Moment of Inertia

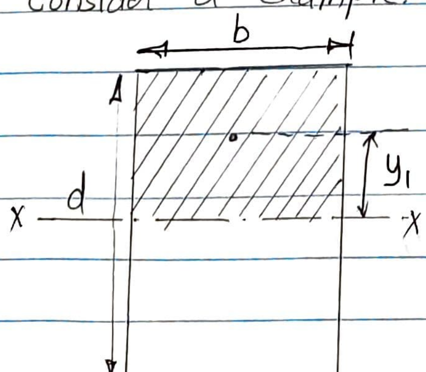
We have already discussed that the moment of a force (P) about a point, is the product of the force and perpendicular distance between the point and the line of action of the force (i.e. $P.x$). This moment is also called first moment of force.

If the moment is again multiplied by the perpendicular distance (x) between the point and the line of action of the force i.e Px^2, then this quantity is called moment of the moment of a force or second moment of force or moment of inertia.

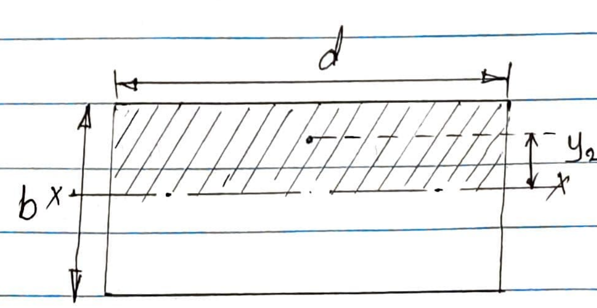
Sometimes instead of force, area or mass of a figure or body is taken into consideration. Then the second moment is known as second moment of area or second moment of mass. But all such second moments are broadly termed as moment of inertia. Now we will discuss the moment of inertia of planes areas only.

* Importance of Area Moment of inertia:-

- It is the quantitative estimate of the relative distribution of area with respect to some reference axis.
- The area moment of inertia (M.I.) of a beam's cross-sectional area measures the beam's ability to resist bending.
- Larger the moment of inertia, less the beam will bend.
- M.I is the geometrical property of beam and depends on the reference axis.
- Consider a example.



Section- I



Section - II

There are two cross-sections I and II having same area.
Taking moments @ x-x-axis ;
M.I. about x-x axis = Second moment of area about x-x axis

For Section I :

$$I_{xx} = \left[\left(\frac{A}{2}\right) \cdot y_1^2\right] 2$$

For section II :

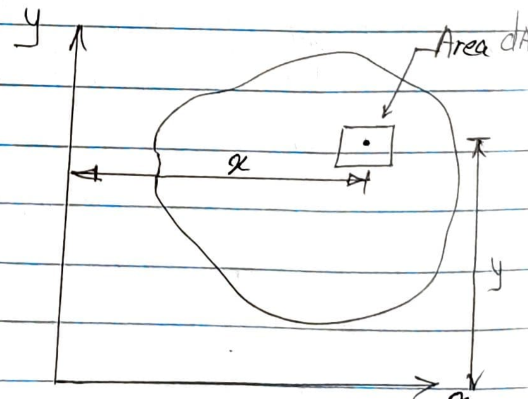
$$I_{xx} = \left[\left(\frac{A}{2}\right) y_2^2\right] \times 2$$

Where A = total area of section

As $y_1 > y_2$, then M.I. Section I is greater than M.I. of section II about x-x axis

∴ Section I is will have more resistance against bending at x-x axis as area is distributed over large distance from the x-x axis.

* Moment of Inertia by Integration :-



Let us consider a plane, as shown in fig. whose moment of inertia is required to be found about x and y axis.

Consider a small area, dA located at a distance x from y-axis and from x-axis.

