ARIMA(R,d,M) Models

Part II

Yule-Walker Equations

$$\gamma_k = E[Y_{t-k}(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_R Y_{t-R} + e_t)]$$

$$when \quad k = 0,1,\ldots,R,\ldots$$

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_R\gamma_R + \sigma_e^2$$

$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 + \cdots + \phi_R\gamma_{R-1}$$

$$\ldots$$

$$\gamma_R = \phi_1\gamma_{R-1} + \phi_2\gamma_{R-2} + \cdots + \phi_R\gamma_0$$

$$\ldots$$

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + \cdots + \phi_R\gamma_{k-R} \quad k > R$$

Yule-Walker Equations

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 + \cdots + \phi_R \rho_{R-1}$$

$$\ldots$$

$$\rho_R = \phi_1 \rho_{R-1} + \phi_2 \rho_{R-2} + \cdots + \phi_R \rho_0$$

$$\ldots$$

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_R \rho_{k-R} \qquad k > R$$

Yule-Walker Equations

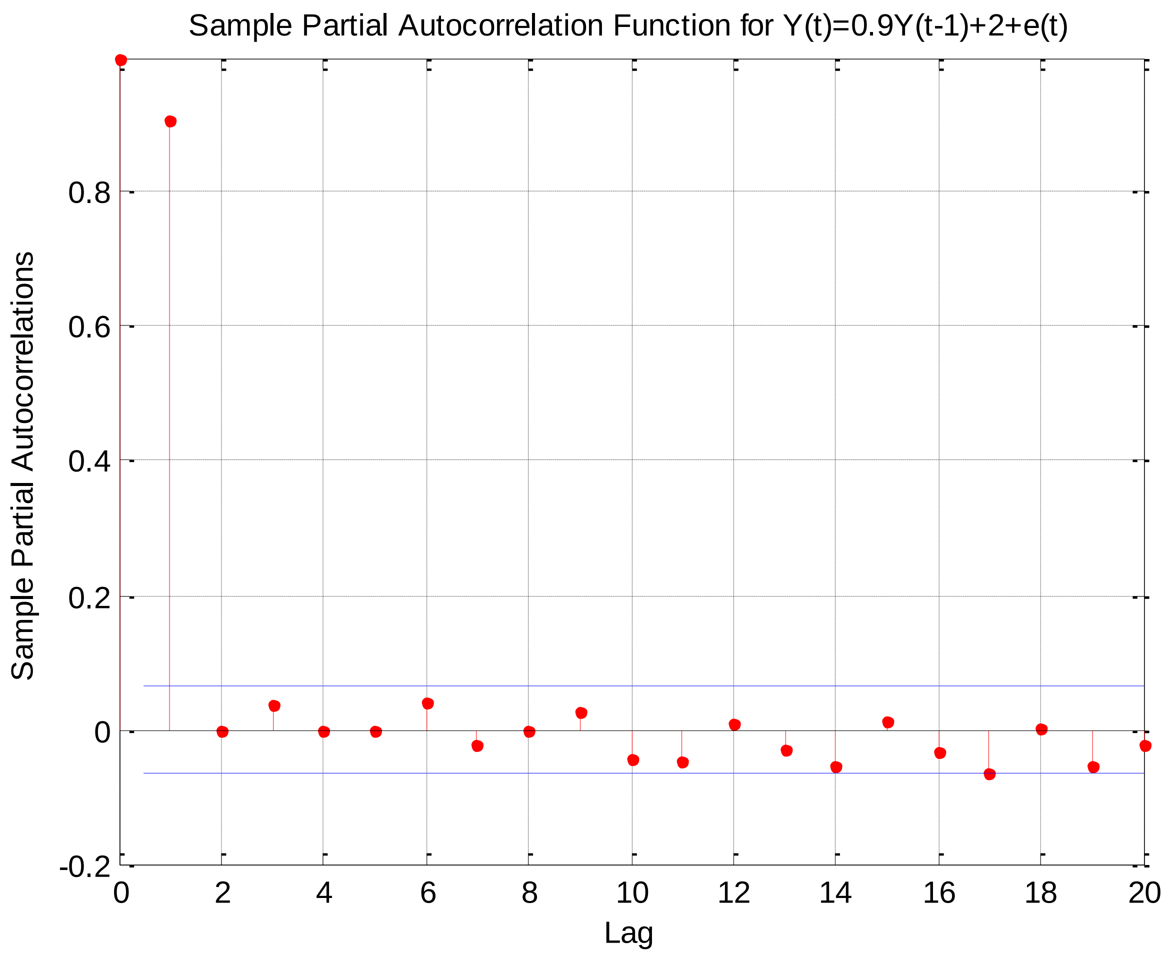
- We can solve the equations for ρ_k, but also for ϕ_i.
- In the second case the estimates of ϕ_i for $i>R$ will have values close to zero

