WEIGHTED BERGMAN KERNELS AND *-PRODUCTS

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ABSTRACT. We calculate the weighted Bergman kernel on a complex domain with a weight of the form $\rho=e^{-\alpha\phi}\mu g$, where α is a positive real number, ϕ is a Kähler potential, g is the determinant of the corresponding Kähler metric and μ is a real-valued positive function. Several *-products related to the Bergman kernel are determined. Explicit formulas are provided up to first order.

1. Introduction

Let Ω be a domain in \mathbb{C}^N $(N \geq 1)$) and ρ a positive smooth weight function on Ω . It is known that the weighted Bergman space $L^2_{\text{hol}}(\Omega, \rho)$ of all holomorphic functions in $L^2(\Omega, \rho)$ has a reproducing kernel $K(x, \overline{y})$, i.e.

$$f(x) = \int_{\Omega} K(x, \overline{y}) f(y) \, \rho(y, \overline{y}) d^{2N} y$$

for $f \in L^{\infty}(\Omega)$, which is called weighted Bergman kernel. Abusing notation, we will often write f(x), $f(\overline{x})$ or $f(x,\overline{y})$ and mean a function, which is holomorphic or anti-holomorphic in the respective argument (and not its value at the points x, y).

The Segal-Bargmann space is the prototypic example for a Bergman space, where $\Omega = \mathbb{C}^N$, the weight function is $\rho(x, \overline{x}) = e^{-\alpha x \overline{x}}$ and the Bergman kernel is $K(x, \overline{y}) = \frac{\alpha^N}{\pi^N} e^{\alpha x \overline{y}}$. See, for example [1], which is a review on weighted Bergman kernels used in mathematical physics. When Ω is the unit disc and the weight function is $\rho(x, \overline{x}) = \frac{\alpha+1}{\pi} (1-|x|^2)^{\alpha}$ where $\alpha > -1$ is a parameter, the Bergman kernel is $K(x, \overline{y}) = (1 - x \overline{y})^{-\alpha - 2}$ Numerous explicit examples of Bergman kernels are known ([2, 3, 4]).

Relaxing the condition of Ω being a domain, one can consider Kähler manifolds M with a Hermitian line bundle $L \to M$ having a curvature that equals the symplectic form ω of the Kähler manifold. Vice versa, a Kähler manifold is said to be quantizable, when its symplectic form has integer cohomology and such an Hermitian line bundle exists. The Bergman space is then the space of holomorphic sections $L^2_{\text{hol}}(M,L)$, which is a subspace of $L^2(M,L)$, the space of square integrable sections with respect to the Hermitian inner product [6,8,9]. In particular, one considers the sequence of Bergman spaces $L^2_{\text{hol}}(M,L^n)$ of the powers of the line bundle in the limit $n \to \infty$. In this context, the weight has locally the form $\rho = e^{-n\phi}g$, where ϕ is a local Kähler potential and $g = \omega^N/N!$ is the determinant of the Kähler metric.

For $n \to \infty$ asymptotic formulas for the Bergman kernel $K_n(x, \overline{x})$ of the the line bundle L^n were firstly derived in [14, 15, 16, 17], and extended to the off-diagonal in [6, 8, 9]. The asymptotic expansion has the form

$$K_n(x,\overline{y}) = \frac{n^N}{\pi^N} e^{n\phi(x,\overline{y})} \sum_{m=0}^{\infty} \frac{1}{n^m} k_m(x,\overline{y})$$
(1.1)

where the functions $k_m(x, \overline{y})$ depend on the respective ϕ . In [18] and [19], this asymptotic expansion is derived with the aid of a path integral and the density matrix projected on the lowest Landau level, respectively.

Returning to the Kähler manifold being a domain Ω in \mathbb{C}^N , it is shown in [13] that for the weight $\rho = e^{-\alpha\phi}g$ with α a positive real number and g the determinant of the Kähler metric for the potential ϕ , the Bergman kernel $K_{\alpha}(x, \overline{y})$ has asymptotic expansion

$$K_{\alpha}(x,\overline{y}) = \frac{\alpha^{N}}{\pi^{N}} e^{\alpha\phi(x,\overline{y})} \sum_{m=0}^{\infty} \frac{1}{\alpha^{m}} k_{m}(x,\overline{y})$$
(1.2)

In [12, 10] it is shown that for the weight $\rho = e^{-n\psi - \phi}$, where ψ and ϕ are plurisubharmonic functions and k is a natural number, the Bergman kernel $K_n(x, \overline{y})$ has asymptotic expansion

$$K_n(x,\overline{y}) = \frac{n^N}{\pi^N} e^{n\phi(x,\overline{y}) + \psi(x,\overline{y})} \sum_{m=0}^{\infty} \frac{1}{n^m} k_m(x,\overline{y})$$
(1.3)

with functions $k_m(x, \overline{y})$ depending on ϕ and ψ .

