Unit 1: chapter-1

Mean Value Theorem

I] Rolle's Theorem:

statment:-

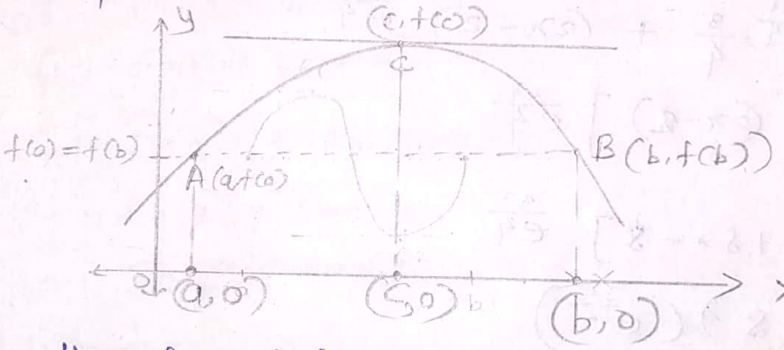
If $f(x)$ is

(i) continuous in $[a,b]$

(ii) derivable in (a,b) &

(iii) $f(a)=f(b)$, then There exist atleast one value $c \in (a,b)$ such that $f'(c)=0$

Geometrically Rolle's Theorem state that the tangent c_1, c_2, c_3 are parallel to x-axis.

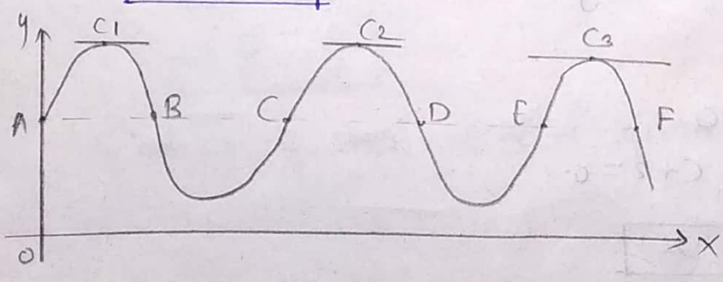


Here $f(a)=f(b)$

$y=f(x)$ is continuous function and $\exists$ Exist a unique tangent parallel to x-axis

i.e slope at $c=0$

i.e $\boxed{f'(c)=0}$



In this figure we get more than one point such that tangent parallel to x-axis.

Example

(1) verify Rolle's Theorem

$F(x) = x(x-2) \cdot e^{\frac{3x}{4}}$ in $(0,2)$

$\Longrightarrow F(x) = (x^2-2x) \cdot e^{\frac{3x}{4}}$

we know that polynomial fun is always continuous and differntiable also exponential fun is continuous and differntiable. and product of two continuous function is again continuous and differntiable

$\Rightarrow F(x)$ is continuous and differntiable

Now $f(0) = 0 = f(2)$

by Rolle's Theorem $\exists$ a point $c \in (0,2)$ such that $f'(c) = 0$

$f(x) = (x^2-2x) \cdot e^{\frac{3x}{4}}$

$f'(x) = (x^2-2x)\, e^{\frac{3x}{4}} \cdot \frac{3}{4} + (2x-2) \cdot e^{\frac{3x}{4}}$

$f'(x) = \left[\frac{3x^2-6x}{4} + (2x-2)\right] e^{\frac{3x}{4}}$

$f'(x) = [3x^2-6x+8x-8] \cdot e^{\frac{3x}{4}}$

$f'(c) = (3c^2+2c-8)(e^{\frac{3c}{4}})$

By Rolle's Theorem

$f'(c) = 0$

$\Rightarrow (3c^2+2c-8)(e^{\frac{3c}{4}}) = 0$

But $e^{\frac{3c}{4}} \neq 0$ for any $c \in (0,2)$

$\Rightarrow 3c^2+2c-8 = 0$

$\Rightarrow (3c-4)(c+2) = 0$

$\Rightarrow (3c-4) = 0$ or $c+2 = 0$

$\Rightarrow \boxed{c = 4/3}$ or $\boxed{c = -2}$

But $c = -2 \notin (0,2)$

$\Rightarrow \boxed{c = 4/3}$

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Ex ② verify Rolle's theorem for

$f(x) = (x+3)(x-4)^2$ in $[-2,1]$

(i) continuity :- The function is polynomial so it is continuous everywhere

(ii) differentiability :- The fun is polynomial so it is differentiable everywhere

