① $P(B) = 1 - P(A) - P(D) = 1 - 0.5 - 0.25 = 0.25.$

$P(B \cup D) = P(B) + P(D) = 0.25 + 0.25 = 0.5$

exclusive of two events.

② $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{P([2,1]) + P([1,2])}{P(C)}$

$P(C) = 1 - P(C^c) = 1 - \dfrac{25}{36} = \dfrac{11}{36}$

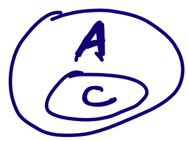
$\therefore P(A|C) = \dfrac{\left(\frac{2}{36}\right)}{\left(\frac{11}{36}\right)} = \dfrac{2}{11}$

$P(B|C) = \dfrac{P(B \cap C)}{P(C)}$

$P(B \cap C) = P([1,7]) + P([7,1]) = \dfrac{2}{36}$

$\therefore P(B|C) = \dfrac{\left(\frac{2}{36}\right)}{\left(\frac{11}{36}\right)} = \dfrac{2}{11}$

$P(A \cap C) = \dfrac{2}{36}.$ $A \to C$



$P(A)P(C) = \dfrac{2}{36} \times \dfrac{11}{36} \neq P(A \cap C)$

$\therefore$ A · C not independent.

$P(B \cap C) = \dfrac{2}{36}.$

$P(B) = \dfrac{6}{36}.$

B: (1,6) (3,4)
(6,1) (4,3)
(2,5)
(5,2)

$\therefore P(B) \cdot P(C) = \dfrac{6}{36} \times \dfrac{11}{36} \neq \dfrac{2}{11}$

$\therefore$ B. C. not independent.

3. $P(C) = P(C|R) \cdot P(R) + P(C|R^c) \cdot P(R^c)$

$= 0.9 \times 0.3 + 0.4 \times 0.7$

$= 0.27 + 0.28 = 0.55.$

$$P(R|C) = \frac{P(C|R)\,P(R)}{P(C)}$$

$$= \frac{0.9 \times 0.3}{0.55} = \frac{0.27}{0.55} \doteq 49\%$$

4. ① $$P(K|C) = \frac{P(C|K)\,P(K)}{P(C)}$$

$P(C) = P(C|K)\,P(K) + P(C|K^c)\,P(K^c)$

$= p \cdot 1 + (1-p) \cdot \frac{1}{c}$

$$P(K|C) = \frac{p}{p + (1-p)\frac{1}{c}}$$

② $P(C_1 \cap C_2 | K_1 \cap K_2^c) + P(C_1 \cap C_2 | K_2 \cap K_1^c)$

using independence

$\Rightarrow P(C_1 \cap C_2 | K_1 \cap K_2^c)$

$$= \frac{P(C_1 \cap C_2 \cap K_1 \cap K_2^c)}{P(K_1 \cap K_2^c)}$$

$$= \frac{P(C_1 \cap K_1) \cdot P(C_2 \cap K_2^c)}{P(K_1)\,P(K_2^c)}$$

$$= P(C_1 | K_1) \cdot P(C_2 | K_2^c)$$

$$= 1 \cdot \frac{1}{c}$$

same value for $P(C_1 \cap C_2 | K_2 \cap K_1^c)$

$$P(K_1 \cap K_2^c \cup K_2 \cap K_1^c | C_1 \cap C_2)$$

$$= P(K_1 \cap K_2^c | C_1 \cap C_2) + P(K_2 \cap K_1^c | C_1 \cap C_2)$$

$$= \frac{P(C_1 \cap C_2 | K_1 \cap K_2^c) \cdot P(K_1 \cap K_2)}{P(C_1 \cap C_2)}$$

$$+ \frac{P(C_1 \cap C_2 | K_2 \cap K_1^c)\, P(K_2 \cap K_1^c)}{P(C_1 \cap C_2)}$$

$$= \frac{\frac{1}{c} \cdot P(1-P) + \frac{1}{c} P(1-P)}{P(C_1 \cap C_2)} \longleftarrow \text{substitute.}$$

$$P(C_1 \cap C_2) = P(C_1) \cdot P(C_2) = \left(p + (1-p)\frac{1}{c}\right)^2.$$

5. $P(S) = 20\%$. S: spam. D: detected.

$$P(S|D) = \frac{P(D|S)P(S)}{P(D)}$$

$$= \frac{90\% \times 20\%}{90\% \times 20\% + 5\% \times 80\%}$$

6. $P(Ace_1) = \frac{4}{52}$.

$$P(Ace_2|Ace_1) = \frac{3}{51}$$

7. decision-making.

the order of games:

$i \Rightarrow j \rightarrow k$ $(i,j,k \in \{1,2,3\})$.

playing vs. i → vs. j → vs. k

let P_i, P_j, P_k be the probability of winning against player type i.

∴ the probability of winning

$$P(win) = \underbrace{P_i \cdot P_j}_{\text{win first two game}} + \underbrace{(1-P_i)P_jP_k}_{\text{win last two games}}$$

$$= P_j\left[P_i + (1-P_i)P_k\right]$$

$$\max_{(i,j,k) \in \text{permutations of } 1,2,3.} P(\text{win})$$

if P_j is the largest, then

say $P_1 < P_2 < P_3$

can we show

$$P_3 [P_1 + (1-P_1) P_2] \overset{?}{>} P_2 [P_1 + (1-P_1) P_3]$$

$$LHS = P_1 P_3 + P_2 P_3 - P_1 P_2 P_3$$

$$RHS = P_1 P_2 + P_2 P_3 - P_1 P_2 P_3$$

$$LHS - RHS = P_1 P_3 - P_1 P_2 = P_1 (P_3 - P_2) > 0$$

because $P_3 > P_2$

$\therefore$ playing $1 \to 3 \to 2$ has a higher winning probability compared to $1 \to 2 \to 3$

compare $3 \to 2 \to 1$ vs. $1 \to 3 \to 2$

$$P(\text{win}) = P_2 (P_3 + (1-P_3) P_1)$$

$$= P_2 P_3 + P_1 P_2 - P_1 P_2 P_3$$

$$P(\text{win} | 1 \to 3 \to 2) - P(\text{win} | 3 \to 2 \to 1)$$

$$= P_1 P_3 + P_2 P_3 - P_1 P_2 P_3 - P_2 P_3 - P_1 P_2 + P_1 P_2 P_3$$

$= p_1(p_3 - p_2) > 0$

$\therefore$ - playing against 3 the second game always has a large winning probability.

- The order of game 1, 2 doesn't matter.