Stochastic Processes - Homework
Brownian Motion (More)

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In what follows, $(B_t)_{t\geq 0}$ is a Brownian motion.

1. (Brownian Bridge). In our representation for Brownian motion

$$B_t = \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)$$

we have $\Delta_0(1) = 1$ and $\Delta_n(1) = 0$ for all $n \geq 1$; so, if we leave off the first term in our representation for Brownian motion to define

$$U_t = \sum_{n=1}^{\infty} \lambda_n Z_n \Delta_n(t)$$

we see that U_t is a continuous process on $[0, 1]$ such that $U_0 = 0$ and $U_1 = 0$. This process is called a Brownian bridge, and it is often useful when we need to model a quantity that starts at some level and that must return to a specified level at a specified future time.

   (a) Show that we can write $U_t = B_t - tB_1$ for $0 \leq t \leq 1$

   (b) Show that we have $\mathrm{Cov}\,(U_s, U_t) = s(1 - t)$ for $0 \leq s \leq t \leq 1$

   (c) Let $X_t = g(t)B_{h(t)}$, and find functions g and h such that X_t has the same covariance as a Brownian bridge.

   (d) Show that the process defined by $Y_t = (1 + t)U_{t/(1+t)}$ is a Brownian motion on $[0, \infty)$. Note: since we have a direct construction of the Brownian bridge U_t, this observation gives us a third way to build Brownian motion on $[0, \infty)$.

2. (Time Inversion of Brownian Motion). We will prove that the process $\{Y_t\}$ defined by the time inversion of Brownian motion,

$$Y_t = \begin{cases} 0 & \text{if } t = 0 \\ tB_{1/t} & \text{if } t > 0 \end{cases}$$

is again a standard Brownian motion. One can easily check that $\{Y_t\}$ is a Gaussian process with covariance function $\min(s, t)$, so the only sticky point to showing that $\{Y_t\}$ is Brownian motion is to prove that it is continuous at zero. Give a verification of this fact by completing the following program:

   (a) Show that for all $\epsilon > 0$ the process defined by $\{X_t = Y_t - Y_\epsilon : t \geq \epsilon\}$ is a martingale.

   (b) Check that for all $0 < s \leq t$ we have $E\left[(Y_t - Y_s)^2\right] = t - s$.

   (c) Use Doob's maximal inequality (in continuous time) to give a careful proof of the fact that

$$P\left(\lim_{t\to 0} Y_t = 0\right) = 1$$

3. Use the martingale

$$X_t = \exp\left(\alpha B_t - \alpha^2 t/2\right)$$

(prove it is indeed a martingale) to calculate $\phi(\lambda) = E[\exp(-\lambda\tau)]$, where $\tau = \inf\{t : B_t = A \text{ or } B_t = -A\}$. Use this result to calculate $E\left[\tau^2\right]$. What difficulty do we face if we try to calculate $E\left[\tau^2\right]$ when the the boundary is not symmetric?

4. (L^1- Bounded Martingales Need Not Be Uniformly Integrable). Consider $X_t = \exp(B_t - t/2)$ and show that X_t is a continuous martingale with $E(|X_t|) = 1$ for all $t \geq 0$. Next, show that X_t converges with probability one to $X = 0$. Explain why this implies that X_t does not converge in L^1 to X and explain why X_t is not uniformly integrable, despite being L^1 -bounded.