We consider the composite variational inequality in the form:

$$\text{Find } x^* \in \mathbb{R}^d : R(x^*) = 0 \text{ with } R(x) := Q(x) + P(x), \tag{1}$$

where $Q(x), P(x) : \mathbb{R}^d \to \mathbb{R}^d$.

Assumption 1. *$R(x)$ satisfies minty assumption:*

$$\exists x^* \in \mathbb{R}^d : \ \forall x \in \mathbb{R}^d \hookrightarrow \langle R(x), x - x^* \rangle \geq 0.$$

Assumption 2. *$Q(x)$ is L_q-Lipschitz:*

$$\forall x_1, x_2 \in \mathbb{R}^d \hookrightarrow ||Q(x_1) - Q(x_2)|| \leq L_q ||x_1 - x_2||.$$

Assumption 3. *$P(x)$ is L_p-Lipschitz:*

$$\forall x_1, x_2 \in \mathbb{R}^d \hookrightarrow ||P(x_1) - P(x_2)|| \leq L_p ||x_1 - x_2||.$$

Algorithm 1 Extragradient Sliding

1: **Input:** $x^0 \in \mathbb{R}^d$
2: **Parameters:** $\eta, \theta > 0, K \in \mathbb{N}$
3: **for** $k = 0, 1, 2, \ldots, K-1$ **do**
4: Find $u^k \approx \tilde{u}^k$ where $\tilde{u}^k$ is solution for
Find $\tilde{u}^k \in \mathbb{R}^d : B_\theta^k(\tilde{u}^k) = 0$ with $B_\theta^k(x) := P(x^k) + Q(x) + \frac{1}{\theta}(x - x^k)$
5: $x^{k+1} = x^k - \eta R(u^k)$
6: **end for**
7: **Output:** x^K

Theorem 1. *Consider Algorithm 1 for Problem 1 under Assumptions 1, 2, 3, with the following tuning:*

$$\theta = \frac{1}{2L_p}, \eta = \frac{\theta}{4}. \tag{2}$$

Assume that u^k (line 4) satisfies:

$$||B_\theta^k(u^k)||^2 \leq \frac{L_p^2}{3} ||x^k - \tilde{u}^k||^2. \tag{3}$$

Then, we have the following inequality:

$$\min_{0 \leq j \leq K-1} ||R(u^j)||^2 \leq \frac{16 L_p^2 ||x_0 - x^*||^2}{K}. \tag{4}$$

Inequality 4 means sublinear convergence in non-convex non-concave case.

Lemma 1. *Consider Algorithm 1. Let θ be defined as $\theta = \dfrac{1}{2L_p}$. Then, under Assumptions 1, 2, 3, the following inequality holds:*

$$2\langle x^* - x^k, R(u^k) \rangle \leq -\theta ||R(u^k)||^2 + 3\theta \left(||B_\theta^k(u^k)||^2 - \frac{L_p^2}{3} ||x^k - \tilde{u}^k||^2 \right). \tag{5}$$

Proof of Lemma 1.

Using Minty assumption 3, we get

$$\begin{aligned} 2\langle x^k - x^*, R(u^k) \rangle &= 2\langle x^* - u^k, R(u^k) \rangle + 2\langle u^k - x^k, R(u^k) \rangle \\ &\leq 2\langle u^k - x^k, R(u^k) \rangle = 2\theta \left\langle \frac{1}{\theta}(u^k - x^k), R(u^k) \right\rangle \end{aligned}$$

The definition of $B_\theta^k(x)$ (line 4) gives

$$\begin{aligned} 2\theta \left\langle \frac{1}{\theta}(u^k - x^k), R(u^k) \right\rangle &= \theta \left\| \frac{1}{\theta}(u^k - x^k) + R(u^k) \right\|^2 - \frac{1}{\theta} ||u^k - x^k||^2 - \theta ||R(u^k)||^2 \\ &= -\frac{1}{\theta} ||u^k - x^k||^2 - \theta ||R(u^k)||^2 + \theta ||B_\theta^k(u^k) - P(x^k) + P(u^k)||^2. \end{aligned}$$

Using of Cauchy–Bunyakovsky–Schwarz inequality and L_p-Lipschitzness of $P(x)$, we get

$$\begin{aligned} 2\langle x^k - x^*, R(u^k)\rangle &\leq -\frac{1}{\theta}\|u^k - x^k\| - \theta\|R(u^k)\|^2 + 2\theta\|B_\theta^k(u^k)\|^2 + 2\|P(u^k) - P(x^k)\|^2 \\ &\leq -\frac{1}{\theta}\|u^k - x^k\| - \theta\|R(u^k)\|^2 + 2\theta\|B_\theta^k(u^k)\|^2 + 2\theta L_p^2\|u^k - x^k\|^2 \\ &= -\frac{1}{\theta}\left(1 - 2\theta^2 L_p^2\right)\|u^k - x^k\|^2 - \theta\|R(u^k)\|^2 + 2\theta\|B_\theta^k(u^k)\|^2. \end{aligned}$$

