Lecture 5: Decision Rules

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EEL 3850

Outline

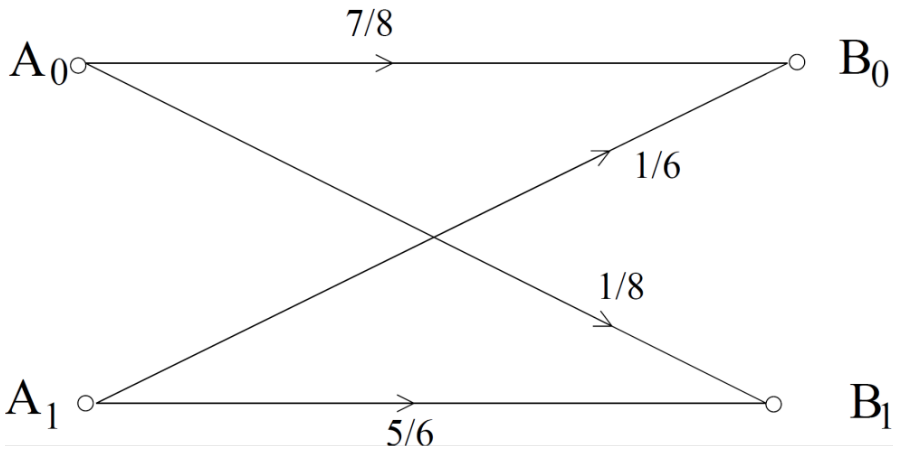
- Maximum-Likelihood Estimation Decision Rules
- Maximum A Posteriori (MAP) Decision Rule
- The probability of error using decision rule

Motivating problem: Binary Communication System

A transmitter Tx sends A_0 and A_1.

- A receiver Rx processes the output of the channel into one of two values B_0 and B_1.

- The channel is a noisy channel that determines the probabilities between A_0, A_1 and B_0, B_1.



if B_0
then decide __

if B_1
then decide __

The receiver must use a decision rule to determine from the output B_0 or B_1 whether the symbol that was sent was most likely A_0 or A_1.**

Frequentist vs Bayesian

- Frequentist statistician: draws conclusions from data by computing relative frequency of events in the data.
- Bayesian statistician: draws conclusions from data by testing out the hypothesis and computing their observed probability from data.
  - It computes the uncertainty of hypothesis by inducing prior beliefs.
  - It tests hypothesis.

two hypothese: A_0 A_1

Decision Problem and Decision Rules

A **decision rule** tells how to choose an input given an observed output.

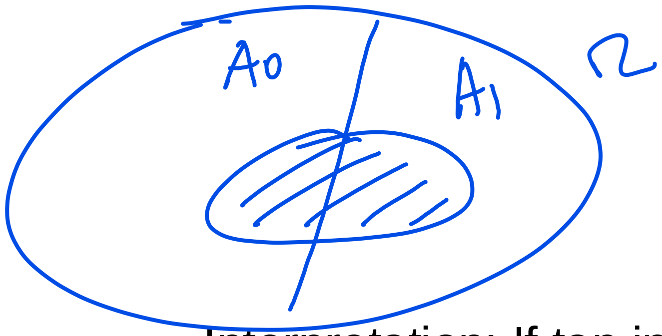
$\{A_0, A_1\}$ $\{B_0, B_1\}$

- **optimal decision problem**:
    1. Maximize the likelihood of the input.
    2. Choose the input that minimizes the probability of error.

Maximize Likelihood Estimate(MLE) decision rule

- MLE: Maximize the likelihood of the input

If the set of events $\{A_0, A_1\}$ partitions Ω, and assuming $P(A_i) > 0$, for all i. Then the Maximum Likelihood decision rule is given by:



? likelihood of B given A0 ; ? likelihood of B given A1

$$P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)$$

Interpretation: If top inequality holds, choose A0, if the bottom inequality holds, choose A1

Alternatively, the MLE RULE:

$\arg\max\limits_{X_2} \{X_0=1, X_1=2, X_2=5\} = 5$

$\{A_0 \dots A_n\}$ hypothesis.

$$\widehat{A}_i, \text{ where } i = \arg \max_{i \in \{0,1,\dots,n\}} P(B_j|A_i).$$

evaluate $P(B_j | A_i)$
find the maximal one.