For studying the Bergman space, in [6] Töplitz operators generalizing the kernel (1.1) have been introduced. These kernels are called covariant Töplitz operator kernels in [8]. For a possibly solely formal sum

$$f(x,\overline{y}) = \sum_{m=0}^{\infty} \frac{1}{n^m} f_m(x,\overline{y})$$
 (1.4)

where the $f_m(x, \overline{y})$ are functions on $M \times M$, the covariant Töplitz operator kernel is

$$T_f(x,\overline{y}) = \frac{n^N}{\pi^N} e^{n\phi(x,\overline{y})} \sum_{m=0}^{\infty} \frac{1}{n^m} f_m(x,\overline{y})$$
 (1.5)

Under further assumptions, it is shown in in [6] and [8] that the product of two covariant Töplitz operators is again a covariant Töplitz operator. The Bergman kernel (1.1) becomes a covariant Toeplitz operator with symbol $k = \sum k_n$.

Starting point of the author for the present work was a physical problem, relating to the quantization of a system on a domain Ω in \mathbb{C}^N with weight $\rho = e^{-\alpha\phi}$ (without the factor g), which for α being a natural number is a special case of (1.3). In particular, when the term of first order of the asymptotic expansion of the Bergman kernel is known, the semi-classical limit of the quantized system, which is mainly encoded in its Poisson bracket, can be derived. However, only for the asymptotic expansions (1.1) and (1.2) explicit formulas were provided for the first few k_m . These expressions can be derived, from recurrence relations for the functions k_m , allowing to explicitly compote the function k_m from the functions k_0, \ldots, k_{m-1} , see [13, 6, 9], for example.

In the following, we will define covariant Töplitz operators for more general weights. In particular, as standing assumptions, $\Omega \subset \mathbb{C}^N$ is a domain, and we will consider weights of the form

$$\rho(x,\overline{x}) = e^{-\alpha\phi(x,\overline{x})}\mu(x,\overline{x})g(x,\overline{x}) \tag{1.6}$$

where $\phi(x, \overline{x})$ is a Kähler potential, i.e. a strictly plurisubharmonic function, and $\mu(x, \overline{x})$ is a real-valued positive function. We further assume that ϕ and μ are real-analytic and have a holomorphic extension to a neighbourhood of the diagonal of $\Omega \times \Omega$. g is the determinant of the Kähler metric $g_{i\overline{i}} = \partial_i \partial_{\overline{i}} \phi$ associated with ϕ . α is a positive real number.

Based on the weight (1.6), it shows that it is beneficial to define the kernel of the covariant Töplitz operator T_f of the symbol f (see section 2) to be

$$T_f(x, \overline{y}) = \frac{\alpha^N e^{\alpha \phi(x, \overline{y})}}{\pi^N \mu(x, \overline{y})} f(x, \overline{y})$$

where we have included the function μ in the denominator.

In general, a "symbol" can be considered as a function with good properties, such that the Töplitz operator T_f exists. Let us assume that f_1 , f_2 , f_3 are symbols and that the product operator $T_{f_1}M_{f_2}T_{f_3}$ is also a covariant Töplitz operator. M_{f_2} denotes the multiplication operator with the symbol f_2 from the left. This means that there is a triple symbol $S(f_1, f_2, f_3)$ with

$$T_{S(f_1,f_2,f_3)} = T_{f_1} M_{f_2} T_{f_3}$$

Let f and h be symbols, then the Berezin-Töplitz- \star -product of them is

$$(f_1 \star f_2)(x, \overline{x}) = S(f_1, 1, f_2)(x, \overline{x}) \tag{1.7}$$

It then follows that for the corresponding Töplitz operators

$$T_f T_h = T_{S(f,1,h)} = T_{f \star h}$$

Since the operator product is associative, the same applies to the \star -product. The symbol k of the Bergman kernel is the unit of the Berezin-Töplitz- \star -product. When one assumes that it exists, the corresponding Toeplitz operator T_k , i.e. the Bergman kernel is a projector

$$T_k T_f = T_{k \star f} = T_{f \star k} = T_f T_k = T_f$$

The corresponding subspace onto which T_k projects is the Bergman space.

