Eigen value and Eigen vector.

consider the eqⁿ

$$AX = \lambda X \quad \Rightarrow AX = I\lambda X$$

where λ is called eigenvalue

$$AX - \lambda I X = 0$$

$$X(A - \lambda I) = 0$$

where $|A - \lambda I|$ is characteristics determinant, and $[A - \lambda I]$ is characteristics matrix of A

Note

① $a_0\lambda^n + a_1\lambda^{n-2} + a_2\lambda^{n-2} + \cdots + a_n = 0$ is called characteristics eqⁿ of A

② Degree of characteristics eqⁿ of matrix is equal to order of that matrix

③ For matrix of order 3×3, highest power of λ is 3.

* Properties of eigen values.

① Trace of A: The sum of the entries on the main digonal of an $n\times n$ matrix A is called the trace of A, thus

Trace of $A = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

② The sum of eigen value of a matrix is the sum of the elements of the principle digonal

i.e. Trace $A = a_{11} + a_{22} + \cdots + a_{nn}$

$= \lambda_1 + \lambda_2 + \lambda_3$.

③ The eigenvalue of an upper and lower triangular matrix are the elements on its main digonal

(4) The product of the eigen values of a matrix equals the determinant of the matrix i.e

$$\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n = |A|.$$

(5) If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen value of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_n}$ are eigen value of A^{-1}

(6) The matrix kA has the eigen values $k\lambda_1, k\lambda_2, \cdots, k\lambda_n$.

(7) The matrix A^m (m is non-negative integer) has the eigen values $\lambda_1^m, \lambda_2^m, \cdots, \lambda_n^m$.

(8) The matrix $(A-KI)$ has the eigen values: $\lambda_1-k, \lambda_2-k, \cdots, \lambda_n-k$

(9) The eigenvalue of symmetric matrix are real.

(10) The eigenvalue of A and A' are the same.

10.3 Properties of Eigen vectors.

(1) If x is an eigen vector of a matrix A corresponding to an eigen value λ, so is kx with any $k \neq 0$

Thus, the eigen vector corresponding to an eigen value is not unique.

Proof:

$$Ax = \lambda x$$
$$k(Ax) = k(\lambda x)$$
$$A(kx) = \lambda(kx)$$

$\Rightarrow$ kx is also eigen vector of A.

(2) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen value of an $n \times n$ matrix then corresponding eigen vector $X_1, X_2, \ldots, X_n$ form a linearly independant set
i.e $x_1, x_2, \ldots, x_n$ are linearly independant

(3) corresponding to n distinct eigen value, we get n independant eigen vectors. But when two or more eigen values are equals, it may or may not be possible to get L.I eigen vector corresponding to the repeated root.

(4) An $n \times n$ matrix may have L.I eigen vectors or it may have fewer than n

(5) Eigen vector of square matrix cannot correspond to two distinct eigen value.

(6) orthogonal eigen vectors:- Two eigen vectors x_1 and x_2 are said to be orthogonal if $x_1 \cdot x_2 = 0$

e.g $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $x_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

$$\therefore x_1' \cdot x_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$= 1(3) + 2(0) + 3(-1)$$

$$= 0$$

$\therefore x_1, x_2$ are orthogonal.

(7) Eigen vector of a symmetric matrix corresponding to diff eigen value are orthogonal.

Note:- Eigen value may be zero, an eigen vector may not be zero vector.

10.4 short cut method: Method of finding Eigen value of A.

① Let A be a matrix of order 3×3 say

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$ then write down directly the eigen value of A as $\lambda = 4, 5, 6$

② If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ then eigen value are $\lambda = 1, 3, 5$

③ If $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ then eigen values are $\lambda = 1, 2, 3$

④ For a square matrix of order 2×2 say

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the characteristic eq of A is

$|A - \lambda I| = 0$

i.e $\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = 0$

Now expansion of this determinant is written by using short-cut method as fallow.