The first moment of this area about x-axis = $y \cdot dA$

Second moment of area, i.e. moment of inertia of this area about x-axis is $I_{xx} = y^2 . dA$

$\therefore$ M.I. for the whole area about x-axis is;

$$I_{xx} = \int y^2 . dA$$

Similarly, moment of inertia of the whole area about y-axis is given by

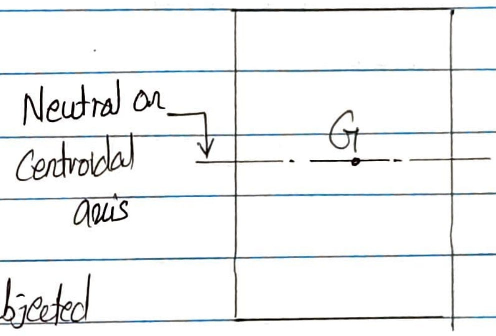
$$I_{yy} = \int x^2 . dA$$

I_{xx} = M.I. about x-axis

I_{yy} = M.I. about y-axis.

* Neutral Axis:

- The line passing through the centroid or centre of gravity of the section is known as 'neutral axis' or also it is known as "centroidal axis."

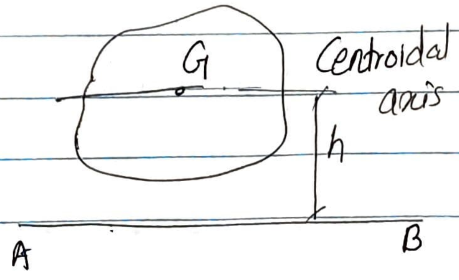


- The portion above neutral axis subjected to compressive forces and below neutral axis tensile forces when the beam bends in "U" shape.
- If it bends in "∩" shape, top portion is subjected to tensile forces and the lower portion compressive forces.
- The forces or stresses along the 'neutral axis" will be zero.

* Parallel Axis Theorem:

It states that, the M.I. of plane area A about any axis AB, which is parallel to the centroidal axis located at a distance 'h' is given by,

$$I_{AB} = I_G + Ah^2$$



OR

It states, if the M.I of a plane area about an axis through centre of gravity is denoted by I_G, then moment of inertia of the area about any other axis AB, parallel to the first and at a distance h from the centre of gravity is given by:

$$I_{AB} = I_G + Ah^2$$

where,

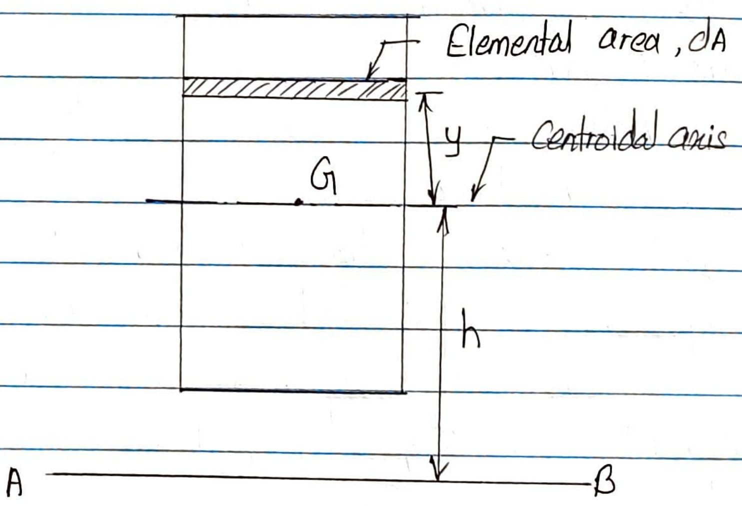
I_{AB} = moment of inertia of the area about an axis AB.

I_G = moment of inertia of the area about its centre of gravity

A = area of the section

h = distance between centre of gravity of the section and axis AB

Proof :-



Consider a strip whose area is dA.

y = distance of the strip from centroidal axis.

AB = Axis about which M.I is to be found.

h = distance between the centroidal axis and parallel axis.

A = Total area of the fig.

