CONVERGENCE

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In the following paragraph, Ω is a space and τ is a topology on Ω, τ is a filter on Ω.

1 Filter

A **filter** is a non-empty collection F of subset in Ω s.t.

1. $A \in F, A \subset B \implies B \in F$
2. Closed under finite intersection.
3. $\notin F$

Note the definition of F is independent with topology τ.

A collection B of subset in Ω is a **base** for the fliter if

1. $B \subset F$
2. $\forall V \in F, \exists W \in B \ni W \subset V$

We say B generates F. For example, suppose A is any non-empty subset of Ω, all the subsets of Ω include A is a filter while $\{A\}$ is a base for it. What's more, suppose $a \in \Omega$ then all neighbourhoods is a filter on E, that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. To assert a collection is a base, we have

> **Theorem 1** Let B be a collection of nonempty subsets. Then B is a filter base, that is, B may generates a filter iff 1. The intersection of each finite family of sets in B inclueds a set in B 2. B is non-empty and $\notin B$.
>
> **Proof**
> $$F = \{X \in \mathcal{P}(\Omega) : \exists A \in B \ni X \supset A\}$$
> F is the filter generated by B. ■

Let A be a collection of subsets of nonempty subsets, then construct A' by taking all finite intersection, if $\notin A'$, it's a base for some filter F, we call F the filter generated by A.

Suppose F and G be filters on Ω. Then

$$X \in F \cap G \iff \exists P \in F \text{ and } Q \in G \ni X = P \cup G$$

$$X \in \{\text{finite intersection in } F \cup G\} \iff \exists P \in F \text{ and } Q \in G \ni X = P \cap Q$$

Suppose R is an order relation on Ω, then Ω is said to be **inductivelt ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Theorem 2 The set of all filters on Ω is inductively ordered by inclusion.
Proof Suppose a collection A of filters is totally ordered, it's a base by theorem 1 since it's finite intersection is just a fliter in A with totally ordered. Then the supremum is just the fliter generates by A. ■

By Zorn's lemma, the set of all filters has maximal filters and we call such fliters **ultrafilters**.

Theorem 3 Let F be an ultrafilter on Ω, if A and B are subsets of Ω s.t. $A \cup B \in F$ then either $A \in F$ or $B \in F$.
Proof If $A \notin F$ and $B \notin F$, suppose $F' = \{X : A \cup X \in F\}$, and easy to verify $F' \supset F$, a contradiction. ■

To assert a filter is ultra, we have:

Theorem 4 Let A be a collection of subsets and F the filter generates by A. If

$$\forall X \subset \Omega, \text{either } X \in A \text{ or } X^c \in A$$

then A is an ultrafilter on Ω.
Proof Suppose F' is an ultrafilter include F, we have $F' \supset A$ clearly. Consider any $X \in F'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in F'$ as $F' \supset F \supset A$ and $X \cap X^c = \emptyset \in F'$ results in a contradiction. It follows that $A \supset F'$ and thus $A = F'$. ■

Suppose any $x \in \Omega$, the filter generates by $\{x\}$ is an ultrafilter as above theorem and we claim that all the ultrafilter may be generates this way.

Theorem 5 Every filter F is the intersection of all the ultrafilter which include F.
Proof We claim that

$$F = \cap\{\text{ultrafilter generates by } \{x\} : x \in \cap F\}$$

■

Theorem 6 Let f be a mapping from Ω to Ω' and B a base for a fliter F on Ω. Then $f(B) = \{f(X)\}_{X \in B}$ is also a base on Ω'. Moreover, if F is ultra then $f(B)$ also generates an ultrafilter.
Proof First assertion is straightforward and the second follows from B is collection of supset for some $\{x\}$, then $f(B)$ generates the fliter that generates by $\{f(x)\}$. ■

Theorem 7 In the same situation as previous theorem. If B' is a base on Ω', then $f^{-1}(B')$ is a base on Ω iff every set in B' meets $f(\Omega)$
Proof We have

$$\Omega \in f^{-1}(B') \implies f(\Omega) \subset X'$$

for some $X' \in f^{-1}(B')$, by definition, $\implies$ is immediately. For $\impliedby$, suppose any finite family $X_i \in B'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}(\bigcap_i X_i) \in f^{-1}(B')$$

