Unit-II

chp. 4 Fourier series.

Introduction:

① The combination of perodic function of same period form a trignometric series. Fourier series are infinite series of sine and cosine terms which are capable of representing almost any perodic single valued function whethere it is continuous or not. moreover, if we have a system, either electric or machanical acted on a perodic force which is not a simple sine or cosine wave, such force can be represented by fourier series.

② fourier series constitute a very important tool for solving problem of ordinary and partial differential equation

③ There are so mammy important application of fourier series, few of them are forced vibrations, power factor and RMS valued for electric circuits, deflection of beam, modeling radiation intensity, single genration and many more.

* Periodic function:

A function f(x) is said to be perodic if it is defined for all real x and if there exist some positive number T such that

$$f(x+T) = f(x) \quad \forall x \qquad \text{Note } T \neq 0$$

The number T is called the period of f(x)

But if n is any positive integer then

$$f(x+nT) = f(x) \quad \text{for all } x$$

Hence $2T, 3T, 4T, \cdots$ are also period of $f(x)$

(3) The smallest period of $f(x)$ is called the primitive period or fandamental period of $f(x)$

Ex sin$x$ and cos$x$ are function with period $2\pi, 4\pi, 6\pi, \cdots$

we know $\sin(x+2\pi) = \sin x$

$\sin(x+4\pi) = \sin x$

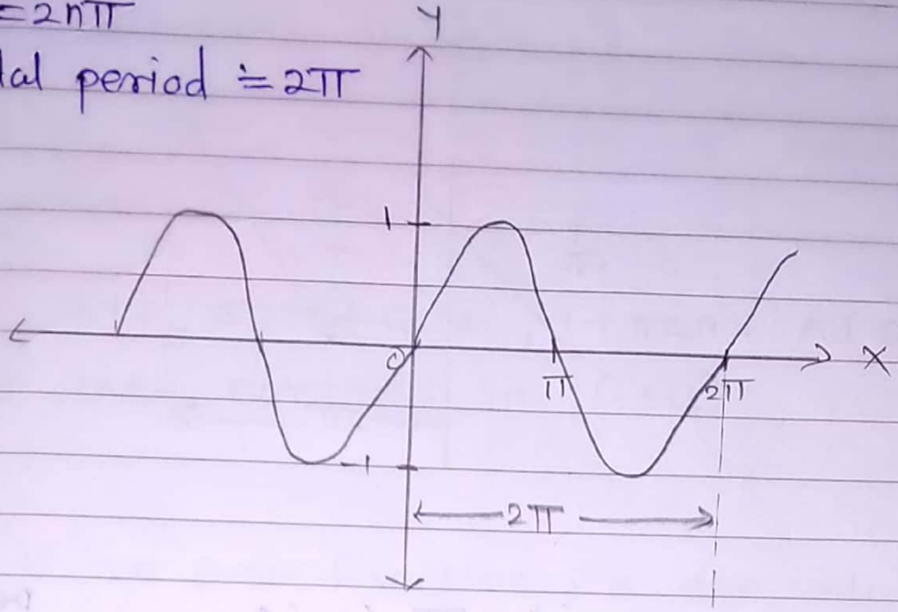
(4) Every positive number is the period of constant function, But it is the only function having no primitive period.

	Function F(x)	Period (2nT)	Fandamental Period T
(1)	sin$x$, cos$x$, sec$x$, cosec$x$	$2n\pi$	2π
(2)	tan$x$, cot$x$	$n\pi$	π
(3)	constant function	Every positive real Number	Not exists
(4)	$f(ax)$	$n \cdot \frac{T}{a}$	$\frac{T}{a}$
(5)	$f(\frac{x}{a})$	naT	aT
(6)	sin$3x$	$n \cdot \frac{2\pi}{3}$	$\frac{2\pi}{3}$
(7)	$\tan(\frac{x}{3})$	$n(3\pi)$	3π

① sinx

Period $= 2n\pi$

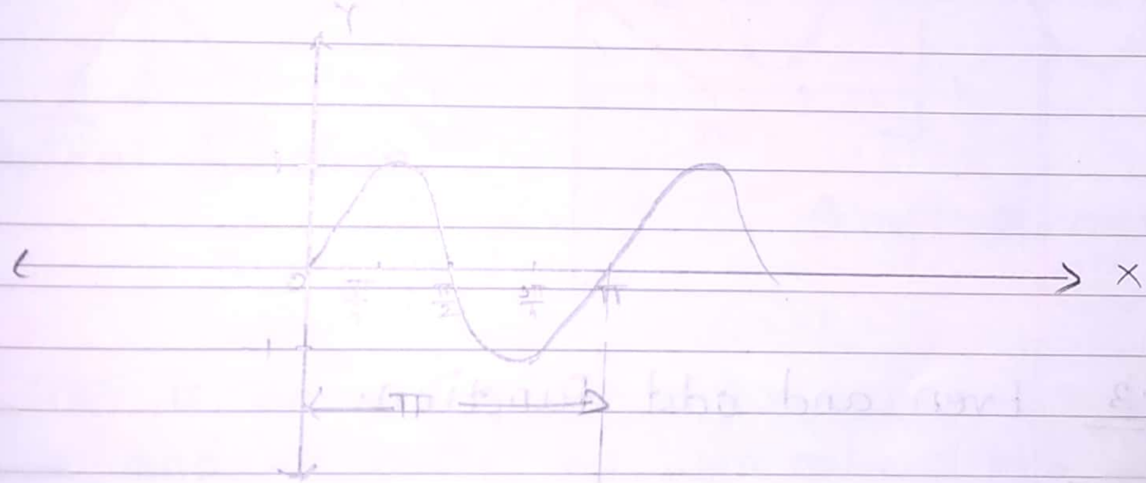
Fandamental period $= 2\pi$



② sin2x

Period $= \frac{2n\pi}{2} = n\pi$

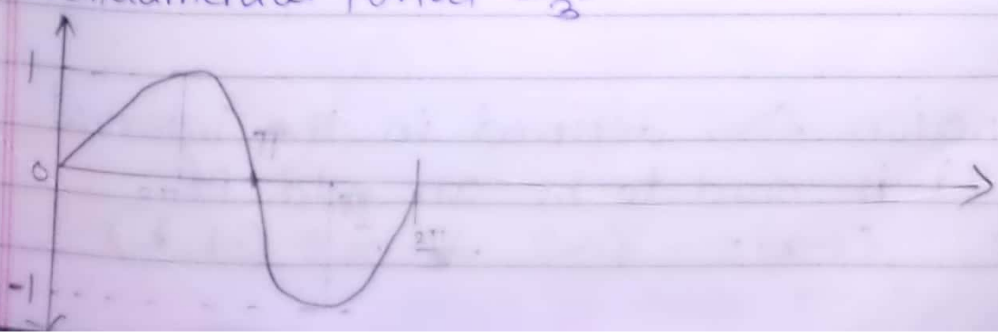
Fandamental period : π



③ sin3x

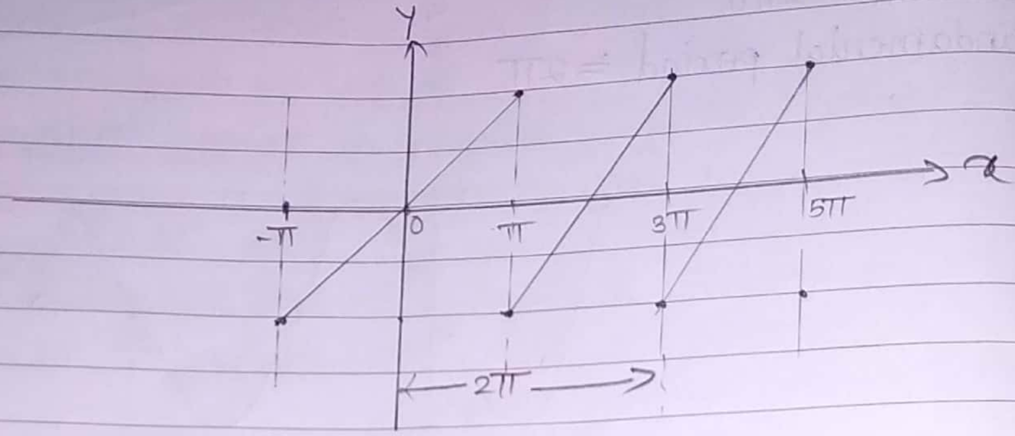
period : $\frac{2n\pi}{3}$

Fandamental period $\frac{2\pi}{3}$



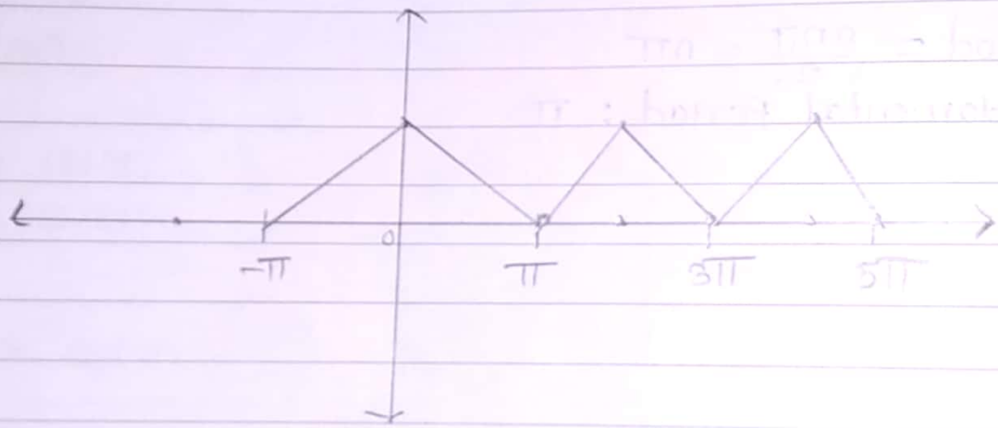
④ $f(x) = x \quad ; \quad -\pi < x < \pi$

