Случайни времеви редове

Стационарност (1/2)

$$E(Y_t) = \mu$$

$$D(Y_t) = E[(Y_t - \mu)^2] = \sigma^2$$

$$Cov(Y_t, Y_{t-k}) = E[(Y_t - \mu)(Y_{t-k} - \mu)] = \gamma_k$$

Стационарност (2/2)

$$\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \bar{Y})^2$$

$$\hat{\gamma}_k = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y}) \quad k = 1,2,\ldots$$

Коефициент на Автокорелация

$$\rho_k = Corr(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{D(Y_t)D(Y_{t-k})}} = \frac{\gamma_k}{\sigma^2}$$

$$\rho_0 = 1$$

$$\rho_{-k} = \rho_k$$

Оценка на автокорелационния коефициент

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\sigma}^2} = \frac{\sum_{t=k+1}^{n}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2} \qquad k = 1,2,\ldots$$

$$E(\hat{\rho}_k) = \rho_k$$

$$D(\hat{\rho}_k) \cong \frac{1}{n}$$

Бял шум

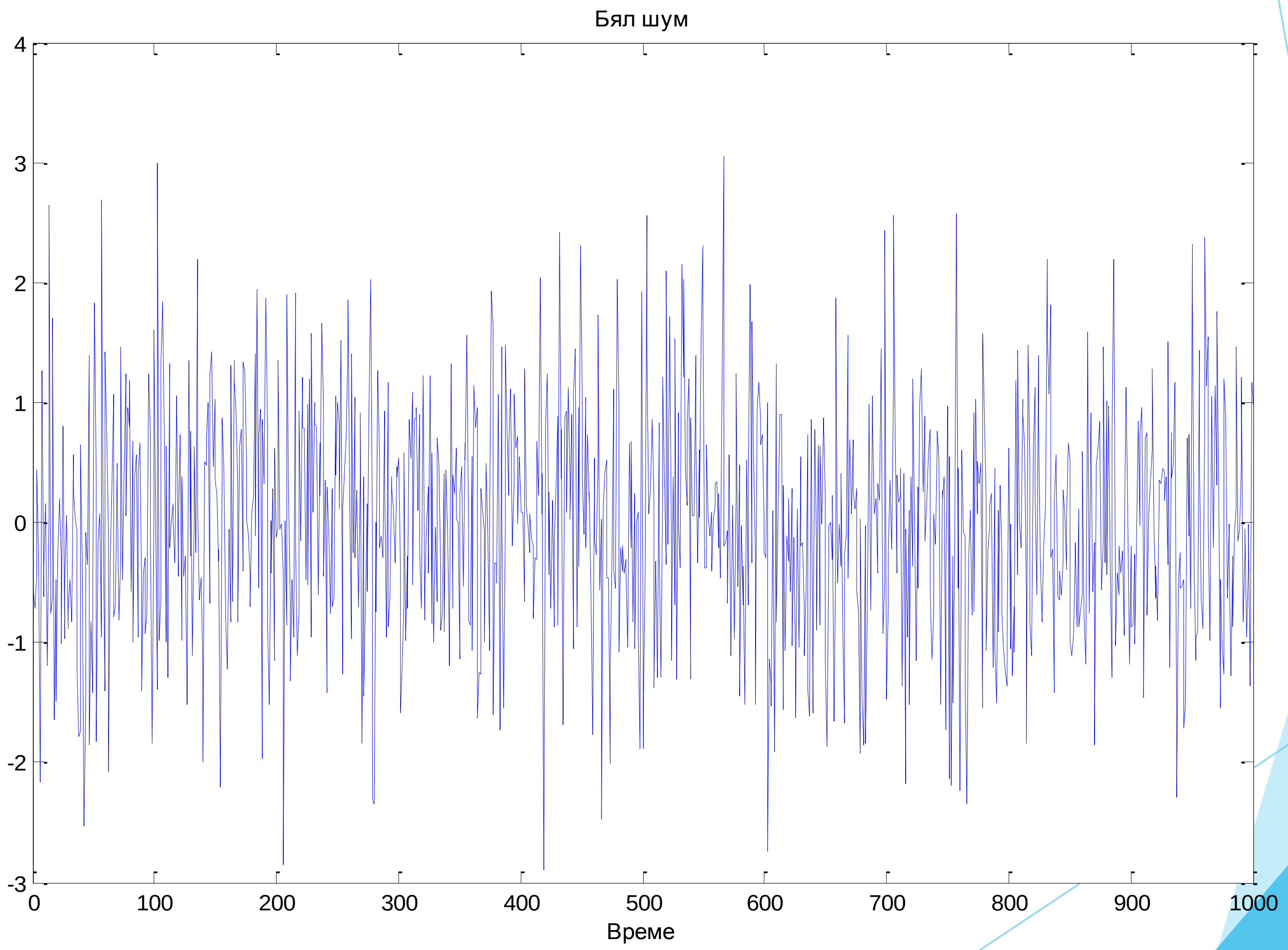
$$Y_t = e_t$$

$$E(e_t) = 0, \quad D(e_t) = \sigma^2,$$

$$Cov(e_t, e_k) = 0 \quad \text{завсички} \quad t \neq k$$

$$\rho_0 = 1, \quad \rho_k = 0 \quad k = 1,2,\ldots$$

Бял шум



Бял шум
4
3
2
1
0
-1
-2
-3
0
100
200
300
400
500
600
700
800
900
1000
Време

Тест на Ljung-Box

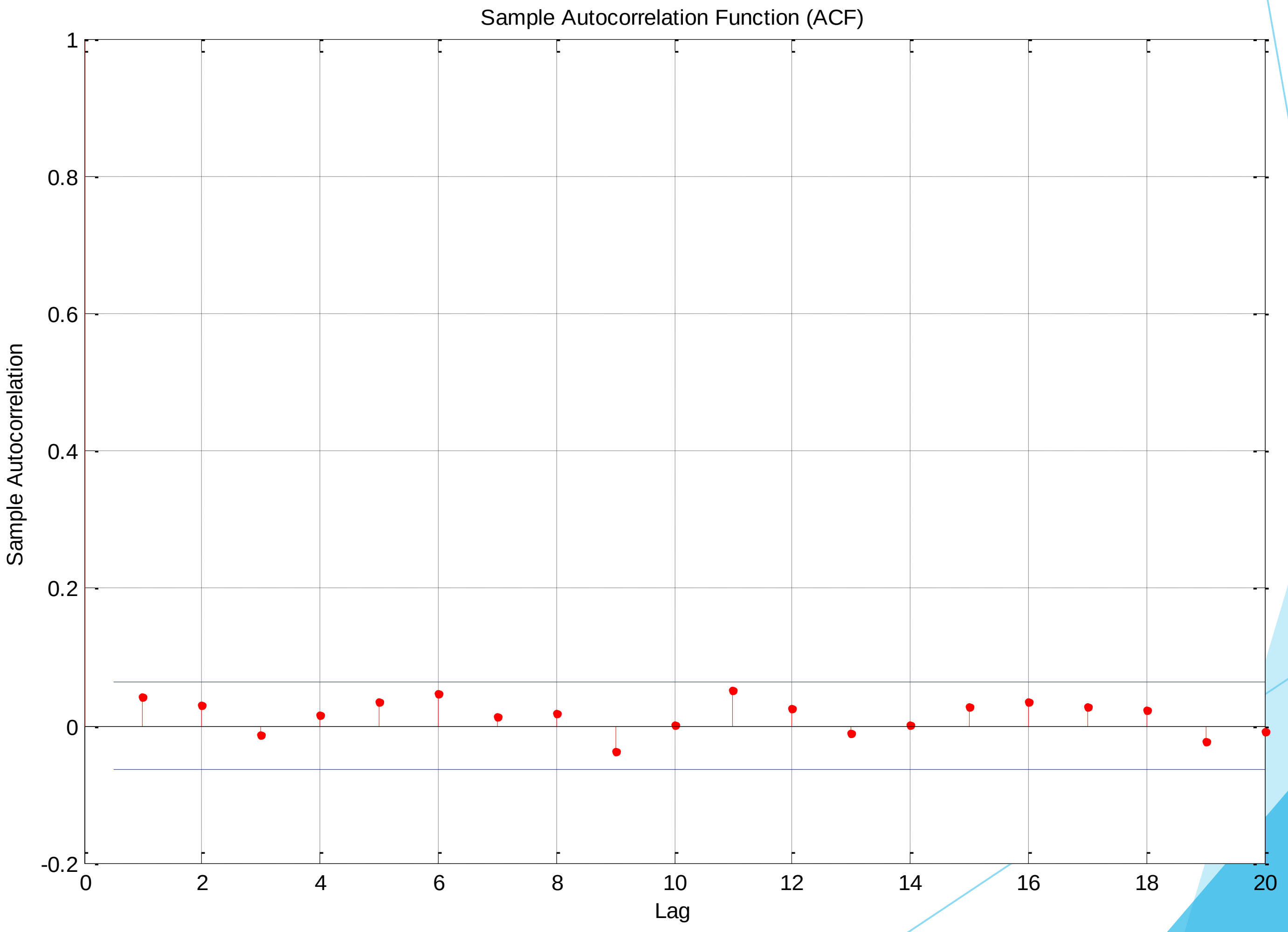
$$Q = n(n+2)\sum_{k=1}^{L}\frac{\hat{\rho}_k^2}{n-k}$$

