UNIT - 2

Application of first order Differential Equation.

*** Trajectory:**

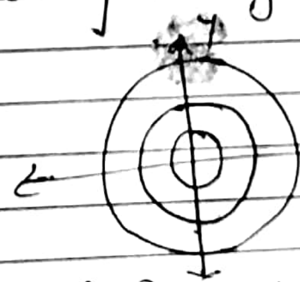
A curve which cuts every member of a given family of curve according to some definite law is called a trajectory of family.

*** Orthogonal Trajectory**

A curve which is cuts every member of given family of curve at right angle is called orthogonal trajectory of family.

*** Orthogonal Trajectories:**

Two family of curve are said to be orthogonal if every member of the either family cut each member of other family at right angle.



Given family consist of a straight line $y = mx$ passing through origin then family of circle $x^2+y^2=a^2$ with centre at $(0,0)$ represent a family of orthogonal trajectories to the family $y = mx$.

*** Working Rule to find the equation of orthogonal Trajectory**

Step 1: Given $f(x,y,a)=0$, where a is variable parameter

Step 2: diff $f(x,y,a)$ w.r.to x and eliminate a

Thus we form diff eq^n $\phi(x, y, \frac{dy}{dx})=0$

Step 3: Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

Then diff eq^n of the family of orthogonal trajectory will be $\phi(x, y, -\frac{dx}{dy})=0$

Step 4: The sol^n of step 3 is family of orthogonal Trajectory.

Q.1 find the orthogonal trajectory of the family of straight line $y = mx$

$\Rightarrow$ $y = mx$ — ①

diff. w.r.to x

$\frac{dy}{dx} = m$

$y = \frac{dy}{dx} \cdot x$

$\frac{dy}{dx} = \frac{y}{x}$ which is diff. eqⁿ of given family

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$-\frac{dx}{dy} = \frac{y}{x}$

$-x\,dx = y\,dy$

$\int x\,dx + y\,dy = \int 0$

$\frac{x^2}{2} + \frac{y^2}{2} = b$

$\boxed{x^2 + y^2 = c^2}$

which is required eqⁿ of orthogonal Trajectory of eqⁿ ①

Q.2 find orthogonal trajectory of the curve given by $x^2 + 2y^2 = c^2$

$\Rightarrow$ Given $x^2 + 2y^2 = c^2$

diff. w.r.to x

$2x + 4y\frac{dy}{dx} = 0$

replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$2x - 4y\cdot\frac{dx}{dy} = 0$$

$$2x = 4y\cdot\frac{dx}{dy}$$

$$x = 2y\cdot\frac{dx}{dy}$$

$$\frac{2}{x}\cdot dx = \frac{1}{y}\cdot dy$$

$$2\cdot\log x = \log y + \log K$$

$x^2 = yK$ is required orthogonal Trajectory

$\boxed{\text{II}}$ For Polar Co-ordinates:-

Step 1: Given $f(r, \theta, a) = 0$, where a is variable parameter.

Step 2: from diff eq^n of the family of the form $\phi(r, \theta, \frac{dr}{d\theta}) = 0$ by eliminating 'a'.

Step 3: replace $\frac{dr}{d\theta}$ by $\left(-r^2\frac{d\theta}{dr}\right)$, whereby the diff. eq^n of the family of orthogonal Trajectory becomes.

$$\phi\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$$

Step 4: solve step 3 which is the family of orthogonal Trajectory

Q.1 find orthogonal Trajectory of the circle defined by $r = a\cos\theta$ which all pass through the origin and have there centres on the initial line, a being the variable diameter

$\Rightarrow$

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$r = a \cdot \cos\theta$ —— ①

$\log r = \log a + \log \cos\theta$

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos\theta}(-\sin\theta)$

$\frac{1}{r} \cdot \frac{dr}{d\theta} = -\tan\theta$

which diff. eqⁿ of family ①

Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$.

$\frac{1}{\cancel{r}}\left(-r^{\cancel{2}} \cdot \frac{d\theta}{dr}\right) = -\tan\theta$

$r \cdot \frac{d\theta}{dr} = \tan\theta$

$\frac{dr}{r} = \frac{d\theta}{\tan\theta}$

$\int \frac{dr}{r} = \int \frac{d\theta}{\tan\theta}$

$\log r = \int \cot\theta \, d\theta + \log c$.

$\log r = \log \sin\theta + \log c$.