Important step:
For 2×2 matrix, characteristic equation is given by

$\lambda^2 - S_1\lambda + |A| = 0$

where S_1 = sum of minors of order 1 along main

Digonal of A

$S_1 = a_{11} + a_{22}$

⑤ For a square matrix of order 3×3 say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic eqⁿ of A is $|A - \lambda I| = 0$

i.e $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

Important step:

for 3×3 matrix, characteristic equation is given by

$$\lambda^3 - S_1\lambda^2 - S_2\lambda - |A| = 0$$

where

S_1 = sum of minors of order one along main digonal of A

$S_1 = a_{11} + a_{22} + a_{33}$

S_2 = sum of minor of order two of the digonal element of A

S_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$$S_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Important Note:

1) For $A_{2\times2}$ matrix, we must have $\lambda_1+\lambda_2 = a_{11}+a_{22}$ and $\lambda_1 \cdot \lambda_2 = |A|$

2) For $A_{3\times3}$ matrix, we must have $\lambda_1+\lambda_2+\lambda_3 = a_{11}+a_{22}+a_{33}$ and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$

Cramer's Rules-

consider

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

Thus by using Cramer's Rule, we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

* find Eigen value and Eigen vector of the following matrix

$$A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$$

Sol^n

Eigen values:

characteristic eq^n for 2x2 matrix is given by

$$\lambda^2 - S_1\lambda + |A| = 0 \quad \text{—— (1)}$$

where S_1 = sum of digonal elements

$$S_1 = 14+(-1)$$
$$\boxed{S_1 = 13}$$

& $|A| = \begin{vmatrix} 14 & -10 \\ 5 & -1 \end{vmatrix} = -14+50 = 36$

∴ From eq^n (1)

$\lambda^2 - 13\lambda + 36 = 0$

$\lambda = 4, 9$ ... Eigen values.

Part 2 : Eigen vectors:

To find eigen vectors consider the matrix eqⁿ

$$(A - \lambda I)x = 0$$

$$\therefore \begin{bmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{---} (2)$$

Now eigen vector $\overline{X_1}$ for eigen valle $\lambda = 4$

put $\lambda = 4$ in eqⁿ (2)

$$\therefore \begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By R_1 $\quad 10x - 10y = 0$

~~put~~ $\Rightarrow x = y$

put $y = t$

$\therefore x = t$

$$\therefore \overline{X_1} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \overline{X_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now Eigen vector $\overline{X_2}$ for eigen value $\lambda = 9$

put $\lambda = 9$ in eqⁿ (2)

$$\therefore \begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By R_1 $5x - 10y = 0$

put $y = t$

$5x = 10t$

$x = 2t$

$$\therefore \bar{X}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\bar{X}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Ex 2) Find Eigen value and Eigen vector corresponding to highest eigen value of the following matrices.

$$A = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ characteristic eqⁿ for 3X3 matrix is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad \text{--- (1)}$$

where S_1 = sum of digonal elements

$S_1 = -2 + 4 + 1$

$\boxed{S_1 = 3}$

S_2 = sum of minor of digonal elements

$$= \begin{vmatrix} 4 & 4 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -12 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -8 \\ 1 & 4 \end{vmatrix}$$

$= 4 - 2 + 0$

$\boxed{S_2 = 2}$

and

$$|A| = \begin{vmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 4 & 4 \\ 0 & 1 \end{vmatrix} + 8\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} + (-12)\begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix}$$

$|A| = -8 + 8 + 0 = 0$

$\therefore$ from eq^n (1)

$$\lambda^3 - 3\lambda^2 + 2\lambda - 0 = 0$$

$$\lambda = 0, 1, 2$$

Eigen vector for highest eigen value:

To find Eigen vector, consider the matrix eq^n

$$(A - \lambda I)X = 0 \quad \text{--- (2)}$$

$$\therefore \begin{bmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$

$$\therefore \begin{bmatrix} -4 & -8 & -12 \\ 1 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -4x - 8y - 12z = 0$$

$$x + 2y + 4z = 0$$

$$0x + 0y - z = 0$$

$\therefore$ By cramer's Rule

$$\frac{x}{\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{-y}{-1} = \frac{z}{0} = t$$

$$x = -2t$$

$$y = t$$

$$z = 0$$

$$\therefore \overline{x_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \overline{x_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Homework

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -1 & 3 \end{bmatrix} \qquad \overline{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{x_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \overline{x_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Ans $\lambda = 1, 2, 4$

