1.3 第五周作业

1.3.1 作业答案

习题 1.10 (第三章第 6 题)

证明：

$$\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

证明 证法一：考虑四阶行列式

$$\begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix}$$

我们用后面两行将前两行前两列的四个元素消为 0：

$$\begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix} \xlongequal[a_{21}r_3+a_{22}r_4\to r_2]{a_{11}r_3+a_{12}r_4\to r_1} \begin{vmatrix} 0 & 0 & a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ 0 & 0 & a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix} \tag{1.1}$$

分别对式 (1.1) 两边的前两行进行拉普拉斯展开得：

$$LHS = (-1)^{1+2+1+2}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \tag{1.2}$$

$$\begin{aligned} RHS &= (-1)^{1+2+3+4}\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix} \cdot \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix} \end{aligned} \tag{1.3}$$

证毕.

证法二：将等式两边行列式完全展开，分别对比 8 项得证 (这种方法只适用于二、三阶这样的低阶行列式，高阶行列式展开项数过多，费时费力)

注 本题即是以下性质的二阶形式，感兴趣的同学可自行证明：设 $A, B \in \mathbb{F}^{n\times n}$，有

$$det(AB) = det(A)det(B) \tag{1.4}$$

习题 1.11 (第三章第 7 题)

证明：

$$\begin{vmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}-1 & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}-1 \end{vmatrix}$$

证明 证法一：类似于上题中的证法一，我们将左边四阶行列式的前两行前两列四个元素消为 0：

$$\begin{vmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{vmatrix} \xlongequal[-a_{21}r_3-a_{22}r_4\to r_2]{-a_{11}r_3-a_{12}r_4\to r_1} \begin{vmatrix} 0 & 0 & 1-a_{11}b_{11}-a_{12}b_{21} & -a_{11}b_{12}-a_{12}b_{22} \\ 0 & 0 & -a_{21}b_{11}-a_{22}b_{21} & 1-a_{21}b_{12}-a_{22}b_{22} \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{vmatrix} \tag{1.5}$$

对式 (1.5) 的右边进行拉普拉斯展开得：

$$\begin{aligned} RHS &= (-1)^{1+2+3+4}\begin{vmatrix} 1-a_{11}b_{11}-a_{12}b_{21} & -a_{11}b_{12}-a_{12}b_{22} \\ -a_{21}b_{11}-a_{22}b_{21} & 1-a_{21}b_{12}-a_{22}b_{22} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}-1 & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}-1 \end{vmatrix} \end{aligned} \tag{1.6}$$

证毕

证法二：这里给出一种笔者能想到的较为简单的展开证法

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{vmatrix} &= \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & 0 & b_{12} \\ 0 & 1 & 0 & b_{22} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & b_{11} & 0 \\ 0 & 1 & b_{21} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & 0 & b_{12} \\ 0 & 1 & 0 & b_{22} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & 0 \\ 1 & 0 & b_{11} \\ 0 & 1 & b_{21} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} & 1 \\ 1 & 0 & b_{12} \\ 0 & 1 & b_{22} \end{vmatrix} \\ &= \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix} - a_{11}b_{11}-a_{12}b_{21}+1-a_{21}b_{12}-a_{22}b_{22} \end{aligned} \tag{1.7}$$

$$\begin{aligned} &\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}-1 & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}-1 \end{vmatrix} \\ =&\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}-1 \end{vmatrix} + \begin{vmatrix} -1 & a_{11}b_{12}+a_{12}b_{22} \\ 0 & a_{21}b_{12}+a_{22}b_{22}-1 \end{vmatrix} \\ =&\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix} + \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & 0 \\ a_{21}b_{11}+a_{22}b_{21} & -1 \end{vmatrix} + \begin{vmatrix} -1 & a_{11}b_{12}+a_{12}b_{22} \\ 0 & a_{21}b_{12}+a_{22}b_{22}-1 \end{vmatrix} \\ =&\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix} - a_{11}b_{11}-a_{12}b_{21}+1-a_{21}b_{12}-a_{22}b_{22} \end{aligned} \tag{1.8}$$