$\log r = \log(c \sin\theta)$

$\boxed{r = c \cdot \sin\theta}$

which is required eqⁿ of orthogonal Trajectories of ①

* Rate of DECAY of radioactive materials:

This law states that disintegration at any instant, is proportional to the ammount of material present

If u is ammount of material at any time t

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then $\frac{du}{dt} = -ku$, where k is constant

Ex Uranium disintegrate at a rate proportional to the amount is present at any instant. If M_1 and M_2 grams of uranium present at time t_1 and t_2 respectively find half life of uranium.

⇒ Let the mass of uranium at any time t be m grams
The eqn of disintegration of uranium is

$$\frac{dm}{dt} = -\mu m \quad , \text{when } \mu \text{ is constant.}$$

$$\int \frac{dm}{m} = -\mu \int dt + c$$

$$\log m = -\mu t + c = c - \mu t$$

$$\log m = c - \mu t \quad \text{——— (1)}$$

Initially when $t=0$ $m = M$ (say) so that

$$c = \log M$$

$$\mu t = \log M - \log m \quad \text{——— (2)}$$

when $t = T_1$ and $m = M_1$
$t = T_2$ and $m = M_2$

$$\mu T_1 = \log M - \log M_1 \quad \text{——— (3)}$$

$$\mu T_2 = \log M - \log M_2 \quad \text{——— (4)}$$

(iv) (iii)

$$\mu (T_2 - T_1) = \log M_1 - \log M_2$$

$$\mu = \frac{\log \left(\frac{M_1}{M_2}\right)}{T_2 - T_1}$$

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Let the mass reduce to half it's initial value in time T

i.e $t = T$, $m = \frac{1}{2}M$

$\therefore$ from eqn (11) $\quad \mu T = \log M - \log\frac{M}{2} = \log 2$

$$T = \frac{\log 2}{\mu} = \frac{(T_2 - T_1)\log 2}{\log(M_1/M_2)}$$

* Newton law of cooling.

According to this law, the temp of body changes at the rate which is propotional to the diff in temperature betn that of the surrounding medium and that of body itself

If θ_0 is temp of the ~~ball~~ surrounding and θ that of the body at any time t, then

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \quad \text{where } k \text{ is constant.}$$

Ex A metal ball is heated to a temperature of 100°c and at time t=0 it is placed in water which is maintained at 40°c. If the temprerature of the ~~surrounding~~ ball is reduced to 60°c in 4 minute find time at which the temp of Ball is 50°c

⇒ let the tempreature of the Ball be T°C at time t min

Then diff. eqn is given by

$$\frac{dT}{dt} = -k(T - 40)$$

$$\frac{dT}{T-40} = -kt$$

on integration

$$-kt = \log(T-40) + \log c$$

AT $t=0$, $T=100^\circ$

$$\log c = -\log 60^\circ$$

$$\therefore -kt = \log(T-40) - \log 60$$

$$-kt = \log \frac{T-40}{60} \quad \text{———} (1)$$

But $T=60$ at $t=4$

$$-4k = \log \frac{60-40}{60} = \log \frac{20}{60} = \log \frac{1}{3}$$

$$-4k = \log(1/3)$$

$$k = -\frac{1}{4}\log(1/3) = \frac{1}{4}\log 3$$

eq^n (1) becomes.

$$\therefore -\frac{t}{4}\cdot\log 3 = \log \frac{T-40}{60}$$

when $T=50$, we obtain

$$t = \frac{4\cdot\log 6}{\log 3} = 6.5 \text{ minutes}.$$

Q According to Newton law of cooling, the rate at which substance cool in moving air is proportional to the diff bet^n the temperature of the substance and that of the air. If the temp of air is 30°c and substance cool from 100°c to 70°c in 15 minutes, find when the temperature will be 40°c

