EEL 3850 Assignment 3 Solution

February 2025

Problem 1 (20 pts)

For the digital communication system with given transition probabilities:

$$P(A_0) = 0.6, \quad P(A_1) = 0.4$$

$$P(B_0|A_0) = 0.5, \quad P(B_1|A_0) = 0.25, \quad P(B_2|A_0) = 0.25$$

$$P(B_0|A_1) = 0.1, \quad P(B_1|A_1) = 0.3, \quad P(B_2|A_1) = 0.6$$

1. Completely specify the MLE decision rule and the MAP decision rule. (10 pts)

(a) **MLE Rule (5 pts)**

The MLE rule selects the transmitted symbol A_i that maximizes $P(B_j|A_i)$.

- If B_0 is received:

$$P(B_0|A_0) = 0.5 > 0.1 = P(B_0|A_1) \Rightarrow \text{Decide } A_0$$

- If B_1 is received:

$$P(B_1|A_1) = 0.3 > 0.25 = P(B_1|A_0) \Rightarrow \text{Decide } A_1$$

- If B_2 is received:

$$P(B_2|A_1) = 0.6 > 0.25 = P(B_2|A_0) \Rightarrow \text{Decide } A_1$$

Thus, the **MLE decision rule** is:

- Decide A_0 if B_0 is received.

- Decide A_1 if B_1 or B_2 is received.

(b) **MAP Rule (5 pts)**

The MAP rule selects A_i that maximizes the posterior probability:

$$P(A_i|B_j) = \frac{P(B_j|A_i)P(A_i)}{P(B_j)}$$

where

$$P(B_j) = P(B_j|A_0)P(A_0) + P(B_j|A_1)P(A_1).$$

Step 1: Compute $P(B_j)$ for each received symbol

$$P(B_0) = (0.5 \times 0.6) + (0.1 \times 0.4) = 0.30 + 0.04 = 0.34$$

$$P(B_1) = (0.25 \times 0.6) + (0.3 \times 0.4) = 0.15 + 0.12 = 0.27$$

$$P(B_2) = (0.25 \times 0.6) + (0.6 \times 0.4) = 0.15 + 0.24 = 0.39$$

Step 2: Compute posterior probabilities $P(A_i|B_j)$

- For B_0:

  $$P(A_0|B_0) = \frac{0.5 \times 0.6}{0.34} = \frac{0.30}{0.34} = 0.882 > P(A_1|B_0) = \frac{0.1 \times 0.4}{0.34} = \frac{0.04}{0.34} = 0.118$$

  $\Rightarrow$ Decide A_0.
- For B_1:

  $$P(A_0|B_1) = \frac{0.25 \times 0.6}{0.27} = \frac{0.15}{0.27} = 0.556 > P(A_1|B_1) = \frac{0.3 \times 0.4}{0.27} = \frac{0.12}{0.27} = 0.444$$

  $\Rightarrow$ Decide A_0.
- For B_2:

  $$P(A_0|B_2) = \frac{0.25 \times 0.6}{0.39} = \frac{0.15}{0.39} = 0.385 < P(A_1|B_2) = \frac{0.6 \times 0.4}{0.39} = \frac{0.24}{0.39} = 0.615$$

  $\Rightarrow$ Decide A_1.

Thus, the **MAP decision rule**:

- Decide A_0 if B_0 or B_1 is received.

- Decide A_1 if B_2 is received.

2. Calculate the overall probability of error under that rule. (10 pts)

(a) **Error Probability under MLE Rule (5 pts)**

$$P_e = P(A_0)P(\text{Error}|A_0) + P(A_1)P(\text{Error}|A_1).$$

- **Error when A_0 is sent**: This occurs if B_1 or B_2 is received.

$$P(\text{Error}|A_0) = P(B_1|A_0) + P(B_2|A_0) = 0.25 + 0.25 = 0.5.$$

- **Error when A_1 is sent**: This occurs if B_0 is received.

$$P(\text{Error}|A_1) = P(B_0|A_1) = 0.1.$$

Thus, the **total error probability under MLE** is:

$$P_e^{MLE} = (0.6 \times 0.5) + (0.4 \times 0.1) = 0.30 + 0.04 = 0.34.$$

(b) **Error Probability under MAP Rule (5 pts)**

$$P_e = P(A_0)P(\text{Error}|A_0) + P(A_1)P(\text{Error}|A_1).$$

- **Error when A_0 is sent**: This occurs if B_2 is received.

$$P(\text{Error}|A_0) = P(B_2|A_0) = 0.25.$$

- **Error when A_1 is sent**: This occurs if B_0 or B_1 is received.

$$P(\text{Error}|A_1) = P(B_0|A_1) + P(B_1|A_1) = 0.1 + 0.3 = 0.4.$$

Thus, the **total error probability under MAP** is:

$$P_e^{MAP} = (0.6 \times 0.25) + (0.4 \times 0.4) = 0.15 + 0.16 = 0.31.$$

Problem 2 (10 pts)