对比式 (1.7),(1.8) 可见命题成立

注 本题即是以下性质的二阶形式，感兴趣的同学可自行证明：设 $A \in F^{m\times n}, B \in F^{n\times m}$，有

$$det(I_n - BA) = det\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} = det(I_m - AB) \tag{1.9}$$

习题 1.12 (第三章第 8 题)

设 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ 为 4 维数组向量. 证明：$det(2\boldsymbol{a}-\boldsymbol{b}, -\boldsymbol{a}+2\boldsymbol{b}-\boldsymbol{c}, -\boldsymbol{b}+2\boldsymbol{c}-\boldsymbol{d}, -\boldsymbol{c}+2\boldsymbol{d}) = 5det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$.

证明 本题有多种变换方法，过程合理即可.

$$
\begin{aligned}
&det(2\boldsymbol{a}-\boldsymbol{b},-\boldsymbol{a}+2\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{b}+2\boldsymbol{c}-\boldsymbol{d},-\boldsymbol{c}+2\boldsymbol{d})\\
\xlongequal[c_3\to c_2]{c_2+c_3+c_4\to c_1}&det(\boldsymbol{a}+\boldsymbol{d},-\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{d},-\boldsymbol{b}+2\boldsymbol{c}-\boldsymbol{d},-\boldsymbol{c}+2\boldsymbol{d})\\
\xlongequal{c_1\to c_2}&det(\boldsymbol{a}+\boldsymbol{d},\boldsymbol{b}+\boldsymbol{c},-\boldsymbol{b}+2\boldsymbol{c}-\boldsymbol{d},-\boldsymbol{c}+2\boldsymbol{d})\\
\xlongequal{c_2\to c_3}&det(\boldsymbol{a}+\boldsymbol{d},\boldsymbol{b}+\boldsymbol{c},3\boldsymbol{c}-\boldsymbol{d},-\boldsymbol{c}+2\boldsymbol{d})\\
\xlongequal{3c_4\to c_3}&det(\boldsymbol{a}+\boldsymbol{d},\boldsymbol{b}+\boldsymbol{c},5\boldsymbol{d},-\boldsymbol{c}+2\boldsymbol{d})\\
=&det(\boldsymbol{a},\boldsymbol{b}+\boldsymbol{c},5\boldsymbol{d},-\boldsymbol{c})\\
=&det(\boldsymbol{a},\boldsymbol{b},5\boldsymbol{d},-\boldsymbol{c})\\
=&5det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})
\end{aligned}
$$

习题 1.13 (第三章第 9 题)

求以下排列的逆序数，并指出其奇偶性. (1) (6,8,1,4,7,5,3,2,9)　(2) (6,4,2,1,9,7,3,5,8)　(3) (7,5,2,3,9,8,1,6,4)

解 (1) $\tau(6,8,1,4,7,5,3,2,9)=19$，为奇排列；
(2) $\tau(6,4,2,1,9,7,3,5,8)=15$，为奇排列；
(3) $\tau(7,5,2,3,9,8,1,6,4)=20$，为偶排列.

习题 1.14 (第三章第 13 题)

用 Cramer 法则求解下列线性方程组：

$$
(1)\begin{cases}x_1-x_2+x_3=3\\x_1+2x_2+4x_3=5\\x_1+3x_2+9x_3=7\end{cases}\qquad
(2)\begin{cases}2x_1+x_2-5x_3+x_4=8\\x_1-3x_2-6x_4=9\\2x_2-x_3+2x_4=-5\\x_1+4x_2-7x_3+6x_4=0\end{cases}
$$

