Unit II

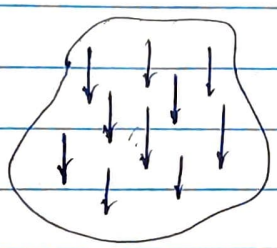
Distributed Forces and Friction

Moment of area, Centroid of plane lamina and wire bends, Moment of Inertia.

Friction - Laws of friction, application of friction on inclined planes wedges and ladders friction. Application to flat belt.

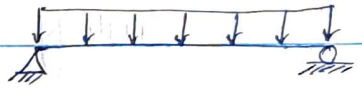
* Centre of Gravity & Centroid.

In reality all the forces are distributed in nature. Forces can be distributed over the entire volume of a body as in the case of force of gravity. (i.e. weight of a body) or distributed over a surface area in contact as in the case of contact forces such as normal reaction or pressure distribution of water against a dam gate, load distribution in beams, stress distribution in deformable bodies etc.



Force of gravity

Water pressure in a retaining wall

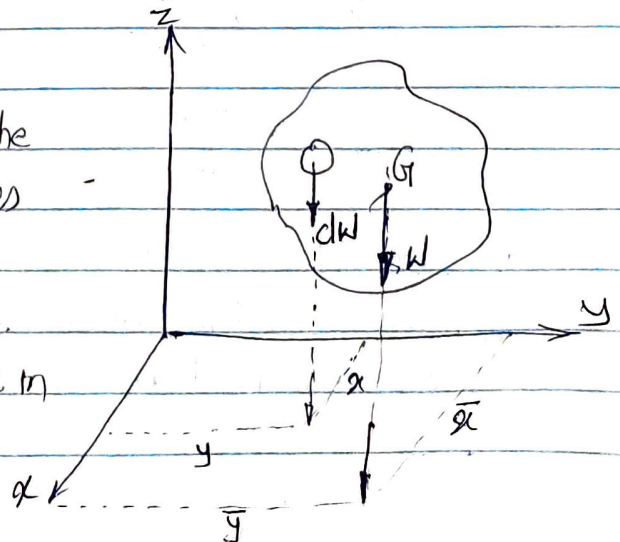


Though these forces are all distributed, for analytical purposes while applying the conditions of equilibrium, it is customary to replace them by a single resultant force, which would produce the same effect as that of the distributed force, which is a point at which the forces are assumed to be concentrated.

* Centre Of Gravity

The point of application of the resultant of the gravitational forces acting on all the particles of a body is called centre of gravity.

Or It is the point ~~at which~~ in the body where entire weight is assumed to be concentrated.



The point of application of W (i.e C.G.) can be obtained by Varignon's theorem. The moment of element about y-axis is $x \cdot dW$, it must be equal to the moment of the resultant force W about y-axis which is $W\bar{x}$.

$$\therefore W\bar{x} = \int x \cdot dW$$

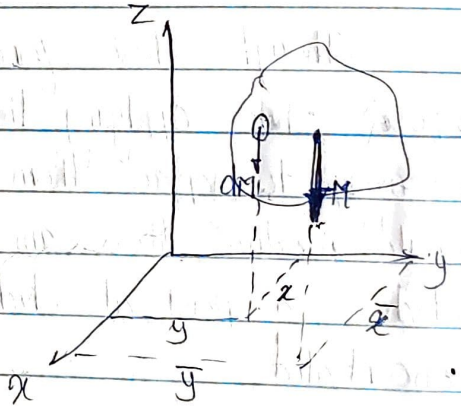
$$\therefore \bar{x} = \frac{\int x \cdot dW}{W}$$

Similarly

$$\bar{y} = \frac{\int y \cdot dW}{W}$$

The term C.G. is applied to three dimensional bodies with cet.

* Centre of Mass.



The point in the body where the entire mass of the body is assumed to be concentrated, is called Centre of mass.

$$\bar{x} = \frac{\int x \cdot dm}{m}$$

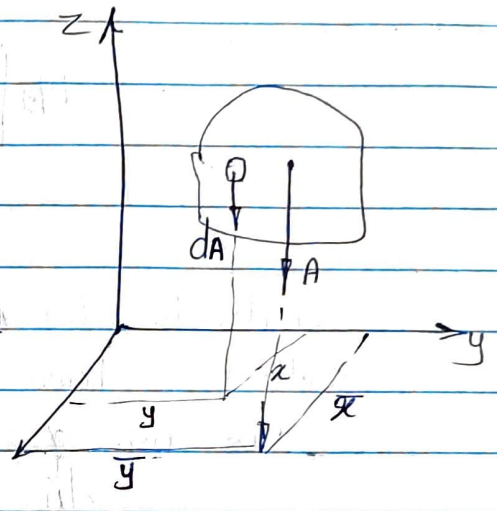
$$\bar{y} = \frac{\int y \cdot dm}{m}$$

* The term centre of gravity of a body & its centre of mass coincide when acceleration due to gravity is uniform throughout the body.

* The centre of mass of a body & its centroid of volume coincide when density is uniform throughout the body.
* If the body is non-homogeneous then the centre of mass does not coincide with the centroid of its volume.
* Centroid :—

It is defined as the point at which entire area is assumed to be concentrated.

i.e. However the geometrical centre of a body in case of volume, area or line, that point is termed as Centroid.



∴ Centroid of a Line.

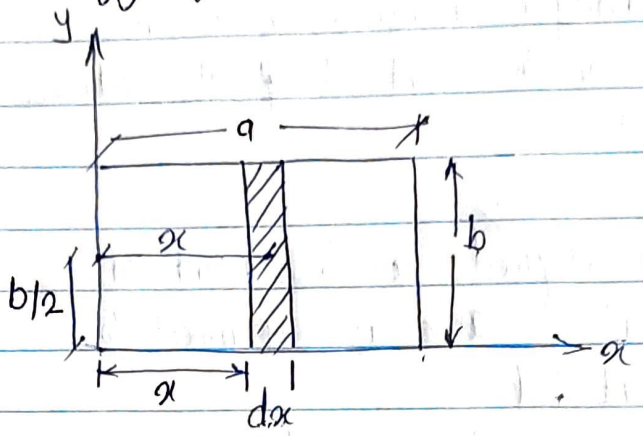
$$\bar{x} = \frac{\int x \cdot dL}{L} \quad \& \quad \bar{y} = \frac{\int y \cdot dL}{L}$$

Centroid of an Area.

$$\bar{x} = \frac{\int x \cdot dA}{A} \quad \& \quad \bar{y} = \frac{\int y \cdot dA}{A}$$

* The term centre of gravity of a homogeneous plate of infinitesimally small thickness is the same as the centroid of the surface area.

* Determine the $\bar{x}$ & $\bar{y}$ co-ordinates of centroid of the rectangle shown in fig from first principles.



Consider the differential element to be a vertical strip of width dx as shown in fig.