Period : $2n\pi$
Fandamental period : 2π



⑤ $f(x) = \pi + x \quad$ if $\quad -\pi < x < 0$
$\quad\;\; = \pi - x \quad$ if $\quad 0 < x < \pi$

period : $2n\pi$
Fandamental period : 2π



4.3 Even and odd function.

① A function $f(x)$ defined in the interval $(-L, L)$ is said to be an even if

$$f(-x) = f(x) \;\; \forall \; x \in (-L, L)$$

② A function $f(x)$ defined in the interval $(-L, L)$ is said to be an odd if

$$f(-x) = -f(x) \;\; \forall \; x \in (-L, L)$$

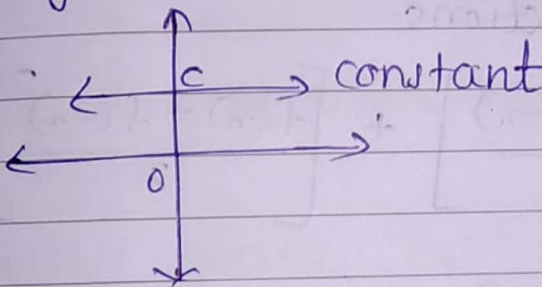
Ex ① $f(x)$ = constant function, x^2, $\cos x$ all are even function in $(-l, l)$

(ii) $f(x) = x$, $\sin x$, $\tan x$, $x \cdot \cos x$, x^3 All are odd function in $(-L, L)$

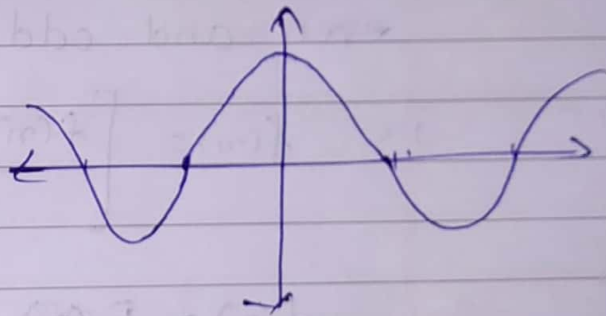
(iii) $f(x) = x+1$, $\sin x + \cos x$, $1 + \tan x$ All are neithere even nor odd in $(-L, L)$

Note

① $f(x)$ is an even function i.e the value of y for x and $-x$ are same, the graph of $y = f(x)$ is symmetric about y-axis.

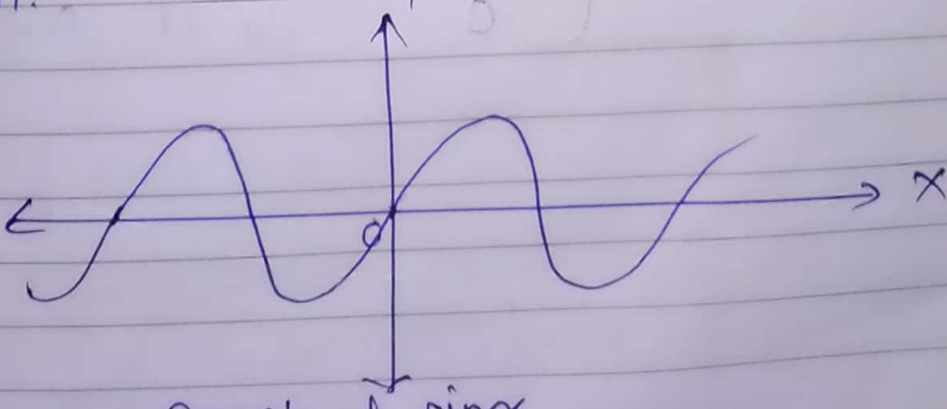


constant function



Graph of $\cos x$

② If $f(x)$ is an odd function i.e value of $y = f(x)$ for x and $-x$ differ by sign only, then the graph of $y = f(x)$ is symmetric about the origin.



Graph of $\sin x$

(3) Algebraic properties of even and odd function.

Sr. NO	f(x)	g(x)	f(x) ± g(x)	f(x) $\overset{\times}{\div}$ g(x)
1	Even	Even	Even	Even
2	odd	odd	odd	Even
3	Even	odd	Neither even nor odd	odd
4	odd	Even	Neither even nor odd	odd.

(4) An function f(x) can be expressed as a sum of even and odd functions.

i.e $f(x) = \left[\frac{f(x) + f(-x)}{2}\right] + \left[\frac{f(x) - f(-x)}{2}\right]$

$f(x) = F_1(x) + F_2(x)$

where $F_1(x)$ is even function

and $F_2(x)$ is an odd function.

(5) $$\int_{-a}^{a} f(x)\cdot dx = \begin{cases} 2\cdot\int_0^a f(x)\cdot dx, & \text{if } f(x) \text{ is even fun} \\ 0 & ; \text{if } f(x) \text{ is odd function} \end{cases}$$

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* Dirichlet conditions:

Any function f(x) defined in the interval $c_1 < x \le c_2$ can be expressed in the fouriers series as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos nx + b_n \cdot \sin nx)$$

where a_0, a_n, b_n are constant provided in the interval and f(x) fallow the following Dirichlet cond^n.

(i) f(x) is defined and single valued in the given interval also $\int_{c_1}^{c_2} f(x) \cdot dx$ exists.

(ii) f(x) may have finite Number of discontinuties.

(iii) f(x) has ~~a~~ finite number of maxima and minima.

4.2 Fourier series (Def^n)

Let f(x) be a periodic fun of period 2L define in the interval $C < x < C+2L$ i.e $(c, c+2L)$ and satiesfying Dirichlet's cond^n, Then f(x) can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right)\right]$$

where a_0, a_n, b_n are called fourier constant and are given by

$$\boxed{a_0 = \frac{1}{L} \int_{C}^{C+2L} f(x) \cdot dx}$$

$$a_n = \frac{1}{L} \int_{C}^{C+2L} f(x) \cdot \cos\left(\frac{n\pi x}{2}\right) \cdot dx$$

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$$\boxed{b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cdot \sin\left(\frac{n\pi x}{L}\right)\cdot dx}$$

Note:
Depending upon value of c and L we get various type of fourier series; which are explain in this chapter.

Type 1: Interval $0 \le x \le 2\pi$

Type 2: Interval $-\pi \le x \le \pi$

Type 3: Interval $0 \le x \le 2L$

Type 4: Interval $-L \le x \le L$

* Type 1: Interval $0 \le x \le 2\pi$

for the interval $0 \le x \le 2\pi$, fourier series can be expressed as.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cdot \cos nx + b_n\cdot \sin nx\right)$$

where

$$a_0 = \frac{1}{L}\int_{c}^{c+2L} f(x)\cdot dx$$

for $0 \le x \le 2\pi$

$C = 0$, $2L = 2\pi \Rightarrow L = \pi$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx}$$

Now $a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cdot \cos\left(\frac{n\pi x}{L}\right)\cdot dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)\cdot dx$$

$$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos(nx)\cdot dx}$$

Now

$$b_n = \frac{1}{L}\int_C^{C+2L} f(x)\cdot \sin\left(\frac{n\pi x}{L}\right)\cdot dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)\cdot dx$$

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin(nx)\cdot dx}$$

Ex① find the fourier series of the function $f(x)=x^2$, $0 \le x \le 2\pi$ and $f(x+2\pi)=f(x)$

$\Longrightarrow$ step① : To find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cdot dx$$

by $\int x^n \cdot dx = \frac{x^{n+1}}{n+1}$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^3 - 0^3}{3}\right]$$

$$= \frac{1}{3\cancel{\pi}}\cdot 8\pi^{\cancel{3}}$$

$$\boxed{a_0 = \frac{8}{3}\pi^2}$$

step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos(nx)\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cdot \cos(nx)\cdot dx$$

Bernoullie Rule

$$\int u\cdot v\cdot dx = (u)[v_1] - (u')[v_2] + (u'')[v_3] - \cdots$$

where u is polynomial, dash = derivative
suffix = integration

Note:- Bernaullies rule is special case of integration by parts. If integration of multiplication of two terms involved one polynomial fun Then we can use Bernallies Rule

Now

$$a_n = u[v_1] - (u')[v_2] + u''[v_3] \mp u'''[v_4] + \cdots$$

$$u = x^2 \Rightarrow u' = 2x, \quad u'' = 2, \quad u''' = 0$$

$$v = \cos nx$$

$$v_1 = \int \cos nx = \frac{\sin(nx)}{n}$$

$$v_2 = \iint \cos nx = \int \frac{\sin(nx)}{n} = \frac{-\cos(nx)}{n^2}$$

$$v_3 = \iiint \cos nx = \int -\frac{\cos(nx)}{n^2} = -\frac{\sin(nx)}{n^3}$$

$$a_n = \frac{1}{\pi}\left\{ x^2 \cdot \left(\frac{\sin(nx)}{n}\right) - (2x)\left(\frac{-\cos(nx)}{n^2}\right) + 2\left(\frac{-\sin(nx)}{n^3}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{ \frac{x^2 \cdot \sin nx}{n} + \frac{2x \cdot \cos nx}{n^2} - \frac{2 \cdot \sin(nx)}{n^3}\right\}_0^{2\pi}$$

