Stochastic Time Series

Stationary TS (weak)

$$E(Y_t) = \mu$$

$$D(Y_t) = E[(Y_t - \mu)^2] = \sigma^2$$

$$Cov(Y_t, Y_{t-k}) = E[(Y_t - \mu)(Y_{t-k} - \mu)] = \gamma_k$$

Does not depend on time *t*

Stationary TS (weak)

Sample Estimators

$$\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \bar{Y})^2$$

$$\hat{\gamma}_k = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y}) \quad k = 1,2,\ldots$$

Autocorrelation Coefficient

$$\rho_k = Corr(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{D(Y_t)D(Y_{t-k})}} = \frac{\gamma_k}{\sigma^2}$$

$$\rho_0 = 1$$

$$\rho_{-k} = \rho_k$$

Sample Autocorrelation Coefficient

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\sigma}^2} = \frac{\sum_{t=k+1}^{n}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2} \qquad k = 1,2,\ldots$$

$$E(\hat{\rho}_k) = \rho_k$$

$$D(\hat{\rho}_k) \cong \frac{1}{n}$$

White noise

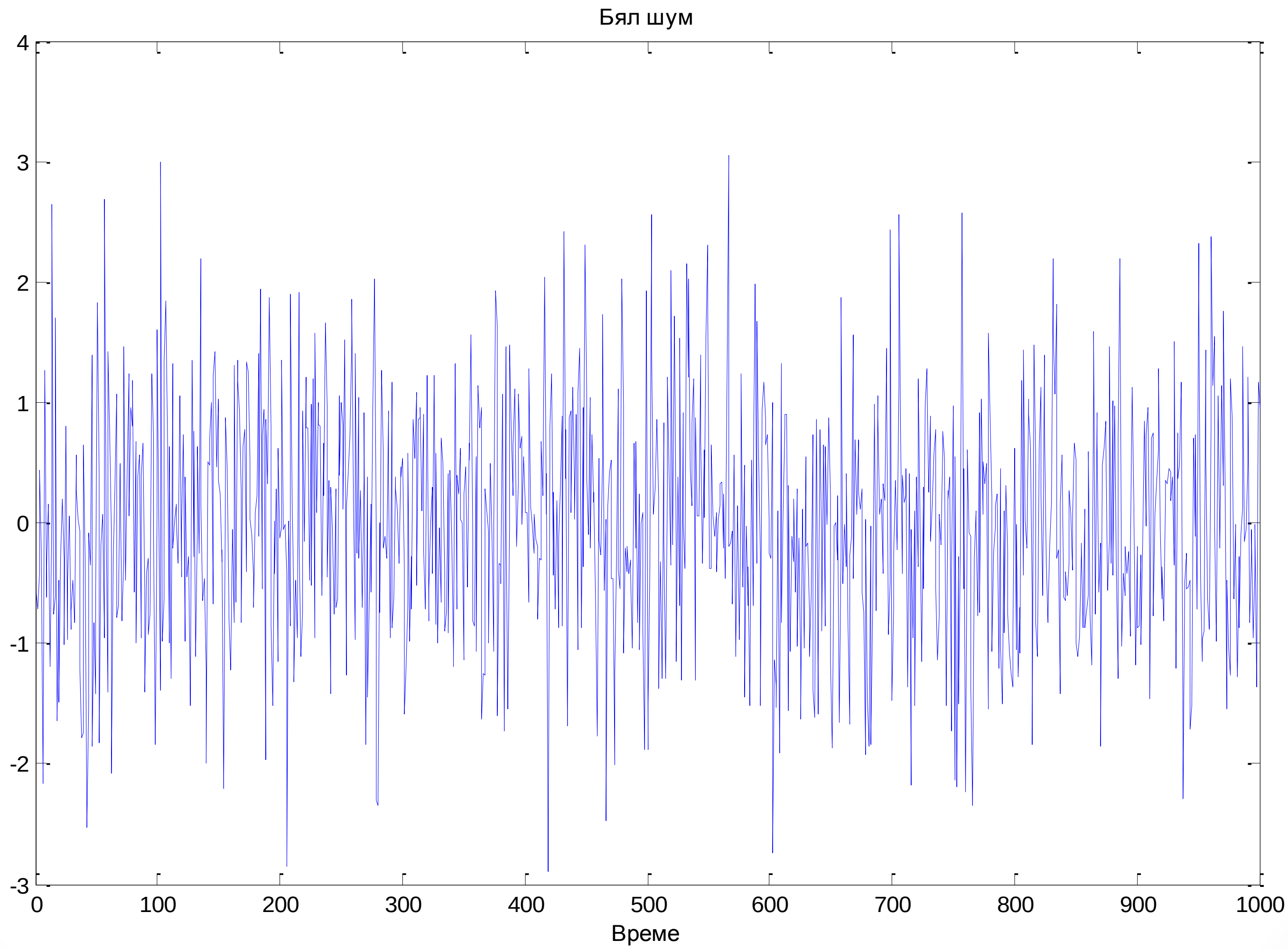
$$Y_t = e_t$$

$$E(e_t) = 0, \quad D(e_t) = \sigma^2,$$

$$Cov(e_t, e_k) = 0 \qquad for\ all \qquad t \neq k$$

$$\rho_0 = 1, \quad \rho_k = 0 \quad k = 1,2,\ldots$$

White noise



Бял шум
4
3
2
1
0
-1
-2
-3
0
100
200
300
400
500
600
700
800
900
1000
Време

Ljung-Box Test

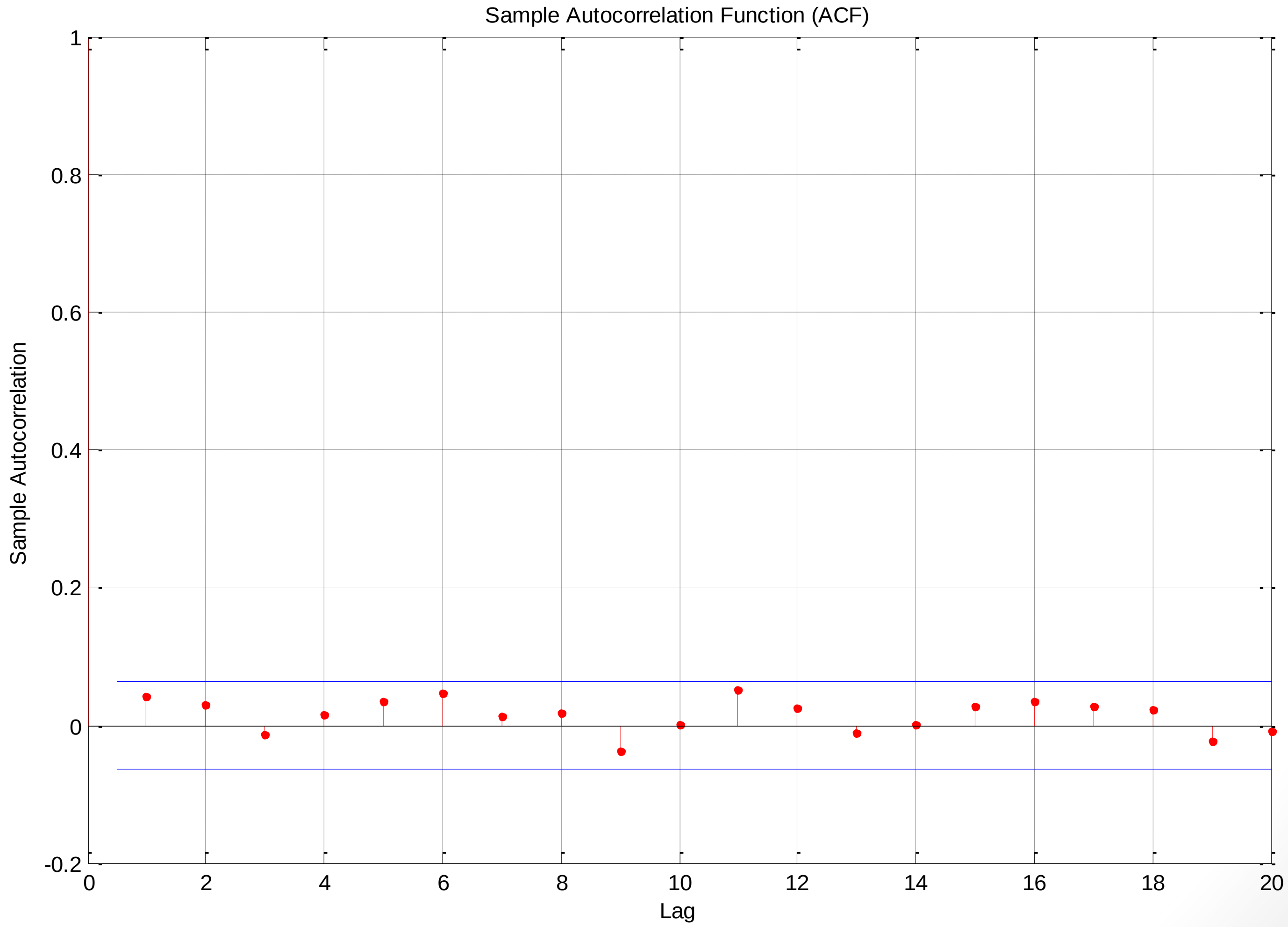
$$Q = n(n+2)\sum_{k=1}^{L}\frac{\hat{\rho}_k^2}{n-k}$$