The area of differential element is

$$dA = b \cdot dx$$

$$\therefore A = \int dA = \int b \cdot dx \quad \therefore$$

$$= b\,(x)_0^a$$

$$\therefore A = ab$$

The distance of centroid of differential element from y-axis is x.

$$\bar{x} = \frac{\int x \cdot dA}{A} = \frac{\int_0^a x \cdot b\,dx}{ab}$$

$$= \frac{b\left(\frac{x^2}{2}\right)_0^a}{ab} = \frac{1}{a} \cdot \frac{a^2}{2}$$

$$\boxed{\bar{x} = \frac{a}{2}}$$

Locate the centroid of the shaded area as shown in fig. with respect to origin O.

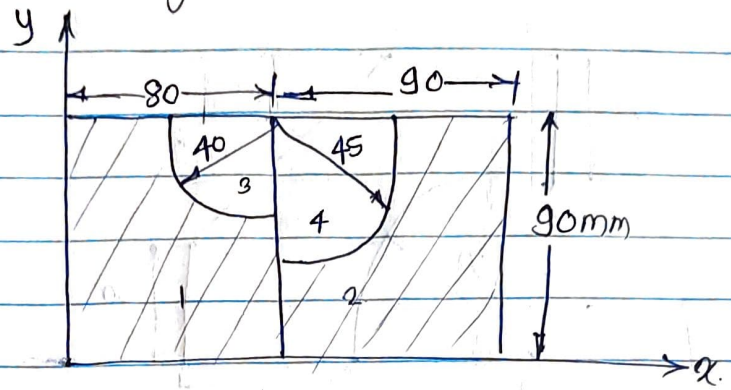


Figure	Area	$\bar{x}$	$\bar{y}$
1. Rectangle	80×90	$80/2$	$90/2$
	$= 7200$	$= 40$	$= 45$
2. Rectangle	90×90	$80 + 90/2$	$90/2$
	$= 8100$	$= 125$	$= 45$
3. Quarter Circle	$\frac{\pi \times 40^2}{4}$	$80 - \frac{4 \times 40}{3\pi}$	$90 - \frac{4 \times 40}{3\pi}$
	$= -1256.637$	$= -63.023$	$= -73.023$
4. Quarter Circle	$\frac{\pi \times 45^2}{4}$	$80 + \frac{4 \times 45}{3\pi}$	$90 - \frac{4 \times 45}{3\pi}$
	$= -1590.431$	$= -99.098$	$= -70.901$
	12452.992		

$$\bar{x} = \frac{7200 \times 40 + 8100 \times 125 - 1256.637 \times 63.023 - 1590.431 \times 99.098}{12452.992}$$

$$= 85.42 \text{ mm}$$

$$\bar{y} = \frac{7200 \times 45 + 8100 \times 45 - 1256.637 \times 73.023 - 1590.431 \times 70.901}{12452.992}$$

$$= 38.864 \text{ mm}$$

Find centroid of following shaded area.

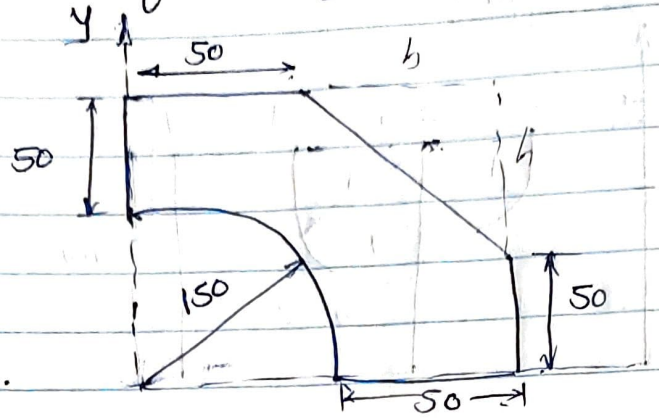


Figure	Area	$\bar{x}$	$\bar{y}$
① Square.	200×200	$200/2$	$200/2$
	$= 40000$	$= 100$	$= 100$
② Right angled Triangle.	$\frac{1}{2} \times 150 \times 150$	$50 + \frac{2}{3} \times 150$	$50 + \frac{2}{3} \times 150$
	$= 11250$	$= 150$	$= 150$
③ Quarter Circle	$\frac{\pi \times 150^2}{4}$	$\frac{4 \times 150}{3\pi}$	$\frac{4 \times 150}{3\pi}$
	$= 17671.46$	$= 63.6$	$= 63.6$

$$\bar{x} = \frac{40000 \times 100 - 11250 \times 150 + 17671.46 \times 63.6}{11078.54} = 107.28 \text{ mm}$$

$\bar{y} =$

$= 107.28$ mm

Find centroid of following shaded area.

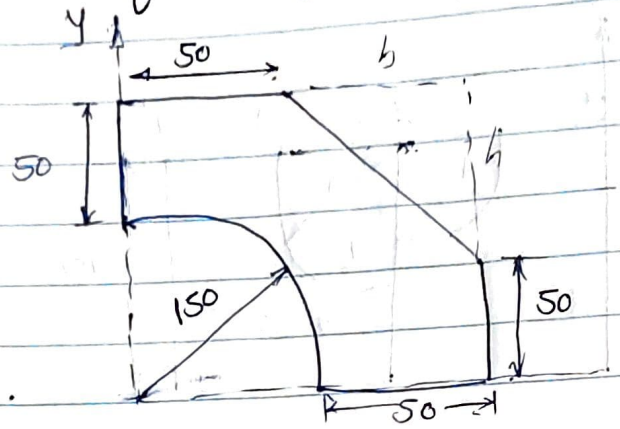


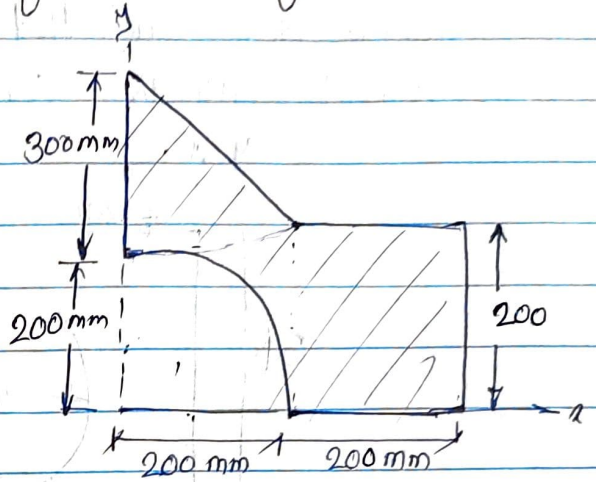
Figure	Area	$\bar{x}$	$\bar{y}$
① Square.	200×200	$200/2$	$200/2$
	$= 40000$	$= 100$	$= 100$
② Right angled Triangle.	$\frac{1}{2} \times 150 \times 150$	$50 + \frac{2}{3} \times 150$	$50 + \frac{2}{3} \times 150$
	$= 11250$	$= 150$	$= 150$
③ Quarter Circle.	$\frac{\pi \times 150^2}{4}$	$\frac{4 \times 150}{3\pi}$	$\frac{4 \times 150}{3\pi}$
	$= 17671.46$	$= 63.6$	$= 63.6$

$$\bar{x} = \frac{40000 \times 100 - 11250 \times 150 + 17671.46 \times 63.6}{11078.54} = 107.28 \text{ mm}$$

$\bar{y} =$

$= 107.28$ mm

Determine the co-ordinates of centroid of the shaded portion shown in fig.



