TOPOLOGY SPACE

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0.1 Topology

Let Ω be as space

Definition: A class of subset $\mathcal{T}$ of Ω is an **topology** if

1. $\emptyset$ and Ω belongs to $\mathcal{T}$.
2. closed under arbitary union.
3. closed under finite intersection.

(X, ρ) is a **metric space**, when ρ defined on $X \times X$ s.t. $\forall x, y, z \in X$: 1. $\rho(x, y) \geq 0$, the equality hold iff $x = y$. 2. $\rho(x, y) = \rho(y, x)$ 3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

ρ is called a **metric**.

Let $E = \mathbb{R}^n$, $l^2 = \sqrt{\sum_1^n (x_i - y_i)^2}$ is called **Euclidean metric**. $l^1 = \sum_1^n |x_i - y_i|$ is called **texi-cab metric** and $l^\infty = \sup\{|x_i - y_i|\}$ is called **sup norm metric**.

Let (E, d) be an metric space. $V(a, r) = \{x \in E, d(x, a) < r\}$ is r**-ball** with center a.

U is **open** relative to d iff $\forall x \in U, \exists r_x > 0 \ni V_d(x, r_x) \subseteq U$. Let T_d be the set of all open subsets of E, we call T_d the **topology induced by** d.

Suppose d is discrete, that is, $d(x, y) = 0$ iff $x = y$, otherwise, $d(x, y) = 1$. Then every subset is open and $T_d = \mathcal{P}(\Omega)$. Such T_d is called **discrete topology**.

Note $d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$ and $d_{l^2}(x, y) \leq \sqrt{n} d_{l^\infty}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$, then d_{l^∞} open $\iff$ d_{l^2} open $\iff$ d_{l^1} open. Hence $T_{d_{l^2}} = T_{d_{l^1}} = T_{d_{l^\infty}}$.

One can change **1** in definition of metric from "iff" to "if" to get a **pseudometric**. A **quasimetric** is measure without **2**. And a **ultrametric** is a metric plus

$$u(x, z) \leq \max(u(x, y), u(y, z))$$

One can check that a triangle in an ultrametric must be a isosceles. The pseudometric, quasimetric, ultrametric can induce topology in a familar way.

Then We can forget metric in some way. (X, Ω) is a topological space if $\mathcal{T}$ is a topology on E. Where E is called as **uderlying set**. The sets in $\mathcal{T}$ are called **open**. If $\mathcal{T}$ can be form by taking union of families in some $\mathcal{B} \subset T$, we call $\mathcal{B}$ the **base** for the topology $\mathcal{T}$.

> $\mathcal{B}$ is a base in $(X, \mathcal{T})$ iff $\forall U \in \mathcal{T}, \forall x \in U, \exists W \in \mathcal{B} \ni x \in W \subset U$.

Proof $\Longrightarrow$: Any U can be written as $U = \cup W_i$ and $x \in U \implies x \in W_i$ for some i and $W_i \in \mathcal{B}$. $\Longleftarrow$: For any $U \in T$, consider arbitary $x \in U$, then there exist W_x such that $x \in W_x \subset U$, thus we have $U = \cup_x W_x$. ■

If $\cup\mathcal{B} = E$ and $\forall W_1, W_2 \in \mathcal{B}, \forall x \in W_1 \cap W_2, \exists W \in \mathcal{B} \ni x \in W \subset W_1 \cap W_2$. Then $\{$union of families of $\mathcal{B}\}$ is a topology and it's the unique topology with B as base.

Proof Let $T = \{$union of families of $\mathcal{B}\}$, then it's sufficient to show that $\mathcal{T}$ is a topology.

Note the families can be empty, $\mathcal{T}$ enjoy **1** and **2** clearly. To show it also satisfy **3**, suppose $U_1, U_2 \in \mathcal{T}$, for any point $x \in U_1 \cap U_2$, we may find some $x \in W_1 \subset U_1$ and $X \in W_2 \subset U_2$. By hypotheseis there exist $W_x \subset W_1 \cap W_2 \subset U_1 \cap U_2$ in B. Hence we may form $U_1 \cap U_2$ by $\cup_x W_x$, thus $U_1 \cap U_2 \in \mathcal{T}$. We skip the discussion of if U_1 or U_2 is empty since it's trival. ■

Let $\mathcal{S}$ be a class of subset in X, the define $\tau(\mathcal{S})$ as all topology contains $\mathcal{S}$. Let $T(S) = \cap\tau(\mathcal{S})$, then $T(\mathcal{S})$ is the smallest topology contains $\mathcal{S}$. We call it the topology **generated** by $\mathcal{S}$.

