

# ARIMA(R,d,M) Models

Part II

# Yule-Walker Equations

$$\gamma_k = E[Y_{t-k} (\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_R Y_{t-R} + e_t)]$$

when  $k = 0, 1, \dots, R, \dots$

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_R \gamma_R + \sigma_e^2$$

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 + \cdots + \phi_R \gamma_{R-1}$$

...

$$\gamma_R = \phi_1 \gamma_{R-1} + \phi_2 \gamma_{R-2} + \cdots + \phi_R \gamma_0$$

...

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_R \gamma_{k-R} \quad k > R$$

# Yule-Walker Equations

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 + \cdots + \phi_R \rho_{R-1}$$

...

$$\rho_R = \phi_1 \rho_{R-1} + \phi_2 \rho_{R-2} + \cdots + \phi_R \rho_0$$

...

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_R \rho_{k-R} \quad k > R$$

# Yule-Walker Equations

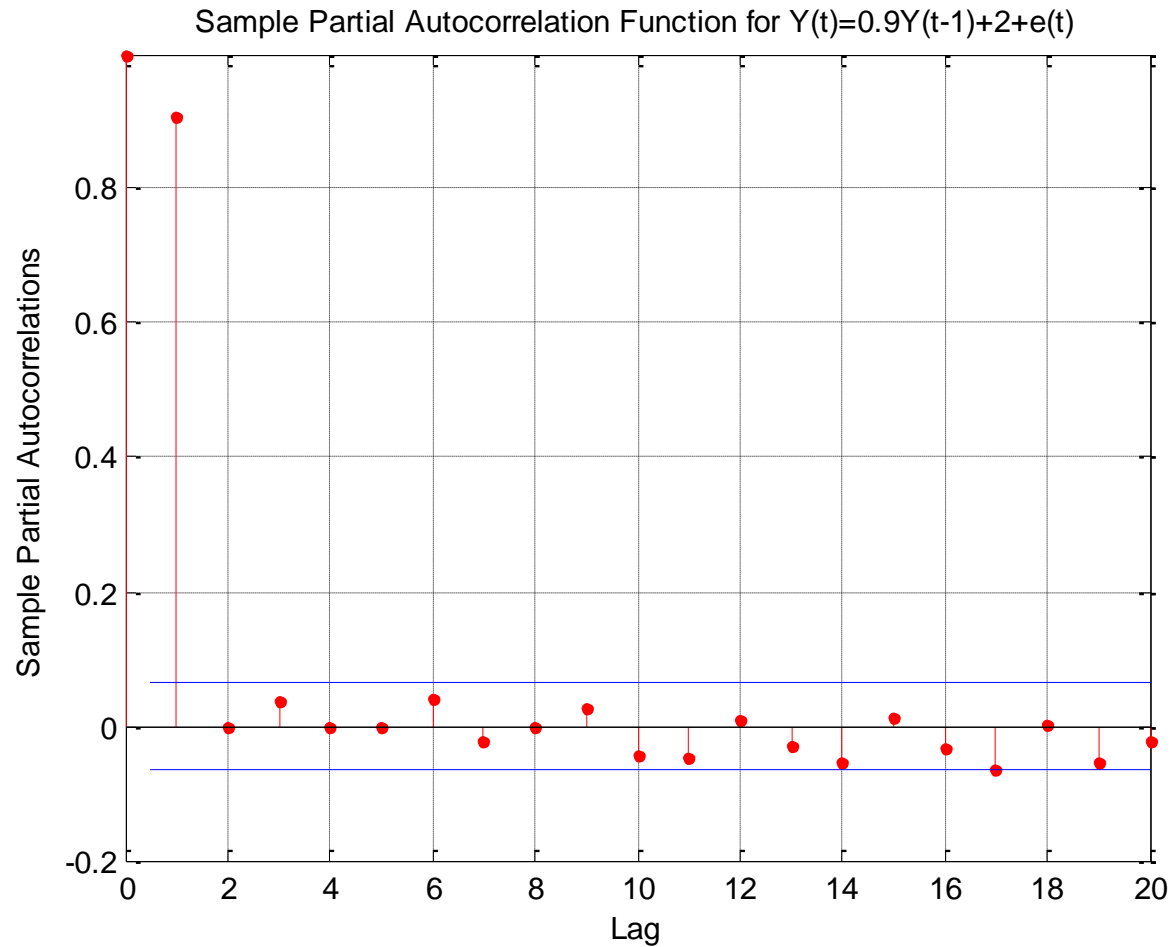
- We can solve the equations for  $\rho_k$ , but also for  $\phi_i$ .
- In the second case the estimates of  $\phi_i$  for  $i > R$  will have values close to zero