Second moment of area of an elemental strip about axis AB

$$d.I_{AB} = (y+h)^2 \cdot dA$$

For whole area;

$$I_{AB} = \int (y+h)^2 \cdot dA = \int (y^2+h^2+2yh)\,dA$$

$$= \int y^2 \cdot dA + \int h^2 \cdot dA + \int 2hy\,dA.$$

$\int y^2 . dA$ = M.I. of an area about centroidal axis, I_G

$h^2 \int dA = h^2 . A$

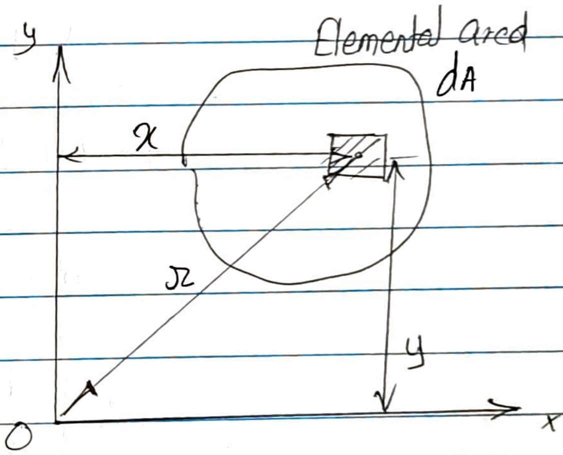
$2h \int y . dA = 2h \times 0 = 0$

$\int y . dA = \bar{y} . A = 0$ $\quad (\because \bar{y} = 0)$

$$\therefore \boxed{I_{AB} = I_G + Ah^2}$$

Note : Parallel axis theorem is used to find M.I. about an axis which is not passing through the C.G. of the section.

* Perpendicular Axis Theorem :



It states that, the M.I of plane area about an axis Ler or normal to the x-y plane. i.e. z-axis is equal to the sum of M.I of the plane area about x and y axes.

$$I_{zz} = I_{xx} + I_{yy}$$

~~Proof:~~

I_{zz} = M.I. of area about z-axis (Also called as polar M.I.)
z-axis is known as polar axis.

I_{xx} = M.I of an area about x-axis.

I_{yy} = M.I. of an area about y-axis.

Proof :-

Let r be the distance of elemental area from z-axis

Second moment of elemental area about z-axis;

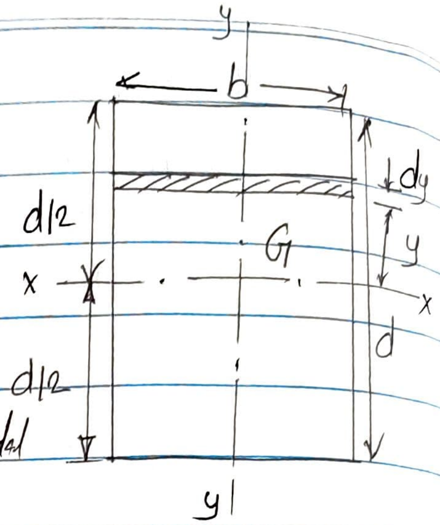
$$d.I_{zz} = r^2 dA.$$

for whole area; $I_{zz} = \int r^2 . dA = \int (x^2+y^2) dA = \int x^2 . dA + \int y^2 . dA$

$\int x^2 . dA$ = M.I of an area at y-axis $\quad \int y^2 dA$ = M.I of an area at x-axis

$$\therefore I_{zz} = I_{xx} + I_{yy}$$

* M.I. of Rectangular Section :



- Consider a rectangle of which 'b' width and 'd' depth as shown in fig.
- Consider a small strip of width 'b' and thickness dy at a distance 'y' from the centroidal x-axis

$$d.I_{xx} = y^2.dA.$$

Total M.I. about x-axis,

$$dA = b.dy$$

$$I_{xx} = \int y^2.dA$$

$$= \int_{-d/2}^{d/2} y^2.b.dy$$

$$= b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{d/2}$$

$$= \frac{b}{3}\left[\left(\frac{d}{2}\right)^3 - \left(-\frac{d}{2}\right)^3\right]$$

