# Lecture 7: Continuous Random Variables

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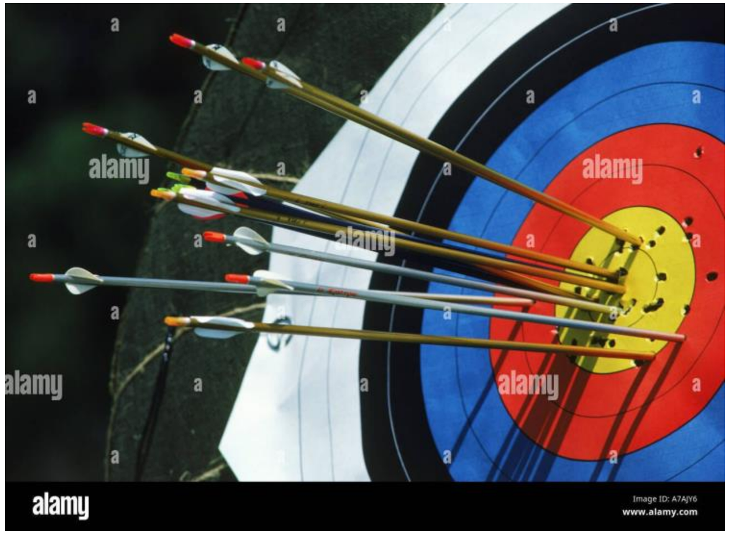
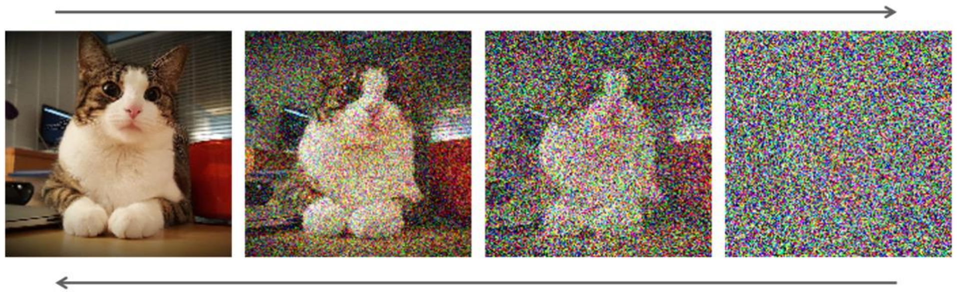
# Content

- Continuous Random Variables and PDFs
- Cumulative Distribution Functions
- Normal Random Variables

# Continuous Random Variables

- Random variables can have a continuous range of possible values.



Diffusion Models to perturb the image with Gaussian noise

- Continuous random variables are useful:
  - finer-grained than discrete random variables
  - able to exploit powerful tools from calculus.
  - Very board applications in robotic localization and mapping, deep learning, stochastic control, etc.

# Continuous r.v. and PDFs

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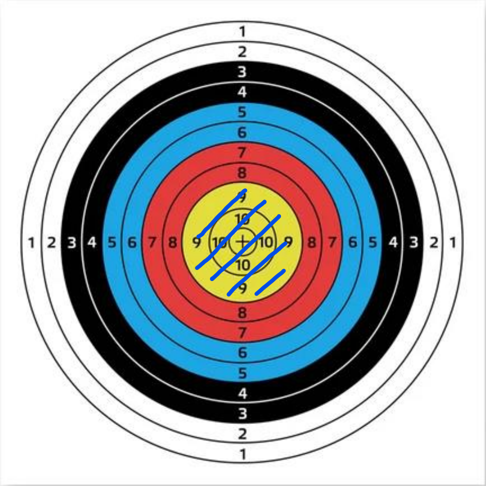
- A random variable $X$ is called continuous if there is a function $f_X \geq 0$, called the probability density function of $X$, or PDF, s.t.

$$P(X \in B) = \int_B \underbrace{f_X(x)}_{\text{p.d.f}} dx$$

for every subset $B \subseteq \mathbb{R}$.

Compare to discrete RV.

pmf: P(X=k)
cdf: P(X<k)