解 (1)

$$\Delta=\begin{vmatrix}1&-1&1\\1&2&4\\1&3&9\end{vmatrix}=12 \tag{1.10}$$

$$\Delta_1=\begin{vmatrix}3&-1&1\\5&2&4\\7&3&9\end{vmatrix}=36 \tag{1.11}$$

$$\Delta_2=\begin{vmatrix}1&3&1\\1&5&4\\1&7&9\end{vmatrix}=4 \tag{1.12}$$

$$\Delta_3=\begin{vmatrix}1&-1&3\\1&2&5\\1&3&7\end{vmatrix}=4 \tag{1.13}$$

于是方程组的解为

$$(x_1,x_2,x_3)=(\frac{\Delta_1}{\Delta},\frac{\Delta_2}{\Delta},\frac{\Delta_3}{\Delta})=(3,\frac{1}{3},\frac{1}{3}) \tag{1.14}$$

(2)

$$\Delta = \begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{vmatrix} = 27 \tag{1.15}$$

$$\Delta_1 = \begin{vmatrix} 8 & 1 & -5 & 1 \\ 9 & -3 & 0 & -6 \\ -5 & 2 & -1 & 2 \\ 0 & 4 & -7 & 6 \end{vmatrix} = 81 \tag{1.16}$$

$$\Delta_2 = \begin{vmatrix} 2 & 8 & -5 & 1 \\ 1 & 9 & 0 & -6 \\ 0 & -5 & -1 & 2 \\ 1 & 0 & -7 & 6 \end{vmatrix} = -108 \tag{1.17}$$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 8 & 1 \\ 1 & -3 & 9 & -6 \\ 0 & 2 & -5 & 2 \\ 1 & 4 & 0 & 6 \end{vmatrix} = -27 \tag{1.18}$$

$$\Delta_4 = \begin{vmatrix} 2 & 1 & -5 & 8 \\ 1 & -3 & 0 & 9 \\ 0 & 2 & -1 & -5 \\ 1 & 4 & -7 & 0 \end{vmatrix} = 27 \tag{1.19}$$

于是方程组的解为

$$(x_1, x_2, x_3, x_4) = (\frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}, \frac{\Delta_3}{\Delta}, \frac{\Delta_4}{\Delta}) = (3, -4, -1, 1) \tag{1.20}$$

习题 1.15 (第三章第 14 题)

设 $x_0, x_1, \cdots, x_n$ 及 $y_0, y_1, \cdots, y_n$ 是任给实数，其中 $x_i (0 \leq i \leq n)$ 两两互不相等. 证明：存在唯一的次数不超过 n 的多项式 $p(x)$ 满足 $p(x_i) = y_i, i = 0, 1, \cdots, n$.

证明 设 $p(x) = a_0 + a_1 x + \cdots + a_n x^n$，命题等价于证明以下 $(n+1)$ 元一次方程组有唯一解 (设 $a_0, a_1, \cdots, a_n$ 为未知数):

$$\begin{cases} a_0 + x_0 a_1 + \cdots + x_0^n a_n = y_0 \\ a_0 + x_1 a_1 + \cdots + x_1^n a_n = y_1 \\ \quad \vdots \\ a_n + x_n a_1 + \cdots + x_n^n a_n = y_n \end{cases} \tag{1.21}$$

注意到方程组 (1.21) 的系数行列式为 (n+1) 阶 Vandermonde 行列式，有

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{vmatrix} = \prod_{0 \leq i < j \leq n} (x_j - x_i) \neq 0 \tag{1.22}$$

由 Cramer 法则知方程组有唯一解 $a_i = \frac{\Delta_{i+1}}{\Delta} \quad (0 \leq i \leq n)$.

注 实际上我们由此得到了拉格朗日插值多项式

习题 1.16 (第三章第 16 题)

计算下列 n 阶行列式

$$(1)\ \begin{vmatrix} a_1 & & & & & b_1 \\ & \ddots & & & \iddots & \\ & & a_n & b_n & & \\ & & c_n & d_n & & \\ & \iddots & & & \ddots & \\ c_1 & & & & & d_1 \end{vmatrix} \qquad (2)\ \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+a_n \end{vmatrix}$$

$$(3)\ \begin{vmatrix} 2cos(\theta) & 1 & 0 & \cdots & 0 \\ 1 & 2cos(\theta) & 1 & \cdots & 0 \\ 0 & 1 & 2cos(\theta) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2cos(\theta) \end{vmatrix}$$

