# unit-4

## chapter 5:- curve Tracing.



* curve divided into five type.

Type 1: curve given by cartesian eqn $(x,y)$ (Explicit relation)

Type 2: Curve given by cartesian eqn $(x,y)$ (Implicit relation)

Type 3: Curve given by polar eqn $(r,\theta)$

Type 4: curve given by polar eqn of the type $r = a\sin n\theta$ and $r = a\cos n\theta$ known as Rose curve

Type 5: curve given by parametric Equations.

* cartesian eqn divided into two part

(i) Explicit equation

In this eqn we take terms of x on one side and terms of y on another side.

e.g $y^2 = \frac{2a-x}{x^3}$

(ii) Implicit function:

we cannot seprate x and y.

e.g $x^2+2xy+y^2=0$

* Rule 1: symmetry

(A) About x-axis

If the given eqn has even power of y everywhere, then curve is symmetrical about x-axis

e.g $y^2=x$.

(B) about Y-axis

If the given $eq^n$ has even power of $x$ everywhere then curve is symmetrical about Y-axis

e.g $x^2=Y$

(C) about line $Y=X$.

replace $x$ by $y$. and $y$ by $X$, if the given $eq^n$ remain unchanged then curve is symmetrical about the line $y=X$

e.g $X \cdot Y = C$

(D) About the line $y=-x$

replace $x$ by $-y$ and $y$ by $-x$, if the $eq^n$ remain unchanged then curve is symmetrical about $y=-x$

(E) In opposite quadrants:

replace $x$ by $-x$ and $y$ by $-y$, if $eq^n$ remain unchanged, the curve is symmetrical opposite quadrant.

Note: If we get symmetry about x-axis or Y-axis there is no need to check other point of symmetry.

Rule 2 A] origin:

put $x=0$, if we get $y=0$ i.e we get $(0,0)$ then curve passing through origin.

B] Tangent at origin:

If the curve passes through origin, then the tangents at origin can be obtained by equating to zero, then lowest degree terms taken together in the given equation.

**Rule 3: ~~Intersection with~~ Co-ordinate axis.**

A] Intersection with x-axis
put $y=0$ and find different value of $x$

B] Intersection with Y-axis
put $x=0$ and find different value of $Y$

(C) Intersection with the line $y=x$ or $y=-x$
If the curve is symmetrical about the line $y=x$ or $y=-x$ then find the point of intersection of the curve with these line by putting $y=x$ or $y=-x$ respectively.

**Note:-** point given by Rule 3 is called special point.

**Rule 4:**
Tangents at special points.
To find tangents at special points. Differentiate the given equation w.r.to i.e find $\frac{dy}{dx}$ and if

(i) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$ then tangent is parallel to x-axis

(ii) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$ then tangent is parallel to Y-axis

(iii) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = +ve$ then tangent make acute angle with X-axis.

(iv) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -ve$ then tangent make obtuse angle with Y-axis.

**Note:** point (iii) and (iv) are only for implicit relation

**Rule 5 :** Asymptotes:

Asymptotes are tangent to the curve at infinity.

[A] Asymptotes parallel to X-axis

Asymptotes parallel to x-axis, can be obtained by equating to zero, coifficent of highest power of x taken togethere

[B] Asymptotes parallel to Y-axis

Asymptotes parallel to Y-axis, can be obtained by equating to zero, coifficent of highest power of y taken togethere

[C] oblique asymptotes (only for implicit relation)

Let $y = mx + c$ be the oblique asymptote

put $y = mx + c$ in the given eq^n and by comparing L.H.S and R.H.S ~~By~~ find the value of m and c

**Rule 6:** Region of absence.

Arrange the given eq^n as $y^2 = f(x)$ or $x^2 = f(y)$ then check -ve value of

(i) odd power x (OR y)

(ii) Addition

(iii) subtraction

**Note:** Any value is -ve if it is less than zero

~~Curve does not~~

e.g. $x+a<0$
$x<-a$
i.e curve does not exist for value less than $-a$.

Type 1: solved example on curves given by cartesian curves. (Explicit Relation).

Q Trace the curve $y^2(2a-x)=x^3$

Rule 1: symmetry:
even power of $y$ everywhere, therefore curve is symmetrical about x-axis.

Rule 2: origin.
(A) origin:
Given $y^2(2a-x)=x^3$
put $x=0$
$y^2\cdot(2a)=0$
$\boxed{y=0}$
when $x=0$, we get $y=0$ i.e $(0,0)$
$\therefore$ curve passes through origin.

(B) Tangent at origin.
To find tangents always arrange the given eqn equal to zero, without any bracket multiplication.

Given $y^2(2a-x)=x^3$
$2ay^2-xy^2=x^3$
$\underbrace{2ay^2}_{2}-\underbrace{xy^2}_{3}-\underbrace{x^3}_{3}=0$ → power of $x$ and $y$.

∴ To find tangent at origin, equate to zero lowest degree term $=0$

$$2ay^2=0$$
$$\therefore y^2=0$$
$$y=0$$

Here we find tangent, so $y=0$ is line not a point

$\therefore y=0$ i.e $x$-axis

$\therefore$ $x$-axis is tangent at origin.

**Rule 3: Intersection with axes:**

A] Interaction with $x$-axis

Given $y^2(2a-x)=x^3$

put $y=0$

we get $0=x^3$

$x=0$

i.e $(0,0) \rightarrow$ origin

B] Interaction with $Y$-axis

Given $y^2(2a-x)=x^3$

put $x=0$

$y^2(2a)=0$

$y=0$

i.e $(0,0)$

**Rule 4: Tangent at special point**

In Rule 3, we are getting only point $(0,0)$ and $(0,0)$ is not a special point, Therefore there is no tangent at special point for this example

Rule 5: ~~Asymptote~~ Asymptote

Asymptotes are tangents at infinity and to find tangent we always arrange the given eqⁿ equal to zero

A] Asymptotes parallel to x-axis.

$\Rightarrow y^2(2a-x) = x^3$

$2ay^2 - xy^2 = x^3$

$\downarrow \quad \downarrow \quad \downarrow$ power of $x$ only

$0 \quad 1 \quad 3$

To find Asymptotes parallel to x-axis, equate to zero. coifficent of higher power of $x$ taken togethore

i.e.

i.e. $-1 = 0$

which is not possible

$\therefore$ NO Asymptote parallel to x-axis.

B] Asymptotes parallel to Y-axis.

$\Rightarrow$ Given $y^2(2a-x) = x^3$

$\therefore 2ay^2 - xy^2 - x^3 = 0$

$\downarrow \quad \downarrow \quad \downarrow$ power of $y$

$2 \quad 2 \quad 0$

Coiff. of higher power of $y$

$2a - x = 0$

$x = 2a$

$\therefore x = 2a$ is the asymptote parallel to Y-axis

## Rule 6: Region of Absence

⇒ Given $y^2(2a-x)=x^3$

i.e $y^2 = \dfrac{x^3}{2a-x}$

then we check -ve value of given by

(i) odd power of $x$

$\therefore x^3 < 0 \quad , x<0$

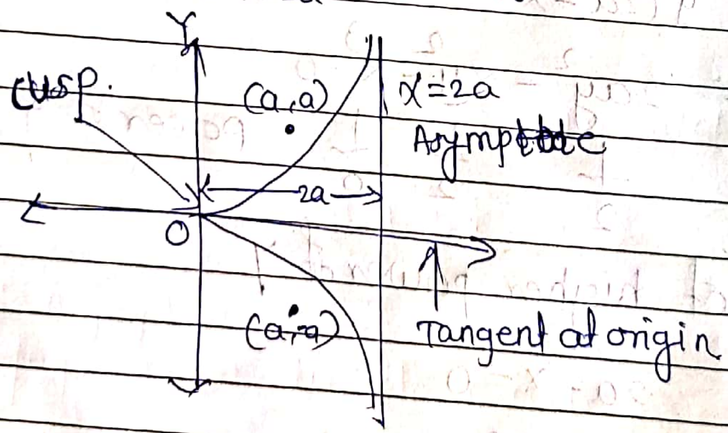
(ii) subtraction

$2a-x < 0 \quad \Rightarrow 2a < x$

i.e $x > 2a$

The scale of Region of Absence as follow

| $-\infty$ | $0$ | | $2a$ | $\infty$ |
|---|---|---|---|---|
| No curve below | | curve betⁿ 0 to 2a | | No curve beyond 2a |



Q. Trace the curve $xy^2 = a^2(a-x)$

Rule 1: Symmetry.

$xy^2 = a^2(a-x)$

$\therefore$ Even power of y everywhere, therefore curve is symmetrical about x-axis.

Rule 2: origin.

A] origin

$xy^2 = a^2(a-x)$

put $x=0$

$0 = a^2(a-x)$

$a^3 = 0$

i.e when put $x=0$ we do not get $y=0$

$\therefore$ curve does not passes through origin.

B] Tangent at origin.

As curve doesnot passes through origin, there wont be any tangent at origin

Rule 3: Intersection with axes.

