Problem Sheet 1

Exercise 1 (Inversion and Rejection)

1. We got Y $Exp(\lambda)$ and $a > 0$. The cumulative distribution function of X:

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x | Y \geq a) = \frac{\mathbb{P}(Y \leq x, Y \geq a)}{\mathbb{P}(Y \geq a)} = \frac{[\mathbb{P}(Y \leq x) - \mathbb{P}(Y \leq a)]\mathbb{1}_{Y \geq a}}{1 - \mathbb{P}(Y \leq a)}$$

$$F_X(x) = \frac{1 - e^{-\lambda x} - (1 - e^{-\lambda a})}{1 - (1 - e^{-\lambda a})}\mathbb{1}_{X \geq a} = 1 - \frac{e^{-\lambda x}}{e^{-\lambda a}} = [1 - e^{\lambda(a-x)}]\mathbb{1}_{X \geq a}$$

The quantile function of X:

$$1 - e^{\lambda(a-x)} = U \Leftrightarrow e^{\lambda(a-x)} = 1 - U \Leftrightarrow ln(e^{\lambda(a-x)}) = ln(1-U) \Leftrightarrow x = a - \frac{ln(1-U)}{\lambda}$$

$$F_X^{-1}(U) = a - \frac{ln(1-U)}{\lambda}$$

2. Letting a and b be given, with $a < b$, we have the cumulative distribution function of $X = Y|(a \leq Y \leq b)$:

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x | a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{\mathbb{P}(a \leq Y \leq b)}$$

$$F_X(x) = \frac{\mathbb{P}(Y \leq x) - \mathbb{P}(Y \leq a)}{\mathbb{P}(Y \leq b) - \mathbb{P}(Y \leq a)}\mathbb{1}_{Y \geq a}$$

Using that $X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U)$, we have:

$$\mathbb{P}(F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U) \leq x) = \mathbb{P}(F_Y(a)(1-U) + F_Y(b)U \leq F_Y(x))$$
$$= \mathbb{P}(F_Y(a) - F_Y(a)U + F_Y(b)U \leq F_Y(x)) = \mathbb{P}(U(F_Y(b) - F_Y(a)) \leq F_Y(x) - F_Y(a))$$
$$= \mathbb{P}(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}$$

Then $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x | a \leq Y \leq b)$, when $X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U)$.