Halin graphs with positive Lin-Lu-Yau curvature

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Abstract

Halin graphs constitute an interesting class of planar and polyhedral graphs. A generalized Halin graph is obtained by connecting all leaves of a planar embedding of a tree via a cycle. A Halin graph is a generalized Halin graph having no vertex of degree two. We classify all generalized Halin graphs with positive Lin-Lu-Yau curvature.

1 Introduction

It is natural to ask how large can a graph with given local structural constraints be. Discrete notions of curvature are useful tools to describe such local structural constraints.

There has been an intensive study on graphs with positive combinatorial curvature. Combinatorial curvature is defined by angle deficiency at each vertex, which is a discrete analogue of Guassian curvature of surfaces. Notice that the definition of combinatorial curvature depends on the choice of embeddings of the graph into a surface. Higuchi [11] conjectured that any planar graph with positive combinatorial curvature is finite. This conjecture has been verified for cubic planar graphs by Sun and Yu [22], and finally confirmed by DeVos and Mohar [8]. Indeed, DeVos and Mohar showed a much stronger result: Let G be a connected graph which is 2-cell embedded into a surface S so that every vertex and face has degree at least 3. If the combinatorial curvature is positive at every vertex, then G is finite and S is homeomorphic to either a 2-sphere or the projective plane. Furthermore, if G is not a prism, antiprism, or the projective planar analogue of one of these, then the number of vertices $|V(G)| \leq 3444$. After Devos and Mohar's work, there has been a continuous effort in improving the upper bound 3444 [5, 19, 21, 23]. In [18], Nicholson and Sneddon constructed graphs with positive combinatorial curvature of order 208 embedded into a 2-sphere. Recently, Ghidelli [9] showed that 208 is the optimal upper bound for the case of graphs embedded in a 2-sphere and, for the case of graphs embedded in a projective plane, the optimal upper bound is 104.

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Ricci curvature plays an important role in the study of Riemannian geometry and geometric analysis. It can be considered as an average of sectional (Gaussian) curvatures. Recently, the study of discrete notions of Ricci curvature of graphs has attracted lots of attention, see [17] and the references therein. Lin-Lu-Yau curvature [12], which is defined via a modification of Ollivier's Ricci curvature [20], has proved to be quite useful in detecting geometric, analytic and combinatorial properties of graphs. The Lin-Lu-Yau curvature is defined on each edge of a graph. The Lin-Lu-Yau curvature of an edge $\{x,y\}$ is positive if and only if the optimal transportation distance between certain neighborhoods of x and y is smaller than the combinatorial distance between x and y. In particular, this definition does not rely on embeddings of the graph into a surface or higher dimensional differential manifold.

In contrast to the case of combinatorial curvature, graphs with positive Lin-Lu-Yau curvature can be infinite. One class of examples is given by anti-trees with suitable growth rate [7].

Restricting to planar graphs, Lu and Wang [15] established an analogue of DeVos and Mohar's result with respect to Lin-Lu-Yau curvature.

Theorem 1.1 (Lu-Wang [15]). Let G = (V, E) be a connected planar graph such that every vertex has degree at least 3. If the Lin-Lu-Yau curvature is positive at every edge, then G is finite. In particular, $|V(G)| \le 17^{544}$.

This upper bound is far from sharp. It seems to be difficult to figure out the optimal upper bound. Restricting further to outerplanar graphs, Liu, Lu and Wang [14] established the following optimal upper bound, improving a previous work [4].

Theorem 1.2 (Liu-Lu-Wang [14]). Let G = (V, E) be a connected outerplanar graph such that every vertex has degree as least 2. If the Lin-Lu-Yau curvature is positive at every edge, then $|V(G)| \leq 10$ and the upper bound is optimal.

Moreover, Liu, Lu and Wang [14] classified all the outerplanar graphs with minimum degree at least 2 and positive Lin-Lu-Yau curvature using Sagemath and Nauty/geng.

