UNIT NO. 1

Linear Algebra: matrices, system of linear equation

chapter 1

Matrices

§1 Definition of matrix

An arrangement of certain number in an array of m row and n column, such as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

is called as matrix of order $m \times n$

where m = No. of row

n = No. of columns.

a_{mn} = suffix mn represent position of an element

e.g a_{11} = 1st row & 1st column.

The element with same row number and column number i.e $a_{11}, a_{22}, a_{33}, \ldots$ are said to be digonal elements.

Type of matrices.

(1) Row matrix :-

A matrix having only one row and n coloumn is called as row matrix

e.g $A = [a_{11} \; a_{12} \; \cdots \; a_{1n}]_{1 \times n}$

$B = [1 \; 2 \; 3 \; 4]_{1 \times 4}$

(2) coloumn matrix.

A matrix having only one column and m row is called as column matrix.

e.g $A = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}_{m \times 1}$ $[A]_{4 \times 1}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4 \times 1}$

(3) Zero matrix or Null matrix

A matrix containing all zero elements is called as zero matrix

e.g. $Z_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$

(4) Square matrix

A matrix containing number of row = Number of column is know as square matrix.

e.g $A_{3\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(5) Transpose of matrix:

A matrix obtained by Interchanging rows & column of matrix is called as Transpose. It is denoted by A^T = or A'

e.g $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

(6) Symmetric matrix:

A square matrix is said to be symmetric if $A = A^T$

e.g $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

$\therefore A = A^T$

Skew symmetric matrix
A matrix is said to be skew symmetric matrix if
$A = -A^T$

e.g. $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$

$-A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$

$\therefore \boxed{A = -A^T}$

(8) Digonal matrix
A square matrix contaning all non-digonal element as zero then it is called as digonal matrix

e.g $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(9) Scalar matrix
If a square matrix has all digonal elements equal i.e $a_{11} = a_{22} = a_{33} \cdots$ then it is called scalar matrix

e.g. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(10) Unit matrix or Identity matrix
A digonal matrix, where all digonal element are unity is called identity matrix

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(11) Upper Triangular matrix

It is square matrix in which all the elements below the principal digonal are zero's

e.g $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$ or $\begin{bmatrix}0 & 2 & 0\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}$

(12) Lower Triangular matrix

A square matrix in which all the elements above principal digonal are zeros.

8.3 Determinant of matrix

$|A|$ = determinant of A

If $A=\begin{bmatrix}1 & -2 & -3\\ 4 & 5 & 2\\ 1 & 1 & 2\end{bmatrix}$

$$|A| = 1\begin{vmatrix}5 & 2\\ 1 & 2\end{vmatrix} - (-2)\begin{vmatrix}4 & 2\\ 1 & 2\end{vmatrix} + (-3)\begin{vmatrix}4 & 5\\ 1 & 1\end{vmatrix}$$

$$= 1\left[10-2\right] + 2\left[8-2\right] - 3\left[4-5\right]$$

$$= 8 + 2\times 6 + (-3)(-1)$$

$$= 8 + 12 + 3$$

$$|A| = 23$$

8.4 minor and co-factor of $|A|$

Let $|A| = \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

i = row position
j = column position

Then minor of an element $|A|$ is a determinant obtained by omitting the row and the column in which the element is present

Minor of element is denoted by M_{ij}

$$\text{minor of } a_1 = M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Minor of } b_1 = M_{12} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{minor of } b_2 = M_{22} = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

* Now cofactor of an element is denoted by

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Ex. cofactor of a_1 from above $|A|$

$$A_{11} = (-1)^{1+1} M_{11}$$
$$= (-1)^2 \cdot M_{11}$$
$$A_{11} = M_{11}$$
$$A_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \text{ and so on...}$$