A] Interaction with x-axis.

Given $xy^2 = a^2(a-x)$

put $y=0$

$\frac{0}{a^2} = a-x$

$x=a$ $\longrightarrow$ This point of intersection

ie $(a,0)$ $\rightarrow$ special point.

B] Intersection with Y-axis

put $x=0$

$0 \cdot y^2 = a^2(a-0)$

$0 = a^3$

Not possible

So No intersection with Y-axis

**Rule4:**

Tangent at special point $(a,0)$

To find tangent we always arrange the given eqn equal to zero

$\therefore$ Given $xy^2 = a^2(a-x)$

$$xy^2 = a^3 - a^2x$$

$$\therefore xy^2 - a^3 + a^2x = 0$$

Diff. w.r.to $x$

$$\cancel{y^2 = 0 + a^2 = 0} \quad x \cdot 2y \cdot \frac{dy}{dx} + y^2 + a^2 = 0$$

$$2xy \cdot \frac{dy}{dx} = -y^2 - a^2$$

$$\frac{dy}{dx} = \frac{-y^2 - a^2}{2xy}$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \frac{-a^2}{2(a)(0)}$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty \rightarrow \text{parallel to Y-axis.}$$

$\therefore$ At special point $(a,0)$ tangent is parallel to Y-axis

**Rule5: Asymptotes:**

A] Asymptotes parallel to x-axis

$$xy^2 = a^2(a-x)$$

$$xy^2 - a^3 + a^2x = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$1 \qquad 0 \qquad 1$$

coiff. of higher power of $x$ taken togethere $= 0$

$$y^2 + a^2 = 0$$

$$y^2 = -a^2$$

$y = \sqrt{-a^2}$ which not possible.

$\therefore$ No asymptotes parallel to $x$-axis.

B] Asymptotes parallel to Y-axis.

$$xy^2 - a^3 + a^2x = 0$$

$$\underset{2}{\downarrow} \quad \underset{0}{\downarrow} \quad \underset{1}{\downarrow} \quad \text{power of } y$$

coifficent of highest power of $y = 0$

$x = 0 \rightarrow$ line (Not a point as we are finding tangent)

$\therefore X = 0 \rightarrow$ Y-axis is asymptote parallel to Y-axis.

Rule 6:- Region of Absence:

Arrange the given eq^n as

$$y^2 = \frac{a^2(a-x)}{x}$$

then check -ve value of

(i) odd power of $x$

i.e $x < 0$

(ii) subtraction

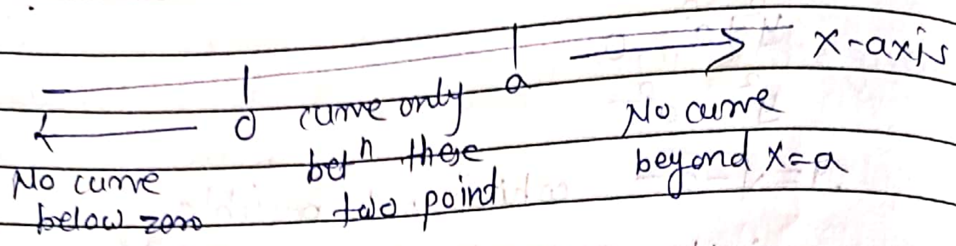
$$a - x < 0$$

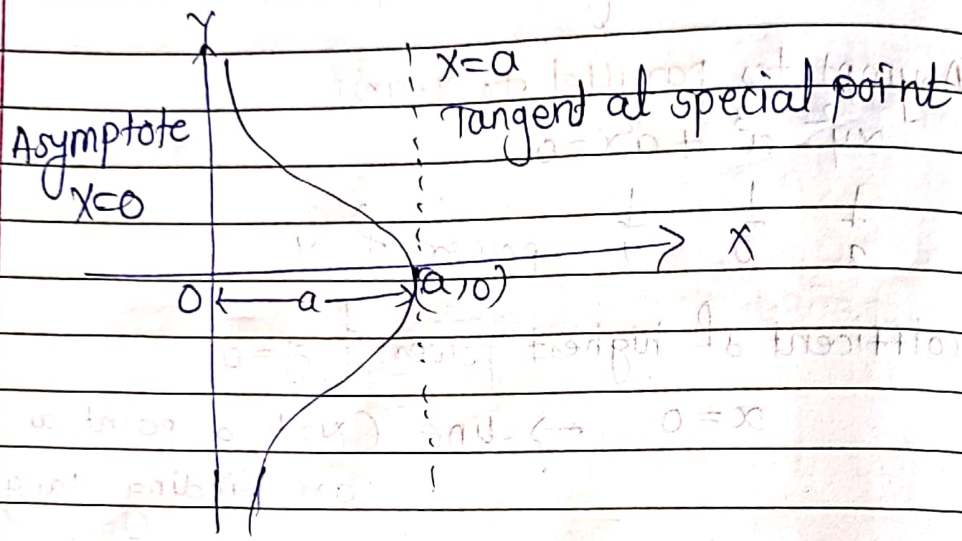
$$a < x$$

i.e $x > a$

i.e there is no curve in the region $x < 0$ and $x > a$

The scale of region of absence is as follow




Q. Trace the curve $a^2x^2 = y^3(2a-y)$

$\Rightarrow$ Rule 1: Symmetry:

Given $a^2x^2 = y^3(2a-y)$

Even powers of x everywhere, curve is symmetrical about Y-axis.

Rule 2: Origin.

A] Origin.

Given $a^2x^2 = y^3(2a-y)$

put $y=0$

$a^2x^2=0$

$\boxed{x=0}$

$\because$ for $y=0$ we get, $x=0$ i.e $(0,0)$

$\therefore$ curve passes through origin.

B] Tangents at origin:

Given $a^2x^2 = y^3(2a-y)$

$\therefore a^2x^2 - 2ay^3 + y^4 = 0$

$\downarrow$ 2 $\quad \downarrow$ 3 $\quad \downarrow$ 4 powers of $x$ and $y$.

To find tangents at origin

lowest degree terms $= 0$

$a^2x^2 = 0$

$x^2 = 0$

i.e $x = 0 \longrightarrow$ Y-axis.

$\therefore$ Y-axis is tangent at origin.

Rule 3 Intersection with axes.

A] Intersection with X-axis.

Given $a^2x^2 = y^3(2a-y)$

put $y=0$

$a^2x^2 = 0$

$x = 0$

i.e $(0,0) \Rightarrow$ origin

B] Intersection with Y-axis.

$\Rightarrow$ Given $a^2x^2 = y^3(2a-y)$

put $x=0$

$0 = y^3(2a-y)$

i.e $y^3 = 0$ or $2a-y=0$

$y = 0$ or $y = 2a$

$\therefore x=0$ we get $y = 0, 2a$.

$\therefore (0,0)$ & $(0,2a)$ are the two point which intersect Y-axis. out of these two points $(0,2a)$ is

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the special point

Rule 4: Tangents at special points:

⇒ $a^2x^2 = y^3(2a-y)$

$a^2x^2 - 2ay^3 + y^4 = 0$

diff. w.r.t. to $x$

$a^2 \cdot 2x - 2a \cdot 3y^2 \cdot \frac{dy}{dx} + 4y^3 \cdot \frac{dy}{dx} = 0$

$-2a^2x = \frac{dy}{dx}(4y^3 - 6ay^2)$

$$\frac{dy}{dx} = \frac{-2a^2x}{4y^3 - 6ay^2}$$

$$\left(\frac{dy}{dx}\right)_{(0,2a)} = \frac{-2a^2(0)}{-6a(2a)^2 + 4(2a)^3}$$

$$\left(\frac{dy}{dx}\right)_{(0,2a)} = 0 \rightarrow \text{parallel to X-axis}$$

∴ At special point $(0,2a)$ tangent is parallel to X-axis.

Rule 5: Asymptotes.

A] Asymptotes parallel to x-axis

we have,

$a^2x^2 - 2ay^3 + y^4 = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$2 \quad 0 \quad 0$ power of $x$

To find asymptotes parallel to x-axis we have coiff. of highest power of $x = 0$

i.e $a^2 = 0$

No meaning

so No Asymptote parallel to X-axis.

B] Asymptote parallel to Y-axis.

we have $a^2x^2 - 2ay^3 + y^4 = 0$

$\downarrow \;\; 0 \quad \downarrow \;\; 3 \quad \downarrow \;\; 4$ power of y

To find parallel to Y-axis, we have coifficent of highest power of y = 0

i.e $1 = 0$

No asymptote parallel to Y-axis.

Rule 6: Region of Absence.

Arrange the given equation as

$$x^2 = \frac{y^3(2a-y)}{a^2}$$

Check for negative value of.