In this paper, we consider another interesting class of planar graphs called *Halin graphs*. Such graphs were studied by Halin [10] as examples of minimally 3-connected planar graphs. A Halin graph is a graph obtained from a planar embedding of a tree graph having at least 4 vertices and no vertex of degree 2 by connecting all the leaves of the tree with a cycle. A Halin graph is also known as a roofless polyhedron. In fact, a graph G with V(G) = n and E(G) = m is Halin if and only if it is polyhedral (i.e., planar and 3-connected) and has a face whose number of vertices equals m - n + 1. Halin graphs have very interesting properties. For example, they are 1-Hamiltonian [1] and almost pancyclic [2].

Indeed, we consider a larger class of graphs, which we call generalized Halin graphs. In a generalized Halin graph, we allow vertices of degree 2. A wheel graph is a typical example of Halin graphs. Let W_n be a wheel graph with order n, i.e. the graph obtained by joining a vertex v_0 to all the vertices of a cycle $C_{n-1} = v_1 \cdots v_{n-1}$. We further denote by W'_n the graph obtained from W_{n-1} by subdividing the edge v_0v_1 , and by W''_n the graph obtained from W_{n-2} by subdividing the two edges v_0v_1 and $v_0v_{\lceil \frac{n-2}{2} \rceil}$. Both W'_n and W''_n are generalized Halin graphs but not Halin.

We identify all generealized Halin graphs with positive Lin-Lu-Yau curvature.

Theorem 1.3. Let G be a generalized Halin graph with positive Lin-Lu-Yau curvature. Then $|V(G)| \le 12$, and G is isomorphic to one of the following graphs: $W_n(4 \le n \le 12)$, $W'_n(5 \le n \le 9)$, $W''_n(6 \le n \le 10)$, and $H_i(1 \le i \le 8)$ depicted in Figure 1.

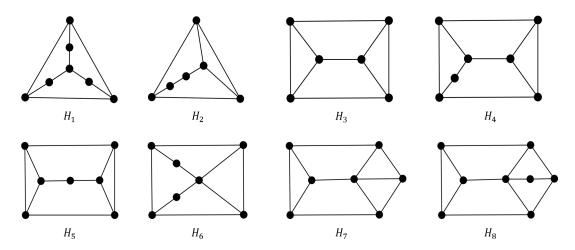


Figure 1: Generalized Halin graphs H_i , $1 \le i \le 8$

Consequently, we classify all Halin graphs with positive Lin-Lu-Yau curvature.

Corollary 1.4. Let G be a Halin graph with positive Lin-Lu-Yau curvature. Then G is isomorphic to one of the graphs: wheel graphs $W_n(4 \le n \le 12)$ and the graphs H_i (i = 3, 7) depicted in Figure 1.

A key step in our proof is to divide generalized Halin graphs with positive Lin-Lu-Yau curvature into subclasses, according to the maximum degree D(T) of the corresponding tree graphs.

Throughout the paper, we use the following notation. Let G = (V, E) be an undirected simple connected graph. For any $x \in V$, let N(x) be the set of neighbors of x and $d_x := |N(x)|$ be its degree. We use N[x] to denote $N(x) \cup \{x\}$. Let D(G) be the maximum vertex degree of G. For any $x, y \in V(G)$, we denote the combinatorial distance between x and y by d(x,y). For a vertex x and a vertex set S, the distance between x and S is denoted by

$$d(x,S) := \min\{d(x,s)|s \in S\}.$$

We write $x \sim y$ if $\{x,y\} \in E$ is an edge. A function f on V(G) is called 1-Lipschitz if $|f(x) - f(y)| \leq d(x,y)$ holds for any $x,y \in V(G)$, and we denote the set of 1-Lipschitz functions by Lip(1).

2 Preliminaries

In this sections, we collect some basics about Lin-Lu-Yau curvature and Halin graphs.

2.1 Ollivier's Ricci curvature and Lin-Lu-Yau curvature

We recall the definitions of Ollivier's Ricci curvature and Lin-Lu-Yau curvature. First, we recall the definition of Wasserstein distance between probability measures on graphs.

