies that

$$v_{p_i}(k) + v_{p_i}(d) - v_{p_i}(l) \equiv v_{p_i}(u) - v_{p_i}(v) \pmod 2.$$

Hence, for every $i = 1, 2, \ldots, s$, there are positive integers x_i and y_i such that

$$v_{p_i}(k) + v_{p_i}(d) + 2x_i - v_{p_i}(l) - 2y_i = v_{p_i}(u) - v_{p_i}(v).$$

Let $M = \prod_{i=1}^{s} p_i^{x_i}$ and $N = \prod_{i=1}^{s} p_i^{y_i}$. Since d is square-free, it follows that the Pell's equation $x^2 - dM^2y^2 = 1$ has infinitely many solutions in positive integers x and y. Then we have $m_0^2 - dM^2n_0 = 1$ for some $m_0, n_0 \in \mathbb{N}$. Let $m = m_0$ and $n = n_0N$. Since $\mathrm{rad}(kdM^2n_0^2) = \mathrm{rad}(lN^2n_0^2)$, we obtain

$$\frac{\varphi(k(m^2-1))}{\varphi(ln^2)} = \frac{\varphi(kdM^2n_0^2)}{\varphi(lN^2n_0^2)} = \frac{kdM^2n_0^2}{lN^2n_0^2} = \frac{kdM^2}{lN^2}.$$

Therefore, for every $i = 1, 2, \ldots, s$, we have

$$\begin{aligned}v_{p_i}\left(\frac{\varphi(k(m^2-1))}{\varphi(ln^2)}\right) &= v_{p_i}\left(\frac{kdM^2}{lN^2}\right) = v_{p_i}(k) + v_{p_i}(d) + 2x_i - v_{p_i}(l) - 2y_i \\ &= v_{p_i}(u) - v_{p_i}(v) = v_{p_i}\left(\frac{u}{v}\right) = v_{p_i}(q).\end{aligned}$$

It follows that $q = \varphi(k(m^2-1))/\varphi(ln^2)$. This completes the proof.

Example 2.1. *Find a positive integer pair* (m, n) *such that*

(i) $\dfrac{5}{12} = \dfrac{\varphi(m^2)}{(\varphi(n^2))^3}$;

(ii) $\dfrac{5}{12} = \dfrac{\varphi(m^2)}{(\varphi(n^2))^5}$.

Solution. (i) Note that $12 \cdot (2-1) \cdot (3-1) \cdot (5-1)/5 = 2^5 \cdot 3^1 \cdot 5^{-1}$ and

$$t \cdot (3-1) > \max\{5, 1, -1\} = 5.$$

It follows that $t > 5/2$. We may let $t = 3$. Since 3456001 is a prime such that $3456001 \equiv 1 (\bmod\ (5!)^3)$, we have

$$\begin{aligned}
\frac{5}{12} &= \frac{\varphi(3456001^4)}{(\varphi(3456001^2))^3} \cdot (3456001-1)^2 \cdot \frac{5}{12} = \frac{\varphi(3456001^4)}{(\varphi(3456001^2))^3} \cdot 5^7 \cdot 3^5 \cdot 2^{18} \\
&= \frac{\varphi(3456001^4)}{(\varphi(3456001^2))^3} \cdot \frac{\varphi(5^8)}{(\varphi(1^2))^3} \cdot \frac{3^5 \cdot 2^{18}}{5-1} = \frac{\varphi(3456001^4)}{(\varphi(3456001^2))^3} \cdot \frac{\varphi(5^8)}{(\varphi(1^2))^3} \cdot 3^5 \cdot 2^{16} \\
&= \frac{\varphi(3456001^4)}{(\varphi(3456001^2))^3} \cdot \frac{\varphi(5^8)}{(\varphi(1^2))^3} \cdot \frac{\varphi(3^6)}{(\varphi(1^2))^3} \cdot 2^{15} \\
&= \frac{\varphi(3456001^4)}{(\varphi(3456001^2))^3} \cdot \frac{\varphi(5^8)}{(\varphi(1^2))^3} \cdot \frac{\varphi(3^6)}{(\varphi(1^2))^3} \cdot \frac{\varphi(2^{16})}{(\varphi(1^2))^3} \\
&= \frac{\varphi\left((3456001^2 \cdot 5^4 \cdot 3^3 \cdot 2^8)^2\right)}{\left(\varphi\left(3456001^2\right)\right)^3}.
\end{aligned}$$

