ch1-Vector-Space-01

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Contents

0.0.0.1 Linear independence and basis

Definition 0.1 (linear independence)**.** A family of vectors $\{x_i\}_{i\in I}$ is called **linear independent** if the vectors x_i are linearly independent i.e.

$$\sum_{i\in I}\alpha_i x_i = 0 \implies \alpha_i = 0 \quad \text{for each } i$$

Definition 0.2 (system of generators)**.** A subset $S \subset E$ is called a system of generators of E if every vector $x \in E$ is a linear combination of vectors in S.

Proposition 0.1.
1. *Every finitely generated non-trivial vector space has a finite basis.*
2. *Suppose that $S = \{x_1, \ldots, x_m\}$ is a finite system of generators of E and that the subset $R \subset S$ by $R = \{x_1, \ldots, x_r\}\,(r \leq m)$ consists of linearly independent vectors. Then there exists a basis T of E s.t. $R \subset T \subset S$.*

Proof. Just need to notice that every basis is the system of generators, and it is a minimal one.

□

Theorem 0.1. *Let E be a non-trivial vector space. Suppose S is a system of generators and R is a family of linearly independent vectors in E s.t. $R \subset S$. Then there exists a basis T of E s.t. $R \subset T \subset S$.*

Proof. Consider the partially order defined between R and S, find some $X \subset E$ s.t.

- $R \subset X \subset S$
- the vectors in X are linearly independent.

We note this partially order as $\mathcal{P}(R, S)$.
Notice that for every chain $\{X_\alpha\} \subset \mathcal{P}(R, S)$ has a maximal element $A = \bigcup_\alpha X_\alpha$. It is obvious that $A \in \mathcal{P}(R, S)$ (Notice that $R \subset A \subset S$ and the property of a chain of the set that contains linearly independent vectors.)
So we prove that every chain $\{X_\alpha\} \subset \mathcal{P}(R, S)$ has a upper bound in $\mathcal{P}(R, S)$, so Zorn's Lemma implies that there exists a maximal element $T \in \mathcal{P}(R, S)$ s.t. vectors in T are linearly independent.
Then we just need to show that T generates E. Give $x \in E$, suppose that x is linearly independent to vectors in T. Notice that S generates E, so

$$x = \sum_{i\in I'}\alpha_i x_i \qquad \text{for some } x_i \in S$$

If x is linearly independent to vectors in T then exists some $i \in I'$ s.t. x_i is linearly independent to vectors in T and note this set as $\{x_j\}_{j\in J} \subset S$, consider the set $\{x_j\}_{j\in J} \cup T \supsetneqq T$ which leads to a contradiction of the maxmality of T. So T is a basis of E.

$\square$

Corollary 0.1. 1. *Every system of generators of E contains a basis. In particular, every non-trivial vector space has a basis.*
2. *Every family of linearly independent vectors of E can be extended to a basis.*

0.0.0.2 Free vector space

Let X be an arbitrary set and consider all maps $f : X \to \mathbb{K}$ s.t. $f(x) \neq 0$ only for finitely many $x \in X$, denoting the set of these maps by $F(X)$, it is easy to show that $F(X)$ is a vector space.

Now give a basis of $F(X)$. For any $a \in X$, let f_a be:

$$f_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

Then $\{f_a\}_{a\in X}$ forms a basis of $F(X)$.

$F(X)$ is called the **free vector space over** X.

0.0.0.3 Linear mappings

Definition 0.3 (linear mapping)**.** Suppose that E and F are vector spaces, and let $\phi : E \to F$ be a set mapping s.t.

$$\phi(x+y) = \phi(x) + \phi(y) \ \text{ for all } \ x, y \in E$$

and

$$\phi(\alpha x) = \alpha\phi(x) \ \text{ for all } \ \alpha \in \mathbb{K}, x \in E$$

Then we call the mapping ϕ satisfying above conditions linear mappings.
Moreover, if $F = \mathbb{K}$, then we called ϕ a **linear function** on E.

Corollary 0.2. *Linear mappings preserve linear relations.*

Proof. Suppose ϕ be a linear mappings, and let $u = \alpha x + \beta y \in E$, then

$$\phi(u) = \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$$

$\square$

Let $\phi : E \to F, \psi : F \to G$ be linear mappings, then the composition of them $\psi \circ \phi : E \to G$ is defined by:

$$(\psi \circ \phi)(x) = \psi(\phi(x))$$

It is easy to show that $\psi \circ \phi$ is still a linear mapping.

Proposition 0.2. *Suppose S is a system of generators of E and $\phi_0 : S \to F$ where F is also a vector space. Then ϕ_0 can be extended in at most one way to linear mapping $\phi : E \to F$.*
And the extension exists iff such an extension is that

$$\sum_i \alpha_i \phi_0(x_i) = 0$$

whenever $\sum_i \alpha_i x_i = 0$.

Proof.

- $\Longrightarrow$: Suppose ϕ to be a linear mapping and it is the extension of ϕ_0, then $\phi\left(\sum_{i=1}^{n}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i \phi\left(x_i\right)$ for each $x_i \in E$.
  And for each $x_i \in S$,
  $$\phi\left(\sum_{i=1}^{n}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i \phi\left(x_i\right) = \sum_{i=1}^{n}\alpha_i \phi_0\left(x_i\right)$$
  so $\phi\left(0\right) = \phi_0\left(0\right) = 0$.
- $\Longleftarrow$: For any $x \in E$, define there exists some $\{x_i\}_{i\in I} \subset S$ s.t. $x = \sum_{i\in I}\alpha_i x_i$. Define
  $$\phi\left(x\right) = \sum_{i\in I}\alpha_i \phi_0\left(x_i\right)$$
  It is obvious that ϕ is that linear mapping.