(i) odd power of y

i.e $y^3 < 0$ i.e $y < 0$

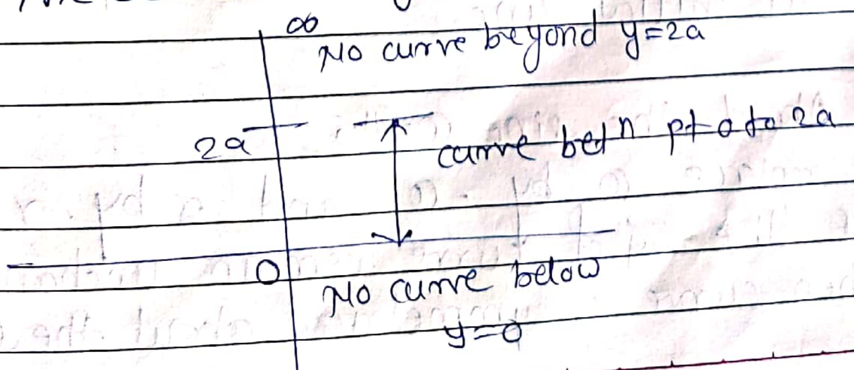
(ii) subtraction

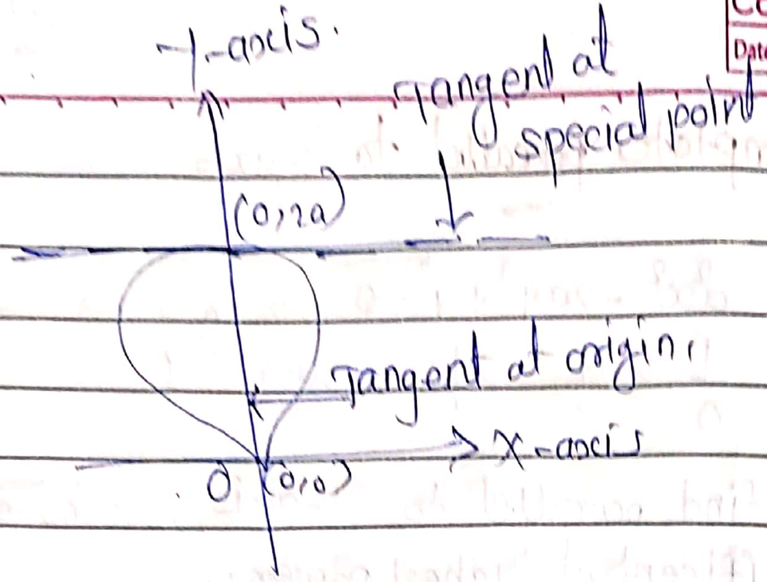
$\therefore 2a - y < 0$

$2a < y$

$\therefore y > 2a$

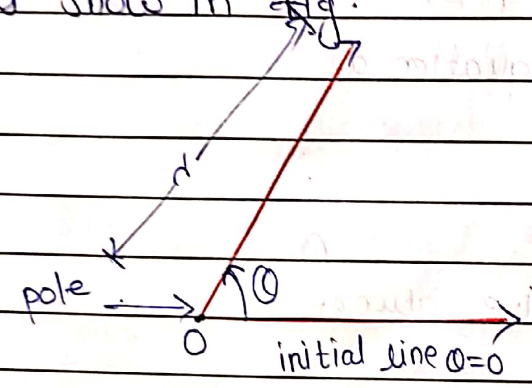
$\therefore$ The scale of Region of Absence is as fallow






## Curves given by Polar Co-ordinates.

In polar co-ordinates system $r$ represent radius vector and $\theta$ represent angle measured positive in anticlockwise direction as show in fig.




Note: In polar co-ordinates, the curve is often given by the eq$^n$ $r=f(\theta)$

Rule 1: symmetry:

A] About Intial line (x-axis)

Replace $\theta$ by $-\theta$, if the equation of the curve remain unchanged then curve is symmetrical about initial line.

B] About the line $\theta=\pi/2$

(1) replace $\theta$ by $-\theta$ and $r$ by $-r$ if the eq$^n$ of curve remain unchanged then curve is symmetrical about the line

Rule 3

| | | | | |
|---|---|---|---|---|
| $\theta =$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| $r =$ | | | | |



through the pole, perpendicular to the initial line

(ii) The same symmetry also exist if the eq$^n$ remain unchanged when $\theta$ replace by $\pi - \theta$

**Rule 2:**

The curve will pass through pole if some value of $\theta$, $r$ become zero

To check that, we put $r=0$ and if we get a valid value of $\theta$, then curve passes through pole.

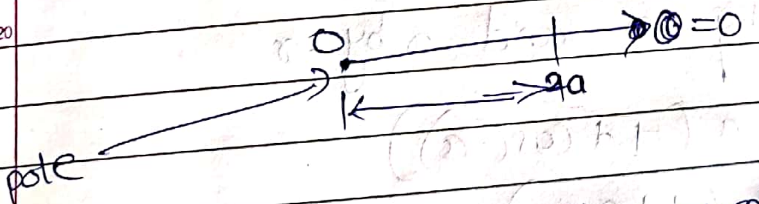
**Rule 4:**

Angle between the radius vector and tangent $\phi$

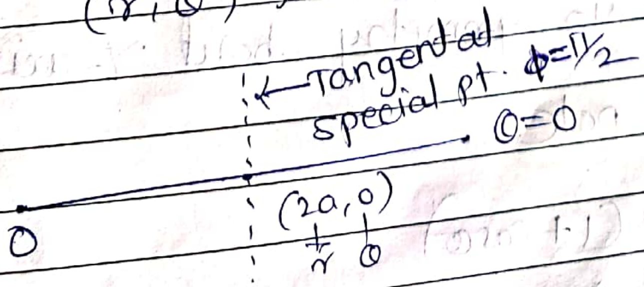
$$\tan\phi = r \cdot \frac{d\theta}{dr} \quad \text{and find } \phi$$

Here $\phi$ is inclination of tangent at special point for Ex $r = a(1+\cos\theta)$

when $\theta = 0$ we get $r = 2a$




Then $\phi = \pi/2$ is a line perpendicular to initial line at $(r, \theta)$ i.e $(2a, 0)$



* curve given by polar co-ordinates.

Ex Trace the curve $r = a(1+\cos\theta)$

Rule 1: symmetry

A] about the initial line

$r = a(1+\cos\theta)$

replace $\theta$ by $-\theta$

$r = a(1+\cos(-\theta))$

$r = a(1+\cos\theta)$

$\therefore$ eqn remain unchanged

$\therefore$ curve symmetrical about the initial line x-axis.

B] about line $\theta = \pi/2$ (i.e y-axis)

Given $r = a(1+\cos\theta)$

replace $\theta$ by $-\theta$ and $r$ by $-r$

$\therefore -r = a(1+\cos(-\theta))$

$\therefore -r = a(1+\cos\theta)$

eqn changes.

$\therefore$ No symmetry about y-axis.

Rule 2: pole.

$r = a(1+\cos\theta)$

put $r=0$

$0 = a(1+\cos\theta)$

$\cos\theta = -1$

$\therefore \theta = \pi, 3\pi, 5\pi \ldots$

but $\theta = 3\pi, 5\pi, \ldots$ are value above $360°$

$\therefore$ we will neglect them

$\therefore \boxed{\theta = \pi}$

$\therefore$ curve passes through pole (origin)

**Rule 3:**

| $\theta$ | $0$ | $90°$ | $180°$ | $270°$ |
|---|---|---|---|---|
| $r$ | $2a$ | $a$ | $0$ | $a$ |

(i) for $\theta = 0$

$r = a(1+\cos\theta) = a(1+1) = 2a$

$\theta = 90°$ $\quad r = a(1+0) = a$

$\theta = 180°$ $\quad r = a(1-1) = 0$

$\theta = 270°$ $\quad r = a(1+0) = a.$

**Rule 4:** Tangent at special points.

$r = a(1+\cos\theta)$

diff. w. r. to $\theta$

$$\frac{dr}{d\theta} = a(-\sin\theta) = -a\sin\theta$$

on reciprocal

$$\therefore \frac{d\theta}{dr} = \frac{1}{-a\sin\theta}$$

multiplying $r$ on both side

$$r \cdot \frac{d\theta}{dr} = \frac{-r}{a\sin\theta}$$

But $r = a(1+\cos\theta)$

$$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{1+\cos\theta}{\sin\theta}$$

$$\therefore r\frac{d\theta}{dr} = \frac{-\cancel{2}\cdot\cos^2(\theta/2)}{\cancel{2}\sin\theta/2\cdot\cos\theta/2} = -\cot\theta/2$$

$$\left(\begin{array}{l} 1+\cos 2\theta = 2\cdot\cos^2\theta \\ \sin 2\theta = 2\sin\theta\cdot\cos\theta \end{array}\right)$$

$$\therefore r\frac{d\theta}{dr} = -\cot\theta/2$$

but $r\cdot\frac{d\theta}{dr} = \tan\phi$

$$\therefore \tan\phi = -\cot\left(\theta/2\right) = \tan\left(\pi/2 + \theta/2\right)$$

$$\therefore \phi = \pi/2 + \frac{\theta}{2}$$

$\therefore$ Table for $r, \theta, \phi$

| | | | | |
|---|---|---|---|---|
| $\theta$ | $0^\circ$ | $90^\circ$ | $180^\circ$ | $270^\circ$ |
| $r$ | $2a$ | $a$ | $0$ | $a$ |
| $\phi$ | $90^\circ$ | $135^\circ$ | $180^\circ$ | $225^\circ$ |

Explanation of $r, \theta, \phi$

Note that, when $\theta = 0^\circ$, we get $\phi = \pi/2 \rightarrow$ i.e tangent at $90^\circ$ with $\theta = 0$ at a distance $r = 2a$.