Partial Autocorrelation Coefficients

- These estimated ϕ_i, from Yule-Walker equations will be called Partial Autocorrelation Coefficients
- The Partial Autocorrelation Coefficients are very useful for determination of the AR model order

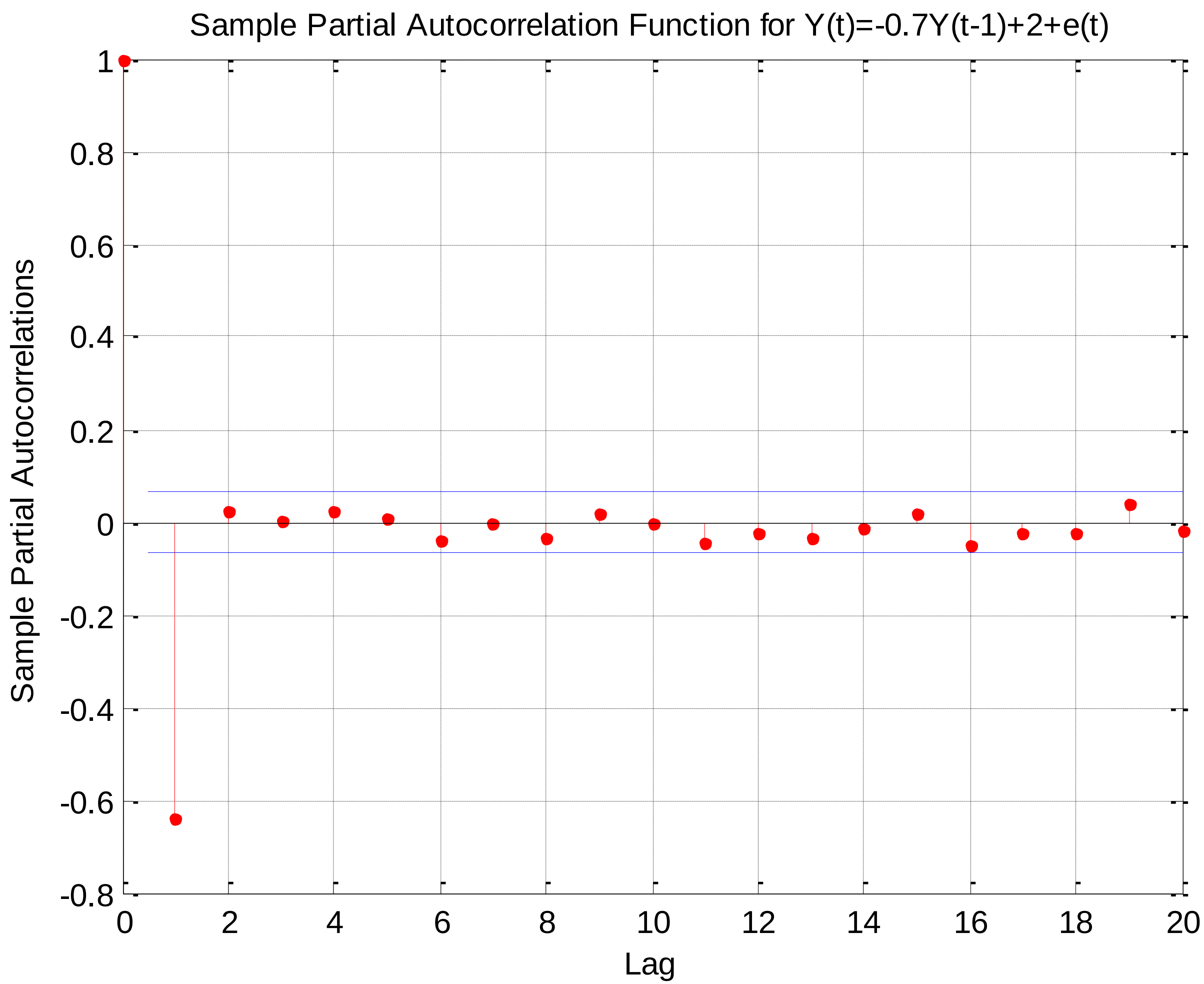
Example 1

$$Y_t = 0.9Y_{t-1} + 2 + e_t$$