8.5 Adjoint of a matrix and A^{-1}

If A is a matrix, then

step 1: find minor matrix i.e M

step 2: find co-factor matrix i.e C

Step 3: Take transpose of matrix C i.e C^T

then $\operatorname{adj} A = C^T$

Step 4 $A^{-1} = \frac{1}{|A|} \cdot Adj\,A$, where $|A| \neq 0$

~~Step 1: Find~~

Ex

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\begin{matrix} 2 & 4 & 1 & 2 \\ 3 & 9 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 2 \end{matrix}$$

Step 1 find minor matrix

$$\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$M_{11} = 18 - 12 = 6(-1)^2$

$M_{12} = 9 - 3 = 6(-1)^3$

$M_{13} = 4 - 2 = 2(-1)^4$

$M_{21} = 9 - 4 = 5(-1)^3$

$M_{22} = 9 - 1 = 8(-1)^4$

$M_{23} = 4 - 1 = 3(-1)^5$

$M_{31} = 3 - 2 = 1(-1)^4$

$M_{32} = 3 - 1 = 2(-1)^5$

$M_{33} = 2 - 1 = 1(-1)^6$

Step 2:

Find co-factor matrix

$$C = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Step 3: $Adj\,A = C^t = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Step 4

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$\therefore |A| = 1(18-12) - 1(9-3) + 1(4-2)$

$= 6 - 6 + 2$

$|A| = 2$

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$\therefore A^{-1} = \frac{1}{|A|} \cdot adj. A$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

* operation of matrices:

① Equality of matrices:

Two matrices A and B are equal, there order is same and there corresponding element are same i.e $a_{ij} = b_{ij}$

i.e $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

② multiplication of matrix by scalar

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$ then $2A = \begin{bmatrix} 4 & 6 \\ 8 & 10 \\ 12 & 14 \end{bmatrix}$

③ Addition of matrix

$$A+B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ -3 & 2 & 9 \\ 11 & -5 & 3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+(-1) & 2+2 & 3+4 \\ 4+(-3) & 5+2 & 6+9 \\ 7+11 & 8+(-5) & 9+3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 7 & 15 \\ 18 & 3 & 12 \end{bmatrix}$$

* multiplication of matrix

If $A = \begin{bmatrix} -1 & 2 & 9 \\ 3 & 4 & 1 \end{bmatrix}_{2\times3}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$

then $A \cdot B = \begin{bmatrix} -1 & 2 & 9 \\ 3 & 4 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times2}$

$A \cdot B = \begin{bmatrix} -1+8+63 & -2+10+72 & -3+12+81 \\ 3+14+7 & 6+20+8 & 9+24+9 \end{bmatrix}_{2\times3}$

$A \cdot B = \begin{bmatrix} 70 & 80 & 90 \\ 26 & 34 & 42 \end{bmatrix}$

Note:

(1) In basic algebra $a \cdot b = 0$
then $a = 0$ or $b = 0$
But in matrices if $A \cdot B = 0$
that does not mean ~~A≠0~~ $A = 0$ or $B = 0$

eg $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$

Here $A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $A \neq 0$, $B \neq 0$

(2) $AB \neq BA$.

87. Elementary transformation of matrix

The following three type transformation performed on any non-zero matrix, are called as elementary transformation.

1. The interchange of i^{th} and j^{th} row / column is denoted by R_{ij} / C_{ij}

2. The multiplication of each element of i^{th} row/column by a non-zero scalar k is denoted by $k \cdot R_i$ / kC_i

3. $R_i + kR_j$ / $C_i + kC_j$

88. Rank of matrix

The matrix is said to be of rank r if there is

1. At least one minor of order r which is not equal to 0 &
2. Every minor of order (r+1) is equal to 0

e.g $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Here $|A| = 0$ But $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$

$\therefore$ Rank cannot be 3

$\therefore \boxed{\rho(A) = 2}$

8.9 Rank By using normal form.

Def^n: By performing elementary transformation any non-zero matrix A can be reduced to one of the following four form, called as normal form.

