UNIT - III

Chapter 5 - Partial Differentiation

(15 marks out of 70)

* Simple Differentiation:

Case 1: Consider a function involving only one variable (x)

e.g. $u = f(x)$

$u = 2 \cdot \sin x$

Diff. w.r.to x

$$\frac{d}{dx}(u) = \frac{d}{dx}(2 \cdot \sin x)$$

$$\therefore \frac{du}{dx} = 2 \cdot \frac{d}{dx}(\sin x)$$

$$\boxed{\frac{du}{dx} = 2 \cdot \cos x}$$

Case 2:

Consider a function involving more than one variab (x, y).

$v = f(x, y)$

$v = x \cdot y$

Diff. w.r.to x,

$$\frac{dv}{dx} = \frac{d}{dx}(xy)$$

As both x and y are variables, we derive it by using uv rule of derivative.

$$\therefore \frac{dv}{dx} = x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x)$$

$$\frac{dv}{dx} = x \cdot \frac{dy}{dx} + y$$

Similarly $\frac{dv}{dy} = x \cdot \frac{d}{dy}(y) + y \cdot \frac{d}{dy} x$

$$= x \cdot 1 + y \cdot \frac{dx}{dy}$$

* Partial Differentiation:

Case 1:

consider the function involving one variable (x)

e.g. $u = f(x)$

$u = 2 - \sin x$

Diff. w.r.to x partially.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2 - \sin x)$$

$$\frac{\partial u}{\partial x} = 2 - \frac{\partial}{\partial x}(\sin x) = 2 - \cos x$$

Case 2:

consider the function involving one variable (x, y)

e.g. $v = f(x, y)$

$v = xy$

Diff w.r.to x partially,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x y)$$

As we diff. w.r.to x, partially, y become constant

$$\therefore \frac{\partial v}{\partial x} = y \cdot \frac{\partial}{\partial x}(x)$$

$$\frac{\partial v}{\partial x} = y(1)$$

$$\frac{\partial v}{\partial x} = y$$

similarly $v = xy$

diff w.r.to y partially, i.e x becomes constant

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(xy)$$

$$\frac{\partial v}{\partial y} = x \cdot \frac{\partial}{\partial y}(y)$$

$$\frac{\partial v}{\partial y} = x(1)$$

$$\boxed{\frac{\partial v}{\partial y} = x}$$

simple differntiation

① $\frac{d}{dx}(2\sin x) = 2\cdot\cos x$

② $\frac{d}{dx}(xy) = x\cdot\frac{dy}{dx} + y$

③ $\frac{d}{dy}(xy) = x + y\cdot\frac{dx}{dy}$

Partial differntiation

(i) $\frac{\partial}{\partial x}(2\cdot\sin x) = 2\cos x$

(ii) $\frac{\partial}{\partial x}(xy) = y$

(iii) $\frac{\partial}{\partial y}(xy) = x$

Conclusion:

The method of finding derivative w.r.to one variable only (i.e treating othere as constant) in a function of two or more than two variable is known as partial differntiation.

However in simple differentiation all the given variables are treated as variables while finding derivative.

* some standard Derivative

① $\frac{\partial}{\partial x}(\text{constant}) = 0$

② $\frac{\partial}{\partial x}(x) = 1$

③ $\frac{\partial}{\partial x}(x^n) = n\cdot x^{n-1}$

④ $\frac{\partial}{\partial x}(x^n) = n\cdot x^{n-1}$

⑤ $\frac{\partial}{\partial x}(a^x) = a^x\cdot\log a$

⑥ $\frac{\partial}{\partial x}e^{ax} = a\cdot e^{ax}$

(7) $\frac{\partial}{\partial x}\frac{1}{x} = -\frac{1}{x^2}$

(8) $\frac{\partial}{\partial x}\log x = \frac{1}{x}$

(9) $\frac{\partial}{\partial x}\sin x = \cos x$

(10) $\frac{\partial}{\partial x}\cos x = -\sin x$

(11) $\frac{\partial}{\partial x}\tan x = \sec^2 x$

(12) $\frac{\partial}{\partial x}\cot x = -\operatorname{cosec}^2 x$

(13) $\frac{\partial}{\partial x}\sec x = \sec x \cdot \tan x$

(14) $\frac{\partial}{\partial x}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$

(15) $\frac{\partial}{\partial x}(u \cdot v) = u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial u}{\partial x}$

(16) $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{\partial u}{\partial x} - u \cdot \frac{\partial v}{\partial x}}{v^2}$

(17) $\frac{\partial}{\partial x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(18) $\frac{\partial}{\partial x}\tan^{-1} x = \frac{1}{1+x^2}$

Type 1: Solved Example on Direct Differentiation

Q.1 If $u = \log(x^3+y^3-x^2y-xy^2)$ prove that,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2}$$

or $$\frac{\partial^2 u}{\partial x^2} + 2\cdot\frac{\partial^2 u}{\partial x \cdot \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$$

$\Rightarrow$ $$u = \log(x^3+y^3-x^2y-xy^2)$$

$$u = \log\left[(x^3-x^2y) + (y^3-xy^2)\right]$$

$$u = \log\left[x^2(x-y) + y^2(y-x)\right]$$

put $y-x = -(x-y)$

$$u = \log\left[x^2(x-y) - y^2(x-y)\right]$$

$$u = \log\left[(x^2-y^2)(x-y)\right]$$

$$u = \log\left[(x-y)(x+y)(x-y)\right]$$

$$u = \log\left[(x-y)^2(x+y)\right]$$

by $\log A \cdot B = \log A + \log B$

$$u = \log(x-y)^2 + \log(x+y)$$

By $\log m^n = n \cdot \log m$

$$u = 2 \cdot \log(x-y) + \log(x+y) \quad \text{——} ①$$

~~step 1: to find~~ $\frac{\partial^2 u}{\partial x^2}$

we have $u = 2 \cdot \log(x-y) + \log(x+y)$

diff. w.r.to x partially.

