UNIT-4

chapter 6: Jacobians

* Concept of Jocobians (J)

Consider a two dimensional function
$u = f_1(x,y)$.
Then we can diff u with respect to both the x and y
i.e $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

simillarly $v = f_2(x,y)$
diff. v w.r.t both the x and y
i.e $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Then the derivative of the above two function viz. u and v can be arranged in the form of determinant called as Jocobians given by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Ex $u = x^2 + y^2$ and $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial u}{\partial y} = 2y$ $\quad \frac{\partial v}{\partial x} = 2y$ $\quad \frac{\partial v}{\partial y} = 2x$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix}$$

$$= 4x^2 - 4y^2$$

similarly.

$u = f_1(x, y, z)$ and $v = f_2(x, y, z)$

$w = f_3(x, y, z)$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Type 1: solved Example on Jocobians.

Q. If $u = x^2 - y^2$ and $v = 2xy$. calculate $\frac{\partial(u, v)}{\partial(x, y)}$

Given $u = x^2 - y^2$, $v = 2xy$

By concept of Jocobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$J = (2x)(2x) - (-2y)(2y)$$

$$J = 4x^2 + 4y^2$$

$$J = 4(x^2 + y^2)$$

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Note:

1. Let

$$u = x+y+z$$
$$v = x^2+y^2+z^2$$
$$w = xy+yz+zx$$

where u is not repeating in othere two equation

simillarly v, ω not repeating in othere two equation

$$J = \frac{\partial(u,v,\omega)}{\partial(x,y,z)} = \frac{\text{Non repeating variable}}{\text{Repeating variables}}$$

2. If $J = \frac{\partial(u,v)}{\partial(x,y)}$

then $J' =$ reciprocal of J

$$J' = \frac{\partial(x,y)}{\partial(u,v)}$$

3. $J \cdot J' = 1$

Ex If $ux = yz$; $vy = zx$; $wz = xy$, find $\frac{\partial(u,v,\omega)}{\partial(x,y,z)}$

$\Rightarrow$ Given $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

Now by concept of Jacobian

$$J = \frac{\partial(u,v,\omega)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix}$$

$$\therefore J = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$\therefore J = \frac{1}{x}\frac{1}{y}\cdot\frac{1}{z} \begin{vmatrix} \frac{-yz}{x} & z & y \\ z & \frac{-zx}{y} & x \\ y & x & \frac{-xy}{z} \end{vmatrix}$$

$$J = \frac{1}{x}\frac{1}{y}\cdot\frac{1}{z}\left\{ \frac{-yz}{x}\begin{vmatrix} \frac{-zx}{y} & x \\ x & \frac{-xy}{z} \end{vmatrix} - z\begin{vmatrix} z & x \\ y & \frac{-xy}{z} \end{vmatrix} + y\begin{vmatrix} z & \frac{-zx}{y} \\ y & x \end{vmatrix} \right\}$$

$$\therefore J = \frac{1}{xyz}\left\{ \frac{-yz}{x}\left(\frac{\cancel{z}x\cdot x\cancel{y}}{\cancel{y}\cancel{z}} - x^2 \right) - z\left[-xy - yx\right] + y\left[zx + zx\right] \right\}$$

$$J = \frac{1}{xyz}\left\{ \frac{-yz}{x}(0) + z(2xy) + y(2zx) \right\}$$

$$J = \frac{1}{xyz}\left\{ 2zxy + 2zxy \right\}$$

$$J = \frac{1}{\cancel{xyz}}\, 4\,\cancel{xyz}$$

$$\boxed{J = 4}$$

If $x = u(1-v)$ and $y = uv$ then prove that $J \cdot J' = 1$

$\Rightarrow$ Given $x = u(1-v)$, $y = u \cdot v$

step 1: To find J.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & u(\cancel{0-1}) \\ v & u \end{vmatrix}$$

$$= (1-v)u + uv$$

$$= u - vu + \cancel{uv}$$

$$\boxed{J = u}$$

step 2: To find J'

$x = u - uv$ and $y = uv$

$f_1 = x - u + uv$ $\quad f_2 = y - uv$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)}$$

$$J' = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$J' = \frac{N}{D} \quad \text{--- (1)}$$

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where $N = \dfrac{\partial(f_1, f_2)}{\partial(x, y)}$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\boxed{N = 1}$$

and $D = \dfrac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -1+v & u \\ -v & -u \end{vmatrix}$

$$D = -u(-1+v) + uv$$

$$= +u - \cancel{uv} + \cancel{uv}$$

$$\boxed{D = u}$$

from eq^n 1

$$J' = \frac{N}{D} = \frac{1}{u}$$

Now. L.H.S $= J \cdot J' = u \cdot \frac{1}{u} = 1 =$ R.H.S.