Shape	Area	$\bar{x}$	$\bar{y}$
1. Rectangle	$400 \times 200 = 80000\ mm^2$	$400/2 = 200$	$200/2 = 100$
2. Triangle	$\frac{1}{2} \times 300 \times 200 = 30000$	$200/3 = 66.667$	$200 + \frac{300}{3} = 300$
3. Quarter Circle	$-\frac{\pi \times 200^2}{4} = -31415.926$	$-\frac{4 \times 200}{3\pi} = 84.88$	$-\frac{4 \times 200}{3\pi} = 84.88$
	78584.073		

$$\bar{x} = \frac{80000 \times 200 + 80000 \times 66.667 - 31415.926 \times 84.88}{78584.073} = 195.121 \text{ mm}$$

$$\bar{y} = \frac{80000 \times 100 + 30000 \times 300 - 31415.926 \times 84.88}{78584.073} = 182.395 \text{ mm}$$

Determine coordinates of centre of circle to be cut from a plate such that centroid of remaining plate will coincide with the centre of circle itself.

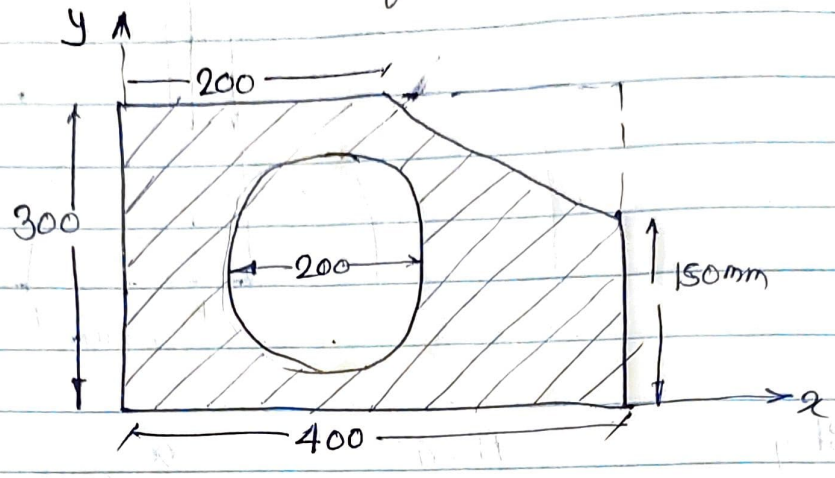
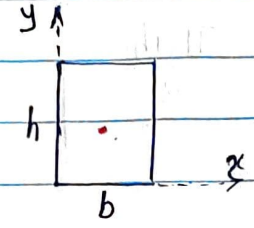
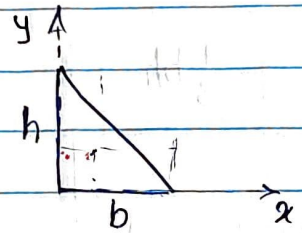
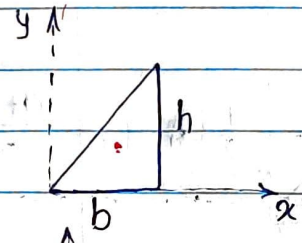
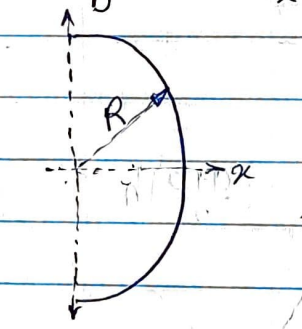
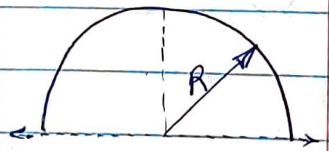
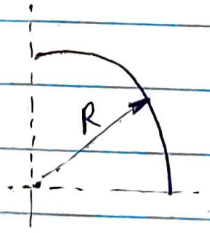
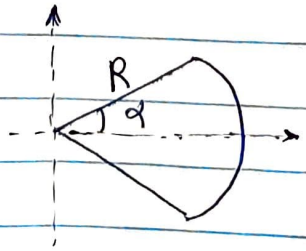


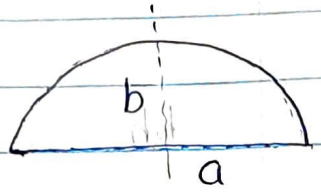
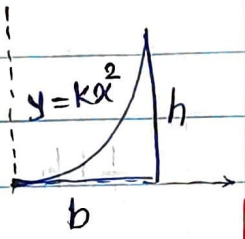
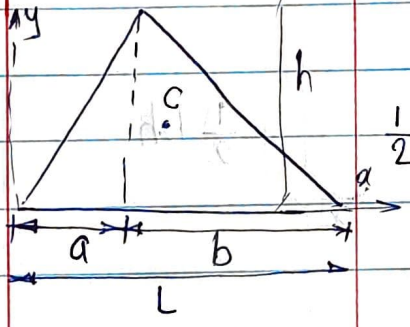
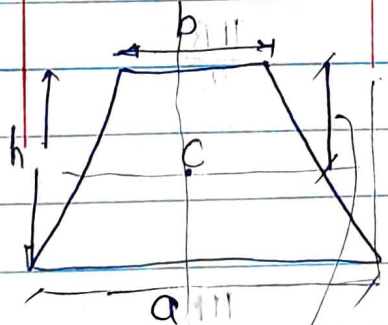
Figure	Area	$\bar{x}$	$\bar{y}$
Rectangle	400×300	$400/2$	$300/2$
	$= 120000$	$= 200$	$= 150$
Triangle	$\frac{1}{2} \times 200 \times 150$	$200 + \frac{2}{3} \times 200$	$150 + \frac{2}{3} \times 150$
	$= -15000$	$= 333.33$	$= 250$
	$1,05,000$		

$$\bar{x} = \frac{120000 \times 200 - 15000 \times 333.33}{1,05,000} = 180.95 \text{ mm}$$

$$\bar{y} = \frac{120000 \times 150 - 15000 \times 250}{1,05,000} = 135.714 \text{ mm}$$

* Centroids of Regular Geometrical Shapes :-

Sr.No.	Shape	Figure	Area	$\bar{x}$	$\bar{y}$
1.	Rectangle		bh	$\frac{b}{2}$	$\frac{h}{2}$
2.	Right-angled Triangle		$\frac{1}{2}bh$	$\frac{b}{3}$	$\frac{h}{3}$
3.	Right-angled Triangle		$\frac{1}{2}bh$	$\frac{2}{3}b$	$\frac{h}{3}$
4.	Semicircle		$\frac{\pi R^2}{2}$	$\frac{4R}{3\pi}$	0
5.	Semicircle		$\frac{\pi R^2}{2}$	0	$\frac{4R}{3\pi}$
6.	Quadrant		$\frac{\pi R^2}{4}$	$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$
7.	Circular Sector		αR^2	$\frac{2R\sin\alpha}{3\alpha}$	0

Sr. No.	Shape	Figure	Area	$\bar{x}$	$\bar{y}$
8.	Semi-elliptical area		$\frac{\pi ab}{2}$	0	$\frac{4b}{3\pi}$
9.	Parabola		$\frac{1}{3}bh$	$\frac{3b}{4}$	$\frac{3h}{10}$
10.	Unsymmetrical Triangle		$\frac{1}{2}Lh$	$\frac{a+L}{3}$	$\frac{h}{3}$
11.	Trapezoid		$\left(\frac{a+b}{2}\right)h$	0	0
					$\left(\frac{2a+b}{a+b}\right)\frac{h}{3}$

Find distances of centroid of shaded area from EF & ED.