⇒ By Newton law of cooling,

$$\frac{dT}{dt} = -k(T-30)$$

$$\frac{dT}{T-30} = -k\,dt$$

on integrating

$$\log(T-30) = -kt + C \quad \text{——} (1)$$

initially when $t=0$, $T=100$

from (1)

$$\log 70 = C$$

$$\therefore \log(T-30) = \cancel{\log} -kt + \log 70$$

$$kt = \log 70 - \log(T-30) \quad \text{——} (2)$$

Also when $t=15$, $T=70$ gives

$$kt = \log 70 - \log(T-30)$$

when $t=15$, $T=70$ give

$$15k = \log 70 - \log 40$$

$$k = \frac{\log 70 - \log 40}{15}$$

eqn (2) become,

$$\frac{(\log 70 - \log 40)t}{15} = \log 70 - \log(T-30)$$

$$\frac{t}{15} = \frac{\log 70 - \log(T-30)}{\log 70 - \log 40} \quad \text{——} (3)$$

when $T=40$

$$\frac{t}{15} = \frac{\log 70 - \log 10}{\log 70 - \log 40}$$

$$\frac{t}{15} = \frac{\log 7}{\log 7/4}$$

$$\frac{t}{15} = 3.48$$

$$t = 3.48 \times 15 = 52.20$$

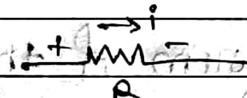
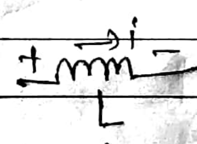
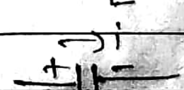
Hence the temperature will be $40^\circ C$ after 52.20 minutes.

* simple electric circuits:

We shall consider circuit made up of

(i) three passive element - resistance, inductance, capacitance

(ii) an active element:- voltage source which may be a battery or a generator

	Element	Symbol	Unit
(i)	Time	t	second
(ii)	current	$I = \frac{dq}{dt}$	ampere (A)
(iii)	Resistance, R		ohm (Ω)
(iv)	electric charge	q	coulomb
(v)	Inductance, L		Henry (H)
(vi)	capacitance, C		farad (F)

7) electromotive force or voltage

unit - Volt

Battery E = constant

(8) variable voltage Generator  Volt

Generator.
E = variable
Voltage

* Basic Relation:

(i) $i = \frac{dq}{dt}$ or $q = \int i \, dt$ (current is rate of flow of electricity)

(ii) Voltage drop across resistance R = Ri (Ohm law)

(iii) Voltage drop across capacitance $C = \frac{q}{C}$

* kirchhoff law:

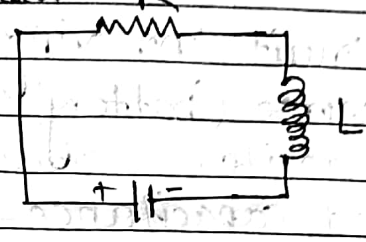
(I) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.

(II) The algebraic sum of the current flowing into (or from) any node is zero

* Differential Equation

(i) Current involving L and R along with a voltage source (battery) E, all in series.

Consider a circuit containing Resistance R and inductance L. Let i be the current flowing in circuit at any time t. Then by kirchoff law we have



$$Ri + L\cdot\frac{di}{dt} = E$$

or $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ which is linear diff. eqn.

$$I.F = e^{\int \frac{R}{L}\cdot dt} = e^{\frac{R}{L}\int 1\cdot dt} = e^{\frac{Rt}{L}}$$

and therefore it's soln is.

$$i\cdot e^{\frac{Rt}{L}} = \int \frac{E}{L} e^{Rt/L}\cdot dt + C$$

$$i\cdot e^{\frac{Rt}{L}} = \frac{E}{L}\frac{e^{Rt/L}}{\frac{R}{L}} + C$$

$$i\cdot e^{\frac{Rt}{L}} = \frac{E}{\cancel{L}}\cdot\frac{\cancel{L}}{R}\cdot e^{Rt/L} + C$$

$$\boxed{i = \frac{E}{R} + C\cdot e^{-\frac{Rt}{L}}} \quad \text{—(1)}$$

at initially there is no current in the circuit i.e $i=0$ when $t=0$

$$\Rightarrow \quad 0 = \frac{E}{R} + C$$

$$\Rightarrow \boxed{C = -\frac{E}{R}}$$

so eq (1) becomes

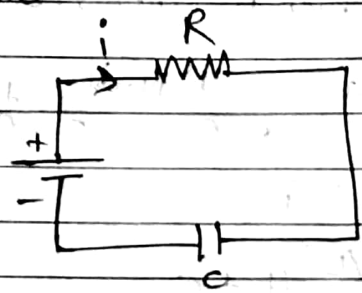
$$i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

As $t \to \infty$

$i = \frac{E}{R}$ which show that i increases with t and attains the maximum value $\frac{E}{R}$.