$$\frac{\partial u}{\partial x} = 2 \cdot \frac{\partial}{\partial x} \log(x-y) + \frac{\partial}{\partial x} \log(x+y)$$

$$= 2 \frac{1}{x-y} \cdot \frac{\partial}{\partial x}(x-y) + \frac{1}{x+y} \cdot \frac{\partial}{\partial x}(x+y)$$

$$= 2 \cdot \frac{1}{x-y}(1-0) + \frac{1}{x+y} \cdot (1+0)$$

$$\frac{\partial u}{\partial x} = \frac{2}{x-y} + \frac{1}{x+y}$$

diff. w.r.to x, again partially,

$$\frac{\partial}{\partial x} \frac{\partial u}{\partial x} = 2 \cdot \frac{\partial}{\partial x}\left(\frac{1}{x-y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right)$$

But $\frac{\partial}{\partial x} \frac{1}{x} = -\frac{1}{x^2}$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2}}$$

Step 2: $\frac{\partial^2 u}{\partial y^2}$:

From eq^n ①

$$u = 2 \cdot \log(x-y) + \log(x+y)$$

Differentiate w.r. to y partially,

$$\frac{\partial u}{\partial y} = 2 \cdot \frac{\partial}{\partial y} \log(x-y) + \frac{\partial}{\partial y} \log(x+y)$$

$$= 2 \frac{1}{x-y} \cdot \frac{\partial}{\partial y}(x-y) + \frac{1}{x+y} \frac{\partial}{\partial y}(x+y)$$

$$= \frac{-2}{x-y}(-1+0) + \frac{1}{x+y}(1+0)$$

$$\frac{\partial u}{\partial y} = \frac{-2}{x-y} + \frac{1}{x+y} \quad (2)$$

diff w.r. to y partially again

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -2 \cdot \frac{\partial}{\partial y}\left(\frac{1}{x-y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right).$$

$$= -2 \frac{-1}{(x-y)^2}(0-1) + -\frac{1}{(x+y)^2}(0+1)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

Part 3:

To find $\frac{\partial^2 u}{\partial x \cdot \partial y}$:

From equation (2)

$$\frac{\partial u}{\partial y} = \frac{-2}{x-y} + \frac{1}{x+y}.$$

Diff. w. r. to x partially.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -2 \cdot \frac{\partial}{\partial x}\left(\frac{1}{x-y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right)$$

$$= -2 \frac{-1}{(x-y)^2}(1-0) + \frac{-1}{(x+y)^2}(1+0)$$

$$\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

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Now

L.H.S $\frac{\partial^2 u}{\partial x^2} + \frac{2 \cdot \partial^2 u}{\partial x \cdot \partial y} + \frac{\partial^2 u}{\partial y^2}$

$$= \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} + 2\left[\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}\right]$$

$$+ \left[\frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2}\right]$$

$$= \cancel{\frac{-2}{(x-y)^2}} - \frac{1}{(x+y)^2} + \cancel{\frac{4}{(x-y)^2}} - \frac{2}{(x+y)^2}$$

$$\cancel{-\frac{2}{(x-y)^2}} - \frac{1}{(x+y)^2}$$

$$= \frac{-4}{(x+y)^2}$$

Ex. If $u = x^2 \cdot \tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot \tan^{-1}\left(\frac{x}{y}\right)$, find u_{xy}.

$\Longrightarrow$ Given $u = x^2 \cdot \tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot \tan^{-1}\left(\frac{x}{y}\right)$

Diff. w.r.t. to y partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2 \cdot \tan^{-1}\left(\frac{y}{x}\right)\right] - \frac{\partial}{\partial y}\left[y^2 \cdot \tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= x^2 \cdot \frac{\partial}{\partial y} \tan^{-1}\left(\frac{y}{x}\right) - \left\{y^2 \cdot \frac{\partial}{\partial y} \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) \cdot \frac{\partial}{\partial y} y^2\right\}$$

$$= x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) - \left\{y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y\right\}$$

$$= x^{\cancel{2}} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{\cancel{x}} - \left\{\cancel{y^2} \cdot \frac{1}{\frac{y^2+x^2}{y^2}} \, x\left(\frac{-1}{\cancel{y^2}}\right) + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y\right\}$$

$$= \frac{x^2}{x^2+y^2} - \left\{\frac{-xy^2}{y^2+x^2} + 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)\right\}$$

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$$= \frac{x^3}{x^2+y^2} + \frac{x \cdot y^2}{x^2+y^2} - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial y} = x - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

Now diff. w.r.to x partially,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}\left(2y \cdot \tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial^2 u}{\partial x \cdot \partial y} = 1 - 2y \cdot \frac{\partial}{\partial x} \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y(y^2)}{x^2+y^2} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$\boxed{u_{xy} = \frac{x^2-y^2}{x^2+y^2}}$$

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Homework:-

If $z^3 - zx - y = 0$ prove that,

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = -\frac{(3z^2 + x)}{(3z^2 - x)^3}$$

* prove that a point of the surface $x^x \cdot y^y \cdot z^z = c$ where $x = y = z$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = -(x \cdot \log e x)^{-1}$$

$\Rightarrow$ Given $x^x \cdot y^y \cdot z^z = c$

Taking log on both side

$$\log(x^x \cdot y^y \cdot z^z) = \log c$$

By $\log(A \cdot B \cdot C) = \log A + \log B + \log C$

$$\therefore \log x^x + \log y^y + \log z^z = \log c$$

By $\log m^n = n \log m$

$$\therefore x \cdot \log x + y \cdot \log y + z \cdot \log z = \log c \quad \text{---} (1)$$

Diff. w.r.to y partially,

$$\frac{\partial}{\partial y}(x \cdot \log x) + \frac{\partial}{\partial y}(y \cdot \log y) + \frac{\partial}{\partial y}(z \cdot \log z) = \frac{\partial}{\partial y}(\log c)$$

$$0 + y \cdot \frac{1}{y} + \log y \cdot (1) + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial y} + \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$1 + \log y + \frac{\partial z}{\partial y}(1 + \log z) = 0$$

$$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{1 + \log z}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{1+\log y}{1+\log z}\right)$$

$$\therefore \frac{\partial^2 z}{\partial x \cdot \partial y} = -(1+\log y)\frac{\partial}{\partial x}\left(\frac{1}{1+\log z}\right) \qquad \left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right)$$

$$= -(1+\log y)\frac{-1}{(1+\log z)^2}\left[0+\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$

$$\therefore \frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{1+\log y}{(1+\log z)^2}\,\frac{1}{z}\cdot\frac{\partial z}{\partial x} \quad \text{--- (2)}$$

