Notes of Probability and Stochastics

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Contents

Chapter 1

Measure and integrations

1.1 Measurable space

1.1.1 σ algebra

Definition 1.1. A nonempty system of subset of Ω is an algebra on Ω if it's

1. Closed under complement: $A^c \in \mathcal{A} \iff A \in \mathcal{A}$
2. Closed under finite union: $\cup_i A_i \in \mathcal{A} \impliedby \forall i, A_i \in \mathcal{A}$

it's an σ algebra on Ω if it's also closed under countable union.

Remark. $\mathcal{A}$ is an algebra auto implies $\varnothing \in \mathcal{A}$ and $\Omega \in \mathcal{A}$. So $\{\varnothing, \Omega\}$ is the minimum algebra on Ω and thus minimum σ algebra while the discrete algebra 2^Ω is maximum.

Let $\left\{\mathcal{A}_\gamma : \gamma \in \Gamma\right\}$ is a collection of σ algebra, then we have

$$\mathcal{A} = \cap_{\gamma\in\Gamma}\mathcal{A}_\gamma$$

is also a σ algebra. Hence we can define the smallest σ algebra as intersection of all σ algebras contains $\mathcal{A}$, that called the σ algebra **generated** by $\mathcal{A}$ and denoted by $\sigma(\mathcal{A})$.

The smallest $\sigma-$algebra generated by the system of all open sets in a topological space (Ω, τ) is called **Borel σ algebra** on Ω and denoted by $\mathcal{B}(\Omega)$, its elements are called **Borel sets**.

1.1.2 π, λ, m systems

A collection of subsets $\mathcal{A}$ is called

1. **m-system** if closed under monotone series, that is if $(A_n) \subset \mathcal{A}$ and $A_n \nearrow A$, then $A \in \mathcal{A}$.
2. $\pi-$**system** is closed under finite intersection
3. $\lambda-$ **system** if
    1. $\Omega \in \mathcal{A}$
    2. closed under proper difference:
    $$A - B \in \mathcal{A} \impliedby B \subset A \text{ and } B, A \in \mathcal{A}$$
    3. is an m-system (cause $\Omega - A_i \downarrow$ whenever $A_i \uparrow$)

1.1.3 Relationships with σ algebra

$$\mathcal{A} \text{ is a } \sigma- \text{ algebra } \iff \mathcal{A} \text{ is a } m\text{-system and } \mathcal{A} \text{ is an algebra}$$
$$\mathcal{A} \text{ is a } \sigma- \text{ algebra } \iff \mathcal{A} \text{ is a } \pi- \text{ system } \And \mathcal{A} \text{ is a } \lambda- \text{ system}$$

Which can be proved as follows:

- $\implies$:
    1. $\Omega \in \mathcal{A}$
    2. $A - B = A \cap B^c \in \mathcal{A}$
    3. is an m-system
- $\impliedby$:
    1. $A^c = \Omega - A \in \mathcal{A}$
    2. $A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$
    3. hence $\mathcal{A}$ is an algebra and $\mathcal{A}$ is a m-system

Similarly, for m, π, λ -system, those properties also hold:

Let $\left\{\mathcal{A}_\gamma : \gamma \in \Gamma\right\}$ is a collection of m, π, λ -system then we have

$$\mathcal{A} = \cap_{\gamma\in\Gamma}\mathcal{A}_\gamma$$

is also a m, π, λ -system

$\forall \mathcal{A} \subset \mathcal{P}(\Omega), \quad \exists m(\mathcal{A}) \quad s.t.$

1. $\mathcal{A} \subset \sigma(\mathcal{A})$
2. $\forall \mathcal{A} \subset \mathcal{B} \in$ m-systemes $\quad m(\mathcal{A}) \subset \mathcal{B}$
3. $m(\mathcal{A})$ is unique.

similarly with $\lambda(\mathcal{A})$ and $\pi(\mathcal{A})$

$$\sigma \iff$$

::: {.theorem name="Simple Approximation Theorem"}

Give a function:

$$d_n(r) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(r) + n1_{[n,\infty]}(r), \qquad r \in \overline{\mathbb{R}}_+$$

Notice that $d_n(r)$ is an increasing right-continuous simple function and $d_n(r) \nearrow r$ as $n \to \infty$.

A positive function f on E is $\mathcal{A}$-measurable iff it is the limit of an increasing sequence of positive simple functions.

:::

Proof. When a lim sup and a lim inf exists for a simple function sequence, then $\lim f_n$ exists and as for $n \in \mathbb{N}_+$, f_n is $\mathcal{A}$-measurable, $\lim f_n$ is $\mathcal{A}$-measurable.
For the converse, let $f_n = d_n \circ f$ is. Then $f_n \nearrow f$ and f_n is a simple function.

□

Definition 1.2 (Integrations).

- $\int f d\mu$ is called the integrations of f according to measure μ, usually noted as μf.
- $\int_A f d\mu$ is called the integrations of f according to μ on $\mathcal{A}$-measurable set A, noted as $\mu(f1_A)$.