10·6 Type 2: A is Non-symmetric. Eigen value are repeated

Find the eigen value and eigen vector for the following matrices'

$$A = \begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & -2 \end{bmatrix}.$$

⟹ characteristic eqⁿ for 3X3 matrix is given by

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{--- (1)}$$

where s_1 = sum of digonal elements'

$$= -17 + 19 - 2$$

$$s_1 = 0$$

s_2 = sum of ~~m~~ digonal minor $= \begin{vmatrix} 19 & -6 \\ 9 & -2 \end{vmatrix} + \begin{vmatrix} -17 & -6 \\ -9 & -2 \end{vmatrix} + \begin{vmatrix} -17 & 18 \\ -18 & 19 \end{vmatrix}$

$$\boxed{s_2 = -3}$$

$$|A| = -2$$

$$\therefore \lambda^3 - 0\lambda^2 + (-3)\lambda - (-2) = 0$$

$$\lambda^3 - 3\lambda + 2 = 0$$

$$\lambda = -2, 1 \qquad \text{(on calculater)}$$

Procedure to find 3^{rd} eigen value

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$$

& $\lambda_1 + \lambda_2 + \lambda_3 = s_1$

$$(-2)(1)\lambda_3 = -2$$

$$\boxed{\lambda_3 = 1}$$

$\therefore \lambda = -2, 1, 1$

Eigen vector for $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -17-\lambda & 18 & -6 \\ -18 & 19-\lambda & -6 \\ -9 & 9 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (2)}$$

put $\lambda = 1$ in above eq^n

$$\begin{bmatrix} -18 & 18 & 6 \\ -18 & 18 & -6 \\ -9 & 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

above system will effectively gives a single equation.

$$-9x + 9y - 3z = 0$$

$$-3x - 3y - z = 0$$

put $z = 0$, $y = t$

$$-3x + 3t = 0$$

$$-3x = -3t$$

$$\boxed{x = t}$$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overline{x_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For repeated $\lambda = 1$

put $y = 0$, $z = t$

$\therefore -3x + 0 - t = 0$

$-3x = t$

$x = -t/3$

$$\therefore \overline{x_3} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t/3 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\overline{x_3} = 3t \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$\therefore \overline{x_3} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

simillorly for $\lambda = 2$

$$\overline{x_1} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

10·11 Cayley-Hamilton Theorem

Statment :- Every square matrix satiesfy it's own characteristic eqn

i.e we can put $\lambda = A$ in characteristic eqn

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{for } 3\times3 \text{ matrix}$$

and

$$\lambda^2 - s_1\lambda + |A| = 0 \quad \text{for } 2\times2 \text{ matrix}$$

Ex $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$S_1 = 2+1 = 3$

$|A| = 2 - 12 = -10$

$\lambda^2 - 3\lambda - 10 = 0$

$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$

$\lambda(\lambda-5) + 2(\lambda-5) = 0$

$(\lambda+2)(\lambda-5) = 0$

$\lambda = -2, 5$

Here put $\lambda = A$

$$A^2 - 3A - 10\cdot I = 0$$

L.H.S $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16 & 9 \\ 12 & 13 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 12 & 3 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 =$ R.H.S.

chapter 11

Digonalization and quadratic Form.

1) Digonalization of matrix
The process of converting any square matrix A into a diagonal matrix with digonal elements as eigen value of matrix A, with the help of modal matrix (P) is called as diagonalization of a matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[\text{help of } P]{\text{With the}} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\Downarrow$ Given matrix A

$\Downarrow$ D matrix

where P consists of eigen vector as columns.

Note: Digonalization of a matrix is quite useful for obtaining power of a matrix

* Working Procedure for Digonalization

1. find the eigen value of the square matrix A :
2. find the corresponding eigen vectors and write the modal matrix P
3. verify the diagonal matrix 'D' by using $D = P^{-1}AP$.
4. obtain A^n from $A^n = PD^n \cdot P^{-1}$

where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

* conditions for Diagonalization of a matrix

For any matrix A of order $n \times n$

1. If it has non-repeated eigen values and independant eigen vectors then it is diagonalizable

2. If it has repeated eigen value and independant eigen vectors then it is digonalizable

3. If it has repeated eigen value and dependant eigen vectors then it is non-digonalizable

* some useful Results:

1. $D = P^{-1} A P$
2. $A = P D P^{-1}$
3. $A^n = P D^n . P^{-1}$
4. we can use different method to find P^{-1} as
   a. Adjoint method
   b. method of reduction.