(ii) Note that $12 \cdot (2-1) \cdot (3-1) \cdot (5-1)/5 = 2^5 \cdot 3^1 \cdot 5^{-1}$ and

$$t \cdot (5-1) > \max\{5, 1, -1\} = 5.$$

It follows that $t > 5/4$. We may let $t = 2$. Since 14401 is a prime such that $14401 \equiv 1 (\bmod\ (5!)^2)$, we have

$$\begin{aligned}
\frac{5}{12} &= \frac{\varphi(14401^6)}{(\varphi(14401^2))^5} \cdot (14401-1)^4 \cdot \frac{5}{12} = \frac{\varphi(14401^6)}{(\varphi(14401^2))^5} \cdot 5^9 \cdot 3^7 \cdot 2^{22} \\
&= \frac{\varphi(14401^6)}{(\varphi(14401^2))^5} \cdot \frac{\varphi(5^{10})}{(\varphi(1^2))^5} \cdot \frac{3^7 \cdot 2^{22}}{5-1} = \frac{\varphi(14401^6)}{(\varphi(14401^2))^5} \cdot \frac{\varphi(5^{10})}{(\varphi(1^2))^5} \cdot 3^7 \cdot 2^{20} \\
&= \frac{\varphi(14401^6)}{(\varphi(14401^2))^5} \cdot \frac{\varphi(5^{10})}{(\varphi(1^2))^5} \cdot \frac{\varphi(3^8)}{(\varphi(1^2))^5} \cdot 2^{19} \\
&= \frac{\varphi(14401^6)}{(\varphi(14401^2))^5} \cdot \frac{\varphi(5^{10})}{(\varphi(1^2))^5} \cdot \frac{\varphi(3^8)}{(\varphi(1^2))^5} \cdot \frac{\varphi(2^{20})}{(\varphi(1^2))^5} \\
&= \frac{\varphi\left((14401^3 \cdot 5^5 \cdot 3^4 \cdot 2^{10})^2\right)}{\left(\varphi\left(14401^2\right)\right)^5}.
\end{aligned}$$

Remark 2.1. Using the method provided by the proof of Theorem 1.1, we must find the positive integer pair (m, n) that satisfies the condition. However, such pairs are not necessarily optimal. For example, we have

$$\begin{aligned}\frac{5}{12} &= \frac{\varphi(241^4)}{\left(\varphi(241^2)\right)^3}\cdot(241-1)^2\cdot\frac{5}{12} = \frac{\varphi(241^4)}{\left(\varphi(241^2)\right)^3}\cdot 5^3\cdot 3^1\cdot 2^6\\
&= \frac{\varphi(241^4)}{\left(\varphi(241^2)\right)^3}\cdot\frac{\varphi(5^4)}{\left(\varphi(1^2)\right)^3}\cdot\frac{3^1\cdot 2^6}{5-1} = \frac{\varphi(241^4)}{\left(\varphi(241^2)\right)^3}\cdot\frac{\varphi(5^4)}{\left(\varphi(1^2)\right)^3}\cdot 3^1\cdot 2^4\\
&= \frac{\varphi(241^4)}{\left(\varphi(241^2)\right)^3}\cdot\frac{\varphi(5^4)}{\left(\varphi(1^2)\right)^3}\cdot\frac{\varphi(3^2)}{\left(\varphi(1^2)\right)^3}\cdot 2^3\\
&= \frac{\varphi(241^4)}{\left(\varphi(241^2)\right)^3}\cdot\frac{\varphi(5^4)}{\left(\varphi(1^2)\right)^3}\cdot\frac{\varphi(3^2)}{\left(\varphi(1^2)\right)^3}\cdot\frac{\varphi(2^4)}{\left(\varphi(1^2)\right)^3}\\
&= \frac{\varphi\left((241^2\cdot 5^2\cdot 3^1\cdot 2^2)^2\right)}{\left(\varphi\left(241^2\right)\right)^3}.\end{aligned}$$