$T(\mathcal{S})$ is unions of families of finite intersections together with Ω

$$\{\bigcup(\bigcap_1^N S_i)\} \cup \Omega$$

A subset F is **closed** if $F^c \in \mathcal{T}$, it has parallel properties with open sets. Countable intersection of open sets is G_σ set and countable union of closed sets is F_δ set. A complement of a G_σ set is F_δ and vice versa.

A subset V is called a **neighborhood** of a if there exists a open set $U \subset V$ contains a. Then we called $V' = V - \{a\}$ **punctured(deleted)** neighborhood. A **neighborhood base** is a collection of neighborhood $BN(a)$ s.t. for any neighborhood V of a, there exist a $W \in BN(a)$ and $W \subset V$.

A subset U is open iff it's a neighborhood for each of its points.

Proof $\Longrightarrow$ is trival. $\Longleftarrow$ follows from $\cup_x G_x = U$ and unions of open set is still open. ■

This suggest a equivalent definition of finear topology:

$T' \subset T \iff$ T' neighborhood is a T neighborhood.

Proof $\Longrightarrow$ any open set G_x satisfy $x \in G_x \subset V$ in T' is still open in T, hence V is T neighborhood. $\Longleftarrow$ Consider any open set $G \in T'$, it's a T' neighborhood for each of its points implies it's a T neighborhood for each of its points and hence G is T open.

The **interior** of A is the union of all open sets which are included A, i.e., the largest open set included in A, we denote it A°. And the **closure** is the intersection of all closed sets which include A and thus the smallest closed set includes A, we denote it $\overline{A}$.

1. $(A \cap B)^\circ = A^\circ \cap B^\circ$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. $A \subset \overline{B} \implies \overline{A} \subset \overline{B}$
4. $A^\circ \subset B \implies A^\circ \subset B^\circ$
5. $\overline{A^c} = (A^\circ)^c$
6. $(\overline{A})^c = (A^c)^\circ$

Proof We only prove **5**, note $(A^\circ)^c$ is closed and

$$A^\circ \subset A \implies (A^c) \subset (A^\circ)^c$$

we have $\overline{A^c} \subset (A^\circ)^c$. On the other hand

$$\overline{A^c} \supset (A^\circ)^c \impliedby (\overline{A^c})^c \subset A^\circ \impliedby (\overline{A^c})^c \subset A \impliedby \overline{A^c} \supset A^c. ■$$

The **frontier** of A is $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ$.

$\overline{A} = A \cup \partial A$ and $A^\circ = A - \partial A$

Proof

$$
\begin{aligned}
A \cup \partial A &= A \cup (\overline{A} \cap \overline{A^c}) \\
&= (A \cup \overline{A}) \cap (A \cup \overline{A^c}) \\
&= (\overline{A}) \cap (A \cup (A^\circ)^c)
\end{aligned}
$$

note $A \cup (A^\circ)^c \supset A^\circ \cup (A^\circ)^c = \Omega$, $A \cup \partial A = \overline{A} \cap \Omega = \overline{A}$. And the $A^\circ = A - \partial A$ follows from substituting $\overline{A} = A \cup \partial A$. ■

x is said to be an **interior point** of A if A is neighborhood of x.

x is said to be an **adherent point** if it's every neighborhood meets A, an ω **accumulation point** of A if every neighborhood of x contains **infinitely** many points of A and is a **condensation point** of A if every neighborhood of x contains **uncountable** many points of A.

x is a **cluster point** or **accumulation point** if every deleted neighborhood of x meets A and is **isolated point** if x is not cluster point.

x is **frontier point** if every neighborhood of x meets both A and A^c.

The points of A° are precisely all the interior points of A and $\overline{A}$ are precisely all the adherent points.

Proof For interior points, consider I as all the interior points, it's sufficient to show that $I = A^\circ$

$$
I \subset \bigcup_{x \in I} G_x \subset A^\circ
$$

where G_x is the corresponding open set. On the other hand we have $A^\circ \subset I$ since every points in A° has A° as their neighborhood.

