Notes of analysis

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Contents

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Chapter 1

Odds and ends

1.1 Space of sequences

Definition 1.1. For $1 \leq p < \infty$, ℓ_p is defined to be the set of all sequences $x. = (x_1, x_2, \cdots)$ for which $\|x\|_p < \infty$. Where

$$\|x\|_p = (\sum_1^\infty |x_i|^p)^{1/p}$$

is the ℓ_p **norm** of the sequences.

While ℓ_∞ is defined as the set of all $\sup\{|x_n|\} \leq \infty$, such norm is called ℓ_∞ **norm**, **supremum norm** or **uniform norm**.

All of these spaces are vector space. And sequence $\{\ell_i\}_{i=1}^\infty$ is increasing.

The space of all convergent sequence is denoted c and all sequences convergent to 0 is denoted c_0. Finally, the collection of sequences with finite nonzero terms is φ. One can check that

$$\varphi \subset \ell_p \subset c_0 \subset \ell_\infty \subset \mathbb{R}^n$$

1.2 Spaces of functions

One can think $\mathbb{R}^n$ as

$$\{f : \{1, 2, \cdots, n\} \to \mathbb{R}\} = \mathbb{R}^n = \mathbb{R}^{\{1,2,\cdots,n\}}$$

Replace $\{1, 2, \cdots, n\}$ by an arbitrary X, then $\mathbb{R}^X$ is all functions from X to $\mathbb{R}$.

For $1 \le p < \infty$, $L_p(\mu)$ is defined to be the set of all μ measurable functions f for which $\|f\|_p < \infty$, where the L_p **norm** is defined as

$$\|f\|_p = (\int_\Omega |f|^p)^{1/p}$$

And the L_∞ **norm**, or **essential supremum** is defined as

$$\|f\|_\infty = \operatorname{ess\,sup} f = \sup\{t : \mu(\{x : |f(x)| \ge t\})0\}$$

1.3 Ordinals

Suppose R is an order relation on Ω, then Ω is said to be **inductively ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Definition 1.2. A set X is **well ordered** by linear $\preceq$ if every nonempty subset has a least element.

Definition 1.3. An **initial segement** of $(X, \preceq)$ is any set of the form $I(x) = \{y \in X : y \le x\}$.

Definition 1.4. An **ideal** in a well ordered X is a subset A s.t. forall $a \in A$, $I(a) \subset A$.

Theorem 1.1 (Well Ordering Principle)**.** *Every nonempty set can be well ordered.*

Proof. Let X nonempty, and let

$$\mathcal{X} = \{(A, \preceq_A) \text{ is well order} : A \subset X\}$$

all well ordered sets, and define $\preceq$ on $\mathcal{X}$ as $(B, \preceq_B) \preceq (A, \preceq_A)$ if B is an ideal in A and $\preceq_A$ extends $\preceq_B$. Suppose every chain $\mathcal{C}$ in $\mathcal{X}$, $(\cup\mathcal{C}, \cup\{\prec_A : A \in \mathcal{C}\})$ clearly an upper bound of $\mathcal{C}$ and well ordered. By Zorn's lemma, there is a maximal element of $\mathcal{X}$ and it's actually X. $\square$

Kind of remarkable and useful well ordered set is exist:

Theorem 1.2. *There exist poset* $(\Omega, \preceq)$ *satisfy*

1. $(\Omega, \preceq)$ *is well ordered.*

2. *Ω has a greast element ω_1*
3. *$I(x)$ is countable for $x < \omega_1$*
4. *$\{y \in \Omega : x \leq y \leq \omega_1\}$ is uncountable.*
5. *Every nonempty subset of Ω has a least upper bound.*
6. *A nonempty subset of $\Omega - \{\omega_1\}$ has greatst element iff it's countable. Every uncountable subset has least upper bound ω_1.*

Proof. Let $(X, \preceq)$ be uncountable well ordered set, and let A

$$A = \{x \in X : I(x) \text{is uncountable}\}$$

w.l.o.g we may assume A is nonempty. Then there is a first element and denoted by ω_1. Then we show that $\Omega = I(\omega_1)$ enjoy all the properies.

1-4 is straightforward and 5 follows from all the upper bound is well ordered and thus least upper bound exist. For 6, suppose there is a countable $C \subset \Omega - \{\omega_1\}$, then $\bigcup_{i=1}^{\infty} I(x_i)$ is countable, so there is some $x < \omega_1 \notin \bigcup_{i=1}^{\infty} I(x_i)$, that is an upper bound. By 5, least upper bound is exist and belong to C. Conversely, if some subset C has some least upper bound $b < \omega_1$, then $C \subset I(b)$ and must countable. □

The elements of Ω are called **ordinals** and ω_1 is called **first uncountable ordinal**. The elements of $\Omega_0 = \Omega - \{\omega_1\}$ is **countable ordinals**. We treat $\mathbb{N}$ as a subset of Ω. Then the first element of $\Omega - \mathbb{N}$ is **first infinite ordinal**.

Theorem 1.3 (Interlacing Lemma)**.** *Suppose sequence $\{x_n\}$ and $\{y_n\}$ in Ω_0 with $x_n \leq y_n \leq x_{n+1}$. Then they share the same least upper bound.*

Proof. Clearly since $x_n \leq y_n \leq x_{n+1}$.

□

Chapter 2

Topology

2.1 Topological spaces

Let Ω be as space

Definition 2.1. A class of subset τ of Ω is an **topology** if

1. $\varnothing$ and Ω belongs to τ.
2. closed under arbitrary union.
3. closed under finite intersection.

(Ω, τ) called a **topological space** where Ω is called as **underlying set**. The sets in τ are called **open** while sets with complement in τ is **closed**. Both open and closed set is called **clopen**.

Definition 2.2. Countable intersection of open sets is $\mathscr{G}_\sigma$ set and countable union of closed sets is $\mathscr{F}_\delta$ set.

Definition 2.3. (X, ρ) is a **semimetric space**, when ρ defined on $X \times X$ s.t. $\forall x, y, z \in X$:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

ρ is called a **semimetric**.

If $\rho(x, y) = 0 \iff x = y$, ρ become a **metric** and (X, ρ) become **metric space**. $B(a, r) = \{x \in E, d(x, a) < r\}$ is r-ball with center a.

U is **open** in (Ω, d) iff $\forall x \in U, \exists r_x 0 \ni B_d(x, r_x) \subseteq U$. Let τ_d be the set of all open subsets of Ω, we call τ_d the **topology generated by** d. A Topological space is **metrizable** if there exist metric d generates it.

Suppose d is discrete, that is, $d(x, y) = 0$ iff $x = y$, otherwise, $d(x, y) = 1$. Then every subset is open hence $\tau_d = \mathcal{P}(\Omega)$ and called **discrete topology**. The zero semimetric, defined by $d(x, y) = 0$ for all $x, y \in \Omega$ generates $\tau_d = \{\varnothing, \Omega\}$ and called **trivial topology**.

Let $\Omega = \mathbb{R}^n$, $l^2 = \sqrt{\sum_1^n (x_i - y_i)^2}$ is called **Euclidean metric**. $l^1 = \sum_1^n |x_i - y_i|$ is called **taxi-cab metric** and $l^\infty = \sup\{|x_i - y_i|\}$ is called **sup norm metric**.

Note $d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$ and $d_{l^2}(x, y) \leq \sqrt{n} d_{l^\infty}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$, then d_{l^∞} open $\iff$ d_{l^2} open $\iff$ d_{l^1} open. Hence $\tau_{d_{l^2}} = \tau_{d_{l^1}} = \tau_{d_{l^\infty}}$.

All topologies on Ω is poset with greatest element $\mathcal{P}(\Omega)$ and least $\{\varnothing, \Omega\}$. If $\tau' \subset \tau$, we say τ' **coarser** than τ while τ finer than τ'.

If τ can be form by taking union of families in some $\mathcal{B} \subset \tau$, we call $\mathcal{B}$ the **base** for the topology τ.

Theorem 2.1. *$\mathcal{B}$ is a base in (X, τ) iff $\forall U \in \tau, \forall x \in U, \exists W \in \mathcal{B} \ni x \in W \subset U$.*

Proof. $\implies$: Any U can be written as $U = \cup W_i$ and $x \in U \implies x \in W_i$ for some i and $W_i \in \mathcal{B}$. $\impliedby$: For any $U \in T$, consider arbitrary $x \in U$, then there exist W_x such that $x \in W_x \subset U$, thus we have $U = \cup_x W_x$.

$\square$

Let $\mathcal{S} \subset \tau$, suppose all topologies include $\mathcal{S}$. Then the intersection of all of them is again a topology, denoted as $\tau(S) = \cap \mathrm{T}$, then $\tau(\mathcal{S})$ is the smallest topology contains $\mathcal{S}$. We call it the topology **generated** by $\mathcal{S}$.

Theorem 2.2. *$\tau(\mathcal{S})$ is unions of families of finite intersections together with Ω, formally:*

$$\{\bigcup(\bigcap_1^N S_i)\} \cup \Omega$$

$\mathcal{S} \subset \tau$ is a **subbase** for τ if $\bigcup \mathcal{S} = \Omega$ then all finite intersections of $\mathcal{S}$ is a base. Note that if $\Omega \in \mathcal{S}$, $\mathcal{S}$ is the subbase of $\tau(\mathcal{S})$.

(Ω, τ) is **second countable** if τ has countable base. Clearly, a topology is second countable iff it has countable subbase.

For any subset X in (Ω, τ), then

$$\tau_X = \{X \cap V : V \in \tau\}$$

form a topology in X and we call (X, τ_X) a **subspace** or **relative topology**. Sets in τ_X are **relative open**. **Relative closed** sets of the form

$$X - (X \cap V) = X - V = X \cap V^c$$

2.2 Neighborhood

A subset V is called a **neighborhood** of a if there exists a open set $U \subset V$ contains a. Then we called $V' = V - \{a\}$ **punctured(deleted)** neighborhood. A **neighborhood base** is a collection of neighborhood $BN(a)$ s.t. for any neighborhood V of a, there exist a $W \in BN(a)$ and $W \subset V$. Clearly, all the neighborhoods is a neighborhood base and denoted as $\mathcal{N}(x)$, which is called **neighborhood system**.

Lemma 2.1. *A subset U is open iff it's a neighborhood for each of its points.*

Proof. $\Longrightarrow$ is trivial. $\Longleftarrow$ follows from $\cup_x G_x = U$ and unions of open set is still open. ■

□

This suggest a equivalent definition of finer topology:

Lemma 2.2. $\tau' \subset \tau \iff$ *τ' neighborhood is a τ neighborhood.*

Proof. $\Longrightarrow$ any open set G_x satisfy $x \in G_x \subset V$ in T' is still open in T, hence V is T neighborhood. $\Longleftarrow$ Consider any open set $G \in T'$, it's a T' neighborhood for each of its points implies it's a T neighborhood for each of its points and hence G is T open.

□

2.3 Closures

The **interior** of A is the union of all open sets which are included A, i.e., the largest open set included in A, we denote it A°. And the **closure** is the intersection of all closed sets which include A and thus the smallest closed set includes A, we denote it $\overline{A}$.

Lemma 2.3. *Following is some useful truth:*

1. $(A \cap B)^\circ = A^\circ \cap B^\circ$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

3. $A \subset \overline{B} \implies \overline{A} \subset \overline{B}$
4. $A^\circ \subset B \implies A^\circ \subset B^\circ$
5. $\overline{A^c} = (A^\circ)^c$
6. $(\overline{A})^c = (A^c)^\circ$

Proof. We only prove **5**, note $(A^\circ)^c$ is closed and

$$A^\circ \subset A \implies (A^c) \subset (A^\circ)^c$$

we have $\overline{A^c} \subset (A^\circ)^c$. On the other hand

$$\overline{A^c} \supset (A^\circ)^c \impliedby (\overline{A^c})^c \subset A^\circ \impliedby (\overline{A^c})^c \subset A \impliedby \overline{A^c} \supset A^c$$

□

The **frontier** of A is $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ$.

x is said to be an **interior point** of A if A is neighborhood of x.

x is said to be an **adherent point** if it's every neighborhood meets A, an ω **accumulation point** of A if every neighborhood of x contains **infinitely** many points of A and is a **condensation point** of A if every neighborhood of x contains **uncountable** many points of A.

x is a **cluster point** or **accumulation point** if every deleted neighborhood of x meets A and is **isolated point** if x is not cluster point. That is, $\{x\}$ is relative open in A. We denoted all the cluster points as A' and called **derived set**.

x is **frontier point** or **boundary point** if every neighborhood of x meets both A and A^c.

It's east to show that the points of A° are precisely all the interior points of A and $\overline{A}$ are precisely all the adherent points. ∂A is precisely points of frontier. We claim that

$$\overline{A} = A^\circ \cup \partial A = A \cup A'$$

A subset A is called **perfect** if it's closed while point in A is cluster points in A, that is $A' = A = \overline{A}$.

2.4 Dense

A is said **dense** if $\overline{A} = \Omega$ and **nowhere dense** if $(\overline{A})^\circ = \varnothing$ ($\mathbb{Q}$ is dense in $\mathbb{R}$ while $\mathbb{Z}$ is nowhere dense.) A is said to be **meagre** or **set of the first category** if it's countable union of nowhere dense. Sets which are not meagre is **set of the second** category set.

Space (Ω, τ) is **first countable** if every point of Ω has countable neighborhood base. The space is said **separable** if Ω has a countable dense subset.

Lemma 2.4. *Second countable space is separable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, by axiom of choice, we may take x_i in I, let $X = \{x_i\}_{i \in I} \subset \Omega$. Then we show that X is dense. For any $x \in \Omega$, it's neighborhood must contain some open G which is unions of $\mathcal{B}$ and thus contains at least one element in X, that is, G meet X. Hence $\overline{X} = \Omega$.

□

Lemma 2.5. *Second countable space is first countable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, for each point $x \in \Omega$, one may take all the sets in $\mathcal{B}$ which contains x as a neighborhood base. To verify it's neighborhood base, if there is a neighborhood N of x, then there is a open G contains x. By the definition of base, G is the union of sets of $\mathcal{B}$ and those sets must at least one contains x and these sets is subset to G.

□

2.5 Mappings

Suppose (Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

Lemma 2.6. *Following is some useful truth for mappings.*

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap N) = f^{-1}(U) \cap f^{-1}(N)$
4. $f^{-1}(U \cup N) = f^{-1}(U) \cup f^{-1}(N)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ *always holds if* f *is injection while* $ff^{-1}(A) = A$ *always holds if* g *is surjection.*
7. *If* f *is bijection,* $(f^{-1})^{-1}(A) = f(A)$ *always hold.*
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$
9. $f^{-1}(A) \subset f^{-1}(B) \impliedby A \subset B$

10. $f(A) \subset f(B) \impliedby A \subset B$

Definition 2.4. f is **continuous** at x if for every neighborhood N' of $f(x)$, there is a neighborhood N of x s.t. $f(N) \subset N'$. It's continuous if it's continuous at every points $x \in \Omega$.

Theorem 2.3. *f is continuous iff*

1. *$f^{-1}(G')$ is open for every open subset G' of Ω'.*
2. *$f^{-1}(F')$ is closed for every closed subset F' of Ω'.*
3. *If $A \subset \Omega'$, then $f^{-1}(A^\circ) \subset (f^{-1}(A))^\circ$*
4. *If $A \subset \Omega$, then $f(\overline{A} \subset \overline{f(A)})$*

Proof. We only prove 1 and 3.

1 $\implies$: For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood N s.t. $f(N) \subset G'$, and

$$x \in N \subset f^{-1}f(N) \subset f^{-1}(G')$$

$\impliedby$: For every neighborhood N', there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $ff^{-1}(G') \subset G'$.

3 $\implies$: $f^{-1}(A^\circ)$ is open and th claim follows from $f^{-1}(A\circ) \subset f^{-1}(A)$. $\impliedby$: Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggest $f^{-1}(A)$ is open.

□

Lemma 2.7 (Glueing Lemma)**.** *Let $X = A \cup B$ and A and B are both closed or both open, then $f : X \to Y$ is continuous iff it's restriction on A and B are both continuous.*

Proof. $\implies$ is trivial.

$\Leftarrow$ Suppose they are both open and U be any open set in Y. Note $f_{|A}^{-1}(U)$ is open in A and thus open in X, thus

$$f^{-1}(U) = (f^{-1}(U) \cap B) \cup (f^{-1}(U) \cap A) = f_{|A}^{-1}(U) + f_{|B}^{-1}(U)$$

is open.

□

Lemma 2.8. *Suppose $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$, $f \circ g$ is continuous if f and g are continuous.*

Proof. Suppose G_3 is open and the claims follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$.

□

Lemma 2.9. *Suppose $f : (\Omega, \tau), (\Omega', \tau(\mathcal{S}))$, f is continuous iff $f^{-1}(S) \in \tau$ for any $S \in \mathcal{S}$.*

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f, s.t. f^{-1} is continuous and such f is called **homeomorphism**. In particular, f is an **embedding** if $f : (\Omega, \tau) \to (f(\Omega), \tau | f(\Omega))$ is a homeomorphism.

f is **open** if $f(G)$ is open for all open set $G \in \tau$ and is **closed** if $f(F)$ is closed for all closed set $F^c \in \tau$.

Lemma 2.10. *Suppose f is bijection, then it's homeomorphism iff it's continuous and either open or closed.*

Proof. By the continuity of f^{-1}, since $(f^{-1})^{-1}(G) = f(G)$ for all open set G.

$$f^{-1} \text{ is continuous } \iff f(G) \text{ is open } \iff f \text{ is open .}$$

□

Lemma 2.11. *Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f continuous.*

Proof. Suppose f is homeomorphism, T_0 is another topology where f is continuous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1},

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0.

□

Note that $\mathcal{P}(\Omega)$ let all f continuous and $\{\varnothing, \Omega\}$ let all $g : \Omega' \to \Omega$ continuous.

2.6 Semicontinuous

$f : \Omega \to \mathbb{R}^*$ is

- **lower semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \leq c\}$ is closed.

- **upper semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \geq c\}$ is closed.

Clearly f is lower semicontinuous iff $-f$ is upper and vice versa. Also, f is continuous iff it's both upper and lower semicontinuous.

Lemma 2.12. *Suppose* $\{f_i\}_{i \in I}$ *is family of lower(upper) semicontinuous function then* $\sup f_i(\inf f_i)$ *is lower(upper) semicontinuous.*

Proof. Note

$$\{x \in \Omega : \sup f_i(x) \leq c\} = \bigcap_{i \in I} \{x \in \Omega : f_i(x) \leq c\}$$

is closed.

□

Lemma 2.13. $f : \Omega \to \mathbb{R}^*$ *is*

- ***lower semicontinuous*** *iff for any net*

$$x. \to x \implies \liminf f(x.) \geq f(x)$$

- ***upper semicontinuous*** *iff for any net*

$$x. \to x \implies \limsup f(x.) \leq f(x)$$

Proof. Suppose f is lower semicontinuous and $x. \to x$. For any $c < f(x)$, then $G = \{\omega \in \Omega : f(\omega)c\}$ is open and thus $x.$ eventually in, that is $x.c$ eventually and thus $\liminf f(x.) \geq c$. This implies that $\liminf f(x.) \geq f(x)$.

Conversely, for any $c \in \mathbb{R}$, consider $F = \{\omega \in \Omega : f(\omega) \leq c\}$. Then we show that F is closed. Suppose $x.$ is nets in F and converges to some $x \in \Omega$. Then $c \geq \liminf f(x.) \geq f(x)$ thus x in F and thus F is closed.

□

Then we can generalizes Weierstrass' Theorem in corollary 2.5.

Theorem 2.4. $f : \Omega \to \mathbb{R}^*$ *on a compact set attains a minimum(maximum) value and set of minima(maxima) is compact if it's lower(upper) semicontinuous.*

Proof. Suppose X is compact and f is lower semicontinuous, then for every $c \in f(X)$, $F_c = \{x \in X : f(x) \leq c\}$ is closed and $\{F_c : c \in f(X)\}$ has FIP clearly. Note X is compact, $\ker\{F_c : c \in f(X)\}$ is nonempty by 2.28. That is just the set of minima and it's compact since it's closed.

□

2.7 Comparing topologies

We list some useful properties when comparing topologies, some of them has been mentioned before and proof omitted.

Lemma 2.14. *Suppose τ' and τ are two topologies on Ω, then the following are equivalent.*

1. $\tau' \subset \tau$
2. *Identity mapping $I : x \mapsto x$ from (Ω, τ) to (Ω', τ') is continuous.*
3. *τ' closed set is closed in τ.*
4. $x_\cdot \xrightarrow{\tau} x \implies x_\cdot \xrightarrow{\tau'} x$
5. $Cl_\tau(A) \subset Cl_{\tau'}(A)$

Lemma 2.15. *Suppose $\tau' \subset \tau$, then*

1. *Every τ compact set is τ' compact.*
2. *Every τ' continuous function is τ continuous.*
3. *Every τ dense set is τ' dense.*

2.8 Filter

Definition 2.5. A **filter** is a non-empty collection $\mathcal{F}$ of subset in Ω s.t.

1. $A \in \mathcal{F}, A \subset \mathcal{B} \implies \mathcal{B} \in \mathcal{F}$
2. Closed under finite intersection.
3. $\varnothing \notin \mathcal{F}$

Note the definition of $\mathcal{F}$ is independent with topology τ. A **free filter** is filter with $\ker \mathcal{F} = \bigcap_{F \in \mathcal{F}} F = \varnothing$. Not free filters are called **fixed**.

Filter can be formed by taking upward closure of a filter base.

Definition 2.6. A collection $\mathcal{B}$ of subset in Ω is a **fiter base** of or **prefilter** if

1. $\mathcal{B} \subset \mathcal{F}$
2. $\forall N \in \mathcal{F}, \exists W \in \mathcal{B} \ni W \subset N$

We say $\mathcal{B}$ generates $\mathcal{F}$, where

$$\mathcal{F} = \mathcal{B}^{\uparrow} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

For example,

- Suppose $x \in \Omega$ then

  $$\mathcal{N}(x) = \{\text{All neighbourhoods of } x\}$$

  is a filter on Ω, that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. Note

  $$\tau(x) = \{X \in \tau : x \in X\}$$

  is a base for $\mathcal{N}(x)$ and thus $\mathcal{N}(x) = \tau(x)^{\uparrow}$.

- Suppose Ω is infinite, the collection of all **cofinite** subsets(subset s with finite complement) is a filter on Ω, such filter is free and called **Frechet filter**.

To assert a collection is a base, we have

Theorem 2.5. *Let $\mathcal{B}$ be a collection of nonempty subsets. Then $\mathcal{B}$ is a filter base, that is, $\mathcal{B}$ may generates a filter iff*

1. *The intersection of each finite family of sets in $\mathcal{B}$ inclueds a set in $\mathcal{B}$*
2. *$\mathcal{B}$ is non-empty and $\varnothing \notin \mathcal{B}$.*

Proof. We claim that

$$\mathcal{F} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

$\mathcal{F}$ is the filter generated by $\mathcal{B}$. □

A family of subsets $\mathcal{F}$ is said to have **finite intersection property** if intersection of every finite subfaimily is nonempty.

Let $\mathcal{A}$ be collection of subsets with finite intersection property, then collection of all finite intersection of $\mathcal{A}$ is a base, we call the filter generated **filter generated by** $\mathcal{A}$. Formally

$$\mathcal{F} = \{\bigcap_{A \in \mathcal{I}} A : \mathcal{I} \subset \mathcal{A} \text{ and } \mathcal{I} \text{ is finite}\}^{\uparrow}$$

A filter $\mathcal{F}$ is **finer** than another $\mathcal{G}$ if $\mathcal{F} \subset \mathcal{G}$. Clearly, the set of all filters on Ω is inductively ordered by inclusion. By Zorn's lemma, the set of all filters has maximal filters and we call such fliters **ultrafilters**.

Lemma 2.16. *Every fixed ultrafilter of the form*

$$\mathcal{U}(x) = \{x\}^{\uparrow}$$

for any $x \in \Omega$. And every free ultrafilter contains no finite subsets.

To assert a filter is ultra, we have:

Theorem 2.6. *Let A be a collection of subsets and $\mathcal{F}$ the filter generates by A. If*

$$\forall X \subset \Omega, \textit{either } X \in A \textit{ or } X^c \in A$$

then A is an ultrafilter on Ω.