and

$$\begin{aligned}\frac{5}{12} &= \frac{\varphi(61^6)}{\left(\varphi(61^2)\right)^5}\cdot(61-1)^4\cdot\frac{5}{12} = \frac{\varphi(61^6)}{\left(\varphi(61^2)\right)^5}\cdot 5^5\cdot 3^3\cdot 2^6\\
&= \frac{\varphi(61^6)}{\left(\varphi(61^2)\right)^5}\cdot\frac{\varphi(5^6)}{\left(\varphi(1^2)\right)^5}\cdot\frac{3^3\cdot 2^6}{5-1} = \frac{\varphi(61^6)}{\left(\varphi(61^2)\right)^5}\cdot\frac{\varphi(5^6)}{\left(\varphi(1^2)\right)^5}\cdot 3^3\cdot 2^4\\
&= \frac{\varphi(61^6)}{\left(\varphi(61^2)\right)^5}\cdot\frac{\varphi(5^6)}{\left(\varphi(1^2)\right)^5}\cdot\frac{\varphi(3^4)}{\left(\varphi(1^2)\right)^5}\cdot 2^3\\
&= \frac{\varphi(61^6)}{\left(\varphi(61^2)\right)^5}\cdot\frac{\varphi(5^6)}{\left(\varphi(1^2)\right)^5}\cdot\frac{\varphi(3^4)}{\left(\varphi(1^2)\right)^5}\cdot\frac{\varphi(2^4)}{\left(\varphi(1^2)\right)^5}\\
&= \frac{\varphi\left((61^3\cdot 5^3\cdot 3^2\cdot 2^2)^2\right)}{\left(\varphi\left(61^2\right)\right)^5}.\end{aligned}$$

Example 2.2. *Find a positive integer pair* (m, n) *such that*

(i) $\dfrac{5}{12} = \dfrac{\varphi(15(m^2-1))}{\varphi(2n^2)}$;

(ii) $\dfrac{5}{12} = \dfrac{\varphi(3(m^2-1))}{\varphi(5n^2)}$.

Solution. (i) Note that $2^5\cdot 3^4\cdot 5^2\cdot 7^2\cdot 11^2 = 19601^2 - 1$. Then we have

$$\begin{aligned}\frac{5}{12} &= \frac{5}{2^2\cdot 3} = \frac{2^5\cdot 3^5\cdot 5^3\cdot 7^2\cdot 11^2}{2^7\cdot 3^6\cdot 5^2\cdot 7^2\cdot 11^2} = \frac{\varphi(2^5\cdot 3^5\cdot 5^3\cdot 7^2\cdot 11^2)}{\varphi(2^7\cdot 3^6\cdot 5^2\cdot 7^2\cdot 11^2)}\\
&= \frac{\varphi(15\cdot(2^5\cdot 3^4\cdot 5^2\cdot 7^2\cdot 11^2))}{\varphi(2\cdot(2^3\cdot 3^3\cdot 5\cdot 7\cdot 11)^2)} = \frac{\varphi(15\cdot(19601^2-1))}{\varphi(2\cdot 83160^2)}.\end{aligned}$$

(ii) Since $5185001 = 2^9\cdot 3^4\cdot 5^3 + 1$, we obtain

$$(2^{10}\cdot 3^4\cdot 5^3+1)^2 - 1 = 2^{11}\cdot 3^4\cdot 5^3\cdot(2^9\cdot 3^4\cdot 5^3+1) = 2^{11}\cdot 3^4\cdot 5^3\cdot 5184001.$$

Note that 5184001 is a prime. Then

$$\begin{aligned}\frac{5}{12} &= \frac{5}{2^2\cdot 3} = \frac{2^{20}\cdot 3^9\cdot 5^6}{2^{22}\cdot 3^{10}\cdot 5^5} = \frac{2^{11}\cdot 3^5\cdot 5^3}{2^{22}\cdot 3^{10}\cdot 5^5}\cdot 2^9\cdot 3^4\cdot 5^3\\ &= \frac{\varphi(2^{11}\cdot 3^5\cdot 5^3)}{\varphi(2^{22}\cdot 3^{10}\cdot 5^5}\cdot\varphi(5184001) = \frac{\varphi(2^{11}\cdot 3^5\cdot 5^3\cdot 5184001)}{\varphi(2^{22}\cdot 3^{10}\cdot 5^5)}\\ &= \frac{\varphi(3\cdot(2^{11}\cdot 3^4\cdot 5^3\cdot 5184001))}{\varphi(5\cdot(2^{22}\cdot 3^{10}\cdot 5^4))} = \frac{\varphi(3\cdot((2^{10}\cdot 3^4\cdot 5^3+1)^2-1))}{\varphi(5\cdot(2^{11}\cdot 3^5\cdot 5^2)^2)}.\end{aligned}$$

3 Further Researches

In this section, we further discuss the problem in Section 1. Let Γ be the set of all positive integer quadruples (a,b,r,s) such that

$$\mathbb{Q}^+ = \left\{ \frac{(\varphi(m^r))^a}{(\varphi(n^s))^b} \,\middle|\, m,n\in\mathbb{N} \right\}.$$

It is obvious that $\gcd(a,b)=1$. Moreover, we have the following facts.