Definition 2.1 (Wasserstein distance). Let G = (V, E) be a locally finite graph with two probability measures m_1 and m_2 on V. The Wasserstein distance between m_1 and m_2 is defined as

$$W(m_1, m_2) := \inf_{\pi} \sum_{y \in V} \sum_{x \in V} \pi(x, y) d(x, y),$$

where the infimum is taken over all maps $\pi: V \times V \to [0,1]$ satisfying

$$m_1(x) = \sum_{y \in V} \pi(x, y)$$
 for any $x \in V$ and $m_2(y) = \sum_{x \in V} \pi(x, y)$ for any $y \in V$.

For a vertex $x \in V$ and any $\alpha \in [0,1]$, the probability measure m_x^{α} is defined as

$$m_x^{\alpha}(v) = \begin{cases} \alpha, & \text{if } v = x, \\ (1 - \alpha)/d_x, & \text{if } v \in N(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the α -Ollivier-Ricci curvature of an edge $\{x,y\} \in E(G)$ is defined as

$$\kappa_{\alpha}(x,y) = 1 - W\left(m_x^{\alpha}, m_y^{\alpha}\right).$$

In particular, we have $\kappa_{\alpha}(x,y) > 0$ if and only if $W(m_x^{\alpha}, m_y^{\alpha}) < 1 = d(x,y)$.

Definition 2.2 (Lin-Lu-Yau curvature). Let G = (V, E) be a locally finite graph. The Lin-Lu-Yau curvature of an edge $\{x, y\} \in E$ is defined as

$$\kappa_{LLY}(x,y) := \lim_{\alpha \to 1} \frac{\kappa_{\alpha}(x,y)}{1-\alpha}.$$

The ratio $\kappa_{\alpha}(x,y)/(1-\alpha)$ is constant when α is large enough. Indeed, it was proved in [3] that

$$\kappa_{LLY}(x,y) = \frac{\kappa_{\alpha}(x,y)}{1-\alpha}, \text{ for any } \alpha \in \left[\frac{1}{\max\{d_x,d_y\}+1},1\right].$$
 (2.1)

Another limit-free formulation of Lin-Lu-Yau curvature is due to Münch and Wojciechowski [16]. For any locally finite graph G, the normalized graph Laplacian Δ is defined as

$$\Delta f(x) := \frac{1}{d_x} \sum_{y: y \sim x} (f(y) - f(x)), \text{ for any } f: V(G) \to \mathbb{R} \text{ and any } x \in V(G).$$

The following identity is proved via a reformulation of the Kantorovich duality for Wasserstein distance.

Theorem 2.3 (Curvature via the Laplacian [16, Corollary 2.2]). Let G be a locally finite graph and let $\{x, y\}$ be an edge. Then

$$\kappa_{LLY}(x,y) = \inf_{\substack{f: N[x] \cup N[y] \to \mathbb{Z} \\ f \in Lip(1) \\ f(y) - f(x) = 1}} (\Delta f(x) - \Delta f(y)).$$

Next we recall a key lemma due to Lin, Lu and Yau [13].

Lemma 2.4 ([13, Lemma 2]). Suppose that an edge $\{x,y\}$ in a graph G is not in any C_3 or C_4 . Then

$$\kappa_{LLY}(x,y) \le \frac{1}{d_x} + \frac{2}{d_y} - 1.$$

Proof. As a warm-up, we give a proof of this lemma using Theorem 2.3. Consider the function $f \in Lip(1)$ defined as $f(z) := d(z, N[x] \setminus \{y\})$. We check that f(y) - f(x) = 1. Then we have by Theorem 2.3

$$\kappa_{LLY}(x,y) \le \Delta f(x) - \Delta f(y) \le \frac{1}{d_x} - \frac{1}{d_y} \left(-1 + \sum_{z: z \sim y, z \ne x} (2-1) \right) = \frac{1}{d_x} + \frac{2}{d_y} - 1.$$

This completes the proof.