$$Q \sim X^2$$

[H,pValue,Qstat,CriticalValue] = lbqtest(y,[20 30 40 50]);

H	pValue	Qstat	CriticalValue
0	0.741	15.6024	31.4104
0	0.8534	22.0165	43.773
0	0.8848	29.6479	55.7585
0	0.8518	39.6861	67.5048

Autocorrelation Function for the White noise Process



Random walk

$$Y_t = Y_{t-1} + e_t$$

$$E(e_t) = \mu,\ D(e_t) = \sigma^2,\ Cov(e_t, e_{t-k}) = 0 \ \ for\ k \neq 0$$

$$Y_0 = 0, \quad t = 0$$

$$Y_1 = e_1, Y_2 = e_1 + e_2, \ldots$$

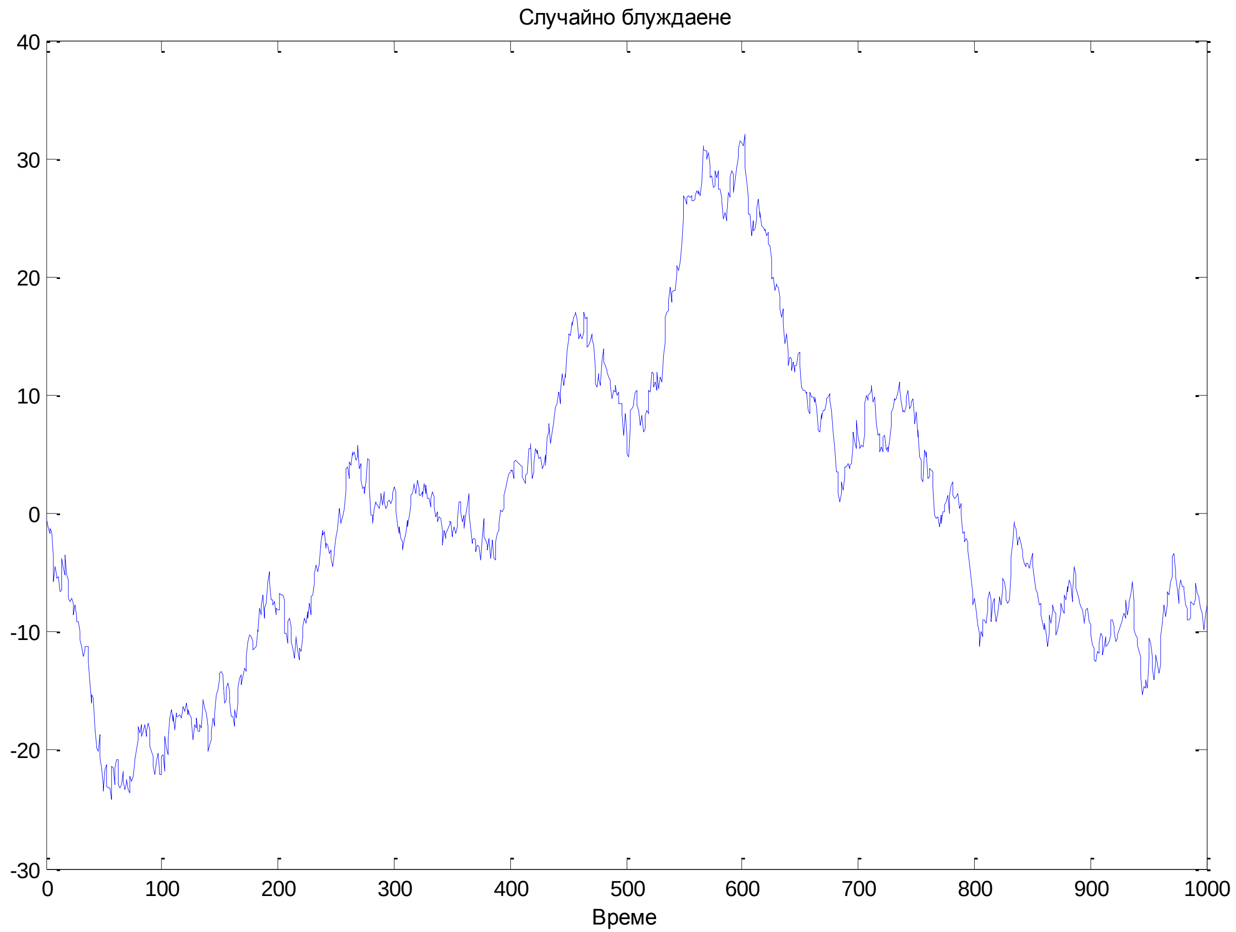
Random walk

$$Y_t = \sum_{i=0}^{t} e_i$$

$$E(Y_t) = E(\sum_{i=0}^{t} e_i) = t\mu$$

$$D(Y_t) = t\sigma^2$$

Random walk



Случайно блуждаене
40
30
20
10
0
-10
-20
-30
0
100
200
300
400
500
600
700
800
900
1000
Време

Dickey-Fuller Test

$$Y_t = \phi Y_{t-1} + e_t$$

$$\Delta Y_t = (\phi - 1) Y_{t-1} + e_t,$$

$$\Delta Y_t = \delta Y_{t-1} + e_t, \quad \delta = (\phi - 1)$$

$$H_0 : \delta = 0$$

[H,pValue,TestStat,CriticalValue] = adftest(y);

H	pValue	Qstat	Critical Value
0	0.936	-1.0419	-3.4145

ARIMA(R,d,M) Models

Part I

Unconditional vs. Conditional Mean

- For a random variable y_t, the *unconditional mean* is simply the expected value, *E*(*yt*).
- In contrast, the *conditional mean of y is the expected value of y_t given a conditioning set of variables, Ω_t.*
- *A conditional mean model specifies a functional form for E(yt|Ωt).*

Static Conditional Mean Models

- For a *static* conditional mean model, the conditioning set of variables is measured contemporaneously with the dependent variable y_t.
- An example of a static conditional mean model is the ordinary linear regression model. Given x_t, a row vector of exogenous covariates measured at time *t*, and β, a column vector of coefficients, the conditional mean of y_t is expressed as the linear combination

$$E(y_t|x_t) = x_t\beta$$

(that is, the conditioning set is Ω*t*=*xt*).