Theorem 1.1 (Monotone Convergence Theorem). *Let (f_n) be an increasing sequence in $\mathcal{A}_+$, then:*

$$\mu(\lim f_n) = \lim \mu f_n$$

Proof. Let $f_n \nearrow f$, then $\mu f \geq \mu f_n$ for each n, so $\mu f \geq \lim \mu f_n$.
Now need to prove that $\lim \mu f_n \geq \mu f$.
Fix $b \in \mathbb{R}_+$ and $B \in \mathcal{A}$, where f is $\mathcal{A}$-measurable. Then $\{f_n > b\} \nearrow \{f > b\}$, where let $B_n = B \cap \{f_n > b\}$, then $B_n \nearrow B$, so

$$\lim \mu(B_n) = \mu(B)$$

and:

$$f_n 1_B \geq f_n 1_{B_n} \geq b 1_{B_n}$$

then

$$\mu(f_n 1_B) \geq \mu(b 1_{B_n}) = b\mu(1_{B_n})$$

Take limits for each side:

$$\lim \mu(f_n 1_B) \geq b\mu(B)$$

Let g be a positive simple function s.t. $f \geq g$ holds on Ω. Let $g = \sum_{i=1}^m b_i 1_{B_i}$ where on each B_i, $f \geq b_i$ holds. Then:

$$\lim \mu(f_n 1_{B_i}) \geq b_i \mu(B_i) \qquad i = 1, 2, \dots, m$$

Let g has the canonical representation, then

$$\lim_n \mu f_n = \lim_n \sum_{i=1}^m \mu(f_n 1_{B_i}) = \sum_{i=1}^m \lim_n \mu(f_n 1_{B_i}) \geq \mu g$$

Let $g = d_k \circ f$ and change g to its canonical representation, then

$$\lim_n \mu f_n \geq \mu(d_k \circ f)$$

holds for every k, then let $k \to \infty$, we get $\lim \mu f_n \geq \mu f$. Now we prove the theorem.

□

Theorem 1.2 (Fatou's Lamma)**.** *Let* $(f_n) \in \mathcal{A}_+$*, then*

$$\mu(\liminf f_n) \leq \liminf \mu f_n$$

Proof. Let $g_m = \inf_{n \geq m} f_n$, notice that g_m is increasing and $g_m \nearrow \liminf f_n$ as $m \to \infty$. By the monotone converge theorem:

$$\mu(\liminf f_n) = \lim \mu g_m$$

Notice that $g_m \leq f_n$ for all $n \geq m$, and $\mu g_m \leq \mu f_n$ for all $n \geq m$, so $\mu g_m \leq \inf_{n \geq m} \mu f_n$, take limit of m,

$$\lim \mu g_m \leq \liminf \mu f_n$$

□

Theorem 1.3 (Almost Everywhere Theorem)**.** *Let* (f_n) *be a sequence of numerical functions on* Ω*. Suppose that for each* n*, there is* $g_n \in \mathcal{A}$ *s.t.* $f_n = g_n$ *a.e. Further suppose for each* n *that* $f_n \geq 0$ *a.e. and* $f_n \leq f_{n+1}$ *a.e. Then,* $\lim f_n$ *exists a.e. is positive a.e. and* $\mu(\lim f_n) = \lim \mu f_n$*.*

Proof. Let

$$N = \bigcup_{n=1}^{\infty} (L_n \cup M_n \cup N_n)$$

where L denotes the $f_n \leq f_{n+1}$'s zero-measure set, and N denote the $f_n = g_n$'s and M denotes $f_n \geq 0$'s.
For every $x \in \Omega - N$, we have

$$0 \leq f_1(x) = g_1(x) \leq f_2(x) = g_2(x) \, ...$$

so $\lim f_n = \lim g_n$ exists on $\Omega - N$. Then define:

$$f(x) = \begin{cases} \lim f_n(x) & x \notin N \\ 0 & x \in N \end{cases}$$

Then $(g_n 1_{\Omega - N}) \nearrow f$ on Ω. So

$$\mu f = \lim \mu(g_n 1_{\Omega - N}) = \lim \mu g_n$$

□

Theorem 1.4 (Integral characterization theorem)**.** *Let $(\Omega, \mathcal{A})$ be a measurable space. Let L be a mapping from $\mathcal{A}_+$ into $\overline{\mathbb{R}}_+$ then there is a unique measure μ on $(\Omega, \mathcal{A})$ s.t. $L(f) = \mu f$ for every $f \in \mathcal{A}_+$ iff: - $f = 0 \implies L(f) = 0$ - $f, g \in \mathcal{A}_+$ and $a, b \in \mathbb{R}_+ \implies L(af + bg) = aL(f) + bL(g)$ - $(f_n) \subset \mathcal{A}_+$ and $f_n \nearrow f \implies L(f_n) \nearrow L(f)$*

Proof. Let there is a function L satisfies above conditions and give a μ and let $\mu(A) = L(1_A)$, then use those conditions we can prove that μ is a measure a $(\Omega, \mathcal{A})$.

□

Definition 1.3 (Image measure)**.** Let $(F, \mathcal{F})$ and $(\Omega, \mathcal{A})$ be measurable spaces. Let ν be a measure on $(F, \mathcal{F})$ and let $h : F \to \Omega$ be measurable relative to $\mathcal{F}$ and $\mathcal{A}$, then define a mapping $\nu \circ h^{-1}(B) = \nu(h^{-1}B), \ B \in \mathcal{A}$.
Then $\nu \circ h^{-1}$ is a measure on $(\Omega, \mathcal{A})$, which is called the image of ν under h.

Theorem 1.5. *For every $f \in \mathcal{A}_+$, we have $(\nu \circ h^{-1})f = \nu(f \circ h)$.*

Proof. Let $L : \mathcal{A}_+ \to \overline{\mathbb{R}}_+$ by letting $L(f) = \nu(f \circ h)$. Then as the property of $\nu(f \circ h)$, f satisfies the properties of integral characterization theorem. Then, $L(f) = \mu f$ for some unique measure μ on $(\Omega, \mathcal{A})$. And

$$\mu(B) = L(1_B) = \nu(1_B \circ h) = \nu(h^{-1}B), \qquad B \in \mathcal{A}$$

□