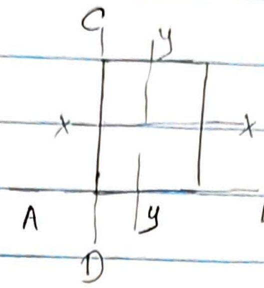
$$= \frac{b}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$= \frac{b}{3} \times \frac{\cancel{2}d^3}{\cancel{8}\,4}$$

$$\boxed{I_{xx} = \frac{bd^3}{12}}$$

Similarly, $\boxed{I_{yy} = \frac{db^3}{12}}$

① M.I. of Rectangle



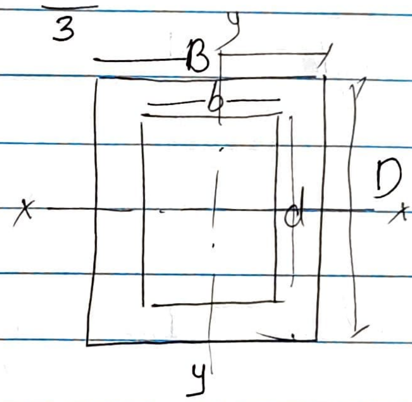
$$I_{xx} = \frac{bd^3}{12}$$

$$I_{yy} = \frac{db^3}{12}$$

$$M.I \text{ @ } AB = I_{AB} = \frac{bd^3}{3}$$

$$M.I \text{ @ } CD = I_{CD} = \frac{db^3}{3}$$

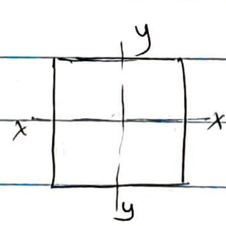
② M.I. of Hollow Rectangle



$$I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12}$$

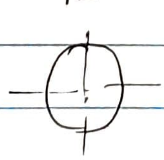
$$I_{yy} = \frac{DB^3}{12} - \frac{db^3}{12}$$