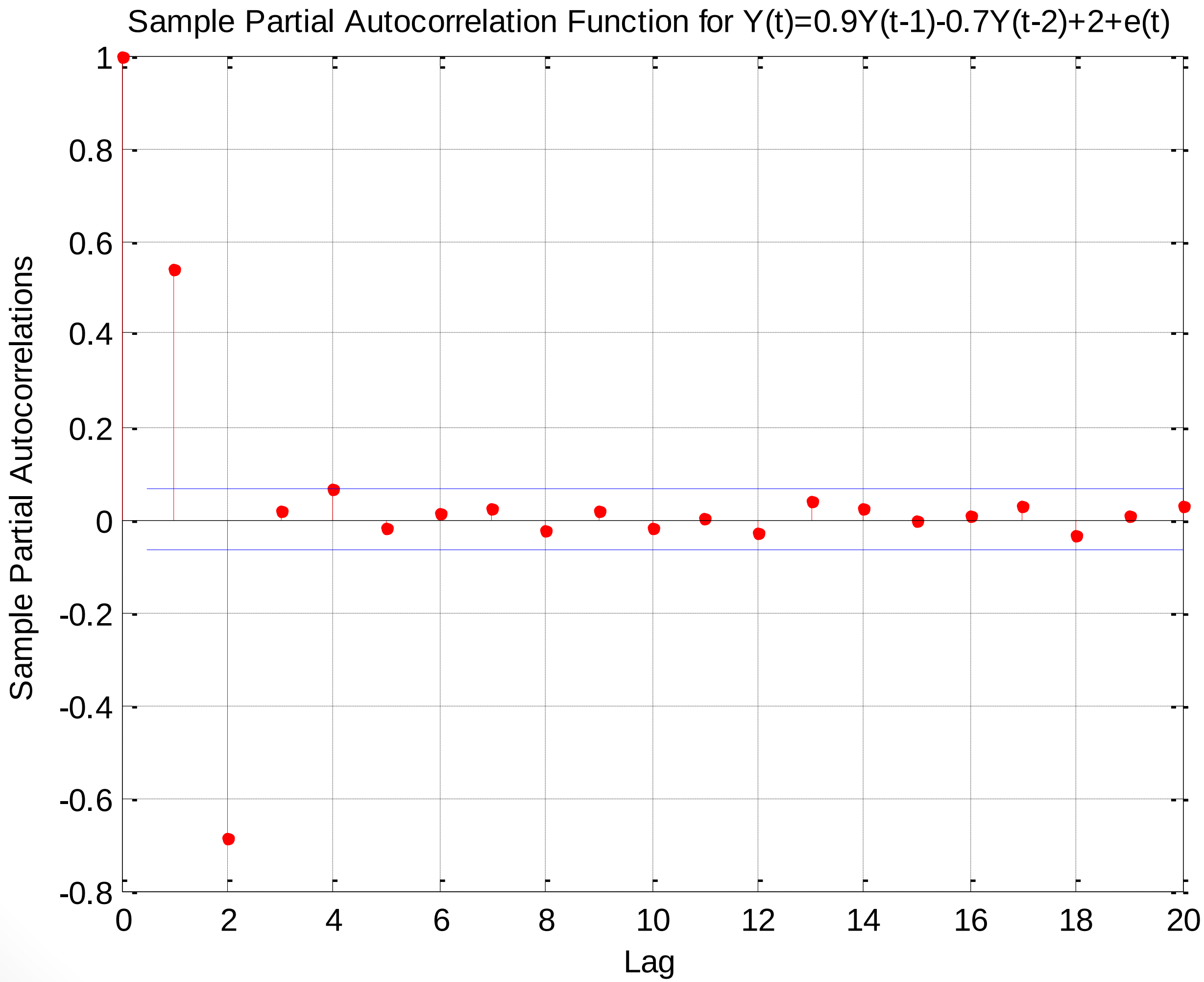
Example 2

$$Y_t = -0.7Y_{t-1} + 2 + e_t$$



Example 3

$$Y_t = 0.9Y_{t-1} - 0.7Y_{t-2} + 2 + e_t$$

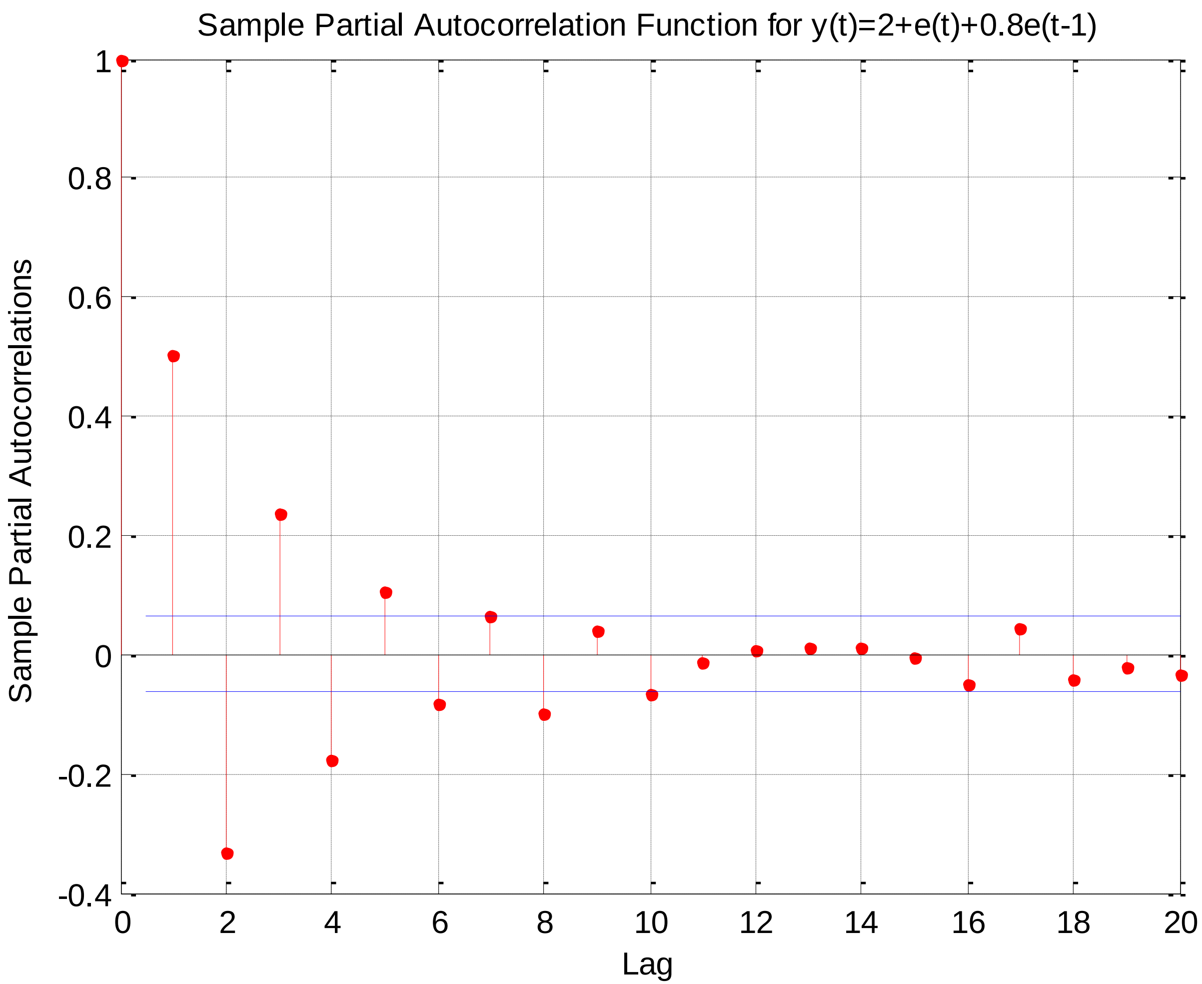


Partial Autocorrelation Coefficients for MA models

- The Partial Autocorrelation Coefficients diminish quickly for MA models

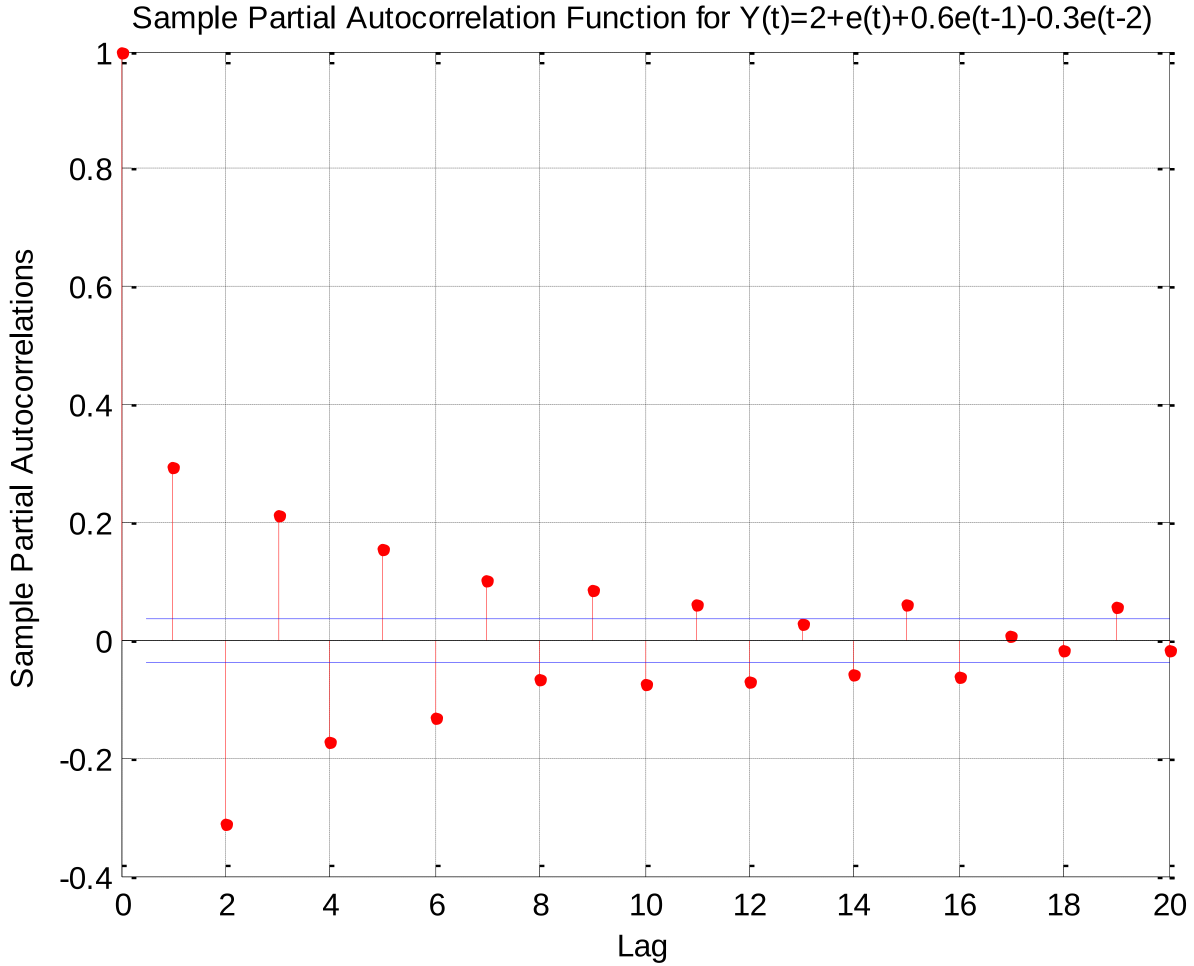
Example 4

$$Y_t = 2 + e_t + 0.8e_{t-1}$$



Example 5

$$Y_t = 2 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$



Combined ARMA(R,M)

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_R Y_{t-R} + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_M e_{t-M}$$

$$\mu = c + \mu + \phi_1 \mu + \phi_2 \mu + \cdots + \phi_R \mu \Rightarrow$$

$$\mu = \frac{c}{1 - \phi_1 - \phi_2 - \cdots - \phi_R}$$

ARMA(1,1) model

$$Y_t = c + \phi_1 Y_{t-1} + e_t + \theta_1 e_{t-1}$$

$$\gamma_0 = E[(\phi_1 Y_{t-1} + e_t + \theta_1 e_{t-1})^2] = \phi_1^2 \gamma_0 + 2\phi_1\theta_1\sigma_e^2 + \sigma_e^2 + \theta_1^2\sigma_e^2 \Rightarrow$$

