ASTR 792 HW 4

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5.3

a

For ${}^{12}\mathrm{C}^{16}\mathrm{O}$, $J = 1 \Rightarrow 0$

$$\nu_0 = 115.27\ GHz \rightarrow \lambda = .261\ cm$$
$$v = 1 \Rightarrow 0 = \lambda = 4.61 \mu m$$

Additionally

$$E = \frac{h^2}{2I} = \frac{h^2}{2\mu r^2}$$
$$\rightarrow E = h\nu \propto \frac{1}{\mu}$$

Here, μ is the reduced mass of the CO molecule
for ${}^{12}\mathrm{C}^{16}\mathrm{O}$

$$\mu_{{}^{12}\mathrm{C}^{16}\mathrm{O}} = m_p \left(\frac{12 \cdot 16}{12 + 16} \right) = 6.86 m_p$$

similarly

$$\mu_{{}^{13}\mathrm{C}^{16}\mathrm{O}} = 7.17 m_p \qquad , \mu_{{}^{12}\mathrm{C}^{17}\mathrm{O}} = 7.03 m_p$$

From ${}^{13}\mathrm{C}^{16}\mathrm{O} \Rightarrow {}^{12}\mathrm{C}^{16}\mathrm{O}$

$$\nu_{{}^{13}\mathrm{C}^{16}\mathrm{O}} \cdot \mu_{{}^{13}\mathrm{C}^{16}\mathrm{O}} = \nu_{{}^{12}\mathrm{C}^{16}\mathrm{O}} \cdot \mu_{{}^{12}\mathrm{C}^{16}\mathrm{O}}$$
$$\rightarrow \nu_{{}^{13}\mathrm{C}^{16}\mathrm{O}} = \nu_{{}^{12}\mathrm{C}^{16}\mathrm{O}} \frac{\mu_{{}^{12}\mathrm{C}^{16}\mathrm{O}}}{\mu_{{}^{13}\mathrm{C}^{16}\mathrm{O}}}$$
$$= 115.27\ GHz \frac{6.86}{7.17}$$
$$= 110.29\ GHz$$

From ${}^{12}\mathrm{C}^{16}\mathrm{O} \Rightarrow {}^{12}\mathrm{C}^{17}\mathrm{O}$

$$
\begin{aligned}
\nu_{^{12}\mathrm{C}^{16}\mathrm{O}} \cdot \mu_{^{12}\mathrm{C}^{16}\mathrm{O}} &= \nu_{^{12}\mathrm{C}^{17}\mathrm{O}} \cdot \mu_{^{12}\mathrm{C}^{17}\mathrm{O}} \\
\rightarrow \nu_{^{12}\mathrm{C}^{16}\mathrm{O}} &= \nu_{^{12}\mathrm{C}^{16}\mathrm{O}} \frac{\mu_{^{12}\mathrm{C}^{16}\mathrm{O}}}{\mu_{^{12}\mathrm{C}^{17}\mathrm{O}}} \\
&= 115.27\ GHz \frac{6.86}{7.03} \\
&= 112.48\ GHz
\end{aligned}
$$

b

Let

$$
E = (n + 1/2)\hbar\omega = (n + 1/2)\hbar\sqrt{\frac{k}{\mu}}
$$

Since $\omega = 2\pi\nu \rightarrow \nu \propto \frac{1}{\sqrt{\mu}}$, this also means $\lambda \propto \sqrt{\mu}$. therefore

$$
\begin{aligned}
\frac{\sqrt{\mu_{^{12}\mathrm{C}^{16}\mathrm{O}}}}{\lambda_{^{12}\mathrm{C}^{16}\mathrm{O}}} &= \frac{\sqrt{\mu_{^{13}\mathrm{C}^{16}\mathrm{O}}}}{\lambda_{^{13}\mathrm{C}^{16}\mathrm{O}}} \\
\rightarrow \lambda_{^{13}\mathrm{C}^{16}\mathrm{O}} &= \lambda_{^{12}\mathrm{C}^{16}\mathrm{O}} \frac{\sqrt{\mu_{^{13}\mathrm{C}^{16}\mathrm{O}}}}{\sqrt{\mu_{^{12}\mathrm{C}^{16}\mathrm{O}}}} \\
&= .261\ cm \frac{\sqrt{7.17}}{\sqrt{6.86}} \\
&= .267\ cm
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{^{12}\mathrm{C}^{17}\mathrm{O}} &= \lambda_{^{12}\mathrm{C}^{16}\mathrm{O}} \frac{\sqrt{\mu_{^{12}\mathrm{C}^{17}\mathrm{O}}}}{\sqrt{\mu_{^{12}\mathrm{C}^{16}\mathrm{O}}}} \\
&= .261\ cm \frac{\sqrt{7.03}}{\sqrt{6.86}} \\
&= .264\ cm
\end{aligned}
$$

c

if the $^{12}\mathrm{C}^{16}\mathrm{O} J = 1 - 0$ line were mistaken for $^{13}\mathrm{C}^{16}\mathrm{O} J = 1 - 0$, then

$$
\begin{aligned}
\frac{\Delta\nu}{\nu} &= (115.27 - 110.29) GHz \\
&= 4.98\ GHz
\end{aligned}
$$

Then

$$\frac{\Delta\nu}{\nu} = \frac{v}{c}$$
$$\rightarrow \frac{4.98\ GHz}{115.27\ GHz} \cdot 3 \cdot 10^8\ m/s = v$$
$$= 1.30 \cdot 10^7\ m/s$$