Stochastic Processes - Homework
Itô formula

Yuri F. Saporito

In what follows, $(B_t)_{t\geq 0}$ is a Brownian motion.

1. Prove using Itô formula that
$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

2. Let $Y_t = \exp(\alpha B_t + ct)$, for α and c constants. Show that Y solves the SDE $dY_t = (c + \alpha^2/2)Y_t dt + \alpha Y_t dB_t$. When is Y a martingale?

3. Check if the following processes are martingales (using Itô Formula):
   (a) $X_t = B_t + 4t$;
   (b) $X_t = B_t^2$;
   (c) $X_t = B_t^2 - t$;
   (d) $X_t = t^2 B_t - 2\int_0^t sB_s ds$;
   (e) $X_t = B_1(t)B_2(t)$, where B_1 and B_2 are independent Brownian motions;
   (f) $X_t = B_t^3 - 3tB_t$.

4. Let $\beta_k(t) = \mathbb{E}[B_t^k]$. Use Itô Formula to find that
$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds,$$
for $k \geq 2$. Show that $\mathbb{E}[B_t^4] = 3t^2$ and compute $\mathbb{E}[B_t^6]$.

5. Let $g \in C^1$. Use Itô Formula to conclude that
$$\int_0^T g(t)dB_t = g(T)B_T - \int_0^T g'(t)B_t dt.$$

6. Define $X_t = e^{-\alpha t}(X_0 + \sigma\int_0^t e^{\alpha s}dB_s)$, where α is a constant. Prove that $dX_t = -\alpha X_t dt + \sigma dB_t$.

7. Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a non-decreasing and continuous function. Find a martingale $(M_t)_{t\geq 0}$ such that $\langle M\rangle_t = f(t)$.

8. (Nice Local Martingale, But Not a Martingale). The purpose of this exercise is to provide an example of a local martingale that is L^2 -bounded (and hence uniformly integrable) but still fails to be an honest martingale.
   (a) First, show that the function $f(x,y,z) = \left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$ satisfies $\Delta f = 0$ for all $(x,y,z) \neq 0$, so that if $\vec{B}_t$ is standard Brownian motion in $\mathbb{R}^3$ (starting from zero), then for $1 \leq t < \infty$ the process defined by
   $$M_t = f\left(\vec{B}_t\right)$$
   is a local martingale.

(b) Second, use direct integration (say, in spherical coordinates) to show

$$E\left(M_t^2\right) = \frac{1}{t}$$

for all $1 \leq t < \infty$.

(c) Third, use the identity above and Jensen's inequality to show that M_t is not a martingale.