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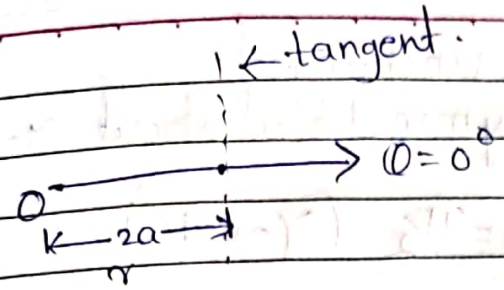
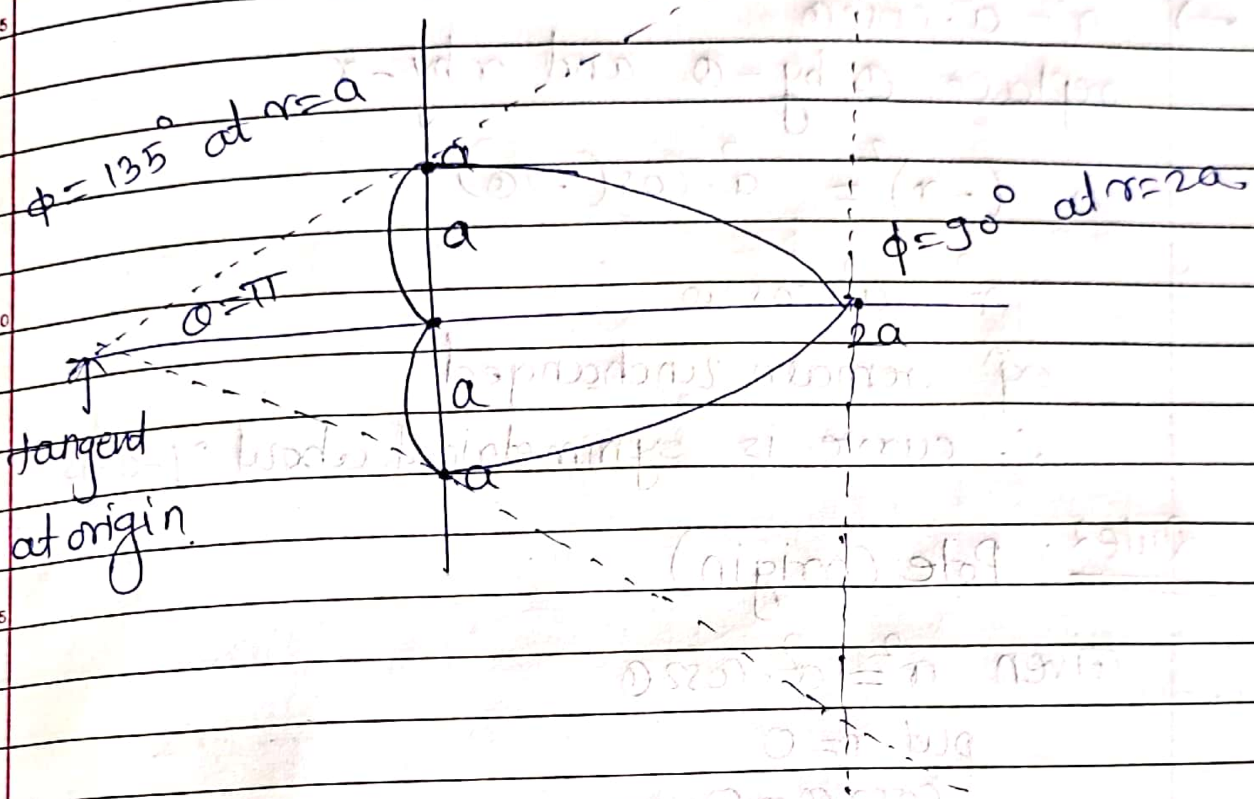



**Homework**

Trace the curve $r = a(1+\sin\theta)$

Q. Trace the curve $r^2 = a^2 \cos 2\theta$.

* Rule I : symmetry

A] about X-axis

Given $r^2 = a^2 \cdot \cos 2\theta$

replace $\theta$ by $-\theta$

$r^2 = a^2 \cdot \cos(-2\theta)$

but $\cos(-\theta) = \cos\theta$

$\therefore r^2 = a^2 \cdot \cos 2\theta$

∴ equation remain unchanged
∴ curve is symmetrical about x-axis

B] About a line $\theta = \pi/2$ (y-axis)

⟹ $r^2 = a^2 \cdot \cos 2\theta$
replace $\theta$ by $-\theta$ and $r$ by $-r$
∴ $(-r)^2 = a^2 \cdot \cos(-2\theta)$

$r^2 = a^2 \cdot \cos 2\theta$
eqn remain unchanged
∴ curve is symmetrical about y-axis.

Rule 2: Pole (origin)

Given $r^2 = a^2 \cdot \cos 2\theta$
put $r = 0$
$\cos 2\theta = 0$
$2\theta = \cos^{-1}(0) = 90^\circ$
$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, 9\pi/4, 11\pi/4$

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$
~~∴ curve is symmetrical about y-axis~~
∴ curve passes through pole

Rule 3

| $\theta$ | 0 | $90^\circ$ | $180^\circ$ | $270^\circ$ |
|---|---|---|---|---|
| $r$ | a | — | a | — |

when $\theta = 0$
$r^2 = a^2 \cdot \cos 2\theta$
$r^2 = a^2 \Rightarrow r = a$

(ii) when $\theta = 90^{\circ}$

$r^2 = a^2 \cdot \cos 2\theta \Rightarrow r^2 = -a^2$ which is not possible

(iii) $\theta = 180^{\circ}$

$r^2 = a^2 \cdot \cos 2\theta = a^2 \cdot \cos(360)$

$r^2 = a^2 (1) = a^2$

$\boxed{r = a}$ radius of $r$ never be -ve

(iv) $\theta = 270^{\circ}$

$r^2 = a^2 \cdot \cos 2\theta = a^2 \cdot \cos(540)$

$r^2 = -a^2$ which is not possible

Rule 4:- Tangent at special points.

$\Rightarrow$ Given $r^2 = a^2 \cdot \cos 2\theta$

diff. w. r. to $\theta$

$$2r \cdot \frac{dr}{d\theta} = -a^2 \cdot \sin 2\theta \cdot 2$$

$$r \cdot \frac{dr}{d\theta} = -a^2 \cdot \sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{-a^2 \cdot \sin 2\theta}{r}$$

on reciprocal

$$\frac{d\theta}{dr} = \frac{r}{-a^2 \sin 2\theta}$$

multiply by $r$ on both side

$$r \cdot \frac{d\theta}{dr} = \frac{r^2}{-a^2 \sin 2\theta}$$

$$r \cdot \frac{d\theta}{dr} = -\frac{\cancel{a^2} \cdot \cos 2\theta}{\cancel{a^2} \cdot \sin 2\theta}$$

$$r \cdot \frac{d\theta}{dr} = -\cot 2\theta$$

but $r \cdot \frac{d\theta}{dr} = \tan\phi$

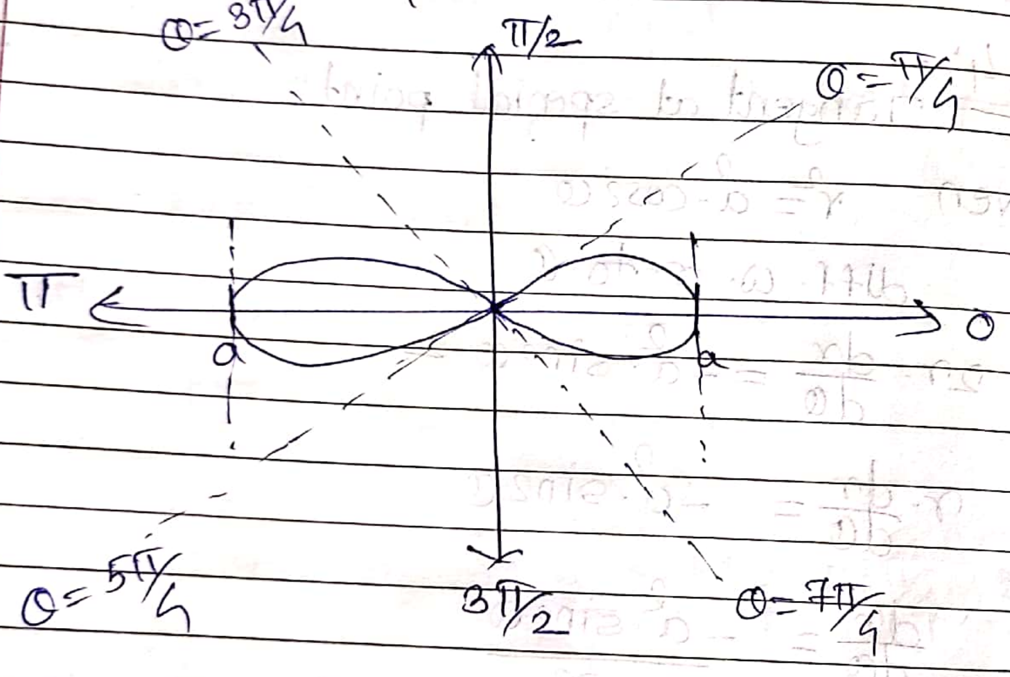
$\therefore \tan\phi = \tan(\pi/2 + 2\theta)$

$\phi = \pi/2 + 2\theta$

when $\theta = 0 \Rightarrow \phi = \pi/2$ → parallel to Y-axis.

when $\theta = 180^\circ$ $\phi = 450^\circ = 360^\circ + 90 = \pi/2$

parallel to Y-axis.