(iii) $f(-2) = (-2+3)(-2-4)^2 = 36$

$f(1) = (4)(-3)^2 = 4\times 9 = 36$

$f(-2) = f(1)$

By Rolle's Theorem $\exists$ a point $c \in (-2,1)$ such that $f'(c)=0$

$f(x) = (x+3)(x-4)^2$

$f'(x) = (x+3)\,2(x-4) + (x-4)^2$

$= (x-4)[2x+6+x-4]$

$f'(x) = (x-4)(3x+2)$

$f'(c) = (c-4)(3c+2)$

But $f'(c) = 0$

$\Rightarrow (c-4)(3c+2) = 0$

$\Rightarrow (c-4)=0$ or $(3c+2)=0$

$\Rightarrow \boxed{c=4}$ or $\boxed{c=-2/3}$

But $4 \notin (-2,1)$

$\Rightarrow \boxed{c=-2/3}$

$\therefore$ at $c = \frac{-2}{3}$ derivative of given function becomes zero

② Lagrange's Mean Value Theorem

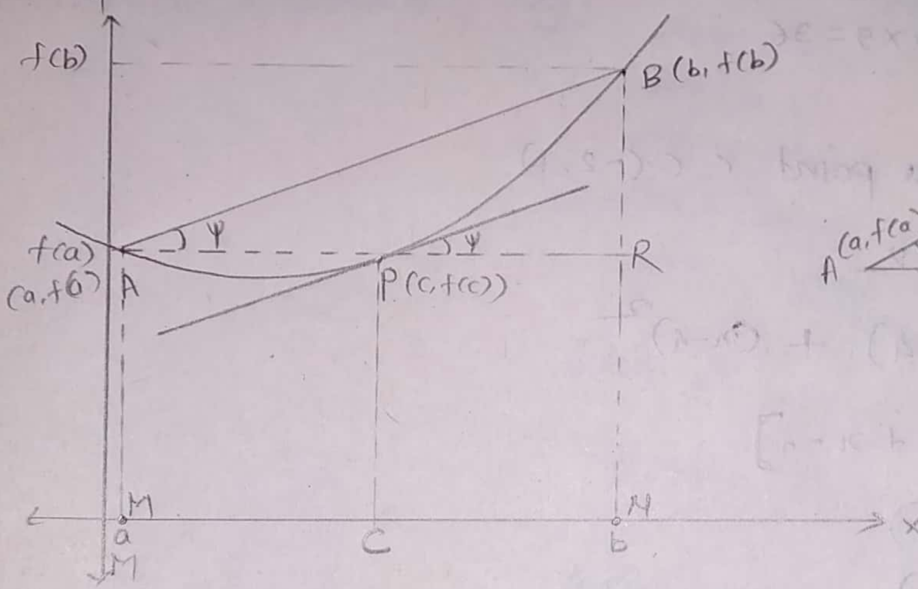
Statments :- Let f be

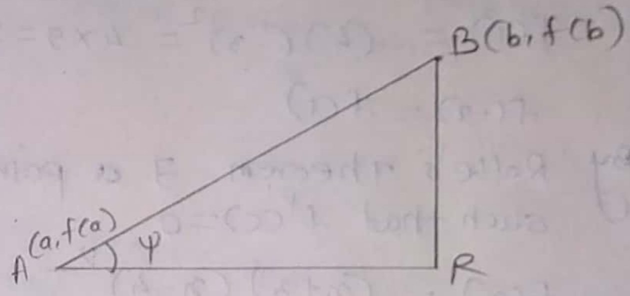
ⓘ continuous in $[a,b]$ ⓘⓘ derivable in (a,b)

Then there exist at least one value $c \in (a,b)$ such that

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

Geometrically LMVT state that the tangent to the curve at P is parallel to the chord AB





we have slope $= \tan\psi = \frac{BR}{AR} = \frac{BN-RN}{AR} = \frac{BN-RN}{MN} = \frac{f(b)-f(a)}{b-a}$

$$\text{slope} = \frac{f(b)-f(a)}{b-a} \quad -①-$$

$$\text{slope at pt. } c = f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Note ⓘ if $f(a) = f(b)$, apply Rolle's theorem

ⓘⓘ if $f(a) \neq f(b)$, apply mean value theorem (LMVT)

Example

1. $f(x) = x^2$ in $(1,5)$

(i) f is continuous, since it is polynomial fun

(ii) f is differntiable everywhere

(iii) $f(1) = 1$

$f(5) = 25$

$f(1) \neq f(5)$

By LMVT there exist $c \in (1,5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f'(c) = 2c \quad \text{---(1)}$$

$$f'(c) = \frac{f(5) - f(1)}{5-1}$$

$$2c = \frac{25-1}{4} = 6$$

$$\boxed{c = 3/2} \in (1,5)$$

2. $f(x) = x^{2/3}$, $(-1, 1)$

LMVT, not applicable because f is not differntiable at $x=0$ $\in (-1,1)$

we know $f(x)$ is differntiable

if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ is exist

Here $\lim_{x \to 0} \frac{f(a+x) - f(a)}{x} = \lim_{x \to 0} \frac{(a+x)^{2/3} - (a)^{2/3}}{x} = \infty$. i.e does not exist.

③ verify LMVT

$f(x) = x - 2\sin x$ on $[-\pi, \pi]$ ⑥

$\Longrightarrow$ (i) subtraction of polynomial and sin x fun is continuous
(ii) It also differentiable

$f(-\pi) = -\pi - 2\cdot\sin(-\pi) = -\pi - 0 = -\pi$

$f(\pi) = \pi - 2\sin\pi = \pi - 0 = \pi$

$f(-\pi) \neq f(\pi)$

By LMVT

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\therefore f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

$$f'(c) = \frac{\pi - (-\pi)}{\pi + \pi} = \frac{2\pi}{2\pi} = 1$$

$f'(c) = 1$ —①—

But $f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x$

$\Rightarrow f'(c) = 1 - 2\cos c$ —②—

put ② in eqn ①

$1 - 2\cos c = 1$

$-2\cos c = 0$

$\cos c = 0$

$\Rightarrow c = 2n\pi \pm \frac{\pi}{2}$, $n = 0, 1, 2 \ldots$

for $n=0$ $c = \pm\frac{\pi}{2} \in [-\pi, \pi]$

for $n = 1, 2, \ldots$, c does not belongs to $[-\pi, \pi]$

so $\boxed{c = \pm\frac{\pi}{2}}$

③ chachy Mean value Theorem (CMVT)

① statment:-

Let $f(x)$ and $g(x)$ be two function defined in $[a,b]$, such that

(i) $f(x)$ and $g(x)$ are continuous in $[a,b]$.

(ii) $f(x)$ and $g(x)$ are differntiable in $[a,b]$.

(iii) $g'(x) \neq 0$, for all $x \in [a,b]$

Then there exist at least one point $c \in (a,b)$ such that.

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

② Geometrical Interpretation of CMVT

The raito of slope of $f(x)$ and $g(x)$ at $x=c$ in (a,b) is equal to the raito of increase of $f(x)$ and $g(x)$ in the interval (a,b)

Example

① verify CMVT for the function $f(x)=e^x$ and $g(x)=e^{-x}$ in the interval $[a,b]$.

$\Rightarrow$ Given that

$f(x)=e^x$ and $g(x)=e^{-x}$ in $[a,b]$

$f'(x)=e^x$ and $g'(x)=-e^{-x}$ in (a,b)

$f(x)$ and $g(x)$ are differntiable in (a,b) and hence continuous in $[a,b]$

$g'(x)=-e^{-x} \neq 0 \quad \forall x \in (a,b)$.