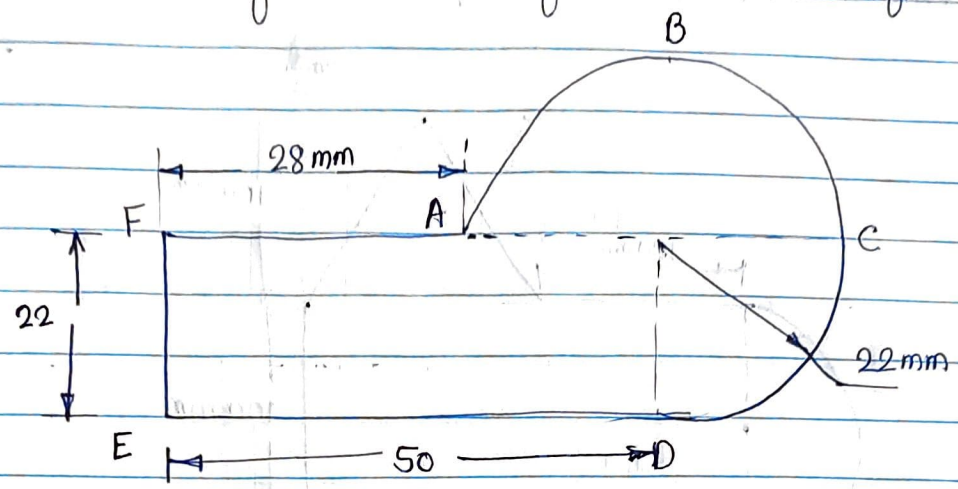
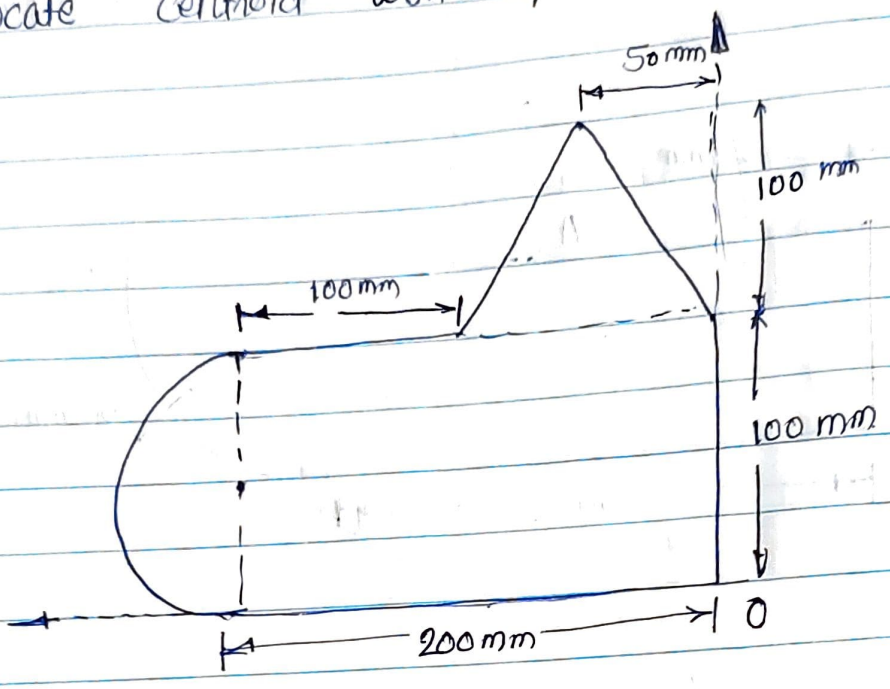


Figure	Area	$\bar{x}$	$\bar{y}$
1. Rectangle	$50 \times 22 = 1100$	$\frac{50}{2} = 25$	$\frac{22}{2} = 11$
2. Quarter Circle	$\frac{\pi \times 22^2}{4} = 380.132$	$50 + \frac{4 \times 22}{3\pi} = 59.387$	$22 - \frac{4 \times 22}{3\pi} = 12.662$
3. Semi Circle	$\frac{\pi \times 22^2}{2} = 760.265$	$28 + 22 = 50$	$22 + \frac{4 \times 22}{3\pi} = 31.33$
	2240.397		

$$\bar{x} = \frac{1100 \times 25 + 380.132 \times 59.387 + 760.265 \times 50}{2240.397} = 39.309 \text{ mm}$$

$$\bar{y} = \frac{1100 \times 11 + 380.132 \times 12.662 + 760.265 \times 31.33}{2240.397} = 18.180 \text{ mm}$$

Locate centroid with respect to O.



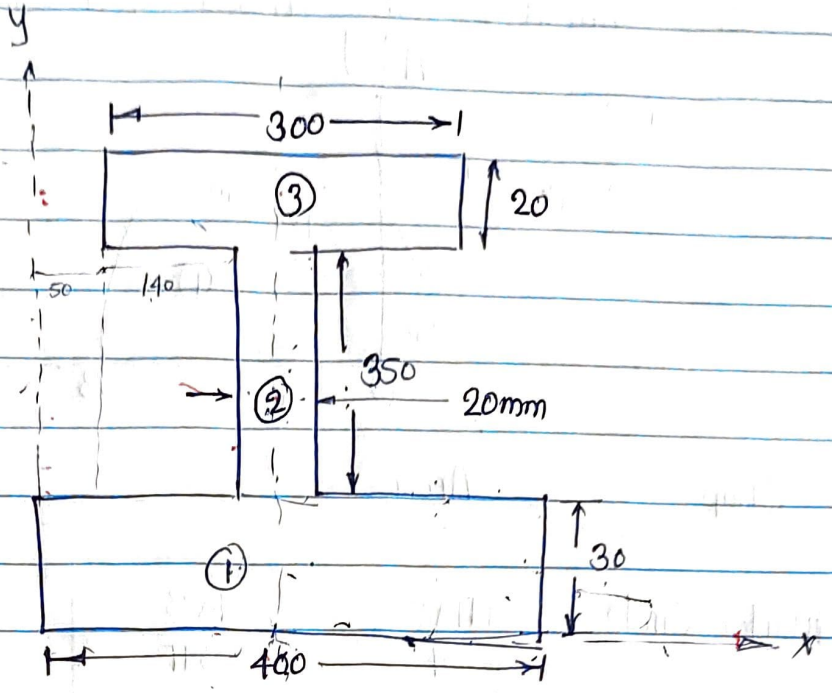
	Shape	Area	$\bar{x}$	$\bar{y}$
1.	Rectangle	200×100 $= 20000$	$\frac{-200}{2}$ $= -100$	$\frac{100}{2}$ $= 50$
2.	Triangle	$\frac{1}{2} \times 100 \times 100$ $= 5000$	-50	$100 + \frac{100}{3}$ $= 133.33$
3.	Semicircle	$\frac{\pi \times 50^2}{2}$ $= 3926.99$	$200 + \frac{4 \times 50}{3\pi}$ $= -221.22$	50
		28926.99		