Dynamic Conditional Mean Models

- In time series econometrics, there is often interest in the dynamic behavior of a variable over time.
- A *dynamic* conditional mean model specifies the expected value of y_t as a function of historical information. Let H_{t-1} denote the history of the process available at time t. A dynamic conditional mean model specifies the evolution of the conditional mean, $E(y_t \mid H_{t-1})$.
- Examples of historical information are:
  - Past observations, $y_1, y_2, \ldots, y_{t-1}$
  - Past innovations, $e_1,\ e_2, \ldots\ e_{t-1}$,

Moving Average Models - MA(M)

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_M e_{t-M}$$

$$E(e_t) = 0,\ D(e_t) = \sigma^2,\ Cov(e_i, e_j) = 0 \ \ for\, all\, i \neq j$$

$$E(Y_t) = \mu$$

$$D(Y_t) = E[(Y_t - \mu)^2] = E(e_t^2 + \theta_1^2 e_{t-1}^2 + \cdots + \theta_M^2 e_{t-M}^2 + 2\theta_1 e_t e_{t-1} + \cdots) \Rightarrow$$

$$D(Y_t) = \sigma_e^2 + \theta_1^2 \sigma_e^2 + \cdots + \theta_M^2 \sigma_e^2 \Rightarrow$$

$$D(Y_t) = \sigma_e^2 (1 + \theta_1^2 + \cdots + \theta_M^2)$$

MA(1)

$$Y_t = \mu + e_t + \theta_1 e_{t-1}$$

$$E(Y_t) = \mu, \quad D(Y_t) = \gamma_0 = \sigma_e^2 (1 + \theta_1^2)$$

$$\gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)] = E[(e_t + \theta_1 e_{t-1})(e_{t-1} + \theta_1 e_{t-2})] \Rightarrow$$

$$\gamma_1 = \theta_1 \sigma_e^2$$

$$\gamma_k = E[(e_t + \theta_1 e_{t-1})(e_{t-k} + \theta_1 e_{t-k-1})] = 0 \quad when\ k > 1$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\theta_1}{1+\theta_1^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

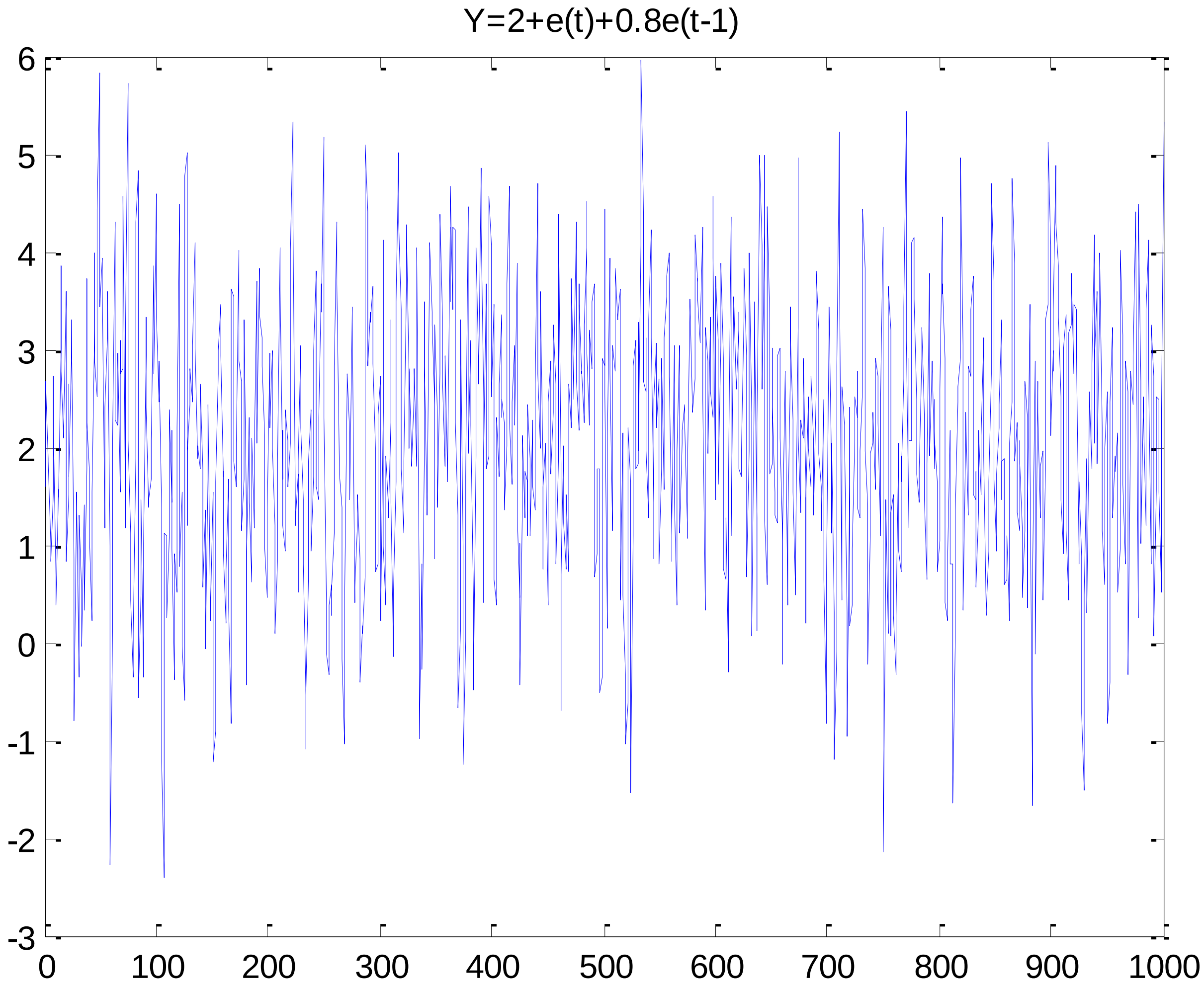
Example 1

$$Y_t = 2 + e_t + 0.8e_{t-1}$$

Example 1

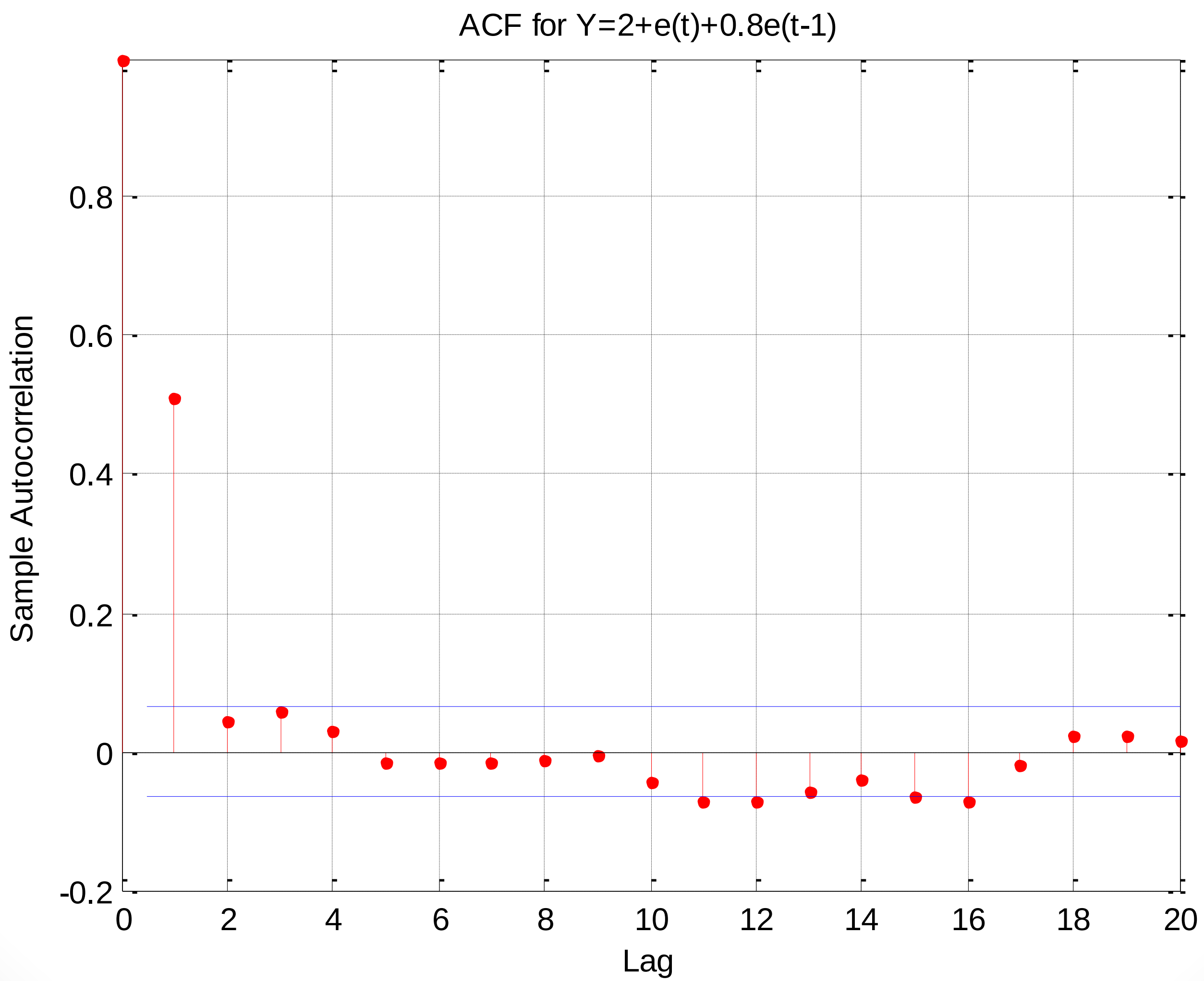
```
ma1=arima('ma',0.8,'c',2, 'var',1)
ma1 =
   ARIMA(0,0,1) Model:
   --------------------
   Distribution: Name = 'Gaussian'
            P: 0
            D: 0
            Q: 1
       Constant: 2
           AR: {}
          SAR: {}
           MA: {0.8} at Lags [1]
          SMA: {}
       Variance: 1

y=simulate(ma1,1000);
```