Fact 3.1 (2022, Vu [5])**.** *If* $\gcd(ar,bs)=1$*, then* $(a,b,r,s)\in\Gamma$*.*

Fact 3.2. $(1,1,r,s)\in\Gamma$ *if and only if* $r=s=2$*.*

Proof. This fact follows by the work of Krachun and Sun [2] and Li, Yuan and Bai [3]. □

Fact 3.3. *If* $\gcd(ar,bs)=d>1$ *and* $d\nmid(a-b)$*, then* $(a,b,r,s)\notin\Gamma$*.*

Proof. We have just seen that the statement holds for $\gcd(a,b)>1$. Now we consider $\gcd(a,b)=1$, which implies that at least one of the positive integers r and s is greater than 1. Without loss of generality, we may assume that $r>1$ when $d>2$. When $d=2$, we may assume that $2\nmid a$ and $2\mid b$, which implies that $r>1$.

When $d>2$ or $d=2$ with $2\nmid a$ and $2\mid b$, we can choose a positive integer k such that $d\nmid(a+k)$ and $d\nmid(a-b+k)$. It suffices to show that there are no positive integers m and n such that

$$2^k = \frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}. \tag{2}$$

Clearly, it is impossible when $m=n=1$. Let p and q be the maximal prime factor of m and n, respectively.

If $m=1$ and $n>1$, then $v_2\left((\varphi(m^r))^a/(\varphi(n^s))^b\right) = -v_2(\varphi(n^s))\,b\leqslant 0$. It follows that

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = -v_2(\varphi(n^s))\,b \neq v_2\left(2^k\right),$$

which contradicts with (2).

If $m > 1$ and $n = 1$, since $r > 1$ and $d \nmid (a+k)$, then there are no positive integer x such that $(rx-1)a = v_p(2^k)$. It follows that

$$v_p\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_p(m)-1)a \neq v_p\left(2^k\right),$$

which contradicts with (2).

If $m > 1$ and $n > 1$, then we distinguish three cases as follows.

Case 1: $p > q \geqslant 2$

In this case, since $r > 1$, we have

$$v_p\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_p(m)-1)a \neq v_p(2^k),$$

which contradicts with (2).

Case 2: $q > p \geqslant 2$

In this case, if $s > 1$, then we have

$$v_q\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = -(sv_q(n)-1)b \neq v_q(2^k),$$

which contradicts with (2). If $s = 1$, then $d \mid b$ and $d \nmid a$. Since $d \nmid a$ and $d \nmid (a+k)$, there are no positive integers x and y such that

$$(rx-1)a - by = v_p(2^k).$$

It follows that

$$v_p\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_p(m)-1)a - v_p\left(\varphi(n^s)\right)b \neq v_p(2^k),$$

which contradicts with (2).

Case 3: $p = q \geqslant 2$

In this case, since $d \nmid (a-b)$ and $d \nmid (a-b+k)$, then there are no positive integers x and y such that

$$(rx-1)a - (sy-1)b = v_p(2^k).$$

Thus, we have

$$v_p\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_p(n)-1)a - (sv_p(n)-1)b \neq v_p(2^k).$$

which contradicts with (2) and the proof is completed. □

Fact 3.4. *If* $\gcd(ar, bs) = d > 2$ *and* $d \mid (a-b)$, *then* $(a, b, r, s) \notin \Gamma$.

Proof. We have just seen that the statement holds for $\gcd(a, b) > 1$. Now we consider $\gcd(a, b) = 1$. Since $d \mid (a-b)$, it follows that $d \nmid a$ and $d \nmid b$. Thus, $r > 1$ and $s > 1$. By Fact 3.2, we have $a \neq b$. Without loss of generality, we may assume that $a < b$.

Since $d > 2$, we can choose a positive integer k such that $d \nmid k$ and $d \nmid (a+k)$. It suffices to show that there are no positive integers m and n such that

$$2^k = \frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}. \tag{3}$$

Clearly, it is impossible when $m = n = 1$. Let p and q be the maximal prime factor of m and n, respectively.

If $m = 1$ and $n > 1$, then $v_2\left((\varphi(m^r))^a / (\varphi(n^s))^b\right) = -v_2(\varphi(n^s))\, b \leqslant 0$. It follows that

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = -v_2(\varphi(n^s))\, b \neq v_2\left(2^k\right),$$

which contradicts with (3).

