

Set theory

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Set limitations

$$\begin{aligned}\{A_n, i.o.\} &= \limsup_n A_n = \lim_{k \rightarrow \infty} \bigcup_{n=k}^{\infty} A_n \\ \{A_n, ult.\} &= \liminf_n A_n = \lim_{k \rightarrow \infty} \bigcap_{n=k}^{\infty} A_n\end{aligned}$$

i.o. means elements in $\{A_n, i.o.\}$ is occurs in A_n infinitely often and ult. means it will always occurs ultimately. Hence we have

$$\liminf A_n \subset \limsup A_n$$

One sequence is converges to A iff

$$\liminf_n A_n = \limsup_n A_n = A$$

$$A_i \uparrow \implies A_n \rightarrow A = \bigcup_{k=1}^{\infty} A_k = \lim A_n$$

$$A_i \downarrow \implies A_n \rightarrow A = \bigcap_{k=1}^{\infty} A_k = \lim A_n$$

Topology

Let Ω be as space

Definition: A class of subset of Ω is an **topology** if

1. E and Ω belongs to T .
2. closed under countable union.
3. closed under finite intersection.

Let

Algebras

Let Ω be as space

Definition

Definition: A nonempty class of subset of Ω is an semi algebra if

1. Closed under inter: $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$
2. Complement can be written as finite disjoint union: $A \in \mathcal{S} \Rightarrow \exists A_i \in \mathcal{S}, A_i \cap A_j = \emptyset, i \neq j, \text{ s.t. } A^c = \sum_{i=1}^n A_i$

Definition: A nonempty class of subset of Ω is an algebra on Ω if

1. Closed under complement: $A^c \in \mathcal{A} \iff A \in \mathcal{A}$
2. Closed under finite union: $\cup_i A_i \in \mathcal{A} \iff \forall i, A_i \in \mathcal{A}$

Note that algebra is closed by finite union and we can prove that is Equivalent to it is closed by finite intersection

Definition: A nonempty class of subset of Ω is an σ algebra on Ω if

1. is an algebra
2. Closed under countable union.

Remark: \mathcal{A} is an algebra auto implies $\emptyset \in \mathcal{A}$ and $\Omega \in \mathcal{A}$. So $\{\emptyset, \Omega\}$ is the minimum algebra on Ω and thus minimum σ algebra.

Generate algebras

The pair (Ω, \mathcal{A}) is called a **measurable space**. The sets of \mathcal{A} are called **measurable sets**.

a semi-algebra \mathcal{S} can generate algebras by take all finite disjoint unions of sets, i.e

$$\overline{\mathcal{S}} = \{\text{finite disjoint unions of sets in } \mathcal{S}\} = \mathcal{A}(\mathcal{S})$$

is an algebra.

Generated classes

Let $\{\mathcal{A}_\gamma : \gamma \in \Gamma\}$ is a collection of σ algebra, then we have

$$\mathcal{A} = \cap_{\gamma \in \Gamma} \mathcal{A}_\gamma$$

is also a σ algebra. Hence we can define the smallest σ algebra as intersection of all σ algebras contains \mathcal{A} . That is

$$\forall \mathcal{A} \subset \mathcal{P}(\Omega), \quad \exists \sigma(\mathcal{A}) \quad \text{s.t.}$$

1. $\mathcal{A} \subset \sigma(\mathcal{A})$
2. $\forall \mathcal{A} \subset \mathcal{B} \in \sigma\text{-algebras} \implies \sigma(\mathcal{A}) \subset \mathcal{B}$
3. $\sigma(\mathcal{A})$ is unique.

We have

$$\sigma(\mathcal{S}) = \sigma(\bar{\mathcal{S}})$$

Which can be proved by show that

$$\begin{aligned}\mathcal{S} \subset \sigma(\bar{\mathcal{S}}) &\implies \sigma(\mathcal{S}) \subset \sigma(\bar{\mathcal{S}}) \\ \bar{\mathcal{S}} \subset \sigma(\mathcal{S}) &\implies \sigma(\bar{\mathcal{S}}) \subset \sigma(\mathcal{S})\end{aligned}$$

The smallest σ -algebra generated by the class of all open intervals on the real line $\mathcal{R} = (-\infty, \infty)$ is **Borel σ algebra**, denoted by \mathcal{B} i.e.

$$\begin{aligned}\mathcal{A} &= \{(a, b) : -\infty < a < b < \infty\} \\ \mathcal{B} &= \sigma(\mathcal{A})\end{aligned}$$

, whose elements are called **Borel sets**, $(\mathcal{R}, \mathcal{B})$ is called **Borel measurable space**

Monotone class

m-class is closed under monotone op.