The \star -product (1.7) has the disadvantage that instead of the function 1, the symbol k of the Bergman kernel is its unit. For quantization, a \star -product with the function 1 as unit is more useful. To this end, Berezin [20] introduced the Berezin \star -product. This is a \star -product equivalent to the Berezin-Töplitz- \star -product (1.7) by the Berezin transform. In particular, the contravariant symbol $\psi(f)$ of the symbol f is defined to be

$$\psi(f) = S(k, f, k) \tag{1.8}$$

where k is the symbol of the Bergman kernel and S is the triple symbol from above. ψ is called *Berezin transform*. Due to $\psi(1) = S(k, 1, k) = k \star k = k$, the symbol k of the Bergman kernel is the contravariant symbol of 1. When one assumes that the contravariant symbol exists, the corresponding Toeplitz operator is

$$T_{\psi(f)} = T_{S(k,f,k)} = T_k M_f T_k$$

This means that the contravariant symbol is the projection of the multiplication operator on the Bergman space. The Berezin *-product or contravariant *-product is defined by

$$f \star_{\text{con}} h = \psi^{-1} (\psi(f) \star \psi(h)) \tag{1.9}$$

and has the function 1 as unit, since $\psi(1) = k$.

A further possibility to define a \star -product with 1 as unit starts with the covariant symbol of a symbol f, which is

$$\varphi(f) = \frac{f}{k}$$

where k is the symbol of the Bergman kernel. The covariant \star -product is defined by

$$f \star_{\text{cov}} h = \varphi(\varphi^{-1}(f) \star \varphi^{-1}(h)) = \frac{1}{k} ((kf) \star (kg))$$
(1.10)

which also has the function 1 as unit.

All the relations above can be derived, when one knows the triple symbol $S(f_1, f_2, f_3)$. In the following, we will define formal symbols and a formal triple symbol, which are power series in a formal parameter \hbar , which can be identified with $\frac{1}{\alpha}$. With the formal triple symbol, formal *-products and a formal Bergman kernel can be defined. In particular in section 2, we will use analytic properties of the triple symbol to show that it fulfils a generalized associativity law and in section 3, we will use the generalized associativity law to show the following theorem.

Theorem 1.1. Let $\Omega \subset \mathbb{C}^N$ be a domain and ρ a weight of the form (1.6). Let R_n be the differential operators defined by (2.10) occurring in the asymptotic expansion of the integral (2.9). Furthermore, let f and h be formal symbols (see Definition 2.1). Then the formal Berezin-Töplitz-*-product is

$$(f \star h)(x, \overline{x}) = \sum_{n=0}^{\infty} \hbar^n R_n (f^{x\overline{y}} h^{y\overline{x}} \tilde{\mu}^{x\overline{x}y\overline{y}})(x, \overline{x})$$
(1.11)

($\tilde{\mu}$ is a function depending on μ and is defined in (2.8). (1.11) is an associative product having the formal symbol k of the Bergman kernel as unit.

Up to first order, the formal Berezin-Töplitz ⋆-product is

$$f \star h = fh + \hbar \left(g^{i\bar{j}} \bar{f}_{,\bar{j}} h_{,i} + fh(\Delta \mu + \frac{1}{2}R)\right) + \mathcal{O}(\hbar^2)$$
(1.12)

where $g^{i\bar{j}}$ is the inverse of the Kähler metic induced by ϕ . The formal symbol k of the Bergman kernel is

$$k = 1 - \hbar \left(\Delta \mu + \frac{1}{2}R\right) + \mathcal{O}(\hbar^2) \tag{1.13}$$

The contravariant \star -product is

$$f \star_{con} h = fh - \hbar g^{i\overline{j}} f_{,i} h_{,\overline{j}} + \mathcal{O}(\hbar^2)$$
(1.14)

The covariant \star -product is

$$f \star_{cov} h = fh + \hbar \left(g^{i\overline{j}} f_{\overline{j}} h_{,i}\right) + \mathcal{O}(\hbar^2)$$
(1.15)

For two functions f, h, the Poisson structure of the classical limit is

$$\{f,h\} = g^{i\overline{j}} \left(f_{,\overline{j}} h_{,i} - f_{,i} h_{,\overline{j}} \right) \tag{1.16}$$

The Poisson structure of the classical limit is the first order of the commutator of the \star -products defined above. It can be shown that equivalent \star -products have the same Poisson structure.

Note that μ is not present up to first order in the contravariant \star -product and the covariant \star -product, and therefore in the Poisson structure of the classical limit. This means that the quantization of the Kähler manifold Ω with the weight $\rho = e^{-\alpha\phi}\mu g$ is independent of the function μ .

2. Töplitz operators

In this section, we define formal symbols, which are formal power series of functions and analytic symbols, which are functions having an asymptotic expansion in $\frac{1}{\alpha}$. The parameter \hbar of the formal power series can be identified with $\frac{1}{\alpha}$, when the corresponding power series converges.