Note: sufficent $cond^n$ for Diagonalizability.

If a matrix A of order $n \times n$ has n distinct (i.e non repeated) Eigen value, that is, a guarrantee that A is digonalisable.

But matrix A must be have independant eigen vector.

Ex Diagonalize the following matrix

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$$ if possible

characteristics eq^n for 2x2 matrix is given by

$\lambda^2 - S_1\lambda + |A| = 0$ —— ①

S_1 = sum of diagonal element $= a_{11} + a_{22} = 3+7 = 10$

and $|A| = \begin{vmatrix} 3 & 4 \\ -1 & 7 \end{vmatrix} = 21-(-4) = 25$

substitute value in eq^n ①

we get $\lambda^2 - 10\lambda + 25 = 0$

solving above quadratic eq^n, we get

$\lambda_1 = 5$ or $\lambda_2 = 5$

The eigen value are repeated.

(since, eigen value are repeated, then matrix A may or may not be diagonalizable)

step2: let us to calculate eigen vector for the matrix A

consider the matrix eq^n

$[A - \lambda I]X = 0$

$\begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— ②

finding eigen vector $\overline{x_1}$ for eigen value of $\lambda = 5$

put $\lambda = 5$ in eq^n ②

$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— ③

by R_1

$-2x + 4y = 0$

put $y = t$

$-2x = -4t$

$\boxed{x = 2t}$

$\therefore \overline{x_1} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \overline{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

similarly $\overline{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Finding rank from eq^n (3)

r = No. of row − No. of zero row

$= 2 - 1$

$= 1$

No. of unknow = 2

$n - r = 2 - 1 = 1$

which represent same eigen vector for repeated eigenvalue $\lambda = 5$.

$\therefore$ eigen vector are linearly dependant.

Hence, Given matrix A is not diagonalizable

Ex:2 Find a modal matrix P which diagonalise the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. verify that $P^{-1}AP = D$, where D is a Diagonal matrix. Hence find A^6.

$\Rightarrow$

$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

characteristic eq^n for 2x2 matrix

$\lambda^2 - S_1\lambda + |A| = 0$

$S_1 = 4 + 3 = 7$

$|A| = 12 - 2 = 10$

$\therefore \lambda^2 - 7\lambda + 10 = 0$

$\therefore \lambda = 2, 5$.

$\therefore$ Digonal matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

step2. eigen vector and modal matrix

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{——— (2) ———}$$

(i) eigen vector $\overline{x_1}$ for eigen value $\lambda = 2$

put $\lambda = 2$ in eq^n (2)

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\cancel{2x+y}=0$ $\quad R_2 \to R_2 - R_1$

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x + y = 0$

put $y = t$

$x = -t/2$

$$\therefore \overline{x_1} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = 2t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore \overline{x_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(ii) eigen vector $\overline{x_2}$ for eigen value $\lambda = 5$

put $\lambda = 5$ in eq^n (2)

$$\therefore \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -x+y=0$

let $y=t$

$-x=-t$

$\boxed{x=t}$

$\therefore \bar{x_2}=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}t\\t\end{bmatrix}=t\begin{bmatrix}1\\1\end{bmatrix}$

$\therefore \boxed{\bar{x_2}=\begin{bmatrix}1\\1\end{bmatrix}}$

Now arrange these eigen vector as column of a matrix, we get modal matrix P

$\therefore P=\begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix}$

Step 3: verify $P^{-1}AP=D$

To find P^{-1} (By adjoint method)