(ii) Circuit involving R and C along with a voltage source (battery) E all in series

Consider a circuit containing Resistance R and capacitance C in series with a voltage source (battery) E



Let i be a current flowing in the circuit at any time t. Then by kirchoffs first law, we have

sum of voltage across R and C = E (e.m.f)

i.e $Ri + \frac{q}{C} = E$

since $i = \frac{dq}{dt}$ (i.e rate change of charge)

$$R \cdot \frac{dq}{dt} + \frac{q}{C} = E$$

$$\frac{dq}{dt} + \left(\frac{1}{RC}\right) q = \frac{E}{R}$$

which is linear diff eqn

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$$I.F = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

~~and therefore sol^n is~~

$$q \cdot e^{\frac{t}{RC}} = \int \frac{E}{R} \cdot e^{\frac{t}{RC}} \cdot dt + B$$

$$= \frac{E}{R} \cdot \frac{e^{\frac{t}{RC}}}{\frac{1}{RC}} + B$$

$$= \frac{E}{\cancel{R}} \cdot \cancel{R}C \cdot e^{\frac{t}{RC}} + B$$

$$q \cdot e^{\frac{t}{RC}} = EC \cdot e^{t/RC} + B$$

$$q = EC + B \cdot e^{-t/RC}$$

Assume $q = q_0$ and $t = 0$

$$q_0 = EC + B$$

$$\Rightarrow B = q_0 - EC$$

$$q = EC + (q_0 - EC) \cdot e^{-t/RC}$$

$$q = (1 - e^{-t/RC}) EC + q_0 \cdot e^{-t/RC}$$

$$i = \frac{dq}{dt} = EC \cdot \frac{(0 - e^{-t/RC})}{-RC} + \frac{q_0 \cdot e^{-t/RC}}{} \times -\frac{1}{RC}$$

$$i = \frac{E\cancel{C} \cdot e^{-t/RC}}{R\cancel{C}} - \frac{q_0 \cdot e^{-t/RC}}{RC}$$

$$\boxed{i = \left(\frac{E}{R} - \frac{q_0}{RC}\right) e^{-t/RC}}$$

(iii) circuit involving L and C both in series after removing source Applied e.m.f.

consider a circuit contaning inductance L and capacitance C in series without applied e.m.f. let i be the current flowing in the circuit at any time t. Then by kirchhoff's law we have sum of voltage drop across L & C = 0

$$\text{i.e.}\quad L\cdot\frac{di}{dt}+\frac{q}{C}=0$$

$$\frac{di}{dt}=\frac{-q}{LC}$$

$$\frac{di}{dq}\cdot\frac{dq}{dt}=\frac{-q}{LC}$$

$$\cancel{\int i\,di}\quad \frac{di}{dq}\cdot i=\frac{-q}{LC}$$

$$\int i\cdot di=-\int\frac{q}{LC}\cdot dq+A$$

$$\frac{i^2}{2}=-\frac{q^2}{2LC}+A$$

$$i^2=\frac{-q^2}{LC}+B$$

Assuming $i=0,\ q=q_0$ when $t=0$

$$\therefore\ \boxed{B=\frac{q_0^2}{LC}}$$

$$i^2=\frac{-q^2}{LC}+\frac{q_0^2}{LC}$$

$$i^2=\frac{(q_0^2-q^2)}{LC}$$

$$i = \pm \frac{\sqrt{q_0^2 - q^2}}{\sqrt{LC}}$$

since q decrease as t increase

$$i = \frac{dq}{dt} = -\frac{1}{\sqrt{LC}}\sqrt{q_0^2 - q^2}$$

$$-\frac{dq}{\sqrt{q_0^2 - q^2}} = \frac{dt}{\sqrt{LC}}$$

Integrating $\cos^{-1}\left(\frac{q}{q_0}\right) = \frac{t}{\sqrt{LC}} + C$

Assuming $q = q_0$ when $t = 0$ $\therefore C = 0$

$$\left(\cos^{-1}(1) = C \Rightarrow C = 0\right)$$

$$\frac{t}{\sqrt{LC}} = \cos^{-1}\left(\frac{q}{q_0}\right)$$

$$\boxed{q = q_0 \cos\left(\frac{t}{\sqrt{LC}}\right)}$$

* Useful Formulae.

