Littlewood's three principles

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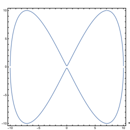{height=10%}

Littlewood's three principles

Littlewood's three principles are as follows:

1. Every measurable set is nearly a finite sum of intervals.
2. Every integrable function is nearly continuous
3. Every pointwise convergent sequence of functions is nearly uniformly.

Littlewood's first principles

Suppose $E \subset \mathbb{R}$ and $m(E) < \infty$, then E is almost finite open interval sum, formally, there exist finite class of open interval I_i, s.t., $m(\cup I_i \Delta E) = m(\cup I_i - E) + m(E - \cup I_i) < \epsilon$ for any ϵ.

Proof Suppose

$$I_1 = B(0,1), I_2 = B(0,2) - B(0,1), I_n = B(0,n) - B(0,n-1), \cdots$$

then $\cup_1^\infty I_i = \mathbb{R}$ and thus

$$\begin{aligned} m(E) &= m(E \cap \mathbb{R}) \\ &= m(E \cap (\cup_1^\infty I_i)) \\ &= m(\cup_1^\infty (E \cap I_i)) \\ &= \sum_1^\infty m(E \cap I_i) \end{aligned}$$

since $m(E)$ is finite, $\sum_1^\infty m(E \cap I_i)$ is converge, thus we may find N_ϵ s.t. for all $n > N_\epsilon$, $m(E) - \sum_1^n m(E \cap I_i) < \epsilon$, noting every $\sum_1^n m(E \cap I_i)$ correspond to a bounded measurable subset of E, we have show that E is almost bounded since there exist $A \subset E$ s.t. $m(E - A) < \epsilon$.

For any measurable A, we may find closed $F \subset A$ s.t. $m(A - F) < \epsilon$, i.e., A is almost closed and thus E is almost compact. Then the results is immediately by the Heine-Borel theorem. ■

Littlewood's second principle

> Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable, then $\forall \epsilon > 0$, there exists
>
> 1. An integrable simple funtion g s.t.
> 2. An integrable step funtion g s.t.
> 3. A continuous g with compact support s.t.
>
> $$\int |f - g| < \epsilon$$

Proof 1 is immediately by the definition of integral of non-negative function then spliting measurable $f = f^+ - f^-$ and adjusting ϵ.

2 is sufficient to show this when f is simple and again sufficient to show this when f is indicator fuction of a finite measure set(since integrable), say, $f = \chi_E$. By littlewood first principle, E is nearly finite sum of open interval where we may define the step function: $g = \chi_{\cup I_i}$. Then the claim follows from

$$\int \chi_E - \int g = m(E) - m(\cup I_i) < m(E - \cup I_i) < m(E \Delta \cup I_i) < \epsilon$$

3 is sufficient to show when f is indicator on a interval E, then one may define

$$g = \max(1 - \delta d(x, E), 0)$$

where g is continous and $g = 1 - \delta d(x, E)$ at $A_\delta = \{x : 0 < d(x, E) < \frac{1}{\delta}\}$ and $g = \chi_E$ otherwise.

$$\int g - \int f = \int_{A_\delta} g + (\int_{A_\delta^c} g - \int f) = \int_{A_\delta} g \leq m(A_\delta)$$

Note we may choose δ s.t. $m(A_\delta) < \epsilon$(since $\lim_{\delta \to \infty} A_\delta = \emptyset$) and g has compact support $\overline{A_\delta + E}$. ■

Lusin's theorem

Another version of littlewood's second principle is known as **Lusin's throrem**.

> Let f be integrable and $\epsilon > 0$, there exist a E s.t. the restriction of f_E is continous and $m(R^d - E) < \epsilon$.

Proof By the littlewood's second principle, there is a series of continous function f_n with compact support s.t.

$$\int |f - f_n| \leq \frac{\epsilon}{4^n}$$

Let $A_n = \{|f(x) - f_n(x)| > \frac{1}{2^n}\}$, by markov inequality:

$$\frac{\epsilon}{4^n} \geq \int |f - f_n| \geq m(A_n)/2^n \implies m(A_n) \leq \frac{\epsilon}{2^n}$$

Note $|f(x) - f_n(x)| \leq \frac{1}{2^n}$ outside of A_n and $m(A := \cup_1^\infty A_n) \leq 1$, we conclude that $f \to f_n$ almost uniformly (by ignoring an arbitary small set A). Then the claim follows from uniformly limit of continous function is continuous. ■

Remark By the the inner regularity of measurable set, the restriction may be compact. Then by **Tietze theorem**

> There exist a extend g of function f from any closed subset C to Ω, s.t. g is continous on Ω and $g|_c = f$

There exist a continous g agree with f outside an arbitary set and g is bounded by M if so does f.