$\therefore$ By CMVT $c \in (a,b)$ s.t

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} = (e^b - e^a)\frac{1}{\frac{1}{e^b} - \frac{1}{e^a}}$$

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$$-e^{2c} = (e^{b}-e^{a})\left(\frac{1}{\frac{e^{a}-e^{b}}{e^{a}\cdot e^{b}}}\right)$$

$$-e^{2c}\left(\frac{e^{a}-e^{b}}{e^{a}\cdot e^{b}}\right) = e^{b}-e^{a}$$

$$-e^{2c}\frac{(e^{a}-e^{b})}{e^{a+b}} = (e^{b}-e^{a})$$

$$\Rightarrow \frac{e^{2c}(\cancel{e^{b}-e^{a}})}{e^{a+b}} = \cancel{(e^{b}-e^{a})}$$

$$\Rightarrow e^{2c} = e^{a+b}$$

$\Rightarrow 2c = a+b$ (Base are equal $\Rightarrow$ power are equal)

$\Rightarrow c = \frac{a+b}{2} \in (a,b)$

$\therefore$ c is arithmatic mean of two number ie bet^n them.

(2) verify CMVT

(i) $f(x) = x^3$, $g(x) = x^4$ in $[0,2]$

$\Rightarrow$ polynomial function always continuous and differntiable

so $f(x)$ and $g(x)$ are continuous and differntiable in $(0,2)$

$f'(x) = 3x^2$ and $g'(x) = 4x^3$

$g'(x) \neq 0$; $\forall x \in (0,2)$

so CMVT is satiesfy and $\exists$ $c \in (0,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f(2)-f(0)}{g(2)-g(0)} \Rightarrow \frac{3c^2}{4c^3} = \frac{8-0}{16-0} = \frac{1}{2}$$

$$\Rightarrow \frac{3}{4c} = \frac{1}{2} \Rightarrow \boxed{c = \frac{3}{2}} \in (0,2)$$

Hene CMVT is verified.

Mean Value Theorem

Rolle's MVT

$f(x)$ is continuous in $[a,b]$

$f(x)$ is differentiable in (a,b)

$f(a) = f(b)$

$\exists\ c \in (a,b)$ s.t

$f'(c) = 0$

Lagranges MVT

$f(x)$ is continuous in $[a,b]$

$f(x)$ is differentiable in (a,b)

$f(a) \neq f(b)$

$\exists\ c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy MVT

$f(x)$ and $g(x)$ are continuous in $[a,b]$ and diff in (a,b)

$g'(x) \neq 0\ \forall\ x \in (a,b)$

$\exists\ c \in (a,b)$ s.t

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Unit-1 chapter 2
Expansion of Function.

Power series

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots \quad —(1)—$$

where $b_0, b_1, b_2 \cdots$ and a are constant and x varie around a is called power series of $f(x)$ in power of $(x-a)$

if $a=0$ then

$f(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$ is called power series of $f(x)$ in power of ~~f(x)~~ x.

Taylor's Theorem :-

Let $f(x)$ be any function defined in $[a, a+h]$ such that

(i) $f(x), f'(x), f''(x), \cdots, f^{n-1}(x)$ be continuous in $[a, a+h]$

(ii) $f^n(x)$ exists $\forall\ x \in (a, a+h)$

Then there exist at least one number θ, $0<\theta<1$ such that

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} \cdot f^{n-1}(a) + R_n$$

where R_n = Remainder after n terms.

$$R_n = \frac{h^n}{n!} f^n(a+\theta h)$$

Maclaurin's Theorem:-

Maclaurins Theorem is special case of Taylor's theorem when $a=0$ and $h=x$

$$\therefore f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(\theta x) \quad , \text{ where } 0<\theta<1$$

i.e any function of x i.e $f(x)$ can be expressed in ascending (i.e increasing) power of x, where $f(x)$ and it's sucessive derivati finite

Ex ① verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange remainder upto 2 terms in the interval $[0,1]$,

$\Longrightarrow$ By maclaurin's Theorem

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(\theta x) \quad \text{--- ①}$$

with $0 < \theta < 1$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = -5/2\,(1-x)^{3/2} \Longrightarrow f'(0) = -\frac{5}{2}$

$f''(x) = \frac{15}{4}(1-x)^{1/2} \Longrightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$

put all this value in eqⁿ ①

$$(1-x)^{5/2} = 1 + x(-5/2) + \frac{x^2}{2!} \cdot \frac{15}{4}(1-\theta x)^{1/2}$$

But interval $[0,1]$

$\therefore x = 1$

$$0 = 1 + (-5/2) + \frac{15}{8}(1-\theta)^{1/2}$$

$$= -\frac{3}{2} + \frac{15}{8}(1-\theta)^{1/2}$$

$$\frac{3}{2} \times \frac{8}{15} = (1-\theta)^{1/2}$$

$$\frac{4}{5} = (1-\theta)^{1/2}$$

$$\frac{16}{25} = 1-\theta$$

$$\theta = 1 - \frac{16}{25}$$

$$\theta = \frac{9}{25}$$

$$\boxed{\theta = 0.36 \in (0,1)}$$

Some standard expansions.

1. $e^{x} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

2. $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots$

3. $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

4. $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

5. $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots$

6. $\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$

7. $\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$

8. $\tanh x = x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\cdots$

9. $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$

10. $\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots$

11. $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$

12. $\frac{1}{(1+x)} = (1+x)^{-1} = 1-x+x^2-x^3+x^4-\cdots$

13. $\frac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+x^3+x^4+\cdots$

(14) $\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots$

(15) $\cos^{-1}x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots\right]$

(16) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(17) $\sinh^{-1}x = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots$

(18) $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

all above expansion derived by using maclaurin's Theorem

$$f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0)$$

Type (1) Example on Expansion by using maclaurin's statment

Ex

(1) Expand $\sin x$ by maclaurin's theorem upto x^5

$\Rightarrow$ $f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \frac{x^5}{5!}f'''''(0)$ —— (1)

where $f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f''''(x) = -(-\sin x) = \sin x \Rightarrow f''''(0) = 0$

$f'''''(x) = \cos x \Rightarrow f'''''(0) = 1$

$\therefore$ From eqn (1)