Example 1



Y=2+e(t)+0.8e(t-1)
6
5
4
3
2
1
0
-1
-2
-3
0
100
200
300
400
500
600
700
800
900
1000

Example 1



ACF for Y=2+e(t)+0.8e(t-1)
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

MA(2)

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}$$

$$\gamma_1 = E[(e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2})(e_{t-1} + \theta_1 e_{t-2} + \theta_2 e_{t-3})] \Rightarrow$$

$$\gamma_1 = \theta_1 \sigma_e^2 + \theta_2 \theta_1 \sigma_e^2 = \theta_1 (1 + \theta_2) \sigma_e^2$$

$$\gamma_2 = \theta_2 \sigma_e^2$$

$$\gamma_k = 0 \qquad k > 2$$

MA(2)

$$\rho_1 = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$$

$$\rho_2 = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\rho_k = 0 \qquad k > 2$$

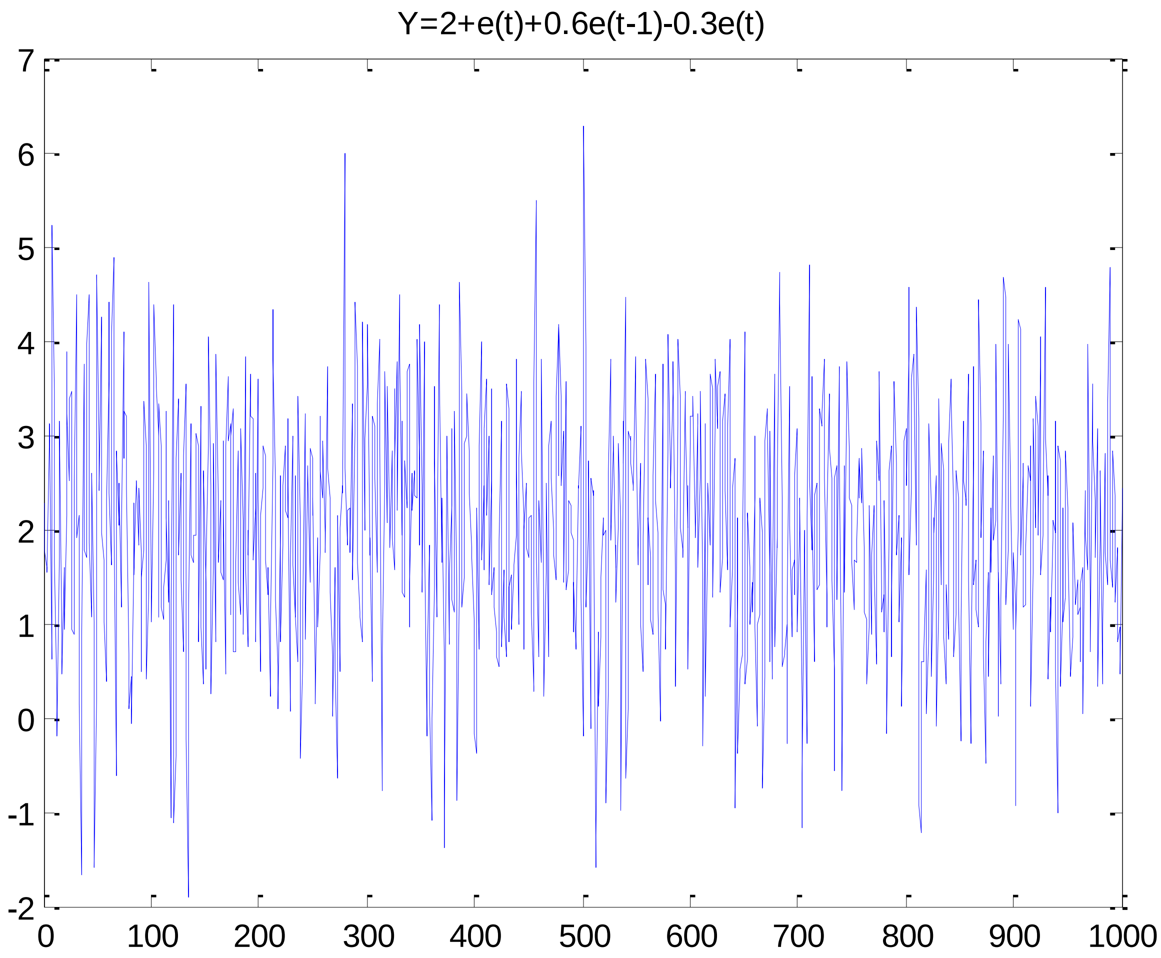
Example 2

$$Y_t = 2 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$

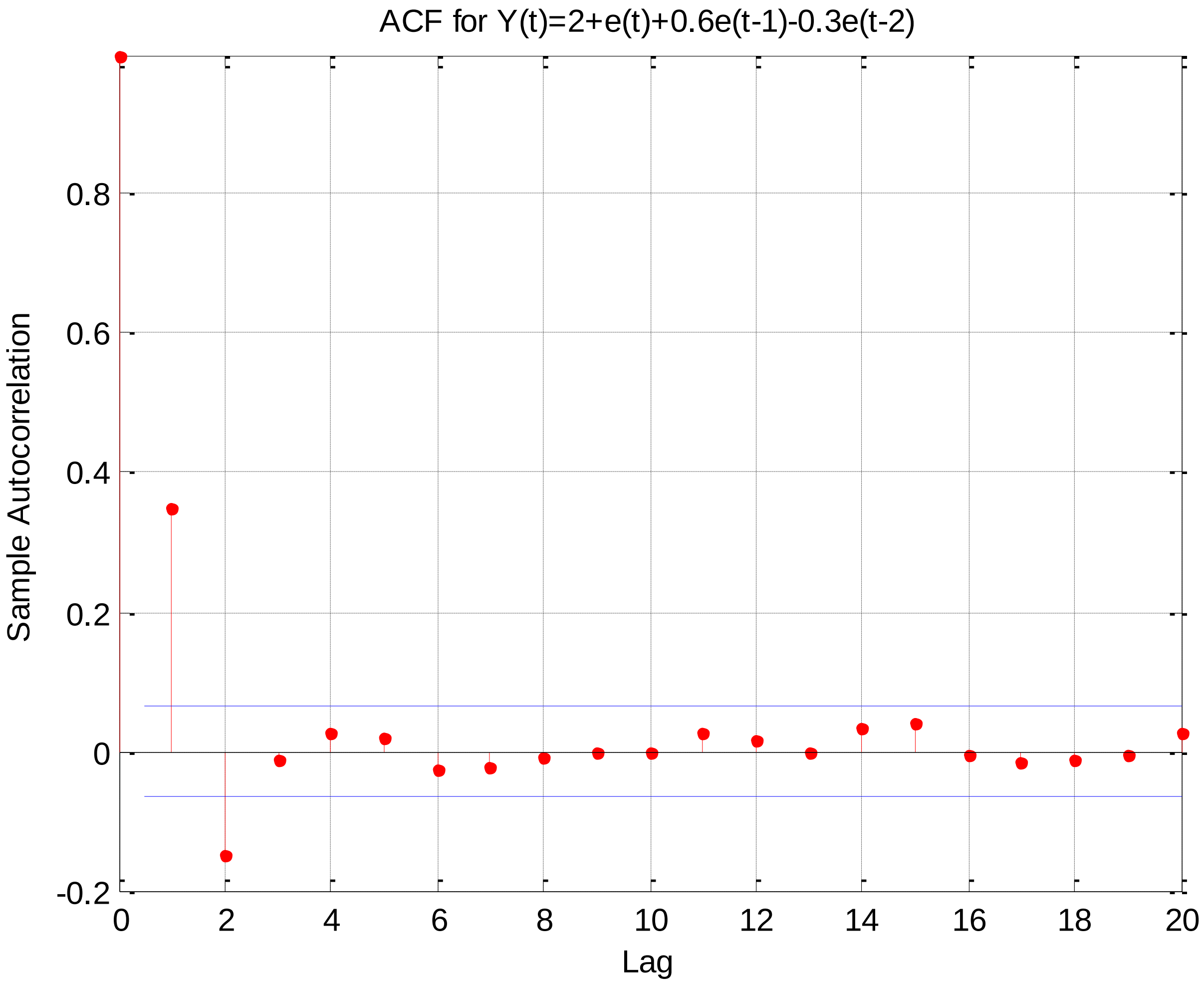
Example 2

```
ma2=arima('ma',[0.6 -0.3],'c',2, 'var',1)

ma1 =

    ARIMA(0,0,2) Model:
    --------------------
    Distribution: Name = 'Gaussian'
               P: 0
               D: 0
               Q: 2
        Constant: 2
              AR: {}
             SAR: {}
              MA: {0.6 -0.3} at Lags [1 2]
             SMA: {}
        Variance: 1
```

Example 2


Y=2+e(t)+0.6e(t-1)-0.3e(t)
7
6
5
4
3
2
1
0
-1
-2
0
100
200
300
400
500
600
700
800
900
1000

Example 2



ACF for Y(t)=2+e(t)+0.6e(t-1)-0.3e(t-2)
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

Autocorrelation Function for MA(M)

$$\rho_k = \begin{cases} \dfrac{\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{M-k}\theta_M}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_M^2} & k = 1, \ldots, M \\ 0 & k > M \end{cases}$$

Autoregression Models

$$Y_t = c + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \cdots + \varphi_R Y_{t-R} + e_t$$

$$E(Y_t) = E(Y_{t-1}) = \cdots = E(Y_{t-R}) = \mu \Rightarrow$$

$$\mu = c + \varphi_1 \mu + \varphi_2 \mu + \cdots + \varphi_R \mu \Rightarrow$$

$$\mu = \frac{c}{1 - \varphi_1 - \varphi_2 - \cdots - \varphi_R}$$

AR(1)

$$Y_t = c + \phi_1 Y_{t-1} + e_t$$

$$\mu = \frac{c}{1-\phi_1}$$

$$Let \quad \mu = 0 \Rightarrow Y_t = \phi_1 Y_{t-1} + e$$

$$\gamma_0 = E[(\phi_1 Y_{t-1} + e_t)^2] = E(\phi_1^2 Y_{t-1}^2 + e_t^2 + 2\phi_1 Y_{t-1} e_t) = \phi_1^2 \gamma_0 + \sigma_e^2 \Rightarrow$$

$$\gamma_0 = \frac{\sigma_e^2}{1-\phi_1^2}$$

$$\gamma_1 = E[Y_{t-1}(\phi_1 Y_{t-1} + e_t)] = \phi_1 \gamma_0 = \phi_1 \frac{\sigma_e^2}{1-\phi_1^2}$$

$$\gamma_2 = E[Y_{t-2}(\phi_1^2 Y_{t-2} + \phi_1 e_{t-1} + e_t)] = \phi_1^2 \gamma_0$$

AR(1)

$$\gamma_k = \phi_1^k \gamma_0 = \frac{\phi_1^k \sigma_e^2}{1 - \phi_1^2}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k$$