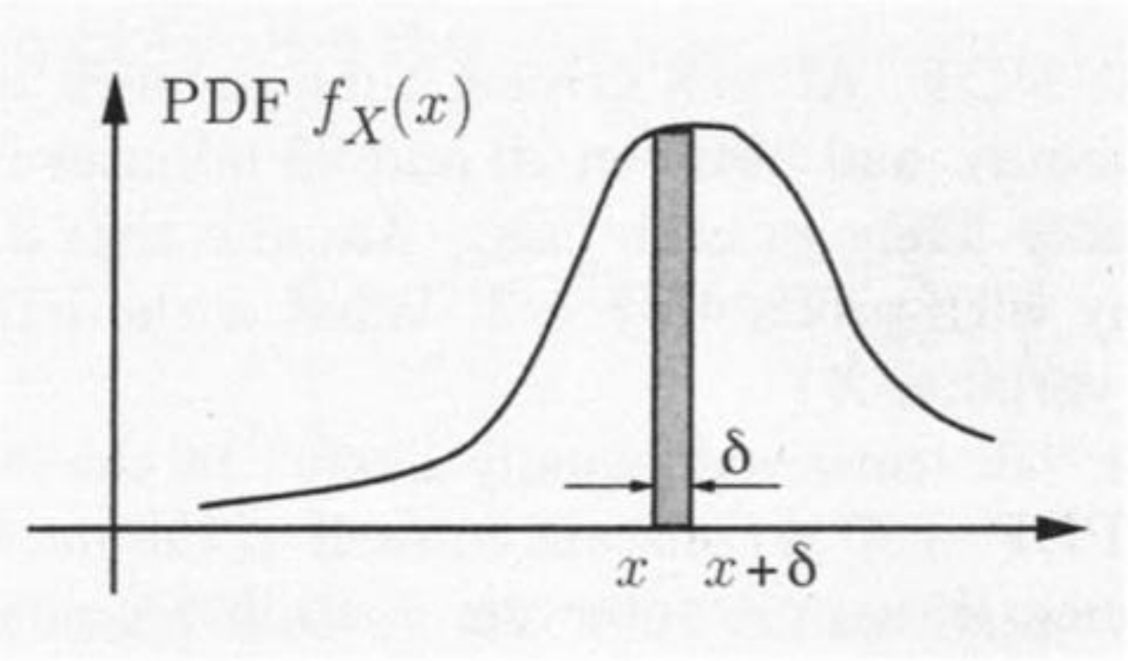



B: subset $\subseteq R^2$
yellow area

$P(X \in B)$

# Interpretation of PDF

- $f_X(x)$: "probability mass per unit length"
- $P([x, x+\delta]) = \int_x^{x+\delta} f_X(t)dt \approx f_X(x) \cdot \delta$



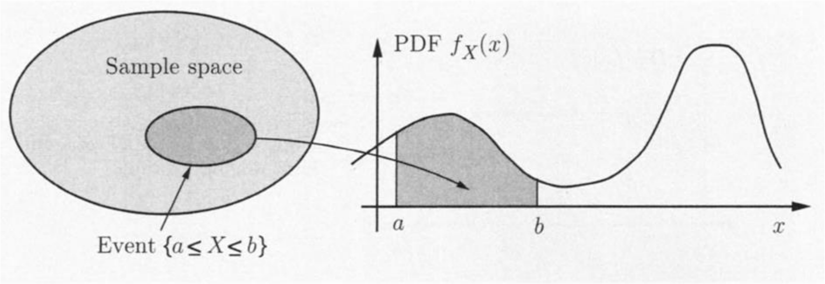
# PDF

$P(X=a)$

- $=P(a \le X \le a) = \int_a^a f_X(x)dx = 0.$
- $P(a \le X \le b) =$ $P(a < X \le b) + P(X=a)$ $=0$ $P(a \le X < b) + P(X=b)$ $=0$

  $= P(a < X \le b) = P(a \le X < b) = P(a < X < b) = \int_a^b f_X(x)dx$
- The entire area under the graph is equal to 1.

$$\int_{-\infty}^{\infty} f_X(x)dx = P(-\infty \le X \le \infty) = 1$$



# PDF can be arbitrarily large

$$\frac{d(\sqrt{x})}{dx} = \frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

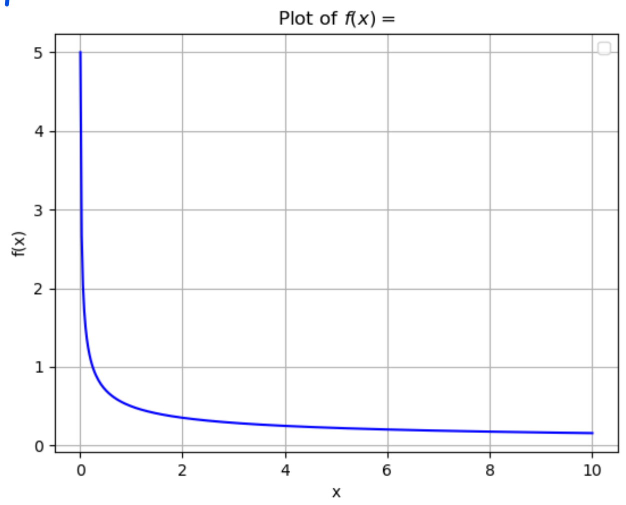
- Consider a random variable $X$ with PDF

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & \text{if } 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

- Note that $\int_{-\infty}^{\infty} f_X(x)dx =$ $\int_{-\infty}^{0} \underbrace{f_X(x)}_{0} dx + \int_{0}^{1} f_X(x) dx + \int_{1}^{+\infty} \underbrace{f_X(x)}_{0} dx$
    - So it's a valid PDF.

$$= \int_0^1 \frac{1}{2\sqrt{x}} dx = \sqrt{x}\Big|_0^1 = \sqrt{1} - \sqrt{0} = 1$$

- *However,* $\lim_{x\to 0^+} f_X(x) = \lim_{x\to 0^+} \frac{1}{2\sqrt{x}} = +\infty$