$f(x) = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$

$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

② expand $\log(1+e^x)$ by maclaurins theorem upto x^3

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \text{ by maclaurin's theorem } \quad \text{—①}$$

where $f(x) = \log(1+e^x)$

$$f'(x) = \frac{1}{1+e^x} \cdot e^x \quad \left(\text{since } \frac{d}{dx}\log x = \frac{1}{x}, \ \frac{d}{dx} e^x = e^x\right)$$

$$\boxed{f'(x) = \frac{e^x}{1+e^x}}$$

$$f''(x) = \frac{(1+e^x) \cdot e^x - e^x(0+e^x)}{(1+e^x)^2} = \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$\therefore \boxed{f''(x) = \frac{e^x}{(1+e^x)^2}}$$

diff. w.r.to x

$$f'''(x) = \frac{(1+e^x)^2 \cdot e^x - e^x \cdot 2(1+e^x) \cdot e^x}{(1+e^x)^4}$$

$$f'''(x) = \frac{(1+e^x)e^x - 2e^{2x}}{(1+e^x)^3} = \frac{e^x + e^{2x} - 2 \cdot e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$\boxed{f'''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3}}$$

$$f(0) = \log(1+e^0) = \log(1+1) = \log 2$$

$$f'(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{2^2} = \frac{1}{4}$$

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$$f'''(0) = \frac{e^0 - e^0}{(1+e^0)^3} = \frac{1-1}{(1+1)^3} = \frac{0}{2^3} = 0$$

$\therefore$ eq^n ① becomes

$$f(x) = \log 2 + x \cdot \left(\frac{1}{2}\right) + \frac{x^2}{2!}\frac{1}{4} + \frac{x^3}{3!}(0) + \cdots$$

$$\boxed{\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{1}{8}x^2 \text{---}}$$

Type 2: Expansion by using standard expansion

Expand $(1+x)^x$ in ascending power of x, expansion being correct upto fifth power of x

$\Longrightarrow$ Let $y = (1+x)^x$

Taking log on both side

$\log y = x \cdot \log(1+x)$ (by $\log m^n = n \log m$)

But $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

$\therefore \log y = x\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right]$

$\log y = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots$

Let $z = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4}$ —①—

$\log y = z$

Taking exponential on both side

$e^{\log y} = e^z$ $\left(\because e^{\log_e a} = a\right)$

$\boxed{y = e^z}$ —②—

But $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ —③—

$\therefore \; y = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ from ② & ③

Resubstituting value of z

$$y = 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)$$
$$+ \frac{1}{2!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)^2$$
$$+ \frac{1}{3!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)^3$$

Type 3: solved example on Expansion by using differentiation and Integration.

Ex ① prove that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

let $y = \tan^{-1}x$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \qquad \left(\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right)$$

$$\therefore \frac{dy}{dx} = (1+x^2)^{-1}$$

By $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 + \cdots$

put $z = x^2$

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$dy = (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

Here y is fun of x, so integrate w.r.to x

$$\int_0^x dy = \int_0^x (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

$$[y]_0^x = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right]_0^x$$

$$\left(\tan^{-1}x\right)_0^x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\tan^{-1}x - \tan^{-1}(0) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots} \quad (\because \tan^{-1}0 = 0)$$

Type 4: Expansion by using substitution.

Ex Expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending power of x

Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put $x = \tan\theta \Rightarrow \tan^{-1}x = \theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$y = \sin^{-1}[\sin 2\theta]$$

Formulae

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$y = 2\theta$$

$$y = 2\tan^{-1}x$$

$$y = 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right]$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots\right]$$

Example Base on Taylor's Theorem.

① Expand $2x^3+7x^2+x-6$ in power of $x-2$

Given $f(x) = 2x^3+7x^2+x-6$ ——①

Let $f(x) = f(x-2+2)$

By Taylor Theorem

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

Note: Here highest power of $f(x)$ is 3, so we expand Taylor's theorem upto power 3.

Let $h = x-2$, $a=2$

$$f(x) = f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$f(x) = f(2) + (x-2) \cdot f'(2) + \frac{(x-2)^2}{2!} \cdot f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \cdots \quad \text{——②}$$

$f(x) = 2x^3+7x^2+x-6$ from eqn ①

$f(2) = 2(2)^3+7(2)^2+2-6 = 16+28+2-6 = 40$

$f'(x) = 6x^2+14x+1 \Rightarrow f'(2) = 6\times4+14\times2+1 = 24+28+1 = 53$

$f''(x) = 12x+14 \Rightarrow f''(2) = 12\times2+14 = 24+14 = 38$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

∴ from eqn ②

$$f(x) = 40 + (x-2)53 + \frac{(x-2)^2}{2!} \cdot 38 + \frac{(x-2)^3}{3!} \times 12$$

$$\boxed{f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3}$$

Ex ② Expand $\log\cos(x+\frac{\pi}{4})$ using Taylor's theorem in ascending power of x and hence find the value of $\log\cos 48^\circ$ correct upto four decimal places.

$\Longrightarrow$

$$f(x+\pi/4) = \log\cos(x+\pi/4)$$

By Taylor Theorem

$$f(x+\pi/4) = f(\pi/4) + x\cdot f'(\pi/4) + \frac{x^2}{2!}f''(\pi/4) + \frac{x^3}{3!}f'''(\pi/4) + \frac{x^4}{4!}f''''(\pi/4) + \cdots \quad (1)$$

h a

Let $x+\pi/4 = t$

$$f(t) = \log\cos t$$

$$f'(t) = \frac{1}{\cos t}(-\sin t)$$

$$f'(t) = -\tan t$$

$$f''(t) = -\sec^2 t$$

$$f'''(t) = -2\sec t\cdot\sec t\cdot\tan t = -2\sec^2 t\cdot\tan t$$

diff. again

$$f''''(t) = -2\sec^2 t\cdot\sec^2 t - 2(2\sec t\cdot\sec t\cdot\tan t)\tan t$$

$$f''''(t) = -2\sec^4 t - 4\sec^2 t\cdot\tan^2 t$$

$$f(\tfrac{\pi}{4}) = \log(\cos\tfrac{\pi}{4}) = \log(\tfrac{1}{\sqrt{2}})$$

$$f'(\tfrac{\pi}{4}) = -\tan(\tfrac{\pi}{4}) = -1$$

$$f''(\tfrac{\pi}{4}) = -(\sec^2 t) = -(\sqrt{2})^2 = -2$$

$$f'''(\tfrac{\pi}{4}) = -2\sec^2(\pi/4)\cdot\tan(\pi/4) = -2(\sqrt{2})^2\cdot 1 = -4$$