解 (1)

$$\begin{vmatrix} a_1 & & & & & b_1 \\ & \ddots & & & \iddots & \\ & & a_n & b_n & & \\ & & c_n & d_n & & \\ & \iddots & & & \ddots & \\ c_1 & & & & & d_1 \end{vmatrix} = (-1)^{2n-2} \begin{vmatrix} a_1 & & & & & & & b_1 \\ c_1 & & & & & & & d_1 \\ & a_2 & & & & & b_2 & \\ & & \ddots & & & \iddots & & \\ & & & a_n & b_n & & & \\ & & & c_n & d_n & & & \\ & & \iddots & & & \ddots & & \\ & c_2 & & & & & d_2 & \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & & & & & & \\ c_1 & d_1 & & & & & & \\ & & a_2 & & & & & b_2 \\ & & & \ddots & & & \iddots & \\ & & & & a_n & b_n & & \\ & & & & c_n & d_n & & \\ & & & \iddots & & & \ddots & \\ & & c_2 & & & & & d_2 \end{vmatrix} = \cdots$$

$$= \begin{vmatrix} a_1 & b_1 & & & & & \\ c_1 & d_1 & & & & & \\ & & a_2 & b_2 & & & \\ & & c_2 & d_2 & & & \\ & & & & \ddots & & \\ & & & & & a_n & b_n \\ & & & & & c_n & d_n \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \times \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \times \cdots \times \begin{vmatrix} a_n & b_n \\ c_n & d_n \end{vmatrix} \\
&= \prod_{i=1}^{n}(a_i d_i - c_i b_i) \qquad (1.23)
\end{aligned}
$$

(2) 分以下三种情况讨论：

a. 当 $a_1, a_2, \cdots, a_n$ 中存在两个及以上的 0，那么此时行列式有两行及以上相等，行列式的值为 0；

b. 当 $a_1, a_2, \cdots, a_n$ 中有一个 0 (设为 $a_k(1 \le k \le n)$)，而其余元素非零，此时行列式第 k 行元素均为 1，有：

$$
\begin{vmatrix}
1+a_1 & 1 & \cdots & \cdots & \cdots & 1 \\
1 & 1+a_2 & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1+a_n
\end{vmatrix}
\xlongequal{-r_k \to r_1, \cdots, r_{k-1}, r_{k+1}, \cdots, r_n}
\begin{vmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & a_2 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_n
\end{vmatrix}
= \prod_{i \neq k} a_i
$$

注 a 和 b 两种情况也可以合并讨论

c. 当 $a_1, a_2, \cdots, a_n$ 均非零：

$$
\begin{vmatrix}
1+a_1 & 1 & \cdots & 1 \\
1 & 1+a_2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1+a_n
\end{vmatrix}
\xlongequal{-r_1 \to r_2, \cdots, r_n}
\begin{vmatrix}
1+a_1 & 1 & \cdots & 1 \\
-a_1 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_1 & 0 & \cdots & a_n
\end{vmatrix}
$$

$$
\xlongequal{-\frac{1}{a_2} r_2 - \cdots - \frac{1}{a_n} r_n \to r_1}
\begin{vmatrix}
1+a_1+a_1\sum_{i=2}^{n}\frac{1}{a_i} & 0 & \cdots & 0 \\
-a_1 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_1 & 0 & \cdots & a_n
\end{vmatrix}
= (\prod_{i=1}^{n} a_i)(1 + \sum_{i=1}^{n} \frac{1}{a_i})
$$