③ M.I. for squar section



$$I_{xx} = I_{yy} = \frac{a^4}{12}$$

④ M.I. of Circle

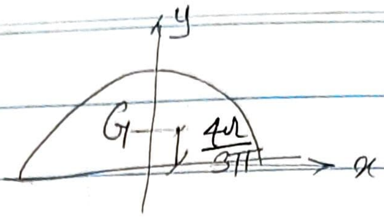


$$I_{xx} = I_{yy} = \frac{\pi D^4}{64}$$

⑤ M.I. of Hollow Circle

$$I_{xx} = I_{yy} = \frac{\pi}{64}(D^4 - d^4)$$

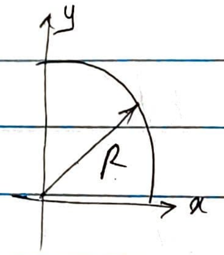
(06) M.I. of Semicircle



$$I_{xx} = I_{yy} = I_{Base} = \frac{\pi R^4}{8} = \frac{\pi D^4}{128}$$

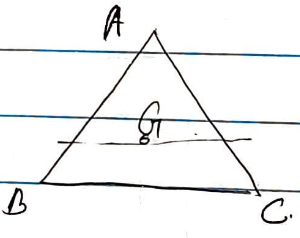
$$I_G = 0.11 R^4$$

(07) M.I. of Quarter Circle



$$I_{xx} = I_{yy} = \frac{\pi R^4}{16} = \frac{\pi D^4}{256}$$

(08) M.I. of triangle

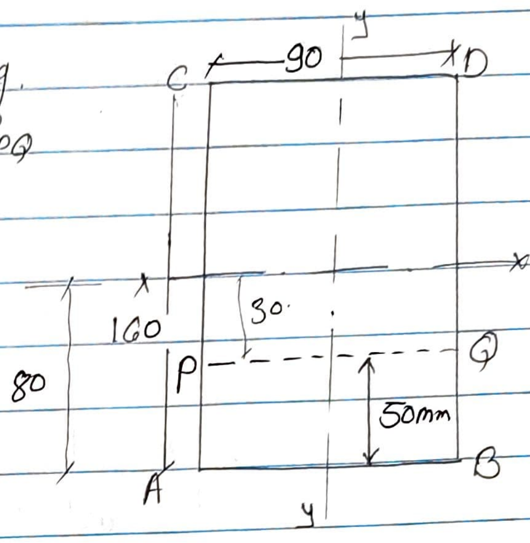


$$\text{M.I @ base} = I_{BC} = \frac{bh^3}{12}$$

$$I_{xx} = \frac{bh^3}{36} \quad ; \quad I_{yy} = \frac{hb^3}{36}$$

$$\text{M.I. @ vertex 'A'} = I_A = \frac{bh^3}{4}$$

(1) For the rectangular section shown in fig. compute the values of I_{xx}, I_{yy}, I_{AB} & I_{PQ}



$$\text{Area} = 90 \times 160 = 14400 \ mm^2$$

$b = 90$

$d = 160$

(a) $I_{xx} = \dfrac{bd^3}{12} = \dfrac{90 \times 160^3}{12} = 30.72 \times 10^6 \ mm^4$

(b) $I_{yy} = \dfrac{db^3}{12} = \dfrac{160 \times 90^3}{12} = 9.72 \times 10^6 \ mm^4$

(c) $I_{AB} = I_{xx} + Ah^2$

$= 30.72 \times 10^6 + 14400 \times 80^2$

$I_{AB} = 122.88 \times 10^6 \ mm^4$

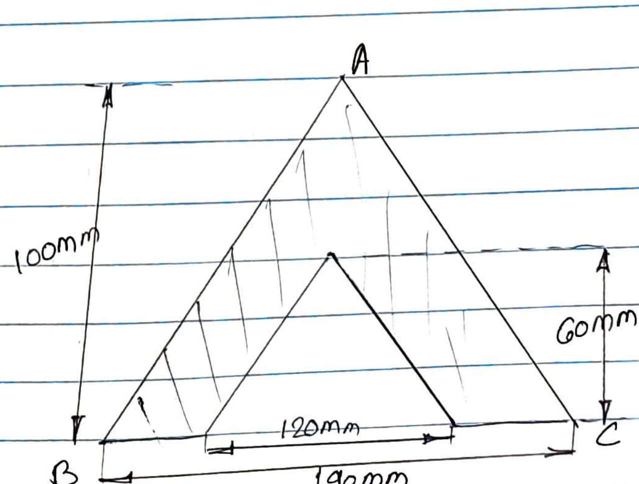
OR $I_{AB} = \dfrac{bd^3}{3} = \dfrac{90 \times 160^3}{3} = 122.88 \times 10^6 \ mm^4$

(d) $I_{PQ} = I_{xx} + Ah^2$

$= 30.72 \times 10^6 + 14400 \times 30^2$

$I_{PQ} = 43.68 \times 10^6 \ mm^4$

(2) A hollow triangular section shown in fig. is symmetrical about vertical axis. Determine the moment of inertia about the base BC.



