COMP2521 19T0
Week 7, Tuesday: A Question of Balance

Jashank Jeremy

jashank.jeremy@unsw.edu.au

radix sort
balanced trees

Sorting

Radix Sorting

Can we decompose our keys?
Radix sorts let us deal with this case.

Keys are values in some base-R number system.
e.g., binary, $R = 2$; decimal, $R = 10$;
ASCII, $R = 128$ or $R = 256$; Unicode, $R = 2^{16}$

Sorting individually on each part of the key at a time:
digit-by-digit, character-by-character, rune-by-rune, *etc.*

Radix Sorting, Most-Significant-Digit First

Consider characters, digits, bits, runes, *etc.*,
from left to right;
partitioning input into R pieces according to `key.0`;
recurse into each piece, using succesive keys —
`key.1`, `key.2`, ..., `key.`w

1019	2301	3129	2122
1019	**2**301	**3**129	**2**122
1019	**23**01	**21**22	**31**29
1019	2122	2301	3129

with $R = 2$, roughly a quicksort.

Radix Sorting, Least-Significant-Digit First

Consider characters, digits, bits, runes, *etc.*,
from right to left;
use a **stable** sort using the dth digit as key,
using (*e.g.*,) key-indexed counting sort.

1019	2301	3129	2122
101**9**	230**1**	312**9**	212**2**
23**0**1	21**2**2	10**1**9	31**2**9
2**3**01	1**0**19	2**1**22	3**1**29
1019	**2**122	**3**129	**2**301
1019	2122	2301	3129

this *will not work* if the sort is not stable!

Radix Sort

Analysis; Summary

Complexity: $O\left(w\left(n+R\right)\right) \approx O(n)$,
where w is the 'width' of data;
the algorithm makes w passes over n keys

LSD

Not in-place: $O(n+R)$ extra space required.
May be stable! Usable on variable length data.

MSD

Not in-place: $O(n+DR)$ extra space required.
(D is the recursion depth.)
May be stable! Usable on variable length data.
Can complete *before* examining all of all keys.

Balanced Trees

Recap: The Search Problem

input a key value

output item(s) containing that key

Common variations:

- keys are unique; key value matches 0 or 1 items
- multiple keys in search, items containing any key
- multiple keys in search/item, items containing all keys

We assume: keys are unique, each item has one key.

Recap: Trees; Binary Trees

Trees are branched data structures,
consisting of nodes and edges, with no cycles.

Each node contains a value.
Each node has edges to $\leq k$ other nodes.
For now, $k = 2$ — binary trees

Trees can be viewed as a set of nested structures:
each node has k (possibly empty) subtrees.

Recap: Binary Search Trees

For all nodes in the tree:

the values in the left subtree
are less than the node value the

values in the right subtree
are greater than the node value

A binary tree of n nodes
is degenerate if its height is
at most $n - 1$.

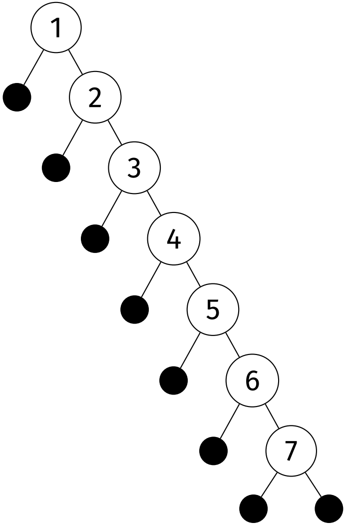
A binary tree of n nodes
is balanced if its height is
at least $\lfloor \log_2 n \rfloor$.

Structure tends to be determined
by order of insertion:
$[4, 2, 1, 3, 6, 5, 7]$ vs $[6, 5, 2, 1, 3, 4, 7]$

Binary Search Trees

The Worst Case

Ascending-ordered or
descending-ordered data
is a pathological case:
we always right- or left-insert
along the spine of the tree.



Binary Search Trees

Performance

Cost for insertion:
balanced $O(\log_2 n)$, degenerate $O(n)$
(we always traverse the height of the tree)

Cost for search/deletion:
balanced $O(\log_2 n)$, degenerate $O(n)$
(worst case, key $\notin \tau$; traverse the height)

We want to build balanced trees.