Now to find $\frac{\partial z}{\partial x}$ consider eq^n (1)

$$\therefore x\cdot\log x + y\cdot\log y + z\cdot\log z = \log c$$

Diff. w.r.to x partially

$$\frac{\partial}{\partial x}(x\cdot\log x) + \frac{\partial}{\partial x}(y\cdot\log y) + \frac{\partial}{\partial x}(z\cdot\log z) = \frac{\partial}{\partial x}(\log c)$$

$$x\cdot\frac{1}{x} + \log x + 0 + z\cdot\frac{1}{z}\frac{\partial z}{\partial x} + 1\cdot\log z\,\frac{\partial z}{\partial x} = 0$$

$$1 + \log x + \left(z\cdot\frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0$$

$$1 + \log x + (1+\log z)\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{1+\log z}$$

put this value in eq^n (2)

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{1+\log y}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\left[\frac{-(1+\log x)}{1+\log z}\right]$$

put $x=y=z$

$\therefore$ converting R.H.S in the terms of x

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\cancel{1+\log x}}{(1+\log x)^{\cancel{2}}} \cdot \frac{1}{x}\left[\frac{-\cancel{(1+\log x)}}{1+\log x}\right]$$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{1}{x} \cdot \frac{-1}{1+\log x}$$

But $\log e = 1$

$$= \frac{-1}{x(\log e + \log x)}$$

$$= \frac{-1}{x \log ex}$$

$$\boxed{\frac{\partial^2 z}{\partial x \cdot \partial y} = -(x \cdot \log ex)^{-1}}$$

Type 2:- Solved example on verification of $\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial^2 z}{\partial y \cdot \partial x}$

Ex If $u = \log(x^2+y^2)$ verify. $u_{xy} = u_{yx}$

Part 1: To find u_{xy} i.e $\frac{\partial^2 u}{\partial x \cdot \partial y}$

$$u = \log(x^2+y^2) \qquad \left(\because \frac{d}{dx}\log x = \frac{1}{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(0+2y)$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

Now diff. w.r.to partially

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x \cdot \partial y} = 2y \cdot \frac{\partial}{\partial x}\left[\frac{1}{x^2+y^2}\right] \quad \left(\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x \cdot \partial y} = 2y \cdot \frac{-1}{(x^2+y^2)^2} \cdot \frac{\partial}{\partial x}(x^2+y^2)$$

$$= 2y\left[\frac{-1}{(x^2+y^2)^2}(2x+0)\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{-4xy}{(x^2+y^2)^2}} \quad (1)$$

Part 2: To find $\frac{\partial^2 u}{\partial x \cdot \partial y} = u_{yx}$

Given $u = \log(x^2+y^2)$

Diff. w.r. to x partially.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\log(x^2+y^2)\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x+0)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$$

diff. w.r. to y, partially.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{2x}{x^2+y^2}\right]$$

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = 2x \cdot \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$= 2x \cdot \frac{-1}{(x^2+y^2)^2} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$= 2x \frac{-1}{(x^2+y^2)^2}(0+2y)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{4xy}{(x^2+y^2)^3}$$

from eqⁿ ① & ②

$$u_{xy} = u_{yx}$$

Derivative of composite functions

① If $u = f(r)$

and $r = f(x, y)$

then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$

and $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y}$

② similarly If $u = f(r)$ and $r = f(x, y, z)$

then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x}$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

Note

① If $r^2 = x^2 + y^2$

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

and $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$

(ii) If $r^2 = x^2 + y^2 + z^2$

then $\frac{\partial r}{\partial x} = \frac{x}{r}$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

Type 3: Example on partial Derivative of composite func.

Ex (i) If $u = f(r)$, where $r = \sqrt{x^2+y^2+z^2}$, then prove that, $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$

$\Rightarrow$ step 1:

Here $r = \sqrt{x^2+y^2+z^2}$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$

Step 2: To find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$

Given $u = f(r)$

Diff. w.r.to x partially,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r)$$

$$= f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot x \cdot r^{-1}$$

(for avoiding $\frac{u}{v}$ rule we write $\frac{1}{r} = r^{-1}$)

Diff. w.r. to x partially, again

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[f'(r) \cdot x \cdot r^{-1} \right]$$

By. $\frac{d}{dx}(uvw) = uv \cdot \frac{\partial w}{\partial x} + uw \cdot \frac{\partial v}{\partial x} + vw \frac{du}{dx}$

$$\frac{\partial^2 u}{\partial x^2} = f'(r) x \cdot \frac{\partial}{\partial x}(r^{-1}) + f'(r) \cdot r^{-1} \cdot \frac{\partial}{\partial x}(x) + x \cdot r^{-1} \frac{\partial}{\partial x}(f'(r))$$

$$= f'(r)(x) \cdot (-r^{-2}) \frac{\partial r}{\partial x} + f'(r) \cdot r^{-1} \cdot (1) + x \cdot r^{-1} f''(r) \frac{\partial r}{\partial x}$$

$$= -\frac{f'(r) \cdot x}{r^2}\left(\frac{x}{r}\right) + \frac{f'(r)}{r} + \frac{x}{r} \cdot f''(r) \cdot \frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{f'(r)}{r^3}\cdot x^2 + \frac{f'(r)}{r} + x^2\cdot\frac{f''(r)}{r^2} \quad \text{--- (1)}$$

similarly,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{f'(r)}{r^3}\cdot y^2 + \frac{f'(r)}{r} + y^2\cdot\frac{f''(r)}{r^2} \quad \text{--- (2)}$$

$$\text{Ily}\quad \frac{\partial^2 u}{\partial z^2} = -\frac{f'(r)}{r^3}\, z^2 + \frac{f'(r)}{r} + z^2\cdot\frac{f''(r)}{r^2} \quad \text{--- (3)}$$

add eqn (1) + (2) + (3)

$$L.H.S = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= -\frac{f'(r)}{r^3}\left[x^2+y^2+z^2\right) + \frac{3f'(r)}{r} + \frac{f''(r)}{r^2}\,(x^2+y^2+z^2)$$