$B = 190 \ mm$ $H = 100 mm$

$b = 120 \ mm$ $h = 60 mm$

$$I_{BC} = \frac{BH^3}{12} - \frac{bh^3}{12}$$

$$= \frac{190 \times 100^3}{12} - \frac{120 \times 60^3}{12}$$

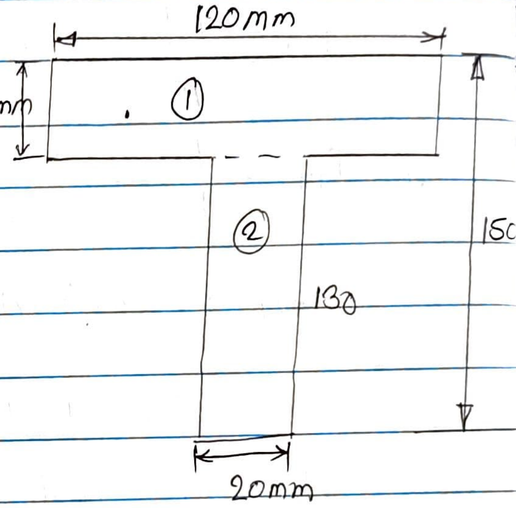
$$= 13.673 \times 10^6 \, mm^4$$

$$I_{BC} = \left[\frac{190 \times 100^3}{36} + \frac{1}{2} \times 190 \times 10\right] - \left[\frac{120 \times 60^3}{36} + \frac{1}{2} \times 120 \times 6\right]$$

$$= 15.833 \times 10^6 - 2.16 \times 10^6$$

$$= 13.673 \times 10^6 \, mm^4$$

(3) Find the moment of inertia of 'T' Section shown in fig. about the centroidal x and y axes.



$a_1 = 120 \times 20 =$

$y_1 = 130 + 10 =$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$a_2 = 20 \times 130$

$y_2 = \frac{130}{2}$

(a) Centroid

Shape	Area	$\bar{x}$	$\bar{y}$
① Rectangle	120×20 $= 2400$	60	$130 + 10$ $= 140$
② Rectangle	130×20 $= 2600$	$50 + 10$ $= 60$	$130/2$ $= 65$
	5000		

$x = 60 \, mm$

$$\bar{y} = \frac{2400 \times 140 + 2600 \times 65}{5000}$$

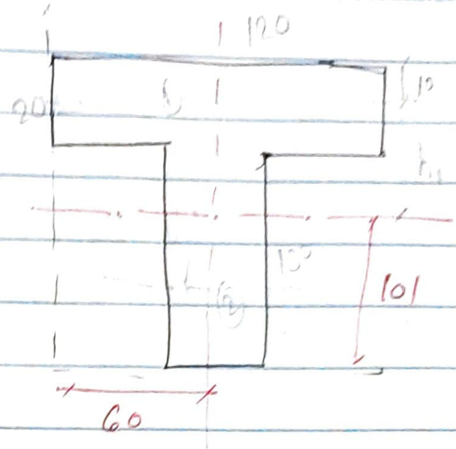
$= 101 \, mm$

$$I_{xx} = I_1 + I_2$$

$$I_1 = I_{G_1} + a_1 h_1^2$$

$$I_1 = \frac{bd^3}{12} +$$

(b) M.I. about centroidal Axis.



Shape	I_{xx}	I_{yy}
① Rectangle	$\frac{120\times 20^3}{12}$ $= 8\times 10^4$	$\frac{20\times 120^3}{12}$ 2.88 $= \cancel{3.66\times 10^6}$
② Rectangle	$\frac{20\times 130^3}{12}$ $= 3.66\times 10^6$	$\frac{130\times 20^3}{12}$ $= 8.66\times 10^4$

$h_1 = 49 - 10 = 39\,mm$

$h_2 = 101 - 65 = 36\,mm$

Using parallel axis theorem;

$I_{xx} = I_G + Ah_1$

$= \left[8\times 10^4 + 120\times 20\times 39^2\right] + \left[3.66\times 10^6 + 20\times 130\times 36^2\right]$

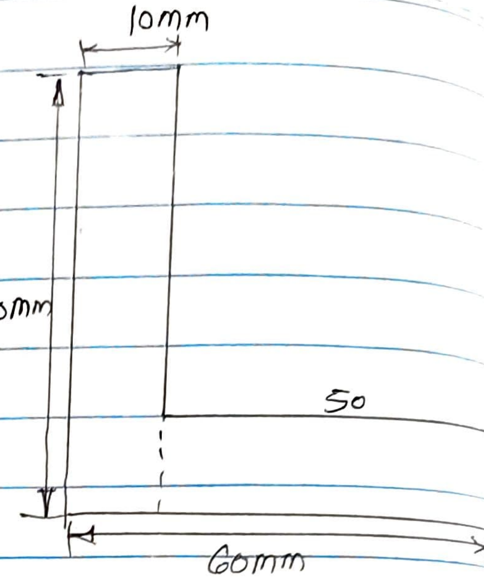
$= 3.7304\times 10^6 + 7.029\times 10^6$