(3) 我们将 n 阶这样的行列式记为 K_n，易知 $K_1 = 2cos\theta, K_2 = 4cos^2\theta - 1$，当 $n \ge 3$ 时，将 K_n 按照第一行展开，得：

$$
K_n = 2cos\theta K_{n-1} - K_{n-2} \qquad (1.24)
$$

方法一：这是一个二阶线性递推数列，其特征方程为：

$$
\lambda^2 = 2cos\theta\lambda - 1 \qquad (1.25)
$$

解特征方程得到特征根：

$$\lambda = e^{\pm i\theta} \tag{1.26}$$

下面分类讨论：

a. $\theta = 2m\pi(m \in Z)$，此时特征根为重根 $\lambda_1 = \lambda_2 = 1$，且 $cos\theta = 1$，数列的递推公式写为：$K_n = 2K_{n-1} - K_{n-2}$，易知 $K_n = n + 1$.

b. $\theta = (2m+1)\pi(m \in Z)$，此时特征根为重根 $\lambda_1 = \lambda_2 = -1$，且 $cos\theta = -1$，数列的递推公式写为：$K_n = -2K_{n-1} - K_{n-2}$，易知 $K_n = (-1)^n(n+1)$.

c. $\theta \neq m\pi(m \in Z)$，此时两个特征根不等：$\lambda_1 = e^{i\theta}, \lambda_2 = e^{-i\theta}, \lambda_1 \neq \lambda_2$

设 $K_n = c_1\lambda_1^n + c_2\lambda_2^n$，将 K_1, K_2 代入解得 $c_1 = \frac{1-e^{i2\theta}}{2-2cos(2\theta)}, c_2 = \frac{1-e^{-i2\theta}}{2-2cos(2\theta)}$，则 $K_n = \frac{cos(n\theta)-cos((n+2)\theta)}{1-\cos(2\theta)} = \frac{2sin((n+1)\theta)sin\theta}{2sin^2\theta} = \frac{sin((n+1)\theta)}{sin\theta}$

方法二：$cos\theta = \pm 1$ 时与方法一类似，不再赘述.

当 $\sin\theta \neq 0$ 时观察可知 $K_1 = \frac{sin(2\theta)}{\sin\theta}, K_2 = \frac{sin(3\theta)}{sin\theta}$，则我们猜测 $K_n = \frac{sin((n+1)\theta)}{sin\theta}$. 下面进行数学归纳法：

假设 $m < n(n \geq 3)$ 时均有 $K_m = \frac{sin((m+1)\theta)}{sin\theta}$ 成立，则

$$\begin{aligned}
K_n &= 2cos\theta K_{n-1} - K_{n-2} \\
&= 2cos\theta\frac{sin(n\theta)}{sin\theta} - \frac{sin((n-1)\theta)}{sin\theta} \\
&= \frac{2cos\theta\sin(n\theta) - sin((n-1)\theta)}{sin\theta} \\
&= \frac{sin((n+1)\theta)}{sin\theta}
\end{aligned} \tag{1.27}$$

由归纳法原理知命题成立.

注 本题为三对角行列式的一种特殊情况，类似本题使用的方法可以计算一般的三对角行列式：

$$\begin{vmatrix} a & b & & \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{vmatrix}_n = \begin{cases} (n+1)(\frac{a}{2})^n & a^2 = 4bc \\ \dfrac{(a+\sqrt{a^2-4bc})^{n+1} - (a-\sqrt{a^2-4bc})^{n+1}}{2^{n+1}\sqrt{a^2-4bc}} & a^2 \neq 4bc \end{cases} \tag{1.28}$$

1.3.2 附录

对于最后一道题用到的知识的一些补充

二阶线性递推数列

一般的二阶线性递推数列 $\{a_n\}$ 的递推关系写为：

$$a_n = pa_{n-1} + qa_{n-2} \quad (n \geq 3, p \neq 0, q \neq 0) \tag{1.29}$$

若已知 a_1, a_2，怎样求通项公式？

我们尝试将递推关系写为等比关系的形式如下：

$$a_n - \lambda_1 a_{n-1} = \lambda_2(a_{n-1} - \lambda_1 a_{n-2}) \tag{1.30}$$

从而可以列出方程组：

$$\begin{cases} \lambda_1 + \lambda_2 = p \\ \lambda_1\lambda_2 = -q \end{cases} \tag{1.31}$$

由韦达定理可知 λ_1, λ_2 是下面二次方程的两个根

$$\lambda^2 - p\lambda - q = 0 \tag{1.32}$$

我们将式 (1.32) 称为特征方程，λ_1, λ_2 称为特征根，可见特征方程与递推关系式类似，只是将 a_n, a_{n-1}, a_{n-2} 分别替换为了 $\lambda^2, \lambda, 1$.