# Partial Autocorrelation Coefficients

- These estimated  $\phi_i$ , from Yule-Walker equations will be called Partial Autocorrelation Coefficients
- The Partial Autocorrelation Coefficients are very useful for determination of the AR model order

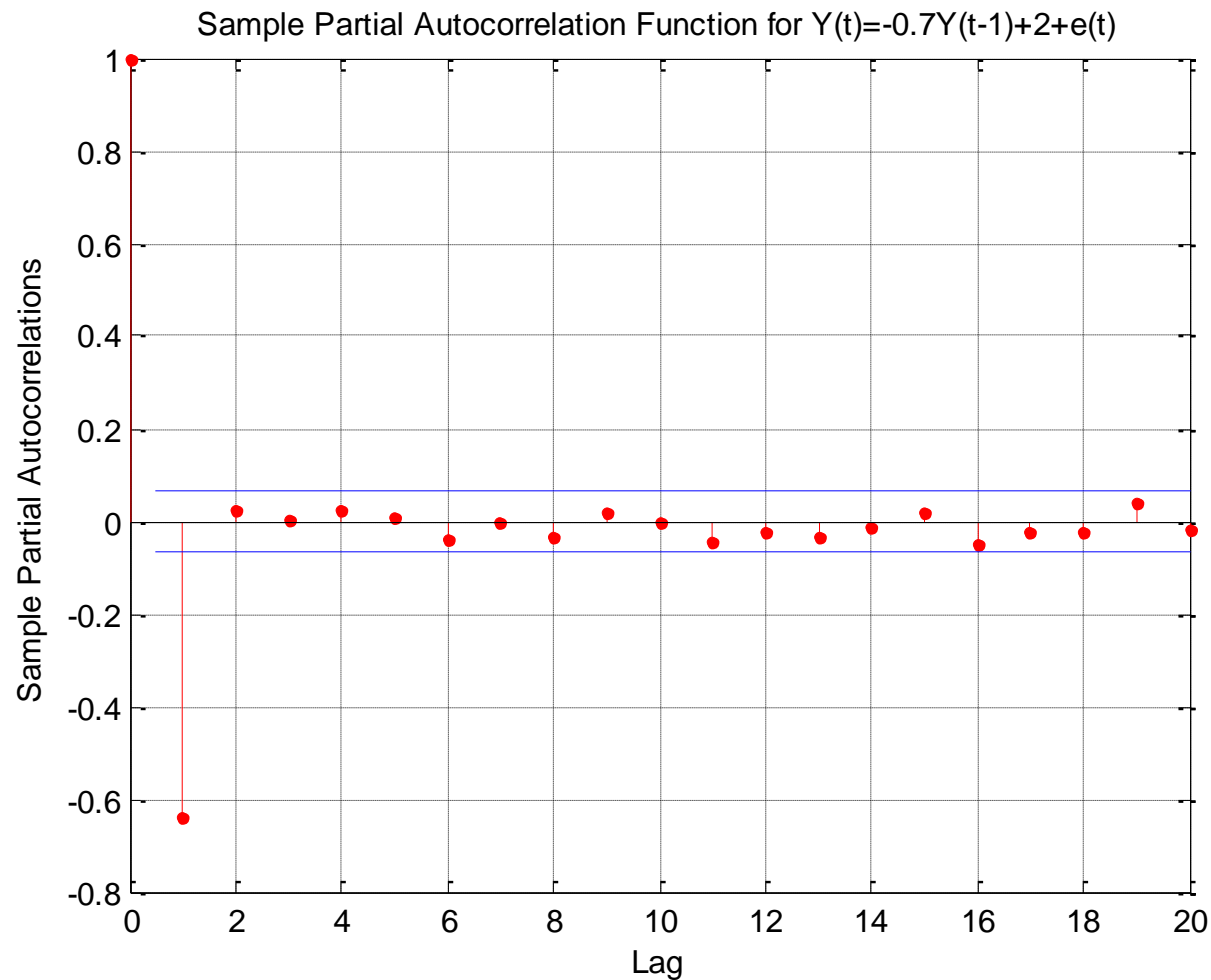
# Example 1

$$Y_t = 0.9Y_{t-1} + 2 + e_t$$