$$\left| \phi_1 \right| < 1$$

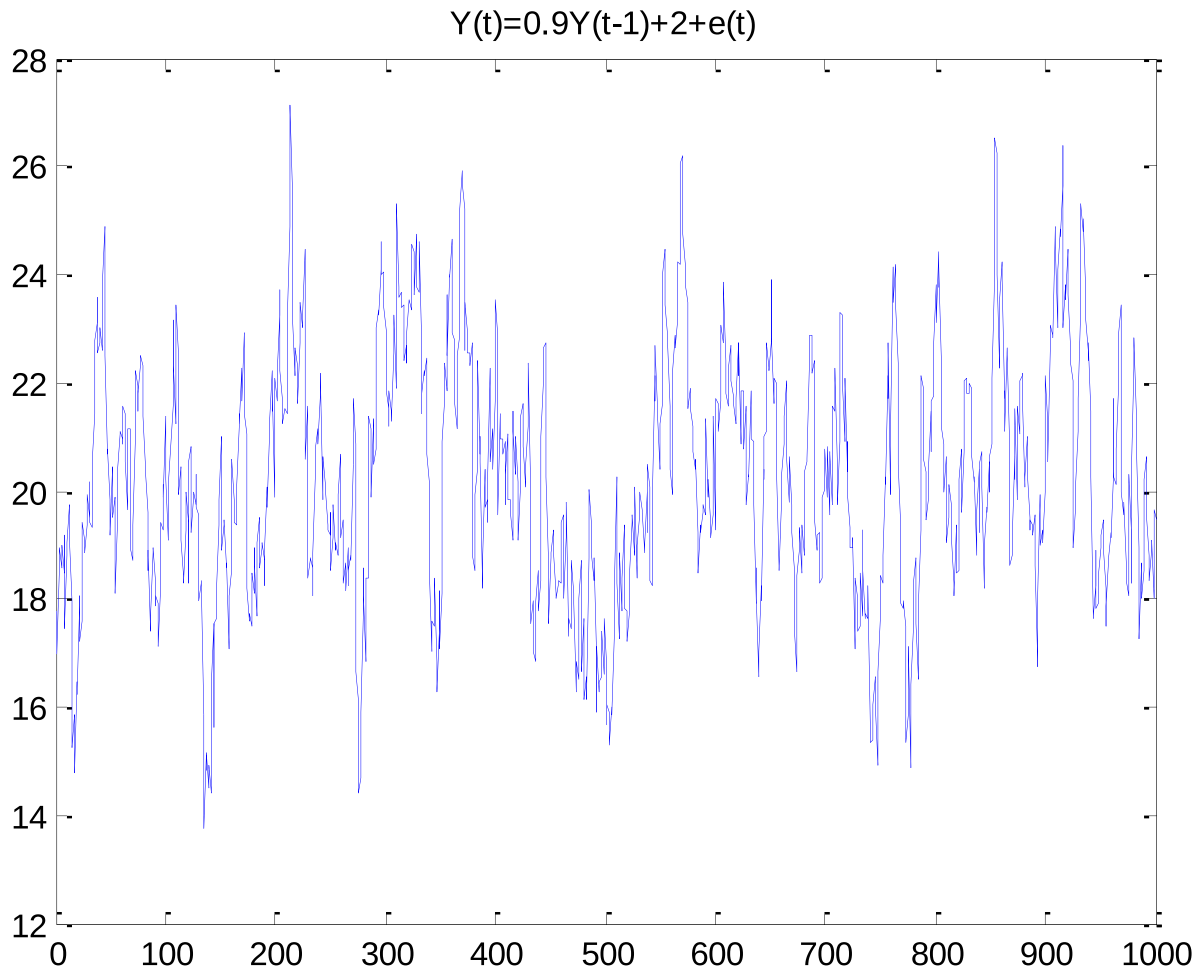
Example 3

$$Y_t = 0.9Y_{t-1} + 2 + e_t$$

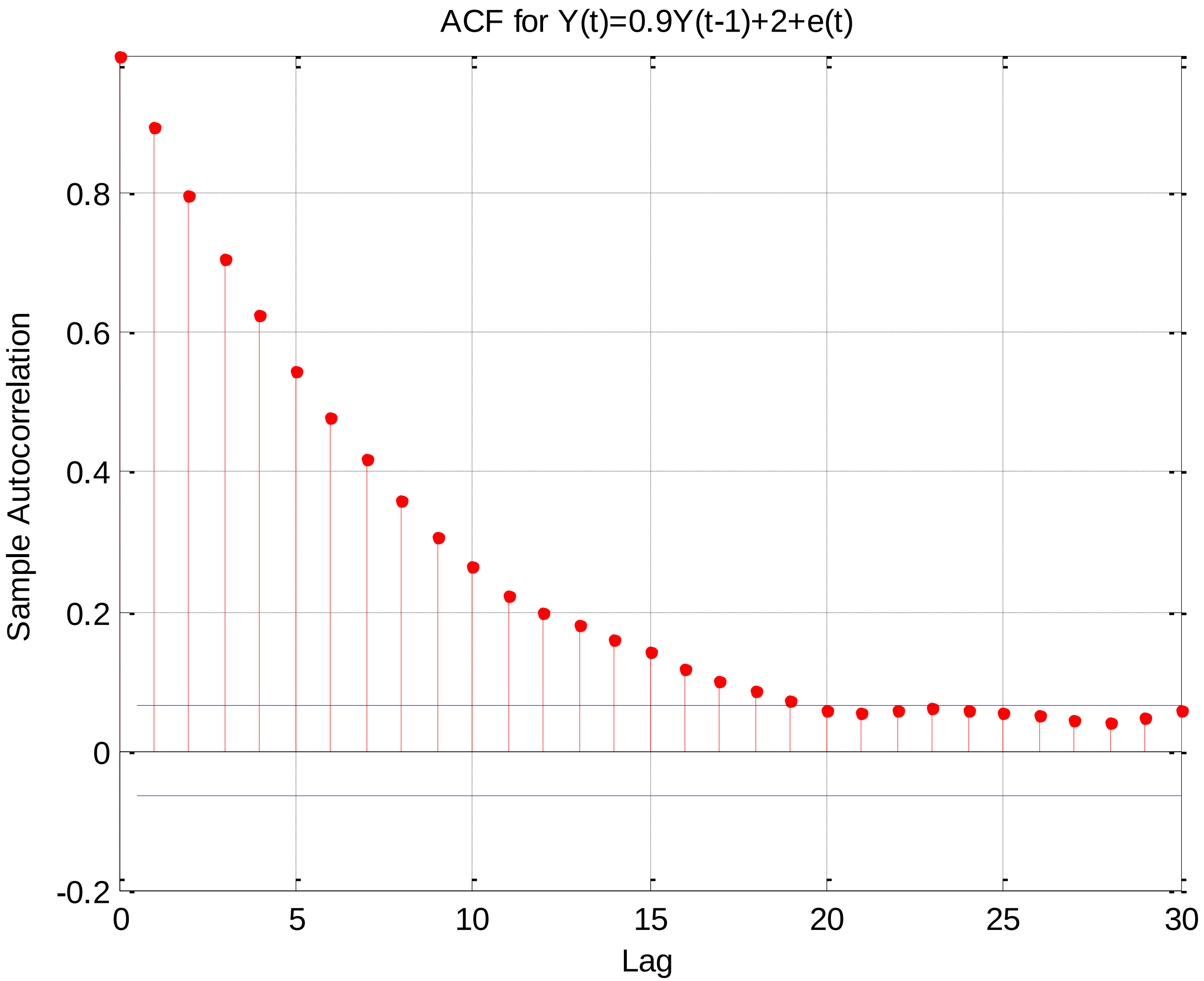
Example 3

```
ar1=arima('c',2, 'var',1,'ar',0.9)

ar1 =

    ARIMA(1,0,0) Model:
    --------------------
    Distribution: Name = 'Gaussian'
               P: 1
               D: 0
               Q: 0
        Constant: 2