Ex If $x = v^2 + w^2$; $y = w^2 + u^2$; $z = u^2 + v^2$, then prov that $J \cdot J' = 1$

solution:

Part I To find J

Given $x = v^2 + w^2$; $y = w^2 + u^2$ and $z = u^2 + v^2$

$$\therefore J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

Taking $2u, 2v, 2w$ from $1^{st}, 2^{nd}, 3^{rd}$ coloumn respectively.

$$J = (2u)(2v)(2w) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 8uvw \{0 - 1(0-1) + 1(1)\}$$

$$= 8uvw(2)$$

$$\boxed{J = 16uvw}$$

Part 2: To find J'

Given $x = v^2 + w^2$; $y = w^2 + u^2$; $z = u^2 + v^2$

$$\therefore f_1 = x - v^2 - w^2$$

$$f_2 = y - w^2 - u^2$$

$$f_3 = z - u^2 - v^2$$

$$\therefore J' = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= (-1)^3 \cdot \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

$$J' = -\frac{N}{D} \quad \text{--- (1)}$$

where $N = \dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\therefore \boxed{N = 1}$

and $D = \frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$

$$D = \begin{vmatrix} 0 & -2v & -2w \\ -2u & 0 & -2w \\ -2u & -2v & 0 \end{vmatrix}$$

$$D = (-2u)(-2v)(-2w) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$D = -8uvw \{ 0 - 1(0-1) + 1(1) \}$$

$$D = -8uvw (2)$$

$$\boxed{D = -16uvw}$$

But from eqⁿ

$$J' = -\frac{N}{D} = -\frac{1}{-16uvw} = \frac{1}{16uvw}$$

Now. L.H.S $= J \cdot J'$

$$= 16uvw \cdot \frac{1}{16uvw}$$

$$\boxed{J' = 1}$$

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6.2 Jacobian of Implicit Functions:

Mainly function are divided into two types.

(i) Explicit function (ii) Implicit function.

(I) Explicit Function:

An explicit function is a function in which one variable can be clearly (explicitly) expressed in terms of the othere

e.g. $x^2+3x^2y+9x^2y^3=1$

above mentioned function is a $f(x,y)$

In this function x can be clearly expressed in terms of y

$$\therefore x^2+3x^2y+9x^2y^3=1$$

$$\therefore x^2(1+3y+9y^3)=1$$

$$x^2=\frac{1}{1+3y+9y^3}$$

$\therefore x^2+3x^2y+9x^2y^3=1$ is an explicit function.

II Implicit Functions:

An implicit function is a function in which one variable cannot be expressed in terms of other i.e no calculation can clearly expressed y in terms of x.

Ex $x^2+2xy+y^2=9$.

* To find Jacobian of Implicit function we will we method of function as mentioned below.

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(I) let $f_1(x,y,u,v)=0$ and $f_2=(x,y,u,v)=0$

then
$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}}$$

(II) If $f_1(x,y,z,u,v,\omega)=0$

$f_2(x,y,z,u,v,\omega)=0$

$f_3(x,y,z,u,v,\omega)=0$ then

$$\boxed{\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,\omega)}}}$$

Example:

If $u^3+v^3=x+y$, $u^2+v^2=x^3+y^3$, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{y^2-x^2}{uv(u-v)}$$

$\Rightarrow$ Given $u^3+v^3=x+y$

$\therefore f_1 = u^3+v^3-x-y$

and $u^2+v^2=x^3+y^3$

$\therefore f_2 = u^2+v^2-x^3-y^3$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \frac{N}{D} \quad \text{--- (1)}$$

where $N = \frac{\partial(f_1, f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}$

$$N = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$N = 3y^2 - (-3x^2)(-1)$$

$$N = 3y^2 - 3x^2$$

$$D = \frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= (3)(2)(u)(v)\begin{vmatrix} u & v \\ 1 & 1 \end{vmatrix}$$

$$= 6uv(u-v)$$

$\therefore$ ~~LHS~~ $D = 6uv(u-v)$

$\therefore$ L.H.S. $= \frac{\partial(u,v)}{\partial(x,y)} = \frac{N}{D} = \frac{3(y^2-x^2)}{6uv(u-v)}$

$$= \frac{1}{2}\,\frac{1}{uv}\,\frac{y^2-x^2}{(u-v)}$$

Note:

For the polynomial $ax^3+bx^2+cx+d=0$, with root α, β, γ we have

$$\alpha+\beta+\gamma = \frac{-b}{a}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

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and $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Ex If $x+y+z=u$ and $y+z=uv$, $z=uvw$, show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

From given relation we have

$f_1 = x+y+z-u$

$f_2 = y+z-uv$

$f_3 = z-uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{(-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} = -\frac{N}{D}$$

where $N = \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$$

$$\therefore N = -1[(-u)(-uv) - 0]$$

$$\boxed{N = -u^2v}$$

Now $D = \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

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$$D = \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}$$

$$D = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$D = 1(1) - 1(0) + 1(0) = 1$$

$$L.H.S = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{N}{D} = \frac{(u^2 v)}{1} = u^2 v = R.H.S.$$

* Partial Derivative of Implicit function using Jacobian

Let $f_1(x, y, u, v) = 0$

$f_2(x, y, u, v) = 0$

Then the formulae to find partial derivative by using Jacobian as fallow.

Ex 1: To find $\frac{\partial u}{\partial x}$

Numerator variable group $\Rightarrow u, v$

Step 1: For $\frac{\partial u}{\partial x}$

$$\therefore \frac{\partial u}{\partial x} = - \frac{\dfrac{\partial(f_1, f_2)}{\text{Numerator variable Group}}}{\dfrac{\partial(f_1, f_2)}{\text{Numerator variable group}}}$$

$$= - \frac{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}$$

step2: Now replace $u \to x$ in numerator only.

$$\frac{\partial u}{\partial x} = \frac{-\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1 f_2)}{\partial(u, v)}}$$

Ex 2:

To find $\frac{\partial y}{\partial u}$

Numerator variable group x, y

step 1: For $\frac{\partial y}{\partial u}$

$$\therefore \frac{\partial y}{\partial u} = -\frac{\frac{\partial(f_1, f_2)}{\text{Numerator variable Group}}}{\frac{\partial(f_1, f_2)}{\text{Numerator variable Group}}}$$

$$= -\frac{\frac{\partial(f_1 f_2)}{\partial(x, y)}}{\frac{\partial(f_1, f_2)}{\partial(x, y)}}$$

Now replace $y \to u$ in numerator

$$\frac{\partial y}{\partial u} = -\frac{\frac{\partial(f_1, f_2)}{\partial(x, u)}}{\frac{\partial(f_1, f_2)}{\partial(x, y)}}$$

* Similarly for two function f_1, f_2 othere formulae can be set up

Now if

$$f_1(x, y, z, u, v, \omega) = 0$$
$$f_2(x, y, z, u, v, \omega) = 0$$
$$f_3(x, y, z, u, v, \omega) = 0$$

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Then the formulae to find partial derivative by using Jacobians is,

e.g ① To find $\frac{\partial u}{\partial x}$

Step 1

For $\frac{\partial u}{\partial x}$ → Numerator Variable Group (NVG) ⇒ u, v, ...

$$\therefore \frac{\partial u}{\partial x} = - \frac{\frac{\partial(f_1, f_2, f_3)}{NVG}}{\frac{\partial(f_1, f_2, f_3)}{NVG}}$$

$$= - \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, \omega)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, \omega)}}$$

Step 2: replace u by x in Numerator

$$= - \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, v, \omega)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, \omega)}}$$

e.g ② To find $\frac{\partial y}{\partial \omega}$

Step 1

For $\frac{\partial y}{\partial \omega}$ → NVG x, y, z

$$\therefore \frac{\partial y}{\partial \omega} = - \frac{\frac{\partial(f_1, f_2, f_3)}{NVG}}{\frac{\partial(f_1, f_2, f_3)}{NVG}} = - \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}$$

Step 2

Replace y → ω in Numerator

$$\frac{\partial y}{\partial w} = \frac{-\frac{\partial(f_1, f_2, f_3)}{\partial(x, w, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}$$