Definition 2.1. The formal power series $f \in C^{\omega}(\Omega)[[\hbar]]$

$$f(x,\overline{x}) = \sum_{m=0}^{\infty} \hbar^m f_m(x,\overline{x})$$
 (2.1)

is a formal symbol, when there is a neighbourhood of the diagonal in $\Omega \times \Omega$, where the real-analytic functions $f_m(x, \overline{x})$ have a holomorphic extension $f_m(x, \overline{y})$. The corresponding analytic symbol of order M is the function

$$f_{\alpha}^{(M)}(x,\overline{x}) = \sum_{m=0}^{M} \frac{1}{\alpha^m} f_m(x,\overline{x})$$
 (2.2)

When $f_{\alpha} = \lim_{M \to \infty} f_{\alpha}^{(M)}$ exists in a neighbourhood of the diagonal, it is called analytic symbol of infinite order or simply analytic symbol. A principal symbol is an analytic symbol of order 0, i.e. it is a function f on Ω , which has a holomorphic extension to a neighbourhood of the diagonal of $\Omega \times \Omega$.

Each analytic symbol of order M has a holomorphic extension in a neighbourhood of the diagonal. Infinite analytic symbols with an additional convergence property are defined in [8, 7].

Definition 2.2. The kernel of the *covariant Töplitz operator* of an analytic symbol f_{α} associated with the weight (1.6) is

$$T_f(x, \overline{y}) = \frac{\alpha^N e^{\alpha \phi(x, \overline{y})}}{\pi^N \mu(x, \overline{y})} f_{\alpha}(x, \overline{y})$$
 (2.3)

This is in analogy with (1.5) and extends to general α . In view of (1.3) we have included the function μ in the prefactor, which will simplify formulas in the following.

We now consider three principal symbols f_1 , f_2 , f_3 . Let M_f be the multiplication operator for left multiplication with the principal symbol f. The triple product operator defined by

$$T_{f_1, f_2, f_3} = T_{f_1} M_{f_2} T_{f_3} \tag{2.4}$$

has kernel

$$T_{f_1,f_2,f_3}(x,\overline{z}) = \int T_{f_1}(x,\overline{y}) f_2(y,\overline{y}) T_{f_3}(y,\overline{z}) \rho(y,\overline{y}) d^{2N}y$$
(2.5)

$$= \frac{\alpha^{2N} e^{\alpha \phi(x,\overline{z})}}{\pi^{2N} \mu(x,\overline{z})} \int e^{-\alpha \tilde{\phi}(x,\overline{z},y,\overline{y})} \tilde{\mu}(x,\overline{z},y,\overline{y}) f_1(x,\overline{y}) f_2(y,\overline{y}) f_3(y,\overline{z}) g(y,\overline{y}) d^{2N} y$$
(2.6)

where

$$\tilde{\phi}(x,\overline{z},y,\overline{y}) = \phi(x,\overline{z}) + \phi(y,\overline{y}) - \phi(x,\overline{y}) - \phi(y,\overline{z})$$
(2.7)

$$\tilde{\mu}(x,\overline{z},y,\overline{y}) = \frac{\mu(x,\overline{z})\mu(y,\overline{y})}{\mu(x,\overline{y})\mu(y,\overline{z})}$$
(2.8)

are generalizations of Calabi's diastasis function.

The integral in (2.5) is of the form

$$J_{\alpha}(\tilde{\phi}, f)(x, \overline{z}) = \int e^{-\alpha \tilde{\phi}(x, \overline{z}, y, \overline{y})} f(x, \overline{z}, y, \overline{y}) g(y, \overline{y}) d^{2N} y$$
 (2.9)

where $f(x, \overline{z}, y, \overline{y})$ is a real-analytic function, which exists in a neighbourhood of the diagonal in $\Omega \times \Omega \times \Omega$. In [13] (Theorem 3), [6], [8] (Propositions, 3.12 and 4.3) it is shown that the

integral (2.9) has an asymptotic expansion in α in a neighbourhood of the diagonal $\Omega \times \Omega$. In [13] this asymptotic expansion is explicitly determined

$$J_{\alpha}(\tilde{\phi}, f)(x, \overline{z}) = \frac{\alpha^{N}}{\pi^{N}} \sum_{n=0}^{\infty} \frac{1}{\alpha^{n}} R_{n}(f)(x, \overline{z})$$
 (2.10)

where R_n are differential operators of order 2n. In general, the function f depends on x, \overline{z} , y and \overline{y} , and the operators R_n contain at least 2n partial derivatives $\partial_i = \frac{\partial}{\partial y^i}$ and $\partial_{\overline{i}} = \frac{\partial}{\partial \overline{y^i}}$ with respect to y and \overline{y} . After application of these partial derivatives, the result is evaluated at x = y and $\overline{z} = \overline{y}$. Below we will give explicit formulas for R_0 and R_1 , see (3.2) and (3.3).