But $x^2+y^2+z^2 = r^2$

$$= -\frac{f'(r)}{r^{\cancel{3}}}\cdot\cancel{r^2} + \frac{3f'(r)}{r} + \frac{f''(r)}{\cancel{r^2}}\,\cancel{r^2}$$

$$= -\frac{f'(r)}{r} + \frac{3f'(r)}{r} + f''(r)$$

$$= \frac{2f'(r)}{r} + f''(r)$$

$= R.H.S.$

Homework

If $u = x\cdot\log(x+r) - r$ where $r^2 = x^2+y^2$

find $u_{xx} + u_{yy}$

Ans $u_{xx} = \frac{1}{r}$, $u_{yy} = \frac{-x}{r(x+r)}$ $\quad u_{xx}+u_{yy} = \frac{1}{x+r}$.

② If $u = \log\sqrt{x^2+y^2+z^2}$ then prove that

$$(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz}) = 1$$

Step 1: Let, $x^2+y^2+z^2 = r^2$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Step 2: Given $u = \log\sqrt{x^2+y^2+z^2}$

$$u = \log\sqrt{r^2}$$

$$u = \log r$$

Diff. w.r.to x partially

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log r$$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\cdot\frac{\partial r}{\partial x} = \frac{1}{r}\cdot\frac{x}{r}$$

$$\frac{\partial u}{\partial x} = x\cdot r^{-2}$$

Diff. w.r.to x partially again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(x\cdot r^{-2})$$

$$= x\cdot\frac{\partial}{\partial x}(r^{-2}) + r^{-2}\cdot\frac{\partial}{\partial x}(x)$$

$$= x(-2r^{-3})\frac{\partial r}{\partial x} + r^{-2}\cdot(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x}{r^3}\left(\frac{x}{r}\right) + \frac{1}{r^2} \quad \text{--- (1)}$$

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{-2y^2}{r^4} + \frac{1}{r^2}$ --- (2)

& $\frac{\partial^2 u}{\partial z^2} = \frac{-2z^2}{r^4} + \frac{1}{r^2}$ --- (3)

step 8:- L.H.S = $(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz})$

$$= (x^2+y^2+z^2)\left(\frac{-2x^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} + \frac{1}{r^2} - \frac{2z^2}{r^4} + \frac{1}{r^2}\right]$$

$$= (x^2+y^2+z^2)\left(-\frac{2}{r^4}(x^2+y^2+z^2) + \frac{3}{r^2}\right)$$

But $x^2+y^2+z^2 = r^2$

$$= r^2\left(-\frac{2}{r^{\cancel{4}}}\cancel{r^2} + \frac{3}{r^2}\right)$$

$$= r^2\left(\frac{-2}{r^2} + \frac{3}{r^2}\right)$$

$$= r^2 \cdot \frac{1}{r^2}$$

$= 1$

$=$ R.H.S

concept of variable treated as constant

If $x = u\tan v$ and $y = u \cdot \sec v$ then we have 4 variables x, y, u, v

But here in variables to be treated as constant we have to deal with only 3 variable at a time

e.g. $\left(\frac{\partial u}{\partial y}\right)_x$ which can be read as partial derivative of u w.r.to y keeping x constant.

For $\left(\frac{\partial u}{\partial y}\right)_x$ we required only 3 variable u, x, y that means v needs to eliminated from the given two equation with algebraic calculation.

Ex ① If $u = 2x + 3y$
$v = 3x - 2y$.
Then find value of $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u$

$\Rightarrow$ Given

$u = 2x + 3y$ — ①

$v = 3x - 2y$ — ②

Part 1: To find $\left(\frac{\partial u}{\partial x}\right)_y$

from eqn ①

$u = 2x + 3y$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial x}(3y)$$

$$= 2 + y \cdot \frac{\partial}{\partial x}(3)$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = 2 + 0 = 2$$

Part 2: To find $\left(\frac{\partial y}{\partial v}\right)_x$

$v = 3x - 2y$.

$v - 3x = -2y$.

$$y = \frac{v - 3x}{-2}$$

$$y = \frac{3x - v}{2}$$

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{\partial}{\partial v}\left(\frac{3x - v}{2}\right) = \frac{1}{2}\left[\frac{\partial}{\partial v}(3x) - \frac{\partial}{\partial v}(v)\right]$$

$$= \frac{1}{2}[0 - 1]$$

$$= -\frac{1}{2}$$

Part 3: To find $\left(\frac{\partial x}{\partial u}\right)_v$

From eqⁿ (1)

$$u = 2x + 3y.$$

$$u - 2x = 3y$$

$$\boxed{\frac{u-2x}{3} = y}$$

put in eqⁿ (2)

$$v = 3x - 2y$$

$$v = 3x - 2\left(\frac{u-2x}{3}\right)$$

$$v = \frac{9x - 2u + 4x}{3}$$

$$\boxed{3v = 13x - 2u}$$

$$3v - 13x = -2u$$

$$\frac{3v + 2u}{13} = x$$

Diff. w.r.to u keeping v constant.

$$\left(\frac{\partial}{\partial u}(x)\right)_v = \frac{\partial}{\partial u}\left(\frac{3v+2u}{13}\right)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \left[0 + \frac{2}{13}\cdot\frac{\partial}{\partial u}(u)\right]$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \left[\frac{2}{13}\right]$$