# Cumulative Distribution Function

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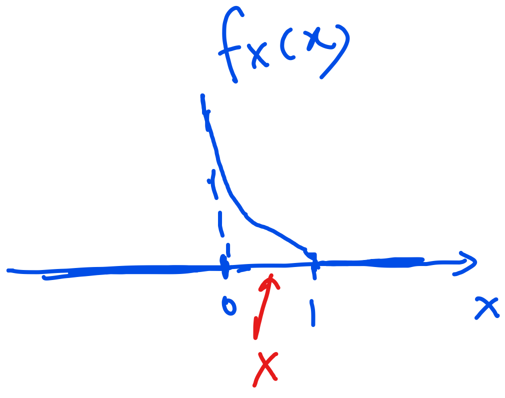
- The cumulative distribution function, or CDF, of a random variable $X$ is

$$F_X(x) = P(X \leq x)$$

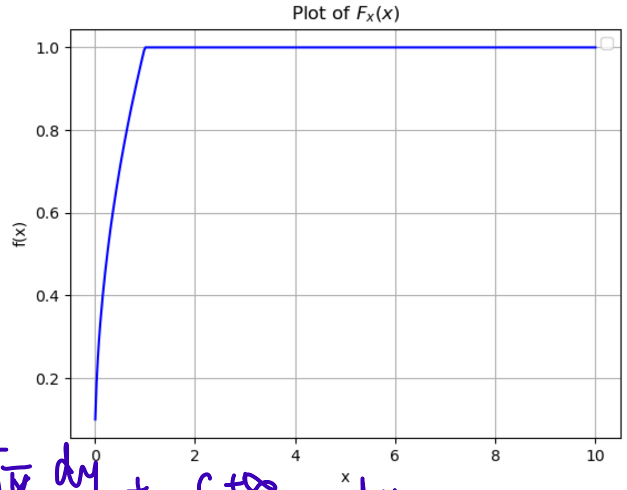
$$= \begin{cases} \sum_{k \leq x} p_X(k), & \text{discrete} \\ \int_{-\infty}^{x} f_X(y)\,dy, & \text{continuous} \end{cases}$$




- The CDF $F_X(x)$ "accumulates" probability "up to" the value $x$.

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & 0 \leq x \leq 1 \\ 0 & \text{o.w} \end{cases}$$

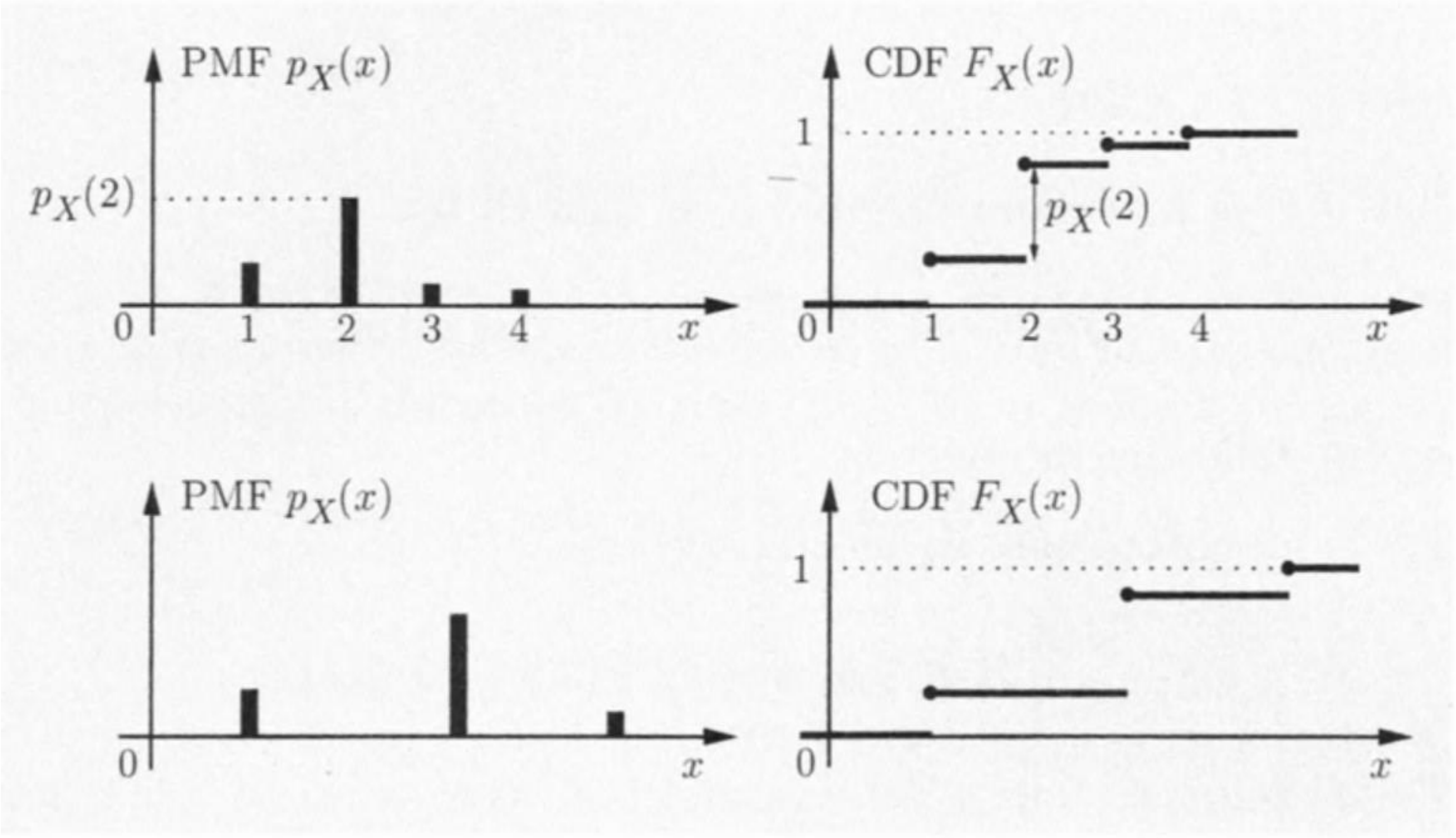
$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \begin{cases} 0, & x < 0 \\ \sqrt{x} & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$