If $m > 1$ and $n = 1$, since $r > 1$ and $d \nmid (a+k)$, then there are no positive integer x such that $(rx-1)a = v_p(2^k)$. It follows that

$$v_p\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_p(m)-1)a \neq v_p\left(2^k\right),$$

which contradicts with (3).

If $m > 1$ and $n > 1$, then we distinguish three cases as follows.

Case 1: $p > q \geqslant 2$

In this case, since $r > 1$, we have

$$v_p\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_p(m)-1)a \neq v_p(2^k),$$

which contradicts with (3).

Case 2: $q > p \geqslant 2$

In this case, since $s > 1$, we have

$$v_q\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = -(sv_q(n)-1)b \neq v_q(2^k),$$

which contradicts with (3).

Case 3: $p = q \geqslant 2$

In this case, since $d \mid (a-b)$ and $d \nmid k$, it follows that $d \nmid a$, $d \nmid b$ and $d \nmid (a-b+k)$. If $p = 2$, then there are no positive integers x and y such that

$$(rx-1)a - (sy-1)b = k.$$

It follows that

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (rv_2(n)-1)a - (sv_2(n)-1)b \neq v_2(2^k),$$

which contradicts with (3). If $p > 2$, then we may assume that $m = p_1^{x_1} m_1$ and $n = p_1^{y_1} n_1$, where $m_1, n_1 \in \mathbb{N}$, $p_1 = p$, $x_1 = v_{p_1}(m)$ and $y_1 = v_{p_1}(n)$. Thus, we have

$$\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b} = \frac{p_1^{\alpha_1}}{(p_1 - 1)^{b-a}} \cdot \frac{(\varphi(m_1^r))^a}{(\varphi(n_1^s))^b},$$

where $\alpha_1 = (rx_1 - 1)a - (sy_1 - 1)b$. By (3), we have $\alpha_1 = 0$. Let p_2 and q_2 be the maximal prime factor of m_1 and n_1, respectively. Since $d \nmid a$ and $d \nmid b$, there are no positive integers x and y such that

$$(rx - 1)a - (b - a)v_{p_2}(p_1 - 1) = 0$$

and

$$-(sy - 1)b - (b - a)v_{q_2}(p_1 - 1) = 0.$$

It follows that $p_2 = q_2$.

Let $m_1 = p_2^{x_2} m_2$ and $n_1 = p_2^{y_2} n_2$, where $m_2, n_2 \in \mathbb{N}$, $x_2 = v_{p_2}(m_1)$ and $y_2 = v_{p_2}(n_1)$. Then we have

$$\begin{aligned} \frac{(\varphi(m^r))^a}{(\varphi(n^s))^b} &= \frac{p_1^{\alpha_1}}{(p_1 - 1)^{b-a}} \cdot \frac{(\varphi(m_1^r))^a}{(\varphi(n_1^s))^b} \\ &= \frac{p_1^{\alpha_1}}{(p_1 - 1)^{b-a}} \cdot \frac{p_2^{\alpha_2}}{(p_2 - 1)^{b-a}} \cdot \frac{(\varphi(m_2^r))^a}{(\varphi(n_2^s))^b}, \end{aligned}$$

where $\alpha_2 = (rx_2 - 1)a - (sy_2 - 1)b$. By (3), we have $\alpha_2 = (b - a)v_{p_2}(p_1 - 1)$. Let p_3 and q_3 be the maximal prime factor of m_2 and n_2, respectively. Since $d \nmid a$ and $d \nmid b$, there are no positive integers x and y such that

$$(rx - 1)a - (b - a)v_{p_3}\left((p_1 - 1)(p_2 - 1)\right) = 0$$

and

$$-(sy - 1)b - (b - a)v_{q_3}\left((p_1 - 1)(p_2 - 1)\right) = 0.$$

It follows that $p_3 = q_3$.

Let $m_2 = p_3^{x_3} m_3$ and $n_2 = p_3^{y_3} n_3$, where $m_3, n_3 \in \mathbb{N}$, $x_3 = v_{p_3}(m_2)$ and $y_3 = v_{p_3}(n_2)$. Then we have