What is 'Balance', then?

PERFECTLY BALANCED

a *weight-balanced* or *size-balanced* tree has, for every node,

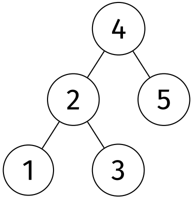
$$\left|\textsc{size}\left(l\right) - \textsc{size}\left(r\right)\right| < 2$$

LESS STRINGENTLY

a *height-balanced* tree has, for every node,

$$\left|\textsc{height}\left(l\right) - \textsc{height}\left(r\right)\right| < 2$$

Balanced or Not? (I)



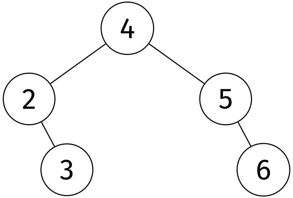
$$\text{SIZE}\left(\tau_4\right) = 5$$
$$\text{SIZE}\left(\tau_2\right) = 3$$
$$\text{SIZE}\left(\tau_5\right) = 1$$
$$\text{SIZE}\left(\tau_1\right) = 1$$
$$\text{SIZE}\left(\tau_3\right) = 1$$
$$\text{SIZE}\left(\tau_2\right) - \text{SIZE}\left(\tau_5\right) = 2$$

NOT SIZE BALANCED

$$\text{HEIGHT}\left(\tau_4\right) = 2$$
$$\text{HEIGHT}\left(\tau_2\right) = 1$$
$$\text{HEIGHT}\left(\tau_5\right) = 0$$
$$\text{HEIGHT}\left(\tau_1\right) = 0$$
$$\text{HEIGHT}\left(\tau_3\right) = 0$$
$$\text{HEIGHT}\left(\tau_2\right) - \text{HEIGHT}\left(\tau_5\right) = 1$$

HEIGHT BALANCED

Balanced or Not? (II)



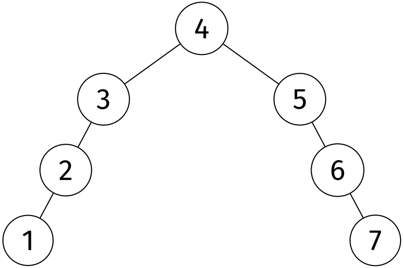
$$\text{SIZE}\left(\mathcal{T}_4\right) = 5$$
$$\text{SIZE}\left(\mathcal{T}_2\right) = 2$$
$$\text{SIZE}\left(\mathcal{T}_5\right) = 2$$
$$\text{SIZE}\left(\mathcal{T}_3\right) = 1$$
$$\text{SIZE}\left(\mathcal{T}_6\right) = 1$$

SIZE BALANCED

$$\text{HEIGHT}\left(\mathcal{T}_4\right) = 2$$
$$\text{HEIGHT}\left(\mathcal{T}_2\right) = 1$$
$$\text{HEIGHT}\left(\mathcal{T}_5\right) = 1$$
$$\text{HEIGHT}\left(\mathcal{T}_3\right) = 0$$
$$\text{HEIGHT}\left(\mathcal{T}_6\right) = 0$$

HEIGHT BALANCED

Balanced or Not? (III)



Let's look at τ_3.

$$\text{SIZE}\left(\tau_2\right) = 2$$
$$\text{SIZE}\left(\tau_\varnothing\right) = 0$$
$$2 - 0 = 2 \not< 2$$

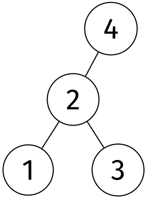
NOT SIZE BALANCED

Let's look at τ_5.

$$\text{HEIGHT}\left(\tau_\varnothing\right) = 0$$
$$\text{HEIGHT}\left(\tau_6\right) = 1$$
$$|0 - 1| = 1 < 2$$

HEIGHT BALANCED

Balanced or Not? (IV)



Let's look at τ_4.

$$\text{SIZE}\left(\tau_2\right) = 3$$

$$\text{SIZE}\left(\tau_\varnothing\right) = 0$$

$$3 - 0 = 3 \nless 2$$

NOT SIZE BALANCED

Let's look at τ_4.