With $\theta = \frac{1}{2L_p}$ and CBS inequality in the form $-\|a\|^2 \leq \|b\|^2 - \frac{1}{2}\|a+b\|^2$, we have:

$$\begin{aligned} 2\langle x^* - x^k, R(u^k)\rangle &\leq -\theta\|R(u^k)\|^2 + 2\theta\|B_\theta^k(u^k)\|^2 - \frac{1}{2\theta}\|u^k - x^k\|^2 \\ &\leq -\theta\|R(u^k)\|^2 + 2\theta\|B_\theta^k(u^k)\|^2 + \frac{1}{2\theta}\|u^k - \tilde{u}^k\|^2 - \frac{1}{4\theta}\|x^k - \tilde{u}^k\|^2. \end{aligned}$$

One can observe that $B_\theta^k(x)$ is $\frac{1}{\theta}-$ strongly monotone. It gives that

$$\frac{1}{\theta}\|x-y\|^2 \leq \langle B_\theta^k(x) - B_\theta^k(y), x-y\rangle \leq \|B_\theta^k(x) - B_\theta^k(y)\|^2\|x-y\|^2.$$

With $B_\theta^k(\tilde{u}^k) = 0$ ($\tilde{u}^k$ is the solution of line 4), we get

$$\frac{1}{2\theta^2}\|u^k - \tilde{u}^k\|^2 \leq \frac{1}{2}\|B_\theta^k(u^k)\|^2.$$

Applying this to the upper inequality, we finalize the proof:

$$\begin{aligned} 2\langle x^* - x^k, R(u^k)\rangle &\leq -\theta\|R(u^k)\|^2 + \frac{5}{2}\theta\|B_\theta^k(u^k)\|^2 - \frac{1}{4\theta}\|x^k - \tilde{u}^k\|^2 \\ &\leq -\theta\|R(u^k)\|^2 + 3\theta\|B_\theta^k(u^k)\|^2 - \frac{3\theta}{12\theta^2}\|x^k - \tilde{u}^k\|^2 \\ &= -\theta\|R(u^k)\|^2 + 3\theta\left(\|B_\theta^k(u^k)\|^2 - \frac{L_p^2}{3}\|x^k - \tilde{u}^k\|^2\right). \end{aligned}$$

Proof of Theorem 1.

Line 5 of Algorithm 1 gives

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^{k+1} - x^k\|^2 + 2\langle x^{k+1} - x^k, x^k - x^*\rangle + \|x^k - x^*\|^2 \\ &= \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 - 2\eta\langle R(u^k), x^k - x^*\rangle \end{aligned}$$

Using the Lemma 1 and assumption on the norm $\|B_\theta^k(u^k)\|^2$ 3, we get

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 - \eta\theta\|R(u^k)\|^2 + 3\eta\theta\left(\|B_\theta^k(u^k)\|^2 - \frac{L_p^2}{3}\|x^k - \tilde{u}^k\|^2\right) \\ &\leq \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 - \eta\theta\|R(u^k)\|^2. \end{aligned}$$

From line 5 it follows:

$$\|x^{k+1} - x^*\|^2 \leq \eta^2\|R(u^k)\|^2 + \|x^k - x^*\|^2 - \eta\theta\|R(u^k)\|^2.$$

Let us substitute $\theta = \frac{1}{2L_p}$, $\eta = \frac{\theta}{2}$:

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \eta(\theta - \eta)\|R(u^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - \frac{\theta^2}{4}\|R(u^k)\|^2. \end{aligned}$$

Summing from 1 to $K-1$, we get

$$\begin{aligned}\sum_{j=0}^{K-1}\frac{\theta^2}{4}\|R(u^j)\|^2 &\le \sum_{j=0}^{K-1}\left(\|x^j-x^*\|^2-\|x^{j+1}-x^*\|^2\right)\\ &= \|x^0-x^*\|^2-\|x^K-x^*\|^2\le\|x^0-x^*\|^2.\end{aligned}$$

Thus we have:

$$\sum_{j=0}^{K-1}\|R(u^j)\|^2\le 16L_p^2\|x^0-x^*\|^2.$$

And finally we get:

$$\min_{0\le j\le K-1}||R(u^j)||^2\le\frac{16L_p^2\|x^0-x^*\|^2}{K}.$$