MAPPINGS

Xie zejian

xiezej@gmail.com

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1 Continuous

(Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$
4. $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ always holds if f is injection while $ff^{-1}(A) = A$ always holds if g is surjection.
7. If f is bijection, $(f^{-1})^{-1}(A) = f(A)$ always hold.
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$
9. $f^{-1}(A) \subset f^{-1}(B) \iff A \subset B$
10. $f(A) \subset f(B) \iff A \subset B$

Proof see Terence tao *Analysis 1*. ■

f is **continuous** at x if for every neighborhood V' of $f(x)$, there is a neighborhood V of x s.t. $f(V) \subset V'$. It's continuous if it's continuous at every points $x \in \Omega$.

f is continuous iff $f^{-1}(G')$ is open for every open subset G' of Ω'.
Proof $\implies$: For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood V s.t. $f(V) \subset G'$, and

$$x \in V \subset f^{-1}f(V) \subset f^{-1}(G')$$

$\impliedby$: For every neighborhood V', there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $ff^{-1}(G') \subset G'$. ■

f is continuous iff

$$f^{-1}(A^\circ) \subset (f^{-1}(A))^\circ$$

for all $A \subset \Omega'$.
Proof $\implies$: $f^{-1}(A^\circ)$ is open and th claim follows from $f^{-1}(A\circ) \subset f^{-1}(A)$. $\impliedby$: Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggets $f^{-1}(A)$ is open. ■

Suppose $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_1$, $f \circ g$ is continuous if f and g are continuous.
Proof Suppose G_3 is open and the claims follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$. ■

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f, s.t f^{-1} is continuous and such f is called **homeomorphism**.

f is open if $f(G)$ is open for all open set $G \in \tau$ and is closed if $f(F)$ is closed for all closed set $f(F)^c \in \tau$.

Suppose f is bijection, then it's homeomorphism iff it's continuous and either open or closed.
Proof By the continuity of f^{-1}, since $(f^{-1})^{-1}(G) = f(G)$ for all open set G.

$$f^{-1} \text{ is continuous } \iff f(G) \text{ is open } \iff f \text{ is open .}\blacksquare$$

Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f continuous.
Proof Suppose f is homeomorphism, T_0 is another topology where f is continuous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1},

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0. $\blacksquare$

Remark $\mathcal{P}(\otimes)$ let all f continuous and $\{\emptyset, \Omega\}$ let all $g : \Omega' \to \Omega$ continuous.

2 Induced topology

Suppose $\{(\Omega_i, \mathcal{T}_\rangle)\}_{i\in I}$ a family of topological space and $(f_i)_{i\in I}$ w.r.t $\{(\Omega_i, \mathcal{T}_\rangle)\}_{i\in I}$. Let A be the set of all the topologies s.t. f_i is continuous for all i. We call $\cap A$, i.e., the corest topology in A the **topology induced** on Ω by $(f_i)_{i\in I}$.

Theorem The topology generated by $(f_i)_{i\in I}$ is the topology $\tau(\mathcal{S})$ generated by

$$\mathcal{S} = \{X \in \mathcal{P}(\Omega) : \exists G_i \in \mathcal{T}_\rangle \ni X = f_i^{-1}(G_i)\}$$

Proof Clearly $\mathcal{S} \subset \cap A$ and thus $\tau(\mathcal{S}) \subset \cap A$. On the other hand, $\tau(\mathcal{S}) \in A$ and thus $\tau(\mathcal{S}) \supset \cap A$. $\blacksquare$.

Theorem g is (τ', τ) continuous iff $f_i \circ g$ continuous for each f_i. Where τ is $\tau(S)$ in above theorem.
Proof $\implies$ is immediately. $\impliedby$, suppose $G \in \tau$,by above theorem, this implies

$$G = \cup_I \cap_F X = \cup_I \cap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continuous and thus $g^{-1}(G) = \cup_I \cap_F g^{-1}f^{-1}(G) = \cup_I \cap_F (f \circ g)^{-1}(G)$. $\blacksquare$

The **subspace topology(relative topology)** τ_A in (Ω, τ) induced by subset $A \subset \Omega$ is just topology induced by identical mapping w.r.t (A, τ), that is

$$\tau_A = \{A \cap U : U \in \tau\}$$

It's clear that a subset $B \in \tau_A$ is closed iff $B = A \cap F$ for some F closed in τ. In additionally,