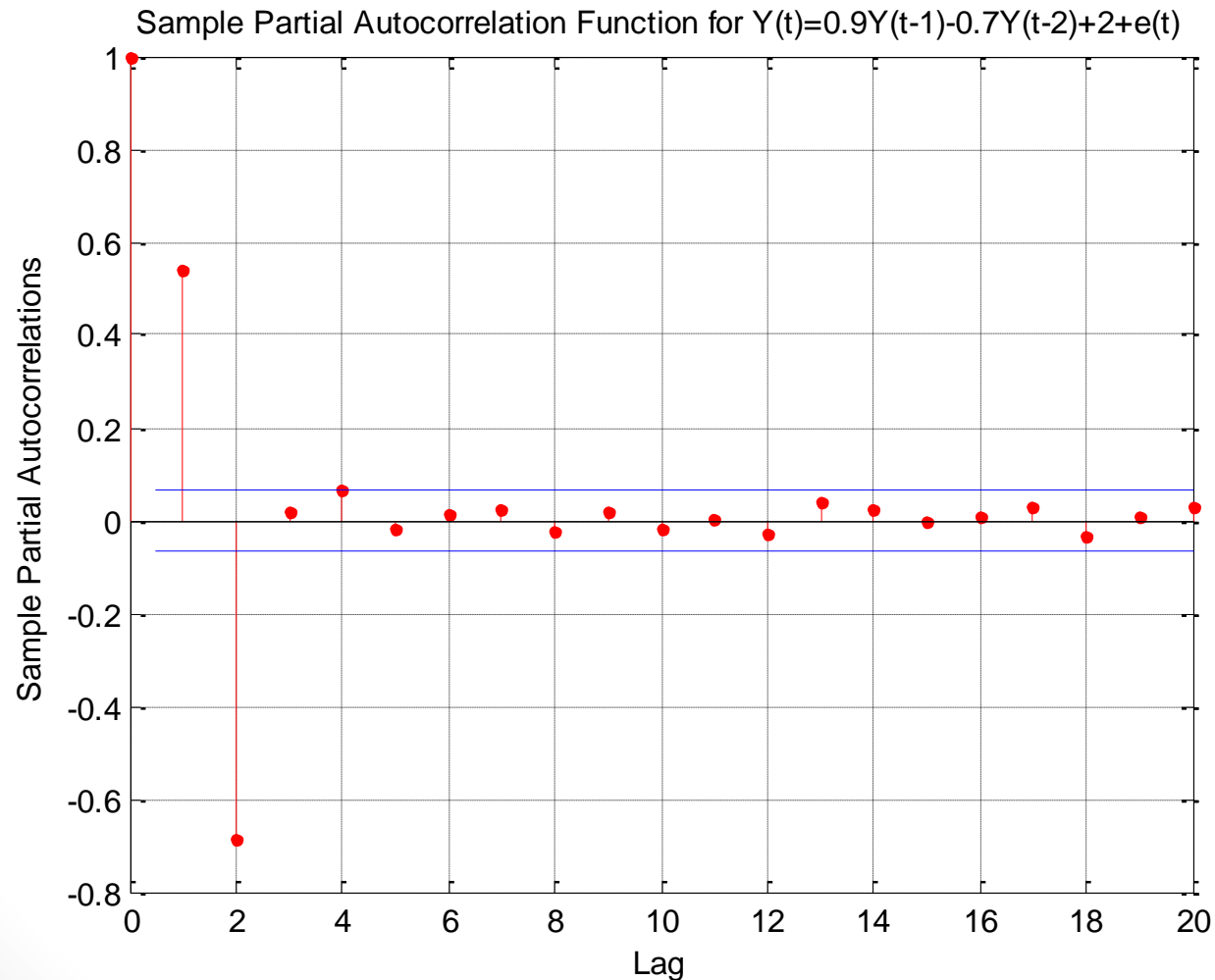
# Example 2

$$Y_t = -0.7Y_{t-1} + 2 + e_t$$



# Example 3

$$Y_t = 0.9Y_{t-1} - 0.7Y_{t-2} + 2 + e_t$$



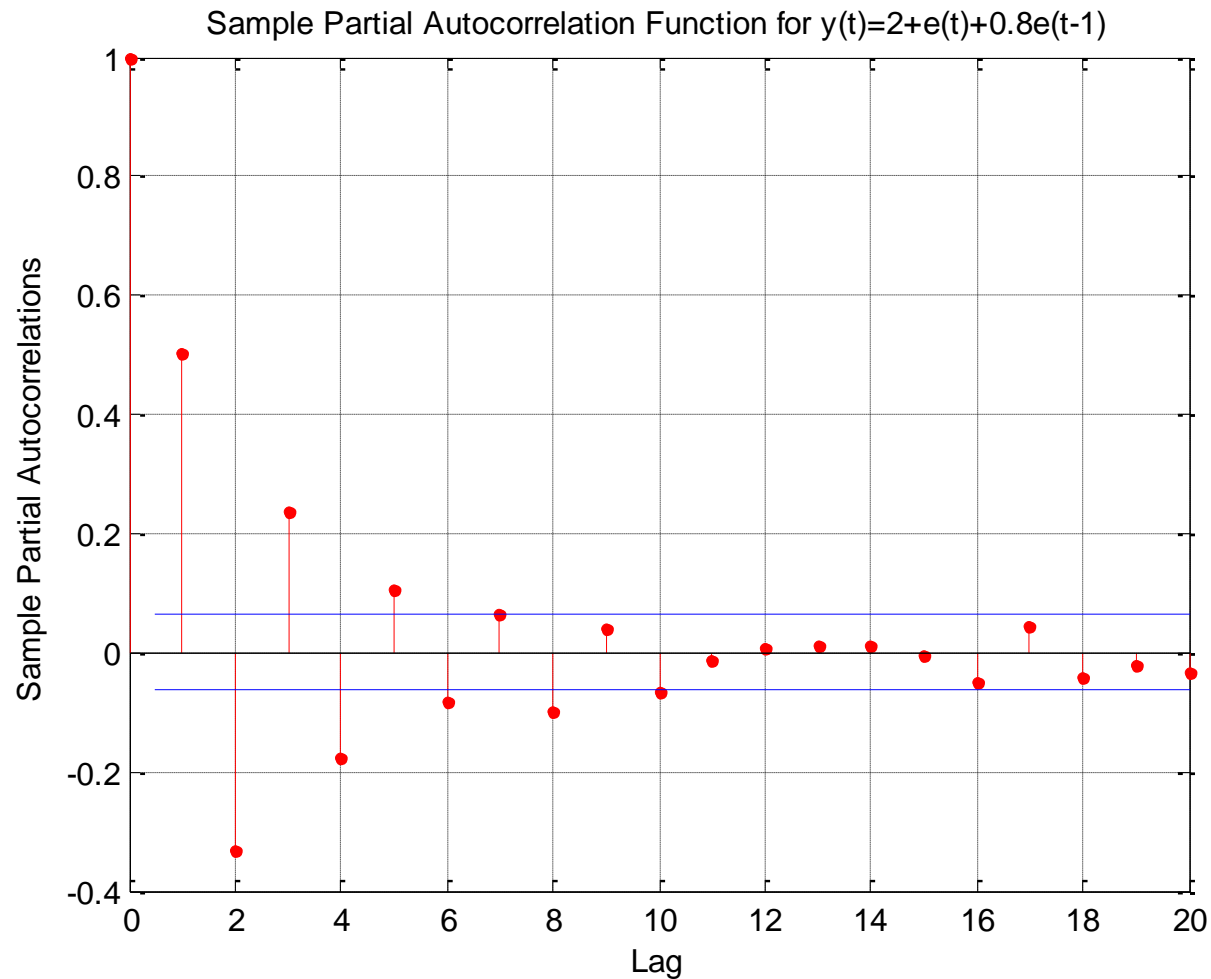


# Partial Autocorrelation Coefficients for MA models

- The Partial Autocorrelation Coefficients diminish quickly for MA models