Part 4: To find $\left(\frac{\partial v}{\partial y}\right)_u$

from eqⁿ (1)

$$u = 2x + 3y$$

$$2x = u - 3y$$

$$x = \frac{u-3y}{2}$$

put in eqⁿ ⓐ

$$V = 3x - 2y$$
$$= 3\left(\frac{u-3y}{2}\right) - 2y$$

$$V = \frac{3u - 9y}{2} - 2y$$

$$V = \frac{3u - 9y - 4y}{2} = \frac{3u - 13y}{2}$$

$$\boxed{V = \frac{3u - 13y}{2}}$$

Diff. w.r.to y keeping u is constant,

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{\partial}{\partial y}\left(\frac{3u - 13y}{2}\right) = \frac{1}{2}\frac{\partial}{\partial y}(3u - 13y)$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{1}{2}\left[\frac{\partial}{\partial y}(3u) - \frac{\partial}{\partial y}(13y)\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{1}{2}\left[0 - 13\right]$$

$$\boxed{\left(\frac{\partial v}{\partial y}\right)_u = \frac{-13}{2}}$$

Now given diff. expression is,

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u = (2)\left(-\frac{1}{2}\right)\left(\frac{2}{13}\right)\left(\frac{-13}{2}\right)$$
$$= 1$$

Homework:

If $u = ax + by$

$v = bx - ay$

Then find value of $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u$.

Ans 1:

(2) If $x = \frac{r}{2}(e^{\theta} + e^{-\theta})$

$y = \frac{r}{2}(e^{\theta} - e^{-\theta})$, then show that

$$\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}.$$

$\Rightarrow$ we know that

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \quad \& \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$x = r \cdot \cosh\theta \quad \text{—— (1)}$$

$$y = r \sinh\theta \quad \text{—— (2)}$$

$$\frac{y}{x} = \frac{\cancel{r} \cdot \sinh\theta}{\cancel{r} \cdot \cosh\theta} = \tanh\theta.$$

$$\frac{y}{x} = \tanh\theta \quad \text{—— (3)}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\frac{x^2}{r^2} - \frac{y^2}{r^2} = 1$$

$$x^2 - y^2 = r^2 \quad \text{—— (4)}$$

part 1 : $\left(\frac{\partial x}{\partial r}\right)_{\theta}$.

From eqn (1)

$$\left(\frac{\partial x}{\partial r}\right)_{\theta} = \cosh\theta \cdot \frac{\partial}{\partial r}(r)$$

$$\left(\frac{\partial x}{\partial r}\right)_{\theta} = \cosh\theta \quad \text{—— (5)}$$

part 2: $\left(\frac{\partial r}{\partial x}\right)_{y}$

From eqn (4)

$x^2 - y^2 = r^2$

diff. w.r. to x keeping y is constant,

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\cancel{2}r \cdot \frac{\partial r}{\partial x} = \cancel{2}x$$

$$r \cdot \frac{\partial r}{\partial x} = x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

But $\frac{x}{r} = \cosh\theta$

$$\left(\frac{\partial r}{\partial x}\right)_y = \cosh\theta \quad \text{--- (6)}$$

from (5) and (6)

$$\boxed{\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta}$$

Homework:-

If $x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$

Then prove that

(i) $\left(\frac{\partial y}{\partial r}\right)_x \left(\frac{\partial y}{\partial r}\right)_\theta = 1$

(ii) $\left(\frac{\partial x}{\partial \theta}\right)_r = r^2 \left(\frac{\partial \theta}{\partial x}\right)_y$.

Euler's Theorem:

Statment 1:

If z is a homogeneous function of two variables x, y of degree n then

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

Statment 2:-

If u is a homogeneous function of x, y of degree 'n' and $z = f(u)$ then

(i) $$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}$$

(ii) $$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where $$g(u) = n \cdot \frac{f(u)}{f'(u)}$$

Statment 3:

If u is homogeneous function of three variables x, y and z of degree n then,

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = nu$$

* Basic Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

* Half angle Formulae

1. $\sin 2\theta = 2\sin\theta \cdot \cos\theta$
2. $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$
3. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(4) $\cos 2\theta = 1 - \sin^2\theta$

(5) $\cos 2\theta = 2\cos^2\theta - 1$

(6) $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$.

Ex If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, then prove that

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 1 - 4\cdot\sin^2 u \,(\sin 2u)$$

$\Longrightarrow$ $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}.$$

$$\tan u = \frac{x^3\left(\frac{x^3}{x^3} + \frac{y^3}{x^3}\right)}{x\left(\frac{x}{x} - \frac{y}{x}\right)}$$

$$\tan u = \frac{x^2 \left(1 + \frac{y^3}{x^3}\right)}{\left(1 - \frac{y}{x}\right)}$$

$$\tan u = x^2 \cdot f(y/x)$$

$\therefore$ $f(u) = \tan u$, and $n = 2$

Now by Euler's Theorem,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{n \cdot f(u)}{f'(u)}$$

$$= \frac{2 \cdot \tan u}{\sec^2 u}$$

$$= 2 \cdot \frac{\sin u}{\cancel{\cos u}} \cdot \cos^{\cancel{2}} u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2 \cdot \sin u \cdot \cos u$$

$\therefore\ x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\sin 2u \Rightarrow g(u)$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\cdot\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]$$

$$=\sin 2u\left[2\cdot\cos 2u-1\right]$$

$$=\sin 2u\left[2\left(1-2\sin^2 u\right)-1\right]$$

$$=\sin 2u\left[2-4\sin^2 u-1\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\cdot\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u\left[1-4\sin^2 u\right]$$

EX If $u=\sin^{-1}(x^2+y^2)^{1/5}$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\cdot\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{25}\tan u\left(2\tan^2 u-3\right)$$

$\Rightarrow$ Given

$$u=\sin^{-1}(x^2+y^2)^{1/5}$$

$$\sin u=(x^2+y^2)^{1/5}$$

$$\sin u=x^{2/5}\left(1+\frac{y^2}{x^2}\right)^{1/5}$$

$$\sin u=x^{2/5}\cdot f\left(\frac{y}{x}\right)$$

$f(u)=\sin u$ and $n=\frac{2}{5}$

Now By Euler's Theorem,

$$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{n\cdot f(u)}{f'(u)}$$

$$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{2}{5}\,\frac{\sin u}{\cos u}$$

$$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{2}{5}\tan u \Rightarrow g(u)$$

$$\therefore x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]$$

$$= \frac{2}{5}\cdot\tan u\left[\frac{2}{5}\sec^2 u - 1\right]$$

$$= \frac{2}{5}\tan u\left[\frac{2\sec^2 u - 5}{5}\right]$$

But $\sec^2\theta = 1 + \tan^2\theta$

$$= \frac{2}{25}\tan u\left[2(1+\tan^2 u) - 5\right]$$

$$= \frac{2}{25}\tan u\left[2\cdot\tan^2 u - 3\right]$$

$$x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} = \frac{2}{25}\tan u\cdot\left[2\tan^2 u - 3\right]$$

Ex If $u = x^8 f\left(\frac{y}{x}\right) + \frac{1}{y^8}\phi\left(\frac{x}{y}\right)$.