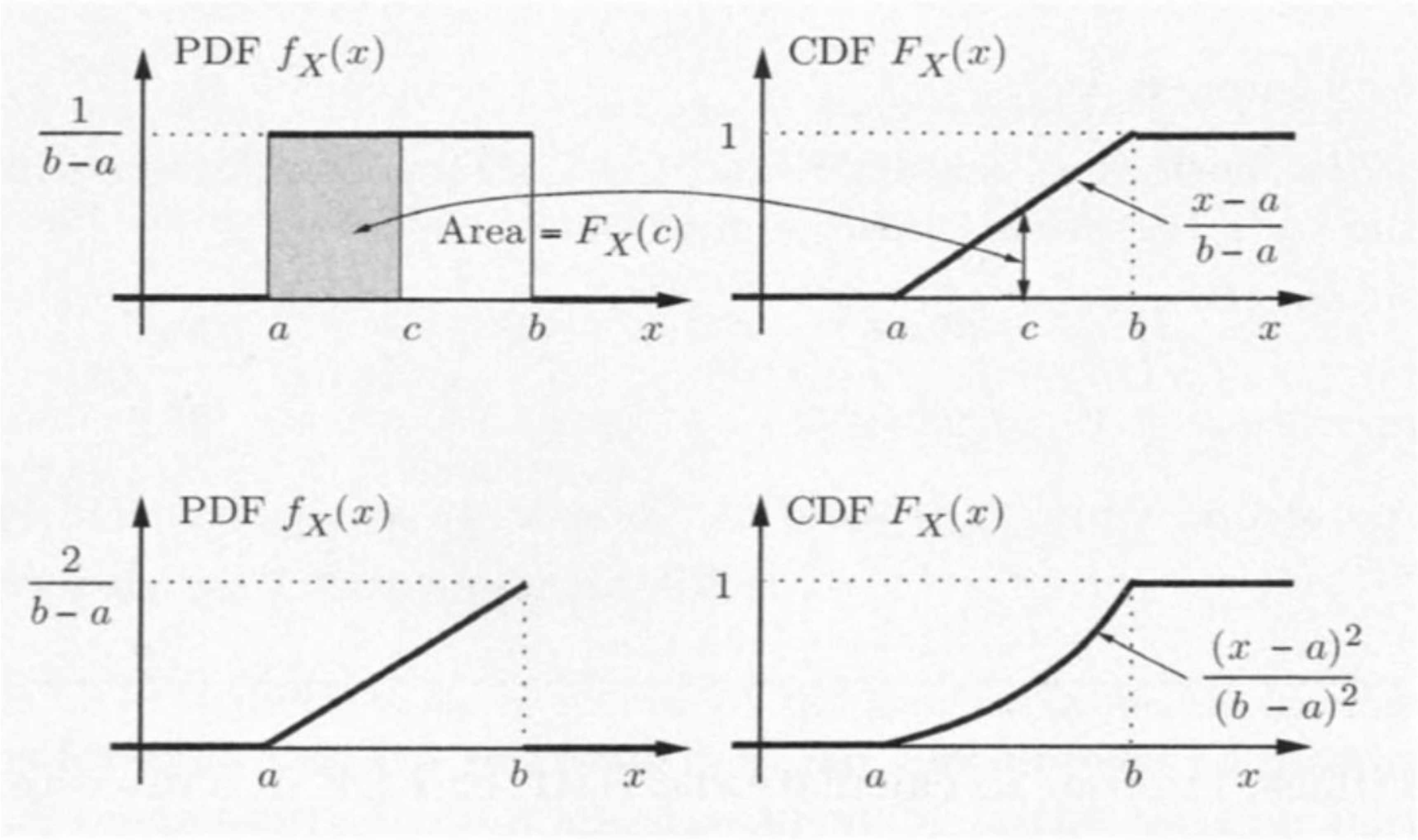

$$\int_{-\infty}^{0} f_X(y)\,dy + \int_{0}^{x} \frac{1}{2\sqrt{y}}\,dy = \sqrt{y}\Big|_0^x = \sqrt{x}$$

$$;\ \int_{-\infty}^{0} 0 \cdot dy + \int_{0}^{1} \frac{1}{2\sqrt{y}}\,dy + \int_{1}^{+\infty} 0 \cdot dy$$

