Lecture 9: Estimation

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EEL 3850

Motivating problem

X : height of 3rd students in population

$\mathbb{E}(X) = \mu_X$

- Estimating the average height of 3rd grade students in schools

Data sample 1:

53.49	51.59	53.94	56.57	51.30	51.30
56.74	54.30	50.59	53.63	50.61	50.60
52.73	46.26	46.83	50.31	48.96	52.94
49.28	47.76	56.40	51.32	52.20	47.73

Average = 51.51 inches

$6 \times 4 = 24$ students.

Data sample 2:

52.33	48.55	53.13	50.20	51.12	50.19
57.56	51.96	48.83	54.47	48.34	52.63
46.12	48.02	52.59	54.22	52.51	51.65
51.10	47.56	49.84	50.62	55.17	53.03

Average = 52.56 inches

X_i: ith 3rd grader's height. $\mu_X = \mathbb{E}(X_i) \quad \forall i$

$$M_n = M_{24} = \frac{1}{24} \sum_{i=1}^{24} X_i$$

Classical inference

- unknown parameter θ as a deterministic (not random!) but unknown quantity.
  - Average height.

X_i μ_X = Average height.

"constant" that we don't know.

- The observation X is random and its distribution
  - $p_X(x;\theta)$ if X is discrete
  - $f_X(x;\theta)$ if X is continuous
    - depends on the value of θ.

$X_i \sim N(\mu_X, \sigma_X^2)$: $\theta := (\mu_X, \sigma_X)$

parameters used to define the distribution.

$B \sim \text{Bernoulli}(p)$. $\theta := p$

Classical inference

$$\underbrace{\theta \to f_X(x;\theta)}_{\text{discrete RV. / continous RV. / complex}} \xrightarrow{x_1, x_2, \ldots, x_n} \text{Estimator} \to \underbrace{\hat{\theta}}_{\text{estimated value for } \theta}$$

- Given observations $X = (X_1, \ldots, X_n)$, an estimator $\widehat{\Theta} = g(X)$ is function of X.

- Thus, $\widehat{\Theta}$ is a random variable.

- Let n be the number of observations, the mean and variance of $\widehat{\Theta}_n$ are denoted $E_\theta[\widehat{\Theta}_n]$ and $var_\theta[\widehat{\Theta}_n]$, respectively.

X: mean, variance, [interval estimate] that contains the mean

Terminology regarding estimators

output of estimator

Random variable (e.g. M_n)

- The underlying parameter θ to be estimated is a constant.
- Estimation error: $\tilde{\Theta}_n = \hat{\Theta}_n - \theta$.
- Bias of the estimator: $b_\theta(\hat{\Theta}_n) = E_\theta[\hat{\Theta}_n] - \theta$, is the expected value of the estimation error.

$$= \mathbb{E}(\tilde{\Theta}_n)$$

e.g. $\mathbb{E}(M_n) - M_X = 0$

Estimation of the Mean

unknown RV. X ; μ_X mean

"sample Average"

- Suppose that the observations $X_1, \dots, X_n$ are i.i.d., with an unknown common mean μ_X.

$M_n =$

- $\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$ is unbiased estimator
  - for any n, the expected value of the average is equal to the true mean.

```
heights= np.array([121.92, 132.64, 113.31, 97.20, 140.94, 139.04, 115.98, 128.27, 121.84, 97.73])
```

The average height estimate

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average = np.mean(heights)= 122.185
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$$M_n \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right)$$

$$(X_1, X_2, \dots X_n) \longrightarrow \boxed{\text{Estimator}} \longrightarrow \hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Properties of the Estimator of the mean

- $\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$ is unbiased estimator
  - for any n, the expected value of the average is equal to the true mean.

$$\text{bias}(\hat{\mu}_X) = E(\hat{\mu}_X - \mu_X)$$

$$= E(\hat{\mu}_X) - \underbrace{E(\mu_X)}$$

constant.

$$= E(\hat{\mu}_X) - \mu_X = \mu_X - \mu_X = 0$$

$$\hat{\mu}_X \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right)$$

Average error $\Rightarrow 0$

Estimating the variance X. $Var(X) = \mathbb{E}((X - \mu_X)^2]$

Let σ_X^2 denote the variance of the random variables. Then there are two cases that should be considered for estimating the variance.