$\square$

Notice that if S is a basis of E, let ϕ_0 be a set map from S to E, then ϕ_0 can be extended in a unique way to a linear mapping $\phi : E \to F$.

Proposition 0.3. *Let $\phi : E \to F$ be a linear mapping and $\{x_\alpha\}$ be a basis of E. Then ϕ is a linear isomorphism iff the vectors $y_\alpha = \phi\left(x_\alpha\right)$ form a basis for F.*

Proof.

- $\Longrightarrow$: As ϕ is a linear isomorphism, so for any $y \in F$, there exists a unique $x \in E$ s.t. $x = \phi^{-1}\left(y\right)$. Notice that $\{x_\alpha\}$ is a basis, so $x = \sum_\alpha a_\alpha x_\alpha$ for some a_α, so $y = \phi\left(x\right) = \phi\left(\sum_\alpha a_\alpha x_\alpha\right) = \sum_\alpha a_\alpha \phi\left(x_\alpha\right)$. That means $\{\phi\left(x_\alpha\right)\}$ generates F. Then we need to prove the linear independence.
  Let $\sum_\alpha \lambda_\alpha x_\alpha = 0$, then $\lambda_\alpha = 0$ for each α. Then let $\sum_\alpha \gamma_\alpha \phi\left(x_\alpha\right) = 0$, then
  $$\sum_\alpha \gamma_\alpha \phi\left(x_\alpha\right) = \phi\left(\sum_\alpha \gamma_\alpha x_\alpha\right) = 0$$
  so $\sum_\alpha \gamma_\alpha x_\alpha = 0$ which means $\gamma_\alpha = 0$ for each α. So $\{\phi\left(x_\alpha\right)\}$ is a basis of F.
- $\Longleftarrow$: Let $\{y_\alpha = \phi\left(x_\alpha\right)\}$ be a basis of F, then for each $y \in F$, there exists a unique components (λ_α) s.t. $\sum_\alpha \lambda_\alpha y_\alpha = y$. Then we have
  $$\sum_\alpha \lambda_\alpha \phi\left(x_\alpha\right) = \phi\left(\sum_\alpha \lambda_\alpha x_\alpha\right) = \phi\left(x\right)$$
  for some unique $x \in E$.

$\square$

0.0.0.4 Subspace and factor space

Definition 0.4 (Subspace)**.** Let X be a vector space and let $A \subset X$ be a subset of X. Then A is called a subspace if A is also a vector space.

Let S be a non-empty subset of X and there exists a set, noting as X_S, is the linear combination of any vectors in S, X_S is truly a subspace which is called **the subspace generated by** S or **linear closure** of S.

Proposition 0.4. *Let A_1, A_2 be two subspaces of the vector space X and suppose that $A_1 \cap A_2 \neq \varnothing$ then $A_1 \cap A_2$ is still a subspace of X.*

Definition 0.5 (sum of subspace)**.** Let A_1, A_2 be two subspaces of a vector space X, then $\{x = x_1 + x_2 : x_1 \in A_1, x_2 \in A_2\}$ is called the **sum of** A_1 **and** A_2, denote as $A_1 + A_2$. It is easy to determine that $A_1 + A_2$ is still a subspace of X.

Notice that the decomposition is not determined uniquely.
Let $x = x_1 + x_2 = x_1' + x_2'$, then $x_1 - x_1' = x_2 - x_2' = z \in A_1 \cap A_2$. Only if $A_1 \cap A_2 = \{0\}$, then $x = x_1 + x_2$ is uniquely determined. In this time, we called that sum as **direct sum** of A_1 and A_2, denote as $A_1 \oplus A_2$.

Proposition 0.5.

- *Let A_1, A_2 be subspaces of X and let S_1, S_2 be systems of generators of A_1 and A_2, then $S_1 \cup S_2$ generates $A_1 + A_2$.*
- *Suppose that $A_1 \cap A_2 = \{0\}$ and T_1, T_2 are basis of A_1, A_2, then $T_1 \cup T_2$ is the basis of $A_1 \oplus A_2$.*

Proof. Give any $x \in A_1 + A_2$, then $x = x_1 + x_2$ for some $x_1 \in A_1, x_2 \in A_2$. $x_1 = \sum_\alpha \lambda_\alpha x_\alpha$ for some $x_\alpha \in S_1$ and $x_2 = \sum_\beta \gamma_\beta x_\beta$ for some $x_\beta \in S_2$, so $x = \sum_\alpha \lambda_\alpha x_\alpha + \sum_\beta \gamma_\beta x_\beta$, notice that every $x_\alpha, x_\beta \in S_1 \cup S_2$, so $S_1 \cup S_2$ generates $A_1 + A_2$.

Now we need to prove that $T_1 \cup T_2$ is linearly independent.
Notice that $T_1 \subset A_1, T_2 \subset A_2$, $A_1 \cap A_2 = \{0\}$, so $T_1 \cap T_2 = \{0\}$. So consider $x \in A_1 \oplus A_2$, $x = \sum_\alpha \lambda_\alpha x_\alpha + \sum_\beta \gamma_\beta x_\beta = 0$, then $A_1 \ni x_1 = \sum_\alpha \lambda_\alpha x_\alpha = -\sum_\beta \gamma_\beta x_\beta = x_2 \in A_2$, so $x_1 = x_2 = 0$, then as the property of basis, $\lambda_\alpha = 0$ for all α and $\gamma_\beta = 0$ for all β.

□