$\forall B \in \tau_A$:

$$\underset{\tau_A}{\mathrm{Cl}}(B) = A \cap \underset{\tau}{\mathrm{Cl}}(B)$$

and

$$\underset{\tau}{\mathrm{Int}}(B) \subset \underset{\tau_A}{\mathrm{Int}}(B)$$

Proof Note

$$\underset{\tau}{\mathrm{Cl}}(B) = \cap\{F : F \supset B \text{ and } F^c \in \tau\}$$

thus

$$A \cap \underset{\tau}{\mathrm{Cl}}(B) = A \cap (\cap\{F : F \supset B \text{ and } F^c \in \tau\} = \cap\{A \cap F : F \supset B \text{ and } F^c \in \tau\}$$

that is just $\mathrm{Cl}_{\tau_A}(B)$. However

$$\underset{\tau}{\mathrm{Int}}(B) = \cup\{G : G \subset B \text{ and } G \in \tau\}$$

thus

$$\operatorname*{Int}_{\tau_A}(B)A \cap \operatorname*{Int}_{\tau}(B) = \cup\{A \cap G : G \subset B \text{ and } G \in \tau\} \subset \operatorname*{Int}_{\tau_A}(B)$$

The difference result from

$$A \cap G \text{ and } G \subset B \implies A \cap G \subset B$$

while

$$A \cap F \text{ and } F \supset B \iff A \cap F \supset B$$

If A is dense, suppose N is a neighborhood of $a \in A$ in τ_A, then $\overline{N}$ is a neighborhood of a in τ.
Proof By definition, there is some $G \in \tau$ s.t.

$$a \in G \cap A \subset N$$

It's sufficient to show that $a \in G \subset \overline{N}$, that is $\forall x \in G, x \in \overline{N}$. Consider any neighborhood W of x in τ, then $W \cap G$ is also neighborhood, note A is dense, $x \in \overline{A}$, which follows $W \cap G \cap A \neq \implies W \cap N \neq$, thus $x \in \overline{N}$. ■

Suppose $A \cup B = \Omega$ and $M \in \tau_A$ and $M \in \tau_B$, then $M \in \tau$.
Proof By definition, there is $G_A, G_B \in \tau$ s.t.

$$M = A \cap G_A = B \cap G_B$$

It's sufficient to show that $M = G_A \cap G_B$:

$$G_A \cap G_B = (G_A \cap G_B) \cap (A \cup B) = (G_A \cap G_B \cap A) \cup (G_A \cap G_B \cap B) = M \cup M = M$$

Suppose (Ω, τ) is separable, then (A, τ_A) is also separable.
Proof Suppose $\overline{D} = \Omega$ and D is countable, we claim that $\mathrm{Cl}_{\tau_A}(D \cap A) = A$. Which follows from any neighborhood of N in τ_A for any $x \in A$,

$$N \cap D \cap A = (N \cap A) \cap D \neq .\blacksquare$$

2.1 Product topology

Let $((\Omega_i, \tau_i))_{i \in I}$ be family of topological spaces, let $\Omega = \prod_{i \in I} \Omega_i$ and π_i be projection mappings from Ω to Ω_i. The topology τ induced by $(\pi_i)_{i \in I}$ is called **product topology** on Ω and denoted by $\prod_{i \in I} \tau_i$. (Ω, τ) is called **topological product**. A base of this topology is

$$\{\bigcup \bigcap (\prod_{i \in I} X_i)\}$$

where $X_i = \Omega_i$ for all i but one.

Suppose $G \in \prod \tau_i$, then $\pi_i(G) = \Omega_i$ except a finite set in I.
Proof By definition,

$$G = \bigcup_I \bigcap_F (\prod_{i \in I} X_i)$$

where $X_i = \Omega_i$ for all i but one. Note there is a finitely intersection, that is

$$G = \bigcup_I (\prod_{i \in I} X_i)$$

where $X_i = \Omega_i$ for all i but finite exception.

Suppose A_i is subset of each i, then

$$\operatorname*{Cl}_{\tau}(\prod A_i) = \prod (\operatorname*{Cl}_{\tau_i}(A_i))$$

3 Coinduced topology