But $\sin 2n\pi = 0$, $\cos 2n\pi = 1$

$$a_n = \frac{1}{\pi}\left\{ 0 + \frac{2(2\pi) \cdot 1}{n^2} - \frac{2 \cdot 0}{n^3}\right\}$$

$$a_n = \frac{1}{\pi}\left\{ 0 + \frac{4\pi}{n^2} - 0\right\}$$

$$a_n = \frac{1}{\cancel{\pi}} \times \frac{4\cancel{\pi}}{n^2} = \frac{4}{n^2}$$

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step 3 To find bn

$$bn = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin(nx)\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cdot \sin(nx)\cdot dx$$

By Bernaulli's Rule

$u = x^2, \quad u' = 2x, \quad u'' = 2$

$V = \sin(nx), \quad v_1 = -\frac{\cos nx}{n} \quad v_2 = \frac{-\sin nx}{n^2}$

$v_3 = \frac{\cos nx}{n^3}$

$$bn = \frac{1}{\pi}\left\{ u[v_1] - u'[v_2] + u''[v_3] - \cdots \right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{ x^2\cdot\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{ -\frac{x^2\cdot \cos nx}{n} + \frac{2x\cdot \sin nx}{n^2} + \frac{2\cdot \cos(nx)}{n^3}\right\}_0^{2\pi}$$

$\cos 2n\pi = 1, \quad \sin 2n\pi = 0$

$$= \frac{1}{\pi}\left\{\left[\frac{(2\pi)^2\cdot 1}{n} + 0 + \frac{2\cdot 1}{n^3}\right] - \left[0 + 0 + \frac{2}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{ \frac{-4\pi^2}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}} \right\}$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{-4\pi^{\cancel{2}}}{n}\right) = \frac{-4\pi}{n}$$

step 4: Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos nx + b_n \cdot \sin nx)$$

$$f(x) = \frac{1}{2}\left(\frac{8\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cdot \cos nx + \left(\frac{-4\pi}{n}\right)\sin nx\right]$$

$$\therefore \boxed{x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4 \cdot \cos nx}{n^2} - \frac{4\pi \cdot \sin nx}{n}\right]}$$

Q. obtained fourier series expansion for the function $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ and $f(x) = f(x+2\pi)$

Hence, deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

$\Rightarrow$ Given

$$f(x) = \left(\frac{\pi - x}{2}\right)^2$$

$$f(x) = \frac{\pi^2 - 2\pi x + x^2}{4} \qquad \text{by } (a+b)^2 = a^2 + 2ab + b^2$$

step 1: To find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi^2 - 2\pi x + x^2}{4}\,dx$$

$$a_0 = \frac{1}{4\pi}\left[\pi^2 x - \frac{2\pi x^2}{2} + \frac{x^3}{3}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{4\pi}\left\{\left[\pi^2\cdot(2\pi) - \frac{2\pi(2\pi)^2}{2} + \frac{(2\pi)^3}{3}\right] - \left[\pi^2\cdot 0 - \frac{2\pi 0^2}{2} - \frac{0^3}{3}\right]\right\}$$

$$= \frac{1}{4\pi}\left\{2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3} - 0\right\}$$

$$a_0 = \frac{1}{4\pi}\left\{\frac{6\pi^3 - 12\pi^3 + 8\pi^3}{3}\right\}$$

$$a_0 = \frac{1}{\cancel{4\pi}_2}\left\{\frac{\cancel{2}\pi^{\cancel{3}}}{3}\right\}$$

$$a_0 = \frac{\pi^2}{6}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(nx)\cdot dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi^2 - 2\pi x + x^2}{4}\right)\cdot\cos(nx)\cdot dx$$

$$a_n = \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2 - 2\pi x + x^2\right)\cdot\cos(nx)\cdot dx$$

By Bernaulli's Rule

$$u = \pi^2 - 2\pi x + x^2$$

$$u' = -2\pi + 2x$$

$$u'' = 2$$

$$v_1 = \frac{\sin nx}{n}$$

$$v_2 = \frac{-\cos nx}{n^2}$$

$$v_3 = \frac{-\sin nx}{n^3}$$

$$a_n = \frac{1}{4\pi}\left\{(\pi^2 - 2\pi x + x^2)\left[\frac{\sin(nx)}{n}\right] - (0 - 2\pi + 2x)\left[\frac{-\cos(nx)}{n^2}\right] + (0+2)\left[\frac{-\sin(nx)}{n^3}\right]\right\}_0^{2\pi}$$

$$a_n = \frac{1}{4\pi}\left\{\frac{(\pi^2 - 2\pi x + x^2)\sin nx}{n} + \frac{(-2\pi + 2x)\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right\}_0^{2\pi}$$

But $\cos 2n\pi = 1$ & $\cos 0 = 1$
$\sin 2n\pi = 0$ & $\sin 0 = 0$

$$a_n = \frac{1}{4\pi}\left\{\left(\frac{-2\pi + 2\cancel{x}2\pi}{n^2}\right) - \left(\frac{-2\pi + 2x}{n^2}\right)\right\}$$

$$a_n = \frac{1}{4\pi}\left\{\left(\frac{-2\pi + 4\pi}{n^2}\right) - \left(\frac{-2\pi + 2\times 0}{n^2}\right)\right\}$$

$$= \frac{1}{4\pi}\left\{\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right\}$$

$$a_n = \frac{1}{\cancel{4\pi}}\left\{\frac{\cancel{4\pi}}{n^2}\right\}$$

$$\boxed{a_n = \frac{1}{n^2}}$$

Step3 To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(nx)\cdot dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi^2 - 2\pi x + x^2}{4}\right)\cdot\sin nx\, dx$$

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$$b_n = \frac{1}{4\pi}\int_0^{2\pi} \underbrace{(\pi^2 - 2\pi x + x^2)}_{u} \cdot \underbrace{\sin(nx)\, dx}_{v}$$

By Bernaulli Rule

$$b_n = \frac{1}{4\pi}\left\{(\pi^2 - 2\pi x + x^2)\left[\frac{-\cos nx}{n}\right] - (0 - 2\pi + 2x)\left[\frac{-\sin nx}{n^2}\right] + 2\left[\frac{\cos(nx)}{n^3}\right]\right\}_0^{2\pi}$$

$$b_n = \frac{1}{4\pi}\left\{-\frac{(\pi^2 - 2\pi x + x^2)\cos(nx)}{n} + \frac{(-2\pi + 2x)\sin nx}{n^2} + \frac{2\cdot\cos nx}{n^3}\right\}_0^{2\pi}$$

but $\cos 2n\pi = 1$, $\cos 0 = 1$

$\sin 2n\pi = 0$ $\quad \sin 0 = 0$

$$b_n = \frac{1}{4\pi}\left\{\left[-\frac{(\pi^2 - 2\pi(2\pi) + (2\pi)^2)\cdot 1}{n} + \frac{2}{n^3}\right] - \left[-\frac{(\pi^2 - 0 + 0)\cdot\cos 0}{n} + 0 + \frac{2\cdot\cos(0)}{n^3}\right]\right.$$

$$b_n = \frac{1}{4\pi}\left\{\left[-\frac{\pi^2 - \cancel{4\pi^2} + \cancel{4\pi^2}}{n} + \frac{2}{n^3}\right] - \left[\frac{-\pi^2}{n} + \frac{2}{n^3}\right]\right\}$$

$$b_n = \frac{1}{4\pi}\left\{\cancel{\frac{-\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right\}$$

$$b_n = \frac{1}{4\pi}(0) = 0$$

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$\boxed{b_n = 0}$

step 4: Fourier series expansion

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \right]$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{1}{2} \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos(nx) + 0 \cdot \sin(nx)]$$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos(nx) \quad \text{—— (A)}$$

(i) put $x = 0$ in eqⁿ (A)

$$\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos 0}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots \quad \text{—— (i)}$$

(ii) put $x = \pi$ in eqⁿ (A)

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos(n\pi)$$

But $\cos(n\pi) = (-1)^n$

$\cos \pi = -1$

$\cos 2\pi = 1 = \cos(0)$

$$\Rightarrow -\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

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$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\therefore \frac{(-1)^1}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \frac{(-1)^4}{4^2} \cdots = \frac{-\pi^2}{12}$$

But $(-1)^{even} = 1$

and $(-1)^{odd} = -1$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \cdots = \frac{-\pi^2}{12}$$

multiply by '$-$' sign on both sides,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \text{--- (2)}$$

add eq^n (1) & (2)

$$2\left[\frac{1}{1^2}\right] + 2\left[\frac{1}{3^2}\right] + 2\left[\frac{1}{5^2}\right] = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Note: For such deduction in any example always put (i) $x=0$ (ii) $x=\pi$ in fourier expansion of $f(x)$.