1) $[I_r]$ ② $[I_r \quad 0]$ ③ $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ ④ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

where r represent rank of matrix

Working Rule for Normal form.

Step 1: Make $a_{11} = 1$

$$\begin{bmatrix} 1 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Step 2 Get zero below $a_{11}=1$ by Row transformation and R_1 only

$$\begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Step 3:

Get zero to the right of $a_{11}=1$ by column transformation and C_1 only.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Step 4: Make $a_{22}=1$ (without using R_1 and C_1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & * \end{bmatrix}$$

step 5: Get zero's below $a_{22} = 1$ by row transformation and R_2 only.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$$ and so on.

Q. Find the rank of following matrix by using normal form.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

→

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 - R_1$ and $R_3 = R_3 - 3R_1$

$R_2 = R_2 - R_1$
$= 1 - (1) = 0$
$= (-1) - (1) = -2$
$= 2 - (-1) = 3$
$= (-1) - 1 = -2$

$R_3 = R_3 - 3R_1$
$= 3 - 3(1) = 0$
$= 1 - 3(1) = -2$
$= 0 - 3(-1) = 3$
$= 1 - 3(1) = -2$

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix}$$

$C_2 = C_2 - C_1$
$C_3 = C_3 + C_1$
$C_4 = C_4 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix}$$

$C_2 = -\frac{1}{2} C_2$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 3C_2$, $C_4 = C_4 + 2C_2$

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$ by normal form.

Ex find the rank of following matrix by using Normal Form.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 7 & 5 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$\Rightarrow$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 7 & 5 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$ & $R_3 = R_3 - 3R_1$

$= 3 - 3(1) = 0$ $\qquad = 3 - 3(1) = 0$

$= 2 - 3(1) = -1$ $\qquad = 3 - 3(1) = 0$

$= 7 - 3(2) = 1$ $\qquad = 6 - 3(2) = 0$

$= 5 - 3(3) = -4$ $\qquad = 9 - 3(3) = 0$

$= 12 - 3(5) = -3$ $\qquad = 15 - 3(5) = 0$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & -1 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = C_2 - C_1$

$C_3 = C_3 - 2C_1$

$C_4 = C_4 - 3C_1$

$C_5 = C_5 - 5C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & -1 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow -1(R_2)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 + C_2$, $C_4 \rightarrow C_4 - 4C_2$, $C_5 \rightarrow C_5 - 3C_2$

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$ by Normal form.

Homework:

(i) $A = \begin{bmatrix} 2 & -3 & 4 & 4 \\ 1 & 1 & 1 & 2 \\ 3 & -2 & 3 & 6 \end{bmatrix}$ Ans 3

(ii) $A = \begin{bmatrix} 2 & -1 & 1 & 3 \\ 2 & 4 & -1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ Ans 3

(iii) $A = \begin{bmatrix} 8 & -6 & 4 & -3 & 2 \\ 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

8.10 Echelon form (eh-shuh-lawn)

In Echelon form we have to make elements below the digonals are zeros. Digonal may be or may not be equal to 1 unlike in normal form where digonal must be equal to 1.

Note: For Echelon form only row transformation is allowed

Matrix A must be reduced to form such as

$$\begin{bmatrix} 3 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ex find the rank of following matrix by using Echelon form

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & 1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$, $R_3 = R_3 - 3R_1$, $R_4 = R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 13 \end{bmatrix}$$

$R_2 : R_2 - R_3$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 4R_2$, $R_4 \Rightarrow R_4 - 9R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 \Rightarrow R_4 - 2R_3$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

② Find the rank of

$$A=\begin{bmatrix}3 & -2 & 0 & -1\\ 0 & 2 & 2 & 7\\ 1 & -2 & -3 & 2\\ 0 & 1 & 2 & 1\end{bmatrix}$$