Binary Communication System

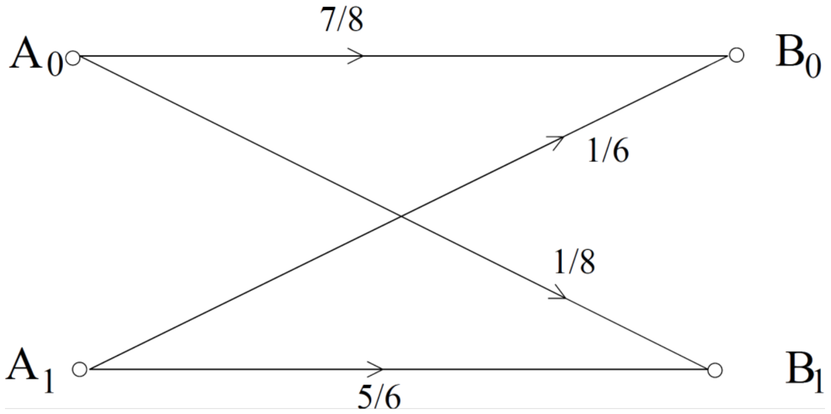
$$P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)$$

- MLE

1. B0
2. B1

Two cases:

1. $\cancel{P(A_0) = \frac{2}{5},\ P(A_1) = \frac{3}{5}}$
2. $\cancel{P(A_0) = \frac{1}{10},\ P(A_1) = \frac{9}{10}}$



if B_0:

then: A_0

$P(B_0|A_0) = \frac{7}{8} > P(B_0|A_1) = \frac{1}{6}$

if B_1:

then A_1

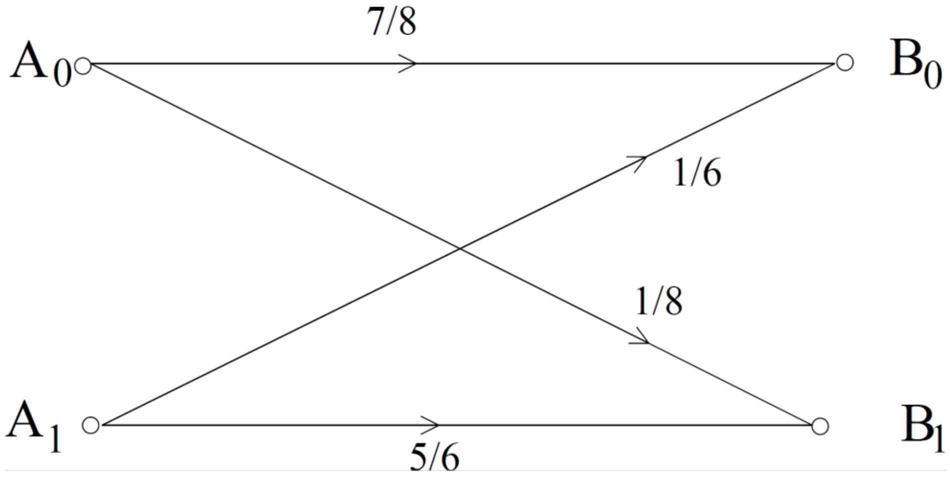
$P(B_1|A_0) = \frac{1}{8} < P(B_1|A_1) = \frac{5}{6}$

Binary Communication System

If we receive B_0, using the MLE rule, which one to select?

$$\boxed{P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)}$$

- MLE: if the probability of A0 given B0 is greater than the probability of A1 given B0, then we select A0, Otherwise, we select A1.

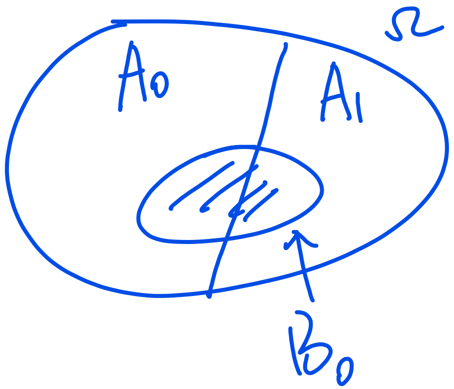


Maximum a Posterior (MAP) rule

- we choose the input that maximizes the a posteriori probability (APP), we call this a maximum a posteriori (MAP) decision rule.

If the set of events $\{A_0, A_1\}$ partitions Ω, and assuming $P(A_i) > 0$, for all i. Then the Maximum A Posteriori decision rule is given by:

$P(A_0 | B_0)$ $\quad$ $P(A_1 / B_0)$



$$P(A_0|B) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B)$$

$$\iff \frac{P(B|A_0)P(A_0)}{P(B)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B|A_1)P(A_1)}{P(B)}$$

$P(B_0|A_0)$ likelihood. given.