Proof. Suppose $\mathcal{F}'$ is an ultrafilter include $\mathcal{F}$, we have $\mathcal{F}' \supset A$ clearly. Consider any $X \in \mathcal{F}'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in \mathcal{F}'$ as $\mathcal{F}' \supset \mathcal{F} \supset A$ and $X \cap X^c = \varnothing \in \mathcal{F}'$ results in a contradiction. It follows that $A \supset \mathcal{F}'$ and thus $A = \mathcal{F}'$. □

Theorem 2.7. *Every filter $\mathcal{F}$ is the intersection of all the ultrafilter which include $\mathcal{F}$.*

Proof. We claim that

$$\mathcal{F} = \cap\{\text{ultrafilter generates by } \{x\} : x \in \cap\mathcal{F}\}$$

□

Suppose mappings on a filter:

Theorem 2.8. *Let f be a mapping from Ω to Ω' and $\mathcal{B}$ a base for a fliter $\mathcal{F}$ on Ω. Then $f(\mathcal{B}) = \{f(X)\}_{X \in B}$ is also a base on Ω'. Moreover, if $\mathcal{F}$ is ultra then $f(\mathcal{B})$ also generates an ultrafilter.*

Proof. First assertion is straightforward and the second follows from $\mathcal{B}$ is collection of supset for some $\{x\}$, then $f(\mathcal{B})$ generates the fliter that generates by $\{f(x)\}$. □

Theorem 2.9. *In the same situation as previous theorem. If $\mathcal{B}'$ is a base on Ω', then $f^{-1}(\mathcal{B}')$ is a base on Ω iff every set in $\mathcal{B}'$ meets $f(\Omega)$*

Proof. We have

$$\Omega \in f^{-1}(\mathcal{B}') \implies f(\Omega) \subset X'$$

for some $X' \in f^{-1}(\mathcal{B}')$, by definition, $\implies$ is immediately.

For $\impliedby$, suppose any finite family $X_i \in \mathcal{B}'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}(\bigcap_i X_i) \in f^{-1}(\mathcal{B}')$$

Then the claim follows from theorem 2.5. □

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the fliter $\mathcal{F}$ and $\mathcal{F}$ is said to **converge** to x, or $\mathcal{F} \to x$, if the neighborhood filter $\mathcal{N}(x) \subset \mathcal{F}$. For filter base $\mathcal{B}$, we define on the filter generated by $\mathcal{B}$, that is, if $\mathcal{N}(x) \subset \mathcal{B}^{\uparrow}$.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff \mathcal{N}_\tau(x) \supset \mathcal{N}_{\tau'}(x) \iff \mathcal{F} \to a \implies \mathcal{F}' \to a$$

also, an equivalent definition of continuity as follows:

Theorem 2.10. *$f : (\Omega, \tau) \to (\Omega', \tau')$ is continous at x iff*

$$\forall \mathcal{F} \to x, f(\mathcal{F}) \to f(x)$$

Proof. By definition, $f(\mathcal{F}) \to f(x)$ if

$$\mathcal{N}(f(x)) \subset f(\mathcal{F})^{\uparrow}$$

That is, for any neighbourhood $N' \in \mathcal{N}(f(x))$, there exist some $A \in \mathcal{F}$ s.t. $f(A) \subset N'$, as $\mathcal{N}(x) \subset \mathcal{F}$ and f is continous at x, such A is always exists. Conversely, take $\mathcal{F} = \mathcal{N}(x)$ then the claim is follows □

A point $x \in \Omega$ is said to be an **adherent point** of $\mathcal{F}$ if x is an adherent point of every set in $\mathcal{F}$. The **adherence** of $\mathcal{F}$, $\mathrm{Adh}_\tau(\mathcal{F})$ or $\overline{\mathcal{F}}$ is the set of all adherent points, thus

$$\overline{\mathcal{F}} = \bigcap_{X \in F} \overline{X}$$

Define similarly on filter base $\mathcal{B}$ by the filter generated. By definition, we have

$$\overline{\mathcal{B}} = \bigcap_{X \in B} \overline{X}$$

Lemma 2.17. *Suppose A be a subset of Ω, then $x \in \overline{A}$ iff there is a filter $\mathcal{F}$ s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.*

Theorem 2.11. *Suppose $BN(x)$ a neighbourhood base of x, then*

1. *$\mathcal{B}$ converges to x iff every set in $BN(x)$ includes a set in $\mathcal{B}$.*
2. *$x \in \overline{\mathcal{B}}$ iff every set in $BN(x)$ meets every set in $\mathcal{B}$.*

As consequence, we have

Corollary 2.1. *x is adherent to a filter $\mathcal{F}$ iff there is $\mathcal{F}' \supset \mathcal{F}$ and converges to x*

Proof. $\Longrightarrow$ follows from taking $\mathcal{F} = BN(x)$. Conversely, $\forall N \in BN(x)$, we have $X' \subset N$ for some $X' \in \mathcal{F}'$, thus for any $X \in \mathcal{F}$, $N \cap X \subset X' \cap X \neq \varnothing$ as $X', X \in \mathcal{F}'$. ☐

Corollary 2.2. *Every limit point of $\mathcal{F}$ is adherent to $\mathcal{F}$*

Proof. Clearly holds by applying theorem 2.11.1 and 2.11.2. ☐

Corollary 2.3. *Every adherent point of an ultra-filter is a limit point of it.*

Proof. Clearly as kernel of ultrafilter is a one point set. ☐

Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a **limit point** of f relative to $\mathcal{F}$ if $f(\mathcal{F}) \to x$.
2. an **adherent point** of f relative $\mathcal{F}$ if it's adherent point of $f(\mathcal{F})$.

Theorem 2.12. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$*

1. *x' is a limit point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, we have $f^{-1}(N') \in \mathcal{F}$.*
2. *x' is an adherent point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, it meets $f(X)$ for any $X \in \mathcal{F}$.*

Proof. x' is limit is equivalent to

$$\mathcal{N}(x') \subset f(\mathcal{F})^{\uparrow}$$

That is, there exist some $A = f(X) \subset N'$ for any N', followed by $X \subset f^{-1}f(X) \subset f^{-1}(N')$, then the claim follows from the definition of filter.

By theorem 2.11, x' is adherent to $f(\mathcal{F})$ iff

$$\forall N' \in BN(x'), \forall X \in \mathcal{F}, f(X) \cap N' \neq \varnothing$$

note for any $N' \in N'(x')$, there exist $N' \in BN(x') \ni N' \subset N'$, thus $f(X) \cap N' \neq \varnothing$ also holds. Conversely, making use of $BN(x') \subset N'(x')$. ☐

For example, suppose $f : (\mathbb{N}, \tau) \to (\Omega', \tau')$ and $\mathcal{F}$ the frechet filter on $\mathbb{N}$. Then x' is limit of f relative to $\mathcal{F}$ iff for all $N' \in N'(x')$, $f^{-1}(N') \in \mathcal{F} \iff f^{-1}(N')^c \subset [0, k] \iff f^{-1}(N') \supset \{n \in \mathbb{N} : n \geq k\}$ for some k, that is, $f(n) \in N'$ for any $n \geq k$.

Theorem 2.13. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ and let $\mathcal{F} = \mathcal{N}(x)$. By theorm 9, x' is limit of f relative to $\mathcal{N}(x)$ iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{N}(x) \iff N \subset f^{-1}(N') \iff f(N) \subset N'$ for some $N \in \mathcal{N}(x)$. That is, iff $x' = f(x)$, f is continous at x. Such limit points also called limit points of f at x.*

2.9 Net

$(D, \preceq)$ is called a **directed set** if every couple $\{x, y\}$ in which has an upper bound.

If $\{D_i\}_{i \in I}$ is family of directed set then $D = \prod_{i \in I} D_i$ is also directed under **product direction** defined by $(a_i)_{i \in I} \succeq (b_i)_{i \in I}$ for all $i \in I$.

Definition 2.7. Let $(D, \preceq)$ be a directed set, $\nu : D \to \Omega$ is called a **net** in Ω with domain D. The directed set is called **index set** of the net and members of D are **indexes**. We often write ν as $x.$ or $\{x_\alpha\}$.

Suppose A a subset of Ω, we say $x.$ **eventually in** A if there exist some $k \in D$ s.t. $x_n \in A$ for all $n \succeq k$. And we say ν is **frequently** in A if for all $n \in D$, there exist an $n' \succeq n$ s.t. $x_{n'} \in A$.

Lemma 2.18. *If $x.$ not frequently in A, then $x.$ eventually in A^c. Thus, for any $X \in \Omega$, $x.$ frequently in either X or X^c.*

Suppose $x \in \Omega$, then $x.$ is said **converge** to x, or $x. \to x$ if $x.$ eventually in N for all $N \in \mathcal{N}(x)$, i.e., $\mathcal{N}(x) \subset \mathcal{F}(x.)$. The point x is **adherent** to $x.$ if $x.$ frequently in N for all $N \in \mathcal{N}(x)$.

Theorem 2.14. *Suppose $A \in (\Omega, \tau)$, then $x \in \overline{A}$ iff it's the limit of some net in the set.*

Proof. $\Longleftarrow$ is clear. $\Longrightarrow$ follows from we may find a associated net taking value in A(since each neighborhood meets A) and such net converges to x. $\square$

As with sequence, if $x.$ is bounded, there is

$$\liminf x. = \sup \inf x. \preceq \limsup x. = \inf \sup x$$

Subnet generalizes subsequence.

Definition 2.8. Suppose D is directed, a subset B of D is called **cofinal** if for any $a \in D$, there exist $b \in B$ s.t. $a \preceq b$. A map $f : D \to A$ is **final** if $f(D)$ is cofinal of A.

Let $x.$ and $x.'$ are two nets in Ω with domains D and D' respectively. We say that $x.'$ is a **subnet** of $x.$ if there exists a final mapping $\varphi : D' \to D$ s.t. $x'_\alpha = x_{\varphi(\alpha)}$.

Theorem 2.15. *Let $\mathcal{A}$ be a collection of subsets that $x.$ is frequently in. If $\mathcal{A}$ is closed under finite intersection, then there exists a subnet $x'.$ of $x.$ and $x.'$ eventually in every member of $\mathcal{A}$*

Lemma 2.19. *Suppose $x.'$ is subnet of $x.$, we have*

1. $x. \to x \implies x.' \to x$
2. *x adherent to $x.'$ $\implies$ x adherent to $x.$.*

Theorem 2.16. *A point x is adherent to $x.$ iff there is a subnet converges to x. While $x. \to x$ iff every subnet converges to x.*

Proof. $\implies$ is clear by lemma2.19. Conversely, suppose a is not adherent to x, there exist a neighborhood N that $x.$ not frequently in, i.e., exist k s.t. $x_n \notin N$ for any $n \geq k$, thus there is no subnet eventually in N.

For the second part, $\implies$ is also clear by lemma2.19 and $\impliedby$ comes from taking subnet as itself. □

A net $x.$ is called **ultranet** or **universal net** if for all $X \in \Omega$, we have either $x.$ eventually in X or $x.$ eventually in X^c. Clearly, subnet of ultranet is ultra and

Lemma 2.20. *Every net has a ultra subnet.*

Proof. Consider collection of $\mathcal{Q}$ s.t. $x.$ is frequently in every member and closed under finite intersection. By Zorn's Lemma, there is a maximal $\mathcal{Q}_0$. By theorem 11, $x.$ has a subnet $x.'$ which eventually in every member of $\mathcal{Q}_0$. We claim that this subnet is ultra since, $\mathcal{Q}_0$ is maximal and thus either $X \in \mathcal{Q}_0$ or $X^c \in \mathcal{Q}_0$. □

2.10 Nets and filters

Let

$$\mathcal{F}(x.) = \{X \in \mathcal{P}(\Omega) : x. \text{ is eventually in } X\}$$

Then $\mathcal{F}(x.)$ is a filter and we call it the **filter associated with the net** $x.$.

Theorem 2.17. *Associated filter is the upward closure of the net's tail, that is*

$$\mathcal{F}(x.) = \{\{x_b : b \succeq a\} : a \in D\}$$

Motivated by the definition of filter that filter is closed under pairwise intersection, let $X \preceq Y \iff X \supset Y$, then any mapping $\nu : \mathcal{F} \to \Omega$ s.t. $\nu(X) \in X$ is a **net associated with the filter** $\mathcal{F}$.

By definition, we claim that $\mathcal{F}$ is the associated filter of every associated net and $x.$ is an associted net of the associated fiter.

Theorem 2.18. *Filter $\mathcal{F} \to x$ iff $x. \to x$ for any $x.$ associated with $\mathcal{F}$.*

Proof. Note

$$\forall N \in \mathcal{N}(x), x. \text{ eventually in } N \iff \mathcal{N}(x) \subset \mathcal{F}(x.)$$

Then is sufficient to show that $\mathcal{F}(x.) = \mathcal{F}$. It's follows from for any $X \in \mathcal{F}$, $x.$ eventually in X. □

Theorem 2.19.

$$x. \to x \iff \mathcal{F}(x.) \to x$$

Proof. Both side is equivalent to $\mathcal{N}(x) \subset \mathcal{F}(x.)$ □

Theorem 2.20. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, then f is continous at x iff $\forall x. \to x$, $f(x.) \to f(x)$.*

Proof. By theorem 2.19,2.18 and 2.13. □

By above theorems, we have

$$\mathrm{Adh}(\mathcal{F}(x.)) = \mathrm{Adh}(x.), \mathrm{Lim}(\mathcal{F}(x.)) = \mathrm{Lim}(x.)$$

and similarly results holds for any filter and one of associated nets.

Lemma 2.21. *If $x.$ is ultra then the associated filter $\mathcal{F}(x.)$ is also ultra and if $\mathcal{F}$ is ultra, every associated net is ultra.*

Proof. Directly from theorm 2.6. □

2.11 Convergence

If $\mathcal{F}$ is collection of functions on X, X can be seen as functions on $\mathcal{F}$ by $e_x(f) = f(x)$ for each $x \in X$, such functions are called **evaluation functional**.

The product topology on $\mathbb{R}^X$ is also called **topology of pointwise convergence** on X because a net $f. \to f$ iff $e_x(f.) \to e_x(f) \iff f.(x) \to f(x)$ for each $x \in X$.

There also exist induced topology $\sigma(\mathcal{F}, X)$ on $\mathcal{F}$, which is identical to the subspace $\mathbb{R}^X|_{\mathcal{F}}$ endowed the product topology. Formally

$$\sigma(\mathcal{F}, X) = \sigma(\mathbb{R}^X, X)|_{\mathcal{F}}$$

Lemma 2.22. *If $\mathcal{F}$ is total, the function*

$$x \mapsto e_x : (X, \sigma(X, \mathcal{F})) \to (\mathbb{R}^{\mathcal{F}}, \sigma(\mathbb{R}^{\mathcal{F}}, \mathcal{F}))$$

is injective and thus an embedding.

Proof. It's remain to show the continuity.

$$\begin{aligned} x. \to x &\iff \forall f \in \mathcal{F}, f(x.) \to f(x) \\ &\iff \forall f \in \mathcal{F}, e_f(e_{x.}) \to e_f(e_x) \\ &\iff e_{x.} \to e_x \end{aligned}$$

□

By Tychonoff theorem 2.44, $\mathcal{F}$ is compact iff $\forall x \in X$, $\{f(x)\}_{f \in \mathcal{F}}$ it's closed and pointwise bounded by borel theorem.

Definition 2.9. A net $f.$ converges uniformly to $f \in \mathbb{R}^X$ iff $|f.(x) - f(x)| < \epsilon$ eventually for each $x \in X$ after some f_α for any ϵ.

Theorem 2.21. *The uniform limit of a continuous net is continuous.*

Proof. Suppose $f. \to f$ uniformly, then for any $x \in X$, for any $\alpha > \alpha_0$

$$|f_\alpha(x) - f(x)| < \epsilon$$

as f_α is continuous, for any $x. \to x$, for any $\lambda > \lambda_0$

$$|f_\alpha(x_\lambda) - f_\alpha(x)| < \epsilon$$

also, there is

$$|f_\alpha(x_\lambda) - f(x_\lambda)| < \epsilon$$

Hence, we have

$$|f(x_\lambda) - f(x)| < 3\epsilon$$

Thus, $f(x.) \to f$ and continuity follows.

$\square$

Theorem 2.22 (Dini's Theorem)**.** *If continuous real function net $f.$ on a compact set converges monotonically to f pointwise, then the net converges to f uniformly.*

Proof. Let $g. = f. - f$, we have $g. \to 0$, $|g.|$ is decreasing as monotone. Then it's sufficient to show that $g. \to g$ uniformly. Note $|g.(x)| < \epsilon$ eventually for any $x \in X$ after, say, α_x. By continuity and compactness:

$$X = \bigcup_{x \in X} |g_{\alpha_x}|^{-1}(B(0,\epsilon)) = \bigcup_{x \in J} |g_{\alpha_x}|^{-1}(B(0,\epsilon))$$

Then we may pick $\alpha_0 \geq \alpha_x$ for all $x \in J$, and for any $\alpha \geq \alpha_0$ and any $x \in X$, suppose $x \in |g_{\alpha_{x_j}}|^{-1}\left(B(0,\epsilon)\right)$

$$\epsilon > |g_{\alpha_{x_j}}(x)| > |g_\alpha(x)|$$

by monotone and thus $g. \to 0$ uniformly.

$\square$

2.12 Separation

Definition 2.10. Space (Ω, τ) is said to be T_0 or **kolmogorov** if for every pair $(x, y) \in \Omega^2$, either there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ or $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Lemma 2.23. *τ isn't T_0 iff there exist pair (x, y), s.t:*

1. $\mathcal{N}(x) = \mathcal{N}(y)$.
2. $\overline{\{x\}} = \overline{\{y\}}$.

Proof. 1 If every $N \in \mathcal{N}(x)$ contains y, then $N \in \mathcal{N}(y) \implies \mathcal{N}(x) \subset \mathcal{N}(y)$, thus $\mathcal{N}(x) = \mathcal{N}(y)$.

2 If some point $a \in \overline{\{x\}}$, then every $N \in \mathcal{N}(a)$ also is neighborhood of x and thus neighborhood of y, hence $a \in \overline{\{y\}}$. $\square$

Definition 2.11. Space (Ω, τ) is said to be T_1 or **Frechet** if for every pair $(x, y) \in \Omega^2$,there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ and $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Theorem 2.23. *Following statements are equivalent:*

1. τ *is* T_1.
2. *Singetons are closed.*
3. $\ker \mathcal{N}(x) = \{x\}$ *holds for any* $x \in \Omega$.

Proof. $1 \implies 2$ If there exist a singeton $\{x\}$ not closed, there is $y \in \overline{\{x\}}$, hence every neighborhood of y contains x, contradiction.

$2 \implies 3$ Suppose $\ker \mathcal{N}(x)$ contains y diifer x, that implies any neighborhood of x contains y and contradict 2.

$3 \implies 1$ is straightforward. □

Lemma 2.24. *Suppose* (Ω, τ) *with a finite base is* T_1*, then* Ω *is finite and* τ *is discrete.*

Definition 2.12. A topology (Ω, τ) is T_2, or **Hausdorff** or **separated** if every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $U \cap V = \varnothing$.

Theorem 2.24. *Following statements are equivalent:*

1. τ *is* T_2.
2. *Intersection of family of closed neighborhoods of* x *is* x.
3. *If a filter(net) converges to some point* x*, then* $Adh(\mathcal{F}) = \{x\}$
4. *Every net(filter) converges to at most one point.*

Proof. $1 \implies 2$ For any pair (x, y), by definition, there is $y \notin \overline{U}$, hence intersection of family of closed neighborhoods of x can only contains x.

$2 \implies 3$ follows from a point adherent to a filter converges to x must be in every closed neighborhood of x.

$3 \implies 4$ is clearly.

$4 \implies 1$ If there is a net $x.$ converges to both x and y, then $\mathcal{N}(x) \subset \mathcal{F}(x.)$ and $\mathcal{N}(y) \subset \mathcal{F}(x.)$, that is, U and V meets for any $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$. □

Definition 2.13. Space (Ω, τ) is said to be $T_{2.5}$ or **Completely Hausdorff** if for every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $\overline{U} \cap \overline{V} = \varnothing$.

Two nonempty sets are caled **separated by open sets** if they are included in disjoint open sets, and they are **separated by continous functions** if there is continos f taking values in $[0, 1]$ and assign 0 on one set and 1 on the other.

Space (Ω, τ) are said to be **regular** if every singeton and any closed A disjoint from it can be separated by open sets.

Definition 2.14. Space (Ω, τ) is said to be T_3 if it's T_1 and regular.

Space (Ω, τ) are said to **Completely regular** if every singeton and any closed A disjoint from it can be separated by continous function.

Definition 2.15. Space (Ω, τ) is said to be $T_{3.5}$ or **Tychonoff space** if it's T_1 and completely regular.

Theorem 2.25 (Tychonoff's Embedding Theorem)**.** *Space (Ω, τ) is $T_{3.5}$ iff it's homeomorphic to a subspace of $([0,1]^n, \tau_{d_{l^1}})$.*

Space (Ω, τ) is said to be **normal** if two disjoint closed subsets can be separated by open sets.

Definition 2.16. Space (Ω, τ) is said to be T_4 if it's normal and T_1.

Theorem 2.26 (Urysohn's Lemma)**.** *Following statements are equivalent:*

1. *(Ω, τ) is normal.*
2. *For any $U \in \tau$ and any closed $A \subset U$, there is a $U' \in \tau$ s.t. $A \subset U'$ and $\overline{U'} \subset U$.*
3. *Every two disjoint closed subsets can be separated by continous function.*

Proof. $1 \implies 2$ Apply normal property to A and U^c, there is a U' include A and V include U^c, as $U' \cap V = \varnothing \implies U' \subset V^c \implies \overline{U'} \subset V^c \subset U$.

$2 \implies 3$ Suppose A and B are two disjoint closed subset, apply 2 to A and $U_1 = B^c$ we have $A \subset U_0$ and $\overline{U_0} \subset U_1$. Apply again for $\overline{U_0}$ and U_1 to generates $U_0 \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$, repeat such process, that is, apply 2 to $\overline{U_{\frac{j}{2^k}}}$ and $U_{\frac{j+1}{2^k}}$ to generates $U_{\frac{2j+1}{2^{k+1}}}$. Finally, we construct a open strictly increasing squence U_r. where r is any dyadic rational in $[0,1]$, i.e., $r \in DR \cap [0,1]$.

Then define f as

$$f = \begin{cases} 1 & x \in B \\ \inf\{r : x \in U_r\} & x \in B^c \end{cases}$$

Then it's sufficient to show that f is continous. Note subspace $[0,1]$ of $\mathbb{R}$ can be generated by collection of $[0,s)$ and $(t,1]$ and

$$f^{-1}[0,s) = \bigcup_{r \in DR \cap [0,s)} U_r$$
$$f^{-1}(t,1] = \bigcup_{r \in DR \cap (t,1]} \overline{U_r}^c$$

Then the claim follows from lemma 2.9.

$3 \implies 1$ By taking any disjoint open set A contains 0 and B contains 1 and looking $f^{-1}(A)$ and $f^{-1}(B)$. $\square$

Theorem 2.27 (Tietze's Extension Theorem)**.** *Let (Ω, τ) be normal, F any closed subset and I any bounded closed interval of $\mathbb{R}$. Then any continous $f : F \to I$ can be extended to $f' : \Omega \to I$ and remain continous.*

Proof. Suppose $I = [-1, 1]$, then $A = f^{-1}[-1, -\frac{1}{3}]$ and $f^{-1}[\frac{1}{3}, 1]$ are disjoint and closed. By Urysohn's Lemma, there is $g : \Omega \to [-\frac{1}{3}, \frac{1}{3}]$ s.t. $g(A) = \{-\frac{1}{3}\}$ and $g(B) = \frac{1}{3}$. Set $f_0 = f, g_0 = g, f_1 = f - g|_F$. Then we can show that $|f_1|$ is bounded by $\frac{2}{3}$.

Repeat such process, we have series of

$$\begin{aligned} f_n &: F \to [-(\frac{2}{3})^n, (\frac{2}{3})^n] \\ g_n &: E \to [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n] \\ f_{n+1} &= (f_n - g_n)|_F \end{aligned}$$

Then we show that $g = \sum_{i=0}^{\infty} g_i$ is the extension of f. That is g is continous and $f = g$ in F. Note for any x

$$|\sum_{i=m}^{n} g_i(x)| \leq \sum_{i=m}^{n} |g_i(x)| \leq \sum_{i=m}^{n} \frac{1}{3}(\frac{2}{3})^i \leq (\frac{2}{3})^m \to 0$$

Thus $\{\sum_{i=0}^{n} g_i\}_{n=0}^{\infty}$ converges uniformly by Cauchy's criterion, followed by g is continous. And $f = g$ on F follows from

$$|f(x) - \sum_{i=0}^{n} g_i(x)| = |f_0(x) - \sum_{i=0}^{n} g_i(x)| = |f_1(x) - \sum_{i=1}^{n} g_i(x)| = |f_{n+1}(x)| \leq (\frac{2}{3})^{n+1} \to 0$$

□

2.13 Compactness

Definition 2.17. A **cover** of a set K is collection of sets whose union includes K. A **subcover** is subcollection of a cover and also covers K.