$$f''''(\pi/4) = -2\sec^4(\pi/4) - 4\sec^2(\pi/4)\cdot\tan^2(\pi/4)$$

$$= -2(\sqrt{2})^4 - 4(\sqrt{2})^2\cdot(1)^2$$

$$= -2\times 4 - 4\times 2 = -8-8 = -16$$

From eq^n (1)

$$f\left(x+\frac{\pi}{4}\right) = \log\frac{1}{\sqrt{2}} + x(-1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(-4) + \frac{x^4}{4!}(-16) \cdots$$

$$\therefore \log\cos\left(x+\frac{\pi}{4}\right) = \log\left(\frac{1}{\sqrt{2}}\right) - x - x^2 - \frac{4}{3}x^3 - \frac{2}{3}x^4 + \cdots \quad (2)$$

Now $\cos(48^\circ) = \cos(45^\circ + 3^\circ)$

but $3^\circ = \frac{3\pi^c}{180} = 0.05235$

$= \cos(45^\circ + 0.05235)$

$\therefore$ put $x = 0.05235$ in eq^n (2)

$$\log\cos\left(0.05235+\frac{\pi}{4}\right) = \log\frac{1}{\sqrt{2}} - (0.05235) - (0.05235)^2 - \frac{4}{3}(0.05235)^3 - \frac{2}{3}(0.05235)^4 + \cdots$$

$\log\cos\left(0.05235+\frac{\pi}{4}\right) = -0.40176$. (Approximately)

$$\boxed{\log\cos 48^\circ = -0.40176.}$$

(Q3) Use Taylor's theorem to find $\sqrt{25.15}$

Here $f(a+h) = \sqrt{a+h}$ —— (1)

By Taylor Theorem

$f(t) = \sqrt{t}$

$f(t) = t^{1/2}$ $\qquad f(a) = a^{1/2}$

$f'(t) = \frac{1}{2}t^{-1/2}$ $\qquad f'(a) = \frac{1}{2}a^{-1/2}$

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$f''(t) = -\frac{1}{4} t^{-3/2}$ $\quad f''(a) = -\frac{1}{4} a^{-3/2}$

$f'''(t) = \frac{3}{8} t^{-5/2}$ $\quad f'''(a) = \frac{3}{8} a^{-5/2}$

By Taylor formulae

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\sqrt{a+h} = a^{1/2} + \frac{h}{2} \cdot a^{-1/2} + \frac{h^2}{2!} \frac{a^{-3/2}}{4} + h^3 \frac{3 a^{-5/2}}{24} + \cdots$$

$$\sqrt{25+0.15} = (25)^{1/2} + \left(\frac{0.15}{2}\right)(25)^{-1/2} - \frac{(0.15)^2 (25)^{-3/2}}{8}$$

$$+ \frac{1}{8} (0.15)^3 (25)^{-5/2} + \cdots$$

$$\sqrt{25+0.15} = 5.0149$$

$$\underline{\sqrt{25.15} = 5.0149}$$

Unit 1 : Chapter 3

Indeterminate forms.

Indeterminant form mean we "cannot determine"

In determinant form refer to a value of function at $x=a$ which reduces to $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞.

* L-Hospital Rule:-

L Hospital Rule is used to evaluate indeterminant form where limit tends to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

i.e $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$

Then we can take derivative of numerator and denominator seprate and evaluate the limit.

i.e $\lim\limits_{x\to a} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

Note:- For $\frac{0}{0}$ form, we can apply L-Hospital Rule for any number of times until we get determined value of the function

* some important limits

(i) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

(ii) $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$

(iii) $\lim\limits_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(iv) $\lim\limits_{x\to 0} (1+x)^{1/x} = e$

(v) $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \log a$

(vi) $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$

Type 1 $\frac{0}{0}$ form

Ex (1) evaluate $\lim\limits_{x\to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}$

~~put $x=0$~~

(i) $L = \lim_{x\to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$

put $x=0$

$L = \dfrac{e^{0} - e^{-0}}{\log(1+0)} = \dfrac{1-1}{\log 1} = \dfrac{0}{0}$

$L = \frac{0}{0}$ form

By L-Hospital Rule

$L = \lim_{x\to 0} \dfrac{\frac{d}{dx}(e^{ax} - e^{-ax})}{\frac{d}{dx}\log(1+bx)} = \lim_{x\to 0} \dfrac{e^{ax}(a) - e^{-ax}(-a)}{\frac{1}{1+bx} \times b}$

put $x=0$

$L = \dfrac{a+a}{b} = \dfrac{2a}{b}$

$\boxed{L = \dfrac{2a}{b}}$

(ii) $\lim_{x\to 0} \dfrac{x\cdot e^{x} - \log(1+x)}{x^2}$ find L

Let $L = \lim_{x\to 0} \dfrac{x\cdot e^{x} - \log(1+x)}{x^2}$ —— ①

put $x=0$

$L = \dfrac{0\cdot e^{0} - \log(1+0)}{0}$ $(\because \log 1 = 0)$

$L = \frac{0}{0}$ form

By L Hospital Rule, diff Numerator and Denominator of eqⁿ ① w.r.to x

$L = \lim_{x\to 0} \dfrac{\frac{d}{dx}(x\cdot e^{x} - \log(1+x))}{\frac{d}{dx}(x^2)} = \lim_{x\to 0} \dfrac{x\cdot e^{x} + e^{x} - \frac{1}{1+x}(0+1)}{2x}$