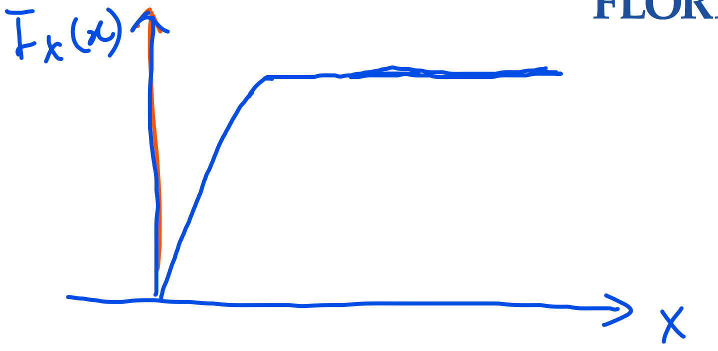
# CDF for discrete case



# CDF for continuous case



# Properties




- $F_X$ is monotonically increasing:

$$\text{if } x \leq y, \text{ then } F_X(x) \leq F_X(y).$$

- $\lim_{x\to-\infty} F_X(x) = 0$, $\lim_{x\to+\infty} F_X(x) = 1$
- If $X$ is discrete, $F_X$ is piecewise constant (step)
- If $X$ is continuous, $F_X$ is continuous and

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt\,, \qquad f_X(x) = \frac{dF_X}{dx}(x).$$

CDF PDF

# Important Continuous R.V.

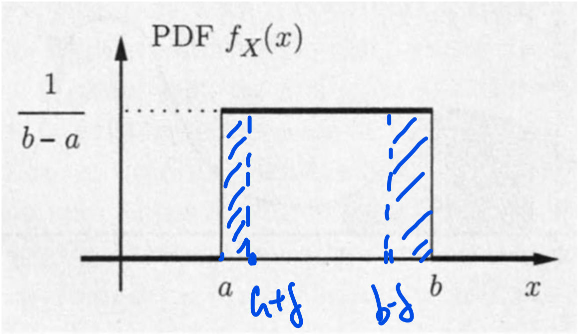
- Uniform
- Exponential
- Gaussian/Normal

# Uniform

- Consider a random variable $X$ takes value in interval $[a, b]$.
- Any subintervals of the same length have the same probability.
- It is called uniform random variable.

$$P(a \le X \le a+\delta)$$

$$P(b-\delta \le X \le b)$$

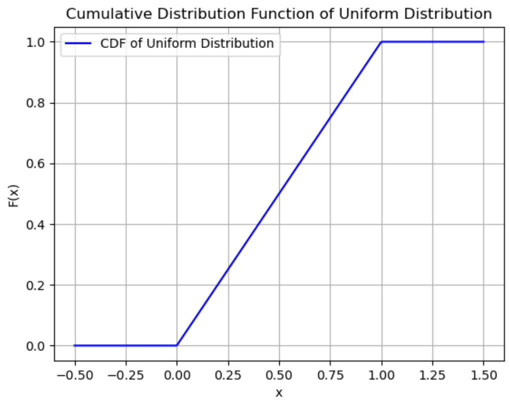



$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

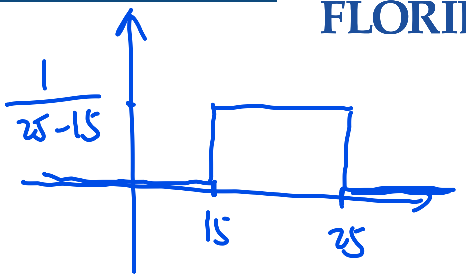
$$\int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a} x \Big|_a^b = 1$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} x & a \le x \le b \\ 1 & x > b \end{cases}$$



# Example

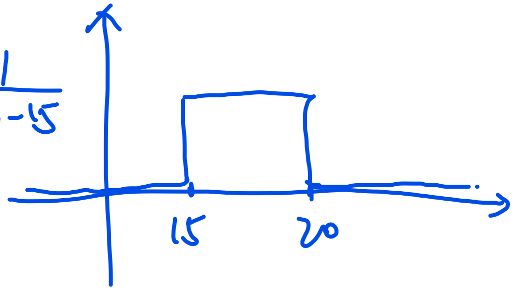
- When sunny, driving time is 15-20 minutes. Assume uniform $\frac{1}{10} = \frac{1}{25-15}$




- When rainy, driving time is 15-25 minutes. Assume uniform.
- Today is sunny, and I start drive at 8:00am, what is the probability that I will be in office between 8:15am to 8:18am?

8:18 am to 8:20 am

$\frac{1}{5} = \frac{1}{20-15}$




$X^s$: driving time

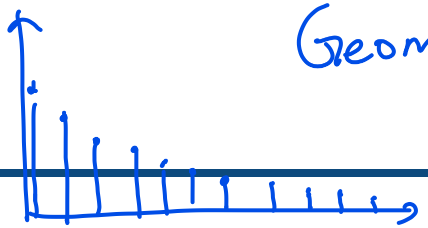
$$P(15 \le X^s \le 18) = \int_{15}^{18} \frac{1}{5}\,dx = \frac{1}{5} \cdot 3 = 60\%$$

- Today is rainy, and I start drive at 8:00am, what is the probability that I will be in office by 8:18am?