For interior points, suppose $x \in \overline{A}$ but is not an adherent point, then there is a open G contains x and $G \cap A = \emptyset$. Hence $A \subset G^c$, note G^c is closed and thus $G^c \supset \overline{A}$, which is contradict to $x \in \overline{A}$. On the other hand, suppose x is adherent but not in $\overline{A}$. Then $\overline{A}^c$ is a neghborhood of A and disjoint to $\overline{A}$, a contradition. ■.

By above theorem, we have

∂A is precisely points of frontier.

Proof By definition, point of frontier is both adherent point of A and A^c and thus all the points of frontier are

$$
\overline{A} \cap \overline{A^c} = \partial A
$$

For any subset X, define $\alpha(X) = (\overline{A})^\circ$, then

1. $X \subset Y \implies \alpha(X) \subset \alpha(Y)$
2. If X is open, $X \subset \alpha(X)$
3. $\alpha(\alpha(X)) = \alpha(X)$
4. If X and Y are disjoint open then $\alpha(X)$ and $\alpha(Y)$ are also.

If $\alpha(X) = X$, X is said to be **regular open**

Proof 2 follows from $X \subset \overline{X} \implies X \subset \alpha(X)$.

To estabish **3**, we show that A° is regular open when A is closed and $\overline{A}$ is regular open when A is open. When A is closed, $\partial A \subset A$, then

$$
\overline{A^\circ} = (A - \partial A) \cup \partial A = A \implies \alpha(A^\circ) = A^\circ
$$

Hence $\alpha(X) = (\overline{A})^\circ$ is regular open since $\overline{A}$ is closed.

For **4**, suppose there is $x \in \alpha(X) \cap \alpha(Y)$, then

$$\alpha(X) \cap \alpha(Y) = (\overline{X} \cap \overline{Y})^\circ \subset \overline{X} \cap \overline{Y}$$

hence x is adherent to both X and Y, note X is neighborhood of x and X meets Y by definition, a contradiction. ■

Finite intesection of regular open sets is regular open

Proof Let $(G_i)_{i \in I}$ be a finite family of regular open sets. We have

$$\bigcap_{i \in I} G_i \subset \alpha(\bigcap_{i \in I} G_i) \subset \alpha(G_i) = G_i$$

holds for all G_i, hence $\alpha(\bigcap_{i \in I} G_i) \subset \bigcap_{i \in I} G_i$, then the claim follows. ■

1. $\partial(\overline{A}) \subset \partial A$ and $\partial(A^\circ) \subset \partial A$
2. $\partial(A \cup B) \subset \partial A \cup \partial B$

Proof:

2: Suppose $x \in \partial(A \cup B)$, then any neighborhood N meet $A \cup B$ and $A^c \cap B^c$. W.L.O.G, we assume N meet A, since N also meet A^c, $x \in \partial A \subset \partial A \cup \partial B$. ■

A is said **dense** if $\overline{A} = \Omega$ and **nowhere dense** if $(\overline{A})^\circ = \emptyset$ ($\mathbb{Q}$ is dense in $\mathbb{R}$ while $\mathbb{Z}$ is nowhere dense.) A is said to be **meagre** or **set of the first category** if it's countable union of nowhere dense. Sets which are not meagre is **set of the second** category set.

Space $(\Omega, \mathcal{T})$ is **first countable** if every point of Ω has countable neighborhood base and is **second countable** if $\mathcal{T}$ has countable base. The space is said **separable** if Ω has a countable dense subset.

Second countable space is separable

Proof Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, by axiom of choice, we may take x_i in I, let $X = \{x_i\}_{i \in I} \subset \Omega$. Then we show that X is dense. For any $x \in \Omega$, it's neighborhood must contain some open G which is unions of $\mathcal{B}$ and thus contains at least one element in X, that is, G meet X. Hence $\overline{X} = \Omega$. ■

Second countable space is first countable

Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, for each point $x \in \Omega$, one may take all the sets in $\mathcal{B}$ which contains x as a neighborhood base. To verify it's neighborhood base, if there is a neighborhood N of x, then there is a open G contains x. By the definition of base, G is the union of sets of $\mathcal{B}$ and those sets must at least one contains x and these sets is subset to G. ■