$\boxed{I_{xx} = 10.759\times 10^6\ mm^4}$

$I_{yy} = 2.88\times 10^6 + 8.66\times 10^4$

$\boxed{I_{yy} = 2.966\times 10^6\ mm^4}$

④ Find the moment of inertia about the centroidal x- and y-axes of the angle section shown in fig.

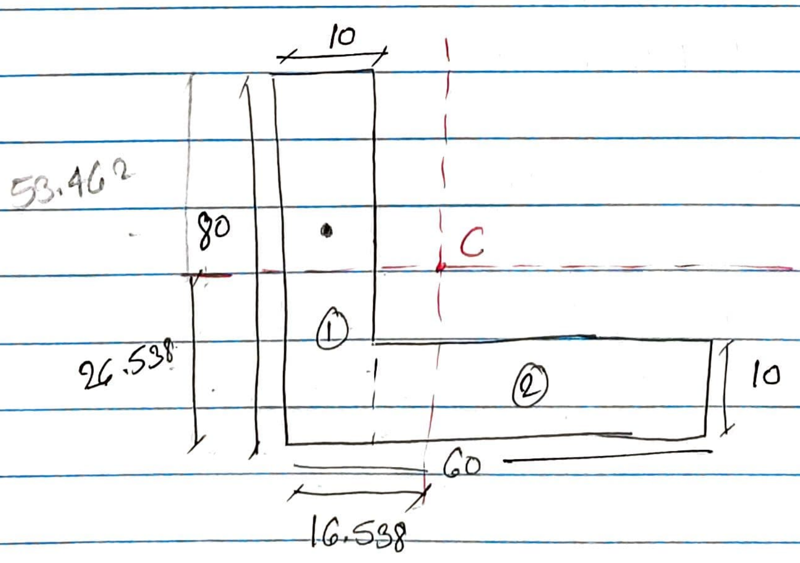


ⓐ Centroid

Shape	Area	$\bar{x}$	$\bar{y}$
① Rectangle	$80\times10 = 800$	$\frac{10}{2} = 5$	$\frac{80}{2} = 40$
② Rectangle	$50\times10 = 500$	$10+50/2 = 35$	$\frac{10}{2} = 5$
	1300		

$$\bar{x} = \frac{800\times5 + 500\times35}{1300} = 16.538 \text{ mm}$$

$$\bar{y} = \frac{800\times40 + 500\times5}{1300} = 26.538$$



(b) M.I. about centroidal axis.

Shape	I_{Gx}	I_{Gy}
① Rectangle	$\frac{10 \times 80^3}{12} = 0.426 \times 10^6$	$\frac{80 \times 10^3}{12} = 6.66 \times 10^4$
② Rectangle	$\frac{50 \times 10^3}{12} = \cancel{4.166} \times 10^4$ (0.4166)	$\frac{10 \times 50^3}{12} = 0.1041 \times 10^6$

(c) For I_{xx}

$53.462 - 40 = 13.462$

$h_1 = \cancel{16.538 - 5 = 6.538}$ $40 - \cancel{26.538 = 13.462\ mm}$

$h_2 = 26.538 - 5 = 21.538\ mm$

$$I_{xx} = \left[0.426 \times 10^6 + 10 \times 80 \times 13.462^2\right] + \left[\overset{0.4166}{\cancel{4.166}} \times 10^4 + 50 \times 10 \times 21.538^2\right]$$