$$Q \sim X^2$$

[H,pValue,Qstat,CriticalValue] = lbqtest(y,[20 30 40 50]);

H	pValue	Qstat	CriticalValue
0	0.741	15.6024	31.4104
0	0.8534	22.0165	43.773
0	0.8848	29.6479	55.7585
0	0.8518	39.6861	67.5048

Автокорелационна функция на “бял шум”



Случайно блуждаене (Random walk) (1/2)

$$Y_t = Y_{t-1} + e_t$$

$$E(e_t) = \mu,\ D(e_t) = \sigma^2,\ Cov(e_t, e_{t-k}) = 0 \ \ for\ k \neq 0$$

$$Y_0 = 0, \quad t = 0$$

$$Y_1 = e_1, Y_2 = e_1 + e_2, \ldots$$

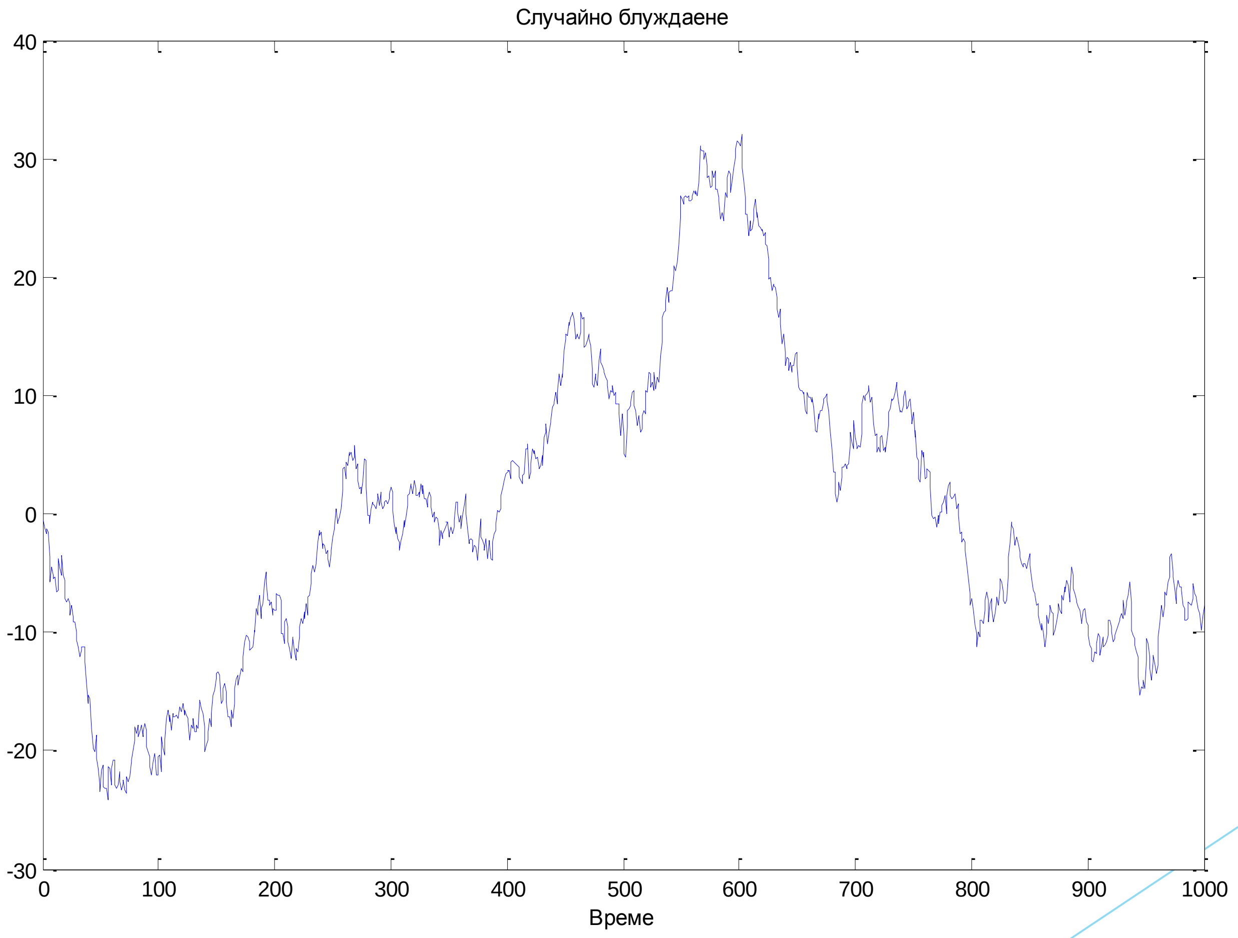
Случайно блуждаене (Random walk) (2/2)

$$Y_t = \sum_{i=0}^{t} e_i$$

$$E(Y_t) = E(\sum_{i=0}^{t} e_i) = t\mu$$

$$D(Y_t) = t\sigma^2$$

Случайно блуждаене



Тест на Dickey-Fuller

$$Y_t = \phi Y_{t-1} + e_t$$

$$\Delta Y_t = (\phi - 1) Y_{t-1} + e_t,$$

$$\Delta Y_t = \delta Y_{t-1} + e_t, \quad \delta = (\phi - 1)$$

$$H_0 : \delta = 0$$

[H,pValue,TestStat,CriticalValue] = adftest(x);

H	pValue	Qstat	Critical Value
0	0.936	-1.0419	-3.4145

Линейни модели на времеви редове – ARIMA(R,d,M)

Първа част

Безусловна и условна средна

- За случайната променлива y_t, *безусловната средна стойност* е математическото очакване $E(y_t)$.
- *Условната средна на y очакваната стойност на y_t при зададено обуславящо множество Ω_t.*
- *Моделите на условната средна специфицират функционалната форма за $E(y_t \mid \Omega_t)$.*

Статични модели на условната средна

- При статичните модели на условната средна условното множество от променливи се измерва едновременно със зависимата променлива y_t.
- Пример за статичен модел на условната средна е класическия линеен регресионен модел. При зададени x_t – вектор ред от екзогенни променливи в момента t, и β – вектор колона от коефициенти, условната средна на y_t се задава като линейна комбинация:

$$E(y_t|x_t) = x_t\beta$$

(т.е. обуславящото множество е Ω_t=x_t).