$R_1 \rightarrow R_3$

$$A=\begin{bmatrix}1 & -2 & -3 & 2\\ 0 & 2 & 2 & 7\\ 3 & -2 & 0 & -1\\ 0 & 1 & 2 & 1\end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$$A=\begin{bmatrix}1 & -2 & -3 & 2\\ 0 & 2 & 2 & 7\\ 0 & 4 & 9 & -7\\ 0 & 1 & 2 & 1\end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A=\begin{bmatrix}1 & -2 & -3 & 2\\ 0 & 1 & 2 & 1\\ 0 & 4 & 9 & -7\\ 0 & 2 & 2 & 7\end{bmatrix}$$

$R_3 = R_3 - 4R_2$; $R_4 = R_4 - 2R_2$

$$A=\begin{bmatrix}1 & -2 & -3 & 2\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & -11\\ 0 & 0 & -2 & 5\end{bmatrix}$$

$R_4 = R_4 + 2R_3$

$R_4 = R_4 + 2R_3$

$$A=\begin{bmatrix}1 & -2 & -3 & 2\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & -11\\ 0 & 0 & 0 & -17\end{bmatrix}$$

$\therefore \rho(A) =$ Total No. of Non-zero row

$\rho(A) = 4$ by Echelon form

<u>Homework</u> $A=\begin{bmatrix}6 & 1 & 3 & 8\\ 4 & 2 & 6 & -1\\ 10 & 3 & 9 & 7\\ 16 & 4 & 12 & 15\end{bmatrix}$ <u>Ans</u> $\rho(A) = 2$

8.11 Finding Non-singular matrices P and Q such that PAQ is in Normal form

Q. Find non-singular matrix P and Q such that

$$A=\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\\ 3 & 0 & 5 & -10\end{bmatrix}$$

is reduced to normal form. Also find rank of A

$$\underbrace{I_{3\times3}\ \underbrace{A_{3\times4}\ I_{4\times4}}_{4\times4}}_{3\times4}$$

$$\underbrace{3\times4 \quad 4\times4}_{3\times4}$$

→ for a given matrix A

Total No. of row = 3

∴ consider unit matrix I_3

Total No. of column = 4

∴ consider unit matrix I_4

we can write $A_{3\times4} = I_3 \cdot A\, I_4$

∴ we can write $A_{3\times4} = I_3\, A\, I_4$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

our aim is to reduce matrix A to normal form using both rows and column transformation

Note 1

① Apply row transformation on L.H.S and I_3 (keep I_4 unchanged)

② Apply column transformation on L.H.S & I_4 (keep I_3 unchanged)

$R_2 \to R_2 - 2R_1$, $R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 = C_2 - 2C_1$; $C_3 = C_3 - 3C_1$

$C_4 = C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 = -\frac{1}{3} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 2 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -3 & -4 \\ 0 & \frac{-1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore R_3 = R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -3 & -4 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 = C_3 + 2C_2$ & $C_4 = C_4 + 5C_2$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{-5}{3} & \frac{-2}{3} \\ 0 & -\frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 = -\frac{1}{12} C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -\frac{5}{3} & \frac{1}{18} \\ 0 & -\frac{1}{3} & -\frac{2}{3} & \frac{5}{36} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \longleftrightarrow C_4$

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$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{12} & -\frac{5}{3} \\ 0 & -\frac{1}{3} & \frac{5}{36} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3$

$\therefore$ PAQ is normal form

and $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{12} & -\frac{5}{3} \\ 0 & -\frac{1}{3} & \frac{5}{36} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Note: ① P and Q are not unique. They vary depending upon elementary transformation used while using the process.
However, we can cross check if $PAQ = I$, then answer is correct.

