

## Problem Sheet 1

### Exercise 1 (Inversion and Rejection)

1. We got  $Y \sim \text{Exp}(\lambda)$  and  $a > 0$ . The cumulative distribution function of  $X$ :

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x | Y \geq a) = \frac{\mathbb{P}(Y \leq x, Y \geq a)}{\mathbb{P}(Y \geq a)} = \frac{[\mathbb{P}(Y \leq x) - \mathbb{P}(Y \leq a)]\mathbb{1}_{Y \geq a}}{1 - \mathbb{P}(Y \leq a)}$$

$$F_X(x) = \frac{1 - e^{-\lambda x} - (1 - e^{-\lambda a})}{1 - (1 - e^{-\lambda a})}\mathbb{1}_{X \geq a} = 1 - \frac{e^{-\lambda x}}{e^{-\lambda a}} = [1 - e^{\lambda(a-x)}]\mathbb{1}_{X \geq a}$$

The quantile function of  $X$ :

$$1 - e^{\lambda(a-x)} = U \Leftrightarrow e^{\lambda(a-x)} = 1 - U \Leftrightarrow \ln(e^{\lambda(a-x)}) = \ln(1 - U) \Leftrightarrow x = a - \frac{\ln(1 - U)}{\lambda}$$

$$F_X^{-1}(U) = a - \frac{\ln(1 - U)}{\lambda}$$

2. Letting  $a$  and  $b$  be given, with  $a < b$ , we have the cumulative distribution function of  $X = Y | (a \leq Y \leq b)$ :

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x | a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{\mathbb{P}(a \leq Y \leq b)}$$

$$F_X(x) = \frac{\mathbb{P}(Y \leq x) - \mathbb{P}(Y \leq a)}{\mathbb{P}(Y \leq b) - \mathbb{P}(Y \leq a)}\mathbb{1}_{Y \geq a}$$

Using that  $X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U)$ , we have:

$$\begin{aligned} \mathbb{P}(F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U) \leq x) &= \mathbb{P}(F_Y(a)(1 - U) + F_Y(b)U \leq F_Y(x)) \\ &= \mathbb{P}(F_Y(a) - F_Y(a)U + F_Y(b)U \leq F_Y(x)) = \mathbb{P}(U(F_Y(b) - F_Y(a)) \leq F_Y(x) - F_Y(a)) \\ &= \mathbb{P}(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} \end{aligned}$$

Then  $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x | a \leq Y \leq b)$ , when  $X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U)$ .