Definition 2.18. K is **compact** if every open cover has a finite subcover and called **relatively compact** if it's closure is compact. A topology (Ω, τ) is **compact** if Ω is compact.

Compactness is a "topological" property. That is, subset compactness in a subspace iff it's also compact in full space.

Theorem 2.28. *Let* (Ω, τ) *be a space, TFAE:*

1. (Ω, τ) *is compact.*
2. *Every filter(net) has at least one adherent point.*
3. *Every ultrafilter(ultranet) converges.*
4. $\ker \mathcal{F} \neq \varnothing$ *For every collection* $\mathcal{F}$ *of closed sets having FIP.*

Proof. $4 \iff 1$ Taking contrapositive:

$$\neg \ker \mathcal{F} \neq \varnothing \equiv \ker \mathcal{F} = \varnothing \equiv \bigcup_{F \in \mathcal{F}} F^c = \Omega$$

And

$$\neg \forall \bigcap_i^n F_i = \varnothing \equiv \exists \bigcup_i^n F_i^c = \Omega$$

note that's precisely the definition of compactness.

$1 \implies 2$ Suppose filter $\mathcal{F}$, then

$$\{\overline{F} : F \in \mathcal{F}\}$$

Enjoy finite intersection property by definition, then $\overline{F}$ has at least one adherent point since $\ker\{\overline{F} : F \in \mathcal{F}\} = \overline{\mathcal{F}} \neq \varnothing$ by 4

$2 \implies 3$ Clearly by corollary 2.3.

$3 \implies 1$ Suppose $\mathcal{A}$ a family of closed subsets with finite intersection property. Then the filter generates by $\mathcal{A}$ has an ultrafilter with a limit point x. Note x is also adherent to $\mathcal{U}$ and thus adherent to $\mathcal{F}$, followed by $x \in A$ for any $A \in \mathcal{A}$, hence $\ker \mathcal{A} \supset \{x\}$. Then the claim follows from 4.

□

Theorem 2.29. *Let* (Ω, τ) *be Hausdorff, then every compact subset and disjoint singleton can be separated by open sets. In particular, compact subset is closed.*

Proof. Suppose $F \subset \Omega$ is compact, for any $x \in \Omega$ not in F, by Hausdorff, there is $x \notin U_y$ and $y \notin V_y$. Then $\bigcup_{y \in F} U_y$ cover F, there is subcover $U = \bigcup_i^n U_{y_i}$ and $V = \bigcup_i^n V_{y_i}$ selected from the same family separated F and $\{x\}$.

□

Theorem 2.30. *Closed subset is compact in compact topological space.*

Proof. Note any open cover of F plus F^c become a open cover of Ω.

□

Theorem 2.31. *Every compact Hausdorff space is normal.*

Proof. Suppose A and B are closed and thus compact by theorem 2.30. For any point $x \in A$, there exist disjoint $V_x \supset B$ and $x \in U_x$ by theorem 2.29. Note $\bigcup_{x\in A} U_x$ cover A, there exist subcover $U = \bigcup_i^n U_{x_i} \supset A$ and $V = \bigcap_i^n V_{x_i} \supset B$ separated A and B.

□

Theorem 2.32. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ is continuous, then $f(A)$ is compact if A is compact.*

Proof. For any open cover of $f(A)$:

$$\cup G_i \supset f(A) \implies f^{-1}(\cup G_i) = \cup f^{-1}(G_i) \supset f^{-1}f(A) \supset A$$

Thus there exist subcover s.t.

$$\cup_1^n f^{-1}(G_i) = f^{-1}(\cup_1^n G_i) \supset A \implies \cup_1^n G_i \supset ff^{-1}(\cup_1^n G_i) \supset f(A)$$

Which shows that $f(A)$ is compact.

□

Corollary 2.4. *Let X be compact and Y be Hausdorff and $f : X \to Y$ is continuous bijection, then f is closed.*

Proof. Note F is closed and thus compact as theorem 2.30 then $f(F)$ is compact as theorem 2.32 and thus closed by theorem 2.29.

□

As consequence:

Corollary 2.5 (Extreme value theorem)**.** *A continuous real valued function defined on a compact space achieves its maximum and minimum values.*

Theorem 2.33. *Let X be compact and Y be Hausdorff and $f : X \to Y$ is continuous bijection. Then f is homeomorphism.*

Proof. By lemma 2.10 and corollary 2.4.

□

2.13.1 Sequentially compact

A subset A of a topological space is **sequentially compact** if every sequence in A has a subsequence converging to an element of A. A topological space is sequentially compact if itself is a sequentially compact set.

Example 2.1. The open interval $(0, 1)$ is not sequentially compact because $\{\frac{1}{n}\}$ has no convergent subsequence.

2.14 Locally compact spaces

Definition 2.19. A topological space is **locally compact** if every point has a compact neighborhood.

Definition 2.20. Subset $A \subset X$ is said **precompact** if $\overline{A}$ is compact.

Theorem 2.34 (Compact neighborhood base)**.** *Let X be Hausdorff, TFAE*

1. *X is locally compact.*
2. *Every $x \in X$ has a precompact neighborhood.*
3. *X has a basis of precompact open sets, i.e., there exist $x \in K^\circ \subset K \subset N$.*

Proof. It's clear that $3 \Rightarrow 2 \Rightarrow 1$ even without Hausdorff, so we show that $1 \Rightarrow 3$.

Begin by open G and compact K neighborhood for x s.t. $A := K - G \neq \varnothing$. For any $y \in A$, there is $U_y \cap W_y = \varnothing$ by Hausdorff, where $y \in U_y$ and $x \in W_y \subset K$. Note A is also compact and then there exist:

$$U = \bigcup_{i=1}^{k} U_{y_i} \supset A$$

Respectively, consider $W = \bigcap_{i=1}^{k} W_{y_i}$, and we claim that $\overline{W}$ is compact and included in G. Compactness is clear as $\overline{V} \subset K$. By theorem 2.29, $\overline{W} \cap U = \varnothing$. Consequently,

$$\overline{W} \cap G^c = \overline{W} \cap K \cap G^c = \overline{W} \cap A \subset \overline{W} \cap U = \varnothing$$

hence $\overline{W} \subset G$.

□

Consequently, that imply the existence of a compact neighborhood base.

Corollary 2.6. *Suppose G is open and F is closed in a locally compact Hausdorff space, then $G \cap F$ is locally compact. That implies every closed and open set is locally compact.*

Proof. Let $x \in G \cap F$, and $N \cap G \cap F$ be neighborhood of x in the subspace, by theorem 2.34, there exist K s.t.

$$x \in K^\circ \subset K \subset N \cap G$$

Then $F \cap K$ is compact as it's closed in compact Hausdorff subspace K.

□

Corollary 2.7. *If K is compact in a locally compact Hausdorff space and G is an open set including K, then there is an open V with compact closure s.t.*

$$K \subset V \subset \overline{V} \subset G$$

Proof. For any $x \in K$, by theorem 2.34, we have

$$x \in V_x \subset \overline{V_x} \subset G$$

then note

$$K \subset \bigcup_{i=1}^{k} V_{x_i} = V$$

we claim that V is desired. Since

$$\overline{V} = \overline{\bigcup_{i=1}^{k} V_{x_i}} = \bigcup_{i=1}^{k} \overline{V_{x_i}}$$

is compact and included in G.

□

2.14.1 Compactification

Locally compact Hausdorff space is very close to a compact Hausdorff space

Definition 2.21. A **Compactification** of a space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

Definition 2.22. Let (X, τ) be a space and define $\hat{X} = X \cup \{\infty\}$, with topology $\hat{\tau}$ consisting of sets that:

1. $G \in \tau$.
2. $\infty \in G$ and $\hat{X} - G = X - G \subset X$ is compact.

Theorem 2.35. *If X is Hausdorff and noncompact, then $\hat{X}$ is a compactification.*

Proof. Firstly we show that $\hat{X}$ is a space. By definition, $\varnothing$ and $\hat{X}$ are open clearly. To show it's closed under countable intersection, it suffices to show that $U_1 \cap U_2$ is open when U_1 and U_2 are so. We classify cases by whether ∞ occurs.

1. If $\infty \notin U_1 \cup U_2$, $U_1 \cap U_2 \in \hat{\tau}$ as $U_1 \cap U_2 \in \tau$.
2. If $\infty \in U_1$ and $\infty \notin U_2$, then $X - U_1$ is compact, as X is Hausdorff, $X - U_1$ is closed in X and thus $X - (X - U_1) = U_1 - \{\infty\}$ is open in X, it follows that $U_1 \cap U_2 = (U_1 - \{\infty\}) \cap U_2$ and the same as 1.
3. If $\infty \in U_1 \cap U_2$, then
$$X - (U_1 \cap U_2) = (X - U_1) \cup (X - U_2)$$
is compact as it's union of compact sets and thus $U_1 \cap U_2$ is open.

Now we turn to show closed under union. Suppose $\bigcup_{i \in I} U_i$ is a collection of open sets. If none contain ∞, $\bigcup_{i \in I} U_i$ is open clearly as it's open in X. If $\infty \in U_j, \forall j \in J$ for some $J \subset I$. Then

$$X - \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i)$$

is closed subset of any compact Hausdorff space $X - U_j$ and thus compact. It follows that $\bigcap_{i \in I} U_i$ is open.

Next, we show that $\iota : X \to \hat{X}$ is an embedding. It's injective and open clearly and it suffices to show it continuity by lemma 2.10. For open sets G in $\hat{X}$:

$$\iota^{-1}(G) = \begin{cases} G & \infty \notin G \\ G - \{\infty\} & \infty \in G \end{cases}$$

is also open as $G - \{\infty\} = X - (X - G)$ is open have shown above.

To see $\iota(X)$ is dense, it suffices to see $\{\infty\}$ is not open and that follows from definition of $\hat{X}$.

Finally, we show that $\hat{X}$ is compact. Let $\mathcal{G}$ be open cover, then there is some $G \in \mathcal{G}$ contains ∞. Note remaining of $\mathcal{G}$ still cover $X - G$ and thus have a finite cover then claim follows easily,

□

Lemma 2.25. *If noncompact X is Hausdorff and locally compact, $\hat{X}$ is also Hausdorff.*

Proof. Let x_1 and x_2 in $\hat{X}$. If neither is ∞, we have desired disjoint neighborhood immediately. If $x_2 = \infty$, let $x_1 \in U \subset K$ then U and $V = \hat{X} - K$ are what we desired.

□

Lemma 2.26. *$\hat{X}$ is not Hausdorff if there is no subset G and K of X s.t. $G \subset K$.*

Proof. Suppose $\hat{X}$ is Hausdorff, then there is $\infty \in U$ s.t. $K = X - U$ is compact and disjoint to some V open in X, note

$$\begin{aligned} U \cap V = \varnothing &\Rightarrow (U - \{\infty\}) \cap V = \varnothing \\ &\Rightarrow (X - K) \cap V = \varnothing \\ &\Rightarrow V \subset K \end{aligned}$$

□

Example 2.2. $\hat{\mathbb{Q}}$ is non Hausdorff as any open sets G of the form $(a, b) \cap \mathbb{Q}$, if it's contained in a compact subset K, then $\overline{G}$ would be compact, which contradict to $[a, b] \cap \mathbb{Q}$ is not compact.

Theorem 2.36. *X is locally compact iff X is open of $\hat{X}$.*

Proof. $\Longleftarrow$ comes from corollary 2.6.

$\Longrightarrow$ Suppose $(\hat{X}, \hat{\tau})$ is compactification of Hausdorff (X, τ). For any $x \in X$, we may pick $x \in G \subset K$, where G is open and K is compact in τ. Consider $W \in \hat{\tau}$ where $W \cap X = G$, we have

$$W = W \cap \hat{X} = W \cap \overline{X} \subset \overline{W \cap X} = \overline{G} \subset K \subset X$$

that implies $x \in X^\circ \implies X^\circ = X$, i.e. X is open.

□

Lemma 2.27. *Let X be a locally compact Hausdorff space and $f : X \to Y$ a compactification, then f is open.*

Proof. As f is an embedding, we can pretend $X \subset Y$ and f is just inclusion. Then it suffices to show that X is open and that follows from theorem 2.36.

□

Theorem 2.37 (Universal property of compactification)**.** *Let X be a locally compact Hausdorff space and $f : X \hookrightarrow Y$ be a compactification. Then there is a unique quotient map $q : Y \to \hat{X}$ s.t. $q \circ f = \iota$.*

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\exists! q"] & & \hat{X} \\
& X \arrow[ul, "f"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Let X be locally compact and Hausdorff and let $f: X \hookrightarrow Y$ be a compactification. Then there is a unique quotient map q: $Y \to \hat{X}$ s.t. $q \circ f = \iota$.

2.15 Weak topology

Suppose $\{(Y_i, \tau_i)\}_{i \in I}$ a family of topological space and $f_i : X \to Y_{i_{i \in I}}$. Let $\mathcal{F}$ be the set of all the topologies s.t. f_i is continuous for all i. We call $\cap \mathcal{F}$, i.e., the coarsest topology the **induced topology** or **weak topology** or **initial topology** on X by $\{f_i\}_{i \in I}$. The topology induced by $\{f_i\}_{i \in I}$ is generated by

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \tau_i\}$$

or

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \mathcal{S}_i\}$$

where $\mathcal{S}_i$ is a subbase for τ_i.

Lemma 2.28. *A net $x. \to x$ in the weak topology iff $f_i(x.) \to f_i(x)$ for each i.*

Proof. $\implies$ is immediately. Conversely, noting sets of the form $\bigcap_1^n f_i^{-1}(V_i)$ consist a neighborhood base.

□

Theorem 2.38. *g is (τ', τ) continuous iff $f_i \circ g$ continuous for each f_i. Where τ is $\tau(S)$ in above .theorem.*

Proof. $\implies$ is immediately. $\impliedby$, suppose $G \in \tau$,by above .theorem, this implies

$$G = \cup_I \cap_F X = \cup_I \cap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continuous and thus $g^{-1}(G) = \cup_I \cap_F g^{-1} f^{-1}(G) = \cup_I \cap_F (f \circ g)^{-1}(G)$.

□

If the family $\mathcal{F}$ consists of real function on X, the weak topology is denoted $\sigma(X, \mathcal{F})$. A subbase for $\sigma(X, \mathcal{F})$ consist of

$$U(f, x, \epsilon) = f^{-1}(B(f(x), \epsilon)) = \{y \in X : |f(y) - f(x)| < \epsilon\}$$

where $f \in \mathcal{F}, x \in X, \epsilon > 0$. $\mathcal{F}$ is said **total** if $\forall f \in \mathcal{F}, f(x) = f(y) \implies x = y$. $\sigma(X, \mathcal{F})$ is Hausdorff iff $\mathcal{F}$ is total.

Lemma 2.29. *Let A be a subset, then*

$$(A, \sigma(A, \mathcal{F}|_A)) = (A, \sigma(X, \mathcal{F})|_A)$$

Proof. Nets converges in $(A, \sigma(X, \mathcal{F})|_A)$ also converges in $(X, \sigma(X, \mathcal{F}))$, that is $\forall f, f_i(x.) \to x.$ and thus the same as nets converges in $\sigma(A, \mathcal{F}|_A)$. That implies identical mapping is a homeomorphism since $x. \to x \iff I(x.) \to I(x)$.

□

The weak topology generated by $C(X)$ is also generated by $C_b(X)$ by noting for any $f \in C(X)$,

$$g(y) = \min\{f(x) + \epsilon, \max\{f(x) - \epsilon, f(y)\}\}$$

is bounded by $B(f(x), \epsilon)$ and $U(g, x, \epsilon) = U(f, x, \epsilon)$.

Theorem 2.39. *(X,) is completely regular iff $\tau = \sigma(X, C(X))$*

Suppose $\tau = \sigma(X, \mathcal{F})$ and is completely regular, then we claim that $\mathcal{F} = C(X)$.

2.16 Product topology

Theorem 2.40 (Universal property of the Cartesian product)**.** *Let X, Y and Z be any space and given $f_X : Z \to X$ and $f_Y : Z \to Y$, there exist unique function $f : Z \to X \times Y$ s.t.*

$$f_X = \pi_X \circ f \text{ and } f_Y = \pi_Y \circ f$$

and f is just (f_X, f_Y).

$$\begin{tikzcd}
& & X \\
Z \arrow[r, dashed, "\exists! f"'] \arrow[urr, bend left, "f_X"] \arrow[drr, bend right, "f_Y"'] & X \times Y \arrow[ur, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
& & Y
\end{tikzcd}$$

Lemma 2.30. *Suppose $\varphi : X \times Y \to Z$ is continuous, for each $x \in X$, define $\hat{\varphi} : Y \to Z$ by $\hat{\varphi}_x(y) = \varphi(x, y)$, then φ_x is continuous.*

Proof. Note $\hat{\varphi}_x$ is composition by $Y \stackrel{i_x}{\to} X \times Y \stackrel{\varphi}{\to} Z$, so it suffices to show that i_x is continuous. And that is just the product of constant map $Y \to X$ and identity map $Y \to Y$. Then the claim follows as both is continuous. □

Also, φ is continuous if $\hat{\varphi}$ is continuous as φ is composition by

$$X \times Y \xrightarrow{\hat{\varphi} \times i} \mathcal{C}(Y, Z) \times Y \xrightarrow{eval} Z$$

Where we should use the truth that product of continuous function is continuous:

Theorem 2.41. *Let $f : X \to Y$ and $f' : X' \to Y'$ be continuous. Then the product $f \times f' : X \times X' \to Y \times Y'$ is also continuous.*

Proof. Clearly as the factor $X \times X' \to Y$ is the composition $X \times X' \xrightarrow{\pi_X} X \stackrel{f}{\to} Y$ □

Let $((\Omega_i, \tau_i))_{i \in I}$ be family of topological spaces, let $\Omega = \prod_{i \in I} \Omega_i$ and π_i be projection mappings from Ω to Ω_i. The topology τ induced by $(\pi_i)_{i \in I}$ is called **product topology** on Ω and denoted by $\prod_{i \in I} \tau_i$. (Ω, τ) is called **topological product**.

A subbase of this topology is all the sets of the form $\pi_i^{-1}(U_i) = \prod_{i \in I} X_i$ where $X_j = \Omega_j$ for all $i \neq j$ and $X_i = U_i$.

Lemma 2.31. *Suppose $G \in \prod \tau_i$, then $\pi_i(G) = \Omega_i$ except a finite set in I.*

Proof. By definition,

$$G = \bigcup_I \bigcap_F (\prod_{i \in I} X_i)$$

where $X_i = \Omega_i$ for all i but one. Note there is a finitely intersection, that is

$$G = \bigcup_I (\prod_{i \in I} X_i)$$

where $X_i = \Omega_i$ for all i but finite exception. And the claim is easily follows. □

The product topology satisfy similar universal property if I is finite, that is

Theorem 2.42. *Given any space Z and $\{f_i : Z \to \Omega_i\}_{i \in I}$, there exist unique continuous $f : Z \to \prod_{i \in I} \Omega_i$ s.t. $\forall i \in I, \pi_i \circ f = f_i$.*

Proof. Existence is clear as we may define f by $f(z)_i = f_i(z)$ and $\pi_i \circ f = f_\alpha$ suggests the uniqueness. Then it suffices to show that continuity. Note the product topology has subbasis $\pi_i^{-1}(U_i)$ and

$$f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i) = f_i^{-1}(U_i)$$

is open as f_i is continuous.

$\square$

We call the topology generated by $\{\prod_{i \in I} U_i\}$ **box topology** and it's finer than product topology unless I is finite and can't enjoy universal property. But they still share following property.

Lemma 2.32. *Let $A_i \subset \Omega_i$ for each $i \in I$, then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}$$

in both product and box topology.

Proof. $\subset$: Let $(x_i)_{\{i \in I\}} \in \prod_{i \in I} \overline{A_i}$, and $U = \prod_{i \in I} U_i$ be a open neighborhood of which, then U_i is neighborhood of x_i and thus U_i meet A_i in, say, y_i, then we may find $(y_i) \in U \cap \prod_{i \in I} A_i$ and thus $(x_i) \in \overline{\prod_{i \in I} A_i}$.

$\supset$: Note product closed set is closed as

$$\left(\prod_{i \in I} F_i\right)^c = \bigcup_{i \in I} \prod_{i = I} X_i$$

Where $X_j = \Omega_j$ for $j \neq i$ and $X_i = F_i^c$, that is open clearly. And the claim follows as closure is minimum.

$\square$

Lemma 2.33. *Ω_i is Hausdorff for each i iff so is $\prod_{i \in I} \Omega_j$ in both product and box topology.*

Proof. $\Rightarrow$: Pick any different (x_i) and (x_i') in $\prod_{i \in I} \Omega_i$ and suppose $x_\ell \neq x_\ell'$ for particular ℓ and they can be separated by U_ℓ and U_ℓ'. Then (x_i) and $(x_i)'$ can be separated by $\pi_\ell^{-1}(U_i)$ and $\pi_\ell^{-1}(U_i')$ and thus Hausdorff. For box topology, it's Hausdorff clearly as it's finer than product topology.

$\Leftarrow$: Note Hausdorff property is hereditary and we may treat factor Ω_ℓ as subspace by define embedding

$$f_\ell(x)_j : \Omega_\ell \to \prod_{i\in I} \Omega_i = \begin{cases} x & j = \ell \\ y_j & j \neq \ell \end{cases}$$

where y_j is any fixed point for each j. It's continuous and injective certainly, to see it's embedding, it suffices to show that it's open. Suppose any open $U_\ell \subset \Omega_\ell$, then

$$f_\ell(U_\ell) = \pi_\ell^{-1}(U_\ell) \cap f_\ell(\Omega_\ell)$$

is open in subspace $f_\ell(\Omega_\ell)$.

$\square$

Thus,$\{(x_i^\alpha)\}_{\{i\in I\}}$ in X converges to some $(x_i)_{i\in I}$ iff its every components converges to the components respectably. A function is called **jointly continuous** if it's continuous w.r.t. the product topology.

Theorem 2.43 (Closed Graph Theorem)**.** *Function $f : (X, \tau) \to (Y, \tau)$ where Y is compact Hausdorff is continuous iff its graph $\mathrm{Gr}\, f$ is closed.*

Proof. $\implies$. For any net $(x., y.) \to (x, y)$, we show that $(x, y) \in \mathrm{Gr}\, f$. Note $f(x.) = y. \to y$, also, $f(x.) \to f(x)$ by continuity. It follows by $f(x) = y$ since Hausdorff and we finished.

$\impliedby$. Since Y is compact and Hausdorff, $f(x.)$ converges to precisely one point and denoted as y. As $\mathrm{Gr}\, f$ is closed, $y = f(x)$ and hence f is continuous.

$\square$

Suppose A_i is subset of each i, then

$$\operatorname*{Cl}_{\tau}(\prod A_i) = \prod(\operatorname*{Cl}_{\tau_i}(A_i))$$

Thus we have an alternative definition of semicontinuous:

$f : X \to \mathbb{R}^*$ is

- lower semicontinuous iff its epigraph $\{(x, c) : c \geq f(x)\}$ is closed.
- upper semicontinuous iff its hypograph $\{(x, c) : c \leq f(x)\}$ is closed.

Theorem 2.44 (Tychonoff Product Theorem)**.** *The product topology of a family of topologies $\tau = \prod_{i\in I} \tau_i$ is compact iff τ_i is compact for every $i \in I$.*

Proof. $\implies$ is clearly as projection is continuous.

$\impliedby$, suppose $\mathcal{U}$ is ultrafilter in τ, then $\pi_i(\mathcal{U})$ is ultra base and thus converges to some point, say x_i, then we claim that $\mathcal{U} \to x = (x_i)_{i \in I}$. Suppose V any neighborhood of x, there is

$$a \in \bigcap_{i \in J} \pi_i^{-1}(X_i) \subset V$$

where X_i is neighborhood of x_i and thus belong to $\pi_i(\mathcal{U})^{\uparrow}$, that implies there is $U \in \mathcal{U}$ s.t. $\pi_i(U) \subset X_i$, note $U \subset \pi_i^{-1}\pi_i(U) \subset \pi_i^{-1}(X_i)$, then $\pi_i^{-1}(X_i) \in \mathcal{U}$ and thus $V \in \mathcal{U}$. It followed by x is adherent to $\mathcal{U}$ and thus $\mathcal{U} \to x$ as $\mathcal{U}$ is ultra.