② For square matrix A, $PAQ = I \Rightarrow A^{-1} = QP$.
where, P, Q, A are non-singular matrices
(i.e $|A| \neq 0$, $|P| \neq 0$, $|Q| \neq 0$)

UNIT. V

Chapter 9: System of Linear Algebraic eqⁿs

9.1 Simultaneous Eqⁿ:

consider an eqⁿ

$2x = 4$

This is one eqⁿ in one unknown (x)

we solve it, we get

$x = \frac{4}{2} = 2$

Now, consider two simultaneous eqⁿ

$2x + y = 3$

$x + 2y = 9$

There are two eqⁿ in two unknowns (x, y)

we solve these eqⁿ to find unknowns x and y by various methods.

1. Elimination method
2. matrix method.

(1) Elimination method.

$2x + y = 3$ —— (1)

$x + 2y = 9$ —— (2) $\times 2$

$\cancel{2x} + y = 3$

$- \cancel{2x} + 4y = 18$

$-3y = -15$

$\boxed{y = 5}$

$x + 2y = 9$

$x + 2 \times 5 = 9$

$x = 9 - 10$

$\boxed{x = -1}$

Solution of system is $x = -1, y = 5$.

9.2 Representation of simultaneous equation in matrix form

$$A \quad X = B$$

A ↓ coifficent matrix

X ↓ unknow variable matrix

B ↓ constant matrix

Ex

$2x+y-z=2$

$x+y+3z=9$

$x+y+z=5$

$AX=B$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix}$$

Note while actualy solving the equation by matrix method, we only consider coiff. and constant matrix seprated by a line called as augumented form.

Augumented form $= (A|B)$

$$(A|B) = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 1 & 3 & 9 \\ 1 & 1 & 1 & 5 \end{array}\right]$$

9.3 Type of eqⁿ

Equation

- Non-homogeneous eqⁿ
  - $x+y+z=3$
  - $x-y+2z=4$
  - $2x+3y-z=0$
- Homogeneous Eqⁿ
  - $x+y+z=0$
  - $x-y+2z=0$
  - $2x+3y-z=0$

Note.

① For Non-homo. eq^n B matrix is a non-zero matrix i.e R.H.S must contain at least one non-zero Number

② For homo. eq^n B matrix is a zero matrix i.e R.H.S is all zero.

9.4 Flow chart to solve system of Algebraic Eq^n

$$AX = B$$

consistant if $\rho(A) = \rho(A|B)$

- unique sol^n if $r = n$
- Infinite sol^n if $r < n$

Inconsistant if $\rho(A) \neq \rho(A|B)$

- $\boxed{\text{No sol}^n}$

where r = rank
n = no. of unknowns.

<u>Note:</u>

As we using Echelon form to reduce the matrix, we can only use elementry row transformation.

Ex: Solve the following system of equations by matrix method

$x+y+z=0$

$2x+3y-z=-5$

$x-y+z=4$

$\Rightarrow$ The system of equation in matrix form

$AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix}$$

Agumented form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & -1 & -5 \\ 1 & -1 & 1 & 4 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -2 & 0 & 4 \end{array}\right]$$

$R_3 \to R_3 + 2R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -6 & -6 \end{array}\right]$$

$\therefore \rho(A) = 3$

$\rho(A|B) = 3$

$\therefore \rho(A) = \rho(A|B) = r = 3$

$\therefore$ system is consistant

but $n=3$ (number of unknow)
x, y, z

$\therefore r = 3 = n$

system processes unique sol^n.

Q.2 Is the following system of eqⁿ consistent? If so find the solution.

$$2x - 3y + 5z = 1$$
$$3x + y - z = 2$$
$$x + 4y - 6z = 1$$

→ Given system of eqⁿ in matrix form

$$AX = B$$

$$\therefore \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$[A|B] = \begin{bmatrix} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$, $R_3 = R_3 - 3R_1$

$$\therefore [A|B] = \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\therefore (A|B) = \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$; $\rho(A|B) = 2$

$\therefore \rho(A) = \rho(A|B) = 2 = r$

system is consistant

but $n = 3$, (No. of unknowns x, y, z)

$\therefore r < n$

$\therefore$ system possesses infinite solution.