Then the claim follows from theorem 1. ■

2 Limit

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the fliter F and F is said to **converge** to x if the neighborhood filter $V(x) \subset F$. For filter base B, we define similarly on the filter generated by B.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff V_\tau(x) \supset V_{\tau'}(X) \iff F \text{ converges to } a \text{ in } \tau \implies F \text{ converges to } a \text{ in } \tau'$$

Then we may define continuous as:

$f : (\Omega, \tau) \to (\Omega', \tau')$ is continous at x if for every filter F converges to x, $f(F)$(a filter base) converges to $f(x)$.
Proof By definition, $f(F)$ converges to $f(x)$ implies
$$V'(f(x)) \subset \{X \in \mathcal{P}(\Omega) : \exists A \in f(F) \ni X \supset A\}$$
That is, for any neighbourhood V' of $f(x)$, there exist some $A \in f(F)$ s.t. $A \subset V'$, note $A = f(V)$ for some neighbourhood V of x. Then the claim follows from definition of continuity.

A point $x \in \Omega$ is said to be an **adherent point** of F if x is an adherent point of every set in F. The **adherence** of F, $\mathrm{Adh}_\tau(F)$ or $\overline{F}$ is the set of all adherent points, thus

$$\overline{F} = \bigcap_{X \in F} \overline{X}$$

Define similarly on filter base B by the filter generated. By definition, we have

$$\overline{B} = \bigcap_{X \in B} \overline{X}$$

Suppose A be a subset of Ω, then $x \in \overline{A}$ iff there is a filter F s.t. $A \in F$ and F converges to x.
Proof If $x \in \overline{A}$ then $F = V(x) \cup \{A\}$ generates a fliter as required. Conversely,
$$V(x) \in F \implies V \cap A \neq \forall V \in V(x)$$
Then the calim follows. ■

Theorem 8 Suppose $BN(x)$ a neighbourhood base of x, then

1. B converges to x iff every set in $BN(x)$ includes a set in B.
2. $x \in \overline{B}$ iff every set in $BN(x)$ meets every set in B.

Proof Directly from definition. ■

As consequence, we have

Corollary 1 x is adherent to a filter F iff there is $F' \supset F$ and converges to x
Proof By above argument in finer topology, we have F converges to x, that is, x is a limit point. Then the calim follows from corollar 2.
Corollary 2 Every limit point of F is adherent to F
Proof Clearly holds by applying corollary 1 and corollary 2.
Corollary 3 Every adherent point of an ultra-filter is a limit point of it.
Proof An ultrafilter can only converges to one point and it's a adherent point by corollary 2. ■

Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a limit point of f relative to F if x' is a limit point of the filter base $f(F)$.
2. an **adherent point** of f relative F if it's adherent point of $f(F)$.

Theorem 9

1. x' is a limit point of f relative to F iff for any τ' neighbourhood $V' \in V'(x)$, we have $f^{-1}(V') \in F$.
2. x' is an adherent point of f relative to F iff for any τ' neighbourhood $V' \in V'(x')$, it meets $f(X)$ for any $X \in F$.

Proof x' is limit is equivalent to
$$V'(x) \subset \{X \in \mathcal{P}(\Omega) : \exists A \in f(F) \ni X \supset A\}$$
That is, there exist some $A = f(X) \subset V'$ for any V', followed by $X \subset f^{-1}f(X) \subset f^{-1}(V')$, then the claim follows from the definition of filter.
By theorem 8, x' is adherent to $f(F)$ iff
$$\forall N' \in BN(x'), \forall X \in F, f(X) \cap N' \neq \emptyset$$
note for any $V' \in V'(x')$, there exist $N' \in BN(x') \ni N' \subset V'$, thus $f(X) \cap V' \neq \emptyset$ also holds. Conversely, making use of $BN(x') \subset V'(x')$. ■