□

As consequence, we have

Theorem 2.45. *In the same notations, let K_i be compact for each i, G is open in τ and including $\prod_{i \in I} K_i$, then there exist basic open set sandwich by them.*

2.17 coinduced topology

If we turn all of the arrows around in the diagram of product, that is,

Theorem 2.46. *Given space Z and f_X and f_Y, there is a unique map from $X \coprod Y$ to Z:*

$$\begin{tikzcd}
X \arrow[dr, "\iota_X"'] \arrow[drr, bend left, "f_X"] & & \\
& X \coprod Y \arrow[r, dashed, "\exists! f"'] & Z \\
Y \arrow[ur, "\iota_Y"] \arrow[urr, bend right, "f_Y"'] & &
\end{tikzcd}$$

The coproduct of $\{X_i\}_{i \in I}$ is given by

$$\coprod_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\})$$

Clearly, there are nature inclusions $\iota_{X_i} : X_i \hookrightarrow \coprod_{i \in I} X_i = x_i \mapsto (x_i, i)$. We topologize the coproduct by giving it the finest topology s.t. all ι_{X_i} are continuous.

Proof. Suppose $V \subset Z$ is open, then is open in $\coprod_{i \in I} X_i$ if each $\iota_i^{-1} f^{-1}(V)$ is open. Note

$$(f \circ \iota_i)^{-1}(V) = f_i^{-1}(V)$$

is open as each f_i is continuous.

□

Lemma 2.34. *Let X_i be a space for $i \in I$, then $\coprod_{i\in I} X_i$ is Hausdorff iff all X_i are Hausdorff.*

Proof. $\Rightarrow$ is trivial as X_i embeds as a subset. For $\Rightarrow$, suppose $x \neq y$ in $\coprod_{i\in I} X_i$, if x and y come from different X_i, we simple select X_i and X_j they live, otherwise, X_i is Hausdorff and guarantee a disjoint neighborhood.

□

2.17.1 Quotient

Suppose $q : X \to Y$ is any subjective function, we define $\sim$ by $x \sim y$ if $q(x) = q(y)$, then $X/\sim \to Y$ is bijection and we can treat q as function that $X/\sim \to Y$. And that gives the universal property of the quotient.

Definition 2.23. A surjection $q : X \to Y$ is a **quotient map** if $V \subset Y$ is open iff $q^{-1}(V)$ is open in X.

Theorem 2.47 (Universal property of quotient)**.** *Let $q : X \to Y$ be a quotient map and $f : X \to Z$ is continuous and constant on the fiber of q, then there exist a unique continuous $g : Y \to Z$.*

$$\begin{tikzcd}
X \arrow[dr, "q"'] \arrow[drr, "f"] & & \\
& Y \arrow[r, dashed, "\exists! g"'] & Z
\end{tikzcd}$$

Proof. Clearly g must be defined by $g = f \circ q^{-1}$ and it remains to show that g is continuous. Let $G \subset Z$ is open then $g^{-1}(G) \subset Y$ is open iff $q^{-1}(g^{-1}(G)) = (g \circ q)^{-1}(G) = f^{-1}(G)$ is open, and that follows from f is continuous.

□

Lemma 2.35. *Let $q : X \to Y$ be a continuous open surjection, then it's quotient map. The same is true if q is closed instead of open.*

Proof. Open case follows easily. For the other, for $V \subset Y$ s.t. $q^{-1}(V) \subset X$ is open, then $q^{-1}(V^c)$ is closed and thus $q(q^{-1}(V^c)) = V^c$ is close as surjection.

□

However, the converse is not true.

Definition 2.24. Let $q : X \to Y$ be a continuous surjection. We say $U \subset X$ is **saturated** w.r.t. q if $U = q^{-1}(V)$ for some $V \subset Y$, i.e., $q^{-1}(q(U)) = U$.

Lemma 2.36. *Let $q : X \to Y$ be a continuous surjection, then it's a quotient map iff it takes saturated open sets to open sets.*

Proof. Suppose $q^{-1}(V) \subset X$ is open, then it's a saturated open sets, thus $q(q^{-1}(V)) = V$ is open. And the other implication follows from definition of continuity and quotient map.

□

Suppose $\{(\Omega_i, \mathcal{T}_i)\}_{i\in I}$ a family of topological space and $\{f_i : (\Omega_i, \mathcal{T}_i) \to (\Omega, \tau)\}_{i\in I}$. Let A be the set of all the topologies s.t. f_i is continuous for all i. We call the finest of A **topology coinduced** on Ω by $\{(f_i)\}_{i\in I}$.

Let R an equivalence relation on Ω, $\eta : \Omega \to \Omega/R$ the canonical surjection. The coinduced topology on Ω/R by η is denoted by τ/R and $(\Omega/R, \tau/R)$ is the quotient space w.r.t. R.

2.18 Connection

Definition 2.25. Two subset A and B are said to be **separated** if

$$A \cap \overline{B} = B \cap \overline{A} = \varnothing$$

Clearly, if disjoint A and B are both open or closed, they are separated.

Definition 2.26. Two nonempty separated subset A and B are called a **separation** if $A \cup B = X$.

Lemma 2.37. *Separation are both clopen.*

Proof. Suppose A and B is a separation, then

$$\overline{A} = \overline{A} \cap X = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = \overline{A} \cap A = A$$

thus A and B are closed, that implies A and B are open.

□

Definition 2.27. Space X is said to be **connected** if the only clopen set is X and $\varnothing$. Not connected space is said to be disconnection. Subset A is said to be *connected* or *disconnected* according to the connectedness of their subspace (A, τ_A)

Note separation are clopen, thus X is disconnected iff there exist a separation in X.

Theorem 2.48. *Suppose A is connected in X, then every set B s.t. $A \subset B \subset \overline{A}$ is connected.*

Proof. Suppose B is disconnected and separated by X and Y, then

$$A = (A \cap X) \cup (A \cap Y)$$

also construct a separation, as A is connected, we have, say $A \cap X = \varnothing$ and thus $A \subset Y$. It follows that

$$X \subset B \subset \overline{A} \subset \overline{Y}$$

whence contradict to $X \cap \overline{Y} = \varnothing$.

□

Theorem 2.49. *Suppose $\{A_i\}_{i \in I}$ is a family of connected subsets, then $A = \bigcup_{i \in I} A_i$ is connected if $\ker\{A_i\}_{i \in I} \neq \varnothing$.*

Proof. Suppose A is disconnected and separated by X and Y, then

$$A_i = A_i \cap A = (A_i \cap X) \cup (A_i \cap Y)$$

also construct a separation, as A_i is connected, we have $A_i \cap X = \varnothing$ or $A_i \cap Y = \varnothing$, suppose $I_X + I_Y = I$ and $A_i \cap X = \varnothing$ for $i \in I_X$ and $A_i \cap Y = \varnothing$ for $i \in I_Y$. Note $A_i \cap X = \varnothing \Rightarrow A_i \cap Y = A_i$ and thus

$$\begin{aligned}
\varnothing = X \cap Y &\supset (X \cap \bigcap_{i \in I_Y} A_i) \cap (Y \cap \bigcap_{i \in I_X} A_i) \\
&= \left(\bigcap_{i \in I_Y} A_i \right) \cap \left(\bigcap_{i \in I_X} A_i \right) \\
&= \ker\{A_i\}_{i \in I}
\end{aligned}$$

A contradiction.

□

Theorem 2.50. *Suppose $f : X \to Y$ is continuous, then f bring connected set subset $A \subset X$ to connected subset of Y.*

Proof. Suppose $f(A)$ is disconnected and separated by two open set, say, $f(A)\cap U$ and $f(A)\cap V$, where U, V are open in Y. That implies $f(A)\subset U\cup V$, note

$$A\subset f^{-1}f(A)\subset f^{-1}(U\cup V)=f^{-1}(U)\cup f^{-1}(V)$$

thus A is separated by $A\cap f^{-1}(U)$ and $A\cap f^{-1}(V)$, say, $A\cap f^{-1}(U)=\varnothing$, then

$$A\subset f^{-1}(V)\Rightarrow f(A)\subset V\Rightarrow f(A)\cap U=\varnothing$$

A contradiction.

□

Theorem 2.51. *Suppose each of family $\{X_i\}_{i\in I}$ is nonempty, then their product topology $\prod_{i\in I} X_i$ is connected iff each X_i is closed.*

Proof. $\Rightarrow$ follows from π_i is continuous and theorem 2.50(uses each X_i is nonempty).

$\Leftarrow$ Firstly, we should prove that in finite case, i.e., when I is finite. By induction, it suffices to show that $X_1\times X_2$ is connected. Pick fixed $z\in X_2$ we have the embedding $f(x): X_1\to X_1\times X_2=(x,z)$ and thus $D=f(X_1)$ is connected as theorem 2.50. Then for each $x\in X$, define embedding $g_x(y)=(x,y)$, let $D_x=g_x(X_2)\cup C$, it's connected as theorem 2.49, then $X_1\times X_2=\bigcup_{x\in X_1} D_x$ is connected for the same reason.

Now we are ready for the general case. Pick some $(z_i)_{i\in I}\in\prod_{i\in I} X_i$, for each finite collection $S_j\subset I$, let

$$F_{S_j}=\bigcap_{i\notin S_j}\pi_i^{-1}(z_i)\subset\prod_{i\in I} X_i$$

Clearly $F_{S_j}\cong\prod_{i\in S_j} X_i$, so it follows that F_{S_j} is connected and $(z_i)\in F_{S_j}$ for each S_j, so it follows that

$$F=\bigcup_{j\in J} F_{S_j}$$

is connected. Then it remains to show that F is dense in $\prod_{i\in I} X_i$ as lemma **??**. Recall any nonempty basis element of the form $\bigcap_{i\in S_j}\pi_i^{-1}(U_i)$ for some S_j and thus meet F_{S_j} ($X\times\cdots\times X\times U\times\cdots\times U\times X\times\cdots\times X$ and $z\times\cdots\times z\times X\times\cdots\times X\times z\times\cdots\times z$), that implies F must be dense.

□

Definition 2.28. $A\subset X$ is said **path-connected** if every distinction singleton a and b has a **path** $f:[0,1]\to A$ s.t. $f(a)=0$ and $f(b)=1$.

Lemma 2.38. *Path-connected implies connected.*

Proof. Pick any $a_0 \in A$, for each other $b \in A$, there exit a path f_b, then $f_b(I)$ is connected. Then

$$A = \bigcup_{b \in A} f_b(I)$$

is connected as theorem 2.49.

$\square$

Path-connected is quite similar to connected.

Theorem 2.52. *1. Image of path-connected spaces are path-connected.*
2. Overlapping unions of path-connected spaces are path-connected.
3. Product is path-connected iff every factor is path-connected.

Proof. We only prove part 3. $\Rightarrow$ is trivial. To achieve $\Leftarrow$, for any pair (x_i) and (y_i), there exist path f_i for each $i \in I$, and then we get a continuous path $f = (f_i)$ by the universal property.

$\square$

The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 2.29. Let $x \in X$, **connected component** of x is defined as:

$$C_x = \bigcup \{C | C \text{ is connected and } x \in C\}$$

Similarly, the **path-component** is

$$PC_x = \bigcup \{C | C \text{ is path-connected and } x \in C\}$$

Example 2.3. Suppose $\mathbb{Q}$ equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are singletons, so $C_x = \{x\}$. Such a space is said **totally disconnected**

In the light of connected component is maximum, each component C_x is closed as $\overline{C_x}$ is connected.

Definition 2.30. Let X be a space, it's **locally connected** if any neighborhood U of any x contains a connected neighborhood. And we define **locally path connected** in a similar way.

Theorem 2.53. *Let X be a space. TFAE:*

1. *X is locally connected.*
2. *X has a basis consisting of connected open sets.*
3. *For every open set $G \subset X$, any component $C \subset U$ is open in X.*

Proof. $1 \Rightarrow 3$. For any open $G \subset X$ and any $C \subset G$, for any $x \in C$, there exist connected neighborhood $x \in U \subset G$, as C is component, we have $U \subset C$ and thus C is open.

$3 \Rightarrow 1$. Let G be a open neighborhood of x, then the component C_x is the desired neighborhood.

$3 \Leftrightarrow 2$. $3 \Rightarrow 2$ is clear, for the converse, note $2 \Rightarrow 1$ and thus implies 3.

□

The property of path-connected is even better.

Theorem 2.54. *Let X be a space, TFAE:*

1. *X is locally path-connected.*
2. *X has a basis consisting of path-connected open sets.*
3. *For every open $G \subset X$, the path-component of G are open in X.*
4. *For every open set $G \subset X$, every component of G is path-connected and thus a path-component.*

Proof. We only show that $1 \Leftrightarrow 4$. Suppose X is locally path-connected, and let $P \subset C \subset G \subset X$, where P, C, G are path-component, component and open set respectly. Then P is open.

□

Chapter 3

Metric space

Definition 3.1 (metric/distance)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying: - $d(x, y) \geq 0$ and $d(x, x) = 0$ for all $x, y \in X$ - $d(x, y) = 0 \implies x = y$ - $d(x, y) = d(y, x)$ for all $x, y \in X$ - $d(x, y) \leq d(y, z) + d(x, z)$ for all $x, y, z \in X$

A semimetric on X is a function $d : X \times X \to \mathbb{R}$ which satisfies the condition 1,3,4.
If d is a metric on X, then (X, d) is called a metric space.

Give a metric space (X, d), and $A \subset X$, we said the diameter of A is

$$\mathrm{diam} A = \sup\{d(x, y) : x, y \in A\}$$

A set A is bounded if $\mathrm{diam} A < \infty$ while A is unbounded if $\mathrm{diam} A = \infty$.

Definition 3.2. Let (X, d) be a semimetric space. $A \subset X$ is d-open if for each $a \in A$ there exists some $r > 0$ s.t. $B_r(a) \subset A$.
Then consider about the family $\{A \subset X : A \text{ is d-open}\}$, it generates a topology on X, denoted as τ_d.

Lemma 3.1. *Let (X, d) be a semimetric space. Then:*

1. *(X, τ_d) is Hausdorff space iff d is a metric*
2. *A sequence (x_n) in X satisfies $x_n \to x$ in (X, τ_d) iff $d(x_n, x) \to 0$*
3. *Every d-open ball is an open set*
4. *The topology τ_d is first countable*
5. *A point $x \in \overline{A}$ of some $A \subset X$ iff there exists some sequence (x_n) in A with $(x_n) \to x$.*
6. *A closed ball is a closed set.*
7. *The closure of the open ball $B_r(x)$ is included in the closed ball $C_r(x)$.*
8. *If (X, d_1) and (Y, d_2) are semimetric spaces, the product topology on $X \times Y$ is generated by the semimetric*

$$D((x, y), (u, v)) = d_1(x, u) + d_2(y, v)$$

9. *For any four points u, v, x, y the semimetric obeys:*

$$|d(x,y) - d(u,v)| \leq d(x,u) + d(y,v)$$

10. *The real funtion $d : X \times X \to \mathbb{R}$ is jointly continuous.*

Definition 3.3.
- A subset A is called d-open if it there is a open ball $B_r^d(x) \subset A$ for every $x \in A$.
- A topology τ_d is **generated by** d if $\tau_d = \{A \subset X : A \text{ is } d\text{-open}\}$
- Two metrics is called equivalent if the topology they generated are the same.

Lemma 3.2. *A metrizable space is separable iff it is second countable.*

Proof. Let (X, τ) is a second countable space. There exists a topology base $\mathcal{B} = \{B_i : i = 1, 2, ...\}$, let $A = \{x_i : i = 1, 2, ...\}$ where $x_i \in B_i$ is arbitrary get. Then it is easy to show that $\overline{A} = X$ which means every give any point $x \in X$, $U \in \mathcal{N}_x$, U intersects A. Notice that for any open set U, there is some $B_i \in \{B_i\}$ s.t. $B_i \subset U$. Now give some $x \notin A$, let $U_x \in \mathcal{N}_x$, then $B_i \subset U_x$ for some i, and there is at least a point $x_i \in B_i$ s.t. $U_x \cap A \supset \{x_i\} \neq \emptyset$.
Let (X, d) is a metric space and (X, τ_d) is a topological space generated by d. Let $A = \{x_i : i = 1, 2, ...\}$ be a countable dense subset in X. Then the collection $\{B_{\frac{1}{n}}(x) : x \in A, n \in \mathbb{N}\}$ of d-open balls is a countable base for the topology τ.

□

Definition 3.4 (completeness)**.** A **Cauchy Sequence** in a metric space $(X, d)$is a sequence (x_n) s.t. for each $\epsilon > 0$ there exists some n_0 satisfying $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$, or equivalently if $\lim_{n,m\to\infty} d(x_m, x_n) = 0$ or also equivalently if $\lim_{n\to\infty} \operatorname{diam}\{x_n, x_{n+1}, ...\} = 0$.
A metric space (X, d) is **complete** if every Cauchy sequence in X converges in X, in which case we say that d is a **complete metric** on X.
A topological space x is **completely metrizable** if there exists a consistent metric d for which (X, d) is complete. A separable topological space that is completely metrizable is called **Polish space**.

Definition 3.5 (uniform metric)**.** If X is a nonempty set, then the vector space $B(X)$ of all bounded real functions on X is a complete metric space under the **uniform metric** defined by

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

It is clear that a sequence (f_n) in $B(X)$ is d-convergent to $f \in B(X)$ iff it converges uniformly to f.

Proposition 3.1. *Let (X, d) be an arbitrary metric space. Then the metric $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a bounded equivalent metric taking values on $[0, 1)$.*
ρ and d has the same Cauchy sequences, and (X, d) is complete iff (X, ρ) is complete.

Let us say that a sequence (A_n) of nonempty sets has vanishing diameter if

$$\lim_{n\to\infty} \mathrm{diam} A_n = 0$$

Theorem 3.1 (Cantor's Intersection Theorem)**.** *In a complete metric space, if a decreasing sequence of nonempty closed subsets has vanishing diameter, then the intersection of the sequence is a singleton.*

Proof. Let (F_n) be a decreasing sequence which means $F_{n+1} \subset F_n$ holds for every n of nonempty closed subsets of the complete metric space (X, d), and let $\lim_{n\to\infty} \mathrm{diam} F_n = 0$. Give $F = \bigcap_{n=1}^{\infty} F_n$, assume that there are more than one point in F, suppose $a, b \in F$, then $d(a, b) \leq \mathrm{diam} F$, it implies that $d(a, b) = 0$.As d is a metric, $a = b$.
Now we just need to prove that F is a nonempty set. For each n pick $x_n \in F_n$, since $d(x_n, x_m) \leq \mathrm{diam} F_n$ for $m \geq n$, the sequence (x_n) is Cauchy. As X is a complete metric space, there is some $x \in X$ s.t. $(x_n) \to x$. Since (F_n) is decreasing, $x_m \in F_n$ for every $m \geq n$, let $m \to \infty$, as F_n is closed, it contains all its accumulation point, so $\lim_{m\to\infty} x_m \in F_n$ for every n, so $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

□

Continuous images may preserve the vanishing diameter property.

Proposition 3.2. *Let (A_n) be a sequence of subsets in a metric space (X, d) s.t. $\bigcap_{n=1}^{\infty} A_n$ is nonempty. If $f : (X, d) \to (Y, \rho)$ is a continuous function and (A_n) has vanishing d-diameter, then $(f(A_n))$ has vanishing ρ-diameter.*

Proof. Since (A_n) has vanishing diameter and $\bigcap_{n=1}^{\infty} A_n$ is nonempty, then $\bigcap_{n=1}^{\infty} A_n$ must be a singleton, namely $\{x\}$. As f is continuous, give $\epsilon > 0$, there exists $\delta > 0$ when $d(x, z) < \delta$ it implies that $\rho(f(x), f(z)) < \epsilon$.
Also there is some n_0 s.t. for $n \geq n_0$ if $z \in A_n$, $d(z, x) < \delta$, so $f(A_n) \subset B(2\epsilon)$. So the series $(f(A_n))$ has vanishing ρ diameter and $\bigcap_{n=1}^{\infty} f(A_n) = \{f(x)\}$.

□

Definition 3.6 (Uniformly Continuous)**.** A function is called uniformly continuous if for each $\epsilon > 0$, there exists some $\delta > 0$ s.t. $d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$ for every $x, y \in X$.

Definition 3.7 (Lipschitz continuous)**.** A function $f : (X, d) \to (Y, \rho)$ is called Lipschitz continuous if for every $x, y \in X$:

$$\rho(f(x), f(y)) \leq c d(x, y)$$

The number c is called **Lipschitz constant** for f. Every Lipschitz continuous function is uniformly continuous.

Definition 3.8 (isometry)**.** An isometry between (X, d) and (Y, ρ) is a one-to-one function $\phi : X \to Y$ satisfying:

$$d(x, y) = \rho(\phi(x), \phi(y))$$

for all $x, y \in X$. If ϕ is one-to-one and onto, then (X, d) and (Y, ρ) is said to be isometric.
Notice that the isometry is uniform continuous, indeed, Lipschitz continuous.

Proposition 3.3. *Let $\phi : (X, d) \to Y$ to be one-to-one and onto, then ϕ is induces a metric on Y s.t. $\rho(u, v) = d(\phi^{-1}(u), \phi^{-1}(v))$. Furthermore, $\phi : (X, d) \to (Y, \rho)$ is a isometry*

Proposition 3.4. *If X is metrizable and ρ is a compatible metric on X, then the vector space $U_\rho(X)$ of all bounded ρ-uniformly continuous real functions on X is a closed subspace of $U_h o(X)$. Thus $U_\rho(X)$ equipped with the uniform metric is a complete metric space in its own right.*

Proof. Notice that X is metrizable means X is first countable and in a first countable space, a point $x \in A$ which satisfies $x \in \overline{A}$ iff there is a sequence (x_n) in A s.t. $x_n \to x$. And a sequence of uniform continuous function will converge to a uniform continuous function. So $\overline{U_\rho(X)} = U_\rho(X)$ which means $U_\rho(X)$ is closed.

□

Lemma 3.3 (Uniformly continuous extensions)**.** *Let A be a nonempty subset of (X, d). Let $\phi : (A, d) \to (Y, \rho)$ be a uniformly continuous function. Assume that (Y, ρ) is complete. Then ϕ has a uniformly continuous extension ϕ' to the $\overline{A}$. Moreover, the extension $\phi' = \overline{A} \to Y$ is given by*

$$\phi'(x) = \lim_{n\to\infty} \phi(x_n)$$

for any $(x_n) \subset A$ satisfying $x_n \to x$.
In particular, if $Y = \mathbb{R}$, then $\|\phi\|_\infty = \|\phi'\|_\infty$.

Proof. Notice that a sequence $(x_n) \to x$ in (X, d) must be d-Cauchy, for as $(x_n) \to x$, give a $\epsilon > 0$, there exists n_0 when $n > n_0$, $d(x, x_n) < \epsilon$. Now suppose that $m \geq n$, then $d(x_m, x) < \epsilon$ and as the triangle inequality, $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < 2\epsilon$, so $\lim_{m\to\infty, m\geq n} d(x_m, x_n) = 0$.
Then we need to show that a uniformly continuous function carries a d-Cauchy sequence to a ρ-Cauchy sequence. Let ϕ be a uniformly continuous function and $(x_n) \to x$ be a Cauchy sequence. Give $\epsilon > 0$, then there exists $\delta > 0$ when $d(x_n, x_m) < \delta$, $\rho(\phi(x_n), \phi(x_m)) < \epsilon$. Give $n_0(\delta)$, then for all $m, n \geq n_0$, $d(x_m, x_n) < \delta$, which means give $\epsilon > 0$, there exists $n_0(\delta)$ s.t. for any $m, n \geq n_0(\delta)$, $\rho(\phi(x_n), \phi(x_m)) < \epsilon$, which means $(\phi(x_n))$ is ρ-Cauchy.
Then we begin our proof. Let $x \in \overline{A}$ and pick a sequence $(x_n) \to x$ in A. Since

(x_n) converges, (x_n) is d-Cauchy, and as ϕ is uniformly continuous, then $(\phi(x_n))$ is ρ-Cauchy. Since Y is complete, there are some $y \in Y$ s.t. $\phi(x_n) \to y$.
y is independent of particular (x_n). To prove this, let (z_n) be another sequence converging to x. Then $\{x_1, z_1, x_2, z_2, ...\}$ is a new sequence which is d-Cauchy and converges to x. Notice that $\{\phi(x_1), \phi(z_1), \phi(x_2), \phi(z_2), ...\}$ is also ρ-Cauchy and since $\phi(x_n)$ is a convergent subsequence and its limit is y, the sequence above is y again which implies that $(\phi(z_n)) \to y$ too.
It is easy to show that ϕ' is uniformly continuous on $\overline{A}$ by particularly prove that ϕ' on boundary(A) is continuous.