Then prove that

$$x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} + x\cdot\frac{\partial y}{\partial x} + u\cdot\frac{\partial u}{\partial y} = 64u.$$

$\Longrightarrow$ $u = x^8\cdot f\left(\frac{x}{y}\right) + \frac{1}{y^8}\phi\left(\frac{x}{y}\right)$

Let $u = p + q$ —— (A)

Now

$$p = x^8 f\left(\frac{y}{x}\right)$$

$\therefore$ $n = 8$ and $f(p) = p$.

Now, by Euler's theorem,

$$x\cdot\frac{\partial p}{\partial x} + y\cdot\frac{\partial p}{\partial y} = n\cdot\frac{f(p)}{f'(p)}$$

$$x\cdot\frac{\partial p}{\partial x}+y\cdot\frac{\partial p}{\partial y}=\frac{8p}{1}$$

$$x\cdot\frac{\partial p}{\partial x}+y\cdot\frac{\partial p}{\partial y}=8p \quad \text{——— (1) ———}$$

& $$x^2\cdot\frac{\partial^2 p}{\partial x^2}+2xy\frac{\partial^2 p}{\partial x\cdot\partial y}+y^2\cdot\frac{\partial^2 p}{\partial y^2}=g(P)[g'(P)-1]$$

$$=8P[8-1]$$

$$=8P(7)$$

$$x^2\cdot\frac{\partial^2 p}{\partial x^2}+2xy\cdot\frac{\partial^2 p}{\partial x\cdot\partial y}+y^2\cdot\frac{\partial^2 p}{\partial y^2}=56P \quad \text{——— (2) ———}$$

Also,

$$q=\frac{1}{y^8}\phi\left(\frac{x}{y}\right)$$

$$q=y^{-8}\phi\left(\frac{x}{y}\right)$$

$n=-8$ and $f(q)=q$

Now, by Euler's theorem;

$$x\cdot\frac{\partial q}{\partial x}+y\cdot\frac{\partial q}{\partial y}=n\cdot\frac{f(q)}{f'(q)}=-\frac{8q}{1}$$

$$x\cdot\frac{\partial q}{\partial x}+y\cdot\frac{\partial q}{\partial y}=-8q \quad \text{——— (3) ———}$$

$$x^2\cdot\frac{\partial^2 q}{\partial x^2}+2xy\cdot\frac{\partial^2 q}{\partial x\cdot\partial y}+y^2\cdot\frac{\partial^2 q}{\partial y^2}=-8q[-8-1]$$

$$=-8q(-9)$$

$$x^2\cdot\frac{\partial^2 q}{\partial x^2}+2xy\frac{\partial^2 q}{\partial x\cdot\partial y}+y^2\cdot\frac{\partial^2 q}{\partial y^2}=72q. \quad \text{——— (4) ———}$$

$\therefore$ Now eq^n (1)+(2)+(3)+(4)

$$x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} + x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}$$

$$= 56p + 72q + 8p - 8q$$

$$= 64p + 64q$$

$$= 64(p+q)$$

$$= 64u$$

But by eq^n (A)

$$p+q = u$$

$$\boxed{x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} + x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 64u}$$

Type 6: Finding the second order Expression by using first order expression.

Working Rule to find the value of

$$ax^2\cdot\frac{\partial^2 v}{\partial x^2} + (a\pm b)xy\frac{\partial^2 v}{\partial x\cdot\partial y} + by^2\frac{\partial^2 v}{\partial y^2} \quad \text{for } v=f(x,y)$$

Step 1:

Compare the given equation with

$$ax^2\frac{\partial^2 v}{\partial x^2} + (a\pm b)\frac{\partial^2 v}{\partial x\cdot\partial y} + by^2\frac{\partial^2 v}{\partial y^2}$$ and find a, b

Step 2: (i) If sign of $\frac{\partial^2 v}{\partial x\cdot\partial y}$ is positive then find

$$ax\cdot\frac{\partial v}{\partial x} + by\cdot\frac{\partial v}{\partial y}$$

OR

(ii) If sign of $\frac{\partial^2 v}{\partial x\cdot\partial y}$ is negative then find

$$ax \cdot \frac{\partial v}{\partial x} - by \cdot \frac{\partial v}{\partial y} \quad \text{—— (2)}$$

Step 3 Differentiate eqⁿ (1) w.r.to x as well as w.r.to y we will get eqⁿ (2) and eqⁿ (3)

Step 4: Use proper multipliers for eqⁿ (2) and eqⁿ (3)

Step 5 add those equation to get the final answer.

Ex

If $u = f\left(\frac{x^2}{y}\right)$ then show that

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 3xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + 2y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ The Given expression is

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 3xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + 2y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0$$

Step 1 compare with

$$ax^2 \cdot \frac{\partial^2 v}{\partial x^2} + (a \pm b) \cdot \frac{\partial^2 v}{\partial x \cdot \partial y} + by^2 \cdot \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore a = 1, \quad b = 2$

and sign of $\frac{\partial^2 u}{\partial x \cdot \partial y}$ is positive.

