Quiz 2

Financial math, MATH Department

Note: *You are expected to finish the quiz in two hours.*
Score: Q1-4: 10 points, Q5: 20 points.

Denote by B a standard Brownian motion with $B_0 = 0$.

1. Show that $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$.
2. X_t is a continuous local martingale with $[X, X]_t = t$, show that $Z_t = F(X_t, t) = \exp\left\{iuX_t + \left(\frac{u^2}{2}\right)t\right\}$ is a martingale.
3. Let $X_t = e^{-\alpha t}\left(X_0 + \sigma \int_0^t e^{\alpha s} dB_s\right)$. Show that X is a solution to the stochastic differential equation
$$dX_t = -\alpha X_t dt + \sigma dB_t$$
4. Show that $J = \int_0^1 s dB_s$ has a Normal $N(0, 1/3)$ distribution.
5. The Cox-Ingersoll-Ross model for the interest rate process $R(t)$ is given by
$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dB(t), R(0) = r_0,$$
where α, β, and σ are positive constant. Find $\mathbb{E}R(t)$ and $Var(R(t))$.