□

Lemma 3.4. *Let (X, d) be a metric space, let d_1 is a new metric on X. Then d is equivalent to d_1 iff a sequence $(x_n) \to x$ in d iff it converges to x in d_1, namely $d(x_n, x) \to 0 \iff d_1(x_n, x) \to 0$*

Proof. □

Lemma 3.5. *If $f : (X, d) \to (Y, \rho)$ is a continuous function between metric spaces, then there exists an equivalent metric d_1 on X s.t. $f : (X, d_1) \to (Y, \rho)$ is Lipschitz continuous.*

Proof. Define $d_1(x, y) = d(x, y) + \rho(f(x), f(y))$. Give a d-open subset $U \subset X$, it means every point $x \in X$, there exists $r > 0$ s.t. $B_r(x) \subset U$, we would show that

□

3.1 Product Structure

3.1.1 Product Topology

Proposition 3.5 (weak topology)**.** *Suppose there exists a topological space X and a family of topological space $\{Y_s\}_{s\in S}$, and a family of mappings $\{f_s\}_{s\in S}$ where $f_i : X \to Y_i$. In all the topologies on X s.t. f_s is a continuous function for each $s \in S$, there exists a weakest topology which generated by the base consisting of all sets of the form $\bigcap_{i=1}^{k} f_{s_i}^{-1}(V_i)$ where $s_1, \dots, s_k \in S$ and V_i is a open subset of Y_{s_i} for $i = 1, 2, \dots, k$.*

Proof. We only prove that the family which consists all the sets of the form $\bigcap_{i=1}^{k} f_{s_i}^{-1}(V_i)$ is actually a base.

- For any $x \in X$, we only need to find a open set V_s containing $f_s(x)$ and notice that $f_s^{-1}(V) \ni x$.

- Suppose there exists a $x \in X$ s.t.

  $$x \in \left(\bigcap_{i=1}^{k_1} f_{s_i}^{-1}(V_i) \right) \cap \left(\bigcap_{i=1}^{k_2} f_{s_i}^{-1}(V_i) \right)$$

  Notice that $f(x) \in V_i$ for all i, just for each V_i, pick a $U_i \ni f(x)$ s.t. $U_i \subset V_i$ and notice that

  $$x \in \left(\bigcap_{i=1}^{k_1} f_{s_i}^{-1}(U_i) \right) \cap \left(\bigcap_{i=1}^{k_2} f_{s_i}^{-1}(U_i) \right)$$

  and the right side is also a member of the family thus it is a base.

$\square$

Let $\{(X_i, \tau_i)\}$ be a family of topological spaces and let $X = \prod_{i\in I} X_i$ denote its Cartesian product. A typical element $x \in X$ is also denoted as $(x_i)_{i\in I}$ or (x_i). For each $i \in I$, the **projection** $P_i : X \to X_i$ is defined by:

$$P_i(x) = x_i$$

Definition 3.9 (product topology)**.** The product topology on $X = \prod_{i\in I} X_i$ is the weak topology on X generated by the family of projections $\{P_i : i \in I\}$, i.e. τ is the weakest topology that makes P_i continuous for each $i \in I$.

Product topology is also called Tychonoff topology.

Proposition 3.6. *The family of all sets $\prod_{i\in I} V_i$ where V_i is an open subset of X_i and $V_i \neq X_i$ only for finitely many $i \in I$, is a base for Cartesian product $\prod_{i\in I} X_i$.*

Moreover, if for every $i \in I$ a base $\mathcal{B}_i$ for X_i is fixed, then the subfamily consisting of those $\prod_{i\in I} V_i$ with $V_i \in \mathcal{B}_i$ whenever $V_i \neq X_i$, is also a base.

Proof. As the definition of the weak topology, $\bigcap_{i=1}^{k} P_i^{-1}(V_i)$ is open in product topology for $V_i \in \mathcal{B}_i$ and notice that it is a base of $\prod_{i\in I} X_i$. Just need to observe that

$$P_j^{-1}(V_j) = \prod_{i\in I} V_i \text{ where } V_i = X_i \text{ for } i \neq j$$

so $\prod_{i\in I} V_i$ is open and $\bigcap_{i=1}^{k} P_i^{-1}(V_i)$ forms the style that $\prod_{i\in I} V_i$ where $V_i \neq X_i$ for only finitely many V_i.

$\square$

Remark. The base described above is called the canonical base for Cartesian product.

Proposition 3.7. *If* $\{X_s\}_{s\in S}$ *is a family of topological space and* $\{A_s \subset X_s\}_{s\in S}$ *is a family of subspaces, then two topologies defined on* $A = \prod_{s\in S} A_s$*, viz., the topology of the Cartesian product of subspaces* $\{A_s\}_{s\in S}$ *and the topology of subspace of* $\prod_{s\in S} X_s$*, coincide.*

Proof. Consider the restrictions $P_s\mid_A\colon A \to A_s$,

$\square$

Chapter 4

Functional Analysis

4.1 Topology Background in Real Analysis

4.1.1 Meager Set

Definition 4.1. A subset E of a metric space X is said to be **dense in an open set** U if $U \subset \overline{E}$. E is defined to be **nowhere dense** if it is not dense in any open subset $U \subset X$. It means $\overline{E}$ does not contain any open set.

Definition 4.2 (first and second category)**.** A set E is said to be of **first category** in X if it is the union of a countable family of nowhere dense sets.

A set E is said to be a of **second category** in X if it is not the first category set.

Theorem 4.1 (Baire Category Theorem)**.** *A complete metric space X is not the union of a countable family of nowhere dense sets. That is, a complete metric space is of the second category.*

Proof. The proof of the Baire category theorem is to construct a sequence of balls and show that the center of the balls is a Cauchy sequence and find the limit of this sequence is not in X then result in a contradiction.

□

Theorem 4.2 (uniform boundedness theorem)**.** *Let $\mathcal{F}$ be a family of real-valued functions defined on a complete metric space X and suppose*

$$f^*(x) = \sup_{f \in \mathcal{F}} |f(x)| < \infty$$

for each $x \in X$.

Then there exists a nonempty open set $U \subset X$ and a constant M s.t. $|f(x)| \leq M$ for all $x \in U$ and $f \in \mathcal{F}$.

Proof. For each positive $i \in \mathbb{N}$, let

$$E_{i,f} = \{x; |f(x)| \leq i\}, \; E_i = \bigcap_{f \in \mathcal{F}} E_{i,f}$$

Notice that $E_{i,f}$ is closed so is E_i and as the hypothesis, we find that for each $x \in X$, there is a M_x s.t. $f(x) \leq M_x$ for all $f \in \mathcal{F}$, so

$$X = \bigcup_{i=1}^{\infty} E_i$$

And the Baire category theorem implies that there is some $E_M, M \in \mathbb{N}$ is not nowhere dense which means there is some open subset $U \subset E_M$ s.t. for all $x \in U$,and $f \in \mathcal{F}$, $|f(x)| \leq M$.

□

4.1.2 Compactness in Metric Spaces

Lemma 4.1.

- *A convergent sequence in a metric space is Cauchy.*
- *A metric space which all the Cauchy sequence in it is convergence is* ***complete****.*
- *A metric space is a first countable space.*
- *A metric space is separable iff it is a second countable space.*

Proof. Give a sequence $(x_i) \to x$ in X, as X is a metric space, give any $\epsilon > 0$, there exists a $m \in \mathbb{N}$ s.t. for any $n_1, n_2 \geq m$, $d\left(x, x_{n_1}\right) \leq \epsilon/2$, and $d\left(x, x_{n_2}\right) \leq \epsilon/2$, so $d\left(x_{n_1}, x_{n_2}\right) \leq d\left(x_{n_1}, x\right) + d\left(x, x_{n_2}\right) \leq \epsilon$, so (x_i) is Cauchy.

□

Definition 4.3 (totally bounded)**.** If (X, d) is a metric space, a set $A \subset X$ is called totally bounded if for every $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ.

A set A is said to be bounded if there is $M \geq 0$ s.t. $d(x, y) \leq M$ for all $x, y \in A$.

Notice that a totally bounded set is bounded but a bounded set may not be totally bounded.

Definition 4.4 (sequentially compact)**.** A set $A \subset X$ is said to be sequentially compact if every sequence in A has a subsequence that converges to a point $x \in A$.

Also, A is said to have **Bolzano-Weierstrass** property if every infinite subset of A has accumulation point in A.

Theorem 4.3. *If A is a subset of a metric space (X, d), the following are equivalent:*

- *A is compact.*
- *A is sequentially compact.*
- *A is complete and totally bounded.*
- *A has the Bolzano-Weierstrass property.*

Proof. We will give a proof from $1 \implies 2$:

- $1 \implies 2$: Let (x_i) be a sequence in A. Assume that (x_i) 's range is infinite, and suppose (x_i) has no convergent subsequence. Let E denotes the range of (x_i).
  Notice that every subsequence of (x_i) does not converge, so every point $x \in E$, there exists a r_x s.t. $B_r(x) \cap E = \{x\}$. Then as $\overline{E} = E \cup E^*$ where E^* denotes the set of accumulation point of E which is empty, so $\overline{E} = E \implies E$ is closed.
  A is compact and E is closed and $E \subset A$, so E is compact. However, E contains infinite points and every point is isolated, so the open cover $\{B_r(x) : r = r_x\}$ cant have a finite subcover that leads to a contradiction.
- $2 \implies 3$: First we need to show that if a subsequence of a Cauchy sequence converges, then the whole sequence converges.
  Let (x_i) be a Cauchy sequence and let $\left(x_{i(k)}\right)_{k=1}^{\infty}$ be a subsequence of (x_i) s.t. $\left(x_{i(k)}\right) \to x$ which means give a $\epsilon > 0$ there exists a $m(k) \in \mathbb{N}$ for all $k \geq m(k)$, $d\left(x_{i(k)}, x\right) \leq \epsilon/2$. Note that every subsequence of a Cauchy sequence is Cauchy, so there exists a $n(k) \in \mathbb{N}$ for all $k_1, k_2 \geq n(k)$, $d\left(x_{i(k_1)}, x_{i(k_2)}\right) \leq \epsilon/2$, pick $s = i(\max(m(k), n(k)))$, when $i \geq s$, $d(x_i, x) \leq \epsilon$.
  So A must be complete, if not there must be a Cauchy sequence (x_i) in A s.t. there exists a subsequence of (x_i) converges but (x_i) does not converge, which leads to a contradiction of the proposition above.

  About the totally bounded, suppose that A is not totally bounded and there exists a $\epsilon > 0$ s.t. A cannot be covered by finitely many balls of radius ϵ. Then we can choose a sequence in A as follows:
  Pick $x_1 \in A$, Then, since $A - B_\epsilon(x_1) \neq \emptyset$, we can choose $x_2 \in A - B_\epsilon(x_2)$. Note that $d(x_1, x_2) \geq \epsilon$, then similarly we choose

  $$x_i \in A - \bigcup_{j=1}^{i-1} B_\epsilon(x_j)$$

  Then as the cover cannot be finite, so (x_i) is a sequence in A with $d\left(x_i, x_j\right) \geq \epsilon$ when $i \neq j$ so clearly (x_i) does not have any convergent subsequence.

- $3 \implies 4$: Let $A \subset X$ be an infinite subset. Notice that A can be covered by a finite number of balls of radius 1, and there is a B_1 of those balls contains infinite points in A. Let x_1 be one of them. Similarly, there is a ball B_2 of radius 1/2 s.t. $A \cap B_1 \cap B_2$ has infinitely many points, then pick $x_2 \neq x_1$ in it. Then we choose the ball B_i of radius $1/i$ and pick distinct x_k from:
$$\bigcap_{i=1}^{k} A \cap B_i$$
then the sequence (x_k) is Cauchy, then it converges as the completeness, then there is at least one accumulation point of A in A.

- $4 \implies 1$: Omission.

□

Corollary 4.1 (Heine-Borel Theorem). *A compact subset $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.*

Proof. First, compact means totally bounded thus bounded. And a compact subset of Hausdorff space is closed.

For the converse, if A is closed, it is complete. To show this, use the definition of Cauchy sequence and for any closed subset A, $A = \overline{A} = A \cup A^*$ where A^* denotes the set of the accumulation point of A.
Meanwhile, in $\mathbb{R}^n$, bounded means totally bounded. (So, when bounded means totally bounded? Why $\mathbb{R}^n$?).

□

Lemma 4.2 (Lebesgue number). *Let (X, d) be a compact metric space, and let $\{V_i\}_{i \in I}$ be an open cover of X, then there exists some $\delta > 0$, called the **Lebesgue number** of the cover, s.t. for each $x \in X$ we have $B_\delta(x) \subset V_i$ for some $i \in I$.*

Proof. Assume that there is not any $\delta > 0$ satisfies.
Then for each n there exists some $x_n \in X$ s.t. $B_{1/n}(x_n) \cap V_i^c \neq \emptyset$ for each $i \in I$.
If x is the limit point of some subsequence of (x_n), and $x \in X$, then $B_r(x) \ni x_i$ for some i for all $r > 0$ and also $B_r(x) \ni x_j$ where x_j in this subsequence and $j \geq i$. This means give $r > 0$, we can find $1/i \leq \epsilon \leq r/2$ s.t. $x \in B_\epsilon(x_i)$ for some i. Then $B_\epsilon(x_i) \subset B_r(x)$ which means $B_r(x)$ intersects V_i^c for all $i \in I$.
Notice that V_i^c is closed, so $\overline{V_i^c} = V_i^c$ and x is the accumulation point of all the V_i^c, so $x \in \bigcap_{i \in I} V_i^c = \left(\bigcup_{i \in I} V_i\right)^c = \emptyset$ which leads to a contradiction.

□

Theorem 4.4 (Tychonoff product theorem). *If $\{X_\alpha : \alpha \in A\}$ is a family of compact topological spaces and $X = \prod_{\alpha \in A} X_\alpha$ with the **product topology**, then X is compact.*

4.2 Continuous Function and Continuous Function Space

4.2.1 Continuous Function

Definition 4.5 (oscillation)**.** If $f : (X, d) \to (Y, \rho)$ is an arbitrary mapping, then the oscillation of f on a ball $B(x_0)$ is defined by:

$$\operatorname{osc}(f, B_r(x_0)) = \sup\{\rho(f(x), f(y)) : x, y \in B_r(x_0)\}$$

Notice that the oscillation is non-decreasing corresponding to r on each x_0.

Proposition 4.1. *A function $f : X \to Y$ is continuous at x_0 iff*

$$\lim_{r\to 0} osc(f, B_r(x_0)) = 0$$

Theorem 4.5. *Let $f : X \to Y$ be an arbitrary function. Then the set of points at which f is continuous is a G_δ set.*

Proof. Let

$$G_i = \left\{x \in X : \inf_{r>0} \operatorname{osc}(f, B_r(x)) < \frac{1}{i}\right\}$$

so the set that f is continuous is given by:

$$A = \bigcap_{i=1}^{\infty} G_i$$

Now we need to prove that G_i is open. Observe that $x \in G_i$ there exists $r > 0$ s.t. $\operatorname{osc}(f, B_r(x_0)) < 1/i$. Give $y \in B_r(x)$, there exists $t > 0$ s.t. $B_t(y) \subset B_r(x)$, so

$$\operatorname{osc}(f, B_y(t)) \le \operatorname{osc}(f, B_r(x)) \le 1/i$$

which means each point $y \in B_r(x)$ is an element of G_i, that is $B_r(x) \subset G_i$, as the arbitrary picking of x, G_i is thus a open set.

□

Theorem 4.6. *Let f be an arbitrary function defined on* $[0, 1]$ *and let*

$$E = \{x \in [0, 1] : \ f \textit{ is continuous at} x\}$$

Then E cannot be the set of rational numbers in $[0, 1]$.

Proof. Observe that if E is the set of rational numbers, then the set of rational numbers in $[0, 1]$ is a G_δ set which implies that the irrational numbers in $[0, 1]$ is a F_σ set.

Notice that the rational numbers are the countable union of closed set (singletons). And since the rational numbers are dense in $[0,1]$, so if the irrational number set is F_σ, then every closed set in this family cannot have any interiors which means the whole $[0,1]$ is a F_σ set with a family of nowhere dense set, which is contrary with the Baire category theorem.

□

Theorem 4.7. *A continuous functions carries a compact subset into a compact subset.*

Proof. Let X, Y be two topological space and $f : X \to Y$ is continuous, now we prove that if $K \subset X$ is compact, then $f(K) \subset Y$ is compact too.
Notice that $f\mid_K$ is surjective, so $f(f^{-1}(U)) = U$. Then consider a open cover $\mathcal{F}$ of $f(K)$, then the set $\mathcal{E} = \left\{f^{-1}(U) : U \in \mathcal{F}\right\}$ is a open cover of K, then there exists a finite open subcover $\{V_1, \dots, V_n : V_i \in \mathcal{E}\}$ s.t. $\bigcup_{i=1}^n V_i \supset K$ where $V_i, i = 1, \dots, n$ is $f^{-1}(U_i)$ for some $U_i \in \mathcal{F}$, so there exists some i s.t. $\bigcup_{i=1}^n f^{-1}(U_i) \supset K$, then

$$f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) = \bigcup_{i=1}^n f\left(f^{-1}(U_i)\right) = \bigcup_{i=1}^n U_i$$

Notice that

$$f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) \supset f(K)$$

so $f(K) \subset \bigcup_{i=1}^n U_i$.

□

Definition 4.6 (uniformly continuous)**.** A function $f : (X, d) \to (Y, \rho)$ is said to be uniformly continuous on X if for each $\epsilon > 0$, there exists $\delta > 0$ s.t. when $d(x, y) \le \delta$, $\rho(f(x), f(y)) \le \epsilon$ for all $x, y \in X$.

An equivalent formulation of uniform continuity can be stated in oscillation. For each $r > 0$, let

$$\omega_f(r) = \sup_{x \in X} \operatorname{osc}\left(f, B_r(x)\right)$$

The function ω_f is called the modulus of continuity of f. Observe that f is uniformly continuous if

$$\lim_{r \to 0} \omega_f(r) = 0$$

Proof. Give a $\epsilon > 0$, there exists a $\delta > 0$, when $r \le \delta$, $\omega_f(r) \le \epsilon$. Then

$$\sup_{x \in X} \operatorname{osc}(f, B_r(x)) \le \epsilon$$

so when $d(x, y) \le r \le \delta$, $\sup_{x \in X} \rho(f(x), f(y)) \le \epsilon$ which means uniform continuity.

□

Theorem 4.8. *Let $f : X \to Y$ be a continuous mapping. If X is compact, then f is uniformly continuous on X.*

Proof. From 4.6, we notice that if $\lim_{r\to 0} \omega_f(r) = 0$, then f is uniformly continuous.
Choose $\epsilon > 0$, the collection

$$\mathcal{F} = \left\{ f^{-1}(B_{\epsilon/2}(y))) : y \in Y \right\}$$

is a open cover of X, then there exists a Lebesgue number $\delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_{\epsilon/2}(y))$ for all $x \in X$ follows from 4.2.
So $f(B_\delta(x)) \subset B_{\epsilon/2}(y)$ for some $y \in Y$ which means $\omega_f(\delta) \le \epsilon$ for arbitrary ϵ, so f is uniformly continuous.

□

4.2.2 Continuous Function Space

Theorem 4.9. *Let K be a compact topological space and let $(Y, \|\cdot\|_Y)$ be a normed vector space. Then $\mathcal{C}(K;Y)$ is a vector space with the norm $\|\cdot\| : \mathcal{C}(K;Y) \to \mathbb{R}$:*

$$\|f\|_{\mathcal{C}} = \sup_{x\in K} \|f(x)\|_Y$$

*for each $f \in \mathcal{C}(K;Y)$. It is called the **sup-norm** on $\mathcal{C}(K;Y)$.*

Proof. Notice that $(Y, \|\cdot\|)$ is a metric space and a compact subset in a metric space is bounded and closed.
- $\sup \|f(x)\|_Y < \infty$ and $\sup \|f(x)\|_Y \ge 0$ - $\sup \|\alpha f(x)\|_Y = |\alpha| \sup \|f(x)\|_Y$ - $\sup \|f+g\|_Y \le \sup \|f\|_Y + \sup \|g\|_Y$

□

Definition 4.7 (converge uniformly)**.** A sequence $(f_n)_{n=1}^\infty$ of functions $f_n \in \mathcal{C}(K;Y)$ is said to **converge uniformly** if $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{C}} = 0$. It means

$$\lim_{n\to\infty} \left(\sup_{x\in K} \|f_n(x) - f(x)\|_Y \right) = 0$$

Theorem 4.10. *Let X be any set and let $(Y, \|\cdot\|_Y)$ be a normed vector space. Then the set $\mathcal{B}(X;Y)$ of all bounded mappings $f : X \to Y$ i.e. $f(X) \subset Y$ is a bounded subset in Y is a vector space and the function $\|\cdot\|_{\mathcal{B}} : \mathcal{B}(X;Y) \to \mathbb{R}$ defined by:*

$$\|f\|_{\mathcal{B}} = \sup_{x\in X} \|f(x)\|_Y$$

is a norm on $\mathcal{B}(X;Y)$.

Proof. Notice that a bounded function f over $\mathbb{K}$, for any $\alpha \in \mathbb{K}$, αf is still bounded and for $f, g \in \mathcal{B}(X;Y)$, $f+g$ is still bounded.

It is easy to show that $\|f\|_{\mathcal{B}}$ is truly a norm on $\mathcal{B}(X;Y)$.

□

Definition 4.8 (local uniform convergence)**.** Let X be a topological space and Y be a normed vector space. Then a sequence $(f_n)_{n=1}^{\infty}$ of mappings $f_n : X \to Y$ is said to converge locally uniformly to a mapping $f : X \to Y$ as $n \to \infty$ if given any $x_0 \in X$ there exists a neighborhood $V(x_0)$ of x_0 s.t.

$$\lim_{n\to\infty} \left(\sup_{x \in V(x_0)} \|f_n(x) - f(x)\|_Y \right) = 0$$

Notice that local uniform convergence and uniform convergence implies the pointwise convergence.

Theorem 4.11. *Let X is a topological space and Y be a normed vector space, let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous mapping from X to Y that converges locally uniformly to a $f : X \to Y$, then f is continuous on X.*

Moreover, if f_n continuous at x_0 and they locally uniformly convergence to f then f is continuous at x_0.

Proof. Assume that f is continuous at x_0 which means give $\epsilon > 0$, there exists a neighborhood $V(x_0) \in \mathcal{N}_{x_0}$ s.t. for every $x \in V(x_0)$, $\|f(x_0) - f(x)\|_Y \leq \epsilon$.

Now suppose that $\epsilon > 0$ is given. As $(f_n) \to f$ locally uniformly. Then we can choose a $k \in \mathbb{N}$ s.t. for any $i \geq k$, we can find a neighborhood $V(x_0)$ s.t. for any $x \in V(x_0)$,

$$\sup_{x \in V(x_0)} \|f_i(x) - f(x)\|_Y \leq \epsilon/3$$

and as all $f_n : n \in \mathbb{N}$ is continuous at x_0, so we can find a neighborhood of x_0, $U(x_0) \in \mathcal{N}_{x_0}$ s.t. for any $x \in U(x_0)$,

$$\sup_{x \in U(x_0)} \|f_i(x) - f_i(x_0)\|_Y \leq \epsilon/3$$

Then we consider the set $W(x_0) = U(x_0) \cap V(x_0) \in \mathcal{N}_{x_0}$, for any $x \in W(x_0)$:

$$\begin{aligned} \|f(x) - f(x_0)\|_Y &\leq \|f(x) - f_i(x)\|_Y + \|f_i(x) - f_i(x_0)\|_Y + \|f_i(x_0) - f(x_0)\|_Y \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

so if $(f_n) \to f$ locally uniformly, and f_n is continuous at x_0 for every n then f is continuous at x_0. Moreover, if f_n is continuous at every $x \in X$ i.e. continuous at X, then f is continuous at X.

□

Definition 4.9 (equicontinuous)**.** A family $\mathcal{F}$ of functions

4.3 Normed Vector Space

4.3.1 Properties of vector space

Definition 4.10.