2.2 Generalized Halin graphs

Definition 2.5. A generalized Halin graph H(T, C) is a graph constructed by starting from a planar embedding of a tree T with maximum vertex degree $D(T) \geq 3$, and connecting all the leaves of the tree with a cycle C. A Halin graph is a generalized Halin graph with no vertex of degree two.

We remark that a tree T may not decide a unique generalized Halin graph since it can be embedded into a plane in different ways.

3 Proof of Theorem 1.3

For convenience, we first introduce some notations. Let G = H(T, C) be a generalized Halin graph. We denote by $d_T(\cdot, \cdot)$ the combinatorial distance in the tree T. Let x be a vertex in T with maximum degree D(T). Recall that in a generalized Halin graph G = H(T, C), we have a planar embedding of the tree T. We consider the connected components of $T - \{x\}$, and label them clockwise as A_1, A_2, \ldots, A_c . Furthermore, we label the leaves of T in each A_i clockwise as $A_{i,1}, A_{i,2}, \cdots, A_{i,k_i}$. In this way, we label all the vertices on the cycle C of G = H(T, C), which we call outer vertices. We depict an example in Figure 2.

Before presenting our proof, we prepare two lemmas.

Lemma 3.1. Let G = H(T,C) be a generalized Halin graph with positive Lin-Lu-Yau curvature, and A_1, A_2, \ldots, A_c be defined as above. If $|A_i \cap C| \ge 2$ for some $i \in \{1, 2, \ldots, c\}$, then $|A_{i+1} \cap C| = |A_{i-1} \cap C| = 1$, where we use the notations $A_0 = A_c$ and $A_{c+1} = A_1$.

Proof. Suppose that either $|A_{i+1} \cap C| > 1$ or $|A_{i-1} \cap C| > 1$. Without loss of generality, we assume $|A_{i-1} \cap C| > 1$. Consider the edge $\{A_{i,1}, A_{i-1,k_{i-1}}\}$. There exist vertices $A_{i,2} \sim A_{i,1}$ and $A_{i-1,k_{i-1}-1} \sim A_{i-1,k_{i-1}}$. Since $d_x = |D(T)| \geq 3$, we have $d(A_{i,2}, A_{i-1,k_{i-1}}) \geq 2$. Therefore, we observe that the edge $\{A_{i,1}, A_{i-1,k_{i-1}}\}$ is not in any C_3 or C_4 . By Lemma 2.4, we have $\kappa_{LLY}(A_{i,1}, A_{i-1,k_{i-1}}) \leq \frac{1}{3} + \frac{2}{3} - 1 = 0$, which is a contradiction. \square

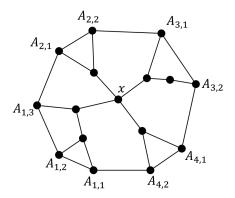


Figure 2: A example of labelling outer vertices in each component of $T - \{x\}$

Lemma 3.2. Let G = H(T, C) be a generalized Halin graph with positive Lin-Lu-Yau curvature, and A_1, A_2, \ldots, A_c be defined as above. For any two adjacent vertices p and q on C which belong to different components A_i and A_j , respectively, we have

$$d_T(x,p) + d_T(x,q) \le 4.$$

Proof. Without loss of generality, we assume $d_T(x,p) \ge d_T(x,q)$. Suppose that $d_T(x,p) + d_T(x,q) \ge 5$. Let y be the neighbor of x in the component A_i and t be the neighbor of x in the component A_j . Consider a function $f: N[x] \cup N[y] \to \mathbb{Z}$ given by

$$f(z) = \min\{d(z, N[x] \setminus \{y\}), d(z, t) - 1\}.$$

Since the minimum of two 1-Lipschitz functions is still 1-Lipschitz, the function f is 1-Lipschitz. By the definition of f, we have f(y) = 1, f(t) = -1 and f(z) = 0 for any $z \in N[x] \setminus \{y, t\}$. It follows that f(y) - f(x) = 1 and $\Delta f(x) = 0$. Let s be the neighbor of y on the unique path in T connecting y and p. Since $d_T(x, p) + d_T(x, q) \geq 5$, we have f(s) = 2. Furthermore, $f(z) \geq 1$ for any $z \in N[y] \setminus \{x\}$. Thus, we derive $\Delta f(y) \geq 0$. It follows by Theorem 2.3 that

$$\kappa_{LLY}(x,y) \le \Delta f(x) - \Delta f(y) \le 0,$$

which leads to a contradiction. Therefore, we have shown $d_T(x,p) + d_T(x,q) \leq 4$.