$$\bar{x} = \frac{20000 \times (-100) + 5000 \times (-50) + 3926.99 \times (-221.22)}{28926.99}$$

$$= -107.813 \text{ mm}$$

$$\bar{y} = \frac{20000 \times 50 + 5000 \times 133.33 + 3926.99 \times 50}{28926.99}$$

$$= 64.403 \text{ mm}$$



	Shape	Area	$\bar{x}$	$\bar{y}$
①	Rectangle	400×30 $= 12000$	$400/2$ $= 200$	$30/2$ $= 15$
②	Rectangle	350×20 $= 7000$	$190 + \frac{20}{2}$ $= 200$	$30 + 350/2$ $= 205$
③	Rectangle	300×20 $= 6000$	$50 + \frac{300}{2}$ $= 200$	$30 + 350 + \frac{20}{2}$ $= 390$

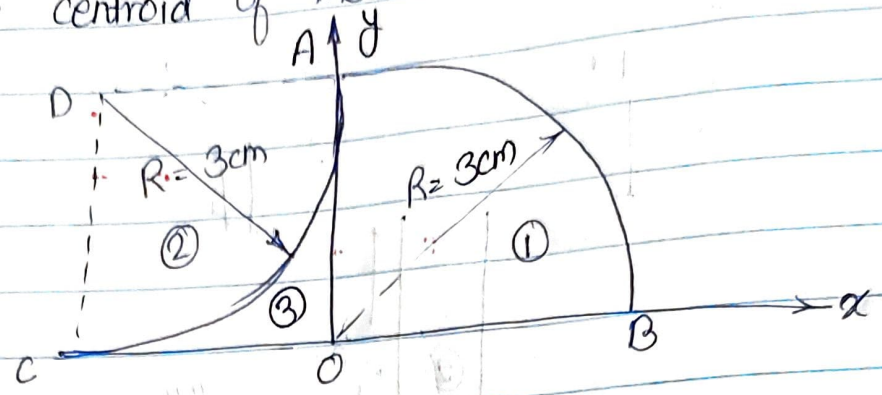
~~17200~~ 25000

$$\bar{x} = \frac{12000 \times 200 + 7000 \times 200 + 6000 \times 200}{\text{~~17200~~ } 25000} = 200 \text{ mm}$$

$$\bar{y} = \frac{12000 \times 15 + 7000 \times 205 + 6000 \times 390}{\text{~~17200~~ } 25000} = \text{~~267.112 mm~~}$$

$$= 158.2 \text{ mm}$$

Find the centroid of the shaded area shown in fig.

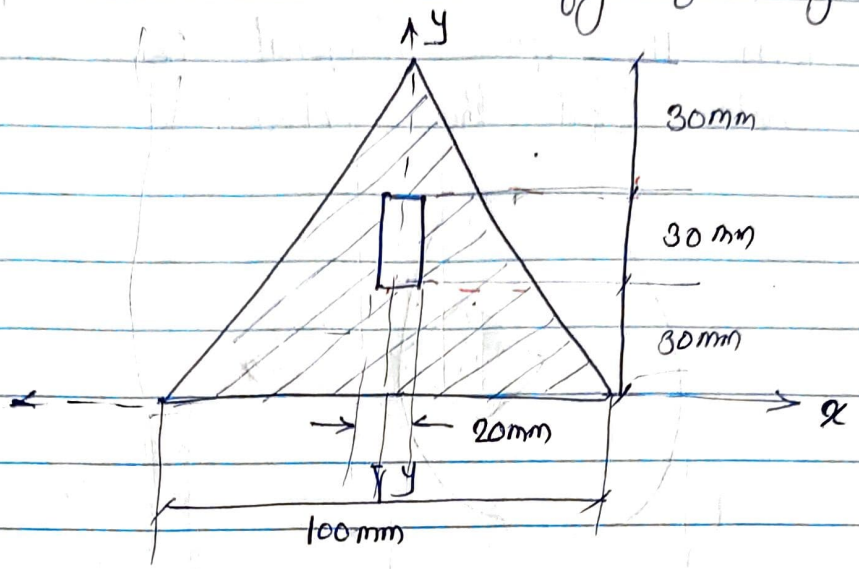


Shape	Area	$\bar{x}$	$\bar{y}$
1. Quarter Circle.	$\frac{\pi \times 3^2}{4}$	$\frac{4\times 3}{3\pi}$	$\frac{4\times 3}{3\pi}$
	$= 7.068$	$= 1.273$	$= 1.273$
2. Square	3×3	$3/2$	$3/2$
	$= 9$	$= -1.5$	$= +1.5$
3. Quarter Circle	$-\frac{\pi \times 3^2}{4}$	$-\left(3-\frac{4\times 3}{3\pi}\right)$	$3-\frac{4\times 3}{3\pi}$
	$= -7.07$	$= -1.73$	$= 1.73$
	8.998		

$$\bar{x} = \frac{7.068 \times 1.273 + 9(-1.5) + (-7.07)\times(-1.73)}{8.998} = 0.8566\,cm$$

$$\bar{y} = \frac{7.07\times 1.27 + 9\times 1.5 + (-7.07)\times 1.73}{8.998} = 1.139\,cm$$

For shaded area shown in fig find $\bar{y}$



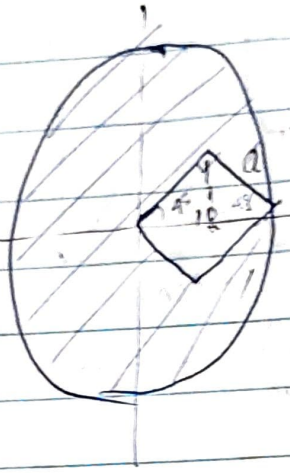
$\bar{x} = 0$

Shape.	Area.	$\bar{y}$
Triangle.	$\frac{1}{2} \times 100 \times 90$	$\frac{90}{3}$
	$= 4500.$	$= 30$
Rectangle	20×30	$30 + \frac{30}{2}$
	$= 600$	$= 45$
	3900	

$$\bar{y} = \frac{4500 \times 30 - 600 \times 45}{3900}$$

$$= 27.692 \text{ mm}$$

A square cut is punched in a circular plate of radius 10cm, the square diagonal being equal to radius of circle. Locate the centroid.



