Stochastic Processes - Homework
Stochastic Differential Equations

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1. Solve the SDE
$$dX_t = (-\alpha X_t + \beta)\, dt + \sigma dB_t$$
where $X_0 = x_0$ and $\alpha > 0$, and verify that the solution can be written as
$$X_t = e^{-\alpha t}\left(x_0 + \frac{\beta}{\alpha}\left(e^{\alpha t} - 1\right) + \sigma \int_0^t e^{\alpha s} dB_s\right)$$
Use the representation to show that X_t converges in distribution as $t \to \infty$, and find the limiting distribution. Finally, find the covariance $\mathrm{Cov}\,(X_s, X_t)$.

2. Solve the SDE
$$dX_t = tX_t dt + e^{t^2/2} dB_t \text{ with } X_0 = 1$$

3. Use appropriate coefficient matching to solve the SDE
$$dX_t = -2\frac{X_t}{1-t}dt + \sqrt{2t(1-t)}dB_t \quad 0 \leq t < 1$$
with $X_0 = 0$. Show that the solution X_t is a Gaussian process. Find the covariance function $\mathrm{Cov}\,(X_s, X_t)$. Compare this covariance function to the covariance function for the Brownian bridge.

4. Show that if X_t is a process that satisfies $X_0 = 0$ and
$$dX_t = a\,(X_t)\, dt + \sigma\,(X_t)\, dB_t$$
where $a(\cdot)$ and $\sigma(\cdot)$ are smooth functions and $\sigma(\cdot) \geq \epsilon > 0$, then there is a monotone increasing function $f(\cdot)$ and a smooth function $b(\cdot)$ such that $Y_t = f\,(X_t)$ satisfies
$$Y_t = Y_0 + \int_0^t b\,(Y_s)\, ds + B_t.$$
The benefit of this observation is that it shows that many one-dimenal SDEs can be recast as an integral equation where there is no stochastic integral (other than a standard Brownian motion). Anyone who wants a hint might consider applying Itô's formula to $f(X_t)$ and making appropriate coefficient matches.