Q(1) If $u^2 + xv^2 = x + y$, $v^2 + yu^2 = x - y$, find $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$

→ from the given relations

$$f_1 = u^2 + xv^2 - x - y$$

$$f_2 = v^2 + yu^2 - x + y$$

part 1: To find $\frac{\partial u}{\partial x}$

we have $$\frac{\partial u}{\partial x} = \frac{-\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

$$\frac{\partial u}{\partial x} = -\frac{N}{D} \quad \text{——— (1)}$$

where
$$N = \frac{\partial(f_1, f_2)}{\partial(x, v)}$$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -1+v^2 & 2xv \\ -1 & 2v \end{vmatrix}$$

$$N = (-1+v^2)2v + 2xv$$

$$N = 2v^3 - 2v + 2xv$$

and $$D = \frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2xv \\ 2yu & 2v \end{vmatrix}$$

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$$D = 4uv - 4xy \cdot uv$$

but from eqn ①

$$\frac{\partial u}{\partial x} = -\frac{N}{D} = -\frac{(2v^3 - 2v + 2xv)}{4uv - 4xy \cdot uv}$$

$$= -\frac{2v(v^2 - 1 + x)}{4uv(1 - xy)}$$

$$\frac{\partial u}{\partial x} = \frac{1 - x - v^2}{2u(1 - xy)}$$

part 2: To find $\frac{\partial v}{\partial y}$

we have $$\frac{\partial v}{\partial y} = \frac{-\frac{\partial(f_1, f_2)}{\partial(u, y)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}} \quad \text{--- ②}$$

$$\frac{\partial v}{\partial y} = \frac{-N}{D} \quad \text{--- ②}$$

where $$N = \frac{\partial(f_1, f_2)}{\partial(u, y)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$N = \begin{vmatrix} 2u & -1 \\ 2uy & 1 \end{vmatrix}$$

$$N = 2u + 2uy$$

$$N = 2u(1 + y)$$

$$D = \frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix}$$

$D = 4uv - 4uvxy.$

$D = 4uv(1-xy)$

But from eqⁿ ②

$$\frac{\partial v}{\partial y} = -\frac{N}{D} = \frac{2u(1+y)}{4uv(1-xy)} = -\frac{(1+y)}{2v(1-xy)}$$

Ex If $u+v = x^2+y^2$ and $u-v = x+2y$ find $\left(\frac{\partial u}{\partial x}\right)_y$.

$\Rightarrow$ $f_1 = u+v-x^2-y^2$

$f_2 = u-v-x-2y$

and $$\left(\frac{\partial u}{\partial x}\right)_y = \frac{-\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} = -\frac{N}{D} \quad \text{①}$$

where $$N = \frac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -2x & 1 \\ -1 & -1 \end{vmatrix}$$

$N = 2x+1$

and $$D = \frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2$$

$\boxed{D = -2}$

But from eqⁿ ①

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{N}{D} = -\frac{(2x+1)}{-2} = \frac{2x+1}{2}.$$

Ex:

If $x = \cos\theta - r\cdot\sin\theta$; $y = \sin\theta + r\cdot\cos\theta$

then prove that $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\Rightarrow$

Let $f_1 = x - \cos\theta + r\cdot\sin\theta$

$f_2 = y - \sin\theta - r\cdot\cos\theta$

$$\frac{\partial r}{\partial x} = \frac{-\frac{\partial(f_1, f_2)}{\partial(x, \theta)}}{\frac{\partial(f_1, f_2)}{\partial(r, \theta)}}$$

$$\therefore \frac{\partial r}{\partial x} = -\frac{N}{D} \quad \text{——— (1)}$$

where $N = \frac{\partial(f_1, f_2)}{\partial(x, \theta)}$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \theta} \end{vmatrix}$$

$$N = \begin{vmatrix} 1 & \sin\theta + r\cdot\cos\theta \\ 0 & -\cos\theta + r\cdot\sin\theta \end{vmatrix}$$

$$N = -\cos\theta + r\cdot\sin\theta$$

and

$$D = \frac{\partial(f_1, f_2)}{\partial(r, \theta)}$$

$$D = \begin{vmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta & \sin\theta + r\cdot\cos\theta \\ -\cos\theta & -\cos\theta + r\cdot\sin\theta \end{vmatrix}$$