Step 2: To find $ax \cdot \frac{\partial u}{\partial x} + by \cdot \frac{\partial u}{\partial y}$

i.e $x \cdot \frac{\partial u}{\partial x} + 2y \cdot \frac{\partial u}{\partial y}$

Let $u = f\left(\frac{x^2}{y}\right)$

Diff. w.r.to x partially

$$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\left(\frac{2x}{y}\right)$$

Again $u=f\left(\frac{x^2}{y}\right)$

Diff. w.r.to y partially

$$\therefore \frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\left(-\frac{x^2}{y^2}\right)$$

$$\therefore x\cdot\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = x\left[f'\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)\right] + 2y\left[f'\left(\frac{x^2}{y}\right)\left(-\frac{x^2}{y^2}\right)\right]$$

$$= f'\left(\frac{x^2}{y}\right)\left(\frac{2x^2}{y}\right) - f'\left(\frac{x^2}{y}\right)\left(\frac{2x^2}{y}\right)$$

$$x\cdot\frac{\partial u}{\partial x} + 2y\cdot\frac{\partial u}{\partial y} = 0 \quad \text{---} (1)$$

Diff. eqn (1) w.r.to x partially.

$$x\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2y\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + \frac{\partial u}{\partial y}(0) = 0$$

$$x\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2y\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} = 0 \quad \text{---} (2)$$

Also diff. eqn (1) w.r.to y partially,

$$\therefore x\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + \frac{\partial u}{\partial x}(0) + 2y\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}(2) = 0$$

$$\therefore x\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + 2y\frac{\partial^2 u}{\partial y^2} + 2\cdot\frac{\partial u}{\partial y} = 0 \quad \text{---} (3)$$

Step 4: eqn (2) $\times x$ + eqn (3) $\times y$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + x\cdot\frac{\partial u}{\partial x} + 2xy\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + xy\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + 2y^2\cdot\frac{\partial^2 u}{\partial y^2} + 2y\frac{\partial u}{\partial y} = 0 + 0$$

$$\therefore \left(x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 3xy \frac{\partial^2 u}{\partial x \cdot \partial y} + 2y^2 \cdot \frac{\partial^2 u}{\partial y^2}\right) + \left(x \cdot \frac{\partial u}{\partial x} + 2y \cdot \frac{\partial u}{\partial y}\right) = 0$$

But $\frac{\partial u}{\partial x} + 2y \cdot \frac{\partial u}{\partial y} = 0$ (from eqⁿ ①)

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 3xy \frac{\partial^2 u}{\partial x \cdot \partial y} + 2y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

Total Derivaties:

If $u = f(x,y)$ where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the value of x and y in $f(x,y)$

Thus we can find the ordinary derivative $\frac{\partial u}{\partial t}$ which is called as total derivative of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Now to find $\frac{du}{dt}$ without actually substituting the values of x and y in $f(x,y)$ we use chain rule as fallows

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Note:

① If $u = f(x,y,z)$ where x, y, z are all function of a variable t, then chain rule is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

② If $f(x,y) = c$ be an implicit relation between x and y which is defined as a differentiable function of x, then

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

which is known as first differentiable coeifficent of an implicit Function.

classmate
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Ex ① find $\frac{du}{dx}$ if $u = x\cdot\log(xy)$ and $x^3+y^3+3xy=0$

$\Longrightarrow$ By Total Derivative Theorem,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \text{①}$$

Step 1: Given,
$$x^3 + y^3 + 3xy = 0$$
Diff. w. r. to x
$$3x^2 + 3y^2\frac{dy}{dx} + 3y + 3x\cdot\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{(3x^2+3y)}{3y^2+3x} = -\left[\frac{x^2+y}{x+y^2}\right]$$

Step. 2: $u = x\cdot\log(xy)$
Diff. w. r. to x, partially:
$$\therefore \frac{\partial u}{\partial x} = \log(xy) + \frac{x}{xy}(y)$$
$$\frac{\partial u}{\partial x} = \log(xy) + 1$$

Again $u = x\cdot\log(xy)$
Differentiate w. r. to y partially,
and $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x\cdot\log x + x\cdot\log y] = \frac{x}{y}$

Step 3: From eqn ①
$$\frac{du}{dx} = \log(xy) + 1 + -\frac{x}{y}\cdot\frac{(x^2+y)}{x+y^2}$$
$$\frac{du}{dx} = [\log(xy)+1] - \frac{x}{y}\frac{[x^2+y]}{[x+y^2]}$$

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classmate

(2) If $f(x,y)=0$, $\phi(z,x)=0$ prove that
$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z}$$

$\Rightarrow$ Given $f(x,y)=0$

$$\therefore \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \quad \text{——— (1)}$$

Also $\phi(z,x)=0$

$$\therefore \frac{dx}{dz}=-\frac{\partial\phi/\partial z}{\partial\phi/\partial x} \quad \text{——— (2)}$$

multiply eqⁿ (1) & (2) we get :

$$\frac{dy}{dx}\cdot\frac{dx}{dz}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\times-\frac{\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial x}}$$

$$\frac{dy}{\cancel{dx}}\cdot\frac{\cancel{dx}}{dz}=\frac{\partial f}{\partial x}\times\frac{\partial f}{\partial y}\times\frac{\partial\phi}{\partial z}\times\frac{\partial x}{\partial\phi}$$

$$\frac{dy}{dz}=\frac{\partial f}{\partial x}\times\frac{\partial f}{\partial y}\times\frac{\partial\phi}{\partial z}\times\frac{\partial x}{\partial\phi}$$

$$\frac{\partial\phi}{\partial x}\times\frac{\partial f}{\partial y}\times\frac{dy}{dz}=\frac{\partial f}{\partial x}\times\frac{\partial\phi}{\partial z}$$

Homework :-
Find $\frac{dz}{dx}$, if $z=x^2y$ and $x^2+xy+y^2=1$

Ans $\frac{dz}{dx}=2xy-x^2\left[\frac{2x+y}{x+2y}\right]$

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* Partial Derivative of composite function:

If z is function of x and y and x and y is function of u and v then z becomes a composite function of x and y

i.e $z \rightarrow x, y \rightarrow u, v$

then $z \xrightarrow[\text{function}]{\text{Composite}} u, v$

Now to find partial Derivative of z w.r.to u and v we use the chain rule as follow.