* Some Important Formulae

① $$\int e^{ax}\cdot\cos(bx)\cdot dx = \frac{e^{ax}}{a^2+b^2}\left(a\cdot\cos bx + b\cdot\sin bx\right)$$

② $$\int e^{ax}\cdot\sin(bx)\cdot dx = \frac{e^{ax}}{a^2+b^2}\left(a\cdot\sin bx - b\cdot\cos bx\right)$$

③ $\cos(n\pm1)2\pi = \cos(2n\pi \pm 2\pi) = 1$

and $\sin(n\pm1)2\pi = \sin(2n\pi \pm 2\pi) = 0$

④ $2\cdot\sin A\cdot\sin B = \cos(A-B) - \cos(A+B)$

where $\cos(A-B) = \cos A\cdot\cos B + \sin A\cdot\sin B$

$\cos(A+B) = \cos A\cdot\cos B - \sin A\cdot\sin B$

⑤ $2\cdot\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$

where $\sin(A+B) = \sin A\cdot\cos B + \cos A\cdot\sin B$

$\sin(A-B) = \sin A\cdot\cos B - \cos A\cdot\sin B$

Ex Find the fourier series of the function $f(x) = e^{-x}$ $0\le x\le 2\pi$ and $f(x+2\pi) = f(x)$

$\Rightarrow$ Given $f(x) = e^{-x}$, $0\le x\le 2\pi$

Step 1: To find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cdot dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

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$$= \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{-1}\right] - \left[\frac{e^{0}}{-1}\right]\right\}$$

But $e^0 = 1$

$$a_0 = \frac{1}{\pi}\left(-e^{-2\pi} + 1\right)$$

$$\boxed{a_0 = \frac{\left(1 - e^{-2\pi}\right)}{\pi}}$$

Step 2: To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \cos(nx) \cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cdot \cos nx \cdot dx$$

But

$$\int e^{ax} \cdot \cos bx \cdot dx = \frac{e^{ax}}{a^2+b^2}\left(a \cdot \cos bx + b \cdot \sin bx\right)$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cdot \cos nx \cdot dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2 + n^2}\left(-\cos nx + n \cdot \sin nx\right)\right\}_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{\frac{e^{-2\pi}}{1+n^2}\left(-\cos 2n\pi + n \cdot \sin 2n\pi\right) - \frac{e^{-0}}{(-1)^2+n^2}\left(-\cos 0 + 0\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{e^{-2\pi}}{1+n^2}\left(-1 + 0\right) - \frac{1}{1+n^2}(-1)\right\}$$

(since $\cos 2n\pi = 1$, $\sin 2n\pi = 0$)

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$$a_n = \frac{1}{\pi} \frac{1}{1+n^2} \left(-e^{-2\pi} + 1\right)$$

$$\boxed{a_n = \frac{1 - e^{-2\pi}}{\pi(1+n^2)}}$$

step 3: To find bn

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin(nx) \cdot dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cdot \sin(nx) \cdot dx$$

But $\int e^{ax} \cdot \sin(bx) = \frac{e^{ax}}{a^2+b^2} \left\{ a \cdot \sin bx - b \cdot \cos bx \right\}$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cdot \sin(nx) \cdot dx$$

$$= \frac{1}{\pi} \left\{ \frac{e^{-x}}{(-1)^2 + n^2} \left((-1) \cdot \sin(nx) - n \cdot \cos(nx) \right) \right\}_0^{2\pi}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{e^{-2\pi}}{1+n^2} \left(-1 \cdot \sin 2n\pi - n \cdot \cos 2n\pi \right) \right] \right.$$

$$\left. - \left[\frac{1}{1^2+n^2} \left((-1) \cdot \sin 0 - n \cdot \cos(0) \right) \right] \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{e^{-2\pi}}{1+n^2} (0 - n) \right] - \left[\frac{1}{1+n^2} (0 - n) \right] \right\}$$

$$= \frac{1}{\pi} \cdot \frac{1}{(1+n^2)} \left(-n \cdot e^{-2\pi} + n \right) = \frac{n\left(1 - e^{-2\pi}\right)}{\pi(1+n^2)}$$

Step 4: Fourier series Expansion

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos(nx) + b_n \cdot \sin(nx)]$$

$$e^{-x} = \frac{1}{2} \frac{(1-e^{-2\pi})}{\pi} + \sum_{n=1}^{\infty} \left[\frac{(1-e^{-2\pi})}{\pi(1+n^2)} \cdot \cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)} \cdot \sin(nx) \right]$$

Example: obtained the fourier series expansion of the function
$f(x) = x \cdot \sin x$ in the interval $0 \le x \le 2\pi$

$\longrightarrow$

Step 1: To find a_0

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot dx$$

By Bernaullis Rule

$$a_0 = \frac{1}{\pi} \left\{ (x) [-\cos x] - 1 \cdot [-\sin x] \right\}_0^{2\pi}$$

$$a_0 = \frac{1}{\pi} [-x \cdot \cos x + \sin x]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi} \left\{ (-2\pi \cdot \cos 2\pi + \sin 2\pi) - (0 \cdot \cos 0 + \sin 0) \right\}$$

$$a_0 = \frac{1}{\pi} [-2\pi(1) + 0 - 0] = -2$$

$$\boxed{a_0 = -2}$$

step 2: To find a_n

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(nx) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \cos(nx) \cdot dx$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} x \cdot \underline{2\cos(nx) \cdot \sin x} \cdot dx$$

But $2 \cdot \cos nx \cdot \sin x = \sin(nx + x) - \sin(nx - x)$

By using $2 \cdot \cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} x \left[\sin(nx + x) - \sin(nx - x)\right] \cdot dx$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} x \left[\sin(n+1)x - \sin(n-1)x\right] \cdot dx$$

By using Bernaulli's Rule

$$a_n = \frac{1}{2\pi} \left\{ (x) \left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right] - (1) \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right] \right\}_0^{2\pi}$$

for $n \neq 1$, because for $n = 1$, denominator $= 0$ which will give ∞

$$a_n = \frac{1}{2\pi}\left\{ \frac{-x\cdot\cos(n+1)x}{n+1} + \frac{x\cdot\cos(n-1)x}{n-1} + \frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\}_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left\{\left[\frac{-2\pi\cdot\cos(n+1)2\pi}{n+1} + \frac{2\pi\cos(n-1)2\pi}{n-1} + \frac{\sin(n+1)2\pi}{(n+1)^2} - \frac{\sin(n-1)2\pi}{(n-1)^2}\right] - \left[-0 + 0 + \frac{\sin 0}{(n+1)^2} - \frac{\sin 0}{(n-1)^2}\right]\right\}$$

$\cos(n\pm1)2\pi = \cos(2n\pi \pm 2\pi) = 1$

$\sin(n\pm1)2\pi = \sin(2n\pi \pm 2\pi) = 0$

$$a_n = \frac{1}{2\pi}\left\{\left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1} + 0 - 0\right] - [0-0]\right\}$$

$$= \frac{1}{2\pi}(2\pi)\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]$$

$$a_n = \frac{-\cancel{n}+1+\cancel{n}+1}{(n+1)(n-1)}$$

$$a_n = \frac{2}{n^2-1}, \quad n > 1$$

Again, we have

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \cos nx \cdot dx$$

put $n=1$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \cos x \cdot dx$$

$$a_1 = \frac{1}{2\pi} \int_0^{2\pi} x \cdot \underline{2 \sin x \cdot \cos x} \cdot dx$$

$$a_1 = \frac{1}{2\pi} \int_0^{2\pi} x \cdot \sin 2x \cdot dx$$

$$a_1 = \frac{1}{2\pi} \left\{ (x) \cdot \left[\frac{-\cos 2x}{2} \right] - (1) \cdot \left[\frac{-\sin 2x}{4} \right] \right\}_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi} \left\{ \left[(2\pi) \left(\frac{-\cos 2(2\pi)}{2} \right) + \frac{\sin 2(2\pi)}{4} \right] - \left[0 \cdot \left(\frac{-\cos 2(0)}{2} + \frac{\sin 2(0)}{4} \right] \right\}$$

$$a_1 = \frac{1}{2\pi} \left[\left(2\pi \cdot \frac{-1}{2} + 0 \right) - (0+0) \right]$$

$$\boxed{a_1 = -\frac{1}{2}}$$

Step 3: To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(nx)\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\sin(nx)\cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x\cdot\sin(nx)\cdot dx$$

$(2\sin A\cdot\sin B = \cos(A-B) - \cos(A$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\cdot[\cos(nx-x) - \cos(nx+x)]\cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{x}_{u}\cdot\underbrace{[\cos(n-1)x - \cos(n+1)x]}_{v}\cdot dx$$

$\therefore$ By Bernaulli's Rule

$$b_n = \frac{1}{2\pi}\left\{(x)\cdot\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right] - (1)\left[\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right\}_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left\{\frac{x\cdot\sin(n-1)x}{(n-1)} - \frac{x\cdot\sin(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right\}_0^{2\pi}$$

But $\sin(n-1)2\pi = 0$

$\sin(n+1)2\pi = 0$

$$b_n = \frac{1}{2\pi}\left\{\left[0 - 0 + \frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n+1)2\pi}{(n+1)^2}\right] - [0\right.$$

$$+ \frac{\cos 0}{(n-1)^2} - \frac{\cos 0}{(n+1)^2} \Big] \Big\}$$

But $\cos(n \pm 1) 2\pi = 1$

$$bn = \frac{1}{2\pi} \left\{ \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right\}$$

$$bn = \frac{1}{2\pi} \{0\}$$

$$bn = 0 \quad , \quad n > 1$$

Again $$bn = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \sin(nx) \cdot dx$$

put $n = 1$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \sin x \cdot dx = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin^2 x \cdot dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cdot \left(\frac{1 - \cos 2x}{2} \right) \cdot dx$$

$$b_1 = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{x}_{u} \underbrace{(1 - \cos 2x)}_{v} \cdot dx$$

