1.2 第四周作业

习题 1.5 (第三章第 1 题)

计算下列行列式：

$$(1)\begin{vmatrix} 1 & 0 & 1 & -4 \\ -1 & -3 & 0 & -2 \\ 2 & -1 & 4 & 0 \\ 0 & 3 & -3 & 2 \end{vmatrix}\quad (2)\begin{vmatrix} 1 & 4 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ -3 & 3 & -4 & 0 \\ 0 & 1 & -1 & -1 \end{vmatrix}\quad (3)\begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix}$$

$$(4)\begin{vmatrix} & & & a_{1n} \\ & & a_{2,n-1} & a_{2n} \\ & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}\quad (5)\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix}\quad (6)\begin{vmatrix} a_1 & a_2 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 \\ d_1 & d2 & d_3 & d_4 & 0 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

解 (1)

$$\begin{vmatrix} 1 & 0 & 1 & -4 \\ -1 & -3 & 0 & -2 \\ 2 & -1 & 4 & 0 \\ 0 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & -4 \\ 0 & -3 & 1 & -6 \\ 2 & -1 & 4 & 0 \\ 0 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & -4 \\ 0 & -3 & 1 & -6 \\ 0 & -1 & 2 & 8 \\ 0 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} -3 & 1 & -6 \\ -1 & 2 & 8 \\ 3 & -3 & 2 \end{vmatrix} = -40.$$

(2)

$$\begin{vmatrix} 1 & 4 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ -3 & 3 & -4 & 0 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -1 & -1 \\ 0 & -4 & 1 & 2 \\ -3 & 3 & -4 & 0 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -1 & -1 \\ 0 & -4 & 1 & 2 \\ 0 & 15 & -7 & -3 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -4 & 1 & 2 \\ 15 & -7 & -3 \\ 1 & -1 & -1 \end{vmatrix} = -20.$$

(3)

(法一) $$\begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = \begin{vmatrix} x-y & x-y & x-y \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = (x-y)\begin{vmatrix} 1 & 1 & 1 \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = (x-y)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a & b & c \end{vmatrix} = 0.$$

(法二)可以将行列式

$$f(x) \triangleq \begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix}$$

看成系数为 $\mathbb{F}[y,z]$ 关于 x 的多项式. 根据行列式的完全展开式知 $f(x)$ 为关于 x 的一次多项式且有两个根 y, z. 故 $f(x)=0$.

(4)

$$\begin{vmatrix} & & & a_{1n} \\ & & a_{2,n-1} & a_{2n} \\ & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^{n+1}a_{1n}\begin{vmatrix} & & a_{2,n-1} \\ & & \vdots \\ a_{n1} & \cdots & a_{n-1,n-1} \end{vmatrix} = \cdots = (-1)^{\frac{n(n-1)}{2}}a_{1n}a_{2,n-1}\cdots a_{n1}.$$

(5) 考虑如下多项式函数

$$f(x) \triangleq \begin{vmatrix} x^2 & (x+1)^2 & (x+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix}$$

由行列式的完全展开式知 $f(x)$ 为关于 x 的二次多项式. 易知 $f(b)=f(c)=0$, 故可设 $f(x)=\lambda(x-b)(x-c)$.

$$\lambda bc = f(0) = \begin{vmatrix} 0 & 1 & 4 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 4 \\ b^2 & b^2+b & b^2 \\ c^2 & c^2+c & c^2 \end{vmatrix} = -\begin{vmatrix} b^2 & b^2 \\ c^2 & c^2 \end{vmatrix} + 4\begin{vmatrix} b^2 & b^2+b \\ c^2 & c^2+c \end{vmatrix} = 4bc(b-c)$$

故 $\lambda = 4(b-c)$, 于是 $f(x)=4(b-c)(x-b)(x-c)$. 带入 $x=a$ 有

$$f(a) = \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = 4(b-c)(a-b)(a-c).$$