              AR: {0.9} at Lags [1]
             SAR: {}
              MA: {}
             SMA: {}
        Variance: 1
```

Example 3


Y(t)=0.9Y(t-1)+2+e(t)
28
26
24
22
20
18
16
14
12
0
100
200
300
400
500
600
700
800
900
1000

Example 3

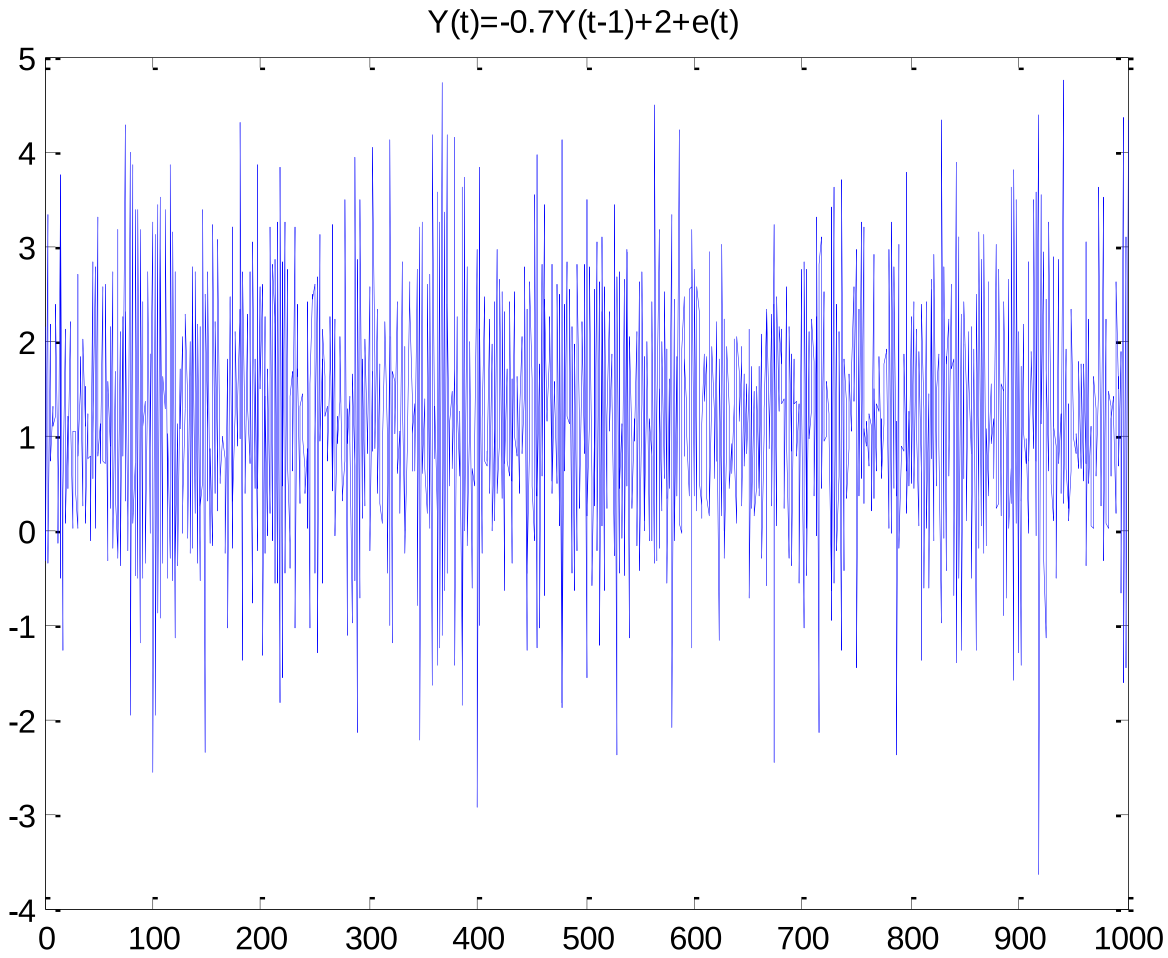


ACF for Y(t)=0.9Y(t-1)+2+e(t)
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
30
Lag