$\therefore$ By Bernaulli's Rule

$$b_1 = \frac{1}{2\pi} \left\{ (x) \left[x - \frac{\sin 2x}{2} \right] - (1) \left[\frac{x^2}{2} + \frac{\cos 2x}{4} \right] \right\}_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi} \left\{ x^2 - \frac{x \cdot \sin 2x}{2} - \frac{x^2}{2} - \frac{\cos 2x}{4} \right\}_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi} \left\{ \left[4\pi^2 - \frac{2\pi \cdot \sin 2(2\pi)}{2} - \frac{4\pi^2}{2} - \frac{\cos 4\pi}{4} \right] - \left[0 - 0 - 0 - \frac{\cos 0}{4} \right] \right\}$$

But $\sin 4\pi = 0$; $\cos 4\pi = 1$; $\cos 0 = 1$

$$b_1 = \frac{1}{2\pi}\left\{4\pi^2 - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right\}$$

$$b_1 = \frac{1}{2\pi}\left\{\frac{4\pi^2}{2}\right\}$$

$$\boxed{b_1 = \pi}$$

Step 4: Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cdot \cos nx + b_n \cdot \sin nx)$$

$$f(x) = \frac{1}{2}a_0 + (a_1 \cdot \cos x + b_1 \cdot \sin x) + \sum_{n=2}^{\infty}(a_n \cdot \cos nx + b_n \cdot \sin nx)$$

$$x \cdot \sin x = \frac{1}{2}(-2) + \left(-\frac{1}{2}\cos x + \pi \cdot \sin x\right) + \sum_{n=2}^{\infty}\left(\frac{2}{n^2-1}\cos nx + 0 \cdot \sin nx\right)$$

$$x \cdot \sin x = -1 - \frac{1}{2}\cos x + \pi \cdot \sin x + \sum_{n=2}^{\infty}\left(\frac{2}{n^2-1} \cdot \cos nx\right)$$

* Homework

① Find fourier series expansion in the interval $0 \le x \le 2\pi$

① $f(x) = x$ Ans $x = \pi - \sum_{n=1}^{\infty}\frac{2}{n} \cdot \sin(nx)$

② $f(x) = \frac{1}{2}(\pi - x)$ Ans $\Rightarrow \frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty}\frac{1}{n} \cdot \sin(nx)$

③

$$f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi \end{cases}$$

Deduce that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$

Ans $f(x) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nx + \frac{1}{2}\sin x$

Ex: what is the fourier expansion of the periodic function whose defination in one period is

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$$

$\Rightarrow$ Given

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$$

step 1 To find a_0:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\cdot dx$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi} -\pi \cdot dx + \int_{\pi}^{2\pi} (x-\pi)\cdot dx \right.$$

$$= \frac{1}{\pi}\left\{ (-\pi x)_0^{\pi} + \left(\frac{x^2}{2} - \pi x\right)_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ (-\pi^2 + 0) + \left[\left(\frac{4\pi^2}{2} - 2\pi^2\right) - \left(\frac{\pi^2}{2} - \pi^2\right)\right] \right\}$$

$$a_0 = \frac{1}{\pi}\left\{-\pi^2 + 0 + 0 + \frac{\pi^2}{2}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = \frac{1}{\pi}\left\{\frac{-2\pi^2 + \pi^2}{2}\right\}$$

$$a_0 = \frac{1}{\cancel{\pi}}\left\{\frac{-\pi^{\cancel{2}}}{2}\right\}$$

$$\boxed{a_0 = \frac{-\pi}{2}}$$

step

step 2:

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(nx)\cdot dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cdot\cos(nx)\cdot dx\right.$$

$$\left. + \int_{\pi}^{2\pi} f(x)\cdot\cos(nx)\cdot dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} -\pi\cdot\cos(nx)\cdot dx + \int_{\pi}^{2\pi}(x-\pi)\cdot\cos(nx)\cdot dx\right\}$$

By Bernaulli's Rule

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\sin(nx)}{n}\right]_0^{\pi} + \left\{(x-\pi)\left[\frac{\sin nx}{n}\right] - (1)\left[\frac{-\cos nx}{n^2}\right]\right\}_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\cdot\sin(n\pi)}{n} + \frac{\pi\cdot\sin(0)}{n}\right] + \left\{\frac{(x-\pi)\cdot\sin(nx)}{n} + \frac{\cos nx}{n^2}\right\}_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{[0+0] + \left\{\left(\frac{(2\pi-\pi)\sin n\pi}{n} + \frac{\cos n(2\pi)}{n^2}\right)\right.\right.$$

$$\left.\left. - \left[(0\cdot\sin\pi) + \frac{\cos n\pi}{n^2}\right]\right\}\right\}$$

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$$an = \frac{1}{\pi}\left\{0 + \frac{1}{n^2} - \left(0 + \frac{\cos n\pi}{n^2}\right)\right\}$$

$$an = \frac{1}{\pi}\left\{\frac{1-\cos n\pi}{n^2}\right\}$$

$$\boxed{an = \frac{1}{n^2\pi}(1-\cos n\pi)}$$

step3 <u>To find bn:</u>

$$bn = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(nx)\cdot dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cdot\sin(nx)\cdot dx + \int_{\pi}^{2\pi} f(x)\cdot\sin(nx)\cdot dx\right.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} -\pi\cdot\sin(nx)\cdot dx + \int_{\pi}^{2\pi}(x-\pi)\cdot\sin(nx)\,dx\right]$$

By Bernaullie Rule

$$bn = \frac{1}{\pi}\left\{\left[\frac{-\pi(-\cos(nx))}{n}\right]_0^{\pi} + \left\{(x-\pi)\left[\frac{-\cos(nx)}{n}\right] - (1)\left[\frac{-\sin(nx)}{n^2}\right]\right\}_{\pi}^{2\pi}\right\}$$

$$bn = \frac{1}{\pi}\left\{\left(\frac{\pi\cdot\cos n\pi}{n} - \frac{\pi\cdot\cos 0}{n}\right) + \left[-\frac{(2\pi-\pi)\cos n(2\pi)}{n} + \frac{\sin 2\pi n}{n^2}\right] - \left[-\frac{(\pi-\pi)\cdot\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right]\right\}$$

$$bn = \frac{1}{\pi}\left\{\left(\frac{\pi\cdot\cos n\pi}{n} - \frac{\pi}{n}\right) - \frac{\pi}{n} + 0 - 0\right\}$$

$$bn = \frac{1}{n\pi}\left(\pi\cdot\cos n\pi - \pi - \pi\right)$$

$$b_n = \frac{1}{n\cancel{\pi}} \cancel{\pi}(\cos n\pi - 2)$$

$$\boxed{b_n = \frac{1}{n} \cdot (\cos n\pi - 2)}$$

Step 4: Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cdot \cos nx + b_n \cdot \sin nx]$$

$$= \frac{-\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi}(1-\cos n\pi) \cdot \cos nx + \frac{1}{n}(\cos n\pi - 2) \cdot \sin nx\right]$$

* Type 2: $-\pi \le x \le \pi$

whenever a function defined in the interval $-a \le x \le a$ we need to check if the function is

(i) even

(ii) odd.

(iii) neither even nor odd.

- To identify above case replace x by $-x$ and if we get

$f(-x) = f(x)$ function is even

$f(-x) = -f(x)$ function is odd

$f(x) \neq f(-x)$ function is neither even nor odd

Even function	odd function	Neither even nor odd.
$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$	$a_0 = 0$	$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$
$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\,dx$	$a_n = 0$	$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$
$b_n = 0$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx$	$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx$

Ex(1) find the power series expansion of the function $f(x) = x$ in the interval $-\pi \le x \le \pi$

step 1: check for even / odd.

Let $f(x) = x$

$f(-x) = -x$

$f(-x) = -f(x)$

$\therefore$ fun is odd.

step 2: To find a_0 and a_n

As the function is odd

$a_0 = 0, \quad a_n = 0$

step 3 To find b_n

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\cdot dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cdot\sin(nx)\cdot dx$$

By Bernaullie's Rule

$$bn = \frac{2}{\pi}\left\{ (x)\cdot\left[\frac{-\cos nx}{n}\right] - (1)\left[\frac{-\sin nx}{n^2}\right] \right\}_0^{\pi}$$

$$bn = \frac{2}{\pi}\left\{ \frac{-x\cdot\cos nx}{n} + \frac{\sin nx}{n^2} \right\}_0^{\pi}$$

$$bn = \frac{2}{\pi}\left\{ \left[\frac{-\pi\cdot\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - [0+0] \right\}$$

$$bn = \frac{2}{\pi}\left\{ \frac{-\pi\cdot\cos n\pi}{n} + 0 \right\}$$

but. $\sin n\pi = 0$; $\sin 0 = 0$. Note $\cos n\pi \neq 1$

$$\therefore bn = \frac{2}{\cancel{\pi}}\left(\frac{-\cancel{\pi}\cdot\cos n\pi}{n} \right)$$

$$\boxed{bn = \frac{-2\cdot\cos n\pi}{n}}$$

ep 4: Fourier series Expansion.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cdot\cos nx + b_n\cdot\sin nx\right]$$

but $a_0 = 0$ and $a_n = 0$.

$$x = \sum_{n=1}^{\infty}\left[\frac{-2\cdot\cos n\pi}{n}\cdot\sin nx \right]$$

Q.2 find the fourier series to represent the function $f(x) = \pi^2 - x^2$ in the interval $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$. Hence deduced that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

$\Rightarrow$ Solution:

To find the fourier series of the given function follow the step given below.