$D = -\sin\theta \cdot \cos\theta + r \cdot \sin^2\theta + \sin\theta \cdot \cos\theta + r\cos^2\theta$

$D = r(\sin^2\theta + \cos^2\theta)$

$\boxed{D = r.}$

From eq^n ①

$$\frac{\partial r}{\partial x} = -\frac{N}{D} = -\frac{(-\cos\theta + r\sin\theta)}{r}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{\cos\theta - r\sin\theta}{r}}$$

* Functional Dependance

If $u = f_1(x,y)$, $v = f_2(x,y)$ and it is possible to write v as a function of u or u as function of v

i.e $v = F(u)$ or $u = f(v)$

then u and v are said to be functionally dependant otherwise independant.

Note

u and v are functionally dependant if $J = 0$.

Ex Examine for function Dependance

$u = \frac{x-y}{x+y}$ and $v = \frac{x+y}{x}$, if functionally dependant find relation between them.

$\Rightarrow$ Let $u = \frac{x-y}{x+y}$

Diff. w.r.to x partially

$$\frac{\partial u}{\partial x} = \frac{(x+y)(1-0) - (x-y)(1+0)}{(x+y)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x+y-x+y}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

and $u = \dfrac{x-y}{x+y}$

Diff. w. r. to y partially,

$$\frac{\partial u}{\partial y} = \frac{(x+y)(0-1) - (x-y)(0+1)}{(x+y)^2}$$

$$= \frac{-x-y-x+y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2x}{(x+y)^2}$$

Now $v = \dfrac{x+y}{x} = 1 + \dfrac{y}{x}$

diff. w. r. to x partially,

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2}$$

and $v = 1 + \dfrac{y}{x}$

Diff. w. r. to y partially

$$\frac{\partial v}{\partial y} = \frac{1}{x}$$

Now $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix}$

$$J = \frac{1}{(x+y)^2} \begin{vmatrix} 2y & -2x \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$J = \frac{1}{(x+y)^2} \left| \frac{2y}{x} - \frac{2xy}{x^2} \right|$$

$$J = \frac{1}{(x+y)^2} \left[\frac{2y}{x} - \frac{2y}{x} \right]$$

$$\boxed{J = 0}$$

$\therefore$ U and V are functionally dependant.

Relation:

~~let~~ $v(u+1) = \frac{x+y}{x} \left(\frac{x-y}{x+y} + 1 \right)$

$$v(u+1) = \frac{x+y}{x} \left(\frac{x - \cancel{y} + x + \cancel{y}}{x+y} \right)$$

$$v(u+1) = \frac{\cancel{x+y}}{x} \cdot \frac{2x}{\cancel{x+y}} = 2$$

* Examine $u = x+y+z$, $V = x-y+z$, $w = x^2+y^2+z^2+2xz$ for functional dependance and find a relation bet^n them if functionally dependant

$\Rightarrow$ $u = x+y+z$, $V = x-y+z$, $w = x^2+y^2+z^2+2xz$

$$\therefore J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$J = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2x+2z & 2y & 2z+2x \end{vmatrix}$$

$C_1 = C_1 - C_2$ and $C_2 = C_2 - C_3$

$$J = \begin{vmatrix} 0 & 0 & 1 \\ 2 & -2 & 1 \\ 2x-2y+2z & 2y-2z-2x & 2z+2x \end{vmatrix}$$

$$J = (4y - 4z - 4x) - (-4x + 4y - 4z)$$

$$J = \cancel{4y} - \cancel{4z} - \cancel{4x} + \cancel{4x} - \cancel{4y} + \cancel{4z}$$

$$J = 0$$

$\therefore$ u, v, w are functionally dependant

Relation:

let $u^2 + v^2 = (x+y+z)^2 + (x-y+z)^2$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$
$$+ x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$$
$$= 2x^2 + 2y^2 + 2z^2 + 4zx$$

$$u^2 + v^2 = 2(x^2 + y^2 + z^2 + 2zx)$$

$$\therefore \boxed{u^2 + v^2 = 2w}$$

Errors and approximation:

Let $f(x,y)$ be a continuous function of x and y. If δx and δy be the increments of x and y, then the new value of $f(x,y)$ will be $f(x+\delta x, y+\delta y)$. Hence change in $f(x,y)$ is given by