(6) 由行列式的 *Laplace* 展开

$$\begin{vmatrix} a_1 & a_2 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 \\ d_1 & d2 & d_3 & d_4 & 0 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_3 & 0 & 0 \\ d_3 & d_4 & 0 \\ e_3 & e_4 & e_5 \end{vmatrix} = (a_1b_2 - a_2b_1)c_3d_4e_5.$$

习题 1.6 (第三章第 2 题)

在三维直角坐标系中，已知点 A,B,C,D 的坐标分别是（1，1，0)，（3，1，2)，（0，1，3)，（2，2，4）. 求四面体 $ABCD$ 的体积及各个面的面积.

解 $\overrightarrow{AB}=(2,0,2), \overrightarrow{AC}=(-1,0,3), \overrightarrow{AD}=(1,1,4), \overrightarrow{BC}=(-3,0,1), \overrightarrow{BD}=(-1,1,2)$.

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 2 \\ -1 & 0 & 3 \end{vmatrix} = (0,-8,0)$$

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 2 \\ 1 & 1 & 4 \end{vmatrix} = (-2,-6,2)$$

$$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-3,7,-1)$$

$$\overrightarrow{BC}\times\overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} = (-1,5,-3)$$

$S_{\triangle ABC} = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = 4, S_{\triangle ABD} = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{11}, S_{\triangle ACD} = \frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{AD}| = \frac{\sqrt{59}}{2}, S_{\triangle BCD} =$

$\frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BD}| = \frac{\sqrt{35}}{2}$.

$$V_{ABCD} = \frac{1}{6}|(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}| = \frac{1}{6}\text{abs}(\begin{vmatrix} 2 & 0 & 2 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix}) = \frac{4}{3}.$$

习题 1.7 (第三章第 3 题)

将行列式

$$\begin{vmatrix} x-2 & 1 & 0 & 0 \\ 1 & x-2 & 1 & 0 \\ 0 & 1 & x-2 & 1 \\ 0 & 0 & 1 & x-2 \end{vmatrix}$$

展开为关于 x 的多项式.

解

$$\begin{vmatrix} x-2 & 1 & 0 & 0 \\ 1 & x-2 & 1 & 0 \\ 0 & 1 & x-2 & 1 \\ 0 & 0 & 1 & x-2 \end{vmatrix} = \begin{vmatrix} 0 & -x^2+4x-3 & 2-x & 0 \\ 1 & x-2 & 1 & 0 \\ 0 & 1 & x-2 & 1 \\ 0 & 0 & 1 & x-2 \end{vmatrix} = \begin{vmatrix} 1 & x-2 & 1 & 0 \\ 0 & x^2-4x+3 & x-2 & 0 \\ 0 & 1 & x-2 & 1 \\ 0 & 0 & 1 & x-2 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-4x+3 & x-2 & 0 \\ 1 & x-2 & 1 \\ 0 & 1 & x-2 \end{vmatrix} = (x^2-4x+3)\begin{vmatrix} x-2 & 1 \\ 1 & x-2 \end{vmatrix} - (x-2)\begin{vmatrix} 1 & 1 \\ 0 & x-2 \end{vmatrix} = (x^2-4x+3)^2-(x-2)^2$$

$= (x^2-3x+1)(x^2-5x+5)$.

习题 1.8 (第三章第 4 题)

A 为 n 阶方阵，λ 为常数. 证明：$\det(\lambda A) = \lambda^n \det(A)$

证明 由矩阵数乘的定义及行列式映射的多重线性性知 $\det(\lambda A) = \lambda^n \det(A)$.

习题 1.9 (第三章第 5 题)

方阵 A 称为反对称方阵，如果它的转置方阵等于 $-A$. 证明：奇数阶反对称方阵的行列式为零.

证明 由第四题知 $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$. 由于 n 为奇数，因此 $\det(A) = -\det(A)$. 故 $2\det(A) = 0$, 即 $\det(A) = 0$.