$P(B_0|A_1)$

Given B_j, choose $\widehat{A}_i$, where $i = \arg\max_{i\in\{0,1\}} P(A_i|B_j)$.

Binary Communication System

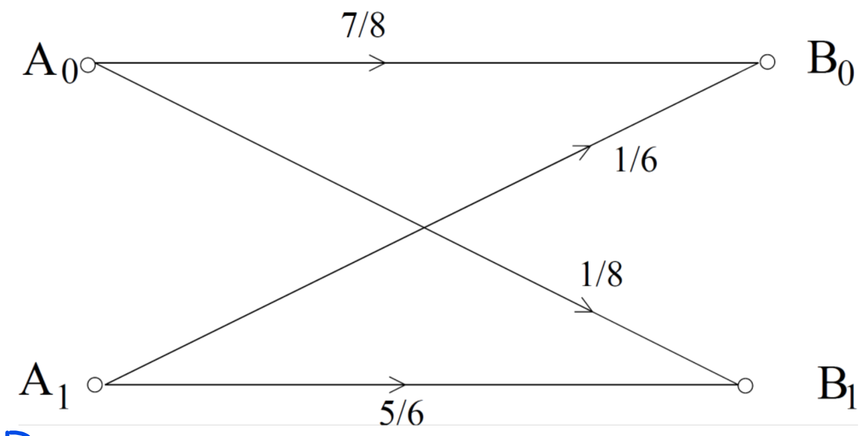
- MAP

1. B0
2. B1

Two cases:

1. $P(A_0) = \frac{2}{5},\ P(A_1) = \frac{3}{5}$
2. $P(A_0) = \frac{1}{10},\ P(A_1) = \frac{9}{10}$

$$P(A_0|B) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B)$$



Case 1 : if B_0 then $\underline{A_0}$

if B_1 then $\underline{A_1}$

$$P(A_0|B_0) = \frac{P(B_0|A_0)P(A_0)}{P(B_0)} \ ?\ P(A_1|B_0) = \frac{P(B_0|A_1)P(A_1)}{P(B_0)}$$

$$\frac{\frac{7}{8}\times\frac{2}{5} = \frac{14}{40}}{P(B_0)} \ ?\ > \frac{\frac{1}{6}\times\frac{3}{5} = \frac{3}{30}}{P(B_0)}$$

$$P(A_0|B_1) = \frac{P(B_1|A_0)P(A_0)}{P(B_1)} = \frac{\frac{1}{8}\times\frac{2}{5}}{P(B_1)} < P(A_1|B_1) = \frac{P(B_1|A_1)P(A_1)}{P(B_1)} = \frac{\frac{5}{6}\times\frac{3}{5}}{P(B_1)}$$

Binary Communication System

- Observing B0

case 2: if B_0

then $\underline{A_1}$

if B_1

then $\underline{A_1}$

$$P(A_0|B_0) = \frac{P(B_0|A_0)P(A_0)}{P(B_0)} = \frac{\frac{7}{8}\times\frac{1}{10}}{P(B_0)} < P(A_1|B_0) = \frac{P(B_0|A_1)P(A_1)}{P(B_0)} = \frac{\frac{1}{6}\times\frac{9}{10}}{P(B_0)}$$

$$P(A_0|B_1) < P(A_1|B_1)$$

? Question: if $P(A_0) = P(A_1)$ MLE $\overset{?}{=}$ MAP rule

- Observing B1

Binary Communication System

MLE

- If a decision rule always selects A0 regardless of what is being received, what is the probability of error?

event.

1. $P(A_0) = \frac{2}{5}, P(A_1) = \frac{3}{5}$
2. $P(A_0) = \frac{1}{10}, P(A_1) = \frac{9}{10}$

if B_0 then A_0 ← decision DA_0

if B_1 then A_1 ← decision DA_1

Question: How to calculate the probability of ERROR?

$$P(E) = P(E|B_0)P(B_0) + P(E|B_1)P(B_1)$$

error: $A_0 \cap DA_1$

$A_1 \cap DA_0$

$E := (A_0 \cap DA_1) \cup (A_1 \cap DA_0)$

Probability of Error for case 1

$$P(E \mid B_0) = P(A_1 \cap DA_0 \mid B_0) = P(A_1 \mid B_0)$$

if B_0 then DA_0

$$P(A_1 \cap DA_0 \mid B_0) = \underbrace{P(A_1 \mid B_0)}\ \underbrace{P(DA_0 \mid B_0)}_{1}$$

"conditionally independent"

Probability of Error for case 2

Continue the same step for scenario 2

- Left as an exercise