(i) minor of matrix

$m=\begin{bmatrix}1 & 2\\1 & -1\end{bmatrix}$

(ii) co-factor of matrix

$C=\begin{bmatrix}1 & -2\\-1 & -1\end{bmatrix}$

(iii) adjoint matrix

$\text{adj}P=C^T$

$\text{adj}P=\begin{bmatrix}1 & -1\\-2 & -1\end{bmatrix}$

(iv) Determinant

$|P|=\begin{vmatrix}-1 & 1\\2 & 1\end{vmatrix}$

$|P|=-1-2$

$\boxed{|P|=-3}$

But $P^{-1}=\frac{1}{|P|}\cdot\text{adj}P \quad =-\frac{1}{3}\begin{bmatrix}1 & -1\\-2 & -1\end{bmatrix}$

Let $P^{-1}AP = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$

$= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -4+2 & 4+1 \\ -2+6 & 2+3 \end{bmatrix}$

$= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 4 & 5 \end{bmatrix}$

$P^{-1}AP = -\frac{1}{3}\begin{bmatrix} -2-4 & 5-5 \\ 4-4 & -10-5 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$

$\therefore \boxed{P^{-1}AP = D}$

step 4: To find A^6

we have $D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

$D^6 = \begin{bmatrix} 2^6 & 0 \\ 0 & 5^6 \end{bmatrix}$

$D^6 = \begin{bmatrix} 64 & 0 \\ 0 & 15625 \end{bmatrix}$

We have

$A^n = PD^n \cdot P^{-1}$

$A^6 = PD^6P^{-1}$

put $n=6$

$A^6 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 64 & 0 \\ 0 & 15625 \end{bmatrix} -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$

$\therefore A^6 = -\frac{1}{3}\begin{bmatrix} -64+0 & 0+15625 \\ 128+0 & 0+15625 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$

$A^6 = \begin{bmatrix} 10438 & 5187 \\ 10374 & 5251 \end{bmatrix}$

11·2 Quadratic Form.

A homogeneous expression of second degree as show in eqⁿ ① is know as quadratic form

$$Q = ax_1^2 + bx_2^2 + cx_3^2 + 2f\cdot x_2\cdot x_3 + 2g\,x_3\cdot x_1 + 2h\,x_1 x_2$$

Some example

(i) $x^2 - y^2$

(ii) $x^2 + y^2 - 9xy$

(iii) $2x_1^2 + 5x_2^2 - 3x_3^2 + 22x_2\cdot x_3 - 16x_3\cdot x_1 - 8x_1\cdot x_2$

Quadratic form can be written in the matrix form as

$$Q = X^T\cdot A\cdot X$$

where $x^T = [x_1, x_2, x_3]$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Note A is real symmetric matrix i.e $A = A^T$

* Reduction of quadratic form to canonical form

let

$$Q = ax_1^2 + bx_2^2 + cx_3^2 + 2f\,x_2\cdot x_2 + 2g\,x_3\cdot x_1 + 2h\,x_1 x_2$$

This expression reduces to canonical form with the help of as eigen value as-

$$\lambda_1 x_1^2 + \lambda_2\cdot x_2^2 + \lambda_3\cdot x_3^2$$

(i.e sum of squares form or principal axes form)

* Nature of A quadratic form

Let $q = X^T A X$ be a quadratic form in n-variables $x_1, x_2, \dots x_n$.

① Index:- Number of positive eigen values.

② signature: signature of quadratic form is the diff. of positive and -ve terms in the canonical form

③ Nature:-

A real quadratic form $X^T A X$ in a variable is said to be

(i) Positive Definite if all eigenvalue are positive.

(ii) Positive semi-definite :- if all eigen value are +ve & atleast one eigenvalue zero.

(iii) Negative definite:- all eigenvalue are -ve

(iv) Negative semi-definite:- if all eigenvalue -ve & atleast one eigenvalue 0.

(v) Indefinite :- If some value are positive and other are negative.