Corollary 3.3. Let G = H(T, C) be a generalized Halin graph with positive Lin-Lu-Yau curvature. Let x be a vertex of T with maximum degree D(T). If T has exactly D(T) leaves and $D(T) \ge 4$, then $d_T(x, y) \le 2$ for any $y \in C$.

Proof. Suppose that there exists a vertex $y \in C$ such that $d_T(x,y) \geq 3$. Due to Lemma 3.2, $d_T(x,y) = 3$ and $d_T(x,y_1) = d_T(x,y_2) = 1$, where y_1, y_2 are the neighbors of y on C. Denote the neighbor of x which is in the same component with y by x'. Then, the edge $\{x,x'\}$ is shared by two 5-faces, which leads to $\kappa_{LLY}(x,x') \leq \frac{1}{2} + \frac{2}{D(T)} - 1 \leq 0$ by Lemma 2.4. Hence, $d_T(x,y) \leq 2$ for any $y \in C$.

Proof of Theorem 1.3. Let G = H(T, C) be a generalized Halin graph with positive Lin-Lu-Yau curvature generated by the tree T and the cycle C. Suppose that x is a vertex with maximum degree D(T) in T. According to the maximum degree D(T) of T, we divide our proof into the following three cases.

Case 1. $D(T) \geq 5$. Notice that T has at least D(T) leaves. We first show that T actually has precisely D(T) leaves. Suppose that T has more than D(T) leaves, then there exists a component A_i of $T - \{x\}$ containing at least two outer vertices. Let $b \neq A_{i,2}$ be the other neighbor of $A_{i,1}$ on the cycle C, and $p \neq A_{i,1}$ be the other neighbor of b on the cycle C. Consider a function $f_1: N[A_{i,1}] \cup N[b] \to \mathbb{Z}$ given by

$$f_1(z) = \begin{cases} -1, & \text{if } z = A_{i,2}; \\ 0, & \text{if } z \in N[A_{i,1}] \setminus \{b, A_{i,2}\}; \\ 1, & \text{if } z \in N[b] \setminus \{A_{i,1}, p\}; \\ 2, & \text{if } z = p. \end{cases}$$

By assumption, T has at least 6 leaves. This ensures $d(A_{i,2}, p) = 3$, and hence $f_1 \in Lip(1)$. Since $f(b) - f(A_{i,1}) = 1$, we have by Theorem 2.3

$$\kappa_{LLY}(A_{i,1}, b) \le \Delta f_1(A_{i,1}) - \Delta f_1(b) = 0,$$

which is a contradiction. Therefore, T has exactly D(T) leaves.

Applying Corollary 3.3, we conclude that $d_T(x,y) \leq 2$ for any y on the cycle C.

Let a and b be two vertices on C with $d_T(x, a) = d_T(x, b) = 2$. If a and b are adjacent, then $\kappa_{LLY}(a, b) \leq 0$ by Lemma 2.4 since the edge $\{a, b\}$ is not in any C_3 or C_4 when $D(T) \geq 5$. Therefore, two vertices a and b on C with $d_T(x, a) = d_T(x, b) = 2$ can not be adjacent. We divide our discussion into two subcases as follows.

Subcase 1.1. There is no vertex y on C with $d_T(x,y) = 2$.