✗ curves given by polar co-ordinates of the type $r = a\sin n\theta$ or $r = a\cos n\theta$ (Rose curve)

Rule 1:

(i) curve given by $r = \cos n\theta$ are symmetrical about initial line (X-axis)

(ii) curve given by $r = a\sin n\theta$ are symmetrical about a line $\perp$er to initial line i.e. Y-axis.

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**Rule 2:**

For $r = a \cdot \sin n\theta$ and $r = a \cdot \cos n\theta$ divides each quandrant into $n$ equal parts.

**Rule 3:**

(i) If $n$ is even $2n$ ~~poles~~ loops.

(ii) If $n$ is odd, $n$ loops.

**Rule 4:**

(i) For $r = a \cdot \cos n\theta$, draw first loop along $\theta = 0^\circ$

(ii) For $r = a \sin n\theta$ draw first loop along $\theta = \frac{\pi}{2n}$

**Rule 5:**

(i) If $n$ is even, draw loops in every sector consectively

(ii) In $n$ is odd, keep two sector vacant bet$^n$ the loop.

**Rule 6:-**

Tangents at origin.
put $r = 0$ and find diff value of $\theta$. Neglet value of $\theta$ beyond $360^\circ = 2\pi$

Note $\cos^{-1}(0) = (2n+1)\pi/2$
and $\sin^{-1}(0) = n\pi$ , $n = 0, 1, 2, 3 \ldots$

**Rule 7:**

Determine then angle $\phi$ of tangent using the formulae $\tan\phi = r \cdot \frac{d\theta}{dr}$.

Q. Trace the curve $r = a.\sin 2\theta$

$\Rightarrow$ Given

$r = a.\sin 2\theta$

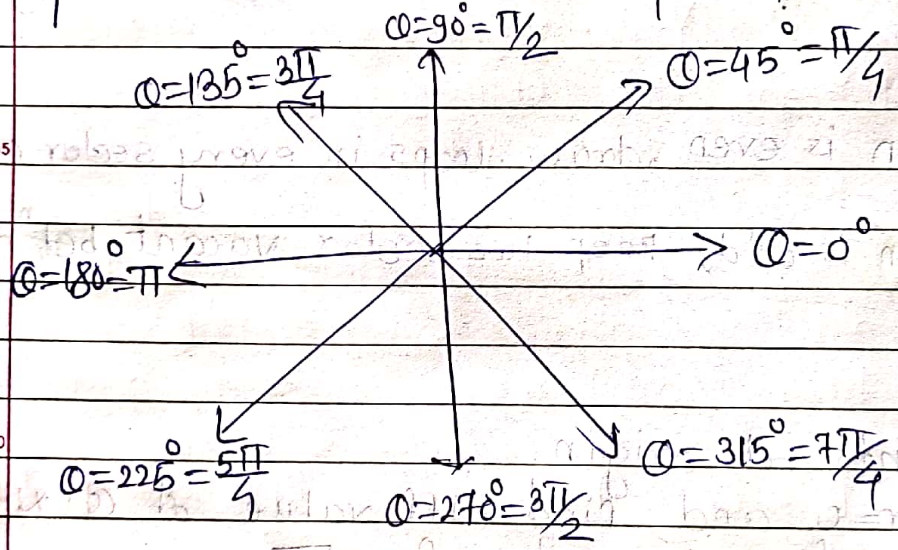
comparing with $r = a\sin n\theta$

$\therefore n = 2$

Rule 1: symmetry

The curve is symmetrical about a line $\perp^{er}$ to initial line i.e Y-axis.

Rule 2: Here $n=2$ therefore divide each quandrant into two equal parts.




Rule 3: Here $n = 2 =$ even, which implies there will be 4 loops.

Rule 4: for $r = a\sin n\theta$ draw first loop along

$\theta = \frac{\pi}{2n} = \frac{\pi}{4} = 45^\circ$

Rule 5: Here $n = 2 =$ even, therefore draw a loop in every sector consecutively.

Rule 6: Tangents at origin

Given $r = a \cdot \sin 2\theta$

put $r = 0$

$0 = a \sin 2\theta$

$\therefore \sin 2\theta = 0$

$2\theta = \sin^{-1}(0)$

$2\theta = n\pi$

$2\theta = 0, \pi, 2\pi, 3\pi, 4\pi \ldots$

$\theta = 0, \pi/2, 3\pi/2, 2\pi \ldots$

These value of $\theta$ are tangent at origin.

Rule 7:

Tangent at special points,

we have $r = a \sin 2\theta$

Diff. w.r.to $\theta$

$$\frac{dr}{d\theta} = 2a \cdot \cos 2\theta$$

on reciprocal

$$\frac{d\theta}{dr} = \frac{1}{2a \cdot \cos 2\theta}$$

multiplying by $r$ on both sides.

$$r \cdot \frac{d\theta}{dr} = \frac{r}{2a \cdot \cos 2\theta} = \frac{a \cdot \sin 2\theta}{2a \cdot \cos 2\theta}$$

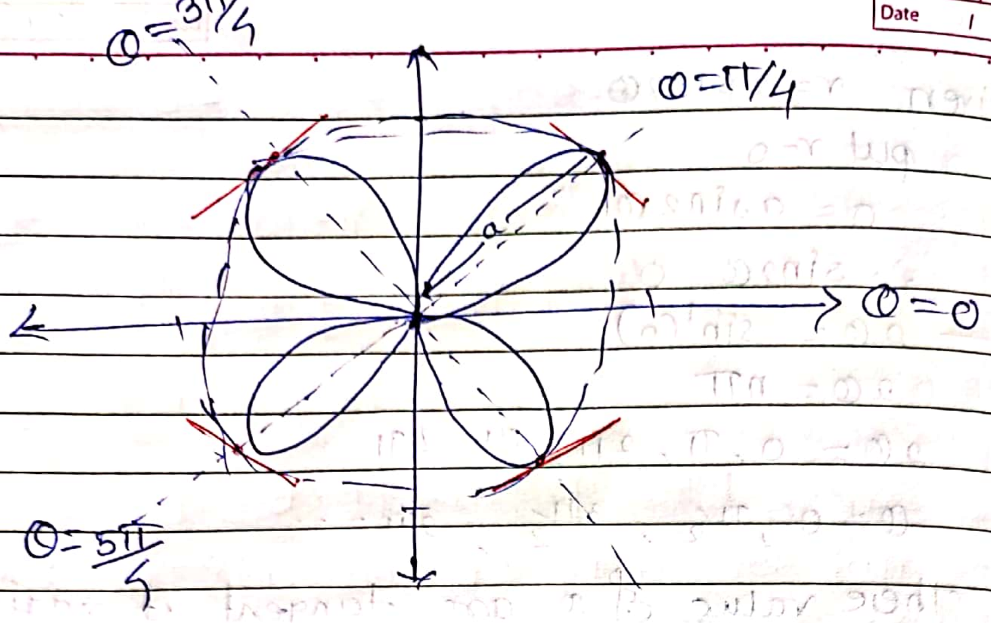
$$\therefore \tan\phi = \frac{1}{2} \cdot \tan 2\theta$$

Rule 8. Table, $r, \theta, \phi$

| $\theta$ | $r$ | $\phi$ |
|---|---|---|
| 0 | 0 | 0 |
| 45 | a | $90^\circ$ |
| 90 | 0 | 0 |
| 135 | -a | $90^\circ$ |
| 180 | 0 | 0 |
| 225 | a | $90^\circ$ |
| 270 | -a | 0 |

| $r$ | $\theta$ | $\phi$ |
|---|---|---|
| $315^\circ$ | -a | $90^\circ$ |

Homework : $r = a\sin 3\theta$




* curve given by parametric eqⁿ

consider a parametric eqⁿ are given by $x = f(t)$ and $y = g(t)$. Then the rule to trace the curves given by parametric eqⁿ are as follow.