$$\gamma_0 = \frac{1 + \theta_1^2 + 2\phi_1\theta_1}{1 - \phi_1^2}\sigma_e^2$$

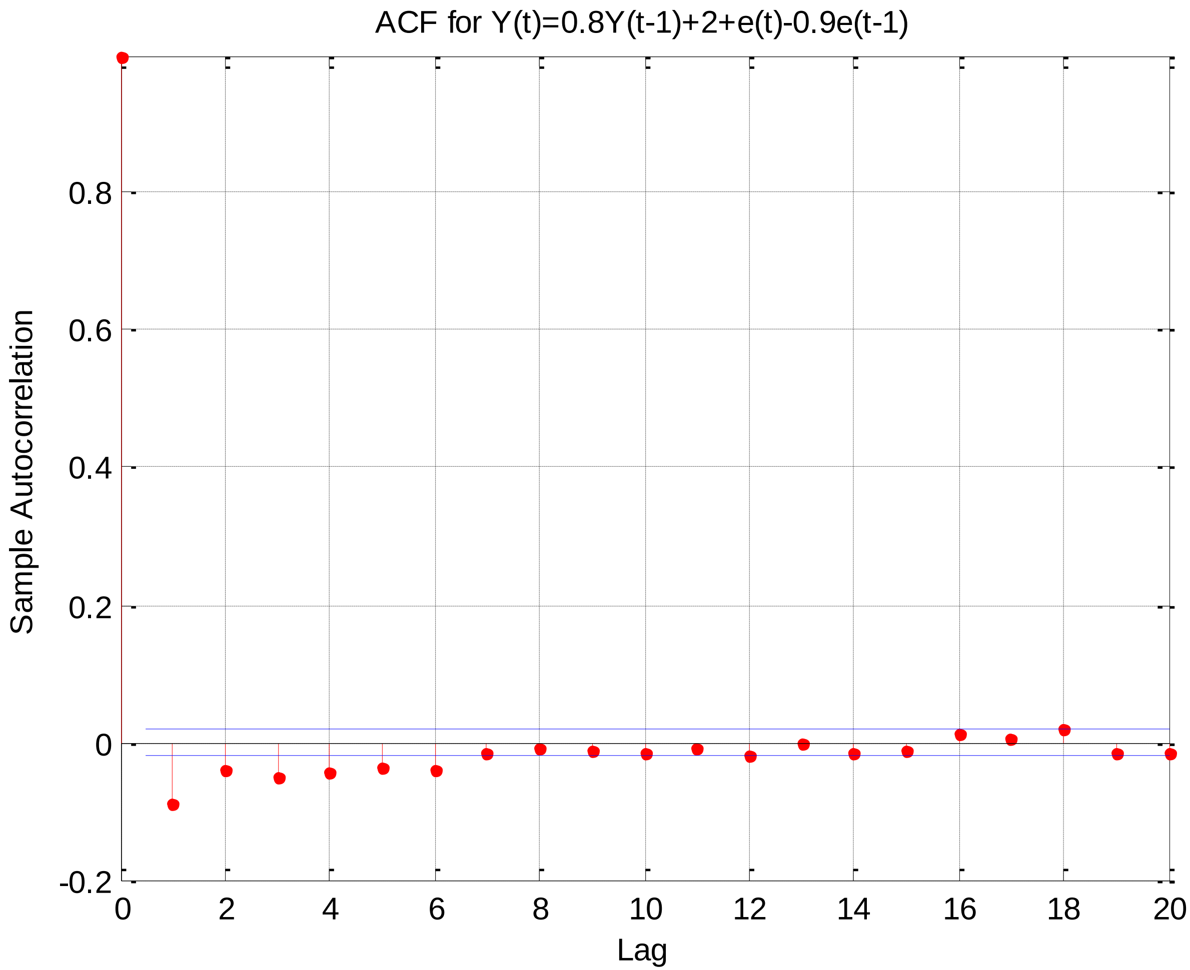
$$\gamma_1 = \frac{(1 + \phi_1\theta_1)(\phi_1 + \theta_1)}{1 - \phi_1^2}\sigma_e^2$$

$$\gamma_2 = \phi_1\gamma_1 \quad \ldots \quad \gamma_k = \phi_1\gamma_{k-1} \quad \mathit{for}\ k > 2$$