Example 4

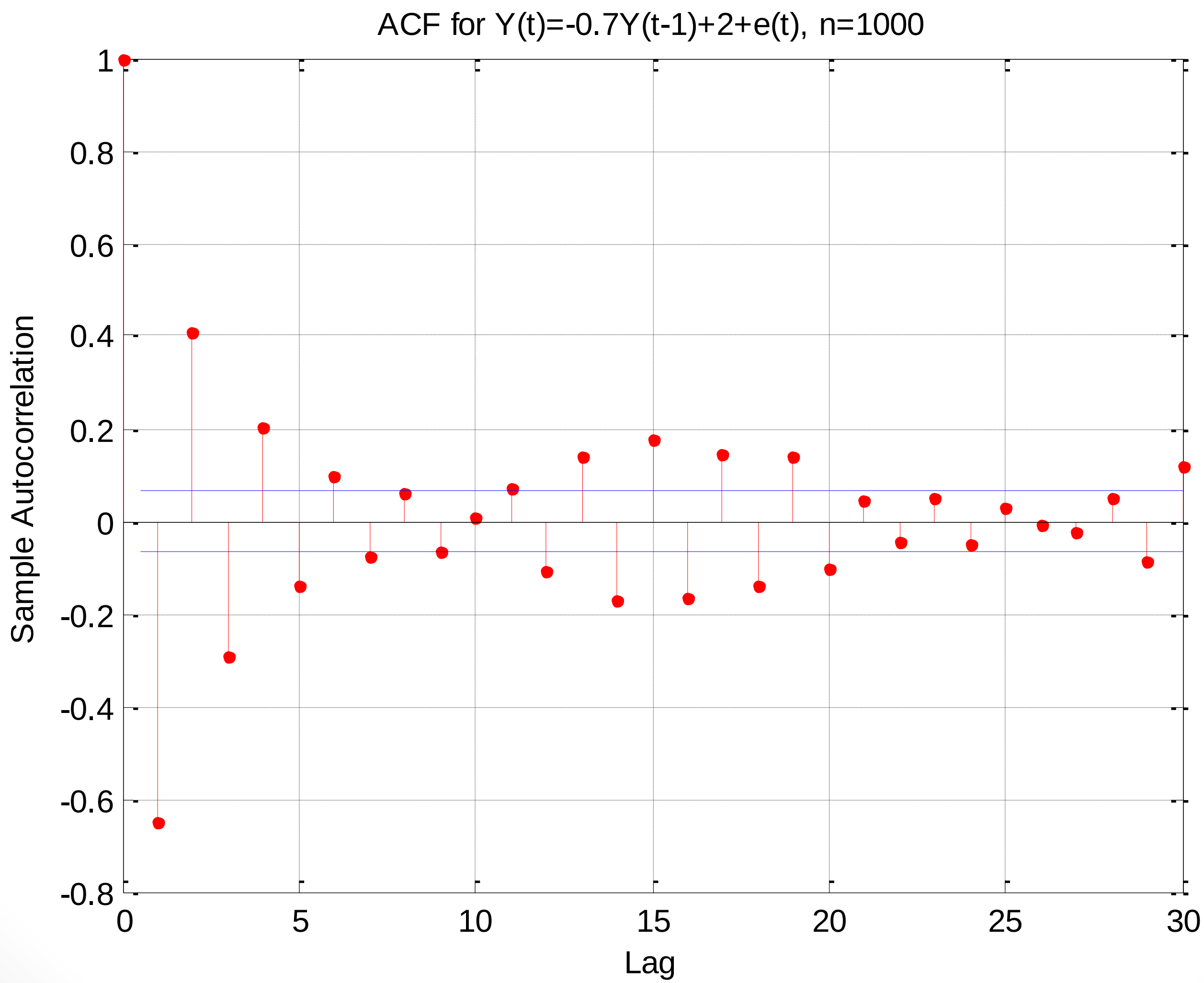
$$Y_t = -0.7Y_{t-1} + 2 + e_t$$

Example 4



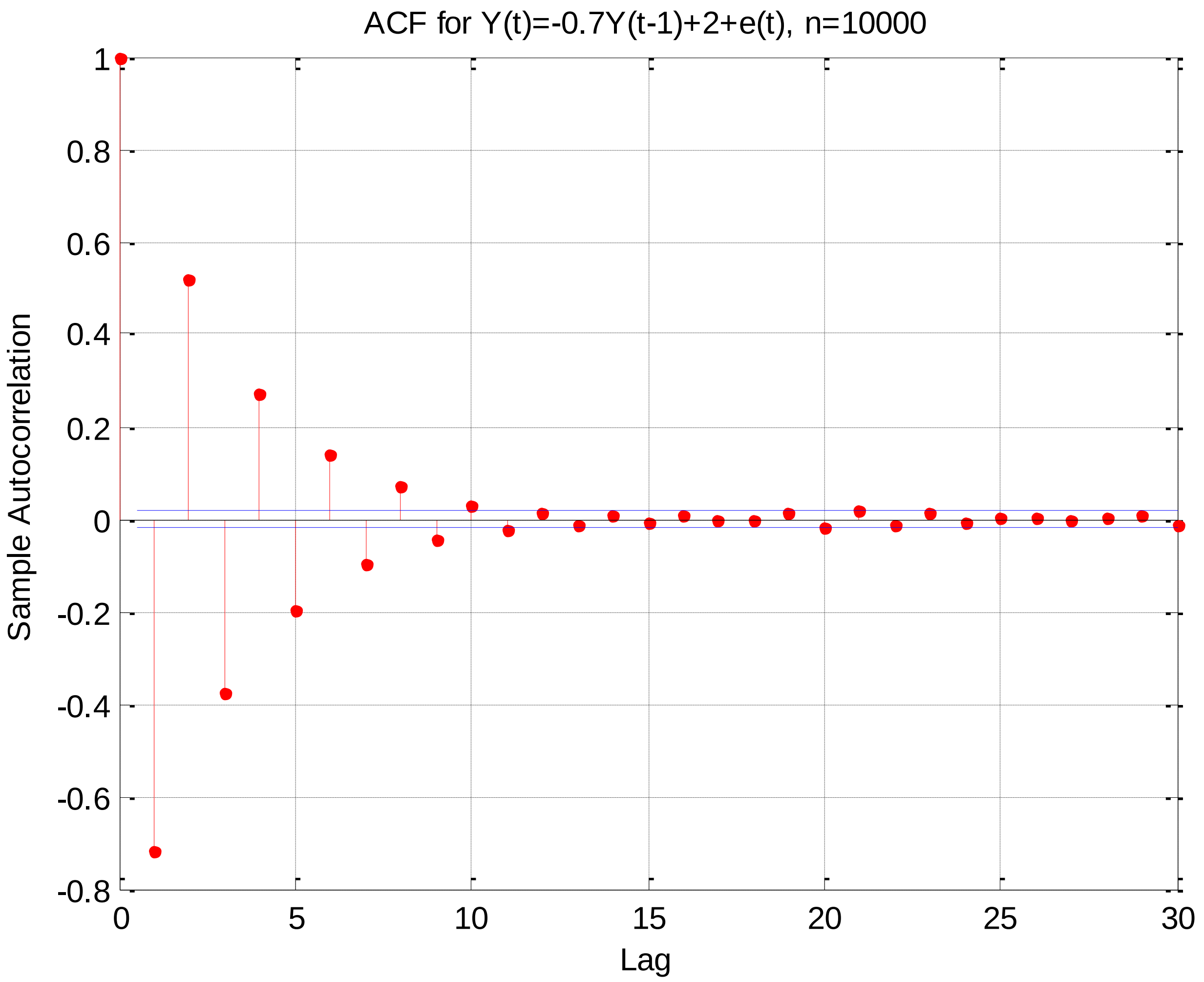
Y(t)=-0.7Y(t-1)+2+e(t)
5
4
3
2
1
0
-1
-2
-3
-4
0
100
200
300
400
500
600
700
800
900
1000