$h = a\cos$

$\cos 45 = \frac{5}{a}$

$a = \frac{5}{\cos 45} = 7.071$

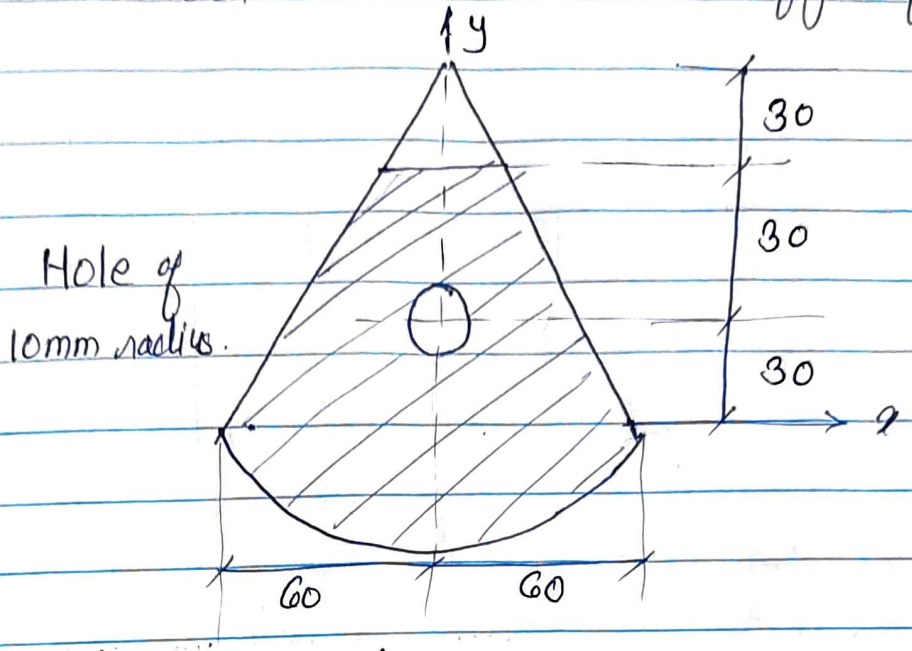
Shape.	Area.	$\bar{x}$
Circle	$\pi \times 10^2$	0
	$= 314.159$	
~~Triangle~~ Square	7.071^2	5
	$= 50$	
	264.159	

$\bar{x} = -0.950\,cm$

$\bar{x} = \frac{-50 \times 5}{264.159} = -0.950\,cm$

3 120 ∴ b = 3

For the shaded area shown in fig. find $\bar{y}$



$$\alpha = \tan^{-1}\frac{60}{90} = 33.69°$$

$$= 33.69 \times \frac{\pi}{180}$$

$$= 0.588 \text{ radian.}$$

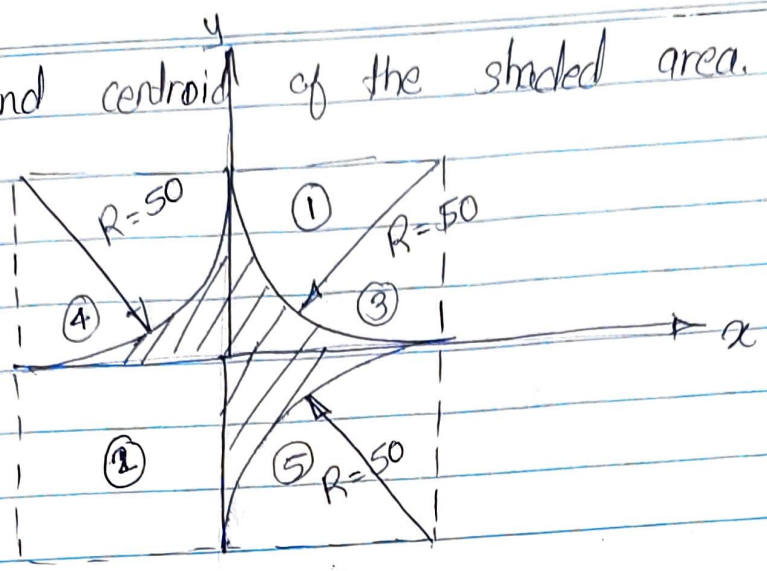
$$r = \sqrt{60^2+90^2} = 108.2 \text{ mm.}$$

By similarity of Δle base of $\Delta le = \frac{1}{3} \times 120 = 40$ mm

Fig.	Area	$\bar{x}$	$\bar{y}$
Sector	$r^2\alpha$	$2/3\frac{r\sin\alpha}{\alpha}$	
	$= 108.2^2 \times 0.588$	$= 90 - \frac{2}{3} \times \frac{108.2 \sin 33.69}{0.588}$	
	6883.857	$= 21.951$	
Triangle	$-\frac{1}{2} \times 30 \times 40$	$60 + \frac{30}{3} = 70$	
	$= -600$		
Circle	$\pi \times 10^2$	60	
	$= -314.159$	30	

$$\bar{y} = 16.698 \text{ mm}$$

Find centroid of the shaded area.



~~Area~~ Shape	Area	$\bar{x}$	$\bar{y}$
1. Square	100×100	0	0
	$= 10000$		
2. Square	-50×50	-25	-25
	$= 2500$		
3. Quarter Circle	$-\frac{\pi \times 50^2}{4}$	$50 - \frac{4 \times 50}{3\pi}$	$50 - \frac{4 \times 50}{3\pi}$
	$= -1963.5$	$= 28.78$	$= 28.78$
4. Quarter Circle	$-\frac{\pi \times 50^2}{4}$	-28.78	28.78
	$= -1963.5$		
5. Quarter Circle	-1963.5	28.78	-28.78
	1609.5		

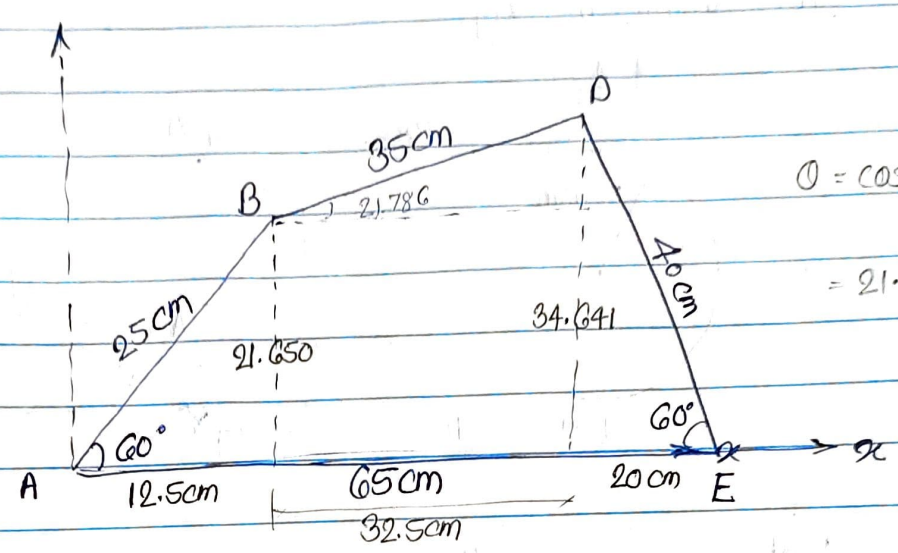
$$\bar{x} = \frac{10000 \times 0 - 2500(-25) - 1963.5(28.78) - 1963.5(-28.78) - 1963.5(28.78)}{1609.5}$$

$$= 3.721$$

$$\bar{y} = \frac{10000 \times 0 - 2500(-25) - 1963.5 \times 28.78 - 1963.5 \times 28.78 - 1963.5(-28.78)}{1609.5}$$

$$= 3.721$$

Find Centroid of the coive bent as shown in fig.



