Normed space

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Contents

Existence of bias

Every non-zero vector space has a basis.

Proof Let $\mathcal{X}$ be the class of all independent subsets of space V. Then $(\mathcal{X}, \subset)$ is a poset. Forall chain $\mathcal{Y} \subset \mathcal{X}$, note $\cup\mathcal{Y} \in \mathcal{X}$ is a upper bound of $\mathcal{Y}$. Apply Zorn's lemma we can find a maximal element $B \in \mathcal{X}$ and $\langle B \rangle = V$, so B forms a basis of V.

Inequality

Young's inequality

Let f be a continues and strictly increasing function with $f(0) = 0$, then we have

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$$

Take $f(x) = x^{p-1}$, then $f^{-1}(x) = x^{q-1}$ if $(p-1)(q-1) = 1 \iff \frac{1}{p} + \frac{1}{q} = 1$. Hence we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Holder's inequality

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{\ltimes}$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum |a_i b_i| = |\mathbf{a}|'|\mathbf{b}| \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q$$

Minkowski's inequality

For $\mathbf{a}, \mathbf{b} \in \mathbb{F}^n$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{a} + \mathbf{b}\|_p \leq \|\mathbf{a}\|_p + \|\mathbf{b}\|_p$$

Normed Vector spaces

Let $(X, +, \cdot)$ be a vector space over $\mathbb{F}$. A norm on X is a function from $X \to \mathbb{R} \geq \nvdash$ satisfy:

1. $\|x\| \geq 0$ and $=$ occurs iff $x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|cx\| = |c|\|x\|$

A vector space with a norm is **normed vector space**.

Let $\mathbf{c}$ is $n \times 1$ and $\mathbf{X} = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \cdots & \mathbf{x_n} \end{bmatrix}$ is $n \times n$ where x_i is n vector. Then

$$\begin{aligned} \|\mathbf{Xc}\| &= \|\sum c_i \mathbf{x_i}\| \\ &\leq \sum \|c_i \mathbf{x_i}\| \\ &= \sum |c_i| \|\mathbf{x_i}\| \\ &= \|\mathbf{X}\| |\mathbf{c}| \end{aligned}$$

where

$$\|\mathbf{X}\| = \begin{bmatrix} \|\mathbf{x_1}\| & \|\mathbf{x_2}\| & \cdots & \|\mathbf{x_n}\| \end{bmatrix}, |\mathbf{c}| = \begin{bmatrix} |c_1| \\ |c_2| \\ \vdots \\ |c_n| \end{bmatrix}$$

Then we give some examples of normed space:

Let $\ell^p, 1 \leq p < \infty$, be collection of sequence satisfying

$$\sum_1^\infty |x_i|^p < \infty$$

It's a vector space and

$$\|x\|_p = (\sum_1^\infty |x_i|^p)^{\frac{1}{p}}$$

defines a norm on ℓ^p

Let ℓ^∞ be the collection of all $\mathbb{F}$ valued bounded sequences, it's a vector space and

$$\|x\|_\infty = \sup_i |x_i|$$

defines a proper norm.

Let $(X, \|\cdot\|)$ be a normed space, define $d(x,y) = \|x-y\|$, one can check d is a metric and is called as induced metric of the form. Then we can talk about convergence in this space. Clearly, the norm is a continuous function and $+$ and $\cdot$ are also continuous.

If $x_n \to x$ in $\|\cdot\|_1 \implies x_n \to x$ in $\|\cdot\|_2$, we say $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$. If they are stonger than each other, we say they are equivalent.

All norm on finite dimensional space are equivalent.

Proof It's sufficient to show that every norm is equivalet to $\|\cdot\|_2$:

$$\|\mathbf{x}\| = \|\mathbf{E}\mathbf{x}\| \leq \|\mathbf{E}\||\mathbf{x}| \leq \|\mathbf{x}\|_2\|(\|\mathbf{E}\|')\|_2 = c\|\mathbf{x}\|_2$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \cdots & \mathbf{e_n} \end{bmatrix} = \mathbf{I}$$

This state that $\|\cdot\|$ stronger than any norm. On the other hand, consider

$$\alpha = \inf\{\|\mathbf{x}\| : \|\mathbf{x}\|_2 = 1\}$$

It's positive since $\{\|\mathbf{x}\|_2 = 1\}$ is compact. Then we have

$$\alpha \leq \|\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\| = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_2} \implies \|\mathbf{x}\| \geq \alpha\|\mathbf{x}\|_2$$

For any abstract space X, $x \in X$ can be presented as linear combinations of bias, say $x = \sum a_i e_i$, then $x \mapsto (a_1, \cdots, a_n)$ is isomorph from X to $\mathbb{R}^{\ltimes}$. And any norm iduced a norm on $\mathbb{R}^{\ltimes}$ by

$$\|x\| = \|(a_1, \cdots, a_n)\widetilde{\|}$$

Hence all norm is equivalent.

Separability

A subset E of (X, d) is a **dense set** if its closure is X:

$$\overline{E} = X$$

A metric space is called **separable** if it has a countable dense subset.

Completeness

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Every metric space has a completion