$\therefore L = \lim_{x\to 0} \dfrac{x\cdot e^{x} + e^{x} - \frac{1}{1+x}}{2x}$ —— ②

put $x=0$

$$L = \frac{0+e^0 - \frac{1}{1+0}}{2(0)} = \frac{1-1}{0} = \frac{0}{0}$$

$L = \frac{0}{0}$ form

By L-Hospital Rule, diff N and D of eqⁿ ②

$$L = \lim_{x\to 0} \frac{\frac{d}{dx}\left[x\cdot e^x + e^x - \frac{1}{1+x}\right]}{\frac{d}{dx}(2x)}$$

$$L = \lim_{x\to 0} \frac{x\cdot e^x + e^x + e^x - \frac{(-1)}{(1+x)^2}(0+1)}{2}$$

$$L = \lim_{x\to 0} \frac{x\cdot e^x + 2e^x + \frac{1}{(1+x)^2}}{2}$$

put $x=0$

$$L = \frac{0\cdot e^0 + 2\cdot e^0 + \frac{1}{(1+0)^2}}{2}$$

$$L = \frac{2+1}{2} = \frac{3}{2}$$

$$\boxed{L = \frac{3}{2}}$$

③ find a, b, c if $\lim_{x\to 0} \frac{a\cdot e^x - b\cdot \cos x + c\cdot e^{-x}}{x\cdot \sin x} = 2$

Given

$$\lim_{x\to 0} \frac{a\cdot e^x - b\cdot \cos x + c\cdot e^{-x}}{x\cdot \sin x} = 2 \quad \text{—①—}$$

put $x=0$

$$\frac{a\cdot e^0 - b\cdot \cos(0) + c\cdot e^{-0}}{0\cdot \sin 0} = 2$$

$$\frac{a-b+c}{0} = 2$$

But
$$\frac{a-b+c}{0} = \infty$$
and finite answer $a-b+c=0$ —(A)—

Apply L-H-Rule for eqⁿ ①

$$\lim_{x\to 0} \frac{\frac{d}{dx}[a\cdot e^{x} - b\cdot \cos x + c\cdot e^{-x}]}{\frac{d}{dx}[x\cdot \sin x]} = 2$$

$$\lim_{x\to 0} \frac{a\cdot e^{x} + b\sin x - c\cdot e^{-x}}{x\cdot \cos x + \sin x} = 2 \quad \text{—②—}$$

put $x=0$

$$\frac{a\cdot 1 + b\cdot 0 - c\cdot 1}{0} = 2$$

This value is finite if $a\cdot 1 + b\cdot 0 - c = 0$ —(B)—

Apply L-Hospital Rule for eqⁿ ②

$$\therefore \lim_{x\to 0} \frac{\frac{d}{dx}[a\cdot e^{x} + b\cdot \sin x - c\cdot e^{-x}]}{\frac{d}{dx}[x\cdot \cos x + \sin x]} = 2$$

$$\lim_{x\to 0} \frac{a\cdot e^{x} + b\cdot \cos x + c\cdot e^{-x}}{-x\cdot \sin x + \cos x + \cos x} = 2$$

put $x=0$

$$\frac{a + b\cdot \cos 0 + c\cdot e^{-0}}{0+1+1} = 2$$

$$\frac{a+b+c}{2} = 2$$

$a+b+c=4$ —(C)—

solve ⓐ, ⓑ, ⓒ

$$\left.\begin{array}{l} a-b+c=0 \\ a+0b-c=0 \end{array}\right\} + \Rightarrow 2a-b=0$$
$$\left.\begin{array}{l} a+0b-c=0 \\ a+b+c=4 \end{array}\right\} + \Rightarrow 2a+b=4$$
$$\left.\begin{array}{l} 2a-b=0 \\ 2a+b=4 \end{array}\right\} + \Rightarrow 4a=4 \Rightarrow \underline{a=1}$$

$\underline{b=2}$, $\underline{c=1}$

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Type 2: $\frac{\infty}{\infty}$ form

Note: If we get $\frac{\infty}{\infty}$ form, then apply L-Hospital Rule only one until we convert it to $\frac{0}{0}$ form by using reciprocal methods.

Ex (1) $\lim_{x\to 0} \frac{\log \tan x}{\log x}$

Let $L = \lim_{x\to 0} \frac{\log \tan x}{\log x}$ —— (1)

put $x=0$

$$L = \frac{\log \tan(0)}{\log(0)} = \frac{\infty}{\infty}$$

$L = \frac{\infty}{\infty}$ form

Apply L-Hospital Rule for eqⁿ (1)

$$\therefore L = \lim_{x\to 0} \frac{\frac{d}{dx}\log(\tan x)}{\frac{d}{dx}\log x} = \lim_{x\to 0} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{\frac{1}{x}}$$

on reciprocal

$$L = \lim_{x\to 0} \frac{x\cdot \cot x}{\cos^2 x} = \lim_{x\to 0} \frac{x}{\cos^2 x}\cdot\frac{\cos x}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x}\cdot\frac{1}{\cos x}$$

But $\lim_{x\to 0} \frac{x}{\sin x} = 1$

$$L = 1\cdot\frac{1}{\cos 0} = 1\cdot 1 = 1$$

$$\boxed{L=1}$$

(9) Evaluate $\lim_{x \to \pi/2} \frac{\tan 3x}{\tan x}$ ——— (1)

put $x = \pi/2$

$L = \frac{\tan 3\pi/2}{\tan \pi/2}$

But $\tan \pi/2 = \infty$ and $\tan 3\pi/2 = \infty$

$L = \frac{\infty}{\infty}$

∴ Apply L-H. Rule to the eqn (1)

$$L = \lim_{x \to \pi/2} \frac{\frac{d}{dx}(\tan 3x)}{\frac{d}{dx}(\tan x)} = \lim_{x \to \pi/2} \frac{\sec^2(3x) \cdot 3}{\sec^2 x}$$

on reciprocal.

$$L = \lim_{x \to \pi/2} \frac{3 \cdot \frac{1}{\cos^2(3x)}}{\frac{1}{\cos^2 x}} = \lim_{x \to \pi/2} \frac{3 \cdot \cos^2 x}{\cos^2(3x)}$$

$$= 3 \left(\lim_{x \to \pi/2} \left[\frac{\cos x}{\cos 3x} \right] \right)^2 \text{ ——— (2)}$$

put $x = \pi/2$

$$L = 3 \left(\frac{\cos \pi/2}{\cos 3\pi/2} \right)^2 \qquad \left(\because \begin{array}{l} \cos \pi/2 = 0 \\ \cos(3\pi/2) = 0 \end{array} \right)$$

$$L = 3 \cdot \left(\frac{0}{0} \right)$$

$$L = \frac{0}{0} \text{ form}$$

Apply L-Hospital Rule for eqn (2)

$$L = 3 \left[\lim_{x \to \pi/2} \frac{\frac{d}{dx}(\cos x)}{\frac{d}{dx}(\cos 3x)} \right]^2$$