Known mean: If the mean of the random variables, μ_X, is known. Let the sample-variance estimator for this case be defined by

$\mathbb{E}(g(X)) = \frac{1}{N} \sum_{i=1}^{N} g(x_i)$

$$\hat{\sigma_X^2} = \frac{1}{N} \sum_{i=1}^{N} (X_i - \underbrace{\mu_X}_{\text{Constant}})^2.$$

sample Approximation

$\mathbb{E}[(X - \mu_X)^2] \approx \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu_X)^2$

sample set $\{x_1, x_2, \dots x_N\}$.

Estimating the variance

- Let's first determine if the sample variance estimator is biased when the true mean is known: (*experiment validate*)

error

$$\text{bias}(\hat{\sigma}_x^2) = \underline{\mathbb{E}(\hat{\sigma}_x^2} - \sigma_x^2) = 0$$

proof.

$$\mathbb{E}(\hat{\sigma}_x^2) = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_x)^2\right) \longrightarrow \sigma_x^2$$

Estimating the variance

Known mean: If the mean of the random variables, μ_X, is known. Let the sample-variance estimator for this case be defined by

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu_X)^2.$$

Unknown mean: it is natural to replace μ_X with its sample estimate $\hat{\mu}_X$:

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^2.$$

See the ADDITIONAL PDF for a proof why the variance estimator above is biased.

Estimating the varaince

- Let's first determine if the sample variance estimator is biased when we replace the true mean with its sample estimate: (*experiment validate*)

Estimating the varaince

Known mean: If the mean of the random variables, μ_X, is known. Let the sample-variance estimator for this case be defined by

$$\hat{\sigma_X^2} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu_X)^2.$$

Unknown mean: unbiased estimator:

$$\hat{\sigma_X^2} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^2.$$

The change in denominator is often referred to as a *degrees-of-freedom (dof) correction*.

Example

- Suppose we have a sample of student scores from an exam, and we want to estimate the population mean score.
- Sample data: array([84, 78, 71, 74, 60, 76, 50, 86, 67, 82, 89, 93, 79, 72, 78, 76, 71, 85, 86, 52, 61, 63, 92, 71, 80, 60, 76, 81, 57, 88])
- Total 30 samples
- Using the same sample data, we want to estimate the population variance.

Point estimate vs interval estimate

- Instead of estimating a single value, an interval estimate is also used:

- For an unknown parameter

$$\theta \to f_X(x;\theta) \xrightarrow{x_1, x_2, \ldots, x_n} \text{Interval Estimator} \to [\hat{\theta}^-, \hat{\theta}^+]$$

An interval contain the unknown parameter with high probability

Confidence intervals (CIs)

- The value of an estimator may not be informative enough
- Let us first fix a desired confidence level, $1 - \alpha$, where α is typically a small number.
- We then replace the point estimator $\hat{\Theta}_n$ by a lower estimator $\hat{\Theta}_n^-$ and an upper estimator $\hat{\Theta}_n^+$, s.t.

$$P\left(\hat{\Theta}_n^- \leq \theta \leq \hat{\Theta}_n^+\right) \geq 1 - \alpha$$

  for every possible value of θ.

- We call $[\hat{\Theta}_n^-, \hat{\Theta}_n^+]$ a $(1 - \alpha)$ confidence interval.

Confidence intervals (CIs)

- $\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$ estimated mean. , point estimator.
- Recall: the observations $X_1, \ldots, X_n$ are i.i.d., with an unknown common mean μ_X

$$\hat{\mu}_X \sim \mathcal{N}(\mu_X, \frac{\sigma^2}{n})$$

- Recall CLT:

lower bound

upper bound.

We call $[\hat{\mu}_X^-, \hat{\mu}_X^+]$ a $(1-\alpha)$ confidence interval if

$\alpha = 0.05$

$1-\alpha = 95\%$,

$$P(\hat{\mu}_X^- \leq \mu_X \leq \hat{\mu}_X^+) > 1-\alpha$$

Confidence intervals (CIs)

$? \; P(\hat{u}_x > u_x + d) \le \frac{\alpha}{2} = 0.025$

standardization: $P\left(\dfrac{\hat{u}_x - u_x}{\sqrt{\sigma^2/N}} > \dfrac{d}{\sqrt{\sigma^2/N}} \right) \le \frac{\alpha}{2}$

