FE Stokes Re-Pair: Game Instructions

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1 Introduction

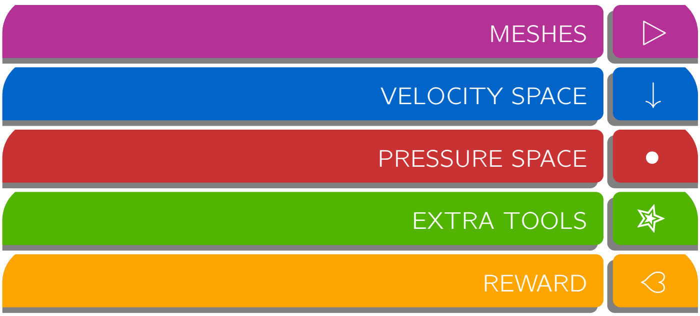
The aim of the game is for all players to collaboratively combine as many stable finite-element discretizations for the Stokes problem as possible from the discretization components available, represented by playing cards.

> Disclaimer: We are applying several restrictions on the setting to keep the game playable and accessible. One strong restriction is that we only consider 2D triangular meshes. Extensions in several directions (basic rules, point system, categories, cards, ...) are possible (we were struggling keeping them out of this version ourselves) and you are invited to try out and suggest some.

2 Rules

2.1 Types of Cards

There are five types of cards in the game:



For the metaphor of the Stokes discretization, we use notation that is explained in Section 3. Used references are listed in Section 5.

2.2 Game Preparation

There are different piles that exist or build up during the game:

- The mesh-type cards are shuffled and placed on one pile (facing down).
- The discretization components where all velocity space, pressure space and extra cards are shuffled and placed on a second pile (facing down).
- The reward cards are placed on a separate pile.
- There are two further (initially empty) piles reserved for the two discretizations that are in construction.

- Finally, two (initially empty) piles are reserved for the finished discretizations (one for stable and convergent discretizations and one for unstable or non-convergent discretizations).

From the pile of discretization components **each player receives 4 cards** at the beginning of the game, which they keep hidden in their hand.
Two mesh-type cards are drawn, revealed, and placed on the piles of the newly formed discretizations that are in construction (no more than two piles are ever used for the active discretizations).
The game starts with one player and continues by turn in clockwise direction.

2.3 Player Turn

The player whose turn it is carries out three actions one after the other:

2.3.1 Action 1: Conclude a discretization

They decide whether they want to complete one or two (or none) of the existing discretization stacks. However, this is only possible if there is already a velocity space card and a pressure space card. If the player closes a discretization pile, it is checked (see below in Section 2.4) whether the Stokes discretization is stable and convergent. If it is, the team scores points (see below in Section 2.4 for details). Otherwise not. In both cases, the cards are removed from play, potential reward cards are gained and a new mesh card is revealed and placed on the free discretization field.

2.3.2 Action 2: Placing a card

Afterwards, the player attaches a card from their hand to one of the two discretization fields. The rule is that never two velocity fields or two pressure fields may lie at a discretization field, but any number of extra cards. Also, duplications of the same extra cards are allowed. If the player is unable to attach a card (because, for example, they only have velocity cards left and there is already a velocity card on each of the discretization stacks), they must unsuccessfully complete one of the discretization stacks (the cards are removed from play and there is no points gained) and a new mesh card is drawn and a card is attached to the new empty discretization field.

2.3.3 Action 3: Drawing a card

Afterwards, the player draws a card, as long as there are still cards left in the game. Otherwise, play continues with the remaining cards until no more cards are left.

2.4 Evaluation of a discretization

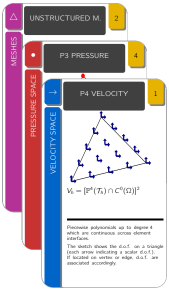
Points for a finished discretization are distributed along the following rules:

Points for a discretization
For the number of points the number on the top right of each card comes into play. It displays the points that the team gets for the card being in the stack of accomplished stable Stokes discretizations in the following way:

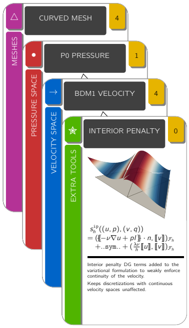
- The number of points that accumulate from the combination of mesh, velocity space, pressure space and extra cards. Note that some extra cards can have negative points.

- Furthermore, additional points can be added if a stable Stokes pair has an additional properties for which *reward cards* exist. If this is the case the corresponding reward card(s) are added to the stack of accomplished discretization cards. Possible rewards can be obtained for:
  - Pressure robustness / pointwise mass conserving[1].
  - Optimal convergence rates: If a discretization achieves the optimal order of convergence in the H^1-(type)-norm for the velocity *and* in the L^2 norm for the pressure, you add the *reward card* for optimal convergence rates.

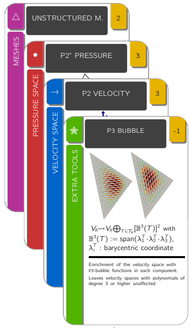
<u>Examples:</u>



stable and convergent (2),
optimal conv. rates (2),
not pressure robust (0),
2+2+0+2+4+1 = <u>11 pts</u>.

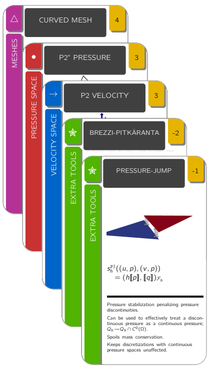


stable and convergent (2),
optimal conv. rates (2),
pressure robust (2),
2+2+2+4+1+4+0 = <u>15 pts</u>.



not stable/convergent (0),

$\rightsquigarrow$ <u>0 pts</u>.



stable and convergent (2),
not optimal conv. rates (0),
not pressure robust (0),
2+0+0+4+2+3-2-1 = <u>8 pts</u>.

2.5 Game End

The game ends when all the players have placed all their cards. The players count the points that they gained from their stable and convergent discretizations. The more points the players have scored, the better they have played. To simplify the scoring – especially for your first game(s) – you can also simply count the number of stable and convergent discretizations that have been achieved.

[1] with the possible Stokes pairs both are equivalent.

3 Notation of the discretization used on the cards

We are setting up the problem where we look for the velocity $u_h \in V_h$ and the pressure $p_h \in Q_h$ so that

$$(\nabla u, \nabla v)_{\mathcal{T}_h} + (\operatorname{div} v, p)_{\mathcal{T}_h} + (\operatorname{div} u, q)_{\mathcal{T}_h} + \sum_{* \in S} s_h^*((u, p), (v, q)) = f(v) + g(q) \quad \forall (v, q) \in V_h \times Q_h,$$

where

- S is the set of added stabilzation terms.
- $\mathcal{T}_h$ is the set of triangular elements in the mesh.

Note that the integrals here are element-wise, so that for discontinuous elements there is no communication between neighboring elements automatically.
Further, we use the following notation:

- $\mathcal{F}_h$ is the set of facets (edges) in the mesh.
- $\mathbb{P}^k(\mathcal{T}_h)$: space of piecewise polynomials of degree k (not necessarily continuous across element boundaries)
- $\mathbb{BDM}^k$: space of piecewise polynomials of degree k which are normal continuous. On curved meshes a Piola transformation from the reference element is applied.
- h: mesh size
- Π_F^0: projection on to constants on the facet F, i.e. the mean value.

4 Checking if a discretization is stable and convergent

See the separate document for now.