$X$: driving time. (length of drive)

$$P(X \le 18) = \int_{-\infty}^{18} \frac{1}{10}\,dx = \int_{15}^{18} \frac{1}{10}\,dx = \frac{3}{10} \simeq 30\%$$

# Exponential



Geom.

the num of trials till first success.

- An exponential random variable: the amount of time until some specific event occurs.
  - the amount of money customers spend in one trip to the supermarket follows an exponential distribution: There are more people who spend small amounts of money and fewer people who spend large amounts of money.
  - Length of long distance business telephone calls: A large number of calls are short.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

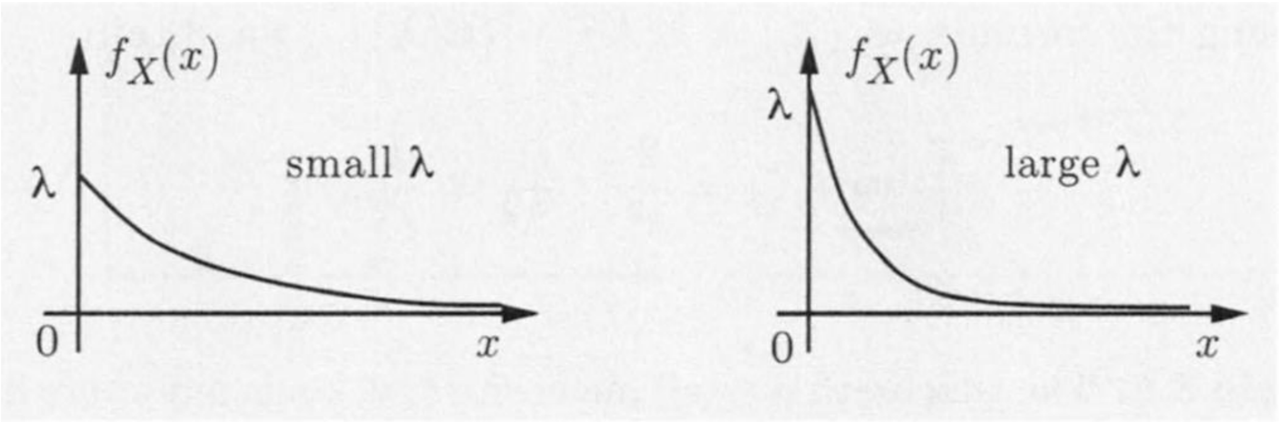
$\lambda$: decay factor.

$$f_X(x) = \begin{cases} \frac{1}{u} e^{-\frac{1}{u}x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$u$: average wait time

$$f_X(0) = \lambda e^{-\lambda \cdot 0} = \lambda$$

$$1/u = \lambda$$




$X \sim \text{Geom}(p)$

$p = \underbrace{\lambda}_{\text{decay rate}} \cdot \underbrace{\Delta T}_{\text{time step}}$ $= 0.001$

$X^E \sim \text{Exp}(\lambda)$

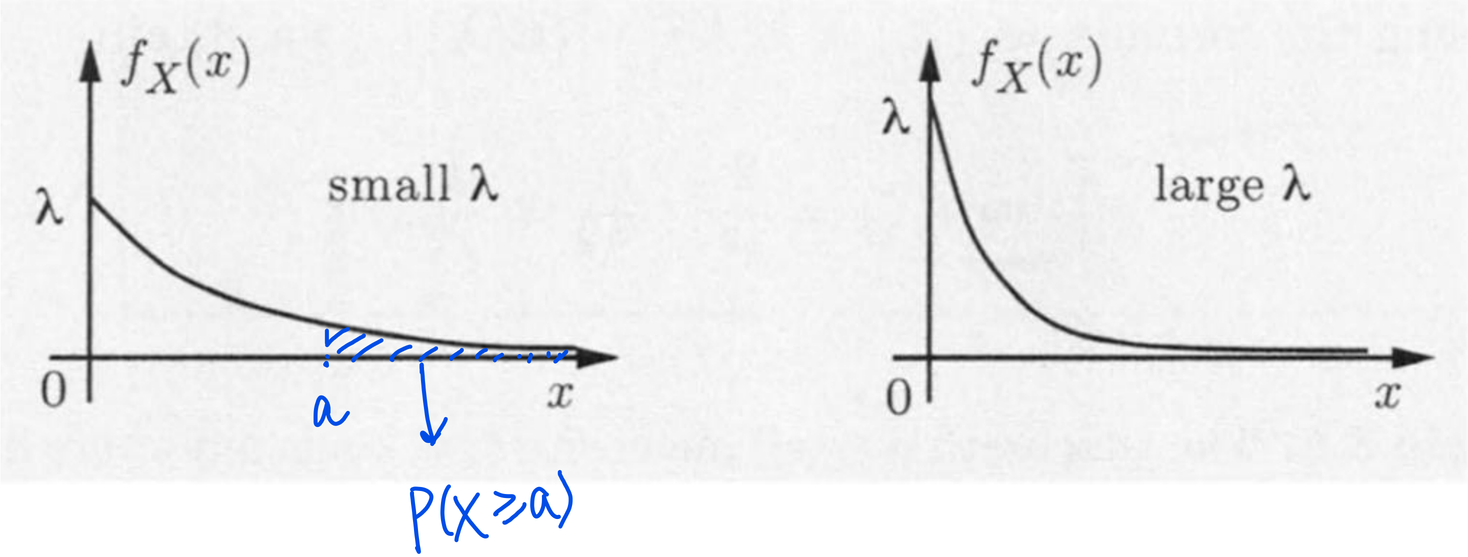
# Exponential

CDF: $P(X \le x) = F_X(x)$

Sf $P(X > x) = 1 - F_X(x)$

- Tail: $P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_a^\infty = e^{-\lambda a}$