- A set X is a **vector space** over $\mathbb{K}$ if there exists two mappings:

$$(x, y) \in X \times X \to x + y \in X$$
$$(\alpha, x) \in \mathbb{K} \times X \to \alpha x \in X$$

  there exists an element of X denoted as 0 s.t. $x + 0 = x$ for all $x \in X$, define $(-x)$ is a vector s.t. $x + (-x) = 0$.
- A **subspace** of a vector space X over $\mathbb{K}$ is any subset of X which is also a vector space over $\mathbb{K}$.
- Let Y and Z be two subspace of X then X is said to be the **direct sum** of Y, Z if any vector $x \in X$ can be written as

$$x = y + z \qquad y \in Y, z \in Z$$

  and such a decomposition is unique.
- A subspace B is called **subspace spanned by a subset** A of X consisting of all finite linear combinations of vectors of A, i.e., $x \in B$ of the form $x = \sum_{i \in I} \alpha_i a_i$ where the set I is finite and $\alpha_i \in \mathbb{K}$, $a_i \in A$, we said that

$$B = \text{span} A$$

- The **Hamel basis** in X is any family $\{e_i\}_{i \in I}$ of vectors $e_i \in X$ satisfying:
  - First, the family is linearly independent. It means that give any finite subfamily of $\{e_j\}_{j \in J}$ and any scalars $\alpha_j \in \mathbb{K}, j \in J$ s.t. $\sum_{j \in J} \alpha_j e_j = 0$ then $\alpha_j = 0, j \in J$.
  - Second, $\text{span}\{e_i\}_{i \in I} = X$.

Theorem 4.12. *Let $X \neq \{0\}$ be a vector space.*
- There exists a Hamel base of X - Let E, F be two Hamel bases of X. Then $cardE = cardF$.

Definition 4.11. A vector space X is finite-dimensional if there exists a finite Hamel basis of X, and its **dimension** denoted as $\text{dim} X$.
If E is a Hamel base of X, then $\text{dim} X = \text{card} E$

Definition 4.12 (norm)**.** Let X be a vector space over $\mathbb{K}$. A norm on X is a mapping $\| \cdot \| : X \to \mathbb{R}$ with: - $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$ - $\|\alpha x\| = \|\alpha\| \|x\|$ for all $\alpha \in \mathbb{K}, x \in X$ - $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

Definition 4.13 (distance in normed vector space)**.** Let $(X, \| \cdot \|)$ be a normed vector space, then the mapping $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ for all $x, y \in X$ is a **distance** on X.

Proof. First we need to show that $\big|\|x\| - \|y\|\big| \leq \|x - y\|$.
Assume that $\|x\| \geq \|y\|$, then consider $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$, so $\|x\| - \|y\| \leq \|x - y\|$, as they all non-negative, $\big|\|x\| - \|y\|\big| \leq \|x - y\|$ holds.
- $d(x, y) = \|x - y\| \geq 0$ - $d(x, y) = 0 \implies \|x - y\| = 0 \implies x = y$ - $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$ - $d(x, y) \leq d(x, z) + d(y, z)$, notice that $\|x - z\| + \|z - y\| \geq \|(x - z) + (z - y)\| = \|x + y\|$, so for any $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(y, z)$

So we find that $d(x, y) = \|x - y\|$ is truly a metric on X, so (X, d) is a metric topological space. It is also called the **norm topology** of X.

□

Theorem 4.13. *Let X be a finite-dimensional vector space over $\mathbb{K}$, and let $(e_i)_{i=1}^n$ denote a basis of X :*

- *For each $p \in [1, \infty]$, the mapping $\|\cdot\|_p$ defined by:*

$$x = \sum_{i=1}^{n} x_i e_i \in X \to \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad \text{if } p \in [1, \infty)$$
$$x = \sum_{i=1}^{n} x_i e_i \in X \to \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \qquad \text{if } p = \infty$$

  is a norm on X.
- *For each $p \in [1, \infty]$, the space $(X, \|\cdot\|_p)$ is separable.*

Theorem 4.14 (Holder's and Minkowski's inequalities).
- *Given a $p \in \mathbb{R}$ s.t. $p > 1$, let the real number q be defined by:*

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad \text{hense } q > 1$$

  and let $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ be two sequences of scalers satisfying

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \text{ and } \sum_{i=1}^{\infty} |y_i|^q < \infty$$

  Then the series $\sum_{i=1}^\infty |x_i y_i|$ converges and Holder's inequality holds:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$$

- *Give a real number $p \geq 1$ s.t.*

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \text{ and } \sum_{i=1}^{\infty} |y_i|^p < \infty$$

Then $\sum_{i=1}^{\infty}|x_i+y_i|^p$ *converges and Minkowski's inequality holds:*

$$\left(\sum_{i=1}^{\infty}|x_i+y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty}|y_i|^p\right)^{1/p}$$

Proof. 1. If $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\alpha\beta \leq \frac{\alpha^p}{p}+\frac{\beta^q}{q} \qquad \text{for all } \alpha>0,\ \beta>0$$

To see this, note that the convexity of exponential function implies that

$$e^{\theta r+(1-\theta)s} \leq \theta e^r + (1-\theta)e^s$$

for all $\theta \in (0,1)$ and $r, s \in \mathbb{R}$. Now let $\theta = \frac{1}{p}, r = p\text{Log}\alpha, s = q\text{Log}\beta$, the first inequality is proved.

2. Let $\|x\|_p = (\sum_{i=1}^{\infty}|x_i|^p)^{1/p}$ and $\|y\|_p = (\sum_{i=1}^{\infty}|y_i|^p)^{1/p}$. Let $\alpha = \frac{|x_i|}{\|x\|_p}$ and $\beta = \frac{|y_i|}{\|y\|_p}$. Then as shown above:

$$\frac{|x_iy_i|}{\|x\|_p\|y\|_q} \leq \frac{|x_i|^p}{p(\|x\|_p)^p}+\frac{|y_i|^q}{q(\|y\|_q)^q}$$

for each $i \in \mathbb{N}, i \geq 1$. Then take sum of above inequality:

$$\sum_{i=1}^{n}\left(\frac{|x_iy_i|}{\|x\|_p\|y\|_q}\right) \leq \sum_{i=1}^{n}\left(\frac{|x_i|^p}{p(\|x\|_p)^p}+\frac{|y_i|^q}{q(\|y\|_q)^q}\right)$$

Notice that the right side of above:

$$\|x\|_p = \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{1/p} \implies \|x\|_p^p = \sum_{i=1}^{\infty}|x_i|^p$$

similar of $\|y\|_q$, so

$$\frac{\sum_{i=1}^{n}|x_i|^p}{p(\|x\|_p)^p} = \frac{\sum_{i=1}^{n}|x_i|^p}{p(\sum_{i=1}^{\infty}|x_i|^p)} \leq \frac{1}{p}$$

and the same as $\|y\|_q$, so the right side is less than $\frac{1}{p}+\frac{1}{q}=1$, so

$$\sum_{i=1}^{n}|x_iy_i| \leq \|x\|_p\|y\|_q$$

holds for every $n \in \mathbb{N}$ and take the limit $n \to \infty$, the holder's inequality holds.

3. Notice that $\frac{1}{p}+\frac{1}{q}=1 \implies p+q=pq \implies p-1=\frac{p}{q}$.

$$\begin{aligned}\sum_{i=1}^{n}(|x_i|+|y_i|)^p &= \sum_{i=1}^{n}|x_i|(|x_i|+|y_i|)^{p-1}+\sum_{i=1}^{n}|y_i|(|x_i|+|y_i|)^{p-1}\\ &\le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/q}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/q}\\ &= \left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/q}\left(\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}\right)\end{aligned}$$

Notice that

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/p}$$

so the Minkowski's inequality holds.

□

Proof. Now we prove that $\|x\|_p$ satisfies the triangle inequality.

$$\|x+y\|_p \le \|x\|_p+\|y\|_p$$

As shown above, when we prove Minkowski's inequality, before letting $n\to\infty$, we find that

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

which means $\|x+y\|_p \le \|x\|_p+\|y\|_p$ holds.

Then we prove that $\|x\|_\infty$ is a norm.
- $\|x\|_\infty = \max_{1\le i\le n}|x_i| \ge 0$ - $\|\alpha x\|_\infty = \max_{1\le i\le n}|\alpha x_i| = |\alpha|\|x\|_\infty$ -

$$\begin{aligned}\|x+y\|_\infty &= \max_{1\le i\le n}|x_i+y_i| \le \max_{1\le i\le n}|x_i| + \max_{1\le i\le n}|y_i|\\ &= \|x\|_\infty+\|y\|_\infty\end{aligned}$$

□

Notice that when $p=2$, $\|x\|_2$ is the Euclidean distance between point $x\in\mathbb{R}^n$ and 0, and the distance generated by $\|x\|_2$, $d(x,y)=\|x-y\|_2$ is the Euclidean distance between x and y.

4.3.2 ℓ^p space and L^p space

Definition 4.14 (ℓ^p space)**.** ℓ^p space is a normed vector space of all the infinite sequences $x = (x_i)_{i=1}^{\infty}$ of scalars $x_i \in \mathbb{K}$ that satisfy:

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \qquad \text{if } p \in [1, \infty)$$
$$\sup_{i \geq 1} |x_i| < \infty \qquad \text{if } p = \infty$$

For each $p \in [1, \infty]$, the set ℓ^p is a vector space with the norm $\|\cdot\|_p$:

$$x = (x_i) \in \ell^p \to \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \qquad \text{if } p \in [1, \infty)$$
$$x = (x_i) \in \ell^\infty \to \|x\|_\infty = \sup_{i \geq 1} |x_i| \qquad \text{if } p = \infty$$

is a norm on ℓ^p space.

Proof. Notice that from Minkowski's inequality, when $p \in [1, \infty)$ and $\sum_{i=1}^{\infty} |x_i|^p < \infty, \sum_{i=1}^{\infty} |y_i|^p < \infty$, $\sum_{i=1}^{\infty} |x_i + y_i|^p$ converges, and for a finite $\alpha \in \mathbb{K}$, $\sum_{i=1}^{\infty} \alpha |x_i|^p = \alpha \sum_{i=1}^{\infty} |x_i|^p$ also converges.

And with Minkowski's inequality, we can also easily to determine that $\|\cdot\|_p$ is a norm.

□

Theorem 4.15.
- *The normed vector space ℓ^p space is separable if $p \in [1, \infty)$*
- *The normed vector space ℓ^p space is not separable if $p = \infty$*

Proof. Let $\mathbb{K} = \mathbb{R}$, and $p \in [1, \infty)$, let

$$A = \bigcup_{n=1}^{\infty} \{(y_i) \in \ell^p; y_i \in \mathbb{Q} \text{ for } i \leq n, y_i = 0 \text{ for } i \geq n+1\}$$

Then we show $\overline{A} = \ell^p$, notice that ℓ^p is a metric space and we only need to show that for any $x \in \ell^p$ and any $\epsilon > 0$, there exists some $y \in A$ s.t. $\|x - y\|_p \leq \epsilon$.

Give any $x = (x_i) \in \ell^p$, there exists a $k \in \mathbb{N}$ s.t. $\sum_{i=k}^{\infty} |x_i|^p \leq \epsilon^p/2$, and there exists some $y \in A$ which means $y_i \in \mathbb{Q}$ for each i s.t. $\sum_{i=1}^{k-1} |x_i - y_i|^p \leq \epsilon^p/2$, then for these $x, y \in \ell^p$, we find that $\|x - y\|_p \leq \epsilon$.

Now give a proof of ℓ^∞ space is not separable.
Give a set

$$B = \{(x_i) \in \ell^\infty; x_i = 0 \text{ or } x_i = 1, i \geq 1\}$$

is an **uncountable set** since there is a one-to-one and onto mapping:

$$(x_i) \in B \to \sum_{i=1}^{\infty} \frac{1}{2^i} x_i$$

It is one-to-one obviously and onto $[0,1]$ by the binary representation of a real number.
Now suppose there is a $C \subset \ell^\infty$ s.t. $\overline{C} = \ell^\infty$. Then give any $x \in B$, there exists a $y(x) \in C$ s.t. $\|y(x) - x\|_\infty < 1/2$ then the mapping $x \in B \to y(x) \in C$ is a injection since if $x_1, x_2 \in B$ with $x_1 \neq x_2$, then $\|x_1 - x_2\|_\infty = 1$, now let $y(x_1) = y(x_2) = y$, we find that $\|x_1 - x_2\|_\infty \leq \|x_1 - y\|_\infty + \|y - x_2\|_\infty$, then we get the contradiction. So if $x_1 \neq x_2$, $y(x_1) \neq y(x_2)$, so this mapping must be one-to-one. It means $\operatorname{card} C \geq \operatorname{card} B$ so C is uncountable.

□

Definition 4.15 ($L^p(\Omega)$)**.** Let Ω is a open subset in $\mathbb{R}^n$ thus measurable. Remember that the $L^1(A)$ consists of all equivalence classes of real Lebesgue-measurable functions, i.e. those measurable functions $f : \Omega \to [-\infty, \infty]$ that satisfy:

$$\int_\Omega |f(x)| dx < \infty$$

Notice that a function $f : \Omega \to \overline{\mathbb{R}}$ is integrable iff $\int_\Omega |f(x)| dx < \infty$.

Now extend this definition. Let $p \in [1, \infty)$, we let $L^p(\Omega)$ denote the set formed by all equivalence classes of measurable functions $f : \Omega \to [-\infty, \infty]$ s.t. $f' = |f|^p \in L^1(\Omega)$ which means:

$$\int_\Omega |f(x)|^p dx < \infty \qquad \text{for some } \; p \in [1, \infty)$$

Theorem 4.16 (Holder and Minkowski's inequality for functions)**.**

- *Holder:*

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\int_\Omega |f(x)|^p dx < \infty \;\; \textit{and} \;\; \int_\Omega |g(x)|^q dx < \infty$$

$$\int_\Omega |f(x)g(x)| dx \leq \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \left(\int_\Omega |g(x)|^p dx \right)^{1/q}$$

- *Minkowski:*

$$\int_\Omega |f(x)|^p dx < \infty \;\; \textit{and} \;\; \int_\Omega |g(x)|^p dx < \infty$$

$$\left(\int_\Omega |f(x) + g(x)|^p dx \right)^{1/p} \leq \left(\int_\Omega |f(x)|^p dx \right)^{1/p} + \left(\int_\Omega |g(x)|^p dx \right)^{1/p}$$

Proof. Replace the sum to integral from the sequence Holder and Minkowski's inequality.

□

As we defined the space $L^p(\Omega)$ above, it is easy to verify that $L^p(\Omega)$ is a vector space and $\|\cdot\|_p : f \to \mathbb{R}$ defined by:

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \qquad p \in [1, \infty)$$

Now we define the space $L^\infty(\Omega)$.

Definition 4.16 ($L^\infty(\Omega)$ space)**.**
- $L^\infty(\Omega)$ space denote the set of all measurable functions $f : \Omega \to [\infty, \infty]$ that satisfy:

$$\inf\{C \geq 0; |f| \leq C \text{ a.e. in } \Omega\} < \infty$$

- The norm $\|\cdot\|_\infty$ on $L^\infty(\Omega)$ is defined:

$$\|f\|_\infty = \inf\{C \geq 0; |f| \leq C \text{ a.e. in } \Omega\}$$

Definition 4.17 (essential supremum)**.** Give a measurable function $f : \Omega \to [-\infty, \infty]$, the extended real number

$$\inf\{C \geq 0; |f| \leq C \text{ a.e. in } \Omega\} \in [0, \infty]$$

is called the **essential supremum** of f.

Notice that $L^\infty(\Omega)$ space consists of all equivalence class of measurable functions whose essential supremum is finite.

Theorem 4.17. *Let Ω is a open subset of $\mathbb{R}^n$, define the space*

$$\mathcal{C}_c(\Omega) = \{g \in \mathcal{C}(\Omega); \text{ supp } g \text{ is compact in } \Omega\}$$

Then, for each $p \in [1, \infty)$, the subspace $\mathcal{C}_c(\Omega)$ is dense in $L^p(\Omega)$.

Proof. To prove that $\mathcal{C}_c(\Omega)$ is a dense set, we need to show that for every $f \in L^p(\Omega)$, give any $\epsilon > 0$, we have some $g \in \mathcal{C}_c(\Omega)$ s.t. $\|f - g\|_p \leq \epsilon$.

1. There exists a measurable simple function $s = s(f, \epsilon)$ s.t.

$$\mu(\{x \in \Omega; s(x) \neq 0\}) < \infty \text{ and } \|f - s\|_p \leq \epsilon/2$$

to achieve this, assume that $f \geq 0$ then there exists a sequence of simple function with:

$$0 \leq s_k \leq f \text{ for all } k \geq 1 \text{ and } (s_k) \nearrow f \text{ pointwise}$$

Notice that $f \in L^p(\Omega)$, which means $\int_\Omega |f(x)|^p dx < \infty$. As $s_k \le f$ holds for every $k \in \mathbb{N}$, $s_k \in L^p(\Omega)$. So $\mu(\{x \in \Omega; s_k(x) \neq 0\}) < \infty$ as the definition of the integral over a simple function.
As $(s_k) \nearrow f$, notice that $|(f - s_k)|^p \le |f|^p$ and $|f - s_k|^p \to 0$ when $k \to \infty$, using Lebesgue's dominated convergence theorem:

$$\int_\Omega \lim_{k\to\infty} |f - s_k|^p = \lim_{k\to\infty} \int_\Omega |f - s_k|^p = 0$$

so we can find some k s.t. $\int_\Omega |f - s_i|^p \le (\epsilon/2)^p$ for all $i \ge k$, so there exists some $s = s(f, \epsilon)$ s.t. $\|f - s\|_p \le \epsilon/2$.

2. Let $s = s(f, \epsilon)$ be the measurable simple function constructed in step 1. Then there exists a function $g = g(s, \epsilon) = g(f, \epsilon) \in \mathcal{C}_c(\Omega)$ s.t.

$$\|s - g\|_p \le \epsilon/2$$

Since $\mu(\{x \in \Omega; s(x) \neq 0\}) < \infty$, Lusin's property implies that there exists a function $g \in \mathcal{C}_c(\Omega)$ that satisfies

$$\sup_{x\in\Omega} |g(x)| \le \|s\|_\infty$$

$$\mu(\{x \in \Omega; g(x) \neq s(x)\}) \le \left(\frac{\epsilon}{4\|s\|_\infty}\right)^p$$

Then

$$\|s - g\|_p = \left(\int_A |s - g|^p\right)^{1/p}$$

Notice that $|s - g| \le 2\|s\|_\infty$ as $\sup|g(x)| \le \|s\|_\infty$, and A denotes the set $\{x \in \Omega; g(x) \neq s(x)\}$, so the integral above is less than $2\|s\|_\infty \cdot \mu A \le \epsilon/2$.

As shown above, give $\epsilon > 0$ and $f \in L^p(\Omega)$ there is a $g(f, \epsilon)$ s.t. $\|f - g\|_p \le \|f - s_k\|_p + \|s_k - g\|_p \le \epsilon/2 + \epsilon/2 = \epsilon$.

□

Theorem 4.18. *1. $L^p(\Omega)$ is separable if $p \in [1, \infty)$*

2. $L^\infty(\Omega)$ is not separable.

Proof. 1. Let a $f \in L^p(\Omega)$ where $p \in [1, \infty)$ then there exists a $g = g(f, \epsilon) \in \mathcal{C}_c(\Omega)$ s.t.

$$\|f - g\|_p \le \epsilon/2$$

Since $K = \operatorname{supp} g$ is compact, there exists a bounded open set U s.t. $K \subset U \subset \Omega$. As U is bounded, $\overline{U}$ is bounded too, so g is uniformly continuous on $\overline{U}$, then there exists $\delta_0 > 0$ s.t.

$$|g(x) - g(y)| \le \frac{\epsilon}{2(\mu(U))^{1/p}} = \epsilon'$$

for all $x, y \in \overline{U}$ s.t. $\|y-x\|_\infty < \delta_0$
As the compactness of K and the property of distance function, there exists $\delta_1 > 0$ s.t.

$$\{y \in \mathbb{R}^n; \|y-x\|_\infty < \delta_1\} \subset U \text{ for all } x \in K$$

Let $\delta \in \mathbb{Q}$ s.t. $0 < \delta \leq \min\{\delta_0, \delta_1\}$.
Let $(B_i)_{i\in I}$ denote the countable family formed by all open balls:

$$\left\{y \in \mathbb{R}^n; \|x-y\|_\infty < \frac{\delta}{2} \text{ with } x_j = p_j\delta \text{ for some } p_j \in \mathbb{Z}, j \in [1,n]\right\}$$

Now pick the subfamily $(B_i)_{i\in I(K)}$ s.t. for any $i \in I(K)$, $B_i \cap K \neq \emptyset$. Then for each $i \in I(K)$, notice that $\delta/2$ makes sure that $\operatorname{diam}(B_i \cap K) \leq \delta \leq \delta_0$, so if $x \in K$, then $B_i \subset U$ and $|g(y_1) - g(y_2)| \leq \epsilon'$ for every $y_1, y_2 \in B_i$ since the property of uniform continuous. If $x \notin K$, then as its minimum is 0, we can also pick some α_i as blow:
we pick some $\alpha_i \in \mathbb{Q}$ s.t.

$$|g(y) - \alpha_i| \leq \epsilon' \text{ for all } y \in B_i$$

Now we can construct a simple function:

$$h = \sum_{i\in I(K)} \alpha_i \mathbf{1}_{B_i}$$

which satisfies that $|h(x) - g(x)| \leq \epsilon'$ for almost all $x \in U$ s.t.

$$\|h-g\|_p = \left(\int_U |h-g|^p\right)^{1/p} \leq (\mu(U))^{1/p} \cdot \frac{\epsilon}{2(\mu(U))^{1/p}} = \frac{\epsilon}{2}$$

Notice that $\|f-g\|_p + \|g-h\|_p \geq \|f-h\|_p$, so $\|f-h\|_p \leq \epsilon$ and as h is simple and $\alpha_i \in \mathbb{Q}$, so h is countable as $I(K)$ is always a finite subset of a countable set and $\mathbb{Q}$ is a countable set. So $L^p(\Omega)$ is separable.

□

4.3.3 More about $L^p(\Omega)$

Definition 4.18 (Locally integrable)**.** Let Ω be a open subset s.t. $\Omega \subset \mathbb{R}^n$. A function $f : \Omega \to [-\infty, \infty]$ is said to be locally integrable if f is measurable and the restriction $f|_K$ of f on any compact subset $K \subset \Omega$ belongs to $\mathcal{L}^1(K)$.

As the usual method, we construct a quotient set

$$L^p_{loc}(\Omega) = \mathcal{L}^p_{loc}(\Omega)/\mathcal{R}$$

where $\mathcal{R}$ is the a.e. equivalence.

Note that in this chapter we change the notations of norm. For example, a $L^p(\Omega)$ norm of f is noted as

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \right)^{1/p}$$

Usually we use the Lebesgue measure μ and if so we don't show it in the integral.

Notice that every $f \in L^p(\Omega), p \in [1, \infty]$ is locally integrable in Ω since for any compact $K \subset \Omega$:

$$\int_K |f(x)| \leq \|f\|_{L^1(\Omega)} < \infty$$

and as Holder's inequality:

$$\begin{aligned}\int_K |f| &\leq \left(\int_K \right)^{1/q} \left(\int_K |f|^p \right)^{1/p} \\ &\leq \left(\int_K \right)^{1/q} \|f\|_{L^p(\Omega)} \\ &< \infty\end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 4.19 (family of mollifiers)**.** A **family of mollifies** in $\mathbb{R}^n$ is a family $(\omega_\epsilon)_{\epsilon>0}$ of functions $\omega_\epsilon : \mathbb{R}^n \to \mathbb{R}$ of the form

$$\omega_\epsilon(x) = \frac{1}{\epsilon^n} \omega\left(\frac{x}{\epsilon} \right), \qquad x \in \mathbb{R}^n$$

where $\omega : \mathbb{R}^n \to \mathbb{R}$ is any functions with following properties:

$$\begin{gathered}\omega \in \mathcal{C}^\infty(\mathbb{R}^n), \ \omega(x) \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n \\ \operatorname{supp} \omega \subset \overline{B_1(0)} \ \text{ and } \int_{\mathbb{R}^n} \omega = 1\end{gathered}$$

Hence for each $\epsilon > 0$,

$$\begin{gathered}\omega_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n), \ \omega_\epsilon \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n \\ \operatorname{supp} \omega_\epsilon \subset \overline{B_\epsilon(0)} \ \text{ and } \int_{\mathbb{R}^n} \omega_\epsilon = 1\end{gathered}$$

Definition 4.20 (regularizing family of f)**.** Let Ω be a open subset of $\mathbb{R}^n$. Give a function $f \in L^1_{loc}(\Omega)$ and a family $(\omega_\epsilon)_{\epsilon>0}$ of mollifiers, define the set Ω_ϵ and $f_\epsilon : \Omega_\epsilon \to \mathbb{R}$:

$$\begin{aligned}\Omega_\epsilon &= \{x \in \Omega; \operatorname{dist}(x, \mathbb{R}^n - \Omega) > \epsilon\} \\ f_\epsilon(x) &= \int_\Omega \omega_\epsilon(x - y) f(y) \, dy \qquad \text{for all } \ x \in \Omega_\epsilon\end{aligned}$$

Then the family $(f_\epsilon)_{\epsilon>0}$ is called a regularizing family of f.