1. $\int e^{at} \cdot \sin bt \cdot dt = \frac{e^{at}}{a^2 + b^2}(a \sin bt - b \cdot \cos bt)$
2. $\int e^{at} \cdot \cos bt \cdot dt = \frac{e^{at}}{a^2 + b^2}(a \cdot \cos bt + b \cdot \sin bt)$
3. $\int e^{at} \sin bt \, dt = \frac{e^{at}}{\sqrt{a^2 + b^2}} \sin(bt - \phi)$ where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$
4. $\int e^{at} \cdot \cos bt \cdot dt = \frac{e^{at}}{\sqrt{a^2 + b^2}} \cos(bt - \phi)$ where $\phi = \tan^{-1}(b/a)$

Ex A resistance of 100 ohms, an inductance of 0.5 henry are conducted in series with battery of 20 volts find the current in a circuit as function of t

⟹ By kirchhoff's law

$$L\cdot\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L} I = E \quad \text{which is linear in } I$$

let $P = \frac{R}{L}$, $Q = \frac{E}{L}$

$$I\cdot F = e^{\int R/L \, dt} = e^{\frac{Rt}{L}}$$

$$G\cdot S \quad I\cdot e^{\frac{Rt}{L}} = \int \frac{E}{L}\cdot e^{\frac{Rt}{L}}\cdot dt + A$$

$$= \frac{E}{L}\left[\frac{e^{\frac{Rt}{L}}}{\frac{R}{L}}\right] + A$$

$$I\cdot e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + A$$

$$I = \frac{E}{R} + A\cdot e^{-\frac{Rt}{L}} \longrightarrow (*)$$

$t=0$, $I=0$

$$0 = \frac{E}{R} + A$$

$$\boxed{A = -\frac{E}{R}} \quad \text{put in eq } (*)$$

$$I = \frac{E}{R} - \frac{E}{R}\cdot e^{-\frac{Rt}{L}}$$

$$I = \frac{E}{R}\left[1 - e^{-\frac{Rt}{L}}\right]$$

Given $R = 100$ ohms
$L = 0.5$ henery
$E = 20$ volts.

$$I = \frac{20}{100}\left(1 - e^{\frac{-100t}{0.5}}\right) = \frac{1}{5}\left(1 - e^{-200t}\right)$$

Ex.2 In a circuit containing inductance L, resistance R, and voltage E, The current I is given by $E = RI + L\cdot\frac{dI}{dt}$
Given $L = 640$ H, $R = 250\,\Omega$, $E = 500$ V
I being zero when $t = 0$ find the time that elapse before it reaches 90% of it's maximum value.

$\Rightarrow$ Given that

$$E = RI + L\cdot\frac{dI}{dt}$$

$$\Rightarrow \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

By privious example sol^n is

$$I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

maximum value of I is obtained when $t \to \infty$

$$I_{max} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \text{ --- (1)}$$

$$= \frac{E}{R}\left(1 - e^{-\infty}\right)$$

$$I_{max} = \frac{E}{R}(1-0) = \frac{E}{R}$$

$$I_{max} = \frac{90}{100}\frac{E}{R} \quad \text{(90% of it's value)}$$

putting $t = t_1$ for 90% of I_{max}

$$\frac{9}{10}\cancel{\frac{E}{R}} = \cancel{\frac{E}{R}}\left(1 - e^{\frac{-Rt_1}{L}}\right)$$

$$\frac{9}{10} = 1 - e^{-\frac{Rt_1}{L}}$$

$$e^{-\frac{Rt_1}{L}} = 1 - \frac{9}{10} = \frac{1}{10}$$

$$e^{-\frac{Rt_1}{L}} = \frac{1}{10}$$

$$-\frac{Rt_1}{L} = \log\frac{1}{10} = -\log 10$$

$$\frac{Rt_1}{L} = \log 10$$

Given $R = 250,\ L = 640$

$$\frac{250 \times t_1}{640} = \log 10$$

$$t_1 = \frac{\log 10 \times 640}{250} = 5.89$$ sec which is required time

Q.3 A resistance of $100\,\Omega$, an inductance of 0.5 henry are connected in series with a battery of 20 volts, find the current in a circuit as function of t

⇒ By kirchhoff's law

$$L\cdot\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

~~we know sol^n is~~

$$I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

$R = 100,\ L = 0.5,\ E = 20$

$$I = \frac{20}{100}\left(1 - e^{-\frac{100 \times t}{0.5}}\right)$$

$$I = \frac{1}{5}\left(1 - e^{-200t}\right)$$

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Q.4 The eq^n of L-R circuit is given by $L\cdot\frac{dI}{dt}+RI=10\sin t$ if $I=0$, $t=0$, express I as fun of t

