High Dimension Probability

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Contents

cvbn

Proposition 0.0.1 (tails of normal distribution)**.** *Let $g \sim \mathcal{N}(0,1)$, then for $t > 0$, we have:*

$$\left(\frac{1}{t} - \frac{1}{t^3}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq \mathbb{P}(g \geq t) \leq \frac{1}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Proof.

$$\begin{aligned}
\mathbb{P}\{g \geq t\} &= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} e^{-ty} e^{-y^2/2} dy \text{ with changing } x = t + y \\
&\leq \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \int_0^\infty e^{-ty} dy \text{ since } e^{-y^2/2} \leq 1 \\
&= \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \left(\frac{1}{t}\right)
\end{aligned}$$

□

Theorem 0.0.1 (Hoeffding's inequality)**.** *Let $X_1, \dots, X_N$ be independent symmetric Bernoulli r.v. and $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ then for any $t \geq 0$ we have:*

$$\mathbb{P}\left\{\sum_{i=1}^N a_i X_i \geq t\right\} \leq \exp\left(-\frac{t^2}{2\|a\|_2^2}\right)$$

Proof. Suppose that $\|a\|_2 = 1$.

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^N a_i X_i \geq t\right\} &= \mathbb{P}\left\{\exp\left(\lambda \sum_{i=1}^N a_i X_i\right) \geq \exp(\lambda t)\right\} \\
&\leq e^{-\lambda t} \mathbb{E} \exp\left(\lambda \sum_{i=1}^N a_i X_i\right)
\end{aligned}$$

Now consider $\mathbb{E} \exp\left(\lambda \sum_{i=1}^N a_i X_i\right)$, from the independency, we find

$$\mathbb{E} \exp\left(\lambda \sum_{i=1}^N a_i X_i\right) = \prod_{i=1}^N \mathbb{E} \exp(\lambda a_i X_i)$$

Since the property of symmetric Bernoulli distribution, notice that for some fixed i,

$$\mathbb{E}\exp(\lambda a_i X_i) = \frac{\exp(\lambda a_i) + \exp(-\lambda a_i)}{2} = \cosh(\lambda a_i)$$

By the Taylor series, we believe that

$$\cosh(x) \leq \exp(\frac{x^2}{2}) \text{ for all } x \in \mathbb{R}$$

so

$$\mathbb{E}\exp(\lambda a_i X_i) \leq \exp\left(\frac{\lambda^2 a_i^2}{2}\right)$$

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^N a_i X_i \geq t\right\} \leq e^{-\lambda t}\prod_{i=1}^N \exp(\lambda^2 a_i^2) &= \exp\left(-\lambda t + \frac{\lambda^2}{2}\sum_{i=1}^N a_i^2\right) \\ &= \exp\left(-\lambda t + \frac{\lambda^2}{2}\right)\end{aligned}$$

Now pick $\lambda = t$, $f(\lambda) = -\lambda t + \frac{\lambda^2}{2}$ get the minimum, then

$$\mathbb{P}\left\{\sum_{i=1}^N a_i X_i \geq t\right\} \leq \exp\left(\frac{t^2}{2}\right)$$

$\square$