Динамични модели на условната средна

- В теорията на времевите редове основно интересът е към динамичното поведение на една променлива във времето.
- Моделите на условната средна специфицират очакваната стойност на y_t като функция от историческа информация. Нека означим с $H_{t\text{-}1}$ историята на процеса налична към момента t. Динамичните модели на условната средна специфицират еволюцията на условната средна във времето: $E(y_t | H_{t\text{-}1})$.
- Примери за историческа информация са:
  - Предишни наблюдения: y_1, y_2,…,$y_{t\text{-}1}$
  - Предишни иновации: e_1, e_2,… e_{t-1},

Модели от тип “пълзящо средно” – MA(M)

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_M e_{t-M}$$

$$E(Y_t) = \mu$$

$$D(Y_t) = E[(Y_t - \mu)^2] = E(e_t^2 + \theta_1^2 e_{t-1}^2 + \cdots + \theta_M^2 e_{t-M}^2 + 2\theta_1 e_t e_{t-1} + \cdots) \Rightarrow$$

$$D(Y_t) = \sigma_e^2 + \theta_1^2 \sigma_e^2 + \cdots + \theta_M^2 \sigma_e^2 \Rightarrow$$

$$D(Y_t) = \sigma_e^2 (1 + \theta_1^2 + \cdots + \theta_M^2)$$

MA(1) модел

$$Y_t = \mu + e_t + \theta_1 e_{t-1}$$

$$E(Y_t) = \mu, \quad D(Y_t) = \gamma_0 = \sigma_e^2(1+\theta_1^2)$$

$$\gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)] = E[(e_t + \theta_1 e_{t-1})(e_{t-1} + \theta_1 e_{t-2})] \Rightarrow$$

$$\gamma_1 = \theta_1 \sigma_e^2$$

$$\gamma_k = E[(e_t + \theta_1 e_{t-1})(e_{t-k} + \theta_1 e_{t-k-1})] = 0 \quad при\ k > 1$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\theta_1}{1+\theta_1^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

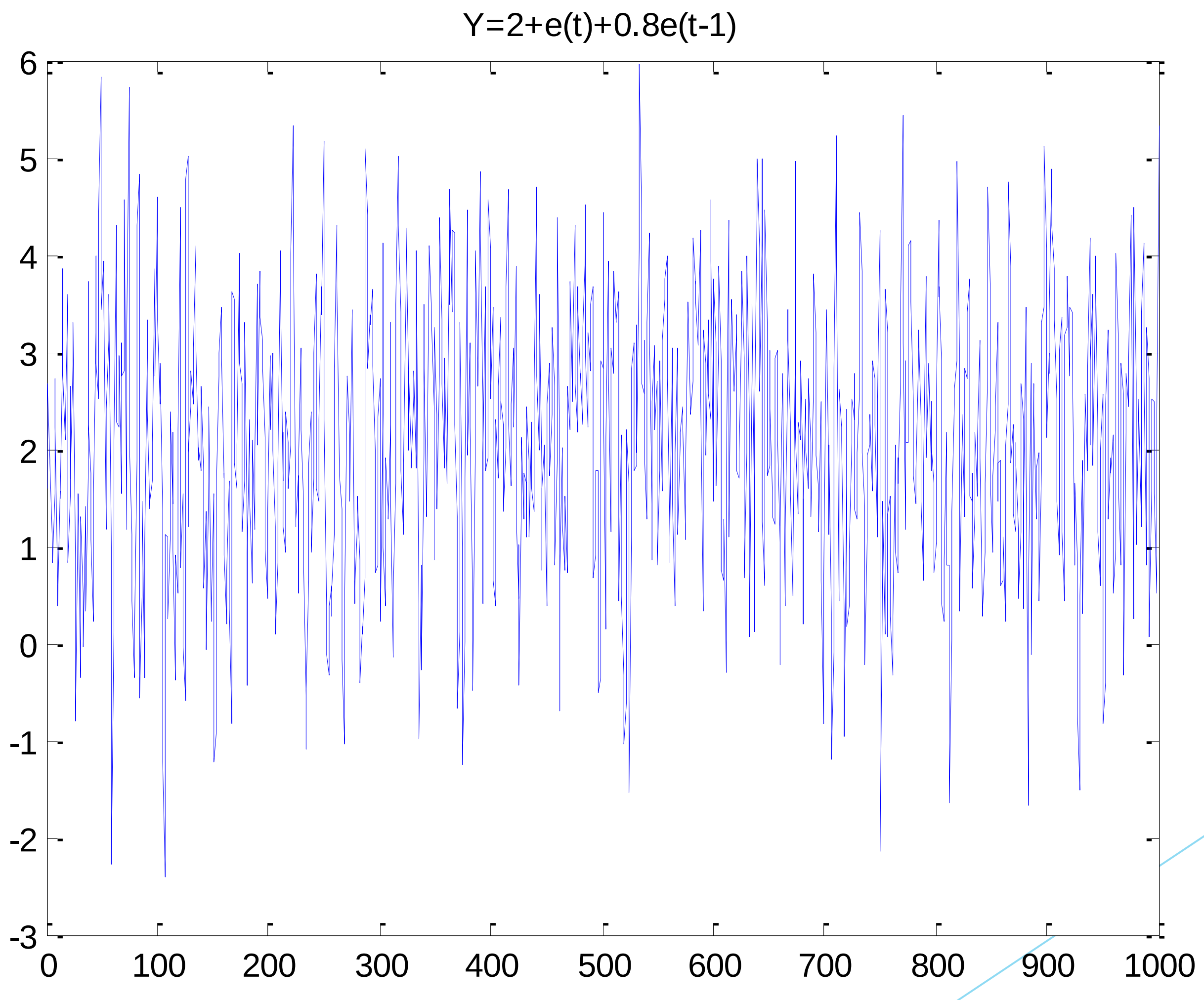
Пример 1

$$Y_t = 2 + e_t + 0.8e_{t-1}$$

Пример 1

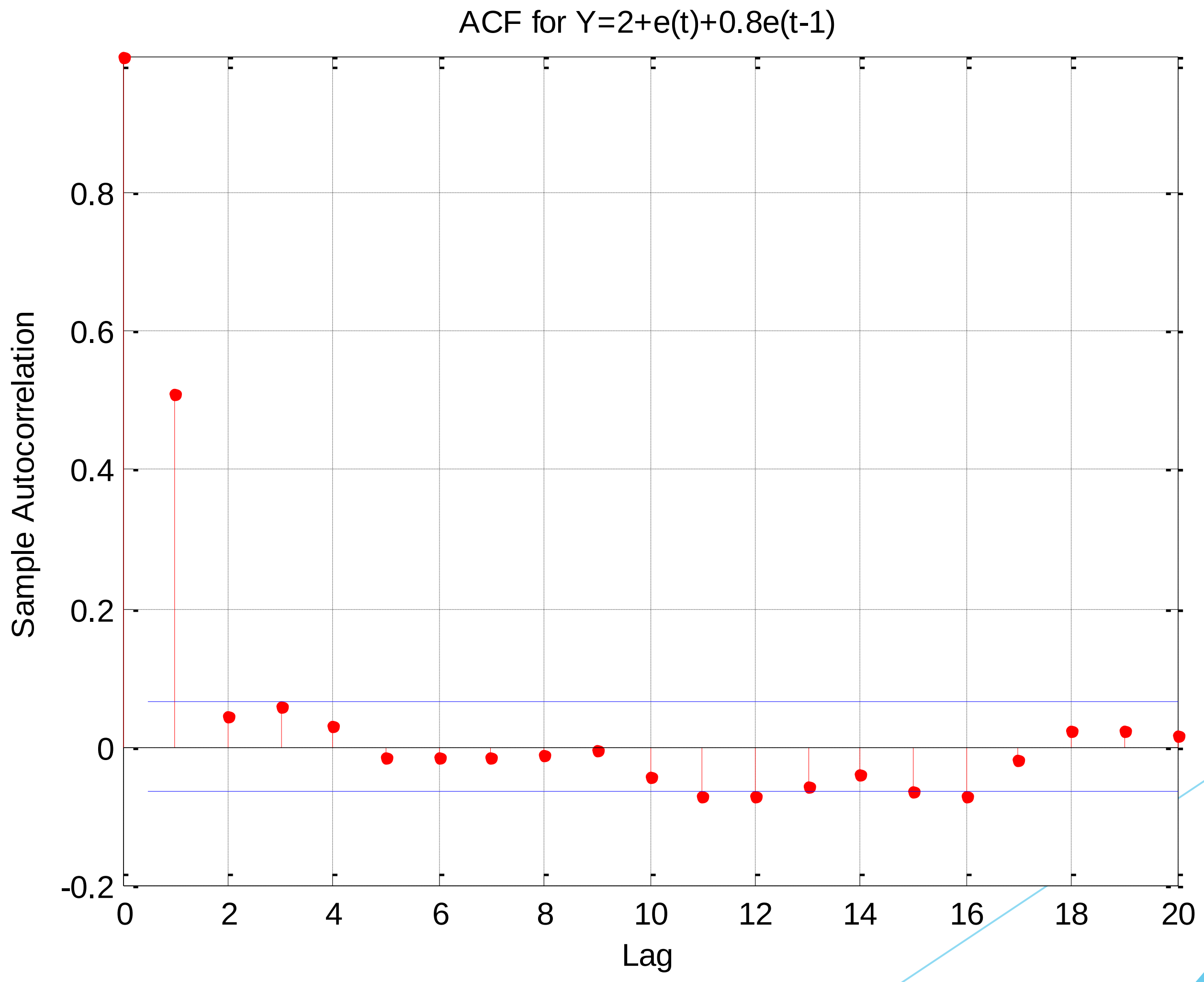
```
ma1=arima('ma',0.8,'c',2, 'var',1)
ma1 =
    ARIMA(0,0,1) Model:
    --------------------
    Distribution: Name = 'Gaussian'
               P: 0
               D: 0
               Q: 1
        Constant: 2
              AR: {}
             SAR: {}
              MA: {0.8} at Lags [1]
             SMA: {}
        Variance: 1

y=simulate(ma1,1000);
```