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$$L = 3\left[\lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{-3\cdot\sin 3x}\right]^2$$

$$L = \frac{3}{9}\left[\lim_{x\to\pi/2} \cdot \frac{\sin x}{\sin 3x}\right]^2$$

$$L = \frac{3}{9}\left(\frac{\sin \pi/2}{\sin 3\pi/2}\right)^2$$

$$L = \frac{3}{9}\left(\frac{1}{-1}\right)^2$$

$$L = \frac{3}{9} = \frac{1}{3}$$

Homework : ① $\lim_{\theta\to\pi/2} \frac{\log\left(\theta - \frac{\pi}{2}\right)}{\tan\theta}$ Ans $L=0$.

* Type 3 : $0\times\infty$ form

Note ① $0\times\infty$ can be converted into $\frac{0}{0}$ form by making following changes $0\times\infty = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$

② If ∞ is obtained due to a logarithmic terms then do not bring it to denominator since it is difficult to differntiate $\frac{1}{\log x}$ function.

③ In such case bring the function other than logarithmic function to denominator.

Ex ① $\lim_{x\to 0} \sin x \cdot \log x$

$L = \lim_{x\to 0} \sin x \cdot \log x$

put $x = 0$

$L = -0 \times \infty$ form ($\because \log 0 = -\infty$, $\sin 0 = 0$)

$$\therefore L = \lim_{x \to 0} \frac{\log x}{\frac{1}{\sin x}}$$

$$L = \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \quad \text{——— (1)}$$

put $x = 0$

$$L = \frac{\log 0}{\operatorname{cosec} 0}$$

$$L = \frac{-\infty}{\infty} \text{ form}$$

Apply L-Hospital Rule to the eq^n (1)

$$L = \lim_{x \to 0} \frac{\frac{d}{dx} \log x}{\frac{d}{dx} \cdot \operatorname{cosec} x} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cdot \cot x}$$

on reciprocal. $\Rightarrow$

$$L = -\lim_{x \to 0} \frac{\sin x \cdot \tan x}{x}$$

$$L = -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \tan x$$

$$L = -(1) \cdot \lim_{x \to 0} \tan x$$

$$L = -\tan 0$$

$$\boxed{L = 0}$$

(2) $\lim_{x \to \frac{\pi}{2}} (1 - \sin x) \cdot \tan x$

let $L = \lim_{x \to \frac{\pi}{2}} (1 - \sin x) \cdot \tan x$

put $x = \frac{\pi}{2}$

$$L = (1 - \sin \pi/2) \cdot \tan \pi/2$$

$$L = (1 - 1) \times \infty$$

$$L = 0 \times \infty \text{ form}$$

$\therefore L = \lim_{x \to \pi/2} \dfrac{1-\sin x}{\frac{1}{\tan x}} = \lim_{x \to \frac{\pi}{2}} \left(\dfrac{1-\sin x}{\cot x} \right)$

$\therefore L = \lim_{x \to \pi/2} \left(\dfrac{1-\sin x}{\cot x} \right)$ —— (1)

put $x = \frac{\pi}{2}$

$L = \dfrac{1-\sin \pi/2}{\cot \pi/2} = \dfrac{1-1}{0} = \dfrac{0}{0}$

$L = \frac{0}{0}$ form

By Applying L-Hospital Rule eqn (1)

$L = \lim_{x \to \pi/2} \dfrac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx} \cot x} = \lim_{x \to \frac{\pi}{2}} \dfrac{(0-\cos x)}{-\operatorname{cosec}^2 x}$

$L = \lim_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\operatorname{cosec}^2 x} = \dfrac{\cos \pi/2}{(\operatorname{cosec} \pi/2)^2} = \dfrac{0}{1^2} = 0$

$\boxed{L = 0}$

Homework:- $\lim_{x \to 0} x \cdot \log x$

Type 4 $(\infty, -\infty)$ form

Ex (1) $\lim_{x \to 1} \left[\dfrac{x}{x-1} - \dfrac{1}{\log x} \right]$

$\Rightarrow$ Let $L = \lim_{x \to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\log x} \right)$

put $x = 1$

$L = \dfrac{1}{0} - \dfrac{1}{\log 1} = \infty - \infty$ $\quad$ ($\because \log 1 = 0 \Rightarrow \frac{1}{\log 1} = \infty$)

$L = \infty - \infty$ form

$\therefore L = \lim_{x \to 1} \left[\frac{x \cdot \log x - (x-1)}{(x-1) \cdot \log x} \right]$ — ①

put $x = 1$

$$L = \frac{\log 1 - (1-1)}{(1-1) \cdot \log 1} = \frac{0}{0} \text{ form}$$

Apply L-Hospital Rule to the eqⁿ ①

$$\therefore L = \lim_{x \to 1} \frac{\frac{d}{dx}\left(x \cdot \log x - (x-1)\right)}{\frac{d}{dx}(x-1) \cdot \log x}$$

$$L = \lim_{x \to 1} \frac{x \cdot \frac{1}{x} + 1 \cdot \log x - (1-0)}{(x-1) \cdot \frac{1}{x} + (1-0) \log x}$$

$$L = \lim_{x \to 1} \frac{\cancel{1} + \log x - \cancel{1}}{1 - \frac{1}{x} + \log x} = \lim_{x \to 1} \frac{\log x}{1 - \frac{1}{x} + \log x} \quad \text{— ②}$$