Then G must be a wheel graph. Select any vertex y on C. Consider a function $f_2: N[x] \cup N[y] \to \mathbb{Z}$ given by

$$f_2(z) = \begin{cases} 1, & \text{if } z \in (N(x) \cap N(y)) \cup \{y\}; \\ 0, & \text{if } z \in N(t) \setminus \{y\} \text{ for some } t \in N(x) \cap N(y); \\ -1, & \text{others.} \end{cases}$$

Observe that $f_2(y) - f_2(x) = 1$ and $f_2 \in Lip(1)$. By Theorem 2.3, $\kappa_{LLY}(x,y) \leq \frac{8}{D(T)} - \frac{2}{3}$. When $D(T) \geq 12$, $\kappa_{LLY}(x,y) \leq 0$. It is direct to check that $W_n(4 \leq n \leq 12)$ has positive Lin-Lu-Yau curvature by (2.1). As an example, we verify that W_5 has positive Lin-Lu-Yau curvature in Section 4. One can also check the curvature by the graph curvature calculator [6], which is a freely accessible interactive app at

https://www.mas.ncl.ac.uk/graph-curvature/

Subcase 1.2. There exists at least one vertex p on C that satisfies $d_T(x,p)=2$.

Let y be the neighbor of x in the component corresponding to p. Consider a function $f_3: N[x] \cup N[y] \to \mathbb{Z}$ given by

$$f_3(z) = \begin{cases} 1, & \text{if } z \in \{y, p\}; \\ 0, & \text{if } z \in (N(p) \cup \{x\}) \setminus \{y\}; \\ -1, & \text{others.} \end{cases}$$

Then, we check that $f_3(y) - f_3(x) = 1$ and $f_3 \in Lip(1)$. By Theorem 2.3, $\kappa_{LLY}(x,y) \le \frac{4}{D(T)} - \frac{1}{2}$. If $D(T) \ge 8$, then $\kappa_{LLY}(x,y) \le 0$. Thus, we only need to check the situations when D(T) = 5, 6, 7.

Recall that any two vertices a and b on the cycle C with $d_T(x,a) = d_T(x,b) = 2$ can not be adjacent. The remaining situations are not so much. We list all of them in Figure 4 in the appendix and find that only $W'_7, W'_8, W'_9, W''_8, W''_9, W''_{10}$ have positive Lin-Lu-Yau curvature by using the graph curvature calculator.

Case 2. D(T) = 3. According to Lemma 3.1, we divide this case into two subcases.

Subcase 2.1. Each component of $T - \{x\}$ has only one outer vertex (|V(C)| = 3).

Denote the three outer vertices by p_1 , p_2 and p_3 . Consider a disordered triplet:

$$(d_T(x, p_1), d_T(x, p_2), d_T(x, p_3)).$$

Due to Lemma 3.2, we can list all the possible triplets as follows:

$$(3,1,1)$$
 $(2,2,2)$ $(2,2,1)$ $(2,1,1)$ $(1,1,1)$.

Using the graph curvature calculator, we check that all the graphs corresponding to the above triplets have positive Lin-Lu-Yau curvature. This means that $G \cong H_2, H_1, W_6'', W_5'$ or W_4 in this subcase.

Subcase 2.2. There is exactly one component of $T - \{x\}$ having at least two outer vertices.

We denote this component by A_1 . If A_1 has at least three outer vertices, then there must exist an edge $\{p,q\}$ on C with $d_T(x,p)=3$ and $d_T(x,q)=1$ such that $p\in A_1$ and q is in another component by Lemma 3.2. Clearly, $\{p,q\}$ is not in any C_3 or C_4 . We have $\kappa_{LLY}(p,q)\leq \frac{1}{3}+\frac{2}{3}-1=0$ by Lemma 2.4. Therefore, A_1 has exactly two outer vertices.

Combining with Lemma 3.2, the remaining graphs are shown in Figure 5 in the appendix, and it tells us $G \cong H_3, H_4$ or H_5 in this subcase.

Case 3. D(T) = 4. We again divide this case into two subcases.

Subcase 3.1. |V(C)| = 4.