pdf



# Example: Exponential

- The number of days ahead travelers purchase their airline tickets can be modeled by an exponential distribution with the average amount of time equal to 15 days. Find the probability that a traveler will purchase a ticket fewer than ten days in advance.

Average wait time : $\mu = 15$

$\frac{1}{\mu} = \lambda = \frac{1}{15}$

tail :

$P(X \geq a) = e^{-\lambda \cdot a}$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} = \frac{1}{15} e^{-\frac{1}{15}x}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$$

$$P(X < 10) = \int_{-\infty}^{10} f_X(x)\,dx = \int_0^{10} \frac{1}{15} e^{-\frac{1}{15}x}\,dx = \left(-e^{-\frac{1}{15}x}\right)\Big|_0^{10}$$

$$= e^{-\frac{2}{3}} + 1$$

$$P(X<10) = 1 - P(X \geq 10) = 1 - e^{-\frac{1}{15}\cdot 10} = 1 - e^{-\frac{2}{3}}.$$

# Normal Random Variable

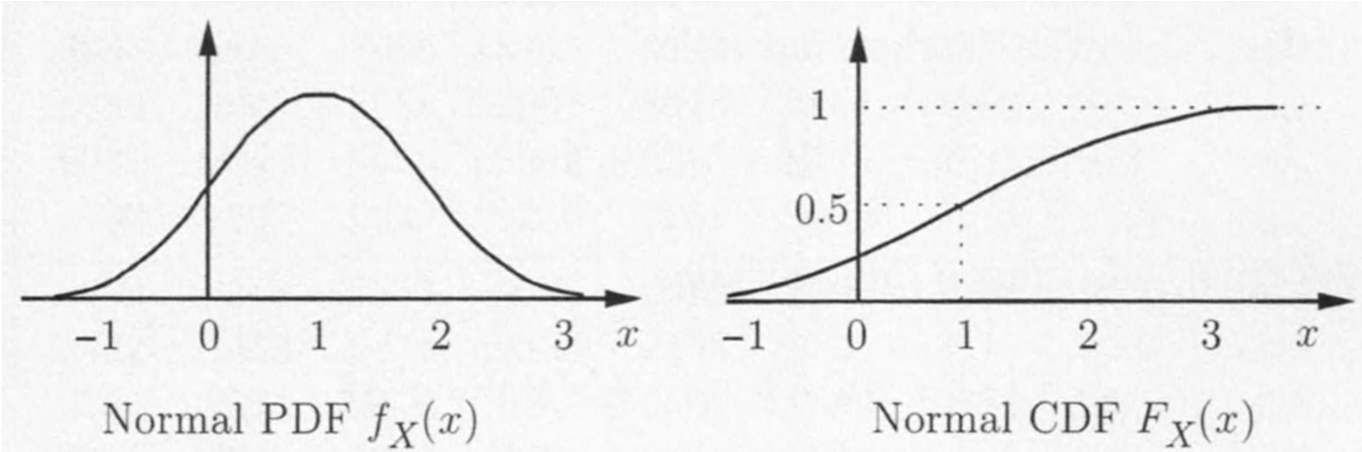
- A continuous random variable $X$ is normal, or Gaussian, if it has a PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for some $\sigma > 0$.

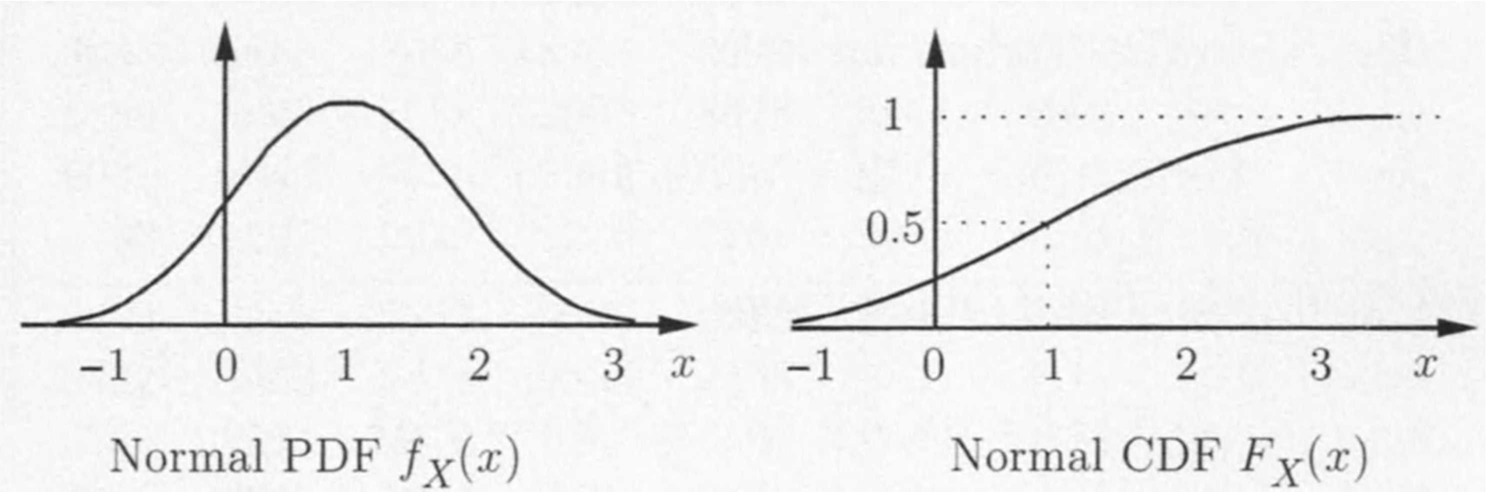
- It can be verified that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$