Rule 1: Symmetry:

(i) If $x = f(t)$ is an even fun i.e $f(-t) = f(t)$ and $y = g(t)$ is an odd fun i.e $[g(-t) = -g(t)]$ then curve is symmetrical about x-axis

(ii) If $x = f(t)$ is an odd fun and $y = g(t)$ is an even fun, then curve is symmetrical in opposite quadrant.

(iii) If $x = f(t)$ and $y = g(t)$ both are odd, then curve is symmetrical in opposite quandrants

(iv) For value of $t$ and $-t$, $x$ remains unchanged But $y$ has equal and opposite values, then curve is symmetrical about x-axis.

(v) Replace $t$ by $\pi - t$, if $y$ has the same value and $x$ has equal and opposite value, then curve is symmetrical about Y-axis.

Rule 2: origin.

If for some value of $t$, both $x$ and $y$ are zero, then curve passes through origin. we can check it by using the process mentioned below.

step 1: put $x=0$ and find $t$

step 2: put this value of $t$ in $y$

step 3: If we get $y=0$ for some value of $t$ then the curve passes through origin.

Rule 3:

Tangent at special point i.e $\frac{dy}{dx}$

find $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Rule 4: Prepare a table of $t, x, y, \frac{dy}{dx}$.

Ex(1) Trace the cycloid $x = a(t+\sin t)$, $y = a(1+\cos t)$

Sol$^n$

$\Rightarrow$ Rule 1: symmetry

let $x = f(t) = a(t+\sin t)$

put $t = -t$

$f(-t) = a[-t+\sin(-t)]$

But $\sin(-\theta) = -\sin\theta$

$f(-t) = a[-t-\sin t]$

$\therefore f(-t) = -a[t+\sin t]$

$f(-t) = -f(t)$,

$\therefore x = f(t)$ is an odd fun.

Also $y = g(t) = a(1+\cos t)$

put $t = -t$

But $\cos(-\theta) = \cos\theta$

$g(-t) = a[1+\cos(-t)]$

$g(-t) = g(t)$

$\therefore y = g(t)$ is an even fun

Now as $x$ is odd and $y$ is even fun, then the given curve is symmetrical about $y$-axis

**Rule 2: origin.**

Note: if we get $x=0, y=0$ for any value of $t$ then curve passes through origin.

we have

$y = a(1+\cos t)$

put $y=0$

$1+\cos t = 0$

$\cos t = -1$

$\cos t = \cos(180^\circ)$

$\therefore t = 180^\circ = \pi$

put $t = 180^\circ = \pi$ in $x = a(t+\sin t)$

$x = a(\pi + \sin\pi)$

But $\sin\pi = 0$

~~Also~~ $x = a\pi$

i.e for $t=\pi$ we do not get $x=0$ and $y=0$

$\therefore$ curve does not pass through origin.

**Rule 3: find $\frac{dy}{dx}$**

we have $y = a(1+\cos t)$

diff. w.r.to t

$$\frac{dy}{dt} = a[0 - \sin t]$$

$$\therefore \frac{dy}{dt} = -a \cdot \sin t \quad \text{--- (1)}$$

Also $x = a(t + \sin t)$

diff. w.r.to t

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{--- (2)}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$\therefore$ from eq (1) & (2)

$$\frac{dy}{dx} = \frac{-a \cdot \sin t}{a(1 + \cos t)}$$

$$\therefore \frac{dy}{dx} = -\frac{2 \cdot \sin(t/2) \cdot \cos(t/2)}{2 \cdot \cos^2(t/2)}$$

$$\therefore \frac{dy}{dx} = -\frac{\sin(t/2)}{\cos(t/2)}$$

$$\frac{dy}{dx} = -\tan(t/2)$$

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Rule 4: Prepare a table of $t, x, y, \frac{dy}{dx}$

| $t$ | $0$ | $\pi/2 = 90^\circ$ | $\pi = 180^\circ$ | $\frac{3\pi}{2} = 270^\circ$ | $2\pi = 360^\circ$ |
|---|---|---|---|---|---|
| $x$ | $0$ | $a(\pi/2+1)$ | $a\pi$ | $a(\frac{3\pi}{2}-1)$ | $2a\pi$ |
| $y$ | $2a$ | $a$ | $0$ | $a$ | $2a$ |
| $dy/dx$ | $0$ | $-1$ | $-\infty$ | $-1$ | $0$ |

(i) for $t=0$

(A) $x = a(t+\sin t)$

$x = a(0+\sin 0)$

$\boxed{x=0}$

(B) $y = a(1+\cos t)$

$y = a(1+\cos 0)$

$y = 2a$

(C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan(0) = 0$

(ii) for $t = \frac{\pi}{2} = 90^\circ$

(A) $x = a(t+\sin t)$

$x = a(\pi/2 + \sin 90)$

$x = a(\pi/2+1)$

(B) $y = a(1+\cos t)$

$y = a(1+\cos 90)$

$y = a(1+0)$

$\boxed{y=a}$

(C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan\left(\frac{90}{2}\right)$

$\frac{dy}{dx} = -\tan(45) = -1$

(iii) for $t = \pi = 180^\circ$

(A) $x = a(t+\sin t)$

$x = a(\pi + \sin\pi)$

$x = a(\pi+0)$

$x = a\pi$

(B) $y = a(1+\cos t)$

$y = a(1+\cos 180)$

$y = a(1-1)$

$\boxed{y=0}$

(C) $\frac{dy}{dx} = -\tan(t/2)$

$= -\tan\left(\frac{180}{2}\right)$

$\frac{dy}{dx} = -\infty$

(iv) $t = \frac{3\pi}{2} = 270^\circ$

(A) $x = a(t+\sin t)$

$x = a\left(\frac{3\pi}{2} + \sin 270\right)$

$x = a\left(\frac{3\pi}{2} - 1\right)$

(B) $y = a(1+\cos t)$

$y = a(1+\cos 270)$

$y = a(1+0)$

$y = a$

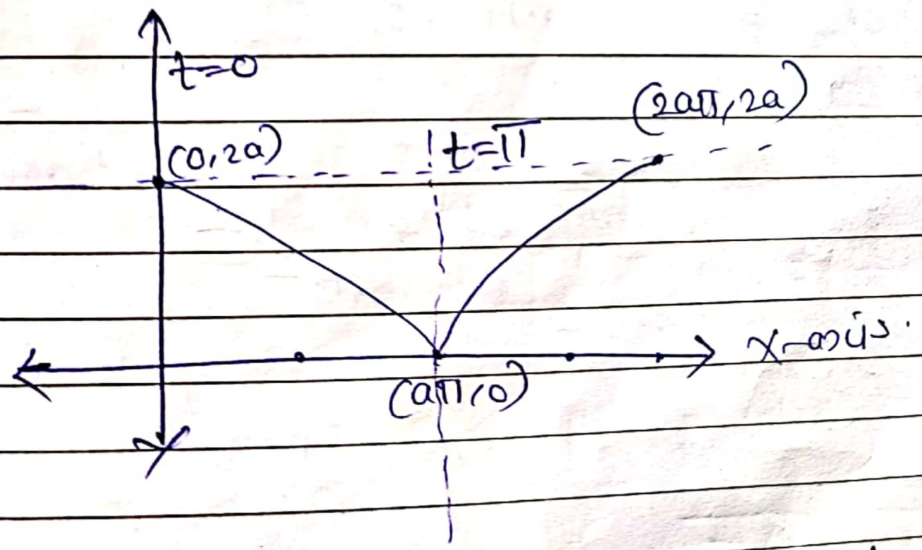
(C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan\left(\frac{270}{2}\right)$