Step 1: Check for even / odd:

$f(x) = \pi^2 - x^2$

put $x = -x$

$f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$

$\therefore$ function is even

Step 2: To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\,dx = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left[\pi^2(\pi) - \frac{(\pi)^3}{3} - 0 - 0\right]$$

$$a_0 = \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right)$$

$$= \frac{2}{\pi}\left(\frac{3\pi^3 - \pi^3}{3}\right)$$

$$= \frac{2}{\cancel{\pi}}\left(\frac{2\pi^{\cancel{3}}}{3}\right) = \frac{4\pi^2}{3}$$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

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step(2) To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underbrace{(\pi^2 - x^2)}_{u}\cdot \underbrace{\cos nx \cdot dx}_{v}$$

$\therefore$ By Bernaullie's Rule

$$a_n = \frac{2}{\pi}\left\{(\pi^2-x^2)\left[\frac{\sin nx}{n}\right] - (0-2x)\left[\frac{-\cos nx}{n^2}\right] + (-2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{(\pi^2-x^2)\cdot \sin nx}{n} - \frac{2x\cdot \cos nx}{n^2} + \frac{2\sin nx}{n^3}\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{(\pi^2-\pi^2)\cdot \sin n\pi}{n} - \frac{2\pi\cdot \cos n\pi}{n^2} + \frac{2\sin n\pi}{n^3}\right] - (0 - 0 + 0)\right\}$$

$$a_n = \frac{2}{\cancel{\pi}}\left\{0 - \frac{2\cancel{\pi}\cdot \cos n\pi}{n^2} + 0\right\}$$

$$\boxed{a_n = \frac{-4\cdot \cos n\pi}{n^2}}$$

Step 4 To find b_n

since $f(x)$ is even function so $b_n = 0$

Step 5: Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{-4\cdot\cos n\pi}{n^2}\cdot\cos(nx) + 0\cdot\sin nx\right]$$

$$f(x) = \frac{1}{2}\frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\cos n\pi\cdot\cos nx}{n^2}$$

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\cdot\cos n\pi\cdot\cos nx}{n^2} \quad \text{--- (A)}$$

Step 6: deduction

put $x=0$ in eq^n (A)

$$\pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\cdot\cos n\pi\cdot\cos n(0)}{n^2}$$

$$\pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\cdot\cos(n\pi)\cdot 1}{n^2}$$

But $\cos(n\pi) = (-1)^n$

$$\pi^2 - \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{(-4)(-1)^n}{n^2}$$

$$\frac{\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{(-1)^1}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \frac{(-1)^4}{4^2} + \cdots$$

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$$\therefore \frac{-\pi^2}{12} = -\frac{1}{1} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{---} (1)$$

(ii) put $x = \pi$ in eqⁿ (A)

$$\pi^2 - \pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4 \cdot \cos n\pi \cdot \cos n\pi}{n^2}$$

$$0 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4 \cdot (\cos n\pi)^2}{n^2}$$

$$\cos n\pi = (-1)^n$$

$$(\cos n\pi)^2 = ((-1)^n)^2 = 1$$

$$\frac{-2\pi^2}{3} = -4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{-2\pi^2}{3(-4)} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \quad \text{---} (2)$$

adding eqⁿ (1) & (2)

we get

$$2\left(\frac{1}{1^2}\right) + 2\left(\frac{1}{3^2}\right) + 2\left(\frac{1}{5^2}\right) + \cdots = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{4}$$

$$\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}}$$

Homework

(i) find fourier series expansion of the function $f(x)=x^2$ in the interval $-\pi < x \le \pi$

Ans $x^2 = \sum_{n=1}^{\infty}\left[\frac{2}{n}\left(-\pi^2+\frac{6}{n^2}\right)\cos n\pi \cdot \sin n\pi\right]$

Ex (3) Find the fouriers series expansion of the function $f(x)=x^2$ in the interval. $-\pi \le x \le \pi$

$\Rightarrow$

Given $f(x)=x^2$; $-\pi \le x \le \pi$

Step 1: Check for even / odd

$f(x)=x^2$

put $x=-x$

$f(-x)=(-x)^2$

$f(-x)=x^2$

Step 2: To find a_0

$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$=\frac{2}{\pi}\int_0^{\pi} x^2\,dx$

$a_0=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$

$a_0=\frac{2}{\pi}\cdot\frac{\pi^3}{3}$

$$\boxed{a_0=\frac{2}{3}\pi^2}$$

Step 3: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx \cdot dx = \frac{2}{\pi}\int_0^{\pi} x^2\cdot\cos(nx)\cdot dx$$

$$a_n = \frac{2}{\pi}\left\{ x^2\cdot\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n^2}\right] + (2)\left[\frac{-\sin nx}{n^3}\right]\right\}$$

$$a_n = \frac{2}{\pi}\left\{ \frac{x^2\cdot\sin nx}{n} + \frac{2x\cdot\cos nx}{n^2} - \frac{2\cdot\sin nx}{n^3}\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi\cdot\cos n\pi}{n^2} - 0\right] - [0+0-0]\right\}$$

$$a_n = \frac{4\cdot\cos n\pi}{n^2}$$

Step 4: Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cdot\cos nx + b_n\cdot\sin nx\right]$$

since $f(x)$ is even, $b_n = 0$

$$x^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\frac{4\cdot\cos n\pi}{n^2}\cdot\cos nx + 0$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cdot\cos n\pi}{n^2}\cdot\cos nx$$

Example 4:

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Step 1 check for even / odd.

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

$$f(-x) = \begin{cases} -\pi & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases}$$

$f(x) \neq f(-x)$

$\therefore$ function is neither even nor odd.

step 2: To find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\cdot dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cdot dx + \int_{0}^{\pi} x\cdot dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\left[-\pi(0) + \pi(-\pi)\right] + \left[\frac{\pi^2}{2} - 0\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi^2}{2}\right\}$$

$$\boxed{a_0 = \frac{-\pi}{2}}$$

step 3: To find an:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\cdot dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cdot\cos nx\cdot dx + \int_{0}^{\pi} f(x)\cdot\cos nx\cdot dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cdot\cos nx\, dx + \int_{0}^{\pi} x\cdot\cos nx\cdot dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\pi\cdot\sin nx}{n}\right]_{-\pi}^{0} + \left\{x\left[\frac{\sin nx}{n}\right] - (1)\left[\frac{-\cos nx}{n^2}\right]\right\}_{0}^{\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-0 + \pi\cdot\sin(-n\pi)}{n}\right] + \left[\frac{x\cdot\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[0 + 0\right] + \left[\frac{x\cdot\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

since $\sin(-n\pi) = -\sin n\pi = 0$

$$a_n = \frac{1}{\pi}\left\{\left[-0+0\right] + \left\{\left[0 + \frac{\cos n\pi}{n^2}\right] - \left[0 + \frac{\cos 0}{n^2}\right]\right\}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{\cos n\pi - 1}{n^2}\right\}$$

$$\boxed{a_n = \frac{\cos n\pi - 1}{n^2\pi}}$$

step 4: To find bn

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\cdot dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cdot\sin nx\cdot dx + \int_{0}^{\pi} f(x)\sin nx\, dx\right\}$$

$$bn = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi \cdot \sin nx \cdot dx + \int_{0}^{\pi} \underset{u}{x} \cdot \underset{v}{\sin nx} \cdot dx\right\}$$

$$bn = \frac{1}{\pi}\left\{\left[\frac{\pi \cdot \cos nx}{n}\right]_{-\pi}^{0} + \left\{(x)\left[\frac{-\cos nx}{n}\right] - (1)\left[\frac{-\sin nx}{n^2}\right]\right\}_{0}^{\pi}\right\}$$

$$bn = \frac{1}{\pi}\left\{\left[\frac{\pi \cdot \cos 0}{n} - \frac{\pi \cdot \cos(-n\pi)}{n}\right] + \left[\frac{-x \cdot \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

~~bn =~~ But $\cos(-n\pi) = \cos n\pi$

$$bn = \frac{1}{\pi}\left\{\left[\frac{\pi}{n} - \frac{\pi \cdot \cos(n\pi)}{n}\right] + \left\{\left[\frac{-\pi \cdot \cos n\pi}{n} + 0\right] - (0+0)\right\}\right\}$$

$$bn = \frac{1}{\pi}\left\{\frac{\pi}{n} - \frac{\pi \cdot \cos n\pi}{n} - \frac{\pi \cdot \cos n\pi}{n}\right\}$$

$$bn = \frac{1}{n}(1 - 2\cos n\pi)$$

Step 5: Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[an \cdot \cos nx + bn \cdot \sin nx]$$

$$f(x) = \frac{1}{2}\left(-\frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(\cos n\pi - 1) \cdot \cos nx + \frac{1}{n}(1 - 2\cos n\pi) \cdot \sin nx\right]$$

Type 3: Interval $0 \le x \le 2L$

Expansion of the function $f(x)$ in the interval $0 \le x \le 2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where $a_0 = \frac{1}{L} \int_0^{2L} f(x) \cdot dx$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot dx$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

ample:- Find the fourier series expansion of the fun $f(x) = 4 - x^2$ in the interval $0 < x < 2$

ep 1 Given interval, $0 < x < 2$

$2L = 2 \Rightarrow L = 1$

2: To find a_0

$$a_0 = \frac{1}{L} \int_0^{2L} f(x) \cdot dx = \frac{1}{1} \int_0^{2} (4 - x^2) \cdot dx = \left[4x - \frac{x^3}{3}\right]$$

$$a_0 = \left[4(2) - \frac{(2)^3}{3}\right] - \left[4(0) - \frac{(0)^2}{2}\right]$$

$$\boxed{a_0 = \frac{16}{3}}$$

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Step 3: To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)\cdot dx$$

$$= \frac{1}{1}\int_0^{2} \underbrace{(4-x^2)}_{u}\cdot\cos\underbrace{\left(\frac{n\pi x}{1}\right)}_{v} dx$$

$\therefore$ By Bernaulli-es Rule

$$a_n = \left\{(4-x^2)\left[\frac{\sin(n\pi x)}{n\pi}\right] - (-2x)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right] + (-2)\left[\frac{-\sin(n\pi x)}{(n\pi)^3}\right]\right\}_0^2$$

$$a_n = \left\{\frac{(4-x^2)\cdot\sin(n\pi x)}{n\pi} - \frac{2x\cdot\cos(n\pi x)}{(n\pi)^2} + \frac{2\sin(n\pi x)}{(n\pi)^3}\right\}_0^2$$

$$a_n = \left\{\left[0 - \frac{2(2)\cdot\cos n\pi 2}{n^2\pi^2} + 0\right] - \left[0 - 0 + 0\right]\right\}$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}}$$

Step 4 To find b_n.