Пример 1



Y=2+e(t)+0.8e(t-1)
6
5
4
3
2
1
0
-1
-2
-3
0
100
200
300
400
500
600
700
800
900
1000

Пример 1



ACF for Y=2+e(t)+0.8e(t-1)
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

MA(2) модел

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}$$

$$\gamma_1 = E[(e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2})(e_{t-1} + \theta_1 e_{t-2} + \theta_2 e_{t-3})] \Rightarrow$$

$$\gamma_1 = \theta_1 \sigma_e^2 + \theta_2 \theta_1 \sigma_e^2 = \theta_1 (1 + \theta_2) \sigma_e^2$$

$$\gamma_2 = \theta_2 \sigma_e^2$$

$$\gamma_k = 0 \quad \text{при } k > 2$$

MA(2) модел

$$\rho_1 = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$$

$$\rho_2 = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\rho_k = 0 \quad npu\; k > 2$$

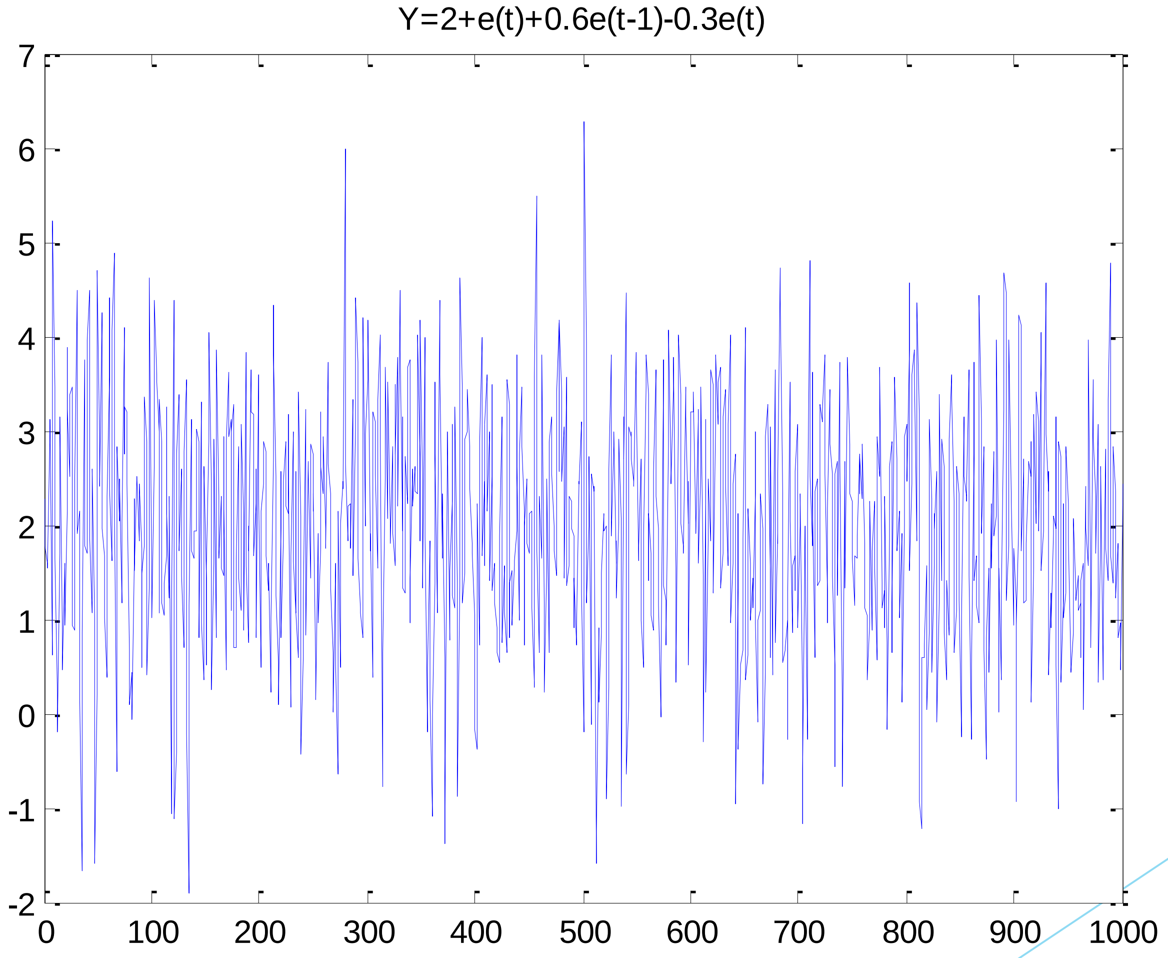
Пример 2

$$Y_t = 2 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$

Пример 2

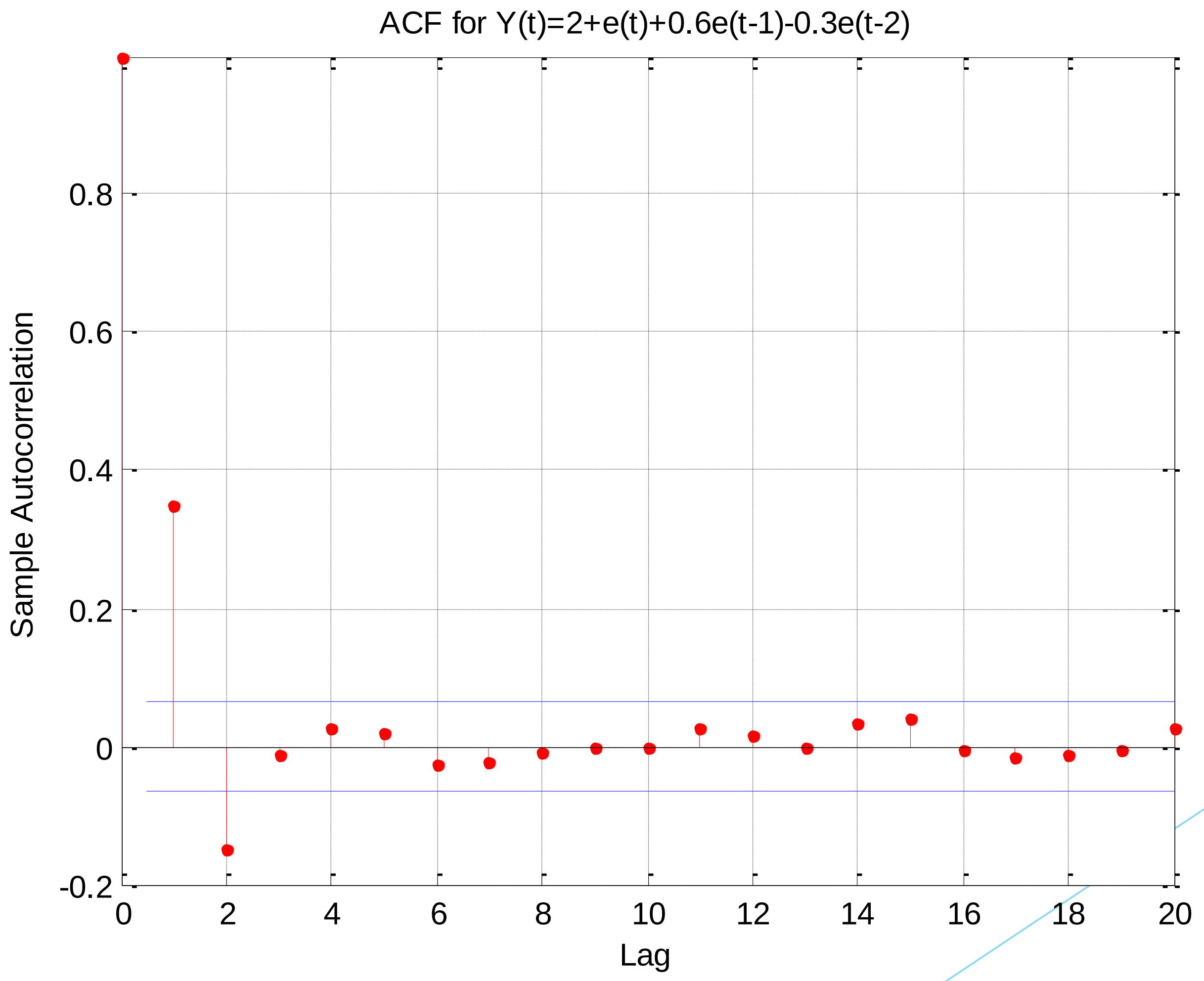
```
ma2=arima('ma',[0.6 -0.3],'c',2, 'var',1)

ma1 =

    ARIMA(0,0,2) Model:
    --------------------
    Distribution: Name = 'Gaussian'
               P: 0
               D: 0
               Q: 2
        Constant: 2
              AR: {}
             SAR: {}
              MA: {0.6 -0.3} at Lags [1 2]
             SMA: {}
        Variance: 1
```