Littlewood like principle

> Absolutely integrable function almost support on bounded set. Formally, let f integrable and $\epsilon > 0$, there exist a ball $B(0, R)$ s.t.
> $$\int_{B^c(0,R)} |f| \leq \epsilon$$

Proof Note

$$\int |f| = \int_{B(0,R)} |f| + \int_{B^c(0,R)} |f| = \int |f\chi_{B(0,R)}| + \int_{B^c(0,R)} |f|$$

and $g_n = |f\chi_{B(0,R)}|$ is increasing to $g = |f|$ and by MCT

$$\lim_{R \to \infty} \int |f\chi_{B(0,R)}| = \int |f|$$

hence the claim follows for suffictently large R. ■

> Measurable function almost locally bounded. Let f support on a finite measure set E and $\epsilon > 0$, there exist a measurable subset A s.t. $m(A) \leq \epsilon$ and f is locally bounded outside A. That is, for every $R > 0$, there exist $M < \infty$ s.t. $|f| \leq M$ for all $x \in B(0, R) - A$

Proof A measurable function is nearly continuous by lusin's theorem and finite measurable set is almost compact. Then this claim follows from continuous function attain maxima on a compact set.

Remark If f is integrable, then f is bounded outside A.

Lusin's theorem is equivalent to Littlewood's second principle. To estabish this, note integrable function is almost bounded. Then it's sufficient to show when f is bounded. Suppose $|f| \leq M$, by the remark above, there exist continous g and $g = f$ outside an arbitary set A, then

$$\int |g - f| = \int_A |g - f| \leq 2Mm(A) \leq 2M\epsilon$$

note integrable functions almost have bounded support, we may restric g again and finished. ■

Littlewood's third principle

Recall that a sequence f_n may converge to f:

1. (**Pointwise convergence**) $f_n \to f$ everywhere.
2. (**Pointwise a.e.**) $f_n \to f$ a.e.
3. (**Uniformly convergence**) $\forall \epsilon > 0, x \in \mathbb{R}$, there exist N s.t. $d(f_n(x), f(x)) < \epsilon$ forall $n \geq N$.

Then we introduce **locally** uniform convergence. A sequence f_n converge locally uniformly to f if it's converges uniformly in every bounded subset $E \in \mathbb{R}$. By the compactness of reals, we have

> $f_n \to f$ locally uniformly iff $\forall x \in \mathbb{R}$, there exist a open neighborhood G s.t. $f_n \to f$ uniformly in G.

Proof $\Longrightarrow$ is immediately by taking $B(x, R)$ for any R. For $\Longleftarrow$, consider any bounded E, by definition, there exist some $B(0, R) \supset E$, then we take its closure $\overline{B(0,R)}$, which is still bounded and thus compact. For any point $x \in \overline{B(0,R)}$, we may find an open set G_x contains x and on which $f_n \to f$ uniformly. Then we have a open cover:

$$\bigcup_{x \in \overline{B(0,R)}} G_x \supset \overline{B(0,R)}$$

By Heine-Borel, we may take some finite $I = \{x_1, x_2, \cdots, x_n\}$, and

$$\bigcup_{x \in I} G_x \supset \overline{B(0,R)}$$

Note that if $f_n \to f$ uniformly for every G_x, it's also converges uniformly in their finite union. Then the results follows from $E \subset \overline{B(0,R)}$. ■

One can recover local uniform by ignoring a small set:

> **(Egorov's theorem)** Suppose $f_n \to f$ a.e., for any m $\epsilon > 0$, there exist a A of measure at most ϵ s.t. $f_n \to f$ locally uniformly outside A.

Proof Since We may take the zero measure set into A, we may assume $f_n \to f$ everywhere. Consider

$$E_{N,m} = \{x : |f_n(x) - f(x)| > 1/m \text{ for some } n \geq N\}$$

It's clearly decreasing with N and

$$\bigcap_{N=0}^{\infty} E_{N,m} = \emptyset$$

thus we have

$$\lim_{N \to \infty} m(E_{N,m} \cap B(0,R)) = 0$$

By the definition of limit, we may find N_m s.t for all $N \geq N_m$.

$$m(E_{N,m} \cap B(0,m)) \leq \frac{\epsilon}{2^m}$$

Then let

$$A = \bigcup_{m=1}^{\infty} E_{N_m,m} \cap B(0,m)$$

Where $m(A) \leq \epsilon$. Note if $x \in B(0,m) - E^c_{N,m}$, $|f_n(x) - f(x)| \leq 1/m$. For any $m_0 \in \mathbb{N}^+$

$$\begin{aligned}
B(0,m_0) - A = B(0,m_0) \cap A^c \\
&= \bigcap_{m=1}^{\infty} (E^c_{N_m,m} \cup B^c(0,m)) \cap B(0,m_0) \\
&= \bigcap_{m=1}^{\infty} (E^c_{N_m,m} \cap B(0,m_0)) \cup (B^c(0,m) \cap B(0,m_0))
\end{aligned}$$

When $m \geq m_0$, $B^c(0,m) \cap B(0,m_0) = \emptyset$. Hence we can always find $\frac{1}{m} < \varepsilon$ and for $n \geq N_m$, $|f_n(x) - f(x)| \leq \varepsilon$ for any $x \in B(0,m_0) - A$. Note every bounded subset $E \subset B(0,m)$ for some m and hence $f_n \to f$ locally uniformly. ■

Remark If all f_n and f support on a fixed E with finite measure, then $f_n \to f$ uniformly not only locally from the above argument.(Since finite masure set is almost compact and thus there exist $B(0,m) \supset E$)