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y) \quad \text{--- (1)}$$

Expanding $f(x+\delta x, y+\delta y)$ by Taylor's Theorem and supposing δx, δy to be so small that their product squares and higher power can be neglected, we have

$$\delta f = \frac{\partial f}{\partial x}\cdot\delta x + \frac{\partial f}{\partial y}\cdot\delta y \quad \text{approximately ---}$$

$$df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy \quad \text{--- (2)}$$

similarly,

If f be a function of variables x, y, z.

$$df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy + \frac{\partial f}{\partial z}\cdot dz + \cdots \text{ approximately} \quad \text{--- (3)}$$

Note 1:

Actually in eqn (2) dx, dy may be taken as actual error (increments) in x and y respectively, while df is approximate error in

Note 2:

If $z = f(x,y)$ we may have $dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy$

Here dx, dy, dz are know as actual error in $x, y, \& z$ respectively.

$\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}$ as relative error in x, y, z respectively and $\frac{100 \cdot dx}{x}$, $\frac{100 \cdot dy}{y}$, $\frac{100 \cdot dz}{z}$ are know as percentage error in x, y, z respectively.

Ex1 find the percentage error in the area of ellips when an error of 1% is made in measuring it's major and minor axis.

$\Rightarrow$ If A is area and $2a$ and $2b$ are the major and minor axes of the ellips having eqn $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then.

Then $A = \pi ab$

$$\log A = \log \pi + \log a + \log b$$

$$\frac{dA}{A} = 0 + \frac{da}{a} + \frac{db}{b}$$

$$\frac{100 \cdot dA}{A} = \frac{100\, da}{a} + \frac{100\, db}{b}$$

Given that % error $\frac{100\, da}{a}$ & $\frac{100\, db}{b}$ each are equal to 1

hence

$$\frac{100 \cdot dA}{A} = 1 + 1 = 2$$

$\therefore$ Percentage error in the Area $A = 2\%$

Maxima and minima of a function of two Independant variable.

(1) maxima and minimum value of a function of single independant variable

Step (1) find $f'(x)$ and equate it to zero then find it root's, suppose root are $a_1, a_2 - - -$

Step 2 find $f''(x)$

~~Step 3 find value of $f''(x)$ at~~ point $a_1, a_2, - - -$

Step 4. If $f''(a_1) < 0$, we have maximum at $x = a_1$

If $f''(a_1) > 0$, we have minimum at $x = a_1$

Step 5 If $f''(a_1) = 0$, we must be find $f'''(x)$ and substitute in it a_1.

If $f'''(a_1) \neq 0$, there is neithere maxima ~~and~~ nor a minimum at a_1

But if $f'''(a_1) = 0$, we must be find $f^{iv}(x)$ and substitute it in a_1.

If $f^{iv}(a_1) < 0$ we get maxima at $x = a_1$.

$f^{iv}(a_1) > 0$ we get minima at $x = a_1$

If it is zero then we find $f^{v}(x)$ and so on

Defination:

A function $z = f(x, y)$ of two independant variables is said to be have maximum for $x = a$ and $y = b$ if $f(a, b)$ is greater than the value of the function for every othere pair of value of x, y in small nbh of a,b i.e if

$$f(a,b) > f(a+h, b+k)$$

for all sufficent small independant value of h and k

similarly for minima

$$f(a,b) < f(a+h, b+k) \quad \forall \text{ value of } h \text{ \& } k$$

Note-1

A maximum and minimum value of function is called extreme value

Note2: A function $f(x,y)$ is said to be stationary at (a,b) or $f(a,b)$ is said to stationary value of the function $f(x,y)$ if $f_x(a,b)=0$ & $f_y(a,b)=0$

Thus every extreme value is stationary value but converse may not be true.

Remark:

$z = f(x,y)$ is regarded as surface, then a point of maximum the ordinates to the surface is the largest in the ~~nbh~~ neighbourhood. It is corresponding to a hill-top from which the surface ~~acesen~~ ascends in every direction.

*** Working Rule of Determining the maxima & minima of a function $z = f(x,y)$**

Step1:

find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate them to zero

i.e $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Step2: solve these simultaneous equation to find different value of x and y
let root are (a_1, b_1) ; (a_2, b_2)

Step 3
calculate
$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y} \quad t = \frac{\partial^2 f}{\partial y^2}$$
for all root (a_1, b_1), (a_2, b_2) and so on.