Note:-

① If $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ is a modal matrix then its normalised modal matrix is given by,

$$\hat{p} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+y_1^2+z_1^2}} & \frac{x_2}{\sqrt{x_2^2+y_2^2+z_2^2}} & \frac{x_3}{\sqrt{x_3^2+y_3^2+z_3^2}} \\ \frac{y_1}{\sqrt{x_1^2+y_1^2+z_1^2}} & \frac{y_2}{\sqrt{x_2^2+y_2^2+z_2^2}} & \frac{y_3}{\sqrt{x_3^2+y_3^2+z_3^2}} \\ \frac{z_1}{\sqrt{x_1^2+y_1^2+z_1^2}} & \frac{z_2}{\sqrt{x_2^2+y_2^2+z_2^2}} & \frac{z_3}{\sqrt{x_3^2+y_3^2+z_3^2}} \end{bmatrix}$$

and $\hat{p}$ is know as matrix of transformation which is an orthogonal matrix

(2) matrix A corresponding to the quadratic form.

$$ax_1^2 + bx_2^2 + cx_3^2 + 2f\,x_2x_3 + 2g\,x_3\cdot x_1 + 2h\,x_1\cdot x_2 \text{ is.}$$

$$A = \begin{bmatrix} \text{coiff. of } x_1^2 & \frac{1}{2}\text{ coiff. of } x_1\cdot x_2 & \frac{1}{2}\text{ coiff of } x_1\cdot x_3 \\ \frac{1}{2}\text{ coiff. of } x_2\cdot x_3 & \text{coiff. of } x_2^2 & \frac{1}{2}\text{ coiff. of } x_2\cdot x_3 \\ \frac{1}{2}\text{ coiff. of } x_3\cdot x_1 & \frac{1}{2}\text{ coiff. of } x_3\cdot x_2 & \text{coiff. of } x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

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Q. Determine the nature, index, and signature of the quadratic form

$$2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3$$

→ Compare with

$$ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2 \cdot x_3 + 2g \cdot x_3 \cdot x_1 + 2h \cdot x_1 x_2$$

$a=2,\ b=2,\ c=3,\ f=-2,\ g=-2,\ h=1$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

Now for eigen value
characteristic eqn is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad (1)$$

where S_1 = sum of digonal element

$= 2+2+3$

$S_1 = 7$

S_2 = sum of minor of digonal element

$$= \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$\boxed{S_2 = 7}$

$|A| = 1$

from eq (1)

$$\lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0$$

$\lambda = 5.82,\ 1,\ 0.1715$ ... eigen value

(i) Index = No. of +ve eigen value = 3

(ii) singnature = (No. of +ve eigen value) − (No. of −ve eigen value)

= 3 − 0

= 3

(iii) <u>Nature</u>:- all eigen value are positive, therefore nature of given matrix is positive definite.

<u>Ex</u> Transform the given quadratic form to cannonical form. Also state matrix for transformation.

$17x^2 - 30x_1 x_2 + 17x_2^2$

⟹ Given quadratic form is $17x^2 - 30x_1 x_2 + 17x_2^2$

compare with $ax_1^2 + 2hx_1 x_2 + bx_2^2$

$a = 17, \quad h = -15, \quad b = 17$

$$\therefore A = \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

* characterotic eqn

$\lambda^2 - s_1\lambda + |A| = 0$

$\lambda^2 - 34\lambda + (289 - 225) = 0$

$\lambda^2 - 34\lambda + 64 = 0$

$\lambda = 2, 32$

∴ The quadratic form reduces to connical form

a

$\lambda_1 x_1^2 + \lambda_2 x_2^2 = 2x_1^2 + 32x_2^2$

* To find matrix of Transformation.

consider the matrix eqn

$(A - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{——— (2)}$$

For eigen vector $\overline{x_1}$ for $\lambda = 2$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$15x_1 - 15x_2 = 0$

put $x_2 = t$

$\boxed{x_1 = t}$

$$\therefore \overline{x_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15t \\ 15t \end{bmatrix} = 15t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \overline{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(ii) Eigen ~~value~~ vector $\overline{x_2}$ for $\lambda = 32$

put $\lambda = 32$ in eq (2)

$$\begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} -15 & -15 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e $-15x_1 - 15x_2 = 0$

put $x_2 = t$

$x_1 = -t$

$$\therefore \overline{x_2} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\overline{x_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore$ modal matrix $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\hat{P} = \begin{bmatrix} \frac{1}{\sqrt{1^2+1^2}} & \frac{-1}{\sqrt{(-1)^2+(1)^2}} \\ \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{(-1)^2+(1)^2}} \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$