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot dx$$

$$= \frac{1}{1}\int_0^{2} (4-x^2)\cdot\sin\left(\frac{n\pi x}{1}\right)\cdot dx$$

$$= \int_0^2 \underbrace{(4-x^2)}_{u} \cdot \underbrace{\sin(n\pi x)}_{v}\, dx$$

$\therefore$ By Bernaullies Rule.

$$bn = \left\{ (4-x^2) \cdot \left[\frac{-\cos(n\pi x)}{n\pi}\right] - (-2x)\left[\frac{-\sin(n\pi x)}{(n\pi)^2}\right] + (-2)\left[\frac{\cos(n\pi x)}{(n\pi)^3}\right] \right\}_0^2$$

$$bn = \left\{ \frac{-(4-x^2)\cos n\pi x}{n\pi} - \frac{2x \sin(n\pi x)}{(n\pi)^2} - \frac{2 \cdot \cos(n\pi x)}{(n\pi)^3} \right\}_0^2$$

$$bn = \left\{ \left[\frac{-(4-2^2) \cdot \cos n\pi 2}{n\pi} - \frac{2 \times 2 \sin(n\pi 2)}{(n\pi)^2} - \frac{2 \cdot \cos n\pi 2}{(n\pi)^3} \right] - \left[\frac{-(4-0) \cdot \cos n\pi(0)}{n\pi} - 0 - \frac{2 \cos n\pi(0)}{(n\pi)^3} \right] \right\}$$

$$bn = \left\{ \left[0 - 0 - \frac{2 \cdot 1}{n^3\pi^3} \right] - \left[\frac{-4}{n\pi} - \frac{2 \cdot 1}{(n\pi)^3} \right] \right\}$$

$$bn = \cancel{\frac{-2}{n^3\pi^3}} + \frac{4}{n\pi} + \cancel{\frac{2}{n^3\pi^3}}$$

$$\boxed{bn = \frac{4}{n\pi}}$$

step 5: Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$4-x^2 = \frac{1}{2}\left(\frac{16}{3}\right) + \sum_{n=1}^{\infty}\left[\left(\frac{-4}{n^2\pi^2}\right)\cdot\cos(n\pi x) + \frac{4}{n\pi}\sin(n\pi x)\right]$$

Type 4 :- Interval $-L \leq x \leq L$

whenever the function is defined in the interval from $-L$ to L. we have to check if the given function is (i) even (ii) odd (iii) neither even nor odd.
and for interval $-L \leq x \leq L$ Apply the formulae accordingly as:

Even function	Odd function	Neither even nor odd.
$a_0 = \frac{2}{L}\int_0^L f(x)\cdot dx$	$a_0 = 0$	$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$
$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx$	$a_n = 0$	$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx$
$b_n = 0$	$b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot dx$	$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx$

Homework

(1) find the fourier series expansion of the function $f(x) = x^2$ in the interval $-1 \leq x \leq 1$

Ans $$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4\cdot\cos n\pi}{n^2\cdot\pi^2}\cdot\cos(n\pi x).$$

* Half Range Expansion:

In some problem it is required to obtained a fourier expansion of the function to hold for range which is half the period of fourier series, i.e to expand f(x) in the range of $(0, \pi)$ in series of period 2π or more genrally in the range $(0, L)$ in a fourier series of period $2L$

We have devided half range expansion in two type

(1) Half range expansion in $0 < x < \pi$ (angular interval)

(2) Half range expansion in $0 < x < L$: (arbitory interval)

Half range expansion again divided into 4 type

(i) Half range cosine expression in $0 < x < \pi$
(ii) Half range sine expression in $0 < x < \pi$

(iii) Half range cosine expression in $0 < x < L$
(iv) Half range sine expression in $0 < x < L$

* Formulae for Half Range cosine expansion in the interval $0 < x < \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cdot \cos nx]$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos nx \cdot dx$$

$$b_n = 0$$

(we can see these formulae are equal to formulae for <u>even function</u> in the interval $-\pi < x < \pi$)

* <u>Formulae for Half range sine expansion in the interval $0 < x < \pi$</u>

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(nx)\cdot dx$$

where $a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin(nx)\cdot dx$$

(we can see that these formulae are equal to the formulae for <u>odd function</u> in the interval $-\pi < x < \pi$)

* <u>Formulae for Half range cosine expansion in the interval $0 < x < L$.</u>

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right).$$

where

$$a_0 = \frac{2}{L}\int_0^{L} f(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cdot \cos\left(\frac{n\pi x}{L}\right)\cdot dx$$

$$b_n = 0$$

(This formulae is same as the formulae for even fun in the interval $(-L < x < L)$

* Formulae for half range sine expansion in the interval $0 < x < L$

$$\therefore \; f(x) = \sum_{n=1}^{\infty} \left[b_n \cdot \sin\left(\frac{n\pi x}{L}\right) \right]$$

where $a_0 = 0$

$$a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

Problem

① find the half range cosine series for $f(x) = \pi x - x^2$ in the interval $0 \le x \le \pi$.

Step 1: To find a_0

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cdot dx$$

$$= \frac{2}{\pi} \left\{ \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right] \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{2} - \frac{\pi^3}{3} - (0 - 0) \right]$$

$$= \frac{2}{\pi} \left[\frac{3\pi^3 - 2\pi^3}{6} \right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}} \times \frac{\pi^{\cancel{3}}}{\cancel{6}\,3} = \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

step2: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\cdot dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cdot\cos(nx)\cdot dx$$

$\quad\quad u \quad\quad\quad\quad v$

$\therefore$ By Bernaullies Rule,

$$a_n = \frac{2}{\pi}\left\{(\pi x - x^2)\left[\frac{\sin(nx)}{n}\right] - (\pi - 2x)\left[\frac{-\cos(nx)}{n^2}\right] + (-2)\left[\frac{-\sin(nx)}{n^3}\right]\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{(\pi x - x^2)\cdot\sin nx}{n} + \frac{(\pi - 2x)\cdot\cos(nx)}{n^2} + \frac{2\cdot\sin(nx)}{n^3}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0 + \frac{(\pi - 2\pi)\cdot\cos n\pi}{n^2} + 0\right] - \left[0 + \frac{\pi}{n^2} + 0\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi(-1)^n}{n^2} - \frac{\pi}{n^2}\right\}$$

$$= \frac{2}{\cancel{\pi}} \times \frac{\cancel{\pi}}{n^2}\left\{-(-1)^n - 1\right\}$$

$$= -\frac{2}{n^2}\left\{1 + (-1)^n\right\}$$

$$a_n = \frac{-2}{n^2}\{1+(-1)^n\}$$

step 3: To find b_n

For half range cosine series $b_n = 0$

step 4 Half range co-sine expression

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(nx)$$

$$\pi x - x^2 = \frac{1}{2}\left(\frac{\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{-2}{n^2}(1+(-1)^n)\right)\cos nx$$

$$\pi x - x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{-2}{n^2}(1+(-1)^n)\cdot \cos nx\right]$$

Homework

Find the half range sine series for the function $f(x) = \pi^2 - x^2$, $0 < x < \pi$

(2) Find the half range cosine series for $f(x) = x - x^2$ in the interval $0 \leq x \leq 1$.

$\Longrightarrow$

step 1: $f(x) = x - x^2$; $0 \leq x \leq 1$

$\because$ comparing the interval with $0 \leq x \leq L$

$\therefore L = 1$

step 2: To find a_0

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{1}\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

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$$a_0 = \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6}$$

$$\boxed{a_0 = \frac{1}{6}}$$

Step 3: To find an

$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi x}{L}\right).dx$$

$$= \frac{2}{1}\int_0^1 (x-x^2)\cdot\cos\left(\frac{n\pi x}{1}\right).dx$$

$$a_n = 2\int_0^1 \underbrace{(x-x^2)}_{u}\cdot\underbrace{\cos(n\pi x)}_{v}.dx$$

$$a_n = 2\left\{\left[(x-x^2)\left[\frac{\sin(n\pi x)}{n\pi}\right] - (1-2x)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right] + (-2)\left(\frac{-\sin(n\pi x)}{(n\pi)^3}\right)\right]\right\}_0^1$$

$$a_n = 2\left\{\frac{(x-x^2)\cdot\sin n\pi x}{n\pi} + \frac{(1-2x)\cdot\cos(n\pi x)}{(n\pi)^2} + \frac{2\sin(n\pi x)}{(n\pi)^3}\right\}_0^1$$

$$a_n = 2\left\{\left[0 + \frac{(1-2)\cdot\cos n\pi}{n^2\pi^2} + 0\right] - \left[\frac{1}{n^2\pi^2}\right]\right\}$$

$$a_n = \frac{-2}{n^2\pi^2}(\cos n\pi + 1)$$

For half range cosine series $b_n = 0$

Step 4: Half range cosine expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cdot \cos nx]$$

$$x - x^2 = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{-2}{n^2 \pi^2} (1 + \cos n\pi) \cdot \cos nx$$

Note

(1) For fourier series expansion $f(x) = 2x - x^2$, $0 \le x$
To find value of L,
we have To compare given interval
$0 \le x \le 2L$, as it is full series expansion

$\therefore 2L = 3$, $L = \frac{3}{2}$

(2) whereas for half range fourier series of
$f(x) = 2x - x^2$ $0 \le x \le 3$
To find value of L, we have compare the given interval with $0 \le x \le L$, as it is half series expansion.