$$\text{HEIGHT}\left(\tau_2\right) = 1 \;\text{HEIGHT}\left(\tau_\varnothing\right) = 0$$

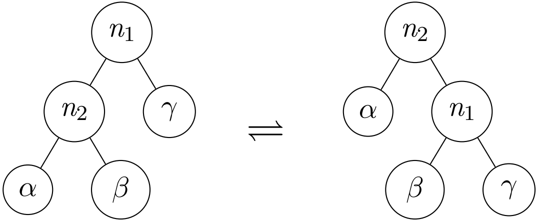
$$1 - 0 = 1 < 2$$

HEIGHT BALANCED

Rotations

Rebalancing Primitives

LEFT ROTATION and **RIGHT ROTATION**:
a pair of ‘primitive’ operations
that change the balance of a tree
whilst maintaining a search tree.



$$(n_1, (n_2, \alpha, \beta), \gamma) \rightleftharpoons (n_2, \alpha, (n_1, \beta, \gamma))$$

Rotations

Rotating Right

```
btree_node *btree_rotate_right (btree_node *n1)
{
    if (n1 == NULL) return NULL;
    btree_node *n2 = n1->left;
    if (n2 == NULL) return n1;
    n1->left = n2->right;
    n2->right = n1;
    return n2;
}
```

n_1 starts as the root of this subtree and is demoted;
n_2 starts as the left subtree of this tree, and is promoted.

Rotations

Rotating Left

```
btree_node *btree_rotate_left (btree_node *n2)
{
    if (n2 == NULL) return NULL;
    btree_node *n1 = n2->right;
    if (n1 == NULL) return n2;
    n2->right = n1->left;
    n1->left = n2;
    return n1;
}
```

n_2 starts as the root of this subtree and is demoted;
n_1 starts as the right subtree of this tree, and is promoted.

Partition

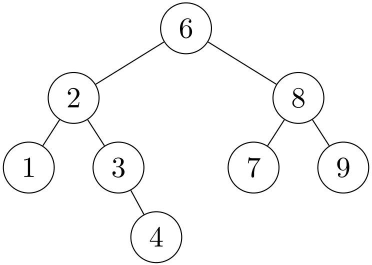
Rotation in Context

A way to brute-force some balance into a tree:
lifting some kth index to the root.

$$
\begin{array}{l}
\textsc{partition} :: \text{BTree} \to \text{Word} \to \text{BTree} \\
\textsc{partition}\ \text{Empty}\ k = \text{Empty} \\
\textsc{partition}\ \ (\text{Node}\ n\ l\ r)\ \ k \\
\quad \mid k < \textsc{size}\, l = \textsc{rotate}\text{R}\ \ (\text{Node}\ n\ \ (\textsc{partition}\, l\, k)\ \ r) \\
\quad \mid \textsc{size}\, l < k = \textsc{rotate}\text{L}\ \ (\text{Node}\ n\ l\ \ (\textsc{partition}\, r\ \ (k - 1 - \textsc{size}\ l))) \\
\quad \mid \textsf{otherwise} = \text{Node}\ n\ l\ r
\end{array}
$$

Partition

Partition in Context



What happens if we partition at index 3 (node 4)?

Partition

C Implementation

```
btree_node *btree_partition (btree_node *tree, size_t k)
{
    if (tree == NULL) return NULL;
    size_t lsize = size (tree->left);
    if (lsize > k) {
        tree->left = btree_partition (tree->left, k);
        tree = btree_rotate_right (tree);
    }
    if (lsize < k) {
        tree->right = btree_partition (tree->right, k - 1 - lsize);
        tree = btree_rotate_left (tree);
    }
    return tree;
}
```

Primitives

With our primitive operations in hand —

$$\textsc{rotate}\mathrm{L} :: \mathrm{BTree} \to \mathrm{BTree}$$
$$\textsc{rotate}\mathrm{R} :: \mathrm{BTree} \to \mathrm{BTree}$$
$$\textsc{partition} :: \mathrm{BTree} \to \mathrm{Word} \to \mathrm{BTree}$$

— let's go balance some trees!

