Lecture 5: Decision Rules

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EEL 3850

Outline

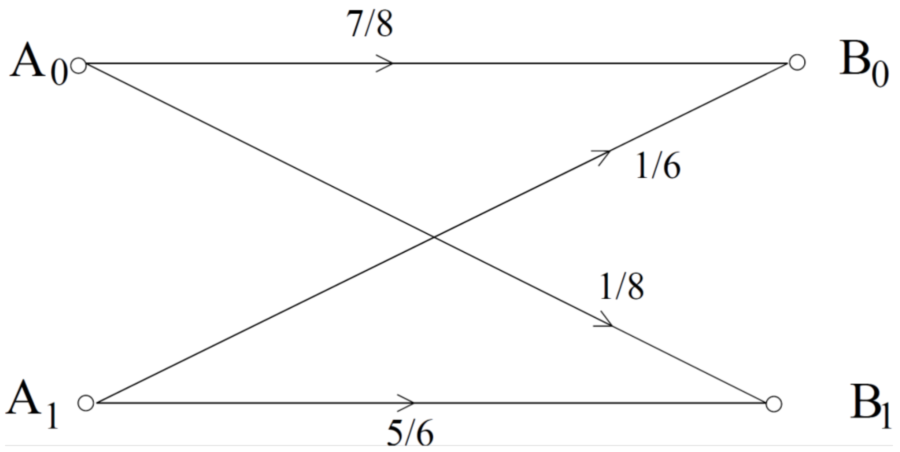
- Maximum-Likelihood Estimation Decision Rules
- Maximum A Posteriori (MAP) Decision Rule
- The probability of error using decision rule

Motivating problem: Binary Communication System

A transmitter Tx sends A_0 and A_1.

- A receiver Rx processes the output of the channel into one of two values B_0 and B_1.

- The channel is a noisy channel that determines the probabilities between A_0, A_1 and B_0, B_1.



The receiver must use a decision rule to determine from the output B_0 or B_1 whether the symbol that was sent was most likely A_0 or A_1.**

Frequentist vs Bayesian

- Frequentist statistician: draws conclusions from data by computing relative frequency of events in the data.
- Bayesian statistician: draws conclusions from data by testing out the hypothesis and computing their observed probability from data.
  - It computes the uncertainty of hypothesis by inducing prior beliefs.
  - It tests hypothesis.

Decision Problem and Decision Rules

A **decision rule** tells how to choose an input given an observed output.

- **optimal decision problem**:
    1. Maximize the likelihood of the input.
    2. Choose the input that minimizes the probability of error.

Maximize Likelihood Estimate(MLE) decision rule

- MLE: Maximize the likelihood of the input

If the set of events $\{A_0, A_1\}$ partitions Ω, and assuming $P(A_i) > 0$, for all i. Then the Maximum Likelihood decision rule is given by:

$$P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)$$

Interpretation: If top inequality holds, choose A0, if the bottom inequality holds, choose A1

Alternatively, the MLE RULE:

$$\widehat{A}_i, \text{ where } i = \arg\max_{i \in \{0,1\}} P(B_j|A_i).$$

Binary Communication System

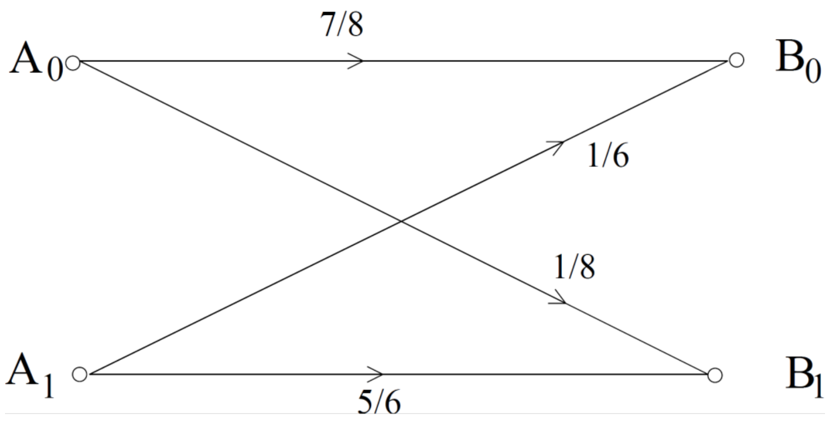
$$P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)$$

- MLE

1. B0
2. B1

Two cases:

1. $P(A_0) = \frac{2}{5}, \ P(A_1) = \frac{3}{5}$
2. $P(A_0) = \frac{1}{10}, \ P(A_1) = \frac{9}{10}$