Applying the asymptotic expansion (2.10) to the kernel (2.5) results in

$$T_{f_1, f_2, f_3}(x, \overline{z}) = \frac{\alpha^N e^{\alpha \phi(x, \overline{z})}}{\pi^N \mu(x, \overline{z})} \sum_{n=0}^{\infty} \frac{1}{\alpha^n} R_n \left(f_1^{x\overline{y}} f_2^{y\overline{y}} f_3^{y\overline{z}} \tilde{\mu}^{x\overline{z}y\overline{y}} \right) (x, \overline{z})$$
(2.11)

where we have abbreviated the arguments of the functions inside the R_n as upper indices to shorten notation. (2.11) is a covariant Töplitz operator, with an analytic symbol of infinite order.

Proposition 2.3. Let f_1 , f_2 and f_3 be principal symbols. Then the triple operator (2.4) is a Töplitz operator with analytic symbol

$$S_{\alpha}(f_1, f_2, f_3)(x, \overline{z}) = \sum_{n=0}^{\infty} \frac{1}{\alpha^n} R_n \left(f_1^{x\overline{y}} f_2^{y\overline{y}} f_3^{y\overline{z}} \tilde{\mu}^{x\overline{z}y\overline{y}} \right) (x, \overline{z})$$
 (2.12)

The triple symbol (3.1) fulfils a generalized associativity law, which we will later use to define an associative product for formal symbols. To shows this, we need

Lemma 2.4. The function

$$\check{\phi}(x,\overline{z},y,\overline{y},w,\overline{w}) = \check{\phi}(x,\overline{z},y,\overline{y}) + \check{\phi}(y,\overline{z},w,\overline{w}) = \check{\phi}(x,\overline{z},w,\overline{w}) + \check{\phi}(x,\overline{w},y,\overline{y}) \\
= \phi(x,\overline{z}) + \phi(y,\overline{y}) + \phi(w,\overline{w}) - \phi(x,\overline{y}) - \phi(y,\overline{w}) - \phi(w,\overline{z})$$

is an analytic phase, such as defined in [8], 3.11. Thus, for every real-analytic function f, the integral

$$J_{\alpha}(\check{\phi}, f)(x, \overline{z}) = \int_{\Omega} e^{-\alpha \check{\phi}(x, \overline{z}, y, \overline{y}, w\overline{w})} f(x, \overline{z}, y, \overline{y}, w, \overline{w}) g(y, \overline{y}) d^{2N} y d^{2N} w$$
 (2.13)

exists for (x, \overline{z}) in a neighbourhood of the diagonal of $\Omega \times \Omega$ and is a real-analytic function. which has an asymptotic expansion in $\frac{1}{\alpha}$.

Proof. Since ϕ has a holomorphic extension on a neighbourhood of the diagonal, the same applies to $\check{\phi}$ and the function

$$\Phi_{\lambda}(y_1, \overline{y}_2, w_1, \overline{w}_2) = \check{\phi}(x, \overline{z}, y_1, \overline{y}_2, w_1, \overline{w}_2)$$

with $\lambda = (x, \overline{z})$ is such as in [8], 3.11. In particular with $X_{\lambda} = (x, \overline{z}, x, \overline{z})$

$$X_{\lambda=0}=0, \qquad \Phi_{\lambda}(X_{\lambda})=0$$
 $\partial_{I}\Phi_{\lambda}(X_{\lambda})=0, \qquad \partial_{IJ}\Phi_{\lambda}(X_{\lambda})$ is positive definite

where $\partial_I = \partial_{y_1}, \partial_{\overline{y}_2}, \partial_{w_1}, \partial_{\overline{w}_2}$. The existence of the integral (2.13) follows from the complex stationary phase lemma, see [8], Proposition 3.12.

Proposition 2.5. Let f_1 , f_2 , f_3 , f_4 and f_5 be principal symbols. Then the operator $\hat{T} = T_{f_1} M_{f_2} T_{f_3} M_{f_4} T_{f_5}$ is a Töplitz operator with analytic symbol

$$S_{\alpha}(f_1, f_2, f_3, f_4, f_5) = S_{\alpha}(f_1, f_2, S_{\alpha}(f_3, f_4, f_5)) = S_{\alpha}(S_{\alpha}(f_1, f_2, f_3), f_4, f_5)$$
(2.14)

where S_{α} with three arguments is the symbol (3.1) of the triple operator.

Proof. We write the symbol (3.1) as integral

$$S_{\alpha}(f_3, f_4, f_5)(x, \overline{z}) = \frac{\alpha^N}{\pi^N} \int \left(e^{-\alpha\tilde{\phi}} \tilde{\mu} \right)^{x\overline{z}y\overline{y}} g^{y\overline{y}} f_3^{x\overline{y}} f_4^{y\overline{y}} f_5^{y\overline{z}} d^{2N} y \tag{2.15}$$

for (x, \overline{z}) in a neighbourhood of the diagonal, which follows directly from (2.10) and (2.9). To shorten notation, we have written the arguments as upper indices.