$\Rightarrow$ sol^n we know

$$L\cdot\frac{dI}{dt}+RI=10\sin t$$

$$\frac{dI}{dt}+\frac{R}{L}\cdot I=\frac{10\sin t}{L}$$

Here $I.F=e^{\frac{Rt}{L}}$

G.S. is $I\cdot e^{\frac{Rt}{L}}=\frac{10}{L}\int e^{\frac{Rt}{L}}\cdot\sin t\cdot dt+B$

$$=\frac{10}{L}\frac{e^{\frac{Rt}{L}}}{\sqrt{\frac{R^2}{L^2}+1}}\sin(t-\phi)+B$$

where $\tan\phi=\frac{L}{R}$

$$I=\frac{10}{\sqrt{R^2+L^2}}\cdot\sin(t-\phi)+B\cdot e^{\frac{-Rt}{L}} \quad \text{--- (1)}$$

when $t=0$, $I=0$

$$\therefore 0=\frac{-10}{\sqrt{R^2+L^2}}\cdot\sin\phi+B$$

$$B=\frac{10}{\sqrt{R^2+L^2}}\cdot\sin\phi$$

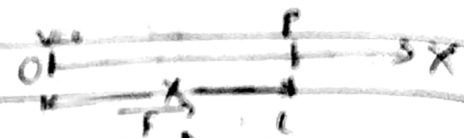
put in (1)

$$I=\frac{10}{\sqrt{R^2+L^2}}\cdot\sin(t-\phi)+\frac{10}{\sqrt{R^2+L^2}}\cdot\sin\phi\cdot e^{\frac{-Rt}{L}}$$

$$\boxed{I=\frac{10}{\sqrt{R^2+L^2}}\left[\sin(t-\phi)+\sin\phi\cdot e^{\frac{-Rt}{L}}\right]}$$

* Rectilinear Motions :-

Rectilinear motions is motion of a body along a straight line let a body of mass m start moving from fixed point O along a straight line OX under the action of a force F. After any time t let it moving at P where OP = x then



(i) its velocity (v) = $\frac{dx}{dt}$

(ii) acceleration (a) = $\frac{dv}{dt}$ or $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v$$

$$a = v\cdot\frac{dv}{dx}$$

so $\boxed{a = \frac{dv}{dt} = v\cdot\frac{dv}{dx} \text{ or } \frac{d^2x}{dt^2}}$

Newton second law of motion states that

$F = \frac{d}{dt}(mv)$ if m is constant then

$$F = m\cdot\frac{dv}{dt} = m\cdot a$$

$$F = m\cdot\frac{dv}{dt} \text{ or } mv\cdot\frac{dv}{dx} \text{ or } m\cdot\frac{d^2x}{dt^2}$$

where F is effective force

* D Alemberts principle

Algebraic sum of the force acting on body along a given direction is equal to the product of mass × acceleration in that direction.

i.e $\boxed{\text{Net force} = \text{Mass} \times \text{Acceleration}}$

* Net force means Algebraic sum of the forces acting along that direction i.e direction of motion.

* Illustrations on Rectilinear Motion:-

A body of mass m, falling from rest is subjected to the force of gravity and an air resistance proportional to the square of velocity 'kv^2'. If it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right)$ where $mg = ka^2$

$\Rightarrow$ The force acting on body are

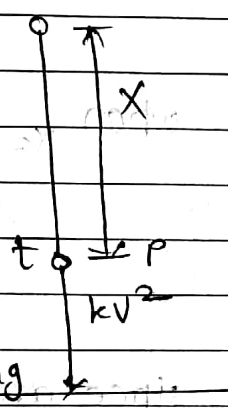
(i) weight acting on downwards $= mg$

(ii) air resistance acting upword $= -kv^2$

Net force on the body $= mg - kv^2$.