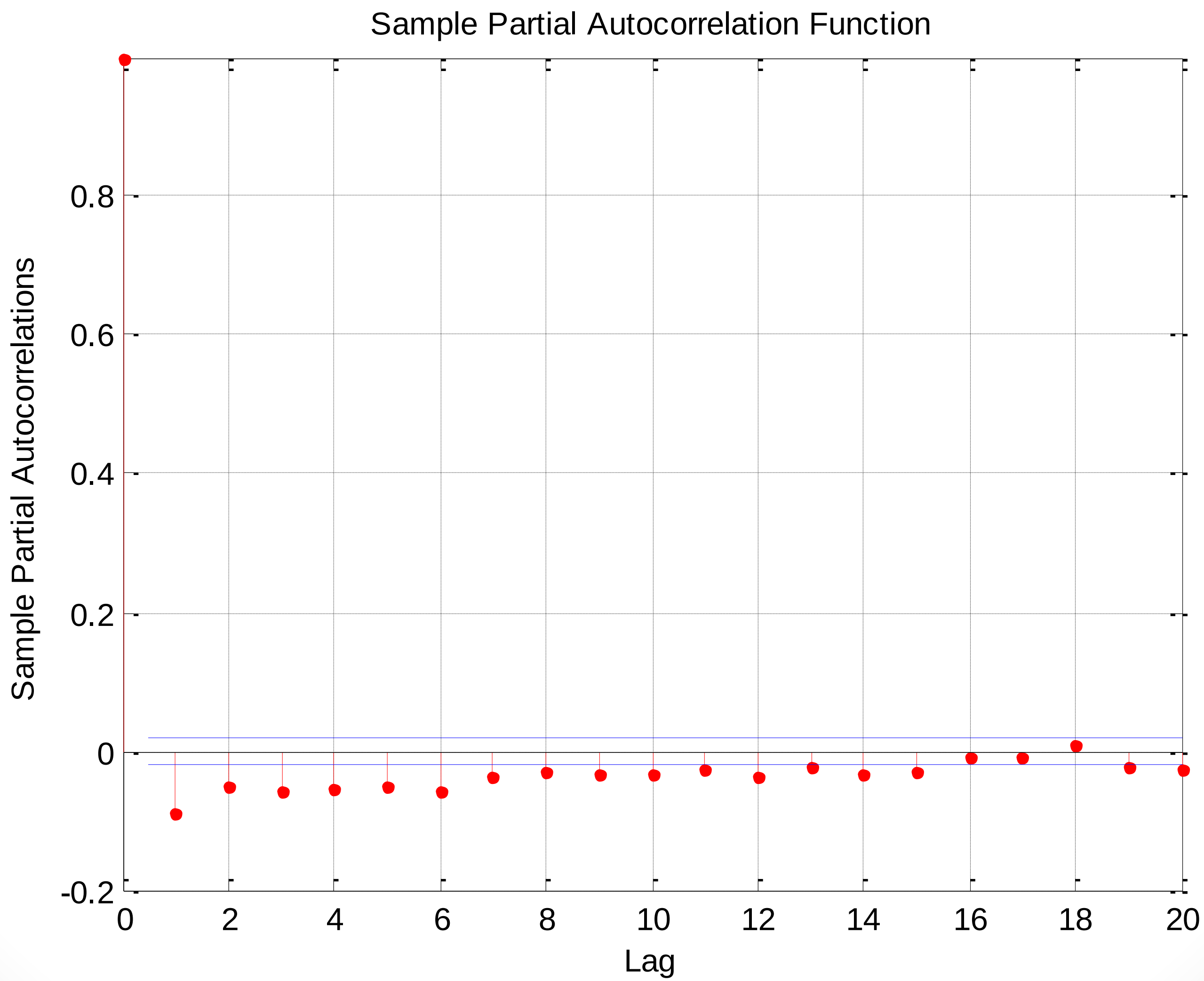
Example 6

$$Y_t = 0.8Y_{t-1} + 2 + e_t - 0.9e_{t-1}$$

ACF for $Y_t = 0.8Y_{t-1} + 2 + e_t - 0.9e_{t-1}$



PACF for $Y_t = 0.8Y_{t-1} + 2 + e_t - 0.9e_{t-1}$



ACF for ARMA(R,M) model when k≥M+1

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_R \gamma_{k-R} \quad k \geq M+1$$

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_R \rho_{k-R} \quad k \geq M+1$$

Lag (Backshift) operator

$$Be_t = e_{t-1},\ B^2 e_t = e_{t-2}, \ldots, B^M e_t = e_{t-M}$$

$$MA(M): \quad Y_t = \mu + (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_M B^M) e_t$$

$$Y_t = \mu + \theta(B) e_t$$

$$AR(R): \quad (1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 - \cdots - \phi_R B^R) Y_t = c + e_t$$

$$\phi(B) Y_t = c + e_t$$

$$ARMA(R, M): \quad \phi(B) Y_t = c + \theta(B) e_t$$

Difference operator

$$\Delta Y_t = Y_t - Y_{t-1}, \quad \Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1}$$

$$\Delta Y_t = (1 - B) Y_t$$

Generalized form for ARIMA models

$$ARIMA(R, d, M):$$

$$\phi(B)\Delta^d Y_t = c + \theta(B)e_t$$

$$\phi(B)(1 - B)^d Y_t = c + \theta(B)e_t$$

Stationarity conditions

$$\tilde{Y}_t = Y_t - \mu$$

$$\phi(B)\tilde{Y}_t = \theta(B)e_t$$

$$\tilde{Y}_t = \phi(B)^{-1}\theta(B)e_t$$

$$1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_R B^R = 0$$

$$z = \frac{1}{B} \Rightarrow$$

$$z^R - \phi_1 z^{R-1} - \phi_2 z^{R-2} - \cdots - \phi_R = 0$$

Stationarity conditions

$$All \ roots \ of \ equation\,(1) \ to \ lie$$
$$out \, of \ the \ unit \ circle$$
$$1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_R B^R = 0 \qquad (1)$$
$$or$$
$$All \ roots \ of \ equation\,(2) \ to \ lie$$
$$in \, the \ unit \ circle$$
$$z^R - \phi_1 z^{R-1} - \phi_2 z^{R-2} - \cdots - \phi_R = 0 \qquad (2)$$