Let $z = t$

$\therefore$ By R_2

$$-11y + 17z = -1$$
$$-11y + 17t = -1$$
$$-11y = -1 - 17t$$
$$\boxed{y = \frac{1 + 17t}{11}}$$

$\therefore$ By R_1

$$x + 4y - 6z = 1$$
$$x + 4\left(\frac{1 + 17t}{11}\right) - 6t = 1$$
$$11x + 4 + 68t - 66t = 11$$
$$11x + 2t = 11 - 4$$
$$11x = 7 - 2t$$
$$\boxed{x = \frac{7 - 2t}{11}}$$

$\therefore$ Solⁿ is

$x = \frac{7 - 2t}{11}$, $y = \frac{1 + 17t}{11}$

$z = t$

Q.3 Examine for consistency of foll. eq^n and solve them if consistant

$$2x-3y+7z=5$$
$$3x+y-3z=13$$
$$2x+19y-47z=32$$

$\Rightarrow$ The Given eq^n in matrix form $AX=B$

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

In Augumented form

$$(A|B)=\left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$R_1 = R_1 - R_2$

$$(A|B)=\left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$R_2 = R_2 + 3R_1$; $R_3 = R_3 + 2R_1$

$$\therefore (A|B)=\left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_2$

$$(A|B)=\left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

$\therefore \rho(A)=2, \ \rho(A|B)=3$

$\rho(A) \neq \rho(A|B)$

$\therefore$ system is Inconsistant

Inconsistant system does not have any sol^n.

Q.4 Investigate the values of λ and μ so that the following eq^n have

(1) unique sol^n (2) Infinite solution

(3) No solution.

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

$\Rightarrow$ matrix form $AX=B$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$[A|B]=\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$R_2 = R_2 - 3R_1$

$$(A|B)=\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 1 & -6 & -17 & -19 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$(A|B)=\left[\begin{array}{ccc|c} 1 & -6 & -17 & -19 \\ 2 & 3 & 5 & 9 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 = R_3 - 2R_1$

$$(A|B)=\left[\begin{array}{ccc|c} 1 & -6 & -17 & -19 \\ 0 & 15 & 39 & 47 \\ 0 & 15 & \lambda+34 & \mu+38 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -6 & -17 & -19 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

(i) unique solⁿ: For unique solⁿ

$\rho(A) = \rho(A|B)$

and $r = n$

$\therefore$ Here $n = 3$

$\therefore$ r must be equal to 3 & for that

$\lambda - 5 \neq 0$ & $\mu - 9$ can have any value

$\therefore$ $\lambda \neq 5$ and μ can have any value

(ii) Infinite solⁿ:-

For infinite solⁿ:

$\rho(A) = \rho(A|B)$ & $r < n$

$\therefore$ Here $n = 3$

$\therefore$ r must be equal to 2

For $r = 2$

$\lambda - 5 = 0$, $\Rightarrow \lambda = 5$

$\mu - 9 = 0$ $\quad \mu = 9$

(iii) No solⁿ.

For No solⁿ $\rho(A) \neq \rho(A|B)$ and for that,

$\lambda - 5 = 0$ & $\mu - 9 \neq 0$

$\therefore$ $\lambda = 5$, $\mu \neq 9$!

Example on Homogeneous system of eqⁿ by using Echelon form

(1) Solve the following system of equation

$x + 2y + 3z = 0$

$2x + 3y + z = 0$

$4x + 5y + 4z = 0$

$x + y - 2z = 0$

matrix form is

$AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 0 \\ 4 & 5 & 4 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 4R_1$

$R_4 = R_4 - R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -3 & -8 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

$R_3 = R_3 - 3R_2$, $R_4 = R_4 -$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $\rho(A) = 3$ and $\rho(A|B) =$

$\therefore$ system is consistant

$\rho(A) = \rho(A|B) = 3 = r = n.$

system possesses unique sol^n.