$$\begin{aligned} \frac{(\varphi(m^r))^a}{(\varphi(n^s))^b} &= \frac{p_1^{\alpha_1}}{(p_1 - 1)^{b-a}} \cdot \frac{(\varphi(m_1^r))^a}{(\varphi(n_1^s))^b} \\ &= \frac{p_1^{\alpha_1}}{(p_1 - 1)^{b-a}} \cdot \frac{p_2^{\alpha_2}}{(p_2 - 1)^{b-a}} \cdot \frac{(\varphi(m_2^r))^a}{(\varphi(n_2^s))^b} \\ &= \frac{p_1^{\alpha_1}}{(p_1 - 1)^{b-a}} \cdot \frac{p_2^{\alpha_2}}{(p_2 - 1)^{b-a}} \cdot \frac{p_3^{\alpha_3}}{(p_3 - 1)^{b-a}} \cdot \frac{(\varphi(m_3^r))^a}{(\varphi(n_3^s))^b}, \end{aligned}$$

where $\alpha_3 = (rx_3 - 1)a - (sy_3 - 1)b$. By (3), we have $\alpha_3 = (b - a)v_{p_3}\left((p_1 - 1)(p_2 - 1)\right)$.

Continuing this procedure, we assume that s is the maximal positive integer such that $p_s > 2$. Then we have

$$m = 2^{\alpha} \prod_{i=1}^{s} p_i^{x_i} \quad \text{and} \quad n = 2^{\beta} \prod_{i=1}^{s} p_i^{y_i},$$

where $p = p_1 > p_2 > \cdots > p_s > 2$ are primes and α, β are nonnegative integers. Thus, we have

$$\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b} = \frac{(\varphi(2^{r\alpha}))^a}{(\varphi(2^{s\beta}))^b} \cdot \prod_{i=1}^{s} \frac{p_i^{\alpha_i}}{(p_i - 1)^{b-a}} = \frac{(\varphi(2^{r\alpha}))^a}{(\varphi(2^{s\beta}))^b} \cdot \frac{1}{2^{t(b-a)}}, \quad \text{where} \quad t \in \mathbb{N}.$$

If $\alpha = 0$ and $\beta = 0$, since $d \nmid k$ or $a < b$, then we have

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = -t(b-a) \neq v_2(2^k),$$

which contradicts with (3).

If $\alpha > 0$ and $\beta = 0$, since $d \nmid (a + k)$, then we have

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (r\alpha - 1)a - t(b-a) \neq v_2(2^k),$$

which contradicts with (3).

If $\alpha = 0$ and $\beta > 0$, since $s > 1$ and $a < b$, then we have

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = -(s\beta - 1)b - t(b-a) \neq v_2(2^k),$$

which contradicts with (3).

If $\alpha > 0$ and $\beta > 0$, since $d \nmid (a - b + k)$, then we have

$$v_2\left(\frac{(\varphi(m^r))^a}{(\varphi(n^s))^b}\right) = (r\alpha - 1)a - (s\beta - 1)b - t(b-a) \neq v_2(2^k),$$

which contradicts with (3) and the proof is completed. □

The remaining case is $\gcd(ar, bs) = 2$ with $2 \nmid ab$. In this case, Theorem 1.1 shows that $(a, 1, 2, 2), (1, b, 2, 2) \in \Gamma$ when a and b are odd integers greater than 1. In fact, using the same methods provided by the proof of Theorem 1.1, we can verify the following fact.

Fact 3.5. $(a, 1, r, 2), (1, b, 2, s) \in \Gamma$, *where a, b are two odd positive integers greater than* 1 *and r, s are two even positive integers.*

Up to now, besides the quadruples listed in Fact 3.2 and Fact 3.5, we still do not know whether there are other quadruples (a, b, r, s) with $\gcd(ar, bs) = 2$ in Γ. This now leads us to propose the following question.

Question 3.1. *Can every positive rational number q can be written in the form*

$$q = \frac{(\varphi(m^2))^3}{(\varphi(n^2))^5}, \quad \textit{where } m, n \in \mathbb{N}\ ?$$

Inspired by Theorem 1.2, we propose the following open problem for further research.

Question 3.2. *Let t be an integer which is not a square. Are there positive integer pairs (k,l) such that every positive rational number q can be written in the form*

$$q = \frac{\varphi(k(m^2-t))}{\varphi(ln^2)}, \text{ where } m,n \in \mathbb{N} \text{ ?}$$

For example, let $t=-1$, $k=1$ and $l=1$. Can every positive rational number q be written in the form

$$q = \frac{\varphi(m^2+1)}{\varphi(n^2)}, \text{ where } m,n \in \mathbb{N} \text{ ?}$$

More specifically, for every positive integer w, can 2^w be written in the form

$$2^w = \frac{\varphi(m^2+1)}{\varphi(n^2)}, \text{ where } m,n \in \mathbb{N} \text{ ?}$$

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