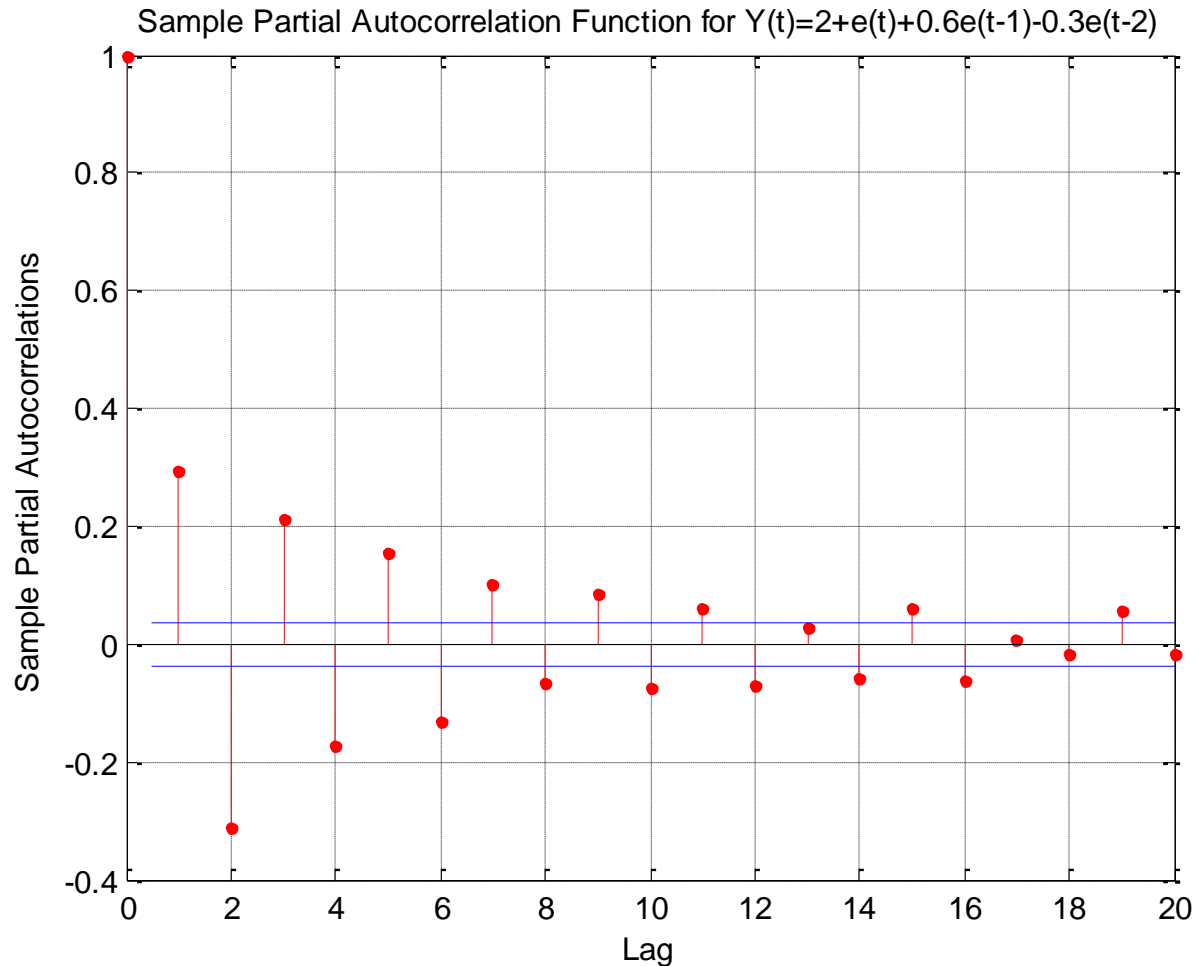
# Example 4

$$Y_t = 2 + e_t + 0.8e_{t-1}$$



# Example 5

$$Y_t = 2 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$



# Combined ARMA(R,M)

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_R Y_{t-R} + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_M e_{t-M}$$

$$\mu = c + \mu + \phi_1 \mu + \phi_2 \mu + \cdots + \phi_R \mu \Rightarrow$$

$$\mu = \frac{c}{1 - \phi_1 - \phi_2 - \cdots - \phi_R}$$

# ARMA(1,1) model

$$Y_t = c + \phi_1 Y_{t-1} + e_t + \theta_1 e_{t-1}$$

$$\gamma_0 = E[(\phi_1 Y_{t-1} + e_t + \theta_1 e_{t-1})^2] = \phi_1^2 \gamma_0 + 2\phi_1 \theta_1 \sigma_e^2 + \sigma_e^2 + \theta_1^2 \sigma_e^2 \Rightarrow$$