put $x = 1$

$$L = \frac{\log 1}{1 - \cancel{1} + \log 1} = \frac{0}{0}$$

$$L = \frac{0}{0} \text{ form.}$$

Apply L-Hospital Rule for eqⁿ ②

$$L = \lim_{x \to 1} \frac{\frac{1}{x}}{0 + \frac{1}{x^2} + \frac{1}{x}}$$

put $x = 1$

$$L = \frac{1}{0 + \frac{1}{1} + \frac{1}{1}} = \frac{1}{2}$$

$$\boxed{L = \frac{1}{2}}$$

Ex ②

Prove that $\lim_{x\to 0}\left[\frac{a}{x} - \cot\frac{x}{a}\right] = 0$

$$L.H.S = \lim_{x\to 0}\left[\frac{a}{x} - \cot\frac{x}{a}\right]$$

put $x = ay$

$\therefore$ when $x \to 0$

$\Rightarrow 0 = ay$

$\Rightarrow y \to 0$

$$L.H.S = \lim_{y\to 0}\left[\frac{1}{y} - \cot y\right]$$

put $y = 0$ $\quad \cot 0 = \infty$

$$= \frac{1}{0} - \cot 0$$

$L.H.S = \infty - \infty$ form

$$L.H.S = \lim_{y\to 0}\left[\frac{1}{y} - \frac{1}{\tan y}\right] \quad \text{—①—}$$

$$= \lim_{y\to 0} \frac{\tan y - y}{y \cdot \tan y}$$

$$= \lim_{y\to 0} \frac{\tan y - y}{y^2 \cdot \frac{\tan y}{y}}$$

$$= \frac{1}{\lim_{y\to 0}\frac{\tan y}{y}} \cdot \lim_{y\to 0}\frac{\tan y - y}{y^2}$$

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But $\lim_{y \to 0} \frac{\tan y}{y} = 1$

put $y=0$

$$L.H.S = \lim_{y \to 0} \frac{\tan y - y}{y^2} \quad \text{—(1)—}$$

put $y=0$

$$L.H.S \ \frac{\tan 0 - 0}{0} = \frac{0}{0} \text{ form}$$

Apply L-Hospital Rule for eqn (1)

$$L.H.S = \lim_{y \to 0} \frac{\sec^2 y - 1}{2y} \quad \text{—(2)—}$$

put $y=0$ $\qquad (\because \sec 0 = 1)$

$$L.H.S = \frac{0}{0} \text{ form}$$

$\therefore$ Apply L-Hospital rule for eqn (2)

$$\therefore L.H.S = \lim_{y \to 0} \frac{2 \sec y \cdot \tan y}{2}$$

put $y=0$ $\quad (\sec 0 = 1 ; \tan 0 = 0)$

$$= 0 = R.H.S.$$

☆ Indeterminate Forms $0^0, \infty^0, 1^\infty$

To evaluate indeterminant form of the type $0^0, \infty^0, 1^\infty$ fallow the below step.

(1) Let L = limit of the form $0^0, \infty^0, 1^\infty$

(2) Take log on both sides.

(3) we get either $0 \times \infty$ or $\infty \times 0$ form

(4) shift suitable function to denominator, so that we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

$$\boxed{\text{by } \log m^n = n \cdot \log m}$$

$$\log L = \lim_{x \to 1} \frac{1}{\log(1-x)} \cdot \log(1-x^2)$$

$$\therefore \log L = \lim_{x \to 1} \frac{\log(1-x^2)}{\log(1-x)} \quad \text{——— (1)}$$

(put $x=1$, $\log 0 = -\infty$)

$\log L = \frac{\infty}{\infty}$ form

Apply L-Hospital Rule for eq^n (1)

$$\log L = \lim_{x \to 1} \frac{\frac{1}{1-x^2}(-2x)}{\frac{1}{1-x}(-1)}$$

$$\log L = \lim_{x \to 1} \frac{\frac{-2x}{1-x^2}}{\frac{-1}{1-x}}$$

on reciprocal

$$\log L = \lim_{x \to 1} \frac{2x(1-x)}{1-x^2}$$

$$= \lim_{x \to 1} \frac{2x(1-x)}{(1-x)(1+x)}$$

put $x=1$

$$\log L = \frac{2(1)}{1+1} = \frac{2}{2} = 1$$

$$\boxed{\log L = 1}$$

$e^{\log L} = e^{1}$

$\Rightarrow \boxed{L = e^{1}}$

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Example (1)

$$\lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$$

$\Longrightarrow$ Let $L = \lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$

Taking log on both sides

$$\log L = \lim_{x \to \frac{\pi}{2}} \log (\cos x)^{\cos x} \quad \left(\because \log m^n = n \cdot \log m\right)$$

$$\therefore \log L = \lim_{x \to \frac{\pi}{2}} \cos x \cdot \log (\cos x)$$

put $x = \frac{\pi}{2}$

$\left(\because \cos \frac{\pi}{2} = 0 \quad , \log (0) = -\infty\right)$

$\therefore \log L = -0 \times \infty$ form

$$\therefore \log L = \lim_{x \to \frac{\pi}{2}} \frac{\log (\cos x)}{\frac{1}{\cos x}}$$

$$\log L = \lim_{x \to \frac{\pi}{2}} \frac{\log \cos x}{\sec x} \quad \text{---} (1) \text{---}$$

put $x = \frac{\pi}{2}$

$$\therefore \log L = \lim_{x \to \frac{\pi}{2}} \frac{\log \left(\cos \frac{\pi}{2}\right)}{\sec \left(\frac{\pi}{2}\right)}$$

But $\sec \frac{\pi}{2} = \sec 90^0 = \frac{1}{\cos 90^0} = \frac{1}{0} = \infty.$

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$\log L = \frac{-\infty}{\infty}$ form

Apply L-Hospital Rule for eqⁿ (1)

$\therefore \log L = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cos x}(-\sin x)}{\sec x \cdot \tan x}$

$\log L = \lim_{x \to \frac{\pi}{2}} \frac{-\tan x}{\sec x \cdot \tan x}$

on reciprocal.

$\log L = -\lim_{x \to \frac{\pi}{2}} \cos x$ $\left(\because \sec x = \frac{1}{\cos x}\right)$

put $x = \frac{\pi}{2}$

$\log L = -\cos\frac{\pi}{2} = 0$

$\log L = 0$

$e^{\log_e L} = e^0$

$L = e^0$

$\boxed{L = 1}$