Пример 2



Y=2+e(t)+0.6e(t-1)-0.3e(t)
7
6
5
4
3
2
1
0
-1
-2
0
100
200
300
400
500
600
700
800
900
1000

Пример 2



ACF for Y(t)=2+e(t)+0.6e(t-1)-0.3e(t-2)
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

Автокорелации при MA(M) модел

$$\rho_k = \begin{cases} \dfrac{\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{M-k}\theta_M}{1+\theta_1^2 + \theta_2^2 + \cdots + \theta_M^2} & k = 1, \ldots, M \\ 0 & k > M \end{cases}$$

Авторегресионни модели

$$Y_t = c + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \cdots + \varphi_R Y_{t-R} + e_t$$

$$E(Y_t) = E(Y_{t-1}) = \cdots = E(Y_{t-R}) = \mu \Rightarrow$$

$$\mu = c + \varphi_1 \mu + \varphi_2 \mu + \cdots + \varphi_R \mu \Rightarrow$$

$$\mu = \frac{c}{1 - \varphi_1 - \varphi_2 - \cdots - \varphi_R}$$

AR(1) модел

$$Y_t = c + \phi_1 Y_{t-1} + e_t$$

$$\mu = \frac{c}{1-\phi_1}$$

$$\text{Нека} \quad \mu = 0 \Rightarrow Y_t = \phi_1 Y_{t-1} + e$$

$$\gamma_0 = E[(\phi_1 Y_{t-1} + e_t)^2] = E(\phi_1^2 Y_{t-1}^2 + e_t^2 + 2\phi_1 Y_{t-1} e_t) = \phi_1^2 \gamma_0 + \sigma_e^2 \Rightarrow$$

$$\gamma_0 = \frac{\sigma_e^2}{1-\phi_1^2}$$

$$\gamma_1 = E[Y_{t-1}(\phi_1 Y_{t-1} + e_t)] = \phi_1 \gamma_0 = \phi_1 \frac{\sigma_e^2}{1-\phi_1^2}$$

$$\gamma_2 = E[Y_{t-2}(\phi_1^2 Y_{t-2} + \phi_1 e_{t-1} + e_t)] = \phi_1^2 \gamma_0$$

AR(1) модел

$$\gamma_k = \phi_1^k \gamma_0 = \frac{\phi_1^k \sigma_e^2}{1-\phi_1^2}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k$$