$\therefore$ By R_3 $\quad 7z = 0$

$\therefore z = 0$

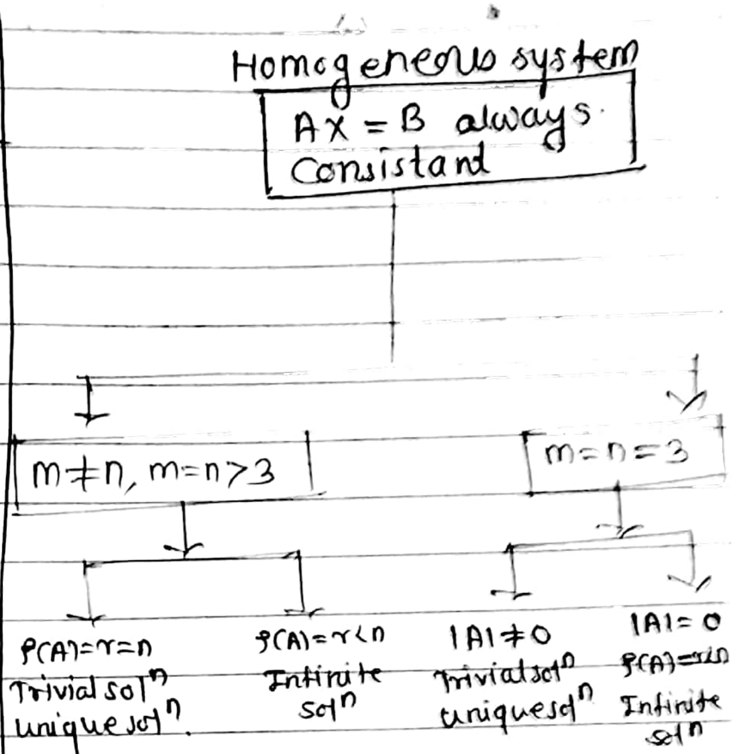
By R_2 $\quad -y - 5z = 0$

$\Rightarrow y = 0$

By R_1 $\quad x + 2y + 3z = 0$

$\Rightarrow x = 0$

Hence $x = y = z = 0$ is a trivial sol^n.



Note:-

① Homogeneous system is always consistent i.e $\rho(A) = \rho(A|B)$

② For homogeneous system, unique sol^n gives all unknow $= 0$ i.e $x = y = z = 0$ is called trivial sol^n.

③ For homogeneous system, infinite sol^n gives infinite sol^n of unknowns (x, y, z) in terms of some arbitrary constant t, is called as non-trivial sol^n.

classmate

Date

Page

9.8 Linear Dependant and Independant vectors.

Let $x_1, x_2, x_3, \dots, x_n$ be a system of n row (or column) matrices of the same order (also called vectors)

If there exist n scalar $c_1, c_2, c_3, \dots, c_n$ not all zero. such that $c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$, then the system is linear dependant.

Whereas, if we get $c_1, c_2, c_3, \dots, c_n$ all zero, then the system is ~~called~~ linearly independant

Ex Examine for linear dependance of the following system of vectors. If ~~in~~ dependant, find the relation betⁿ them.

$X_1 = (3, 1, -4)$, $X_2 = (2, 2, -3)$, $X_3 = (0, -4, 1)$

⟹ Given vector $X_1 = (3, 1, -4)$

$X_2 = (2, 2, -3)$

$X_3 = (0, -4, 1)$

Now consider the matrix equation

$c_1X_1 + c_2X_2 + c_3X_3 = 0$

$\therefore c_1(3, 1, -4) + c_2(2, 2, -3) + c_3(0, -4, 1) = 0$

$\therefore 3c_1 + 2c_2 + 0 \cdot c_3 = 0$

$c_1 + 2c_2 - 4c_3 = 0$

$-4c_1 - 3c_2 + c_3 = 0$

which is homogeneous system

above equation in matrix form

$AX = B$

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & -4 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In Augumented form

