Financial Stochastic Analysis

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Chapter 1

Brownian Motion

Brownian motion at time t is limit of infinite fast random walk W_t^n, it can be equivalently characterized by

1. Any increment $W_{t_1} - W_{t_2}$ is normal distributed with mean 0 and variance $t_1 - t_2$. Disjoint increment are independency.
2. For any time $t_1, t_2, \dots, t_m$, $\mathbf{W} = (W_{t_1}, W_{t_2}, \dots, W_{t_m})$ is normal distributed with zero mean and covariance
$$\mathbf{\Sigma} = \begin{bmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & \dots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \dots & t_m \end{bmatrix}$$
3. $\mathbf{W}$ has MGF
$$\varphi(\mathbf{t}) = \exp\left\{ \sum_{i=1}^{m} \frac{1}{2} \left(\sum_{j=i}^{m} t_j \right)^2 (t_i - t_{i-1}) \right\}$$
where $t_0 = 0$.

Brownian motion is a Markov martingale with variation:

1. $dW_t = \infty$
2. $dW_t dW_t = dt$
3. $dW_t dt = dt^2 = 0$

1.1 Markov Property

Lemma 1.1 (Independence Lemma). *Suppose $X \in \mathcal{A}$, $Y \perp \mathcal{A}$, then*

$$\mathop{\mathbb{E}}_{\mathcal{A}} f(X,Y) = \mathbb{E}\, f(x,Y)|_{x=X}$$

Proof. When $f = g \times h$ for some g, h, then

$$\mathop{\mathbb{E}}_{\mathcal{A}} f(X,Y) = \int K(X,dy) f(X,y) = \int \mu(dy) f(X,y) = \mathbb{E}\, f(x,Y)|_{x=X}$$

since product σ algebra is generated by measurable rectangles, monotone class theorem completes the proof. □

Preceding lemma implies

$$\begin{aligned}
\mathop{\mathbb{E}}_{s} f(W_t) &= \mathop{\mathbb{E}}_{s} f(W_t - W_s + W_s) \\
&= \mathbb{E}\, f(W_t - W_s + x)|_{x=W_s} \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} f(w+x) \exp\left\{-\frac{w^2}{2(t-s)}\right\} dw|_{x=W_s} \\
&\xlongequal{\tau=t-s,y=w+x} \frac{1}{\sqrt{2\pi\tau}} \int_{\mathbb{R}} f(y) \exp\left\{-\frac{w^2}{2\tau}\right\} dw|_{x=W_s} \\
&= \int_{\mathbb{R}} f(y) p(\tau, W_s, y) dy
\end{aligned}$$

where $p(\tau, W_s, y)$ is pdf of $\mathcal{N}(W_s, \tau)$.

1.2 Exponential Martingale

Proposition 1.1. *Suppose W_t is a Brownian Motion with filtration $\mathbb{F}$, then process*

$$Z_t = \exp\left\{\sigma W_t - \frac{1}{2}\sigma^2 t\right\}$$

is a martingale.

Define the first passage time to m as

$$\tau_m = \min\{t \geq 0, W_t = m\}$$

recall the stopped martingale, we have

$$1 = \mathbb{E}\, Z_0 = \mathbb{E}\, Z_{t\wedge\tau_m} = \mathbb{E} \exp\left\{\sigma W_{t\wedge\tau_m} - \frac{1}{2}\sigma^2(t\wedge m)\right\}$$

Taking limit inside expectations:

$$\lim_{t\to\infty} \exp\left\{\sigma W_{t\wedge\tau_m} - \frac{1}{2}\sigma^2(t\wedge m)\right\} = \mathbf{1}_{\tau_m<\infty} \exp\left\{\sigma m - \frac{1}{2}\sigma^2\tau_m\right\}$$

that implies

$$\mathbb{E}\,\mathbf{1}_{\tau_m<\infty} \exp\left\{-\frac{1}{2}\sigma^2\tau_m\right\} = \exp\{-\sigma m\}$$

take $\sigma \searrow 0$, we have τ_m is finite $a.s.$.

And the characteristic function of τ_m is given by taking $t = \frac{1}{2}\sigma^2$:

$$\mathbb{E} \exp\{-t\tau_m\} = \exp\left\{-|m|\sqrt{2t}\right\}$$

1.3 Reflection

By the symmetry of Brownian motion, we have

$$\mathbb{P}\{\tau_m \le t, W_t \le w\} = \mathbb{P}\{W_t \ge 2m - w\}$$

when $0 < m \ge w$. On the other hand:

$$\mathbb{P}\{\tau_m \le t, W_t \ge w\} = \mathbb{P}\{W_t \ge w\}$$

take $m = w$ and adding these two:

$$\mathbb{P}\{\tau_m \le t\} = 2\,\mathbb{P}\{W_t \ge m\}$$

1.3.1 Joint Distribution of Brownian Motion and its maximum

Define maximum process:

$$M_t = \max_{0\le s\le t} W_s$$

clearly, $M_t \ge m$ iff $\tau_m \le t$, thus

$$\mathbb{P}\{M_t \ge m, W_t \le w\} = \mathbb{P}\{W_t \ge 2m - w\}$$

from which we have:

$$f_{M,W}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-\omega)^2}{2t}}$$

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