$$\gamma_0 = \frac{1 + \theta_1^2 + 2\phi_1 \theta_1}{1 - \phi_1^2} \sigma_e^2$$

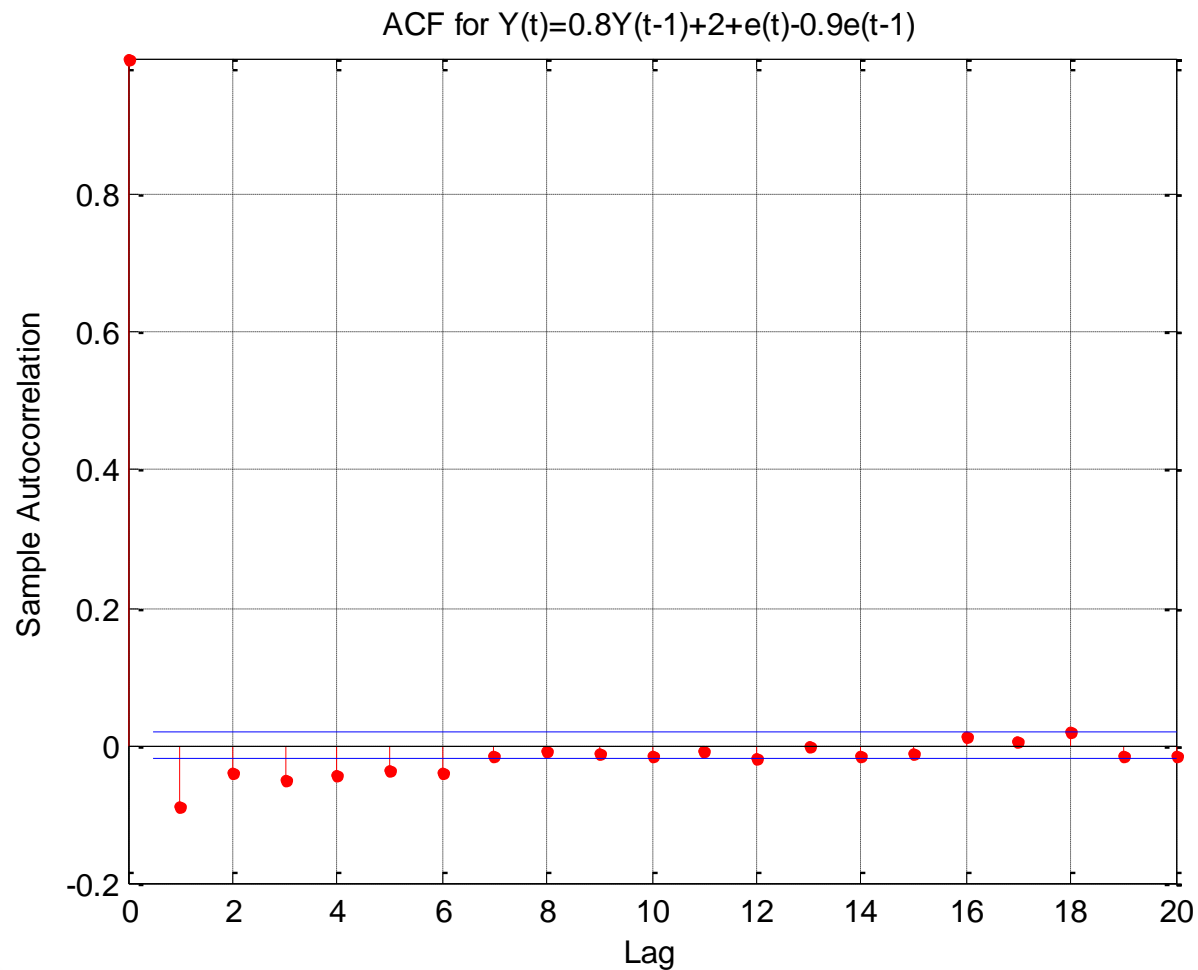
$$\gamma_1 = \frac{(1 + \phi_1 \theta_1)(\phi_1 + \theta_1)}{1 - \phi_1^2} \sigma_e^2$$

$$\gamma_2 = \phi_1 \gamma_1 \quad \dots \quad \gamma_k = \phi_1 \gamma_{k-1} \quad \text{for } k > 2$$