$$|\phi_1| < 1$$

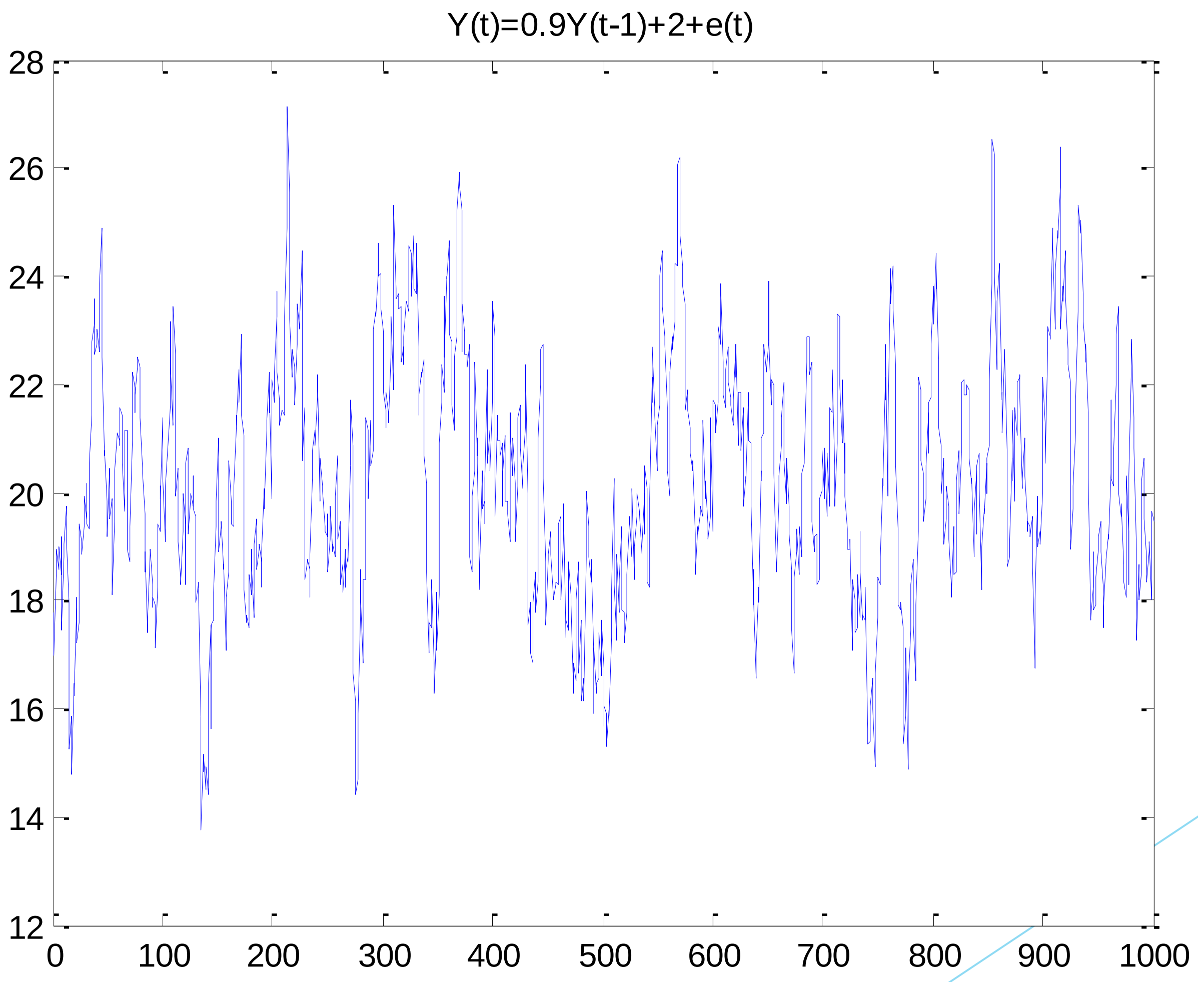
Пример 3

$$Y_t = 0.9Y_{t-1} + 2 + e_t$$

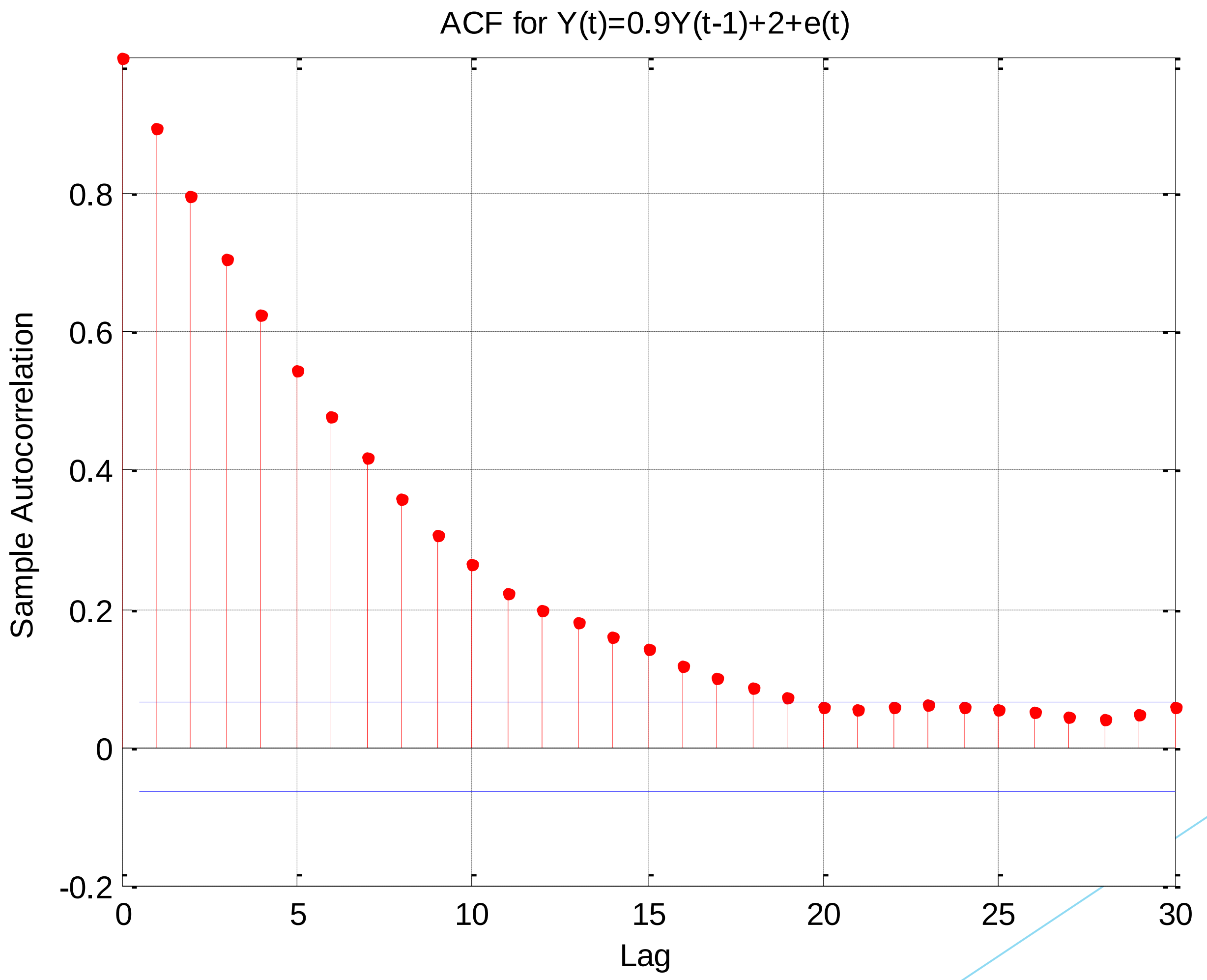
Пример 3

```
ar1=arima('c',2, 'var',1,'ar',0.9)

ar1 =

   ARIMA(1,0,0) Model:
   --------------------
   Distribution: Name = 'Gaussian'
             P: 1
             D: 0
             Q: 0
      Constant: 2
            AR: {0.9} at Lags [1]
           SAR: {}
            MA: {}
           SMA: {}
      Variance: 1
```

Пример 3


Y(t)=0.9Y(t-1)+2+e(t)
28
26
24
22
20
18
16
14
12
0
100
200
300
400
500
600
700
800
900
1000

Пример 3

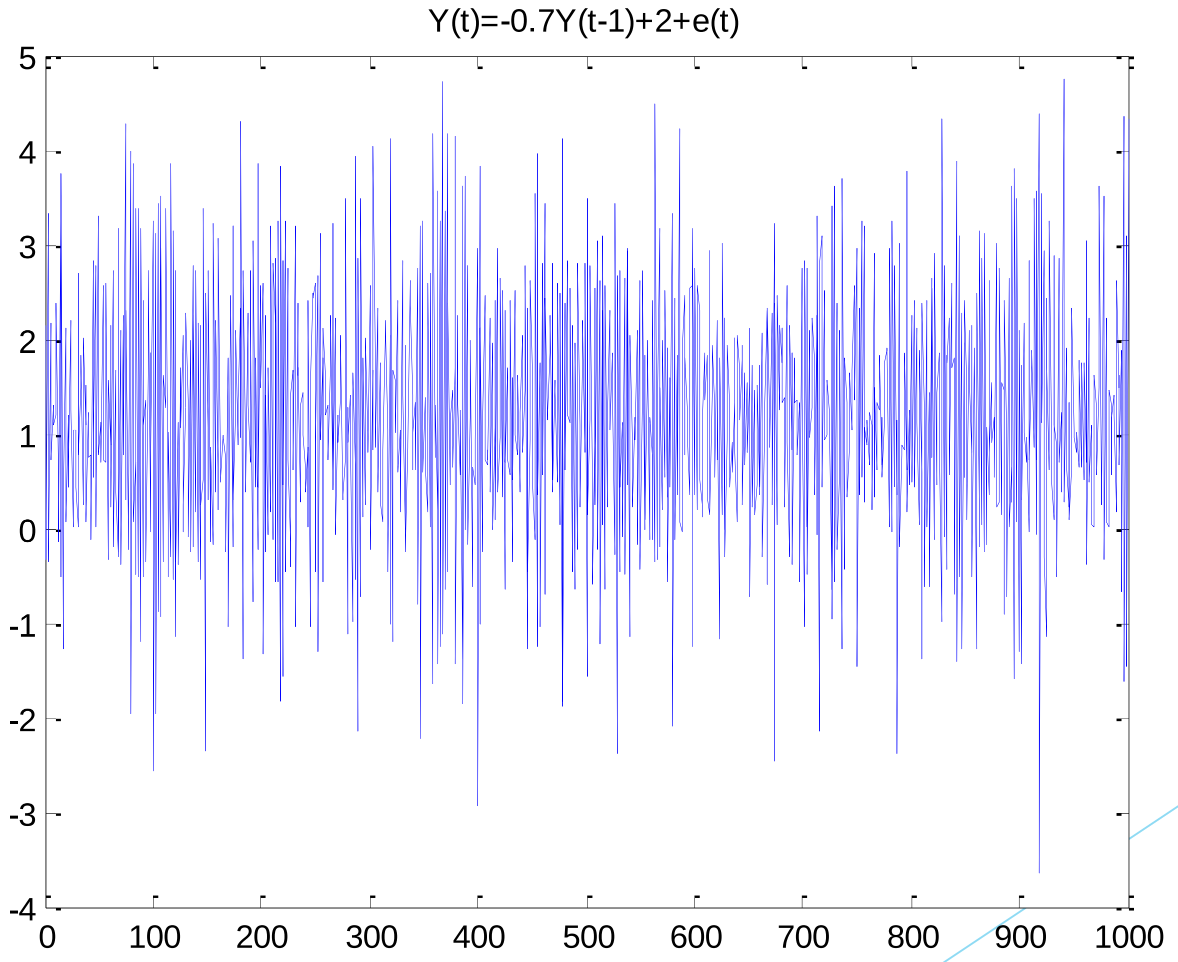


ACF for Y(t)=0.9Y(t-1)+2+e(t)
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
30
Lag