Let X be the number of girls among the 4 natural children. Since each child is equally likely to be a boy or a girl independently, X follows a binomial distribution:

$$X \sim \text{Binomial}(4, \frac{1}{2})$$

The probability mass function (PMF) of X is given by:

$$P(X = k) = \binom{4}{k} \left(\frac{1}{2}\right)^4, \quad k = 0, 1, 2, 3, 4.$$

Since the family has adopted 1 girl, the total number of girls in the family, denoted as Y, is:

$$Y = X + 1.$$

Thus, the **PMF** of Y is:

$$P(Y = n) = P(X = n - 1) = \binom{4}{n-1} \left(\frac{1}{2}\right)^4, \quad n = 1, 2, 3, 4, 5.$$

OR Computing the specific probabilities:

$$P(Y = 1) = \binom{4}{0} \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$P(Y = 2) = \binom{4}{1} \left(\frac{1}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}$$

$$P(Y = 3) = \binom{4}{2} \left(\frac{1}{2}\right)^4 = \frac{6}{16} = \frac{3}{8}$$

$$P(Y = 4) = \binom{4}{3} \left(\frac{1}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}$$

$$P(Y = 5) = \binom{4}{4} \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Thus, the **PMF** of Y is:

$$P(Y = n) = \begin{cases} \frac{1}{16} = 0.0625, & n = 1 \text{ or } 5 \\ \frac{1}{4} = 0.25, & n = 2 \text{ or } 4 \\ \frac{3}{8} = 0.375, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

Problem 3 (10 pts)

A pharmaceutical company is testing a new vaccine against a virus. Based on previous studies, the vaccine is known to be 70% effective, meaning that each vaccinated person has a 70% chance of developing immunity.

In a trial, 50 people receive the vaccine. Let X be the number of people who develop immunity. Since each individual has an independent probability of 0.7 of developing immunity, X follows a binomial distribution:

$$X \sim \text{Binomial}(50, 0.7)$$

1. Probability that exactly 40 people develop immunity (5 pts)

The probability mass function (PMF) for a binomially distributed random variable is given by:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Substituting $n = 50$, $p = 0.7$, and $k = 40$:

$$P(X = 40) = \binom{50}{40} (0.7)^{40} (0.3)^{10}$$

Using Python, we compute:

```
from scipy.stats import binom

n = 50
p = 0.7
prob_40 = binom.pmf(40, n, p)

print(prob_40)
```

$$P(X = 40) \approx 0.03862$$

2. Probability that at least 35 people develop immunity (5 pts)

The probability of at least 35 people developing immunity is:

$$P(X \geq 35) = \sum_{k=35}^{50} P(X = k)$$

Using the cumulative distribution function (CDF):

$$P(X \geq 35) = 1 - P(X \leq 34)$$

Using Python, we compute:

```
prob_at_least_35 = binom.sf(34, n, p)  # sf(34) = 1 - cdf(34)

print( prob_at_least_35 )
```

$$P(X \geq 35) \approx 0.5692$$

Problem 4 (10 pts)

Let X be the number of calls received in a day. Since the call center is open from 8 AM to 5 PM (9 hours), the mean number of calls per day is:

$$\lambda_{\text{daily}} = 5 \times 9 = 45$$

Thus, X follows a Poisson distribution:

$$X \sim \text{Poisson}(45)$$

1. Probability of receiving more than 50 calls in a day (5 pts)

The probability of receiving more than 50 calls is:

$$P(X > 50) = 1 - P(X \leq 50)$$

Using Python, we compute:

```
from scipy.stats import poisson

lambda_daily = 5 * 9  # Mean number of calls per day
prob_more_than_50 = 1 - poisson.cdf(50, lambda_daily)

print(prob_more_than_50)
```

$$P(X > 50) \approx 0.2037$$

2. Probability of having more than 3 busy days in a week (5 pts)

A "busy day" is defined as receiving more than 75 calls:

$$P(X > 75) = 1 - P(X \leq 75)$$

Using Python, we compute:

```
prob_busy_day = 1 - poisson.cdf(75, lambda_daily)
```

$$P(X > 75) \approx= 1.574e^{-05} \approx 0$$

Since a week consists of 5 independent days, let Y be the number of busy days in a week:

$$Y \sim \text{Binomial}(5, P(X > 75))$$

The probability of more than 3 busy days in a week is:

$$P(Y > 3) = 1 - P(Y \leq 3)$$

Using Python, we compute:

```python
from scipy.stats import binom

prob_more_than_3_busy_days = 1 - binom.cdf(3, 5, prob_busy_day)

print(prob_more_than_3_busy_days)
```

$$P(Y > 3) \approx 0$$