Set theory

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Set limitations

$$\{A_n, i.o.\} = \limsup_n A_n = \lim_{k\to\infty} \cup_{n=k}^{\infty} A_n$$
$$\{A_n, ult.\} = \liminf_n A_n = \lim_{k\to\infty} \cap_{n=k}^{\infty} A_n$$

i.o. means elements in $\{A_n, i.o.\}$ is occurs in A_n infinitely often and ult. means it will always occures utimately. Hence we have

$$\liminf A_n \subset \limsup A_n$$

One sequence is converges to A iff

$$\liminf_n A_n = \limsup_n A_n = A$$

$$A_i \uparrow \implies A_n \to A = \cup_{k=1}^{\infty} A_k = \lim A_n$$
$$A_i \downarrow \implies A_n \to A = \cap_{k=1}^{\infty} A_k = \lim A_n$$

Algebras

Let Ω be as space

Definition

Definition: A nonempty class of subset of Ω is an semi algebra if

1. Closed under inter: $A, B \in \mathcal{S} \Rightarrow A \cap B \in S$
2. Complement can be written as finite disjoint union: $A \in S \implies \exists A_i \in \mathcal{S}, \quad A_i \cap A_j = \emptyset, i \neq j, \quad s.t. \quad A^c = \sum_{i=1}^{n} A_i$

Definition: A nonempty class of subset of Ω is an algebra on Ω if

1. Closed under complement: $A^c \in \mathcal{A} \iff A \in \mathcal{A}$
2. Closed under finite union: $\cup_i A_i \in \mathcal{A} \impliedby \forall i, A_i \in \mathcal{A}$

Note that algebra is closed by finite union and we can prove that is Equivalent to it is closed by finite intersection

Definition: A nonempty class of subset of Ω is an σ algebra on Ω if

1. is an algebra
2. Closed under countable union.

Remark: $\mathcal{A}$ is an algebra auto implies $\in \mathcal{A}$ and $\Omega \in \mathcal{A}$. So $\{, \Omega\}$ is the minimun algebra on Ω and thus minimun σ algebra.

Generate algebras

The pair $(\Omega, \mathcal{A})$ is called a **measurable space**. The sets of $\mathcal{A}$ are called **measurable sets**.

a semi-algebra $\mathcal{S}$ can generate algebras by take all finite disjoint unions of sets, i.e

$$\overline{\mathcal{S}} = \{\text{finite disjoint unions of sets in } \mathcal{S}\} = \mathcal{A}(\mathcal{S})$$

is an algebra.

Generated classes

Let $\{\mathcal{A}_\gamma : \gamma \in \Gamma\}$ is a collection of σ algebra, then we have

$$\mathcal{A} = \cap_{\gamma \in \Gamma} \mathcal{A}_\gamma$$

is also a σ algebra. Hence we can define the smallest σ algebra as intersection of all σ algebras contains $\mathcal{A}$. That is

$\forall \mathcal{A} \subset \mathcal{P}(\Omega), \quad \exists \sigma(\mathcal{A}) \quad s.t.$

1. $\quad \mathcal{A} \subset \sigma(\mathcal{A})$
2. $\forall \mathcal{A} \subset \mathcal{B} \in \sigma\text{-algebras} \implies \sigma(\mathcal{A}) \subset \mathcal{B}$
3. $\sigma(\mathcal{A})$ is unique.

We have

$$\sigma(\mathcal{S}) = \sigma(\overline{\mathcal{S}})$$

Which can be proved by show that

$$\begin{aligned} \mathcal{S} \subset \sigma(\bar{\mathcal{S}}) &\implies \sigma(\mathcal{S}) \subset \sigma(\bar{\mathcal{S}}) \\ \bar{\mathcal{S}} \subset \sigma(\mathcal{S}) &\implies \sigma(\bar{\mathcal{S}}) \subset \sigma(\mathcal{S}) \end{aligned}$$

The smallest $\sigma-$algebra generated by the class of all open intervals on the real line $\mathcal{R} = (-\infty, \infty)$ is **Borel σ algebra**, denoted by $\mathcal{B}$ i.e.

$$\mathcal{A} = \{(a, b) : -\infty < a < b < \infty\}$$
$$\mathcal{B} = \sigma(\mathcal{A})$$

, whose elements are called **Borel sets**, $(\mathcal{R}, \mathcal{B})$ is called **Borel measurable space**

Monotone class

m-class is closed under monotone op.

If $A_{1:n} \in \mathcal{A}$ and $A_n \uparrow or \downarrow$

$$\lim_{n\to\infty} A_n \in \mathcal{A}$$

$\pi-$**class** is closed under finite intersection

$$A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$$

$\lambda-$ **class**

1. $\Omega \in \mathcal{A}$
2. closed under proper difference:

$$A - B \in \mathcal{A} \impliedby B \subset A \text{ and } B, A \in \mathcal{A}$$

3. is an m-class (cause $\Omega - A_i \downarrow$ whenever $A_i \uparrow$)

Relationships with σ algebra

$$\mathcal{A} \text{ is a } \sigma- \text{ algebra} \iff \mathcal{A} \text{ is a } m\text{-class and } \mathcal{A} \text{ is an algebra}$$
$$\mathcal{A} \text{ is a } \sigma- \text{ algebra} \iff \mathcal{A} \text{ is a } \pi- \text{ class } \;\&\; \mathcal{A} \text{ is a } \lambda- \text{ class}$$

Which can be proved as follows:

- $\implies$:
  1. $\Omega \in \mathcal{A}$
  2. $A - B = A \cap B^c \in \mathcal{A}$
  3. is an m-class
- $\impliedby$:
  1. $A^c = \Omega - A \in \mathcal{A}$
  2. $A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$
  3. hence $\mathcal{A}$ is an algebra and $\mathcal{A}$ is a m-class

Similarly, for m, π, λ -class, those properties also hold:

Let $\{\mathcal{A}_\gamma : \gamma \in \Gamma\}$ is a collection of m, π, λ -class then we have

$$\mathcal{A} = \cap_{\gamma \in \Gamma} \mathcal{A}_\gamma$$

is also a m, π, λ -class

$\forall \mathcal{A} \subset \mathcal{P}(\Omega), \quad \exists m(\mathcal{A}) \quad s.t.$

1. $\quad \mathcal{A} \subset \sigma(\mathcal{A})$
2. $\forall \mathcal{A} \subset \mathcal{B} \in$ m-classes $\quad m(\mathcal{A}) \subset \mathcal{B}$
3. $m(\mathcal{A})$ is unique.

similarly with $\lambda(\mathcal{A})$ and $\pi(\mathcal{A})$

$$\sigma \iff$$

The Monotone Class Theorem(MCT)

Let $\mathcal{A}$ be an algebra, then

1. $m(\mathcal{A}) = \sigma(\mathcal{A})$
2. if $\mathcal{B}$ is an m class and $\mathcal{A} \subset \mathcal{B}$, then $\sigma(\mathcal{A}) = m(\mathcal{A}) \subset \mathcal{B}$

Similarly, let $\mathcal{A}$ be a π class, then $\lambda(\mathcal{A}) = \sigma(\mathcal{A})$

Then we have **Monotone class theorem**:

$\forall \mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(\Omega), s.t.$:

1. If $\mathcal{A}$ is a π-class, $\mathcal{B}$ is a $\lambda-$class, then $\sigma(\mathcal{A}) \subset \mathcal{B}$
2. If $\mathcal{A}$ is an algebra, $\mathcal{B}$ is a $m-$class, then $\sigma(\mathcal{A}) \subset \mathcal{B}$

Product Spaces

Let $(\Omega_i, \mathcal{A}_\rangle)$ be a measurable space.

n-dim **rectangles** of the product space of $\prod_{i=1}^n \Omega_i$

$$\prod_{i=1}^n A_i := A_1 \times \ldots \times A_n = \{(\omega_1, \ldots, \omega_n) : \omega_i \in A_i \subset \Omega_i, 1 \leq i \leq n\}$$

if $A_i \in \mathcal{A}_\rangle$, they are **measurable rectangles**, let $\mathcal{G}$ denote the class of all measurable rectangles of $\prod_{i=1}^n \Omega_i$, it's easy to check that $\mathcal{G}$ is a π class.

n-dim **product σ-algebra**:

$$\prod_{i=1}^{n} \mathcal{A}_i = \sigma\left(\left\{\prod_{i=1}^{n} A_i : A_i \in \mathcal{A}_i, 1 \leq i \leq n\right\}\right) = \sigma(\mathcal{G})$$

n-dim product **measurable space**:

$$\prod_{i=1}^{n} (\Omega_i, \mathcal{A}_i) = \left(\prod_{i=1}^{n} \Omega_i, \prod_{i=1}^{n} \mathcal{A}_i\right)$$

Theorem Let $\mathcal{A}$ be the set of all finite disjoint union of $\mathcal{G}$, then it's the smallest algebra contains $\mathcal{G}$.

Proof First show that $\mathcal{A}$ is π class, since for element A_i in $\mathcal{A}$, it can be written as disjoint unions of subset E_{ij} of $\mathcal{G}$, i.e. $A_i = \bigcup_{j=1}^{n_i} E_{ij} \in \mathcal{A}$. Then

$$A_1 \cap A_2 = (\bigcup_{j=1}^{n_1} E_{1j}) \cap (\bigcup_{k=1}^{n_2} E_{2k}) = \bigcup\bigcup(E_{ij} \cap E_{2k}) \in \mathcal{A}$$

since $\mathcal{G}$ is already a π class.

Then we show that $\mathcal{A}$ is an algebra, which is suffices to show that it's also closed under complements. Consider element of $\mathcal{G}$:

$$E = E_1 \times E_2 \times \cdots \times E_n$$

we can showt that E^c can be writtena as disjoint union $\cup D_{i:n}$ Then any $A \in \mathcal{A}$

$$A^c = \bigcap E_j^c = \bigcap\bigcup D_{ij} \overset{disjoint}{=\!=\!=} \bigcup\bigcap D_{ij} \in \mathcal{A}$$

Clearly $\mathcal{A}(\mathcal{G}) \subset \mathcal{A}$ and $\mathcal{A} \subset \mathcal{A}(\mathcal{G})$ and hence $\mathcal{A} = \mathcal{A}(\mathcal{G})$. $\blacksquare$

Corollary

$$\prod_{i=1}^{n} \mathcal{A}_i = \sigma(\mathcal{A}) = \sigma(\mathcal{G})$$

Proof $\mathcal{G} \subset \mathcal{A} \subset \sigma(\mathcal{A}) \implies \sigma(\mathcal{G}) \subset \sigma(\mathcal{A})$ and $\mathcal{A} = \mathcal{A}(\mathcal{G}) \subset \sigma(\mathcal{G}) \implies \sigma(\mathcal{A}) \subset \sigma(\mathcal{G})$ and hence $\sigma(\mathcal{A}) = \sigma(\mathcal{G})$. $\blacksquare$