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 1 & 2 & -4 & 0 \\ -4 & -3 & 1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 3 & 2 & 0 & 0 \\ -4 & -3 & 1 & 0 \end{array}\right]$$

$R_2 = R_2 - 3R_1$ and $R_4 = R_4 + 4R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -4 & 12 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right]$$

$R_2 = \frac{R_2}{4}$, $R_3 = \frac{R_3}{5}$

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$R_3 = R_3 + R_2$

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho(A) = \rho(A|B) = r = 2$

But $n = 3 \rightarrow$ No. of unknow c_1, c_2, c_3

$\therefore\ r < n$

$\therefore$ system possesses infinite sol^n

Let ~~ss~~ $c_3 = t$

By R_2

$-c_2 + 3c_3 = 0$

$-c_2 + 3t = 0$

$\boxed{c_2 = 3t}$

By R_1

$c_1 + 2c_2 - 4c_3 = 0$

$c_1 + 2(3t) - 4t = 0$

$\boxed{c_1 = -2t}$

As $c_1, c_2, c_3 \neq 0$, gives system of linearly dependent

Relation:

$c_1x_1 + c_2x_2 + c_3x_3 = 0$

$-2tx_1 + 3tx_2 + tx_3 = 0$

$t(3x_2 + x_3) = t(2x_1)$

$3x_2 + x_3 = 2x_1$

Cross check:

$3(2, 2, -3) + (0, -4, 1) = 2(3, \ldots$

$(6, 6, -9) + (0, -4, 1) = (6, 2, -\ldots$

$(6, 2, -8) = (6, 2, -8)$

9.

9.9 Linear Transformation:

A general linear transformation is represented by

$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n.$

$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n.$

$\vdots$

$y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.$

which in matrix form can be written as:

$$Y = AX$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Ex

Express each of the transformation $x_1 = 3y_1 + 2y_2$; $x_2 = -y_1 + 4y_2$ and $y_1 = z_1 + 2z_2$, $y_2 = 3z_1$, In matrix form and find the composite transformation which express x_1, x_2 in terms of z_1, z_2

→ The transformation

$x_1 = 3y_1 + 2y_2$

$x_2 = -y_1 + 4y_2$

In matrix form can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

i.e $X = AY$ — ①

and the transformation

$y_1 = z_1 + 2z_2$

$y_2 = 3z_1$

In matrix form can be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$y = BZ \quad \text{—— (2)}$$

From eqⁿ (1) & (2)

$$X = A(BZ)$$

$$\therefore \; X = ABZ$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1)+2(3) & 3(2)+2(0) \\ (-1)(1)+(4)(3) & (-1)(2)+4(0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$x_1 = 9z_1 + 6z_2$$

$$x_2 = 11z_1 - 2z_2$$

is the required transformation.

* 9:10 Orthogonal Transformation.

The linear transformation $Y = AX$ where,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be orthogonal if it transforms $x_1^2 + x_2^2 + \cdots + x_n^2$ into $y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2$

Note 1

(1) A square matrix A is said to be orthogonal if $AA^T = I$

(2) If A is orthogonal then $A^{-1} = A^T$

Ex (1) show that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is orthogonal.

Let $A \cdot A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$\therefore A \cdot A^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \boxed{A \cdot A^T = 1}$

Homework

show that A is orthogonal $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

* check wheathere following matrices is orthogonal or Not

$A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

* If A is an orthogonal matrix, prove that $|A| = \pm 1$

$\Rightarrow \quad A \cdot A^T = I$

$\therefore |A| \cdot |A^T| = |I|$

but $|A| = |A^T|$

$|A| \cdot |A| = |I|$

$|A|^2 = 1$

$\boxed{A = \pm 1}$