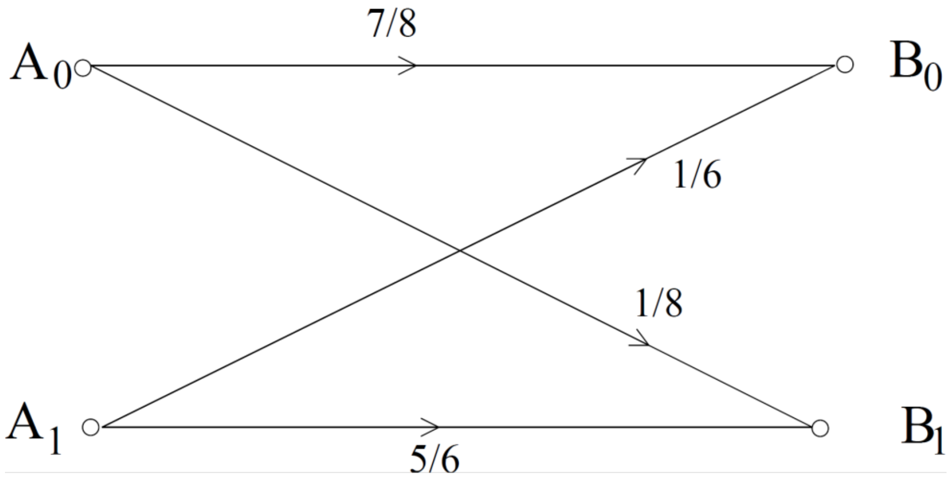


Binary Communication System

If we receive B_0, using the MLE rule, which one to select?

$$P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)$$

- MLE: if the probability of A0 given B0 is greater than the probability of A1 given B0, then we select A0, Otherwise, we select A1.



Maximum a Posterior (MAP) rule

- we choose the input that maximizes the a posteriori probability (APP), we call this a maximum a posteriori (MAP) decision rule.

If the set of events $\{A_0, A_1\}$ partitions Ω, and assuming $P(A_i) > 0$, for all i. Then the Maximum A Posteriori decision rule is given by:

$$P(A_0|B) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B)$$

$$\iff \frac{P(B|A_0)P(A_0)}{P(B)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B|A_1)P(A_1)}{P(B)}$$

$$\text{Given } B_j, \text{ choose } \widehat{A}_i, \text{ where } i = \arg\max_{i\in\{0,1\}} P(A_i|B_j).$$

Binary Communication System

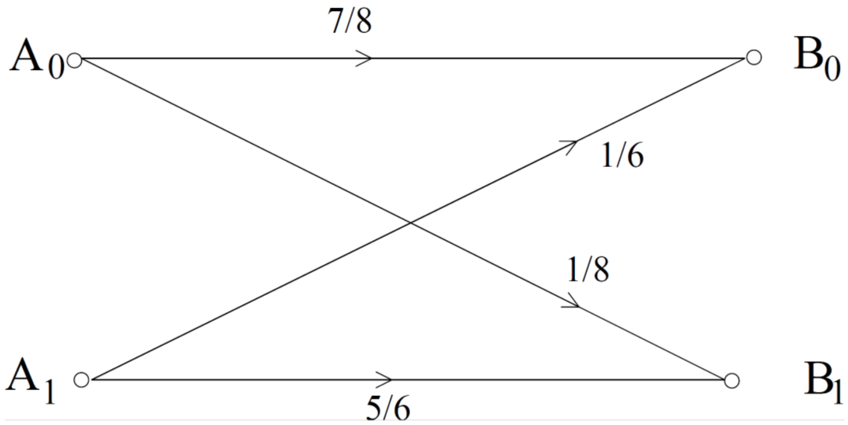
- MAP

1. B0
2. B1

Two cases:

1. $P(A_0) = \frac{2}{5}, \ P(A_1) = \frac{3}{5}$
2. $P(A_0) = \frac{1}{10}, \ P(A_1) = \frac{9}{10}$

$$\boxed{P(A_0|B) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B)}$$



Binary Communication System

- Observing B0

- Observing B1

Binary Communication System

- If a decision rule always selects A0 regardless of what is being received, what is the probability of error?

$$\begin{aligned}
&1.\ P(A_0) = \tfrac{2}{5},\ P(A_1) = \tfrac{3}{5} \\
&2.\ P(A_0) = \tfrac{1}{10},\ P(A_1) = \tfrac{9}{10}
\end{aligned}$$

Question: How to calculate the probability of ERROR?

$$P(E) = P(E|B_0)P(B_0) + P(E|B_1)P(B_1)$$

Probability of Error for case 1

Probability of Error for case 2

Continue the same step for scenario 2

- Left as an exercise