Example 5

$$Y_t = 0.9Y_{t-1} - 0.7Y_{t-2} + e_t$$

```
p1=[1 -0.9 +0.7]

roots(p1)
ans =

   0.4500 + 0.7053i
   0.4500 - 0.7053i
abs(roots(p1))
ans =
   0.8367
   0.8367
```

Example 6

$$Y_t = 0.9Y_{t-1} + 0.5Y_{t-2} + e_t$$

p1=[1 -0.9 -0.5]

roots(p1)
ans =
1.2882
-0.3882
ar2=arima('c',2,'var',1,'ar',[0.9 0.5])
Error using arima/validateModel (line 1306)
The non-seasonal autoregressive polynomial is unstable.

Invertibility conditions

$$\tilde{Y}_t = Y_t - \mu$$

$$\phi(B)\tilde{Y}_t = \theta(B)e_t$$

$$\theta^{-1}(B)\phi(B)\tilde{Y}_t = e_t$$

$$1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_R B^M = 0$$

Invertibility conditions

All roots of equation (1) *to lie out of the unit circle*

$$1+\theta_1 B+\theta_2 B^2+\cdots+\theta_M B^M=0 \qquad (1)$$

or

All roots of equation (2) *to lie in the unit circle*

$$z^M+\theta_1 z^{M-1}+\theta_2 z^{M-2}+\cdots+\theta_M=0 \qquad (2)$$

Example 7

$$Y_t = 2 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$

p2=[1 0.6 -0.3]

roots(p2)

ans =

-0.9245
0.3245

Example 7

$$Y_t = 2 + e_t + 0.7e_{t-1} - 0.3e_{t-2}$$

```
p2=[1 0.7 -0.3]

roots(p2)
ans =

   -1.0000
    0.3000

ma2=arima('c',2,'var',1,'ma',[0.7 -0.3])
Error using arima/validateModel (line 1314)
The non-seasonal moving average polynomial is
non-invertible.
```

ARIMA(R,d,M) Estimation. Box-Jenkins Approach.

Estimation of ARIMA models

$$e_t = Y_t - \sum_{i=1}^{R} \phi_i Y_{t-i} - \sum_{i=1}^{M} \theta_i e_{t-i}$$

$$\sum_{t=1}^{T} e_t^2 = f(\phi_1, \phi_2, \ldots, \phi_R, \theta_1, \theta_2, \ldots \theta_M)$$

$$LLF = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log(\sigma_t^2) - \frac{1}{2}\sum_{t=1}^{T}\frac{e_t^2}{\sigma_t^2}$$

Estimation of ARIMA models

- Initial values for Y_t and e_t
- Initial values for Φ and Θ polynomials
- Optimization parameters

Initial values for Y_t and e_t

$$Y_i = \bar{Y} \qquad for\ i = -R, -R+1, \ldots, 0$$

$$e_i = 0 \qquad for\ i = -M, -M+1, \ldots, 0$$

Initial values for Φ and Θ polynomials

- Yule-Walker equations
- Nonlinear equations for θ
  after filtering the autoregression part

Optimization parameters

- MaxFunEvals
- MaxIter
- TolFun
- TolX
- TolCon

Box-Jenkins Approach

- Determination of ARIMA(R,d,M) model structure
- Estimation and model selection
- Test for model adequacy

Determination of ARIMA(p,d,q) model structure

- Determination of **d.**
- Determination of autoregression order **R.**
- Determination of moving average order **M.**
- Selection of an initial models set.

Estimation and model selection

- Estimation of the selected models.
- Significance test for the estimated parameters.
- Application of AIC and BIC criteria for model selection.
- Selection of appropriate model/models.

Test for model adequacy

- Test for residuals autocorrelation presence .
- Lyung-Box test.
- Final model selection.

Information Criteria

Akaike Criterion

$$-2logL + 2numParam$$

Bayesian Criterion

$$-2logL + numParam * \log(numObs)$$