$\theta = \cos^{-1}\frac{32.5}{35}$

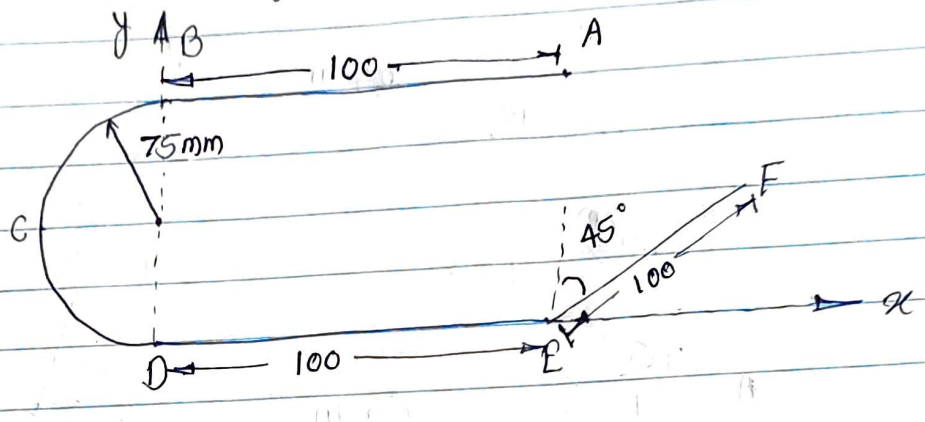
$= 21.786°$

Part	Length	$\bar{x}$	$\bar{y}$
① AB	25	$\frac{25}{2}\cos 60$	$\frac{25}{2}\sin 60$
		$= 6.25$	$= 10.825$
② BD	35	$12.5 + \frac{35}{2}\cos 21.786$	$21.65 + \frac{35}{2}\sin 21.786$
		$= 28.75$	$= 28.144$
③ DE	40	$12.5 + \frac{35}{2} + \frac{40}{2}\cos 60$ 32.5	$\frac{40}{2}\sin 60$
		$= 55$	$= 17.320$
④ AE	65	$\frac{65}{2}$	0
		$= 32.5$	
	165		

$$\bar{x} = \frac{25\times 6.25 + 35\times 28.75 + 40\times 55 + 65\times 32.5}{165} = 33.181 \text{ mm}$$

$$\bar{y} = \frac{25\times 10.825 + 35\times 28.144 + 40\times 17.320}{165} = 11.808 \text{ mm}$$

Find Centroid of homogeneous wire bent into the shape as shown in fig.



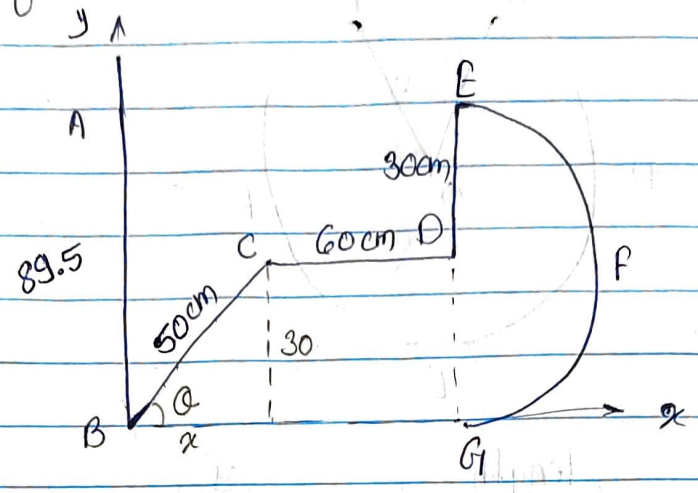
	Part	Length	$\bar{x}$	$\bar{y}$
①	Line AB	100	$\frac{100}{2} = 50$	150
②	Arc BCD	πr	$\frac{\pi r}{2}$ $\quad \frac{2r}{\pi}$	
		$= \pi \times 75$	$= \frac{\pi \times 75}{2} = \frac{2 \times 75}{\pi}$	75
		$= 235.619$	$= 117.809$	
		$= 117.809$	$= -47.746$	
③	Line DE	100	$\frac{100}{2} = 50$	0
④	Line EF	100	$\frac{100 + 100\cos 45}{2}$	$\frac{100 \sin 45}{2}$
			$= 135.355$	$= 35.355$

535.619
117.809

$$\bar{x} = \frac{100 \times 50 + 235.619 \times (-47.746) + 100 \times 50 + 100 \times 135.355}{117.809 \quad 535.619} = 22.981$$

$$\bar{y} = \frac{100 \times 150 + 235.619 \times 75 + 100 \times 35.355}{535.619} = 67.598$$

A bar is bent into shape as shown in fig. find the co-ordinates of centroid.



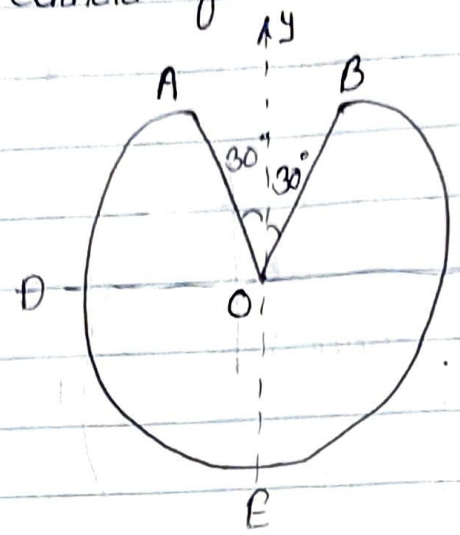
$x = \sqrt{50^2 - 30^2} = 40$ $\quad \theta = \tan^{-1}\frac{30}{40} = 36.869$

	Part	length	$\bar{x}$	$\bar{y}$
①	Line AB	89.5	0	$89.5/2$ $= 44.75$
②	Line BC	50	$\frac{50}{2}\cos 36.869$ $= 20$	$\frac{50}{2}\sin 36.869$ $= 15$
③	Line CD	60	$40 + \frac{60}{2}$ $= 70$	30
④	Line DE	30	$40+60$ $= 100$	$30+30/2$ $= 45$
⑤	Arc EFG	$\pi \times 30$ $= 94.247$	$40+60+\frac{2\times 30}{\pi}$ $= 119.098$	30
		323.747		

$$\bar{x} = \frac{89.5 \times 20 + 60 \times 70 + 30 \times 100 + 94.247 \times 119.098}{323.747} = 60\text{ cm}$$

$$\bar{y} = \frac{89.5 \times 44.75 + 50 \times 15 + 60 \times 30 + 30 \times 45 + 94.247 \times 30}{323.747} = 33.150$$

Find the centroid of wire ADEFBOA as shown in fig.