Example 4



ACF for Y(t)=-0.7Y(t-1)+2+e(t), n=1000
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Sample Autocorrelation
0
5
10
15
20
25
30
Lag

Example 4

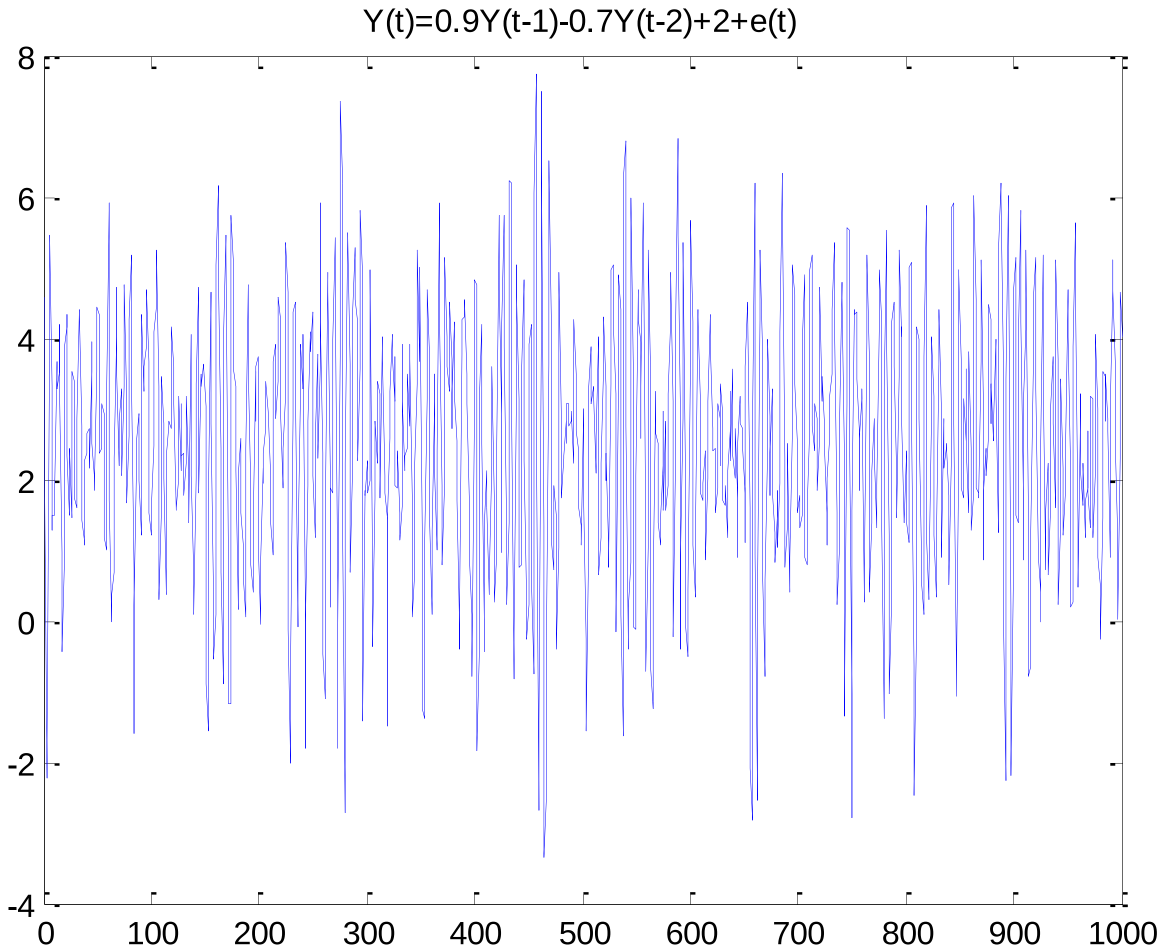


ACF for Y(t)=-0.7Y(t-1)+2+e(t), n=10000
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30

Example 5

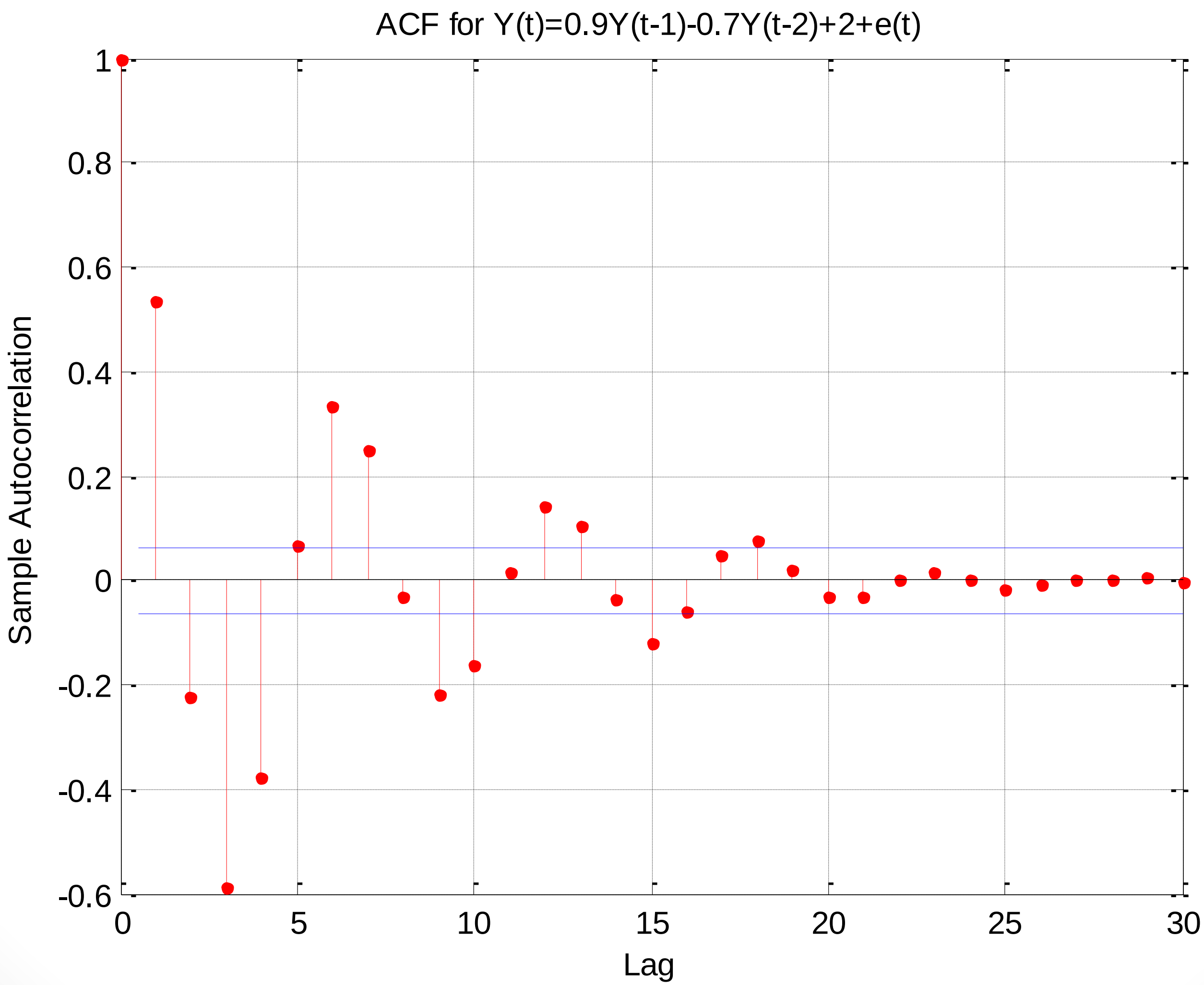
$$Y_t = 0.9Y_{t-1} - 0.7Y_{t-2} + 2 + e_t$$

Example 5



Y(t)=0.9Y(t-1)-0.7Y(t-2)+2+e(t)
8
6
4
2
0
-2
-4
0
100
200
300
400
500
600
700
800
900
1000

Example 5



ACF for Y(t)=0.9Y(t-1)-0.7Y(t-2)+2+e(t)
Sample Autocorrelation
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
5
10
15
20
25
30
Lag