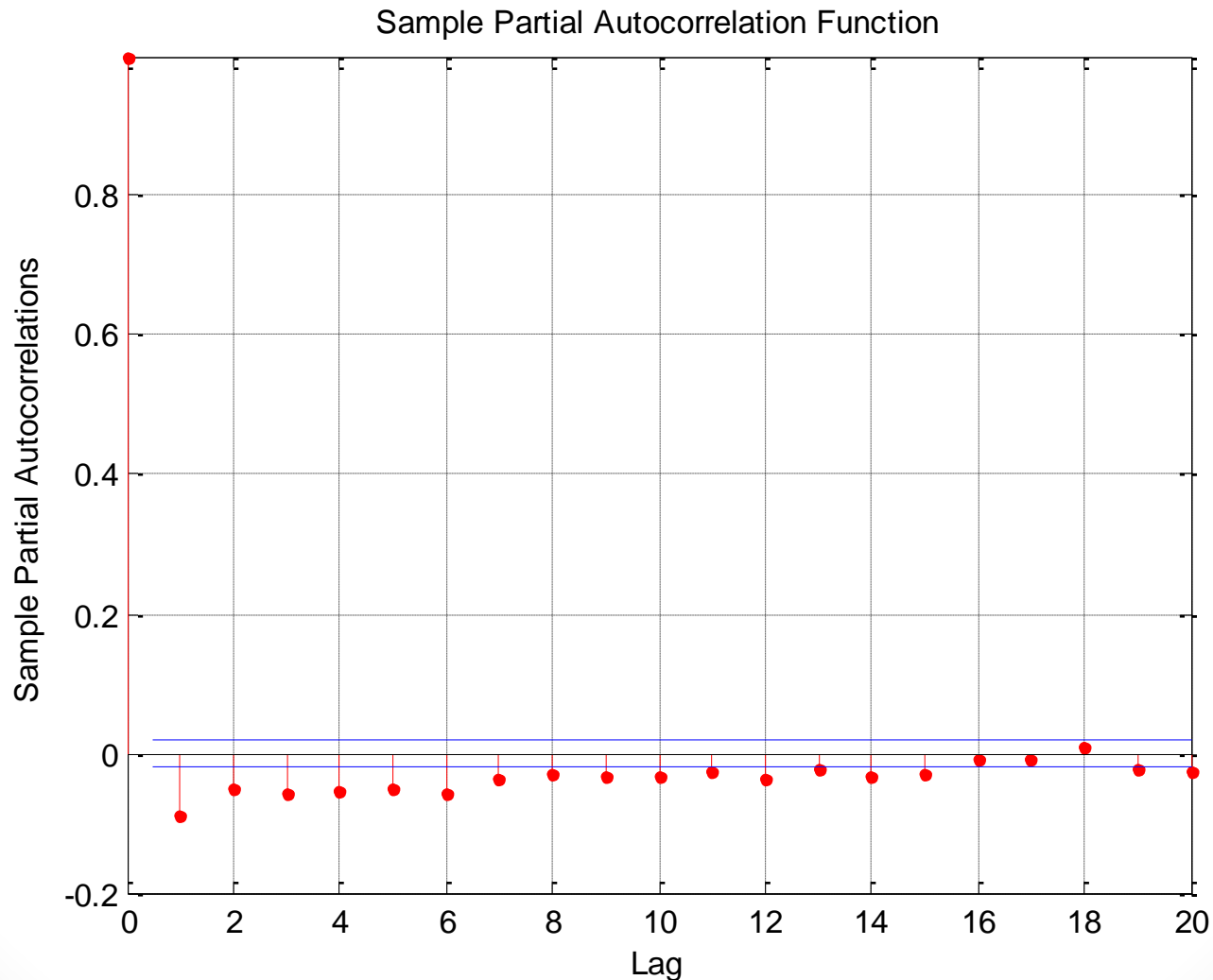
# Example 6

$$Y_t = 0.8Y_{t-1} + 2 + e_t - 0.9e_{t-1}$$

ACF for  $Y_t = 0.8Y_{t-1} + 2 + e_t - 0.9e_{t-1}$



PACF for  $Y_t = 0.8Y_{t-1} + 2 + e_t - 0.9e_{t-1}$





# ACF for ARMA(R,M) model when $k \geq M+1$

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_R \gamma_{k-R} \quad k \geq M+1$$

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_R \rho_{k-R} \quad k \geq M+1$$

# Lag (Backshift) operator

$$Be_t = e_{t-1}, B^2e_t = e_{t-2}, \dots, B^Me_t = e_{t-M}$$

$$MA(M): Y_t = \mu + (1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_M B^M)e_t$$

$$Y_t = \mu + \theta(B)e_t$$

$$AR(R): (1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 - \dots - \phi_R B^R)Y_t = c + e_t$$

$$\phi(B)Y_t = c + e_t$$

$$ARMA(R, M): \phi(B)Y_t = c + \theta(B)e_t$$

# Difference operator

$$\Delta Y_t = Y_t - Y_{t-1}, \quad \Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1}$$

$$\Delta Y_t = (1 - B)Y_t$$

# Generalized form for ARIMA models

*ARIMA*( $R, d, M$ ) :