$\therefore \boxed{L = 3}$

Parseval's Identity for Fourier Coefficient

Theorem:-

If the fourier series for $f(x)$ in $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) \cdot dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

Then $$\int_{-L}^{L} [f(x)]^2 \cdot dx = L\left[\frac{1}{2} a^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

Corollary 1: If $f(x)$ is even fun in $(-L, L)$ then

$$\int_{-L}^{L} [f(x)]^2 \cdot dx = L \cdot \left\{\frac{(a_0)^2}{2} + \sum_{n=1}^{\infty} (a_n)^2\right\}$$

Corollary 2: If $f(x)$ is an odd fun in $(-L, L)$ then

$$\int_{-L}^{L} [f(x)]^2 \cdot dx = L \cdot \left\{\sum_{n=1}^{\infty} (b_n)^2\right\}$$

Corollary 3: If $f(x)$ is function define in interval $(0, 2L)$ then

$$\int_0^{2L} [f(x)]^2 \cdot dx = L \left\{ \frac{(a_0)^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$$

Corollary 4:-

For a half range cosine series in $(0, L)$ of the fun $f(x)$

$$\int_0^{L} [f(x)]^2 \cdot dx = \frac{L}{2} \left\{ \frac{(a_0)^2}{2} + \sum_{n=1}^{\infty} (a_n)^2 \right\}$$

Corollary 5:-

For half range sine series in $(0, L)$ of the function $f(x)$

$$\int_0^{L} [f(x)]^2 \cdot dx = \frac{L}{2} \left\{ \sum_{n=1}^{\infty} (b_n)^2 \right\}$$

Ex. find the fourier series for $f(x) = x^2$ in $(-\pi, \pi)$ we parseval's identity to prove that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$\Rightarrow$ Given $f(x) = x^2$; $-\pi \le x \le \pi$.

step 1 check for even / odd

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ Function is even.

To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot dx = \frac{2}{\pi}\int_0^{\pi} x^2\cdot dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2}{3}\pi^2$$

Step 3: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos nx\cdot dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cdot \cos nx\cdot dx$$

$$a_n = \frac{2}{\pi}\left\{(x^2)\cdot\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n^2}\right] + (2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{\pi^2\cdot\sin n\pi}{n} + \frac{2\pi\cdot\cos n\pi}{n^2} - \frac{2\cdot\sin n\pi}{n^3} - (0-0+0)\right.$$

$$a_n = \frac{2}{\cancel{\pi}}\left\{0 + \frac{2\cancel{\pi}\cdot\cos n\pi}{n^2}\right\}$$

$$\boxed{a_n = \frac{4\cdot\cos n\pi}{n^2}}$$

Step 4: Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cdot\cos nx + b_n\cdot\sin nx$$

Since $f(x)$ is even fun, $b_n = 0$

$$x^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\frac{4\cdot\cos n\pi}{n^2}\cdot\cos nx + 0$$

$$x^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4\cdot\cos n\pi}{n^2} \cdot \cos nx$$

$$x^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cdot \cos nx$$

where $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$

$\therefore$ By using parseval's identity.

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \pi \left\{ \frac{(a_0)^2}{2} + \sum_{n=1}^{\infty} (a_n)^2 \right\}$$

$$\int_{-\pi}^{\pi} (x^2)^2 dx = \pi \left\{ \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2}\right]^2 \right\}$$

$$\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \pi \left\{ \frac{2\pi^4}{9} + \sum_{n=1} \frac{16}{n^4} \right\}$$

$$\left(\frac{\pi^5}{5}\right) - \left(\frac{(-\pi)^5}{5}\right) = \pi \left\{ \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} \right\}$$

$$\frac{2\pi^5}{5} = \pi \left\{ \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \right\}$$

$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{5} - \frac{\pi^4}{9} = 8 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{4\pi^4}{45} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\cancel{4}\pi^4}{45\times\cancel{8}_2} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots}$$

Homework

Find the ~~fourier series~~ fourier sine series for $f(x)=1$ in $(0,\pi)$ and using parsevals identity show that

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

* Practical Hormonic Analysis.

We have discussed the problem of expanding the perodic function $y=f(x)$, defined within its range as fourier series.

$$y=f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n.\cos nx + b_n.\sin nx]$$

By evaluating integrals. for fourier coifficents.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx.dx$$

However, in many practical situation, the fun is to be analysed into hormonic fun which is not in terms of formula but by discreate values in tabular form

In such case, we cannot evaluate the above mentioned integral so we determine approximated value of fourier coifficent by method of approximate integration.

$a_0 = 2 \times$ [mean value of $y = f(x)$ bet^n 0 to 2π]

$a_n = 2 \times$ [mean value of $y \cdot \cos nx$ bet^n 0 to 2π]

$b_n = 2 \times$ [mean value of $y \cdot \sin nx$ bet^n 0 to 2π]

Note:

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cdot \cos nx + b_n \cdot \sin nx]$

then

1. a_0 is known as constant terms
2. For $n=1$, we get $a_1 \cdot \cos x + b_1 \cdot \sin x$ know as first harmonic
3. for $n=2$, we get $a_2 \cdot \cos 2x + b_2 \cdot \sin 2x$ know as second hormonic and so on..

Q Using tabulated values of x and y given in the table, determine the constant term, the coefficients of the first sine and cosine terms in the fourier expansion of y. Also obtain amplitude of the first hormonic

→ Given values of x are not in degree so it represents arbitory interval $0 < x < 2l$

$2l = 6$

$\Rightarrow \boxed{l = 3}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right)$

$y = \frac{a_0}{2} + a_1 \cdot \cos\left(\frac{\pi x}{3}\right) + b_1 \cdot \sin\left(\frac{\pi x}{3}\right)$ —— (1)

x	y	$\cos\left(\frac{\pi x}{3}\right) = \cos\left(\frac{180 \times x}{3}\right)$	$\sin\left(\frac{\pi x}{3}\right) = \sin\left(\frac{180 \times x}{3}\right)$
0	9	1	0
1	18	0.5	0.866
2	24	-0.5	0.866
3	28	-1	0
4	26	-0.5	-0.866
5	20	0.5	-0.866.

$a_0 = 2 \times$ mean value of y

$a_0 = 2 \times \frac{\sum y}{6} = \frac{1}{3}[9+18+24+28+26+20]$

$a_0 = \frac{1}{3} \times 125$

$\boxed{a_0 = 41.66}$

$a_n = 2 \times \left[\text{mean values of } y \cdot \cos\left(\frac{n\pi x}{L}\right)\right]$

put $n=1$, and $l=3$

$a_1 = 2 \times \left[\text{mean value of } y \cdot \cos\left(\frac{\pi x}{L}\right)\right]$

$a_1 = 2 \times \frac{\sum y \cdot \cos\left(\frac{\pi x}{3}\right)}{6}$

$a_1 = \frac{1}{3}\left[(9 \times 1) + (18 \times 0.5) + [24 \times (-0.5)] - (24 \times 0.5) - (28 \times 1) - (26 \times 0.5) + 20 \times 0.5\right]$

$a_1 = \frac{1}{3}(-25)$

$$\boxed{a_1 = -8.33}$$

and $b_n = 2 \times [\text{mean value of } y \cdot \sin(\frac{n\pi x}{l})]$

put $n=1$, $l=3$

$b_1 = 2 \times [\text{mean value of } y \cdot \sin(\frac{\pi x}{3})]$

$$b_1 = 2 \times \frac{\sum y \cdot \sin(\frac{\pi x}{3})}{6}$$

$$b_1 = \frac{1}{3}[(9 \times 0) + (18 \times 0.866) + (24 \times 0.866) + (28 \times 0) - (26 \times 0.866) - (20 \times 0.866)]$$

$$b_1 = \frac{1}{3}(-3.344)$$

$$b_1 = -1.114$$

put value of a_0, a_1 and b_1 in eq^n (1)

$$\therefore y = \frac{41.66}{2} + \left[-8.33 \cos\left(\frac{\pi x}{3}\right) - 1.114 \sin\left(\frac{\pi x}{3}\right)\right]$$

$$y = 20.83 - 8.33 \cdot \cos\left(\frac{x\pi}{3}\right) - 1.114 \cdot \sin\left(\frac{x\pi}{3}\right)$$

$\therefore$ Amplitude of first hormonic $= \sqrt{a_1^2 + b_1^2}$

$$= \sqrt{(-8.33)^2 + (-1.114)^2}$$

$$= 8.4041$$