通过特征方程解出特征根后，有

$$a_n - \lambda_1 a_{n-1} = \lambda_2(a_{n-1} - \lambda_1 a_{n-2}) = \lambda_2^2(a_{n-2} - \lambda_1 a_{n-3}) = \cdots = \lambda_2^{n-2}(a_2 - \lambda_1 a_1) \tag{1.33}$$

观察可知式 (1.30) 可变形为

$$a_n - \lambda_2 a_{n-1} = \lambda_1(a_{n-1} - \lambda_2 a_{n-2}) \tag{1.34}$$

则同理有

$$a_n - \lambda_2 a_{n-1} = \lambda_1^{n-2}(a_2 - \lambda_2 a_1) \tag{1.35}$$

当 $\lambda_1 \neq \lambda_2$ 时，联立式 (2.3)(1.35) 得

$$a_n = \frac{a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2}\lambda_1^{n-1} - \frac{a_2 - \lambda_1 a_1}{\lambda_1 - \lambda_2}\lambda_2^{n-1} \tag{1.36}$$

当 $\lambda_1 = \lambda_2 = \lambda$ 时，有

$$a_n - \lambda a_{n-1} = \lambda(a_{n-1} - \lambda a_{n-2}) = \lambda^2(a_{n-2} - \lambda a_{n-3}) = \cdots = \lambda^{n-2}(a_2 - \lambda a_1) \tag{1.37}$$

此时

$$\begin{aligned} a_n &= (a_n - \lambda a_{n-1}) + \lambda(a_{n-1} - \lambda a_{n-2}) + \lambda^2(a_{n-2} - \lambda a_{n-3}) + \cdots + \lambda^{n-2}(a_2 - \lambda a_1) + \lambda^{n-1}a_1 \\ &= (n-1)\lambda^{n-2}(a_2 - \lambda a_1) + \lambda^{n-1}a_1 \end{aligned} \tag{1.38}$$

观察式 (1.36)(1.38) 可见，通项公式 a_n 是以 $a_1, a_2, \lambda_1, \lambda_2$ 为参数，n 为自变量的函数，我们下面给出求通项公式的一般程式：

(1) 求特征方程 $\lambda^2 - p\lambda - q = 0$ 的根 λ_1, λ_2，在上述证明过程中，我们并未将根限定在实数范围内，实际上，在复数域中，二次方程必有两根.

(2) 若 $\lambda_1 \neq \lambda_2$，通项公式可写为 $a_n = c_1\lambda_1^n + c_2\lambda_2^n$；若 $\lambda_1 = \lambda_2 = \lambda$，通项公式可写为 $(c_1 + c_2 n)\lambda^n$. 其中 c_1, c_2 为待定系数.

(3) 将 $n = 1, 2$ 代入通项公式,得到两个方程构成的方程组,可由此方程组求得 c_1, c_2,其表达式含 $a_1, a_2, \lambda_1, \lambda_2$,即通项公式的参数.

至此，通项公式完全求出.

复数运算

¶ 实系数一元二次方程的复数解

在复数域中，任一实系数二次方程 $ax^2 + bx + c = 0$ 都存在两根，当其判别式 $\Delta = b^2 - 4ac < 0$ 时，根为两共轭复数：

$$x_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad x_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a} \tag{1.39}$$

¶ 欧拉公式

我们知道欧拉公式写为：

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{1.40}$$

从而 $e^{-i\theta} = cos(-\theta) + isin(-\theta) = cos(\theta) - isin(\theta)$，反解得到：

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.41}$$