$$\phi(B)\Delta^d Y_t = c + \theta(B)e_t$$

$$\phi(B)(1-B)^d Y_t = c + \theta(B)e_t$$

# Stationarity conditions

$$\tilde{Y}_t = Y_t - \mu$$

$$\phi(B)\tilde{Y}_t = \theta(B)e_t$$

$$\tilde{Y}_t = \phi(B)^{-1} \theta(B)e_t$$

$$1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_R B^R = 0$$

$$z = \frac{1}{B} \Rightarrow$$

$$z^R - \phi_1 z^{R-1} - \phi_2 z^{R-2} - \dots - \phi_R = 0$$

# Stationarity conditions

*All roots of equation (1) to lie out of the unit circle*

$$1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_R B^R = 0 \quad (1)$$

*or*

*All roots of equation (2) to lie in the unit circle*

$$z^R - \phi_1 z^{R-1} - \phi_2 z^{R-2} - \dots - \phi_R = 0 \quad (2)$$

# Example 5

$$Y_t = 0.9Y_{t-1} - 0.7Y_{t-2} + e_t$$

```
p1=[1 -0.9 +0.7]
```

```
roots(p1)
```

```
ans =
```

```
0.4500 + 0.7053i
```

```
0.4500 - 0.7053i
```

```
abs(roots(p1))
```

```
ans =
```

```
0.8367
```

```
0.8367
```

# Example 6

$$Y_t = 0.9Y_{t-1} + 0.5Y_{t-2} + e_t$$

```
p1=[1 -0.9 -0.5]
```

```
roots(p1)
```

```
ans =
```

```
1.2882
```

```
-0.3882
```

```
ar2=arima('c',2,'var',1,'ar',[0.9 0.5])
```

Error using arima/validateModel (line 1306)

The non-seasonal autoregressive polynomial is unstable.



# Invertibility conditions

$$\tilde{Y}_t = Y_t - \mu$$

$$\phi(B)\tilde{Y}_t = \theta(B)e_t$$

$$\theta^{-1}(B)\phi(B)\tilde{Y}_t = e_t$$

$$1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_R B^M = 0$$

# Invertibility conditions

*All roots of equation (1) to lie out of the unit circle*

$$1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_M B^M = 0 \quad (1)$$

*or*

*All roots of equation (2) to lie in the unit circle*

$$z^M + \theta_1 z^{M-1} + \theta_2 z^{M-2} + \cdots + \theta_M = 0 \quad (2)$$

# Example 7

$$Y_t = 2 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$

p2=[1 0.6 -0.3]

roots(p2)

ans =

-0.9245

0.3245

# Example 7

$$Y_t = 2 + e_t + 0.7e_{t-1} - 0.3e_{t-2}$$

```
p2=[1 0.7 -0.3]
```

```
roots(p2)
```

```
ans =
```

```
-1.0000
```

```
0.3000
```

```
ma2=arima('c',2,'var',1,'ma',[0.7 -0.3])
```

Error using arima/validateModel (line 1314)

The non-seasonal moving average polynomial is non-invertible.

# ARIMA(R,d,M) Estimation. Box-Jenkins Approach.

# Estimation of ARIMA models

$$e_t = Y_t - \sum_{i=1}^R \phi_i Y_{t-i} - \sum_{i=1}^M \theta_i e_{t-i}$$

$$\sum_{t=1}^T e_t^2 = f(\phi_1, \phi_2, \dots, \phi_R, \theta_1, \theta_2, \dots, \theta_M)$$

$$LLF = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^T \log(\sigma_t^2) - \frac{1}{2} \sum_{t=1}^T \frac{e_t^2}{\sigma_t^2}$$

# Estimation of ARIMA models

- Initial values for  $Y_t$  and  $e_t$
- Initial values for  $\Phi$  and  $\Theta$  polynomials
- Optimization parameters

Initial values for  $Y_t$  and  $e_t$

$$Y_i = \bar{Y} \quad \text{for } i = -R, -R + 1, \dots, 0$$

$$e_i = 0 \quad \text{for } i = -M, -M + 1, \dots, 0$$



# Initial values for $\Phi$ and $\Theta$ polynomials

- Yule-Walker equations
- Nonlinear equations for  $\theta$  after filtering the autoregression part

# Optimization parameters

- MaxFunEvals
- MaxIter
- TolFun
- TolX
- TolCon

# Box-Jenkins Approach

- Determination of ARIMA(R,d,M) model structure
- Estimation and model selection
- Test for model adequacy

# Determination of ARIMA(p,d,q) model structure

- Determination of **d**.
- Determination of autoregression order **R**.
- Determination of moving average order **M**.
- Selection of an initial models set.

# Estimation and model selection

- Estimation of the selected models.
- Significance test for the estimated parameters.
- Application of AIC and BIC criteria for model selection.
- Selection of appropriate model/models.

# Test for model adequacy

- Test for residuals autocorrelation presence .
- Lyung-Box test.
- Final model selection.

# Information Criteria

Akaike Criterion

$$-2\log L + 2\text{numParam}$$

Bayesian Criterion

$$-2\log L + \text{numParam} * \text{numObs}$$