Notice that $\operatorname{dist}(x, \mathbb{R}^n - \Omega)$ is a continuous function thus Ω_ϵ is a open set and for every $x \in \Omega_\epsilon$, the ball $\overline{B_\epsilon(x)} \subset \Omega$ which means $f_\epsilon(x)$ is well-defined on Ω. Then

$$\begin{aligned} f_\epsilon(x) &= \int_{B_\epsilon(x)} \omega_\epsilon(x-y) f(y) dy = \int_{B_0(\epsilon)} \omega(z) f(x-z) dz \\ &= \frac{1}{\epsilon^n} \int_{B_1(x)} \omega\left(\frac{x-y}{\epsilon}\right) f(y) dy \end{aligned}$$

Theorem 4.19. *1. Let Ω be an open subset of $\mathbb{R}^n$, and let a function $f \in L^1_{loc}(\Omega)$ and a regularizing family $(f_\epsilon)_{\epsilon>0}$ of f is given. Then:*

$$f_\epsilon \in \mathcal{C}^\infty(\Omega_\epsilon) \text{ for all } \epsilon > 0$$

Moreover,

$$\begin{aligned} \partial^\alpha f_\epsilon(x) &= \int_\Omega \partial^\alpha \omega_\epsilon(x-y) f(y) dy \\ &= \int_{B_\epsilon(x)} \partial^\alpha \omega_\epsilon(x-y) f(y) dy \end{aligned}$$

at each $x \in \Omega_\epsilon$. For any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha| = \sum_{i=1}^n \alpha_i \geq 1$.

2. Assume in addition that $f \in \mathcal{C}^m(\Omega)$ for some integer $m \geq 1$. Then, given any compact subset $K \subset \Omega$ there exists $\epsilon_0 = \epsilon_0(K) > 0$ s.t. $K \subset \Omega_\epsilon$ for all $0 < \epsilon \leq \epsilon_0$, $f_\epsilon(x)$ is well-defined for all $x \in K$ and

$$\sup_{x \in K} |\partial^\alpha f_\epsilon(x) - \partial^\alpha f(x)| \to 0 \text{ for all } |\alpha| \leq m$$

as $\epsilon \to 0$.

Theorem 4.20. *Give an open subset $\Omega \subset \mathbb{R}^n$. For each $p \in [1, \infty)$, the space $\mathcal{C}^\infty_c(\Omega) = \mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

Theorem 4.21 (regularization and approximation in $L^p(\mathbb{R}^n)$)**.** *Let a function $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$ be given, and let $(f_\epsilon)_{\epsilon>0}$ be a regularizing family of f, then*

$$f_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ for all } \epsilon > 0$$

and

$$\|f_\epsilon - f\|_{L^p(\mathbb{R}^n)} \to 0 \text{ as } \epsilon \to 0$$

4.4 Riesz Theorem

To prove the first result, we need to prove a lemma first.

Lemma 4.3. *Two norms on a vector space X are equivalent iff there exists constants C_a, C_b s.t.*

$$\|x\|_a \leq C_b\|x\|_b \text{ and } \|x\|_b \leq C_a\|x\|_a$$

for all $x \in X$.

Proof.
- $\implies$: Assume that $\|\|_a, \|\|_b$ are equivalent. Notice that the identity mapping $\mathrm{id} : (X, \|\|_a) \to (X, \|\|_b)$ is continuous.
  Consider the set $B' = \{y \in X : \|y\|_b < 1\}$ then $\mathrm{id}^{-1}(B') \subset X$ is open. Then exists a $C_a > 0$ s.t. $\{y \in X : \|y\|_a < \frac{1}{C}\}$ is contained in $\mathrm{id}^{-1}(B')$ as $\mathrm{id}(0) = 0 \in B'$. Therefore, we find a C_a s.t.

  $$\|y\|_a \leq \frac{1}{C_a} \implies \|y\|_b \leq 1$$

  So give any nonzero vector $x \in X$, the vector $y = \frac{1}{C_a\|x\|_a}x$ satisfies $\|y\|_a = \frac{1}{C_a}$, so $\|y\|_b = \frac{1}{C_a\|x\|_a}\|x\|_b \leq 1$ which means

  $$\|x\|_b \leq C_a\|x\|_a$$

  The other side is similar.

- $\Longleftarrow$: Assume that $\|x\|_b \leq C_a\|x\|_a$ holds for every $x \in X$. This implies that the closure of any ball centered at any $y \in X$, $B_r(y)$ in the topological space $(X, \|\|_b)$ contains a ball $B_{r/C_a}(y)$ in $(X, \|\|_a)$ hence every open set in $(X, \|\|_b)$ is the open set in $(X, \|\|_a)$, and similar on the other side. Thus we assert that two topology are equivalent.

□

Theorem 4.22.
1. *Any two norms $\|\|$ and $\|\|'$ in a finite-dimensional vector space are equivalent, i.e. the topology they induced are identical.*
2. *Any finite-dimensional vector space is separable.*
3. *A subset of a finite-dim normed vector space is compact iff it is closed and bounded.*
4. *A finite-dim subspace of a normed vector space X is closed in X.*

Proof.
1. As the lemma we proved above, we need to find C and C' s.t. the condition of lemma holds, then we can assert the equivalence.
   Now let $(e_i)_{i=1}^n$ be a basis of X, define $\|\|_1 : x = \sum_{i=1}^n x_i e_i \to \sum_{i=1}^n |x_i|$. Then for any norm $\|\|$,

   $$\|x\| = \left\|\sum_{i=1}^n x_i e_i\right\| \leq C_1\|x\|_1$$

where $C_1 = \max_{1\le i\le n} \|e_i\|$
Then consider the function:

$$f : x \in (X, \|\|_1) \to f(x) = \|x\| \in \mathbb{R}$$

and the set $K = \{y \in X : \|y\|_1 = 1\}$. Notice that f is continuous on X since:

$$|f(x) - f(y)| = |\|x\| - \|y\|| \le \|x - y\| \le C_1\|x - y\|_1$$

for all $x, y \in X$, and K is compact in X since closed and bounded. Then there exists $y_0 \in K$ s.t. $f(y_0) = \inf_{y\in K} f(y)$ and let $\frac{1}{C} = f(y_0) = \|y_0\| > 0$. Then $\|y\|_1 = 1$ implies $\|y\| \ge \frac{1}{C}$.
Give any $x \in X$, let $y = x/\|x\|_1$ s.t. $\|y\|_1 = 1$ so $\|y\| \ge \frac{1}{C}$ which means

$$\frac{\|x\|}{\|x\|_1} \ge \frac{1}{C} \implies \|x\| \ge \frac{1}{C}\|x\|_1$$

so the topology they induced are equivalent.

2. According to 4.15, and 1, we find that the topology induced by $\|\|_p$ is equivalent to any other norm in a finite-dim space. Then we prove the separability.

3. Let K be a closed and bounded set in $(X, \|\|)$. Suppose in $(X, \|\|_1)$, then according to **??**, compact in $(X, \|\|_1)$ means closed and bounded, as (1) proved above, the topology induced by any norm is equivalent, so K is compact in $(X, \|\|)$. The other side is the property of metric spaces.

4. Let $Y \subset X$ be a subspace of X and let a sequence (since metric space) converges to a point in X i.e. $(y_n) \to y \in X$ for each n, $y_n \in Y$, now we need to prove that $y \in Y$.
Notice that for all $k \in \mathbb{N}_+$, $y_k = \sum_{i=1}^n y_{k,i}e_i$ where $(e_i)_{i=1}^n$ is the basis of Y. Then $(y_n) \to y$ means that $(y_{i,n})_{n=1}^\infty$ is a Cauchy sequence for there exists a C s.t.

$$\sum_{i=1}^n |y_{i,k} - y_{i,\ell}| = \|y_k - y_\ell\|_1 \le C\|y_k - y_\ell\|$$

for all $k, \ell \ge 1$. Notice that $(X, \|\|)$ is a metric space and $(y_n) \to y \in X$, so (y_n) is Cauchy which means $C\|y_k - y_\ell\|$ can be arbitrarily small which implies that $(y_{i,n})_{n=1}^\infty$ is Cauchy in $\mathbb{K}$ and as the completeness of $\mathbb{K}$, $(y_{i,n}) \to y_i \in \mathbb{K}$ as $n \to \infty$. Let $y = \sum_{i=1}^n y_i e_i$.
Now prove that $(y_n) \to y$. Notice that there exists a C_1 s.t.

$$\|y_k - y\| \le C_1\|y_k - y\|_1 = C_1 \sum_{i=1}^n |y_{i,k} - y_i|$$

Since $(y_{i,n}) \to y_i$ for each i, $\|y_k - y\|$ can also be arbitrarily small thus convergent.
So all the sequence in Y convergence in Y means $Y = \overline{Y}$ thus closed.

□

Theorem 4.23 (F.Riesz Theorem)**.** *A normed vector space* $(X, \|\|)$ *is finite-dim iff the unit sphere of* X *i.e.* $K = \{x \in X : \|x\| = 1\}$ *is compact.*

Proof.
- Assume that K is compact. Then there exists a finite number of points $x_i \in X$ s.t. $K \subset \bigcup_{i=1}^{n} B_{1/2}(x_i)$.

  Then we need to show that $Y = \text{span}(x_i)_{i=1}^{n}$ is coincide with X and it is enough to show that
  $$\inf_{y \in Y} \|x - y\| = 0$$
  for all $x \in X$ as Y is finite-dim and $\overline{Y} = Y$.

  Let $x \in X$ and $y \in Y$ be given, then let
  $$x' = \frac{x}{\|x - y\|} \text{ and } y' = \frac{y}{\|x - y\|}$$
  Notice that $x' - y' \in K$ thus in some $B_{1/2}(x_{i_0})$. So
  $$\|x - y\|(\|(x' - y') - x_{i_0}\|) \leq \frac{1}{2}\|x - y\|$$
  Now let $y_1 = \|x - y\|(y' + x_{i_0})$, then $\|x - y_1\| \leq \frac{1}{2}\|x - y\|$. Then let $y = y_1$, and by induction, we find
  $$\|x - y_n\| \leq \frac{1}{2^n}\|x - y\|$$
  Notice that $\|x - y\| < \infty$, so there exists a sequence $(y_n) \to x$ for all $y_n \in Y$ and as $Y = \overline{Y}$, $x \in Y$, so $X = Y = \text{span}(x_i)_{i=1}^{n}$ thus finite-dim.

- For the converse, we notice that K is closed and bounded, then compact since 4.22(3).

$\square$

4.5 Continuous Linear Operators

4.5.1 General properties

Definition 4.21 (linear operator)**.** A mapping $A : X \to Y$ is a linear operator from X into Y, or a linear functional if $Y = \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and

- $A(x + y) = A(x) + A(y)$ for all $x, y \in X$
- $A(\alpha x) = \alpha A(x)$ where $x \in X$ and $\alpha \in \mathbb{K}$

Moreover, if $\mathbb{K} = \mathbb{C}$, then a mapping $A : X \to Y$ is semilinear if:

- $A(x+y) = A(x) + A(y)$
- $A(\alpha x) = \overline{\alpha} A(x)$ where $\overline{\alpha}$ denotes the complex conjugate of α.

Definition 4.22 (kernel and image)**.** The kernel of A defined by:

$$\ker A = \{x \in X : Ax = 0\}$$

The image of A defined by:

$$\operatorname{Im} A = \{y \in Y : y = Ax \ \text{ for some } \ x \in X\}$$

Notice that $\ker A$ is a subspace of X and $\operatorname{Im} A$ is a subspace of Y.

Proposition 4.2.

1. *A linear operator $A : X \to Y$ is injective iff* $\ker A = \{0\}$.
2. *If a linear operator $A : X \to Y$ is injective, the inverse mapping $A^{-1} : Im\, A \to X$ of $A\,|_{Im\,A}$ is a linear operator.*

Definition 4.23 (eigenvalue)**.** Let X be a vector space over $\mathbb{K}$ and let $A : X \to Y$ be a linear operator. Then a scalar $\lambda \in \mathbb{K}$ is a eigenvalue of A if there exists a vector $p \in X$ s.t.

$$Ap = \lambda p \ \text{ and } \ p \neq 0$$

and such p is called the eigenvector of A, corresponding to the eigenvalue λ.

For a particular eigenvalue λ, note that the subset $\{p \in X : Ap = \lambda p\} \subset X$ is a subspace, which is called eigenspace corresponding to λ.

Definition 4.24 (continuity of operator)**.** When X, Y are normed vector space which equipped their norm topology, then a mapping $A : X \to Y$ is called continuous linear operator if it is both continuous between their norm topology and keep linearity.

Theorem 4.24. *Let X, Y be normed vector spaces, and $A : X \to Y$ be linear operator, then the following properties are equivalent:*

1. *A is continuous on X.*
2. *A is continuous at $0 \in X$.*
3. *There exists a $C > 0$ s.t.*

$$\|Ax\| \leq C\|x\| \ \text{ for all } \ x \in X$$

4. *The image under A of any bounded subset $K \subset X$, $A(K) \subset Y$ is bounded.*

Proof.

- $1 \implies 2$: Obvious.

- $2 \implies 3$: Consider the closed unit ball $B_1[0] \subset Y$, then there exists a $C > 0$ s.t.
$$A^{-1}(B_1[0]) = B_{1/C}[0] \subset X$$
so for any $x \in X$,
$$\left\| A\left(\frac{x}{C\|x\|}\right) \right\| \leq 1$$
thus $\|Ax\| \leq C\|x\|$ for any $x \in X$.
- $3 \implies 4$: Notice that every bounded set $B \subset X$ is contained in a ball $B_{r(B)}[0]$ and so for every $x \in B_{r(B)}[0]$, $\|Ax\| \leq C\|x\| \leq C \cdot r(B)$ thus bounded in Y.
- $4 \implies 1$: Note that the image of ball $B_1[0] \subset X$ is bounded in Y. In other words, there exists $M \geq 0$ s.t. $\|x\| \leq 1 \implies \|Ax\| \leq M$. Now give an $x_0 \in X$ and a $\epsilon > 0$, let $\delta = \epsilon/M$, then $\|x - x_0\| \leq \delta$ implies that $\frac{1}{\delta}(x - x_0) \in B_1[0]$ so
$$\frac{1}{\delta}\|A(x - x_0)\| = \left\| A\left(\frac{x - x_0}{\delta}\right) \right\| \leq M$$
so $\|Ax - Ax_0\| \leq \delta \cdot M = \epsilon$ which means the continuity.

□

Remark. The above theorem shows that in normed vector spaces, continuous linear operator is equal to the linear bounded operator. So continuous linear operator is also called bounded linear operator.

Let $X \subset Y$ be a subspace of Y, then $X \hookrightarrow Y$ denotes the canonical injection from X into Y. So according to 4.3, we observe that:
$$\|x\|_Y \leq C\|x\|_X$$

Theorem 4.25. *Let X and Y be two normed vector spaces.*

- *Any continuous linear operator from X into Y is uniformly continuous.*
- *If X is finite-dim, any linear operator from X into Y is continuous.*

Proof. Notice that if A is continuous, then there exists a $C \geq 0$ s.t. $\|Ax\| \leq C\|x\|$ for every $x \in X$. So for any $x_1, x_2 \in X$,
$$\|A(x_1 - x_2)\| = \|Ax_1 - Ax_2\| \leq C\|x_1 - x_2\|$$
so A is Lipschitz continuous on X.

If X is finite-dim, then let $(e_i)_{i=1}^n \subset X$ be a basis of X. Observe that for any $x = \sum_{i=1}^n x_i e_i \in X$:
$$\|Ax\| = \left\| A\left(\sum_{i=1}^n x_i e_i\right) \right\| \leq C_1\|x\|_1$$

where $C_1 = \max_{1\le i\le n} \|Ae_i\|$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$ so we can induce that the image of every bounded set is bounded, thus continuous.

□

Theorem 4.26. *Let X, Y be two normed vector spaces, and let $A : X \to Y$ be a linear operator, then the properties are equivalent:*

- *The linear operator A is injective and inverse mapping $A^{-1} : Im\, A \to X$ is a continuous linear operator.*
- *There exists a constant $C > 0$ s.t.*

$$\|x\| \le C\|Ax\| \ \text{ for all } \ x \in X$$

Proof. Suppose the first property holds, then $A' : X \to \operatorname{Im} A$ is a one-to-one and onto function, then there exits a unique $x \in X$ s.t. $A^{-1}(y) = x$ for unique $y \in Y$. Then the continuity of A^{-1} implies that there exists a $C > 0$ s.t.

$$\|A^{-1}(y)\| \le C\|y\| \ \text{ for all } \ y \in \operatorname{Im} A$$

which is equal to the second.

If the second property holds, then $\ker A = \{0\}$ which means A is injective and $\|x\| \le C\|Ax\|$ for every $x \in X$ implies that for every $y \in \operatorname{Im} A$, $\|A^{-1}(y)\| \le C\|y\|$ implies the continuity of A^{-1}.

□

Definition 4.25 ($\mathcal{L}(X;Y)$)**.** Let X and Y be two normed vector spaces over the same field $\mathbb{K}$, then the vector space formed by all continuous linear operator from X into Y, denote by: $\mathcal{L}(X;Y)$ or $\mathcal{L}(X)$ if X to X.

Theorem 4.27 (norm of $\mathcal{L}(X;Y)$)**.** 1. *The mapping defined by:*

$$\|\cdot\| : A \in \mathcal{L}(X;Y) \to \|A\| = \sup_{x\ne 0} \frac{\|Ax\|}{\|x\|}$$

is a norm of $\mathcal{L}(X;Y)$ which is called the sup-norm.

2. *The norm of any $A \in \mathcal{L}(X;Y)$ may be equivalently defined as*

$$\begin{aligned}\|A\| &= \sup_{\|x\|\le 1} \|Ax\| = \sup_{\|x\|<1} \|Ax\| = \sup_{\|x\|=1} \|Ax\| \\ &= \inf\{C > 0 : \|Ax\| \le C\|x\| \ \text{ for all } \ x \in X\}\end{aligned}$$

where the last norm is called the inf-norm.

3. *From 1, we can get $\|Ax\| \le \|A\|\|x\|$ for every $x \in X$, and if X is finite-dim, there exists $0 \ne x_0 \in X$ s.t.*

$$\|A\|\|x_0\| = \|Ax_0\|$$

4. *Let Z be a normed vector space and let $A \in \mathcal{L}(X;Y)$ and $B \in \mathcal{L}(Y;Z)$, then $BA = B \circ A \in \mathcal{L}(X;Z)$ and*

$$\|BA\| \le \|A\|\|B\|$$

Particularly, if $A \in \mathcal{L}(X)$, then

$$\|A^n\| \le \|A\|^n \text{ for any } n \in \mathbb{N}_+$$

5. *If $A \in \mathcal{L}(X)$, then any eigenvalue λ of A satisfies $|\lambda| \le \|A\|$.*

Proof. Now give a proof of 3.

Note that if X is finite-dim, then the unit sphere $\{x \in X : \|x\| = 1\}$ is compact. And the mapping $x \in X \mapsto y \in Y = Ax \mapsto \|Ax\|$ is the composition of two continuous functions thus continuous. Then there exists some $x = x_0 \in \{x \in X : \|x\| = 1\}$ s.t. this continuous function attains its supremum over $\mathbb{K}$ i.e. $\|Ax_0\| = \sup \|Ax\|, \|x\| = 1$ then $\|A\| = \|A\|\|x_0\| = \|Ax_0\|$.

□

4.5.2 Compact Continuous Linear Operator

Definition 4.26 (compact linear operator)**.** A linear operator $A : X \to Y$ is said to be compact if $A(B) \subset Y$ is relatively compact whenever $B \subset X$ is bounded in X.

Theorem 4.28. *Let X and Y be two normed vector space over the same field, and let $A : X \to Y$ be a linear operator.*

1. *If A is compact, then A is continuous.*
2. *The operator A is compact iff given any bounded sequence $(x_n)_{n=1}^{\infty} \subset X$, the sequence $(Ax_n)_{n=1}^{\infty} \subset Y$ contains a subsequence converging in Y.*
3. *If X is finite-dim, A is compact.*
4. *If A is continuous and the image $A(X)$ is finite-dim, then A is compact.*

Proof.

1. Note that compact in metric space means bounded and closed, then we have proved that if A maps a bounded set to a bounded set for any subset $B \subset X$, then A is continuous.

2. Assume that A is compact, and $(x_i)_{i=1}^{\infty}$ is bounded in X, then $(Ax_i)_{i=1}^{\infty}$ is compact in Y then there exists a subsequence converges in Y, particularly in $\{Ax_i\}_{i=1}^{\infty}$as the compactness in metric space means sequentially compact.

   Pick any bounded set $B \subset X$, consider the set $A(B) \subset Y$, notice

$$\lim_{i\to\infty} y_{\varphi(i)} = y \in Y$$

Note that $y_{\varphi(i)} \in A(B)$ for any i, so $y \in \overline{A(B)}$. This implies that $\overline{A(B)}$ is sequentially compact thus compact in Y which means $A(B)$ is relatively compact in Y.

3. Note that if X is finite-dim, then any linear mapping A is continuous, and a continuous mapping carries bounded set to bounded set, i.e. if $B \subset X$ is bounded, then $A(B) \subset Y$ is bounded. And since $A(B) \subset A(X)$ and $A(X)$ is finite-dim, $A(B)$ is bounded and finite-dim. So $\overline{A(B)}$ is closed and bounded in a finite-dim space, thus compact.

4. If $A \in \mathcal{L}(X;Y)$ and $A(X) \subset Y$ is finite-dim, suppose that $B \subset X$ is bounded then $A(B) \subset A(X)$ and $\overline{A(B)}$ is bounded and closed thus compact in a finite-dim space.

□

Remark. If $X \subset Y$ is a subspace in Y, then the notation:

$$X \Subset Y$$

means the canonical injection $x \in X \mapsto x \in Y$ is a compact linear operator. In other words, any bounded sequence in X contains a subsequence converging in Y.

4.5.3 Continuous multilinear mappings

Definition 4.27 (product vector space)**.** Suppose that $k \in \mathbb{N}_+$ and $k \geq 2$, $X_\ell, 1 \leq \ell \leq k$ and Y are vector space over the same field $\mathbb{K}$. Then consider the product space:

$$X = \prod_{i=1}^{k} X_i = X_1 \times X_2 \times \dots \times X_k$$

where $\times$ denotes the Cartesian product. And for $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in X$, define:

$$\begin{aligned} x + y &= (x_1 + y_1, \dots, x_k + y_k) \\ \alpha x &= (\alpha x_1, \dots, \alpha x_k) \ \text{ for } \ \alpha \in \mathbb{K} \end{aligned}$$

easy to see that X is still a vector space over $\mathbb{K}$.

Definition 4.28 (multilinear)**.** A mapping $A : \prod_{i=1}^{k} X_i \to Y$ is said to be multilinear or k-linear mapping if when $(k-1)$ other variables are kept fixed, for any $x_\ell \in X_\ell \mapsto y \in Y$ is linear.If $Y = \mathbb{K}$, it is called the multilinear functional.

Remark. Suppose $X = \prod_{i=1}^{k} X_i$ and Y are vector space over $\mathbb{K}$, then a operator $A : X \to Y$ is said to:

- **linear** if for $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in X :

$$\begin{aligned} A(x+y) &= Ax + Ay = A(x_1, \dots, x_k) + A(y_1, \dots, y_k) \\ A(\alpha x) &= \alpha Ax = \alpha A(x_1, \dots, x_k) \end{aligned}$$

- **multilinear** if for $x, y \in X^2$:

$$\begin{aligned} A(x+y) &= A((x_1, x_2) + (y_1, y_2)) \\ &= A(x_1 + y_1, x_2 + y_2) \\ &= A(x_1, x_2 + y_2) + A(y_1, x_2 + y_2) \\ &= A(x_1, x_2) + A(x_1, y_2) + A(y_1, x_2) + A(y_1, y_2) \end{aligned}$$

  Similar as $A(\alpha x)$:

$$\begin{aligned} A(\alpha x) &= A(\alpha(x_1, x_2)) \\ &= A(\alpha x_1, \alpha x_2) \\ &= \alpha A(x_1, \alpha x_2) \\ &= \alpha^2 A(x_1, x_2) \end{aligned}$$

Definition 4.29 (multilinear operator space)**.** Define:

$$\begin{aligned} (A+B)(x_1, \dots, x_k) &= A(x_1, \dots, x_k) + B(x_1, \dots, x_k) \\ (\alpha A)(x_1, \dots, x_k) &= \alpha A(x_1, \dots, x_k) \end{aligned}$$

and note that $A+B$ is also a multilinear mapping so as αA, so all linear mappings from $\prod_{i=1}^{k} X_i$ to Y over $\mathbb{K}$ form a vector space.