eqn of motion

$$F = mv\frac{dv}{dx}$$



$$mg - kv^2 = mv\cdot\frac{dv}{dx}$$

$(mg = ka^2)$ Given

$$ka^2 - kv^2 = mv\cdot\frac{dv}{dx}$$

$$k(a^2 - v^2) = m\cdot v\cdot\frac{dv}{dx}$$

$$\int\frac{k}{m}dx + C = \int\frac{v\,dv}{a^2-v^2}$$

$$\frac{kx}{m} + C = -\frac{1}{2}\log(a^2-v^2)$$

$\because x=0,\ v=0$

$\therefore\ C = -\frac{1}{2}\log a^2$

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$$\frac{1}{2}\left[\log a^2 - \log(a^2-v^2)\right] = \frac{kx}{m}$$

$$\boxed{\frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right)}$$

Q. A particle is moving in a straight line with an acceleration $k\left[x+\frac{a^4}{x^3}\right]$ directed toword origin. If it start from rest at a distance a from the origin, Prove that it will arrived origin at the end of time $\frac{\pi}{4\sqrt{k}}$

$\Rightarrow$ eqⁿ of motion

$$v\cdot\frac{dv}{dx} = -k\left(x+\frac{a^4}{x^3}\right)$$

$$\int v\cdot dv = -k\int x+\frac{a^4}{x^3}\cdot dx$$

$$\frac{v^2}{2} = -k\left[\frac{x^2}{2} - \frac{a^4}{2x^2}\right] + C$$

when $x=a$, $v=0$,

$$\therefore \boxed{C=0}$$

$$v^2 = k\left(\frac{a^4-x^4}{x^2}\right)$$

since accelaration is directed toword origin

$$v = -\sqrt{k}\cdot\sqrt{\frac{a^4-x^4}{x^2}}$$

$$\frac{dx}{dt} = -\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x}$$

$$\int\frac{x\,dx}{\sqrt{a^4-x^4}} = -\sqrt{k}\int dt$$

put $x^2 = u$ $\therefore x \cdot dx = \frac{du}{2}$

$$\frac{1}{2}\int \frac{du}{\sqrt{a^4-u^2}} = -\sqrt{k}\cdot t + c$$

$$\sin^{-1}\left(\frac{u}{a^2}\right) = -2\sqrt{k}\cdot t + c_1$$

$$\sin^{-1}\left(\frac{x^2}{a^2}\right) = -2\sqrt{k}\cdot t + c_1$$

when $t=0$, $x=a$, $c_1 = \pi/2$

$$\therefore \sin^{-1}\left(\frac{x^2}{a^2}\right) = \frac{\pi}{2} - 2\sqrt{k}\cdot t$$

At $x=0$, $0 = \frac{\pi}{2} - 2\sqrt{k}\cdot t$

$$\boxed{t = \frac{\pi}{4\sqrt{k}}}$$

Q. Assuming that the resistance to movement of a ship through water in the form of $a^2+b^2 \cdot v^2$, where v is the velocity and a and b are constant, write down the diff. eqn for retardation of the ship moving with engine stopped. Prove that the time in which the speed fall to one half it's original value. u is given by $\frac{W}{abg}\cdot \tan^{-1}\frac{abu}{2a^2+b^2u^2}$ where W is weight of ship.

$\Rightarrow$ $m\cdot\frac{dv}{dt} = -(a^2+b^2\cdot v^2)$ but $m = \frac{W}{g}$

$$\frac{W}{g}\cdot\frac{dv}{dt} = -(a^2+b^2\cdot v^2)$$

$$\frac{W}{g}\cdot\frac{1}{a^2+b^2\cdot v^2}dv = -dt$$

since speed is fall ~~from~~ to one half of it's original value, u to $u/2$

$$\frac{W}{g}\int_u^{u/2}\frac{1}{a^2+b^2\cdot v^2}dv = -\int_0 dt$$

$$-\frac{W}{gb^2}\left[\frac{b}{a}\tan^{-1}\frac{bv}{a}\right]_{u}^{u/2} = -t$$

$$t = \frac{W}{abg}\left[\tan^{-1}\frac{bv}{a}\right]_{u/2}^{u}$$

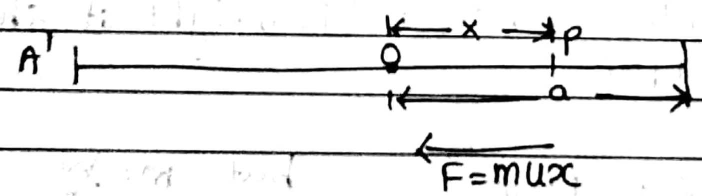
$$= \frac{W}{abg}\left[\tan^{-1}\frac{bu}{a} - \tan^{-1}\frac{bu}{2a}\right]$$

$$= \frac{W}{abg}\tan^{-1}\frac{\left(\frac{bu}{a} - \frac{bu}{2a}\right)}{\left(1+\frac{b^2u^2}{2a^2}\right)}$$

$$\boxed{t = \frac{W}{abg}\tan^{-1}\left(\frac{abu}{2a^2+b^2u^2}\right)}$$

* simple Harmonic motion:

"If a particle moves on straight line, so that the force acting on it is always directed toward a fixed point on the line and proportional to it's distance from the point, the particle is said to move in simple Harmonic motion (S.H.M)



Let O be the fixed point and P be a position of a particle at any time t. Let OP = x. force acting on particle is mux, where m is the mass of particle and u is constant consider it as +ve

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We know
eq^n of motion is

$$m \cdot \frac{d^2x}{dt^2} = F$$

$$m \cdot \frac{d^2x}{dt^2} = -mux \quad \text{or} \quad v \cdot \frac{dv}{dx} = -ux$$

$$v \cdot dv = -u \cdot x \cdot dx$$

$$\frac{v^2}{2} = -u \cdot \frac{x^2}{2} + A$$

$$v^2 = -ux^2 + A \cdot$$

Assuming that particle start from point A ($OA = a$) and it's initial velocity is zero
when $x = a$, $v = 0$;

$$0 = -ua^2 + A \Rightarrow \boxed{A = ua^2}$$

$$\therefore v^2 = -ux^2 + ua^2$$

$$v^2 = u(a^2 - x^2)$$

$$v = \sqrt{u(a^2 - x^2)}$$

$$\therefore \boxed{v = \frac{dx}{dt} = -\sqrt{u(a^2 - x^2)}}$$

(-ve sign is attach because x decreases as t increases.)

$$-\frac{dx}{\sqrt{u(a^2 - x^2)}} = dt$$

$$-\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{u} \cdot dt \qquad \left(\because -\int \frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right)\right)$$

$$\cos^{-1}\left(\frac{x}{a}\right) = \sqrt{u} \cdot t + B$$

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when $t=0$, $x=a$ $\therefore B=0$

we get $\boxed{x = a\cdot\cos(\sqrt{\mu}\cdot t)}$

particle will reach O in time t_1, given by

$$0 = a\cdot\cos(\sqrt{\mu}\cdot t_1)$$

$$\therefore \sqrt{\mu}\cdot t_1 = \cos^{-1}(0)$$

$$\sqrt{\mu}\cdot t_1 = \pi/2$$

$$\boxed{t_1 = \frac{\pi}{2\sqrt{\mu}}}$$

It's velocity at that time will be $\sqrt{\mu}\cdot a$

As soon as it will cross O, the direction of force will changes, partical will move with velocity $\sqrt{\mu}\cdot a$ and ultimately come to rest at A'. $OA = OA'$. and time taken by particle to travel from O and A' will be $\frac{\pi}{2\sqrt{\mu}}$

Due to attraction, particle will start moving toward O. It is oscillatory motion. Due to this is called simple harmonic motion

$\boxed{\text{period of oscillation is } \frac{2\pi}{\sqrt{\mu}}}$

Q. A particle executes S.H.M. when it's 2 cm from mid path, It's velocity is 10 cm/sec. when it's 6 cm from centre of path, it's velocity 2 cm/sec. find it's period. ~~and its greatest acceleration.~~

⇒ Given

$x = 2\ cm$, $V = 10\ cm/sec$

$x = 6\ cm$, $V = 2\ cm/sec$

we know that

$V = \sqrt{\mu(a^2 - x^2)}$ for S.H.M

$10 = \sqrt{\mu(a^2 - 2^2)} \Rightarrow 10 = \sqrt{\mu(a^2 - 4)}$

$\Rightarrow 100 = \mu(a^2 - 4)$ —— (1)

Now $x = 6m$, $V = 2\ cm/s$.

$V = \sqrt{\mu(a^2 - x^2)}$

$2 = \sqrt{\mu(a^2 - 36)}$

$4 = \mu(a^2 - 36)$ —— (2)

solve eqn (1) and (2)

$100 = \cancel{\mu a^2} - 4\mu$

$\underline{-\ 4 = \cancel{\mu a^2} - 36\mu}$
$\ \ -\qquad\ -\qquad +$

$96 = 32\mu$

$\boxed{\mu = 3}$

Period of S.H.M $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\sqrt{3}}$

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