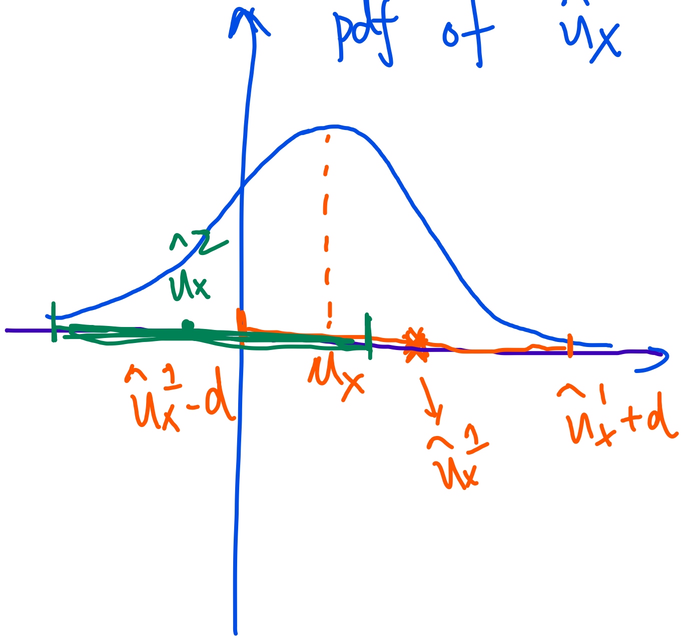
- Suppose $\alpha = 0.05$

$\dfrac{\hat{u}_x - u_x}{\sqrt{\sigma^2/N}} \sim N(0,1)$ $\quad P(G > ?) \le 0.025$

- Let's compute the 95% confidence interval about the mean of unknown RV using the samples.

point estimator $\quad \hat{u}_x = \frac{1}{N}\sum_{i=1}^{N} x_i \rightarrow$ interval estimator $[\hat{u}_x - d, \; \hat{u}_x + d]$ for some constant d.

pdf of $\hat{u}_x$



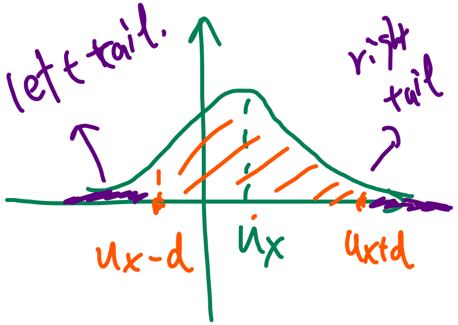
$\hat{u}_x \sim N\left(u_x, \dfrac{\sigma^2}{N}\right)$

Take a sample from $\hat{u}_x$, $\hat{u}_x^1$

$P(\hat{u}_x - d \le u_x \le \hat{u}_x + d) \ge 1 - \alpha$

$\Leftrightarrow P(\hat{u}_x \le u_x + d \text{ and } \hat{u}_x \ge u_x - d) \ge 1 - \alpha$

$\Leftrightarrow P(u_x - d \le \hat{u}_x \le u_x + d) \ge 1 - \alpha$



$P(\hat{u}_x > u_x + d) \le \frac{\alpha}{2}$

https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Confidence interval for mean estimate with unknown variance

- Recall if the variance is unknown, we have an unbiased estimate for the variance

Unknown mean: unbiased estimator:

$$\hat{\sigma_X^2} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^2.$$

$$\frac{\hat{\mu}_X - \mu_X}{\sigma_X/\sqrt{n}}$$

has a Student's t-distribution with $\nu = n-1$ degrees of freedom (dof) t_ν. (Like a Gaussian, but more spread out!)

Summary

1. Point Estimation for Mean with prior knowledge of the population variance

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

2. Standard Error of the Mean with known population variance (not an estimate but can be computed based on the property of variance.)

$$SE = \frac{\sigma}{\sqrt{n}}$$

3. Point Estimation for Mean without prior knowledge of the population variance

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

4. Point Estimation for Variance without prior knowledge of the population variance

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

summary

- Formula to compute confidence interval

For a population mean μ when the sample size is large ($n \geq 30$), the confidence interval is given by:

$$\left(\hat{\mu}_X - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \hat{\mu}_X + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \tag{1}$$

where:

- $\hat{\mu}_X$ = sample mean
- σ = population standard deviation (or sample standard deviation if unknown)
- n = sample size
- $z_{\alpha/2}$ = critical value from the standard normal table for a given confidence level