Definition 4.30 (symmetric and alternate)**.** Let $\mathcal{G}_k$ denote the set of all the permutations of the set $\{1, 2, \dots, k\}$ and suppose $X_i = X$ for $i = 1, \dots, k$,then a multilinear mapping $A : \prod_{i=1}^{n} X_i \to Y$ is said to be:

- **symmetric** if for all $\sigma \in \mathcal{G}_k$ and $x_i \in X_i = X$:

$$A(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = A(x_1, \dots, x_k)$$

- **alternate** if:

$$A(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \epsilon(\sigma) A(x_1, \dots, x_k)$$

  where $\epsilon(\sigma)$ is the signature of σ.

Remark. Notice that the determinant of a $k \times k$ matrix is a alternate multilinear functional.

Theorem 4.29. *Let $X_i, 1 \leq i \leq k$ and Y be normed vector space over $\mathbb{K}$ and let $X = \prod_{i=1}^{k} X_i$ and $A : X \to Y$ be a multilinear mapping. Then the following statements are equivalent:*

1. *The mapping $A : X \to Y$ is continuous.*
2. *The mapping A is continuous at $0 \in X$.*
3. *There exists a constant $C > 0$ s.t.*

$$\|Ax\|_Y \leq C\|x_1\|_{X_1} \cdots \|x_k\|_{X_k} \text{ for all } (x_1, \dots, x_k) \in X$$

4. *The image of any bounded subset of X is bounded in Y.*

Proof. For $x \in X$, define $\|x\|_\infty = \max_{1 \leq \ell \leq k} \|x_\ell\|_{X_\ell}$ and consider the topology induced by $\|\cdot\|_\infty$.

- $1 \implies 2$: Obvious.
- $2 \implies 3$: If 2 holds, the inverse image under A of a closed ball of Y contains a closed ball centered at the origin of X. Let $\alpha > 0$ denote the radius of the ball in X. Then by the definition of $\|\cdot\|_\infty$, if there exists a vector $x = (x_1, \dots, x_k) \in X$ s.t. $\|x_i\|_{X_i} \leq \alpha$ for all $i \in [1, k]$, then $Ax \in B_1[0] \subset Y$.

  Given any vector $x = (x_1, \dots, x_k) \in X$, s.t. $x_i \neq 0$ since if any $x_i = 0$, $Ax = 0 \in Y$ for sure, let $x' = (x'_1, \dots, x'_k)$ with $x'_i = \alpha(\|x_i\|_{X_i})^{-1} x_i$. Then $\|x'_i\|_{X_i} = \alpha$ for all i and thus $\|Ax'\|_Y \leq 1$. Note that $x_i = \frac{1}{\alpha}\|x_i\|_{X_i} x'_i$, so

  $$Ax = \frac{1}{\alpha^k}\|x_1\|_{X_1} \cdots \|x_k\|_{X_k} Ax'$$

  and let $C = 1/\alpha^k$.
- $3 \implies 4$: Assume that 3 holds, note that any bounded subset $B \subset X$ is contained in a ball with radius $r(B)$, i.e. $B \subset B_{r(B)}(0)$, so $A(B) \subset A(B_{Cr(B)^k}(0))$, thus bounded in Y.
- $4 \implies 1$: Assume that 4 holds, then the image of the closed unit ball $B_1[0] \subset X$ is bounded in Y i.e. there exists a $C \geq 0$ s.t. if $\|x_i\|_{X_i} \leq 1$ for all i, $\|Ax\|_Y \leq C$. Therefore as the multilinearity of A,

  $$\|Ax\|_Y \leq C\|x_1\|_{X_1} \cdots \|x_k\|_{X_k}$$

  for all $x = (x_1, \dots, x_k) \in X$.

  Given $x = (x_1, \dots, x_k) \in X$ and $a = (a_1, \dots, a_k) \in X$, $A(x) - A(a)$ can be written as:

  $$\begin{aligned} A(x) - A(a) = {} & A(x_1 - a_1, x_2, \dots, x_k) \\ & + A(a_1, x_2 - a_2, x_3, \dots, x_k) \\ & \vdots \\ & + A(a_1, a_2, \dots, a_{k-1}, x_k - a_k) \end{aligned}$$

Then use the result we proved before:

$$\begin{aligned}\|A(x)-A(a)\|_Y \leq C(&\|x_1-a_1\|_{X_1}\|x_2\|_{X_2}\cdots\|x_k\|_{X_k}\\ &+\|a_1\|_{X_1}\|x_2-a_2\|_{X_2}\cdots\|x_k\|_{X_k}\\ &\vdots\\ &+\|a_1\|_{X_1}\|a_2\|_{X_2}\cdots\|x_k-a_k\|_{X_k})\end{aligned}$$

Let $M=\|a\|_\infty$ and $\delta=\|x-a\|_\infty$, then above:

$$\|A(x)-A(a)\|_Y \leq C\delta\left\{(M+\delta)^{k-1}+M(M+\delta)^{k-2}+\cdots+M^{k-1}\right\}$$

since $\|x\|_\infty \leq \|x-a\|_\infty+\|a\|_\infty = M+\delta$ and the right side of the inequality approaches 0 when $\delta\to 0$, so A is continuous.

□

Remark. For a linear operator:

$$\|Ax\|_Y \leq C\left(\|x_1\|_{X_1}+\|x_2\|_{X_2}+\cdots+\|x_k\|_{X_k}\right)$$

Note that $\|x\|_X=\|x_1\|_{X_1}+\|x_2\|_{X_2}+\cdots+\|x_k\|_{X_k}$ is a norm on X. Or

$$\|Ax\|_Y \leq C\max_{1\leq i\leq k}\|x_i\|_{X_i}$$

And for a multilinear operator:

$$\|Ax\|_Y \leq C\|x_1\|_{X_1}\|x_2\|_{X_2}\cdots\|x_k\|_{X_k}$$

Theorem 4.30. *If $X_i : 1\leq i\leq k$ are all finite-dim, and Y is a normed vector space, any multilinear mapping $A:\prod_{i=1}^k X_i = X\to Y$ is continuous.*

Proof. For each $1\leq \ell\leq k$, $(e^\ell_{i(\ell)})_{i(\ell)=1}^{m(\ell)}$ is a basis. And suppose $x\in X$, $x=(x_1,\dots,x_k)$, there exists:

$$x_\ell=\sum_{i(\ell)=1}^{m(\ell)} x^\ell_{i(\ell)}e^\ell_{i(\ell)}$$

then $\|Ax\|_Y$ can be expanded as:

$$\|Ax\|_Y=\sum_{i(1)=1}^{m(1)}\cdots\sum_{i(k)=1}^{m(k)} x^1_{i(1)}\cdots x^k_{i(k)}A(e^1_{i(1)},\dots,e^k_{i(k)})$$

Note that the sum is finite and $\|A(e^1_{i(1)},\dots,e^k_{i(k)})\|_Y$ are all finite, then there exists a constant C s.t.

$$\|Ax\|_Y \leq C\|x_1\|_\infty\|x_2\|_\infty\cdots\|x_k\|_\infty$$

□

Chapter 5

Real Analysis

5.1 Measurability

5.1.1 Algebra of sets

Definition 5.1 (algebra)**.** A non-empty family of subsets $\mathcal{A}$ is called the algebra of sets X if

- $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
- $A \in \mathcal{A} \implies A^c = X - A \in \mathcal{A}$

Definition 5.2 (σ-Algebra)**.** A family is called σ-Algebra if it is an algebra and

- $(A_n)_{n=1}^{\infty} \subset \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Theorem 5.1. *If $\mathcal{C}$ is a nonempty collection of subsets of X then $\sigma(\mathcal{C}) = \mathcal{A}$ which is the smallest σ-Algebra containing $\mathcal{C}$ satisfies:*

- *If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.*
- *$\mathcal{A}$ is closed under countable intersections.*
- *$\mathcal{A}$ is closed under countable disjoint unions.*

Definition 5.3 (ring)**.** A nonempty family $\mathcal{R}$ of subsets of X is a ring if:

- $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$
- $A, B \in \mathcal{R} \implies A - B \in \mathcal{R}$

Notice that $A - (A - B) = A \cap B$, so $A, B \in \mathcal{R} \implies A \cap B \in \mathcal{R}$.

Definition 5.4 (semiring)**.** A semiring $\mathcal{S}$ is a non-empty family of subsets of X satisfying:

- $\emptyset \in \mathcal{S}$
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- If $A, B \in \mathcal{S}$, then there exists pairwise disjoint sets $C_1, \dots, C_n \in \mathcal{S}$ s.t. $A - B = \bigcup_{i=1}^{n} C_i$

Theorem 5.2. *If $\mathcal{S}_1, \mathcal{S}_2$ are semirings, then $\mathcal{S}_1 \times \mathcal{S}_2$ is also semiring where $\times$ denotes the Cartesian product.*

Proof.
- Let $\emptyset \times \emptyset = \emptyset$ without any doubt.
- Let $A, C \in \mathcal{S}_1$ and $B, D \in \mathcal{S}_2$ then $A \times B, C \times D \in \mathcal{S}_1 \times \mathcal{S}_2$. Notice that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ where $A \cap C \in \mathcal{S}_1, B \cap D \in \mathcal{S}_2$, so $(A \cap C) \times (B \cap D) \in \mathcal{S}_1 \times \mathcal{S}_2$.
- Let $A, C \in \mathcal{S}_1$ and $B, D \in \mathcal{S}_2$.

$$(A \times B) - (C \times D) = (A - C) \times B \cup A \times (B - D)$$

□

5.1.2 Dynkin's Lemma

Definition 5.5 (λ-system)**.** A λ-system or Dynkin system is a nonempty family $\mathcal{A}$ s.t.

- $X \in \mathcal{A}$
- $A, B \in \mathcal{A}, B \subset A \implies A - B \in \mathcal{A}$
- $(A_n) \subset \mathcal{A}, (A_n) \nearrow A \implies A \in \mathcal{A}$

Theorem 5.3. *A nonempty family of subsets of X is a σ-Algebra iff it is both a π-system and a λ-system.*

Theorem 5.4 (Dynkin's Lemma)**.** *If $\mathcal{A}$ is a λ-system and a nonempty family $\mathcal{F} \subset \mathcal{A}$ is closed under finite intersection, then $\sigma(\mathcal{F}) \subset \mathcal{A}$.*

In other words, if $\mathcal{F}$ is a π-system, then $\sigma(\mathcal{F})$ is the smallest λ-system containing $\mathcal{F}$.

5.1.3 Borel σ-Algebra

Definition 5.6 (borel set)**.** The Borel σ-Algebra of a topological space (X, τ), is $\sigma(\tau)$. The set in $\sigma(\tau)$ is called the Borel sets.

Corollary 5.1. *1. The Borel σ-Algebra is the smallest λ-system containing the open sets. It is also the smallest λ-system containing all the closed sets.*

2. The Borel σ-Algebra of a topological space is the smallest family of sets containing all the open sets and all the closed sets that is the closed under countable intersections and countable disjoint unions.

3. The Borel σ-Algebra of a metrizable space is the smallest family of sets that include the open (closed) sets and is closed under countable intersections and countable disjoint unions.

Remark. For the difference between (2) and (3), just notice that in a metrizable space, every closed set is G_δ and every open set is F_σ.

5.1.4 Product Structure

Definition 5.7 (product σ-Algebra)**.** Let $\mathcal{F}_i$ be a σ-Algebra of X_i, $i = 1, \dots, n$. Then the product σ-Algebra denoting as $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$ is the σ-Algebra generated by the product semiring $\prod_{i=1}^n \mathcal{F}_i$.

Theorem 5.5. *For any two topological space X and Y:*

1. $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$
2. *If X, Y are second countable, then $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.*

Proof. For each $A \subset X$, let

$$\mathcal{F}(A) = \{B \subset Y : A \times B \in \mathcal{B}_{X\times Y}\}$$

Then $\mathcal{F}(A)$ satisfies:

1. $\emptyset \in \mathcal{F}(A)$ for $A \times \emptyset = \emptyset \in \mathcal{B}_{X\times Y}$.
2. If $B, C \in \mathcal{F}(A)$ then $B - C \in \mathcal{F}(A)$. Notice that if $B, C \in \mathcal{F}(A)$, then $A \times (B - C) = (A \times B) - (A \times C)$, so $A \times (B - C) \in \mathcal{B}_{X\times Y}$ as the property of σ-Algebra.
3. $\mathcal{F}(A)$ is closed under countable unions. To see this, notice that if $(B_n)_{n=1}^\infty \subset \mathcal{F}(A)$, then $A \times (\bigcup_{n=1}^\infty B_n) = \bigcup_{n=1}^\infty (A \times B_n) \in \mathcal{B}_{X\times Y}$.

From above properties, we find that $\mathcal{F}(A)$ is a σ-ring. If $Y \in \mathcal{F}(A)$, then $\mathcal{F}(A)$ is a σ-Algebra.

Then note that for any open subset $G \in \tau_X$, $U \in \mathcal{F}(G)$ for every $U \in \tau_X$. To see this, just recall the base of product topology of finite Cartesian product of sets which has the form $\prod_{i=1}^n V_i$ where $V_i \subset X_i$ is the open subsets. Notice that $Y \in \tau_Y$ so $Y \in \mathcal{F}(G)$ s.t. $\mathcal{F}(G)$ is a σ-Algebra and $\tau_Y \subset \mathcal{F}(G)$ thus $\mathcal{B}_Y \subset \mathcal{F}(G)$ when G is open.

Now let

$$\mathcal{A} = \{A \subset X : \mathcal{B}_Y \subset \mathcal{F}(A)\}$$

As we discussed above, $\tau_X \subset \mathcal{A}$.

Also note that $\mathcal{A}$ is closed under complementation. To see this, let $A \in \mathcal{A}, B \in \mathcal{B}_Y$, then $A \times B \in \mathcal{B}_{X\times Y}$, as $X \in \tau_X$, then $X \times B \in \mathcal{B}_{X\times Y}$. Therefore, $A^c \times B = (X - A) \times B = (X \times B) - (A \times B) = (X \times B) \cap (A \times B)^c \in \mathcal{B}_{X\times Y}$. As the arbitrary picking of $B \in \mathcal{B}_Y$ we find that $\mathcal{F}(A^c) \supset \mathcal{B}$ thus $A^c \in \mathcal{A}$.

Finally, if $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ and $B \in \mathcal{B}_Y$, we have $A_n \times B \in \mathcal{B}_{X\in Y}$ for each n. As

$$\bigcup_{n=1}^{\infty}(A_n \times B) = \left(\bigcup_{n=1}^{\infty} A_n\right) \times B$$

and $(A_n \times B)_{n=1}^{\infty}$ is a sequence in $\mathcal{B}_{X\times Y}$, so the left side is still belong to $\mathcal{B}_{X\times Y}$, so as the arbitrary picking of $B \in \mathcal{B}_Y$, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ since $\mathcal{B} \subset \mathcal{F}(\bigcup_{n=1}^{\infty} A_n)$.

The above steps show that $\mathcal{A}$ is a σ-Algebra containing τ_X which means $\mathcal{B}_X \subset \mathcal{A}$.

So if $A \in \mathcal{B}_X$, then $A \in \mathcal{A}$ and as the property of $\mathcal{A}$, if there exists $B \in \mathcal{B}_Y$, then $A \times B \in \mathcal{B}_{X\times Y}$ which means $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$.

Note that if X, Y are second countable space, then $X \times Y$ is. Every open subset of $X \times Y$ is the countable union of the form $U \times V$ where $U \in \tau_X, V \in \tau_Y$. So $\mathcal{B}_{X\times Y} = \sigma\{U \times V : U \in \tau_X, V \in \tau_Y\} \subset \sigma(\mathcal{B}_X \times \mathcal{B}_Y) = \mathcal{B}_X \otimes \mathcal{B}_Y$, so $\mathcal{B}_X \otimes \mathcal{B}_Y \supset \mathcal{B}_{X\times Y}$.

□

Theorem 5.6. *Let $(X, \mathcal{F})$ be a measurable space and Y be a second countable Hausdorff space. If $f : X \to Y$ is $(\mathcal{F}, \mathcal{B}_Y)$-measurable, then $Gr\, f \in \mathcal{F} \otimes \mathcal{B}_Y$.*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base of Y, then $f(x) \neq y$ iff there exists some U_i s.t. $f(x) \in U_i$ and $y \notin U_i$. Thus

$$(\mathrm{Gr}\, f)^c = \bigcup_{i=1}^{\infty} f^{-1}(U_i) \times (U_i)^c$$

So $\mathrm{Gr}\, f$ is $\mathcal{F} \otimes \mathcal{B}_Y$-measurable.

□

Definition 5.8 (measurable section)**.** If $A \subset X \times Y$, then we define x-section A^x and y-section A^y as:

$$A^x = \{y \in Y : (x, y) \in A\} \text{ and } A^y = \{x \in X : (x, y) \in A\}$$

Let $(X, \mathcal{F}_X), (Y, \mathcal{F}_Y)$ be measurable spaces, then a subset $A \subset X \times Y$ has measurable sections if $A^x \in \mathcal{F}_Y$ for each $x \in X$ and $A^y \in \mathcal{F}_X$ for each $y \in Y$.

Proposition 5.1. *Every $A \in \mathcal{F}_X \otimes \mathcal{F}_Y$ has measurable sections.*

Proof. Consider the family $\mathcal{A}$ of subsets of $X \times Y$:

$$\mathcal{A} = \{A \subset X \times Y : A^x \in \mathcal{F}_Y \text{ and } A^y \in \mathcal{F}_X\}$$

for each $x \in X$ and $y \in Y$. Then we show that $\mathcal{A}$ is a σ-Algebra.

- $\emptyset^x = \emptyset^y = \emptyset$
- $(X \times Y)^x = Y, (X \times Y)^y = X$
- For each family of subsets $\{A_i\}_{i\in I} \subset X \times Y$, $(\bigcup_{i\in I} A_i)^x = \bigcup_{i\in I}(A_i)^x$ and $(\bigcap_{i\in I} A_i)^x = \bigcap_{i\in I}(A_i)^x$, similar as y.

So if $\mathcal{F}_X$ is a σ-Algebra of X, then $\{A \subset X \times Y : A^y \in \mathcal{F}_X \text{ for all } y \in Y\}$ is also a σ-Algebra and denote it as $\mathcal{A}_y$, similar as $\mathcal{A}_x$. Notice that the intersection of σ-Algebra is also a σ-Algebra. So $\mathcal{A} = \mathcal{A}_x \cap \mathcal{A}_y$ is a σ-Algebra.

Observe that for every measurable rectangles,

$$(A \times B)^x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

similar as y, so every measurable rectangles $A \times B \in \mathcal{A}$, thus $\mathcal{F}_X \otimes \mathcal{F}_Y = \sigma(\mathcal{F}_X \times \mathcal{F}_Y) \subset \mathcal{A}$, so if $A \in \mathcal{F}_X \otimes \mathcal{F}_Y$, A has measurable sections.

□

Definition 5.9. Let $(X, \mathcal{F}_X)$, $(Y, \mathcal{F}_Y)$ and $(Z, \mathcal{F}_Z)$ be measurable spaces. We say a function $f : X \times Y \to Z$ is:

1. jointly measurable if it is $(\mathcal{F}_X \otimes \mathcal{F}_Y, \mathcal{F}_Z)$-measurable.
2. measurable in x if $f^y : (X, \mathcal{F}_X) \to (Z, \mathcal{F}_Z)$ is measurable for each $y \in Y$. Similarly, measurable in y means $f^x : (Y, \mathcal{F}_Y) \to (Z, \mathcal{F}_Z)$ is measurable for each $x \in X$.
3. separately measurable if it is both measurable in x and measurable in y.

Theorem 5.7. *Let $(X, \mathcal{F}_X), (Y, \mathcal{F}_Y), (Z, \mathcal{F}_Z)$ be measurable spaces, then every jointly measurable $f : X \times Y \to Z$ is separately measurable.*

Proof. Give a $y \in Y$, notice that for each $A \in \mathcal{F}_Z$,

$$\begin{aligned}(f^y)^{-1}(A) &= \{x \in X : f^y(x) = f(x, y) \in A\} \\ &= (f^{-1}(A))^y\end{aligned}$$

Note that f is jointly measurable that implies $f^{-1}(A) \in \mathcal{F}_X \otimes \mathcal{F}_Y$, recall that $f^{-1}(A) \in \mathcal{F}_X \otimes \mathcal{F}_Y \implies f^{-1}(A)$ has measurable sections. Thus $(f^{-1}(A))^y \in \mathcal{F}_X$. As the arbitrary picking of $y \in Y$, and similar as x's situation. So it leads to separately measurable.

□

Proposition 5.2. *Let $(X,\mathcal{F}),(X_1,\mathcal{F}_1),(X_2,\mathcal{F}_2)$ be measurable spaces, and let $f_1 : X \to X_1$ and $f_2 : X \to X_2$. Define $f : X \to X_1 \times X_2$ by:*

$$f(x) = (f_1(x), f_2(x))$$

Then f is $(\mathcal{F}, \mathcal{F}_1 \otimes \mathcal{F}_2)$-measurable iff f_1 is $(\mathcal{F},\mathcal{F}_1)$-measurable and f_2 is $(\mathcal{F},\mathcal{F}_2)$-measurable.

Proof. Notice that
$$f^{-1}(A \times B) = f_1^{-1}(A) \cap f_2^{-1}(B)$$
□

5.2 Signed Measure

5.2.1 Signed Measure

Definition 5.10 (signed measure)**.** A finite signed measure μ on a measurable space $(X,\mathcal{F})$ is a function $\mu : \mathcal{F} \to \mathbb{R}$ s.t. $\mu(\emptyset) = 0$ and

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i) \text{ whenever } A = \bigcup_{i=1}^{\infty} A_i$$

for disjoint A_i. And the series has to converge absolutely which means that

$$\sum_{i=1}^{\infty} |\mu(A_i)| < \infty$$

More generally, a signed measure is allowed to take one of the values ∞ and $-\infty$.

Proposition 5.3 (Jordan decomposition and Hahn decomposition)**.**

- *For any signed measure μ there exists unique positive measures μ^+ and μ^- s.t. $\mu = \mu^+ - \mu^-$ and there exists a measurable set A s.t. $\mu^+(A) = \mu^-(A^c) = 0$. The second condition is called mutually singular and μ^+ is called the positive variation of μ, similar as μ^-.*
- *There exists measurable sets P and N s.t. $P \cup N = X$ and $P \cap N = \emptyset$ and $\mu^+(A) = \mu(A \cap P)$ and $\mu^-(A) = -\mu(A \cap N)$ which is called the Hahn decomposition.*

The total variation of μ is $|\mu| = \mu^+ + \mu^-$, we say μ is σ-finite if $|\mu|$ is σ-finite.

Definition 5.11 (integration)**.** Integration with respect to a signed measure is defined by:
$$\int f d\mu = \int f d\mu^+ - \int f d\mu^-$$

Note that a function f is integrable with respect to μ if it is integrable with respect to $|\mu|$ and it is easy to see:

$$\left|\int f d\mu\right| \leq \int |f| \, d\, |\mu|$$

5.2.2 BV functions and LS integrals

Definition 5.12 (total variation function)**.** Let F be a function on $[a,b]$, then the total variation function of F is the function $V_F(x)$ defined on $[a,b]$ by:

$$V_F(x) = \sup\left\{\sum_{i=1}^{n} |F(s_i) - F(s_{i-1})| : a = s_0 < s_1 < ... < s_n = x\right\}$$

Note that the supremum is taken over partitions of $[a,x]$. F has bounded variation on $[a,b]$ if $V_F(b) < \infty$.

$BV[a,b]$ denotes the space of functions with bounded variation on $[a,b]$. A function $f \in BV[a,b]$ is called a BV function over $[a,b]$.

Proposition 5.4. *1. V_f is non-decreasing on $[a,b]$.*

2. f is a BV function iff it is the difference of two bounded non-decreasing functions, and in case f is BV, one way to write decomposition is

$$f = \frac{1}{2}(V_f + f) - \frac{1}{2}(V_f - f)$$

which is called the Jordan decomposition of f.

Definition 5.13 (LS-measure)**.** Suppose f is BV and right-continuous on $[a,b]$. Then there is a unique signed Borel measure μ_f on $(a,b]$ defined by

$$\mu_f(u,v] = f(v) - f(u), \;\; a \leq u < v \leq b$$

where this measure is called Lebesgue-Stietjes measure.