Step 4.

$(rt-s^2)_{(a_1,b_1)}$	$r(a_1,b_1)$	Remark	Max / min value
$rt-s^2>0$	$r<0$	$f(x,y)$ has max value	$f(a_1,b_1) = f_{max}$
$rt-s^2>0$	$r>0$	$f(x,y)$ has min value	$f(a_1,b_1) = f_{min}$
$rt-s^2<0$	—	f has neither maximum nor minimum value	$f(a_1,b_1)$ is a saddle point
$rt-s^2=0$	—	No conclusion	Further investigation is necessary.

Note:
The point which has neither maximum nor minimum value is called as saddle point.

classmate
Date
Page

Ex. Discuss the maxima and minima of

$$f(x,y) = x^2 + y^2 + 6x + 12$$

step 1:
let

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$2x + 6 = 0$ $\qquad$ $2y = 0$

$\boxed{x = -3}$ $\qquad$ $\boxed{y = 0}$

Step 2: root are $(-3, 0)$

Step 3: To find r, s, t

Now $f = x^2 + y^2 + 6x + 12$

Diff. w.r.to x partially,

$$\frac{\partial f}{\partial x} = 2x + 6$$

Diff. w.r.to x partially

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$\Rightarrow \boxed{r = 2}$

Again $f = x^2 + y^2 + 6x + 12$

Diff. w.r.to y partially

$$\frac{\partial f}{\partial y} = 2y$$

Diff. w.r.to x partially

$$\frac{\partial^2 f}{\partial x \cdot \partial y} = 0$$

$\Rightarrow \boxed{s = 0}$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\Rightarrow \boxed{t=2}$$

step 4

$r = 2$

$(r)_{(-3,0)} = 2$

$s = 0 \quad \Rightarrow \quad s_{(-3,0)} = 0$

$t = 2 \quad \Rightarrow \quad t_{(-3,0)} = 2$

$\therefore\ rt - s^2 = (2)(2) - 0^2 = 4$

$\Rightarrow rt - s^2 > 0$ & $r = 2$ i.e $r > 0$

as $rt - s^2 > 0$ & $r > 0$ function has a minimum value at $(-3,0)$

$\therefore$ minimum value is

$f = x^2 + y^2 + 6x + 12$

put $x = -3,\ y = 0$

$f_{min} = (-3)^2 + 0^2 + (6)(-3) + 12$

$f_{min} = 9 - 18 + 12$

$f_{min} = -9 + 12$

$$\boxed{f_{min} = 3}$$

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* Lagrange's method of undetermined multipliers

we use these method to find stationary values or extreme values of a function of several variable.

Note: drawback of this method is that we cannot determine the nature of the stationary value i.e whether they are maximum or minimum.

Method:

Type I: To find stationary value of $f(x,y,z)$ under one condⁿ $\phi(x,y,z)=0$

let $u=f(x,y,z)$ be given fun —— (1)

under a condⁿ $\phi(x,y,z)=0$ —— (2)

construct a function $F=u+\lambda\phi$

where λ is non-zero constant which is called Lagrange's undetermined multiplier.

from the eq

$\frac{\partial F}{\partial x}=0$ —— (3)

$\frac{\partial F}{\partial y}=0$ —— (4)

$\frac{\partial F}{\partial z}=0$ —— (5)

Step II

we eliminate x, y, z and λ using eq (1) & (5) we get equation in terms of u

The root of eq gives the stationary value

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Ex① prove that the stationary value of $x^m \cdot y^n \cdot z^p$ under the condⁿ $x+y+z=a$ is $m^m \cdot n^n \cdot p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}$

let $u = x^m y^n \cdot z^p = f(x,y,z)$ —— ①

under the condⁿ

$$\phi(x,y,z) = 0$$

$$\phi = x+y+z-a = 0 \quad \text{—— ②}$$

construct the fun.

$$F = u + \lambda\phi$$

$$F = x^m \cdot y^n \cdot z^p + \lambda[x+y+z-a]$$

from the eqⁿ.