Approach #1: Global Rebalancing

Move the median node to the root,
by partitioning on $\textsc{size}\,\tau/2$;
then, balance the left subtree,
and balance the right subtree.

```
btree_node *btree_balance_global (btree_node *tree)
{
    if (tree == NULL) return NULL;
    if (size (tree) < 2) return tree;
    tree = partition (tree, size (tree) / 2);
    tree->left = btree_balance_global (tree->left);
    tree->right = btree_balance_global (tree->right);
    return tree;
}
```

Approach #1: Global Rebalancing

Problems

- cost of rebalancing:
  for many trees, $O\left(n\right)$; for degenerate trees, $O\left(n \log n\right)$

- what if we insert more keys?
  - rebalance on every insertion
  - rebalance every k insertions; what k is good?
  - rebalance when imbalance exceeds threshold.

  we either have more costly instertions
  or degraded performance for (possibly unbounded) periods.
  … given a sufficiently dynamic tree, sadness.

Global vs Local Rebalancing

GLOBAL REBALANCING

walks every node, balances its subtree;
$\Rightarrow$ perfectly balanced tree — at cost.

LOCAL REBALANCING

do small, incremental operations
to improve the overall balance of the tree
... at the cost of imperfect balance

Local Rebalancing Approaches

amortisation: do (a small amount) more work now to avoid more work later

randomisation: use randomness to reduce impact of BST worst cases

optimisation: maintain structural information for performance

Root Insertion

How do we insert a node at the root of a tree?
(Without having to rearrange all the nodes?)

We do a leaf insertion ...
... and rotate the new node up the tree.

More work? **No!**
Same complexity as leaf insertion,
but more actual work is done: amortisation.

(Side-effect: recently-inserted items are close to the root.
Depending on what you're doing, this might be very useful!)

Root Insertion

C Implementation

```
btree_node *btree_insert_root (btree_node *tree, Item it)
{
    if (tree == NULL)
        return btree_node_new (it, NULL, NULL);
    if (less (it, tree->value)) {
        tree->left = btree_insert_root (tree->left, it);
        tree = btree_rotate_right (tree);
    } else {
        tree->right = btree_insert_root (tree->right, it);
        tree = btree_rotate_left (tree);
    }
    return tree;
}
```

Randomised Insertion

BSTs don't have control over insertion order.
worst cases — (partially) ordered data — are common.

to minimise the likelihood of a degenerate tree,
we randomly choose which level to insert a node;
at each level, probability depends on remaining tree size.

do a 'normal' leaf insertion, most of the time.
randomly (with a certain probability),
do a root insertion of a value.

Randomised Insertion

C Implementation

```
btree_node *btree_insert_rand (btree_node *tree, Item it)
{
    if (tree == NULL)
        return btree_node_new (it, NULL, NULL);
    if (rand () < (RAND_MAX / size (tree)))
        return btree_insert_root (tree, it);
    else if (less (it, tree->value))
        tree->left = btree_insert_rand (tree->left, it);
    else
        tree->right = btree_insert_rand (tree->right, it);
    return tree;
}
```

Randomised Insertion

Properties

building a randomised BST is equivalent to
building a standard BST with
a random initial permutation of keys.

worst-case, best-case, average-case performance:
same as a standard BST —
but with no penalty for ordering!

Randomised Deletion

We could do something similar for deletion:
when choosing a node to promote,
choose randomly from the
in-order predecessor or successor

Splay Trees

Root insertion can still leave us
with a degenerate tree.

Splay trees vary root-insertion,
by considering *three* levels of the tree
— parent, child, grandchild —
and performing double-rotations based on p-c-g orientation;
the idea: double-rotations improve balance.

No guarantees, but *improved* performance.

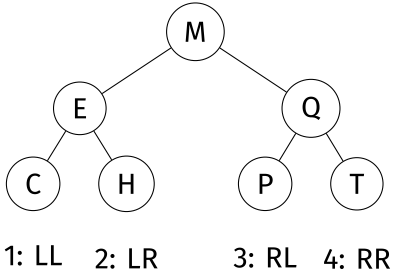
"... their performance is amortised by
the amount of effort required to understand them."