$r = 20$ cm

$\alpha = 30°$

$= 30 \times \frac{\pi}{180} = 0.523$ rad

$\alpha = 150°$

$150 \times \frac{\pi}{180}$

$= 2.617$ rad

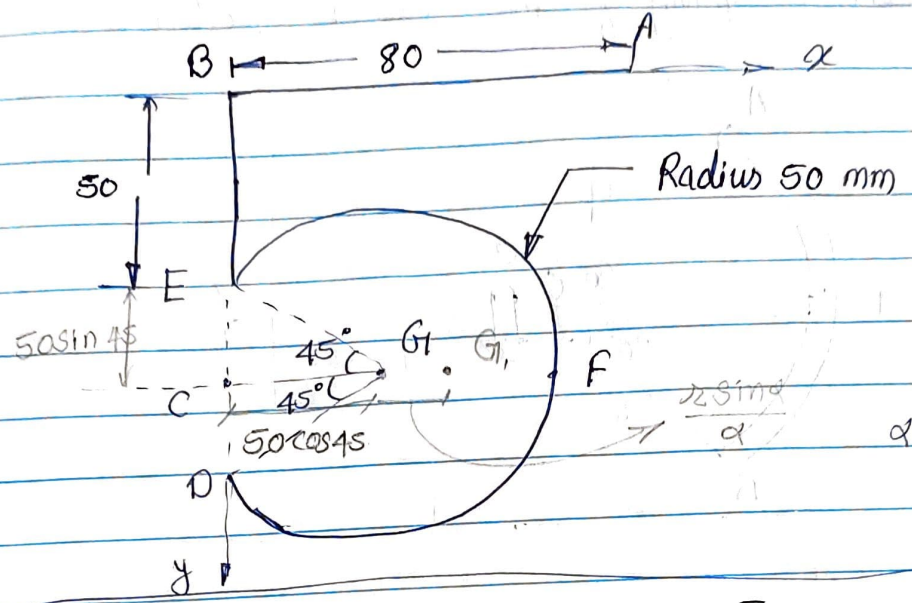
	Part	length	$\bar{y}$
①	Circle	$2\pi \times 20$	0
		$= 125.663$	
②	Arc AB	$-2\alpha r$	$\frac{r \sin\alpha}{\alpha} = \frac{20 \sin 30}{0.523}$
		$= -2 \times 0.523 \times 20$	$= 19.120$
		$= -20.92$	$\frac{20 \sin 150}{2.617} = 3.821$
		$= 104.68$	
③	Line OA	20	$\frac{20}{2} \sin 60 = 8.660$
④	Line OB	20	$\frac{20}{2} \sin 60 = 8.660$
		144.743	

$$\bar{y} = \frac{-20.92 \times 19.120 + 20(-8.660) + 20 \times 8.660}{144.743}$$

$$= -2.763 \text{ cm}$$

$$= 0.866 \text{ cm}$$

Determine position of centroid of a wire bent into shape as shown in fig.



$2\alpha = 270$

$\therefore \alpha = 135°$

$\alpha = 135 \times \frac{\pi}{180} =$

$= 2.356$ rad

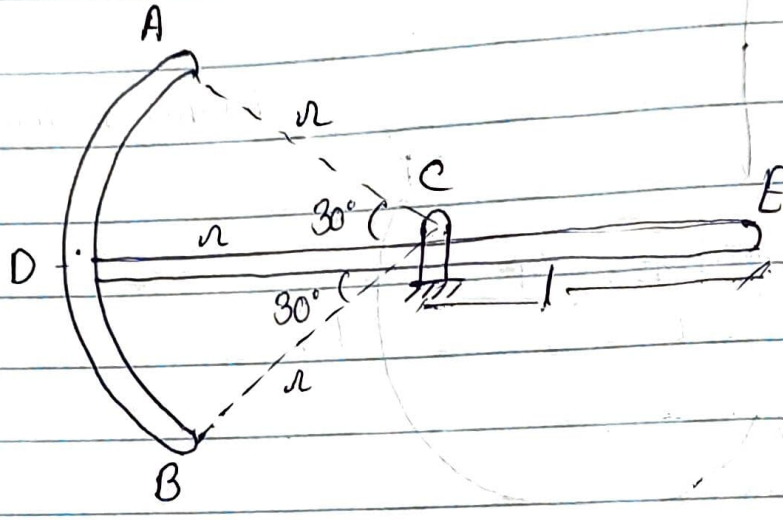
Part	length	$\bar{x}$	$\bar{y}$
Line AB	80	$\frac{80}{2} = 40$	0
Line BE	50	0	$\frac{50}{2} = 25$
Arc EFD	$2\alpha r$ $= 2 \times 50 \times 2.356$ $= 235.6$	$\frac{50\cos 45 + r\sin\alpha}{\alpha}$ $= 85.355 + \frac{50\sin 135}{2.356}$ $= 50.355$	$50 + 50\sin 45$ $= 85.355$
	365.6		

$$\bar{x} = \frac{80 \times 40 + 235.6 \times 50.355}{365.6} = 41.202 \text{ mm}$$

$$\bar{y} = \frac{50 \times 25 + 235.6 \times 85.355}{365.6} = 58.428 \text{ mm}$$

A- homoge

Find the length l of portion CE of the wire for which the centre of gravity of the wire is located at C.



$l_1 =$

$\alpha = 30° = \frac{30 \times \pi}{180} = 0.523 \text{ rad}$

$l_1 = 2r\alpha = 2 \times r \times 0.523$

$= 1.046r$

$\Sigma M_{@C} = 0.$ $\quad \frac{l \times l}{2} - r \times \frac{r}{2} - 1.046r \times \frac{r \sin 30}{0.523} = 0$

$\frac{l^2}{2} = \frac{r^2}{2} + 2r^2 \sin 30$

$1.046r \times \frac{r \sin 30}{0.523} + r \times \frac{r}{2} - l \times \frac{l}{2} = 0$ $\quad \frac{l^2}{2} = \frac{3}{2}r^2$

$2r^2 \cdot \sin 30 + \frac{r^2}{2} = \frac{l^2}{2}$

$2r^2 + \frac{r^2}{2} = \frac{l^2}{2}$

$\frac{3r^2}{2} = \frac{l^2}{2}$

$\frac{4r^2 + r^2}{2} = \frac{l^2}{2}$

$l = \sqrt{3}\, r$

$3r^2 = l^2$

Ou Une