The symbol of $\hat{T} = T_{f_1} M_{f_2} (T_{f_3} M_{f_4} T_{f_5})$ written as double integral is

$$\frac{\alpha^{2N}}{\pi^{2N}} \int \left(e^{-\alpha\tilde{\phi}} \tilde{\mu} \right)^{x\overline{z}y\overline{y}} g^{y\overline{y}} f_1^{x\overline{y}} f_2^{y\overline{y}} \left(\int \left(e^{-\alpha\tilde{\phi}} \tilde{\mu} \right)^{y\overline{z}w\overline{w}} g^{w\overline{w}} f_3^{y\overline{w}} f_4^{w\overline{w}} f_5^{w\overline{z}} d^{2N} w \right) d^{2N} y \tag{2.16}$$

The symbol of $\hat{T} = (T_{f_1} M_{f_2} T_{f_3}) M_{f_4} T_{f_5}$ written as double integral is

$$\frac{\alpha^{2N}}{\pi^{2N}} \int \left(e^{-\alpha\tilde{\phi}} \tilde{\mu} \right)^{x\overline{z}w\overline{w}} g^{w\overline{w}} \left(\int \left(e^{-\alpha\tilde{\phi}} \tilde{\mu} \right)^{x\overline{w}y\overline{y}} g^{y\overline{y}} f_1^{x\overline{y}} f_2^{y\overline{y}} f_3^{y\overline{w}} d^{2N} y \right) f_4^{w\overline{w}} f_5^{w\overline{z}} d^{2N} w \tag{2.17}$$

We note that

$$\left(e^{-\alpha\tilde{\phi}}\tilde{\mu}\right)^{x\overline{z}y\overline{y}}\left(e^{-\alpha\tilde{\phi}}\tilde{\mu}\right)^{y\overline{z}w\overline{w}} = \left(e^{-\alpha\tilde{\phi}}\tilde{\mu}\right)^{x\overline{z}w\overline{w}}\left(e^{-\alpha\tilde{\phi}}\tilde{\mu}\right)^{x\overline{w}y\overline{y}} = \left(e^{-\alpha\tilde{\phi}}\right)^{x\overline{z}y\overline{y}w\overline{w}}\tilde{\mu}^{x\overline{z}y\overline{y}}\tilde{\mu}^{y\overline{z}w\overline{w}}$$

which follows by inserting the definitions (2.7), (2.8) of $\tilde{\phi}$ and $\tilde{\mu}$. Therefore, both integrals (2.16) and (2.17) exist due to Lemma 2.4, are equal and have an asymptotic expansion \tilde{S}_{α} in a neighbourhood of the diagonal. This asymptotic expansion is the analytic symbol of \hat{T} . (2.14) follows by comparing (2.16) and (2.17) with (2.15).

To arrive at a closed algebra of Töplitz operators, one has to show that also the triple operator (2.4) of three analytic symbol (and not only principal symbols) is a covariant Töplitz operator. This is done for the case $\tilde{\mu} = 1$ in [6, 8, 9, 7], where it turns out that the definition of an analytic symbol has to be further restricted.

We will however go in another direction and simply define a formal triple symbol in analogy with (3.1). Nevertheless, under the assumption that the corresponding Töplitz operators and asymptotic expansions converge, the formulas stay the same by simply replacing \hbar with $\frac{1}{\alpha}$.

3. Formal symbols and their *-products

Definition 3.1. Let f_1 , f_2 , f_3 be formal symbols. The formal triple symbol is

$$S(f_1, f_2, f_3)(x, \overline{z}) = \sum_{n.m.v.q=0}^{\infty} \hbar^{n+m+p+q} R_n \left(f_{1,m}^{x\overline{y}} f_{2,p}^{y\overline{y}} f_{3,q}^{y\overline{z}} \tilde{\mu}^{x\overline{z}y\overline{y}} \right) (x, \overline{z})$$
(3.1)

The formal triple symbol is the analytic symbol (3.1) of the triple operator (2.4), where the asymptotic expansion is replaced by the formal power series in \hbar .