Пример 4

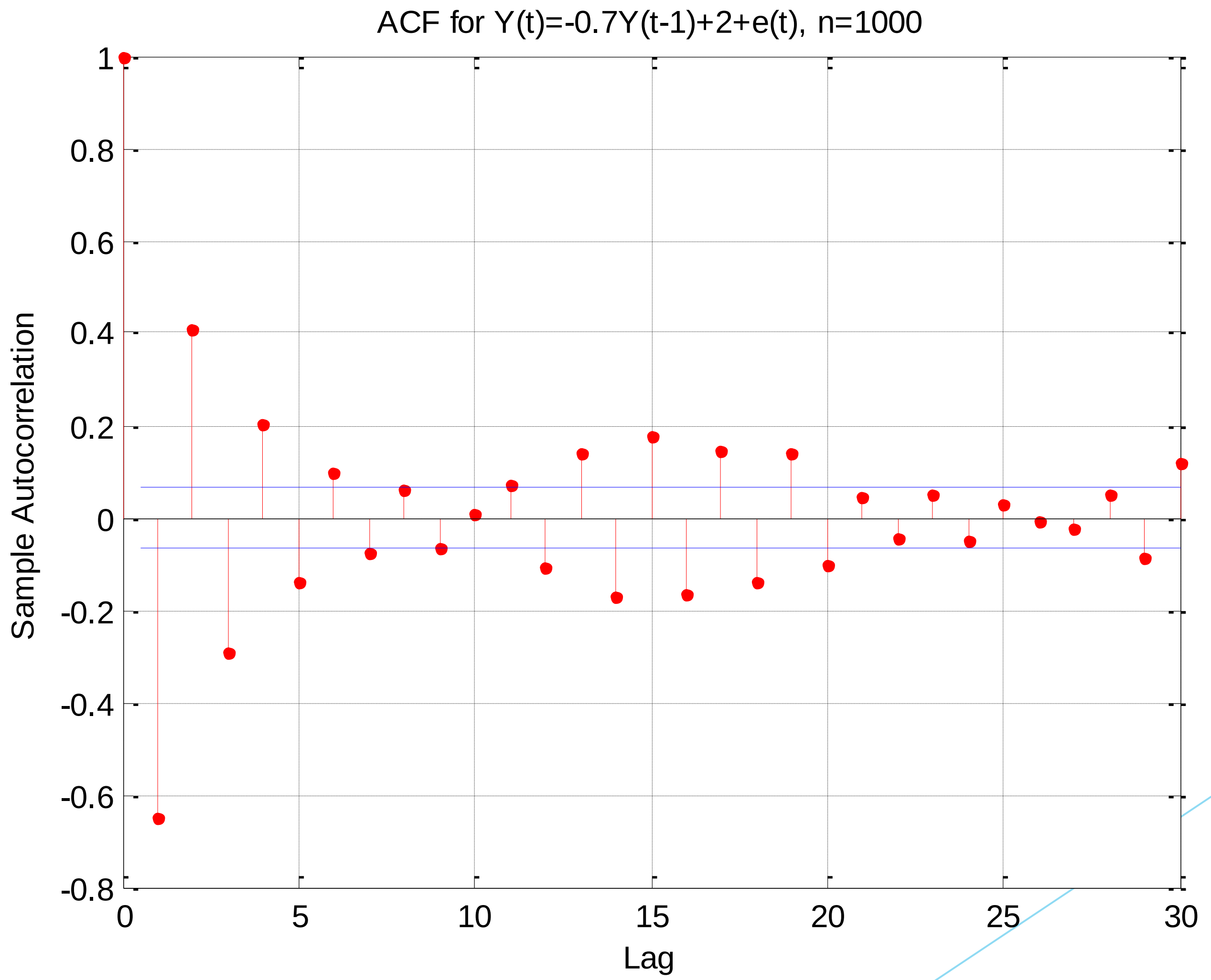
$$Y_t = -0.7Y_{t-1} + 2 + e_t$$

Пример 4



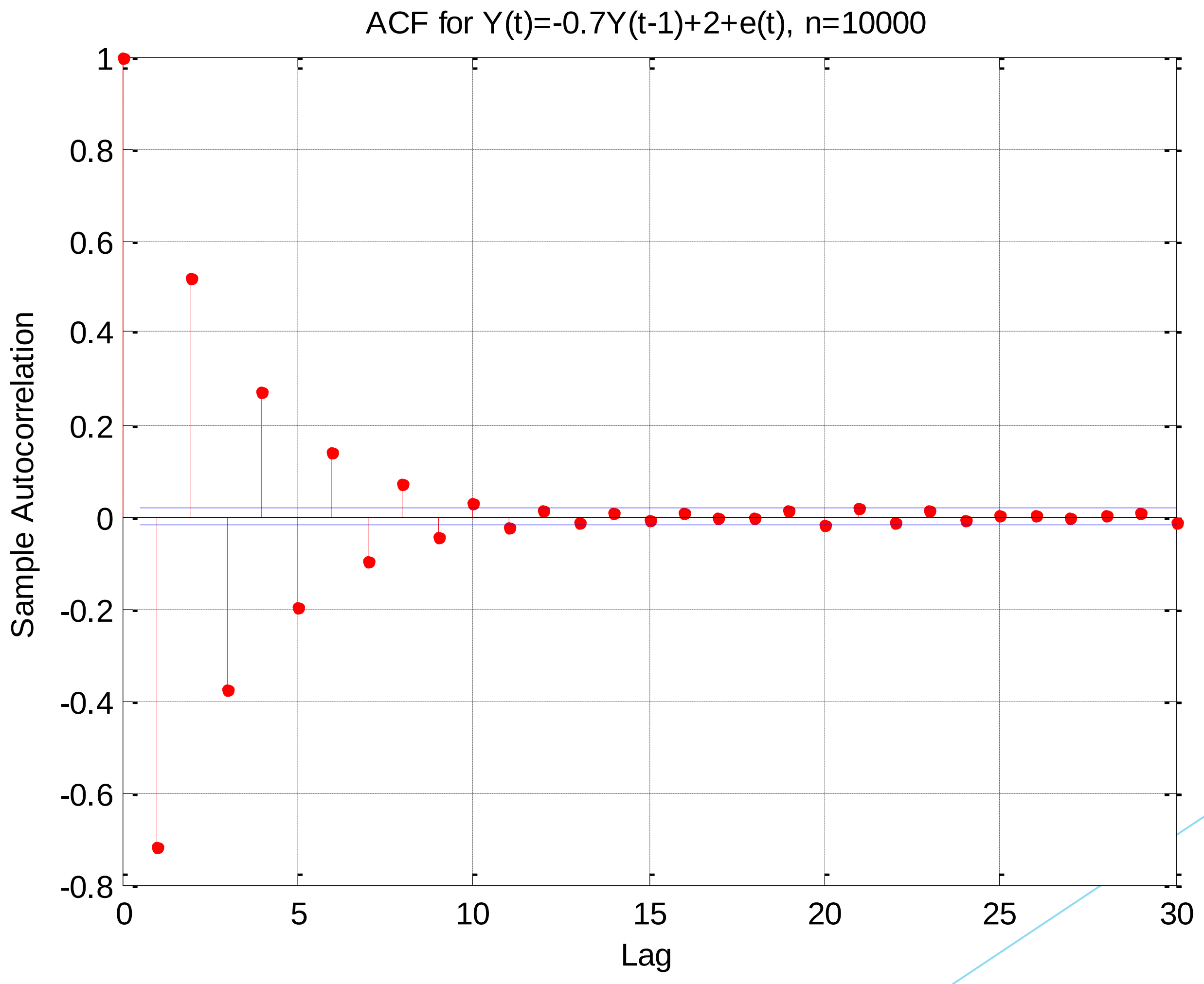
Y(t)=-0.7Y(t-1)+2+e(t)
5
4
3
2
1
0
-1
-2
-3
-4
0
100
200
300
400
500
600
700
800
900
1000