$$\frac{\partial F}{\partial x} = 0$$

$$m x^{m-1} \cdot y^n z^p + \lambda = 0 \quad \text{—— ③}$$

$$\frac{\partial F}{\partial y} = n \cdot x^m \cdot y^{n-1} \cdot z^p + \lambda = 0 \quad \text{—— ④}$$

$$\frac{\partial F}{\partial z} = 0 \quad \therefore \quad p x^m \cdot y^n \cdot z^{p-1} + \lambda = 0 \quad \text{—— ⑤}$$

stepII:

we eliminate x, y, z, λ using eqⁿ ① to ⑤

from eqⁿ ③, ④ & ⑤

$$m \cdot x^{m-1} y^n z^p = n \cdot x^m \cdot y^{n-1} \cdot z^p = p x^m \cdot y^n \cdot z^{p-1}$$

dividing each by $x^m \cdot y^n \cdot z^p$

$$\frac{x}{m} = \frac{y}{n} = \frac{z}{p} = k$$

$$\therefore x = km, \quad y = nk, \quad z = pk \quad \text{—— ⑥}$$

substitute in ② we have

$$km + kn + kp = a$$

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classmate
Date
Page

$$k = \frac{a}{m+n+p}$$

$\therefore$ from eqn (6)

$$x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p}$$

substituting in (1), the stationary value ~~of u is~~

$$\left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p = m^m \cdot n^p \cdot p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}$$

Ex (2)

As the dimension of triangle ABC are varied, show that the maximum value of $\cos A \cdot \cos B \cdot \cos C$ is obtained when the triangle is equilateral.

$\Rightarrow$ Step 1: $u = f(A, B, C) = \cos A \cdot \cos B \cdot \cos C$ —— (1)

$\phi = A + B + C - \pi = 0$ —— (2)

construct the fun

$$F = u + \lambda \phi$$

$$F = \cos A \cdot \cos B \cdot \cos C + \lambda (A + B + C - \pi)$$

From the eqn

$\frac{\partial F}{\partial A} = 0 \quad \therefore \quad -\sin A \cdot \cos B \cdot \cos C + \lambda = 0$ —— (3)

$\frac{\partial F}{\partial B} = 0 \quad \therefore \quad -\cos A \cdot \sin B \cdot \cos C + \lambda = 0$ —— (4)

$\frac{\partial F}{\partial C} = 0 \quad \therefore \quad -\cos A \cdot \cos B \cdot \sin C + \lambda = 0$ —— (5)

step III we eliminates A, B, C & λ using eq (1) to (5) from eq^n (3), (4), (5)

$\sin A \cdot \cos B \cdot \cos C = \cos A \cdot \sin B \cdot \cos C = \cos A \cdot \cos B \cdot \sin C$

dividing by $\cos A \cdot \cos B \cdot \cos C$

we get

$$\tan A = \tan B = \tan C$$

$\Rightarrow A = B = C$

$\Rightarrow \Delta ABC$ is equilateral.

Ex(3) find the point on the surface $z^2 = xy + 1$ nearest to the origin, by using lagrange method.

Step 1: let $P(x, y, z)$ be any point on the surface

$z^2 = xy + 1$

$d(O, P) = \sqrt{x^2 + y^2 + z^2}$

$[d(O,P)]^2 = x^2 + y^2 + z^2$

let $u = f(x, y, z) = x^2 + y^2 + z^2$ —— (1)

under cond^n $\phi = z^2 - xy - 1 = 0$ —— (2)

construct the fun $F = u + \lambda \phi$

$F = x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1)$

$\frac{\partial F}{\partial x} = 0 \quad \therefore \quad 2x - \lambda y = 0$ —— (3)

$\frac{\partial F}{\partial y} = 0 \quad \therefore \quad 2y - \lambda x = 0$ —— (4)

$\frac{\partial F}{\partial z} = 0 \quad \therefore \quad 2z + 2\lambda z = 0$ —— (5)

Step II we eliminate x, y, z and λ using ① to ⑤

from eqⁿ ⑤

$$z(1+\lambda) = 0$$

$$\boxed{\lambda = -1}$$

put in eqⁿ ③ & ④

$$2x + y = 0$$

$$2y + x = 0$$

$\therefore \Rightarrow x = 0, \ y = 0$

substituting in eqⁿ ②

$$z^2 = 1$$

$$\boxed{z = \pm 1}$$

$\therefore (0, 0, \pm 1)$ are the nearest point on the surface from the origin