The PDF is symmetric and has a characteristic bell shape.

# Normal Random Variable



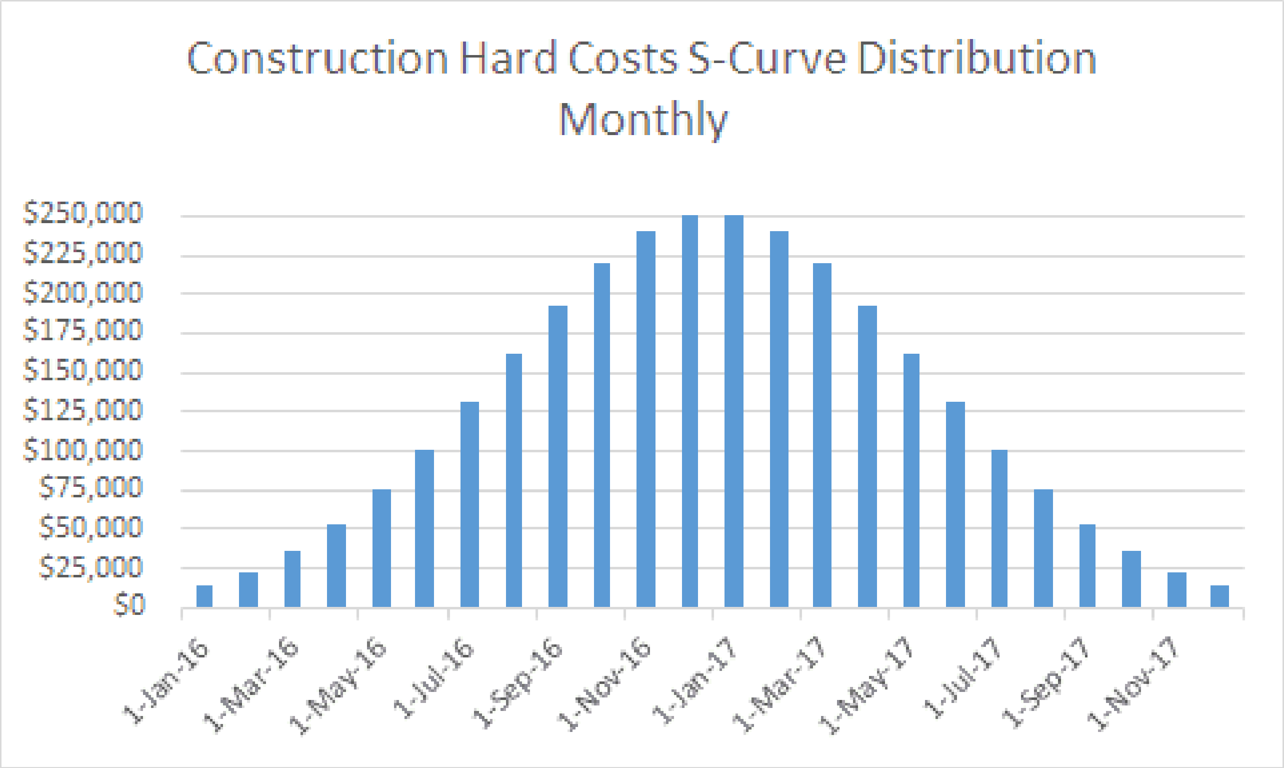

As $x$ gets further from $\mu$. the term $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ decreases very rapidly. In this figure, the PDF is very close to zero outside the interval $[-1,3]$.

- $\sigma$: standard deviation. the larger the $\sigma$ the more spread it is.
- $\mu$: the mean/average value.

# Example

e.g. construction cost

start by spending less per period in the earlier months while you're ramping up construction,
spend more per period in the middle months when construction is humming along, and then less per period in the latter months as construction winds down.



# Example

- if a skilled archer shoots a large number of arrows at a target, the majority of arrows will hit near the center (bullseye), with progressively fewer arrows landing farther away from the center.

https://demonstrations.wolfram.com/ProbabilisticDartboard/