If $A_{1:n} \in \mathcal{A}$ and $A_n \uparrow$ or \downarrow

$$\lim_{n \rightarrow \infty} A_n \in \mathcal{A}$$

π -**class** is closed under finite intersection

$$A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$$

λ - **class**

1. $\Omega \in \mathcal{A}$
2. closed under proper difference:

$$A - B \in \mathcal{A} \iff B \subset A \text{ and } B, A \in \mathcal{A}$$

3. is an m-class (cause $\Omega - A_i \downarrow$ whenever $A_i \uparrow$)

Relationships with σ algebra

$$\begin{aligned}\mathcal{A} \text{ is a } \sigma\text{-algebra} &\iff \mathcal{A} \text{ is a } m\text{-class and } \mathcal{A} \text{ is an algebra} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra} &\iff \mathcal{A} \text{ is a } \pi\text{-class \& } \mathcal{A} \text{ is a } \lambda\text{-class}\end{aligned}$$

Which can be proved as follows:

- \implies :
 1. $\Omega \in \mathcal{A}$
 2. $A - B = A \cap B^c \in \mathcal{A}$
 3. is an m -class
- \impliedby :
 1. $A^c = \Omega - A \in \mathcal{A}$
 2. $A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$
 3. hence \mathcal{A} is an algebra and \mathcal{A} is a m -class

Similarly, for m, π, λ -class, those properties also hold:

Let $\{\mathcal{A}_\gamma : \gamma \in \Gamma\}$ is a collection of m, π, λ -class then we have

$$\mathcal{A} = \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$$

is also a m, π, λ -class

$$\forall \mathcal{A} \subset \mathcal{P}(\Omega), \quad \exists m(\mathcal{A}) \quad s.t.$$

1. $\mathcal{A} \subset \sigma(\mathcal{A})$
2. $\forall \mathcal{A} \subset \mathcal{B} \in m\text{-classes} \quad m(\mathcal{A}) \subset \mathcal{B}$
3. $m(\mathcal{A})$ is unique.

similarly with $\lambda(\mathcal{A})$ and $\pi(\mathcal{A})$

$$\sigma \iff$$

The Monotone Class Theorem(MCT)

Let \mathcal{A} be an algebra, then

1. $m(\mathcal{A}) = \sigma(\mathcal{A})$
2. if \mathcal{B} is an m class and $\mathcal{A} \subset \mathcal{B}$, then $\sigma(\mathcal{A}) = m(\mathcal{A}) \subset \mathcal{B}$

Similarly, let \mathcal{A} be a π class, then $\lambda(\mathcal{A}) = \sigma(\mathcal{A})$

Then we have **Monotone class theorem**:

$$\forall \mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(\Omega), \quad s.t.:$$

1. If \mathcal{A} is a π -class, \mathcal{B} is a λ -class, then $\sigma(\mathcal{A}) \subset \mathcal{B}$
2. If \mathcal{A} is an algebra, \mathcal{B} is a m -class, then $\sigma(\mathcal{A}) \subset \mathcal{B}$

Product Spaces

Let $(\Omega_i, \mathcal{A}_i)$ be a measurable space.

n -dim **rectangles** of the product space of $\prod_{i=1}^n \Omega_i$

$$\prod_{i=1}^n A_i := A_1 \times \dots \times A_n = \{(\omega_1, \dots, \omega_n) : \omega_i \in A_i \subset \Omega_i, 1 \leq i \leq n\}$$

if $A_i \in \mathcal{A}_i$, they are **measurable rectangles**, let \mathcal{G} denote the class of all measurable rectangles of $\prod_{i=1}^n \Omega_i$, it's easy to check that \mathcal{G} is a π class.

n -dim **product σ -algebra**:

$$\prod_{i=1}^n \mathcal{A}_i = \sigma \left(\left\{ \prod_{i=1}^n A_i : A_i \in \mathcal{A}_i, 1 \leq i \leq n \right\} \right) = \sigma(\mathcal{G})$$

n -dim product **measurable space**:

$$\prod_{i=1}^n (\Omega_i, \mathcal{A}_i) = \left(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{A}_i \right)$$

Theorem Let \mathcal{A} be the set of all finite disjoint union of \mathcal{G} , then it's the smallest algebra contains \mathcal{G} .

Proof First show that \mathcal{A} is π class, since for element A_i in \mathcal{A} , it can be written as disjoint unions of subset E_{ij} of \mathcal{G} , i.e. $A_i = \bigcup_{j=1}^{n_i} E_{ij} \in \mathcal{A}$. Then

$$A_1 \cap A_2 = \left(\bigcup_{j=1}^{n_1} E_{1j} \right) \cap \left(\bigcup_{k=1}^{n_2} E_{2k} \right) = \bigcup \bigcup (E_{1j} \cap E_{2k}) \in \mathcal{A}$$

since \mathcal{G} is already a π class.

Then we show that \mathcal{A} is an algebra, which suffices to show that it's also closed under complements. Consider element of \mathcal{G} :

$$E = E_1 \times E_2 \times \dots \times E_n$$

we can show that E^c can be written as disjoint union $\cup D_{i:n}$. Then any $A \in \mathcal{A}$

$$A^c = \bigcap E_j^c = \bigcap \bigcup D_{ij} \stackrel{\text{disjoint}}{=} \bigcup \bigcap D_{ij} \in \mathcal{A}$$

Clearly $\mathcal{A}(\mathcal{G}) \subset \mathcal{A}$ and $\mathcal{A} \subset \mathcal{A}(\mathcal{G})$ and hence $\mathcal{A} = \mathcal{A}(\mathcal{G})$. ■

Corollary

$$\prod_{i=1}^n \mathcal{A}_i = \sigma(\mathcal{A}) = \sigma(\mathcal{G})$$

Proof $\mathcal{G} \subset \mathcal{A} \subset \sigma(\mathcal{A}) \implies \sigma(\mathcal{G}) \subset \sigma(\mathcal{A})$ and $\mathcal{A} = \mathcal{A}(\mathcal{G}) \subset \sigma(\mathcal{G}) \implies \sigma(\mathcal{A}) \subset \sigma(\mathcal{G})$ and hence $\sigma(\mathcal{A}) = \sigma(\mathcal{G})$. ■