EEL 3850 Homework 4 Solution

February 2025

1 Problem 1

Handwritten (Analytical) Solution

(a) $\mathbf{P(X < 1)}$:

- The PDF of X, which is uniformly distributed on $[-5, 5]$, is

$$f_X(x) = \begin{cases} \frac{1}{5-(-5)} = \frac{1}{10}, & -5 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

- Therefore,

$$P(X < 1) = \int_{-5}^{1} \frac{1}{10}\, dx = \frac{1}{10} \times \big(1 - (-5)\big) = \frac{1}{10} \times 6 = 0.6.$$

(b) $\mathbf{P(|X - 1| \ge 1)}$:

- The event $\{|X - 1| \ge 1\}$ is equivalent to $\{X \le 0\} \cup \{X \ge 2\}$.
- Hence,

$$P(|X - 1| \ge 1) = P(X \le 0) + P(X \ge 2).$$

- We compute these separately:

$$P(X \le 0) = \int_{-5}^{0} \frac{1}{10}\, dx = \frac{1}{10} \times \big(0 - (-5)\big) = \frac{5}{10} = 0.5,$$

$$P(X \ge 2) = \int_{2}^{5} \frac{1}{10}\, dx = \frac{1}{10} \times (5 - 2) = \frac{3}{10} = 0.3.$$

- Thus,

$$P\big(|X - 1| \ge 1\big) = 0.5 + 0.3 = 0.8.$$

Coding (Simulation) Solution in Python

~~A simple Monte Carlo simulation in Python to confirm these analytical results:~~

~~Running this code should output values close to:~~

$$\cancel{P(X < 1) \approx 0.6} \quad \text{and} \quad \cancel{P(|X - 1| \geq 1) \approx 0.8.}$$

2 Problem 2

Handwritten (Analytical) Solution

Step 1: Identify the Rate Parameter. Since the mean lifetime is 5 years, for an exponential distribution we have

$$\mathbb{E}[X] = \frac{1}{\lambda} = 5 \quad \Longrightarrow \quad \lambda = \frac{1}{5}.$$

Thus, the probability density function (PDF) of X is

$$f_X(x) = \lambda e^{-\lambda x} = \frac{1}{5} e^{-x/5}, \quad x \geq 0.$$

Step 2: Use Conditional Probability and Memoryless Property. The exponential distribution is memoryless, which means

$$P(X > s + t \mid X > s) = P(X > t),$$

for all $s, t \geq 0$.

In our problem, $s = 3$ years and $s + t = 8$ years, so $t = 5$ years. Then:

$$P(X > 8 \mid X > 3) = P(X > 3 + 5 \mid X > 3) = P(X > 5).$$

Because X is exponential$(\lambda = 1/5)$, we have

$$P(X > 5) = e^{-\lambda \times 5} = e^{-(1/5) \times 5} = e^{-1} \approx 0.3679.$$

Hence,

$$\boxed{P(X > 8 \mid X > 3) = e^{-1} \approx 0.3679.}$$

(Optional) Direct Conditional Probability Calculation. Alternatively, using standard conditional probability,

$$P(X > 8 \mid X > 3) = \frac{P(X > 8 \,\text{and}\, X > 3)}{P(X > 3)} = \frac{P(X > 8)}{P(X > 3)} = \frac{e^{-8/5}}{e^{-3/5}} = e^{-\frac{8}{5} + \frac{3}{5}} = e^{-1}.$$

This matches the memoryless result exactly.

Coding (Simulation) Solution in Python

~~We can verify the same probability via a Monte Carlo simulation:~~

3 Problem 3

Code Solution:

4 Problem 4

1. (4.1) The probability $P(X > 5)$.
2. (4.2) The probability $P(X < 1.25)$.
3. (4.3) A constant c such that $P(X < c) = 0.6$.
4. (4.4) Let $Y = 0.5\,X + 10$. Find $P(Y > 12)$, and clarify the distribution of Y.

Coding Solution

Explanation of Distribution for Y

Since $Y = 0.5X + 10$ and $X \sim \mathcal{N}(2, 16)$:

$$\begin{aligned}\mathbb{E}[Y] &= 0.5 \times \mathbb{E}[X] + 10 \;=\; 0.5 \times 2 + 10 \;=\; 1 + 10 = 11,\\ \mathrm{Var}(Y) &= (0.5)^2 \cdot \mathrm{Var}(X) = 0.25 \times 16 = 4,\end{aligned}$$

so Y is normally distributed with mean 11 and variance 4 (i.e. standard deviation 2).

5 Problem 5 (Additional Exercises)

Lemma. Let X be an exponential random variable with rate $\lambda > 0$. Show that

$$P(X > s + t \mid X > s) \;=\; P(X > t) \quad \text{for all } s, t > 0.$$

Proof

Since X is exponential(λ), we know its cumulative distribution function (CDF) is

$$F_X(x) \;=\; 1 - e^{-\lambda x}, \quad x \geq 0.$$

Hence $P(X > x) = 1 - F_X(x) = e^{-\lambda x}$.

We compute:

$$P(X > s + t \mid X > s) \;=\; \frac{P(X > s + t)}{P(X > s)} \;=\; \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \;=\; e^{-\lambda t}.$$

But $P(X > t) = e^{-\lambda t}$. Therefore,

$$P(X > s + t \mid X > s) \;=\; P(X > t),$$

which is precisely the *memoryless* property of the exponential distribution. $\square$

Does a Uniform Random Variable Have the Memoryless Property?

Now consider a uniform random variable Y. Specifically, suppose Y is uniformly distributed on the interval $[0, 1]$. We will see that

$$P(Y > s + t \mid Y > s) \;\stackrel{?}{=}\; P(Y > t)$$

does *not* hold in general, so the uniform distribution is *not* memoryless.

Counterexample

Take $s = 0.8$ and $t = 0.1$.

- $P(Y > 0.9 \mid Y > 0.8) = \dfrac{P(Y > 0.9)}{P(Y > 0.8)} = \dfrac{1 - 0.9}{1 - 0.8} = \dfrac{0.1}{0.2} = 0.5.$
- On the other hand, $P(Y > 0.1) = 1 - 0.1 = 0.9$.

Thus,

$$P(Y > 0.8 + 0.1 \mid Y > 0.8) = 0.5 \quad \text{while} \quad P(Y > 0.1) = 0.9.$$

Clearly, $0.5 \neq 0.9$. Therefore,

$$P(Y > s + t \mid Y > s) \;\neq\; P(Y > t),$$

which shows that a uniform random variable is **not** memoryless.