(i) $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

(ii) $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Ex (1) If $z = f(x, y)$ where $x = e^u \cdot \cos v$ and $y = e^u \cdot \sin v$, Then prove that

$$y \cdot \frac{\partial z}{\partial u} + x \cdot \frac{\partial z}{\partial v} = e^{2u} \cdot \frac{\partial z}{\partial y}$$

$\Rightarrow$ $z = f(x, y)$ and

$x = e^u \cdot \cos v$ and $y = e^u \cdot \sin v$

By chain Rule for partial derivative of composite function

we have $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \quad \text{--- (a)}$$

Step 2: Let $x = e^u \cdot \cos v$ --- (1)

$y = e^u \cdot \sin v$ --- (2)

squaring and adding eq (1) and (2), we get

$$x^2+y^2 = e^{2u}\cdot\cos^2 v + e^{2u}\cdot\sin^2 v$$
$$x^2+y^2 = e^{2u}[\cos^2 v + \sin^2 v]$$
$$x^2+y^2 = e^{2u}$$
$$\log(x^2+y^2) = 2u\cdot\log e$$
$$u = \frac{1}{2}\log(x^2+y^2) \quad \text{—— (3) ——}$$

~~step 2~~: dividing eqn (2) by eqn (1)

$$\frac{y}{x} = \frac{e^u\cdot\sin v}{e^u\cdot\cos v}$$
$$\tan v = \frac{y}{x}$$
$$v = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{—— (4) ——}$$

Step 3: from eqn (3), we have

$$u = \frac{1}{2}\log[x^2+y^2]$$

Diff. w.r.to y partially,

$$\frac{\partial u}{\partial y} = \frac{1}{2}\,\frac{1}{x^2+y^2}\,(2y)$$
$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = \frac{y}{e^{2u}} \quad \text{---} \left(e^{2u} = x^2+y^2\right)$$
$$\therefore \boxed{\frac{\partial u}{\partial y} = \frac{y}{e^{2u}}}$$

And from eqn (4) we have

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

Diff. w.r.to y partially,

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{e^{2u}} \qquad \left(\because e^{2u} = x^2+y^2\right) \text{ --- } (6)$$

Step 4: Now consider eqⁿ (a)

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

From eqⁿ (5) and (6)

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{y}{e^{2u}} + \frac{\partial z}{\partial v} \cdot \frac{x}{e^{2u}}$$

multiplying e^{2u} throughout

$$\Rightarrow e^{2u} \cdot \frac{\partial z}{\partial y} = y \cdot \frac{\partial z}{\partial u} + x \cdot \frac{\partial z}{\partial v}$$

Ex (2) If $u = f(r,s)$, $r = x^2+y^2$, $s = x^2-y^2$

prove that $y \cdot \frac{\partial u}{\partial x} + x \cdot \frac{\partial u}{\partial y} = 4xy \frac{\partial u}{\partial r}$.

$\Rightarrow$ As, $u \to f(r,s) \to \phi(x,y)$

By chain rule for partial derivative of composite function,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} \text{ --- } (1)$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} \text{ --- } (2)$$

As $r = x^2 + y^2$

$$\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial y} = 2y$$

$s = x^2 - y^2$

$$\frac{\partial s}{\partial x} = 2x$$

$$\frac{\partial s}{\partial y} = -2y$$

From eq^n ① & ② we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot(2x) + \frac{\partial u}{\partial s}(2x)$$

and $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(2y) + \frac{\partial u}{\partial s}(-2y)$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(2y) - \frac{\partial u}{\partial s}(2y)$$

$$\therefore y\cdot\frac{\partial u}{\partial x} + x\cdot\frac{\partial u}{\partial y} = 2xy\frac{\partial u}{\partial r} + 2xy\cdot\frac{\partial u}{\partial s} + 2xy\cdot\frac{\partial u}{\partial r} - 2xy\cdot\frac{\partial u}{\partial s}$$

$$\boxed{y\cdot\frac{\partial u}{\partial x} + x\cdot\frac{\partial u}{\partial y} = 4xy\cdot\frac{\partial u}{\partial r}}$$

Q3 If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, find the value of $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$

$\Rightarrow$ let $l = 2x-3y$

$m = 3y - 4z$

$n = 4z - 2x$

$\therefore u = f(l, m, n) = \phi(x, y, z)$

By chain rule for partial derivative of composite functions,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

But $\frac{\partial l}{\partial x} = 2$, $\frac{\partial m}{\partial x} = 0$, $\frac{\partial n}{\partial x} = -2$

$$\therefore \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\cdot\frac{\partial u}{\partial n}$$

$$\frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \quad \text{--- (1)}$$

similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

But $\frac{\partial l}{\partial y} = -3$, $\frac{\partial m}{\partial y} = 3$, $\frac{\partial n}{\partial y} = 0$

$$\therefore \frac{\partial u}{\partial y} = -3\cdot\frac{\partial u}{\partial l} + 3\cdot\frac{\partial u}{\partial n}$$

$$\therefore \frac{1}{3}\cdot\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial n} \quad \text{--- (2)}$$

similarly,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

But, $\frac{\partial l}{\partial z} = 0$, $\frac{\partial m}{\partial z} = -4$, $\frac{\partial n}{\partial z} = 4$

$$\frac{\partial u}{\partial z} = -4\cdot\frac{\partial u}{\partial m} + 4\cdot\frac{\partial u}{\partial n}$$

$$\frac{1}{4}\frac{\partial u}{\partial z} = -\cancel{4}\cdot\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} \quad \text{--- (3)}$$

By adding (1) + (2) + (3)

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\cdot\frac{\partial u}{\partial y} + \frac{1}{4}\cdot\frac{\partial u}{\partial z} = \cancel{-\frac{\partial u}{\partial n}} + \cancel{\frac{\partial u}{\partial l}} \cancel{-\frac{\partial u}{\partial l}} + \cancel{\frac{\partial u}{\partial m}}$$

$$\cancel{-\frac{\partial u}{\partial m}} + \cancel{\frac{\partial u}{\partial n}}$$

$$\boxed{\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0}$$