$\boxed{\frac{dy}{dx} = -1}$

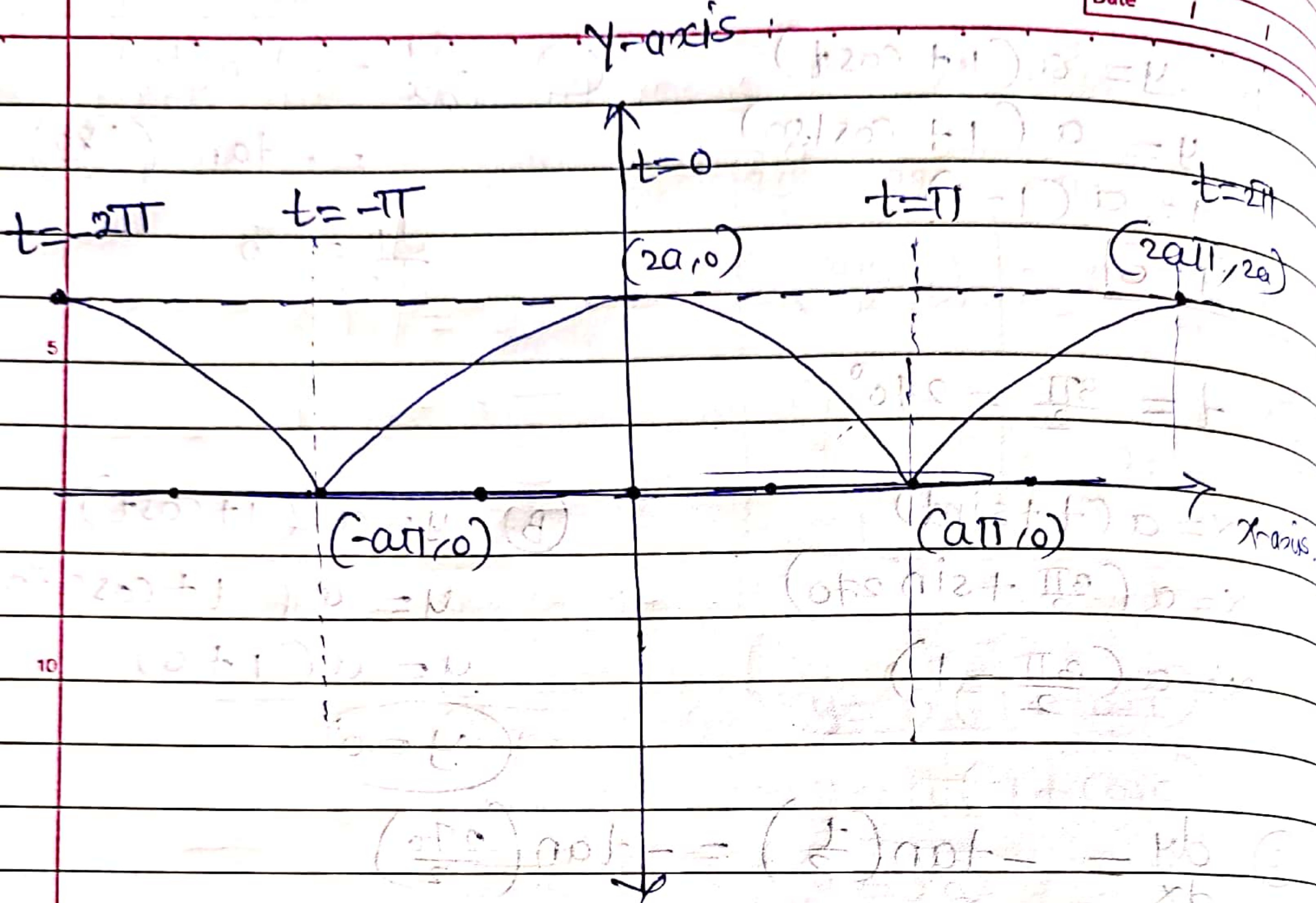
Note (i) $\frac{dy}{dx} = 0 \rightarrow$ tangent parallel to x-axis.

(ii) $\frac{dy}{dx} = \pm\infty \rightarrow$ tangent is parallel to Y-axis.

curve As follow.




But curve is symmetrical about Y-axis.




Homework (1)

Imp

(i) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

(ii) Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

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Q. Trace the curve
$x^{2/3} + y^{2/3} = a^{2/3}$

$\Rightarrow$ This curve is known as Asteroid
Also know as star shape curve
Parametric eq of curve is given by
$x = a\cdot\cos^3 t$ and $y = a\cdot\sin^3 t$.

Rule 1: symmetry.

(i) ~~let~~ $\cancel{x = f(t) = a\cdot\cos^3 t}$

$\cancel{Y = a\cdot\sin^3 t = g(t)}$.

put $t = -t$

$f(-t) = a\cdot[\cos(-t)]^3$

$\cancel{f(-t)} = a\cdot\cos^3 t$

$f(-t) = f(t)$

$\therefore$ $x = f(t)$ is an even fun.

Also
$y = g(t) = a\cdot\sin^3 t$.
$g(-t) = a[\sin(-t)]^3$
$g(-t) = -a\cdot\sin^3 t$
$g(-t) = -g(t)$
$\therefore$ y is odd function.
As x is even fun and y is odd fun, curve is symmetrical about x-axis.

(ii) symmetry about Y-axis.
we have $x = a\cdot\cos^3 t$
Replace t by $\pi - t$
$\cancel{x} = a[\cos(\pi-t)]^3$
$x = -a\cdot\cos^3 t$
$\therefore$ x has opposite value

| $t$ | $0$ | $\pi/2$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $x$ | $a$ | $0$ | $-a$ | $0$ | $a$ |
| $y$ | $0$ | $a$ | $0$ | $-a$ | $0$ |
| $\frac{dy}{dx}$ | $0$ | $-\infty$ | $0$ | $\infty$ | $0$ |

Also $y = a\cdot\sin^3 t$

Replace $t$ by $\pi - t$

~~$y = a\,[\sin(\pi - t)]^3$~~

$y = a\sin^3 t$.

$\therefore$ y has same value

$\therefore$ After replacing $t$ by $\pi - t$, $x$ has opposite value and $y$ has same value. Hence curve is symmetrical about y-axis.

Rule 2: origin.

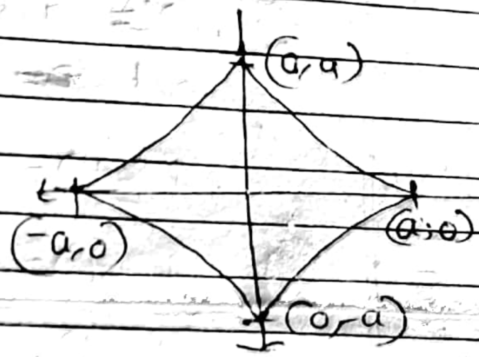
we have $x = a\cdot\cos^3 t$

put $x = 0$

$\cos^3 t = 0$

$\cos t = 0 = \cos\pi/2$

$\therefore t = \pi/2 = 90^\circ$




Now substitute $t = \pi/2$ in

$y = a\sin^3 t$

$y = a(1)^3 = a$.

$\therefore$ for $t = \pi/2$, $x = 0$ and $y = a$ i.e. $(0, a)$.

$\therefore$ curve does not pass through origin.

Rule 3: find $\frac{dy}{dx}$.

we have $y = a\cdot\sin^3 t$

Diff. w.r.to t.

$$\frac{dy}{dt} = a\cdot 3\sin^2 t\cdot\cos t \quad \text{--- (1)}$$

Also $x = a\cdot\cos^3 t$

$$\therefore \frac{dx}{dt} = -a\cdot 3\cos^2 t\,(\sin t) \quad \text{--- (2)}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cdot 3\sin^2 t\cos t}{-a\cdot 3\cos^2 t\cdot\sin t} = -\tan t$$

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# Chapter 6: Rectification of curve



The process of finding the length of an arc of a curve is called as Rectification of curve.

A rectifiable curve is one that has finite length. Rectification is process of finding the length of arc of the curve.

In this chapter, we consider mainly rectifiable curve which have the eqn in cartesian, polar and parametric form and introduce the technique to find length of curve.

| Type | eqn of curve | Formulae to calculate length. |
|---|---|---|
| curve given by cartesian co-ordinates. | ① $y = f(x)$ | $s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ |
| | ② $x = g(y)$ | $s = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ |
| curve given by polar co-ordinates | ③ $r = f(\theta)$ | $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$ |
| | ④ $\theta = f(r)$ | $s = \int_{r_1}^{r_2} \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$ |
| curve given by parametric equation | ⑤ $x = f(t)$ and $y = g(t)$ | $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$ |

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Q. find the circumference of circle of radius a

$\Rightarrow$ eqⁿ of circle with radius a and centre at origin is given by

$$x^2+y^2=a^2$$

$$\therefore y^2=a^2-x^2$$

$$\therefore y=\sqrt{a^2-x^2}$$

Diff. w.r.to $x$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}\cdot\frac{d}{dx}\left(a^2-x^2\right) \quad \left(\because \frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}(0-2x)$$

$$\therefore \frac{dy}{dx}=\frac{-x}{\sqrt{a^2-x^2}}$$

squaring on both side

$$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{a^2-x^2}$$

adding 1 on both side

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{a^2-x^2}$$

$$1+\left(\frac{dy}{dx}\right)^2=\frac{a^2-\cancel{x^2}+\cancel{x^2}}{a^2-x^2}=\frac{a^2}{a^2-x^2}$$

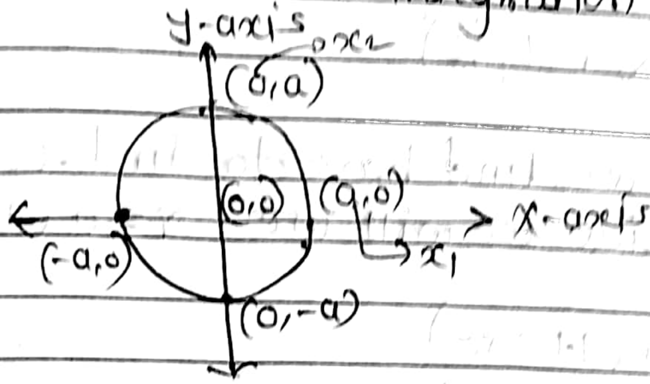
Taking squar root

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{a}{\sqrt{a^2-x^2}}$$

Now to calculate length of curve we hav

$$S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

now To find limit of integration




Now, we find length of the circle only for $1^{st}$ quadrant and then multiply by 4 to calculate the total length

$$S = \int_a^0 \frac{a}{\sqrt{a^2 - x^2}} \cdot dx$$

$$S = a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_a^0$$

$$S = a\left(0 - \pi/2\right)$$

$$S = -a\pi/2$$

$S = a\pi/2$ (length cannot -ve)

$S = 4 \times a\pi/2$ (total length)

$$\boxed{S = 2\pi a}$$

This is required circumference of circle

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Q. find the length of the cardioide $r = a(1+\cos\theta)$ which is lies outside the circle $r = -a\cdot\cos\theta$.