— me, 2016

Splay Trees

Double-Rotation Cases

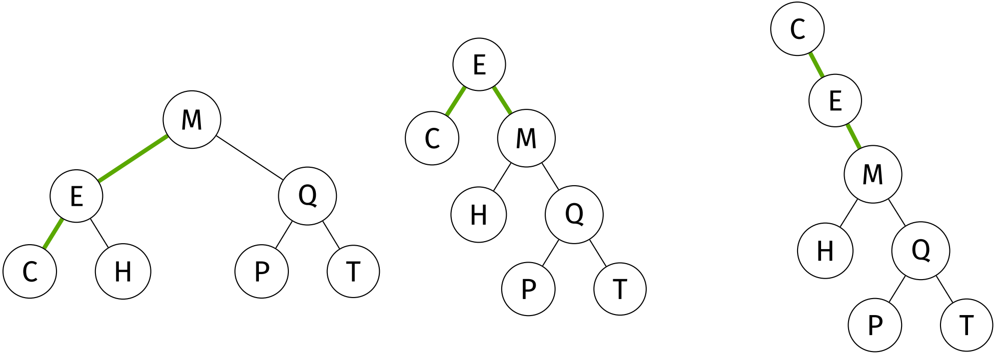
Four choices to consider for a double-rotation:



Splay Rotations

Double-Rotation: Left, Left

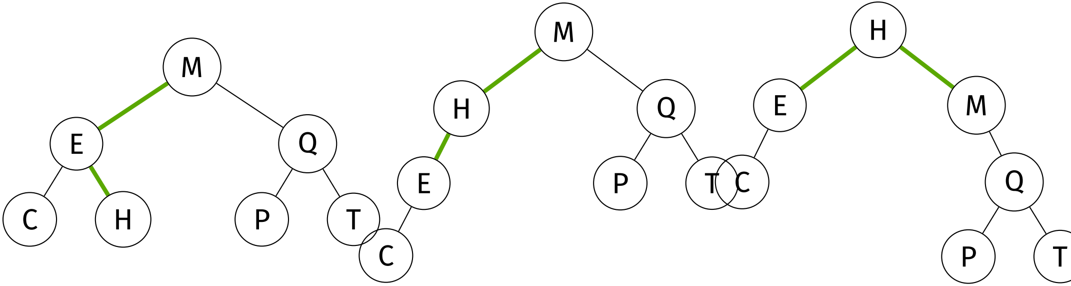
$\text{ROTATER}\ \tau_{\mathsf{M}}$ $\quad$ $\text{ROTATER}\ \tau_{\mathsf{E}}$



Splay Rotations

Double-Rotation: Left, Right

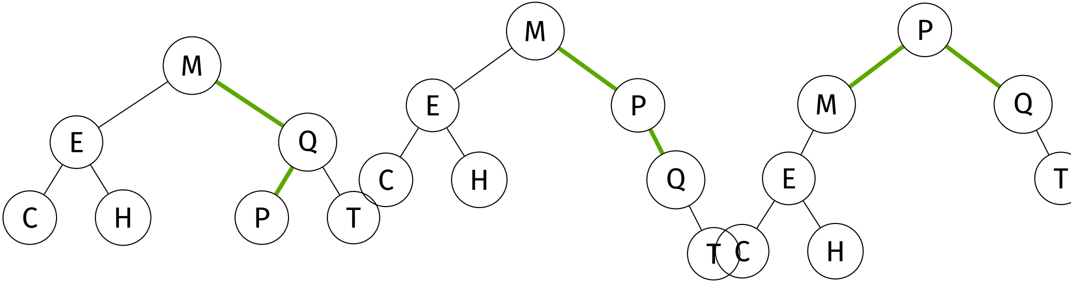
$\textsc{rotateL}\ \tau_E \qquad \textsc{rotateR}\ \tau_M$



Splay Rotations

Double-Rotation: Right, Left

$\textsc{rotateR}\ \tau_Q \qquad \textsc{rotateL}\ \tau_M$



Splay Rotations

Double-Rotation: Right, Right

$\textsc{rotateL}\ \tau_{\mathsf{M}} \qquad \textsc{rotateL}\ \tau_{\mathsf{Q}}$