In [13] it is shown that the coefficients R_n of the expansion (2.10) solely depend on the Riemannian curvature of the metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \phi$ of the Kähler potential ϕ , its contractions

and covariant derivatives thereof. In particular,

$$R_0(f)(x,\overline{z}) = f(x,\overline{z},x,\overline{z}) \tag{3.2}$$

$$R_1(f)(x,\overline{z}) = (\Delta f + \frac{1}{2}Rf)(x,\overline{z},x,\overline{z})$$
(3.3)

for a function $f(x, \overline{z}, y, \overline{y})$, where $R = g^{\overline{j}i} \partial_i \partial_{\overline{j}} (\ln g)$ is the scalar curvature and $\Delta = g^{i\overline{j}} \partial_i \partial_{\overline{j}}$ is the Laplace operator. The derivatives are with respect to g and \overline{g} . In [13] also g (Theorem 5) and g (page 23) are calculated.

Up to first order, the formal triple symbol is explicitly

$$S(f_{1}, f_{2}, f_{3}) = R_{0}(f_{1}^{x\overline{y}} f_{2}^{y\overline{y}} f_{3}^{y\overline{z}} \tilde{\mu}^{x\overline{z}y\overline{y}}) + \hbar R_{1}(f_{1}^{x\overline{y}} f_{2}^{y\overline{y}} f_{3}^{y\overline{z}} \tilde{\mu}^{x\overline{z}y\overline{y}}) + \mathcal{O}(\hbar^{2})$$

$$= f_{1} f_{2} f_{3} (1 + \hbar(\Delta \mu + \frac{1}{2}R))$$

$$+ \hbar g^{i\overline{j}} (f_{1} f_{3} f_{2,i\overline{j}} + f_{3} f_{1,\overline{j}} f_{2,i} + f_{2} f_{1,\overline{j}} f_{3,i} + f_{1} f_{2,\overline{j}} f_{3,i}) + \mathcal{O}(\hbar^{2})$$

$$(3.4)$$

where all functions are evaluated at (x,\overline{z}) and, we have used that $\tilde{\mu}=1$, $\tilde{\mu}_{,I}=0$ and $\tilde{\mu}_{,i\overline{j}}=\mu_{,i\overline{j}}$ at $(x,\overline{z},y,\overline{y})=(x,\overline{z},x,\overline{z})$. We have abbreviated partial derivatives with indices separated by a comma. (Remember that the formal symbols f_1 , f_2 and f_3 are additional formal power series in \hbar .)

When we replace the asymptotic expansion of Proposition 2.5 with a formal power series, we also get a generalized associativity law for the formal triple symbol.

Proposition 3.2. Let f_1 , f_2 , f_3 , f_4 and f_5 be formal symbols. Then

$$S(f_1, f_2, S(f_3, f_4, f_5)) = S(S(f_1, f_2, f_3), f_4, f_5)$$
(3.5)

Proof. When f_1 , f_2 , f_3 , f_4 and f_5 are primary symbols, Proposition 2.5 tells that (2.14) are equal asymptotic expansions, in which we can consider every order in $\frac{1}{\alpha}$ and replace $\frac{1}{\alpha}$ with \hbar . It follows that (3.5) is true for primary symbols. The general case follows by replacing the primary symbols with formal symbols, since for formal power series, the sums in \hbar can be exchanged.

The rest of this section is devoted to proof Theorem 1.1.

Proof. The Berezin-Töplitz \star -product is defined by $f \star h = S(f, 1, h)$ for two formal symbols f and h. Equation (1.11) follows by inserting this definition into (3.1). Equation (1.12) can be directly derived from (3.4).

The Berezin-Töplitz-⋆-product (1.11) is an associative product, since by Proposition 3.2

$$f_1 \star (f_2 \star f_3) = S(f_1, 1, S(f_2, 1, f_3)) = S(S(f_1, 1, f_2), 1, f_3) = (f_1 \star f_2) \star f_3$$
 (3.6)

for three formal symbols f_1 , f_2 and f_3 .

When k is the unit of the Berezin-Töplitz \star -product, it follows that $k \star 1 = 1$ and a linear recurrence relation for k can be derived.

$$1 = \sum_{n,m=0}^{\infty} \hbar^{n+m} R_n \left(k_m^{x\overline{y}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right) = \sum_{n=0}^{\infty} \hbar^n \sum_{m=0}^n R_m \left(k_{n-m}^{x\overline{y}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right)$$
$$= k_0 + \sum_{n=0}^{\infty} \hbar^n \left(k_n + \sum_{m=0}^n R_m \left(k_{n-m}^{x\overline{y}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right) \right)$$

(see [6], [9]) Thus,

$$k_0 = 1,$$
 $k_n = -\sum_{m=1}^n R_m \left(k_{n-m}^{x\overline{y}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right)$

Therefore, the formal symbol k of the Bergman kernel exists and can be calculated by solving the recurrence relation order by order. In particular, by applying (3.2) and (3.3) equation (1.13) follows.