Denote the cycle by $C = q_1q_2q_3q_4$. Consider an ordered quadruple:

$$(d_T(x, q_1), d_T(x, q_2), d_T(x, q_3), d_T(x, q_4)).$$

Due to Corollary 3.3, we know that $d_T(x, q_i) \leq 2$. Combining with Lemma 3.2, we list all the possible quadruples as follows (we identify those ordered quadruples corresponding to isomorphic graphs):

$$(2,2,2,2)$$
 $(2,2,2,1)$ $(2,2,1,1)$ $(2,1,2,1)$ $(2,1,1,1)$ $(1,1,1,1)$.

Actually, both of (2, 2, 2, 2) and (2, 2, 2, 1) lead to at least one edge $\{x, x'\}$ shared by two 5-faces. By Lemma 2.4, $\kappa_{LLY}(x, x') \leq 0$. We directly verify that all the remaining four situations have positive Lin-Lu-Yau curvature by the graph curvature calculator. Therefore, $G \cong W_5, W'_6, W''_7$ or H_6 in this subcase.

Subcase 3.2. $|V(C)| \ge 5$.

Choose a component of $T - \{x\}$ with at least two outer vertices and denote it by A_1 . First, we prove that the neighbor $y \in A_1$ of x is of degree 3. Suppose not, then either $d_y = 2$ or $d_y = 4$.

If $d_y = 2$, then $\{x, y\}$ must be shared by two 5-faces due to Lemma 3.2. In particular, $\{x, y\}$ is not in any C_3 or C_4 . By Lemma 2.4, we have $\kappa_{LLY}(x, y) \leq 0$, a contradiction.

If $d_y = 4$, then we consider a 1-Lipschitz function $f_4: N[x] \cup N[y] \to \mathbb{Z}$ given by

$$f_4(z) = \begin{cases} -1, & \text{if } z = x_2; \\ 0, & \text{if } z \in \{x, x_1, x_3\}; \\ 1, & \text{if } z \in \{y, y_1, y_3\}; \\ 2, & \text{if } z = y_2. \end{cases}$$

Here, y, x_1, x_2, x_3 are the neighbors of x in clockwise order and x, y_1, y_2, y_3 are the neighbors of y in clockwise order, according to the planar embedding of T. Observe that $d(x_2, y_2) = 3$. Hence, $f_4 \in Lip(1)$. Noticing that $f_4(y) - f_4(x) = 1$, we have $\kappa_{LLY}(x, y) \leq 0$ by Theorem 2.3. This proves $d_y = 3$.

A similar argument as in Subcase 2.2 tells us that A_1 has exactly two outer vertices. Let us again denote the neighbors of x in clockwise order by y, x_1, x_2 and x_3 and denote the neighbors of y in clockwise order by x, y_1 and y_2 . Lemma 3.1 tells us that the component containing x_1 or x_3 has precisely one outer vertex. If the component corresponding to x_2 has two outer vertices, then either there exists an edge $\{s_1, s_2\}$ on C which is not in any C_3 or C_4 or $G \cong H'$, which is depicted in Figure 3. In the first case, we have $\kappa_{LLY}(s_1, s_2) \leq 0$ due to Lemma 2.4. In the later case, the graph H' has edges of 0 curvature as shown in Figure 3.

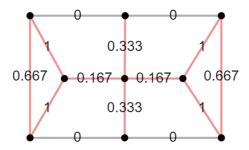


Figure 3: Graph H'

Thus, A_1 is the only component which has at least two outer vertices. To avoid the appearance of an edge $\{s_1, s_2\}$ on C not containing in any C_3 or C_4 , the vertices x_1, x_3, y_1 and y_2 must be the outer vertices. Moreover, $d_T(x, p) \leq 2$ where p is the outer vertex in the component containing x_2 . Otherwise, $\{x, x_2\}$ is shared by two big faces (of size at least 5), which leads to $\kappa_{LLY}(x, x_2) \leq \frac{1}{2} + \frac{2}{4} - 1 = 0$ by Lemma 2.4. There are only two remaining graphs with $d_T(x, p) = 1$ or 2, respectively, which are H_7 and H_8 in Figure 1. We check that both of them have positive Lin-Lu-Yau curvature by the graph curvature calculator. Therefore, we have $G \cong H_7$ or H_8 in this subcase.

