Stochastic Processes - Homework
Itô integral

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In what follows, $(B_t)_{t\geq 0}$ is a Brownian motion.

1. Show that the Itô integral foes not have the monotonicity property, i.e. $X_t \leq Y_t$ a.s. for all $t \in [0,T]$ does NOT imply $\int_0^T X_t dB_t \leq \int_0^T Y_t dB_t$.

2. Prove directly from the definition of Itô integral that
$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

3. Use the Itô isometry to calculate the variances
$$\int_0^t |B_s|^{\frac{1}{2}} dB_s \text{ and } \int_0^t (B_s + s)^2 dB_s$$

4. The integrals
$$I_1 = \int_0^t B_s ds \text{ and } I_2 = \int_0^t B_s^2 ds$$
are not stochastic integrals, although they are random variables. For each ω the integrands are nice continuous functions of s and the ds integration is just the tra- ditional calculus integration. Find the mean and variance of the random variables I_1 and I_2.

5. For any fixed t the random variable B_t has the same distribu- tion as $X_t = \sqrt{t}Z$, where $Z \sim N(0,1)$, but as processes B_t and X_t could not be more different. For example, the paths of X_t are differentiable for all $t > 0$ with probability one, but the paths of B_t are not differentiable for any t with probability one. For another difference (and similarity), show that for a bounded continuous f the process
$$U_t = \int_0^t f(B_s)\, ds \text{ and } V_t = \int_0^t f(X_s)\, ds$$
will have the same expectations but will not in general have the same variance.

6. Show that if X_t is any continuous martingale and ϕ is any convex function, then $Y_t = \phi(X_t)$ is always a local submartingale. Give an example that shows Y_t need not be an honest submartingale.

7. Show that if X_t is a continuous local submartingale such that
$$E\left(\sup_{0\leq s\leq T} |X_s|\right) < \infty$$
then $\{X_t : 0 \leq t \leq T\}$ is an honest submartingale. Show how this result implies our earlier result that a bounded local martingale is a martingale.