Пример 4



ACF for Y(t)=-0.7Y(t-1)+2+e(t), n=1000
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30

Пример 4

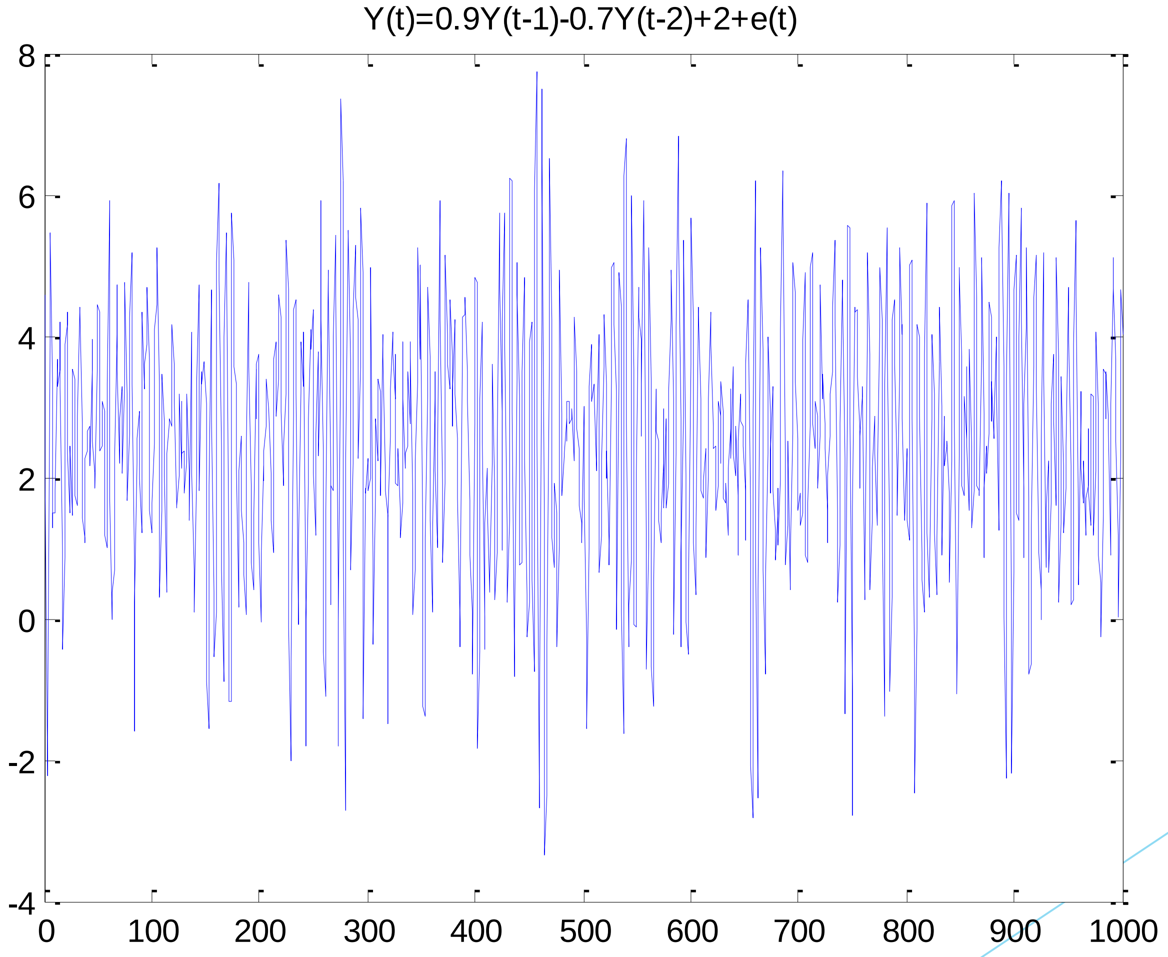


ACF for Y(t)=-0.7Y(t-1)+2+e(t), n=10000
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30

Пример 5

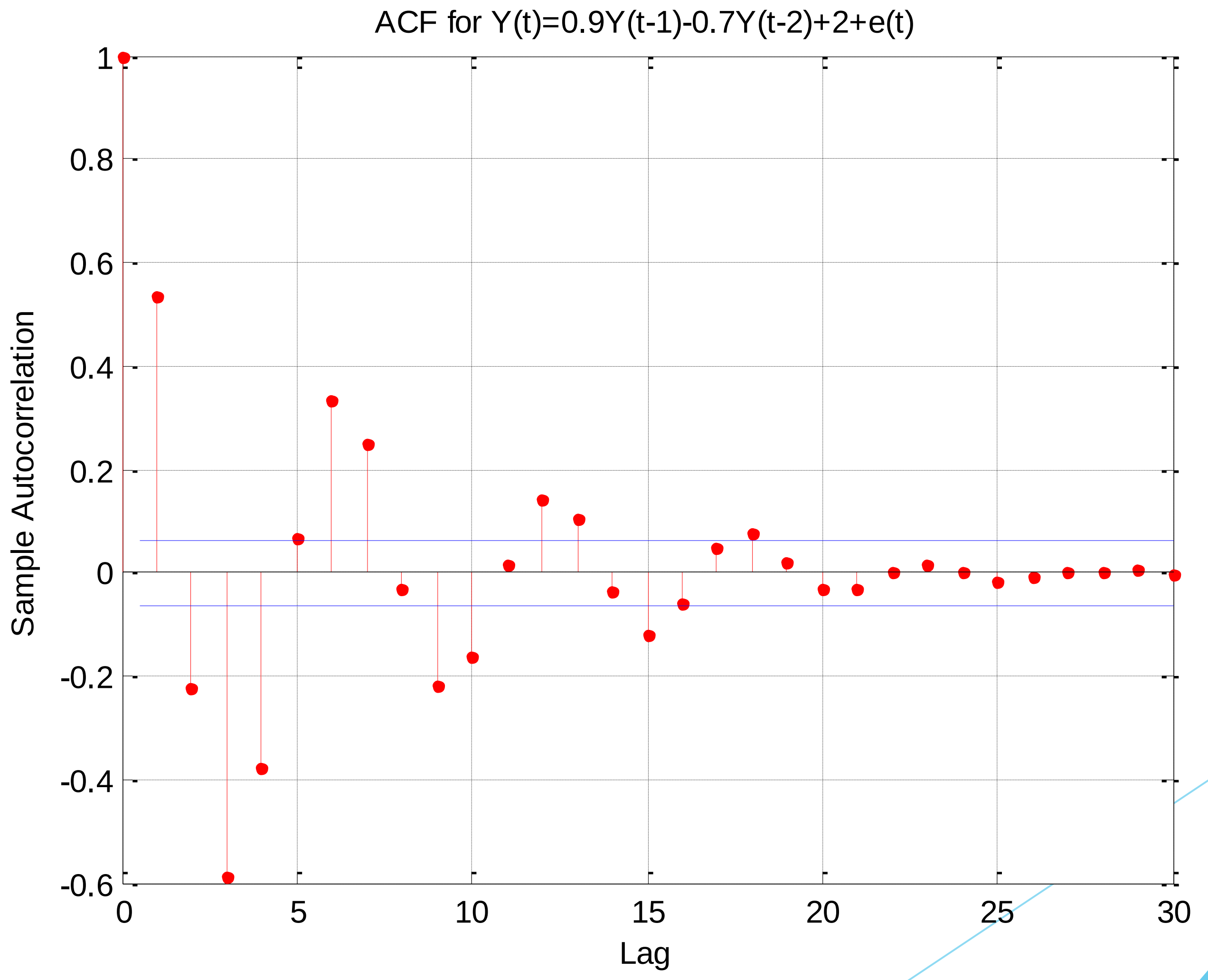
$$Y_t = 0.9Y_{t-1} - 0.7Y_{t-2} + 2 + e_t$$

Пример 5



Y(t)=0.9Y(t-1)-0.7Y(t-2)+2+e(t)
8
6
4
2
0
-2
-4
0
100
200
300
400
500
600
700
800
900
1000

Пример 5



ACF for Y(t)=0.9Y(t-1)-0.7Y(t-2)+2+e(t)
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
5
10
15
20
25
30