This completes our proof.

4 An example

We only calculate the upper bounds of Lin-Lu-Yau curvature in our proof, and utilize the graph curvature calculator to verify whether a graph has positive Lin-Lu-Yau curvature or not for convenience. Below we take W_5 as an example to show how to establish lower bounds for the Lin-Lu-Yau curvature via (2.1).

Denote the center vertex of W_5 by x and the cycle by $C = x_1x_2x_3x_4$. By symmetry, we only need to consider $\kappa_{LLY}(x, x_1)$ and $\kappa_{LLY}(x_1, x_2)$. For $\frac{1}{4} \leq \alpha \leq 1$, consider a map $\pi_1: V \times V \to [0, 1]$ defined as

$$\pi_1(u,v) = \begin{cases} \min\{m_x^{\alpha}(u), m_{x_1}^{\alpha}(u)\}, & \text{if } u = v; \\ \alpha - \frac{1-\alpha}{3}, & \text{if } u = x, v = x_1; \\ \frac{1-\alpha}{3} - \frac{1-\alpha}{4}, & \text{if } u = x_3, v \in \{x_1, x_2, x_4\}; \\ 0, & \text{otherwise.} \end{cases}$$

By (2.1), we derive

$$\kappa_{LLY}(x, x_1) = \frac{\kappa_{\alpha}(x, x_1)}{1 - \alpha} \ge \frac{1 - \sum_{v \in V} \sum_{u \in V} \pi_1(u, v) d(u, v)}{1 - \alpha} = 1.$$

Similarly, for $\frac{1}{4} \leq \alpha \leq 1$, consider another map $\pi_2: V \times V \to [0,1]$ defined as

$$\pi_2(u,v) = \begin{cases} \min\{m_{x_1}^{\alpha}(u), m_{x_2}^{\alpha}(u)\}, & \text{if } u = v; \\ \alpha - \frac{1-\alpha}{3}, & \text{if } u = x_1, v = x_2; \\ \frac{1-\alpha}{3}, & \text{if } u = x_4, v = x_3; \\ 0, & \text{otherwise.} \end{cases}$$

By (2.1), we deduce

$$\kappa_{LLY}(x_1, x_2) = \frac{\kappa_{\alpha}(x_1, x_2)}{1 - \alpha} \ge \frac{1 - \sum_{v \in V} \sum_{u \in V} \pi_2(u, v) d(u, v)}{1 - \alpha} = 1.$$

Therefore, W_5 has positive Lin-Lu-Yau curvature.

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A Appendix

We use the graph curvature calculator to draw the remaining graphs in each subcases of our proof. The number on each edge is the Lin-Lu-Yau curvature.

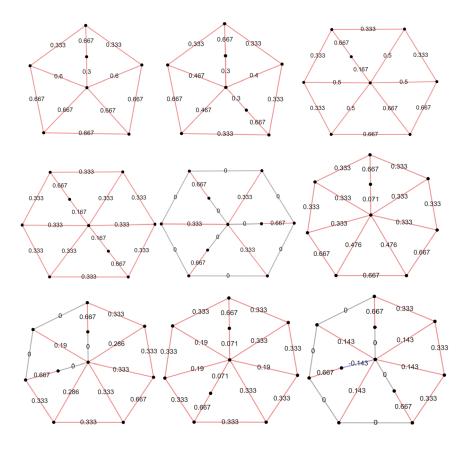


Figure 4: Remaining graphs in Subcase 1.2

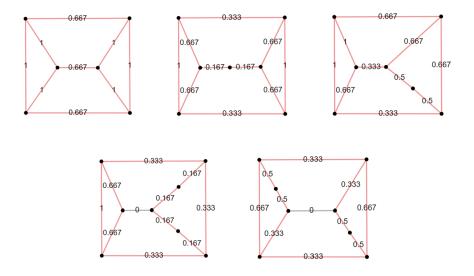


Figure 5: Remaining graphs in Subcase 2.2