As a further option, one can consider $k \star k = k$ from which a quadratic recurrence relation can be derived (see [13]).

$$\sum_{n=0}^{\infty} \frac{1}{\alpha^n} k_n = \sum_{n,m,p=0}^{\infty} \frac{1}{\alpha^{n+m+p}} R_n \left(k_m^{\overline{y}} k_n^y \tilde{\mu}^{y\overline{y}} \right) = \sum_{n=0}^{\infty} \frac{1}{\alpha^n} \sum_{m=0}^n R_m \left(\sum_{p=0}^{n-m} k_p k_{n-p} \tilde{\mu}^{y\overline{y}} \right)$$
$$= k_0^2 + \sum_{n=1}^{\infty} \frac{1}{\alpha^n} \left(2k_0 k_n + \sum_{p=1}^{n-1} k_p k_{n-p} + \sum_{m=1}^n R_m \left(\sum_{p=0}^{n-m} k_p k_{n-p} \tilde{\mu}^{y\overline{y}} \right) \right)$$

Thus,

$$k_0 = 1,$$
 $k_n = -\sum_{p=1}^{n-1} k_p k_{n-p} - \sum_{m=1}^{n} R_m \left(\sum_{p=0}^{n-m} k_p k_{n-p} \tilde{\mu}^{y\overline{y}} \right)$

Using the definition (3.1) of the triple symbol S, the Berezin transform (1.8) is

$$\psi(f)(x,\overline{x}) = \sum_{n=0}^{\infty} \hbar^n R_n \left(k^{x\overline{y}} f^{y\overline{y}} k^{y\overline{x}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right) (x,\overline{x})$$
 (3.7)

Expanding k results in

$$\psi(f) = \sum_{n,m,p=0}^{\infty} \hbar^{n+m+p} R_n \left(k_m^{x\overline{y}} f^{y\overline{y}} k_p^{y\overline{x}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right)$$

and

$$\psi(f)_0 = f$$

$$\psi(f)_1 = 2k_1 f + R_1 \left(\left(f^{y\overline{y}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right) \right)$$

$$= -2 \left(\Delta \mu + \frac{1}{2} R \right) f + \Delta f + f \Delta \mu + \frac{1}{2} f R$$

$$= \Delta f - f \left(\Delta \mu + \frac{1}{2} R \right)$$

In summary,

$$\psi(f) = f + \hbar \left(\Delta f - f\left(\Delta \mu + \frac{1}{2}R\right)\right) + \mathcal{O}(\hbar^2)$$
(3.8)

To calculate $\psi^{-1}(f)$ assume that

$$\psi^{-1}(f) = \sum_{n} \hbar^{n} g_{n}$$

and therefore

$$f = \sum_{n,m,p,q=0}^{\infty} \hbar^{n+m+p+q} R_n \left(k_m^{x\overline{y}} k_p^{y\overline{x}} g_q^{y\overline{y}} \tilde{\mu}^{x\overline{x}y\overline{y}} \right) = \sum_{n=0}^{\infty} \hbar^n \sum_{l+m+p+q=n} R_l \left(k_m^{x\overline{y}} k_p^{y\overline{x}} (g_q^{y\overline{y}} \tilde{\mu})^{x\overline{x}y\overline{y}} \right)$$

which results in a recurrence relation for the g_n

$$g_0 = f,$$
 $g_n = -\sum_{r=0}^{n-1} \sum_{m+n+q=n-r} R_m \left(k_p^{x\overline{y}} k_q^{y\overline{x}} (g_r^{y\overline{y}} \tilde{\mu})^{x\overline{x}y\overline{y}} \right)$

The second relation is

$$g_1 = -2k_1 g_0 - R_1 \left((g_0^{y\overline{y}} \tilde{\mu})^{x\overline{x}y\overline{y}} \right)$$

= $2(\Delta \mu + \frac{1}{2}R)f - (\Delta f + f\Delta \mu + \frac{1}{2}f_0R)$
= $-\Delta f + f(\Delta \mu + \frac{1}{2}R)$

In summary,

$$\psi^{-1}(f) = f - \hbar \left(\Delta f - f\left(\Delta \mu + \frac{1}{2}R\right)\right) + \mathcal{O}(\hbar^2) \tag{3.9}$$

To determine the contravariant *-product (1.9) we calculate up to first order

$$\psi(f)\star\psi(h)=fh+\hbar(h\Delta f+f\Delta h+g^{i\overline{j}}f_{,\overline{j}}h_{,i}-fh\big(\Delta\mu+\frac{1}{2}R\big))+\mathcal{O}(\hbar^2)$$

where we have used (1.12) and (3.8). (1.14) follows by additionally applying (3.9).

For the covariant \star -product (1.10), one determines the formal inverse of k and applies this to the product (1.12) of kf and kh. This results in (1.15).

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