$= 0.570 \times 10^6 + \overset{0.236}{\cancel{0.273}} \times 10^6$

$= \cancel{0.8436 \times} 10^6$

$= 0.807{,}756 \times 10^6\ mm^4$

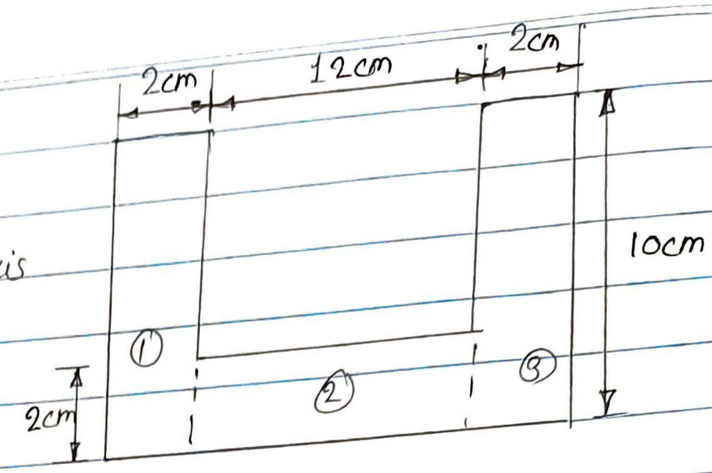
(d) For I_{yy}

$$I_{yy} = \frac{80 \times 10^3}{12} + 80 \times 10 \times (16.54 - 5)^2 + \left[\frac{10 \times 50^3}{12} + 10 \times 50 \times (35 - 16.54)^2\right]$$

$= 1.1320 \times 10^5 + 2.7455 \times 10^5$

$= 3.8775 \times 10^5\ mm^4$

(05) Find the moment of inertia of a section shown in fig. about x-axis passing through its centre of gravity.



:-

From the symmetry of the given fig.

$\bar{x} = 08$ cm

ⓐ Centroid

Shape	Area	$\bar{x}$	$\bar{y}$
① Rectangle	$10 \times 2 = 20$		05
② Rectangle	$12 \times 2 = 24$ cm		01
③ Rectangle	$10 \times 2 = 20$		05
	64		

$$\bar{y} = \frac{20 \times 05 + 24 \times 01 + 20 \times 05}{64}$$

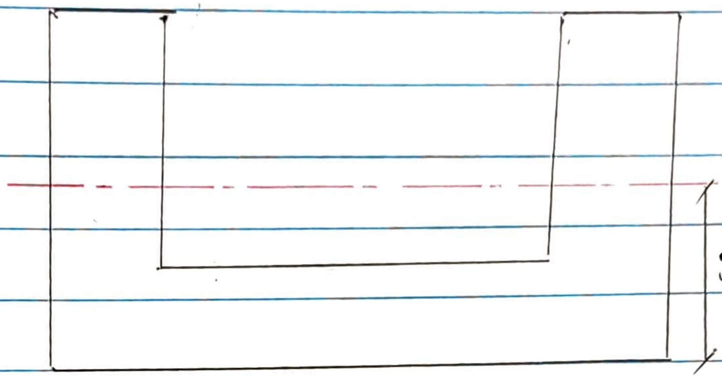
$$= 3.5 \text{ cm}$$

ⓑ M.I. about centroidal axis.

Shape	I_{Gx}	I_{Gy}
① Rectangle	$\frac{2 \times 10^3}{12} = 166.666$	$\frac{10 \times 2^3}{12} = 6.66$
② Rectangle	$\frac{12 \times 2^3}{12} = 8$	$\frac{2 \times 12^3}{12} = 288$

3 Rectangle	$\frac{2 \times 10^3}{12} = 166.66$	$\frac{10 \times 2^3}{12} = 6.66$

~~I_{xx} =~~



$$I_{xx} = 2\left[166.66 + 10 \times 2 \times (5-3.5)^2\right] + \left[8 + 12 \times 2 \times (3.5-1)^2\right]$$

$$= 423.32 + 158$$

$$= \cancel{579.32 cm}$$

$$= 581.32 cm$$