$\Rightarrow$ To find $\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

**Note**: As we ~~find~~ have to find length of the cardioide we will use $r = a(1+\cos\theta)$

Given,

$$r = a(1+\cos\theta)$$

Diff. w. r. to $\theta$

$$\therefore \frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\frac{dr}{d\theta} = -a\cdot\sin\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = +a^2\cdot\sin^2\theta.$$

adding $r^2$ on both sides

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + a^2\cdot\sin^2\theta$$

But $r = a(1+\cos\theta)$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\cdot\sin^2\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\left[1^2 + \cos^2\theta + 2\cdot\cos\theta + \sin^2\theta\right]$$

$$= a^2\left[2 + 2\cdot\cos\theta\right]$$

$$= 2a^2(1+\cos\theta)$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2a^2\cdot 2\cdot\cos^2\frac{\theta}{2}$$

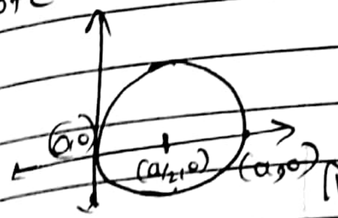
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2 \cdot \cos^2\left(\frac{\theta}{2}\right)$$

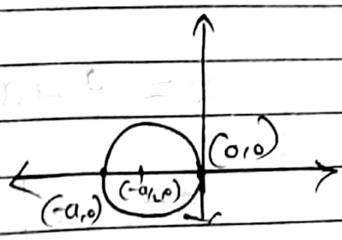
Taking square root

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2a \cdot \cos\left(\frac{\theta}{2}\right)$$

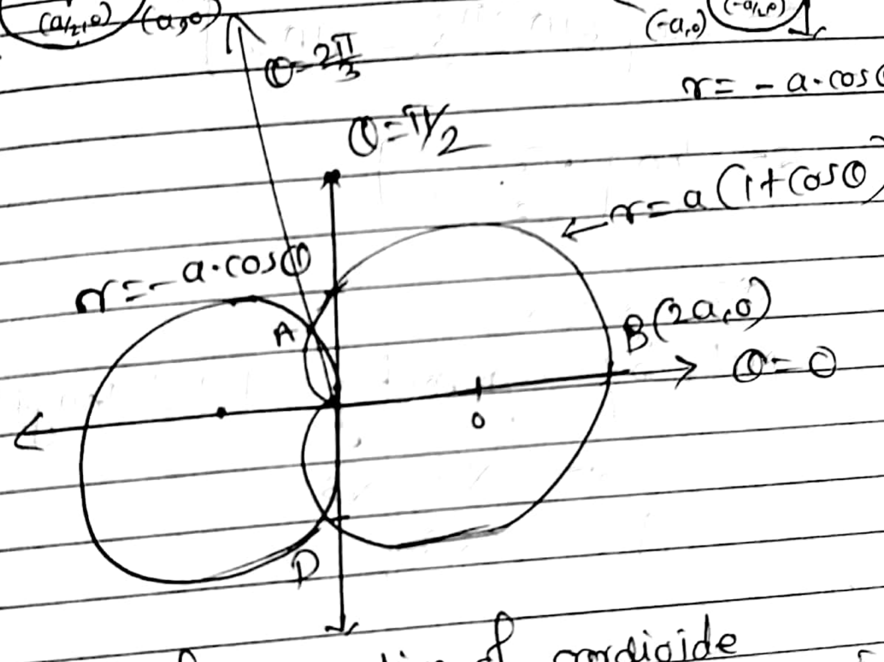
**Step 2:** To find limit of integration $\theta_1$ and $\theta_2$

Note $r = a \cdot \cos\theta$, where $a$ is diameter




$r = -a \cdot \cos\theta$.




The point of intersection of cardioide $r = a(1+\cos\theta)$ and $r = -a\cdot\cos\theta$ is given by

$$a(1+\cos\theta) = -a \cdot \cos\theta$$

$$1 + \cos\theta + \cos\theta = 0$$

$$2\cos\theta = -1$$

$$\cos\theta = -1/2$$

$$\theta = \cos^{-1}(-1/2)$$

$$\theta = 120^0 = \frac{2\pi}{3}$$ from fig

$\therefore \theta_1 = 0$ and $\theta = 2\pi/3$

$$\therefore\ s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ .\ d\theta$$

$$s = \int_0^{2\pi/3} 2a \cdot \cos\left(\frac{\theta}{2}\right) \cdot d\theta = 2a \int_0^{2\pi/3} \cos\left(\frac{\theta}{2}\right) \cdot d\theta$$

$$\therefore s = 2a \left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{2\pi/3}$$

$$s = 4a \left[\sin\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$s = 4a\left[\sin\frac{2\pi/3}{2} - \sin 0\right]$$

$$s = 4a\left[\frac{\sqrt{3}}{2} - 0\right]$$

$$s = 2a\sqrt{3}$$

But by symmetry, total length is

$$s = 2 \times 2a\sqrt{3}$$

$$\boxed{s = 4a\sqrt{3}}$$

Q. Find the complete arc length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$

$\Rightarrow$ The given curve $x^{2/3} + y^{2/3} = a^{2/3}$ is know as astroid and it's parametric eqⁿ is given by

$$x = a \cdot \cos^3\theta \quad \& \quad y = a \cdot \sin^3\theta$$

Step 1:

To find $\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$

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we have $x = a \cdot \cos^3\theta$

Diff. w.r.to $\theta$

$\frac{dx}{d\theta} = -3a \cdot \cos^2\theta \cdot \sin\theta$

squaring on both sides,

$\left(\frac{dx}{d\theta}\right)^2 = 9a^2 \cdot \cos^4\theta \cdot \sin^2\theta$ —— ①

Also $y = a\sin^3\theta$

Diff. w.r.to $\theta$

$\frac{dy}{d\theta} = 3a \cdot \sin^2\theta \cdot \cos\theta$

squaring on both sides,

$\left(\frac{dy}{d\theta}\right)^2 = 9a^2 \cdot \sin^4\theta \cdot \cos^2\theta$ —— ②

Now, adding eqn ① & ② we get

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2 \cos^4\theta \sin^2\theta + 9a^2 \sin^4\theta \cdot \cos^2\theta$

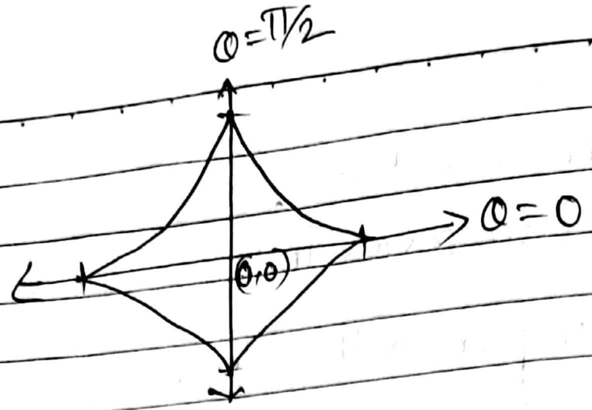
$= 9a^2 \cos^2\theta \cdot \sin^2\theta \left(\cos^2\theta + \sin^2\theta\right)$

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2 \cdot \cos^2\theta \cdot \sin^2\theta$

$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 3a \cdot \cos\theta \cdot \sin\theta$

**Step 2: find $\theta_1$ and $\theta_2$**

Astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is given by




$\theta_1 = 0, \quad \theta_2 = \pi/2$

Step 3: To calculate length

$$S = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$S = \int_0^{\pi/2} 3a \cdot \sin\theta \cdot \cos\theta \cdot d\theta$$

$$S = 3a \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

By Reduction Formula

$$S = 3a\left[\frac{(1)(1)}{2} \cdot 1\right]$$

$$S = \frac{3a}{2}$$

But by symmetry, total length is

$$S = 4 \times \frac{3a}{2}$$

$$\boxed{S = 6a}$$

Homework

Q. find the arc length of the cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ from one cup to another cup.