decision

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1 Introduction to Decision Theory

Definition 1.1. A **weak preference** over A is a complete and transitive binary relation

Definition 1.2 (Decision Problem)**.** A decision problem is a pair $(A, \succsim)$ where A is a set and $\succsim$ is a weak preference over A.

Let $(X, \succsim)$ be a decision problem, for each pair $a, b \in A$:

- Strict preference $\succ$ is defined as $a \succ b$ iff $b \not\succsim a$.
- Indifference $\sim$ is defined as $a \sim b$ iff $a \succsim b$ and $b \succsim a$.

Lemma 1.1. *Let $(X, \succsim)$ be a decision problem, then*

1. *The strict preference is asymmetric and transitive.*
2. *The indifference is an equivalence relation which is reflexive, symmetric and transitive.*

That implies for each $a, b \in A$, either $a \succ b$, $b \succ a$ or $a \sim b$.

For each decision problem $(A, \succsim)$ it's equivalent to another one with antisymmetric weak preference by taking $(A/\sim, \succsim_a)$.

1.1 Ordinal Utility

Definition 1.3 (Utility Function)**.** Let $(X, \succsim)$ be a decision problem, a utility function function representing $\succsim$ is a function: $u : A \to \mathbb{R}$ s.t. for each $a, b \in A$, $a \succsim b \iff u(a) \geq u(b)$.

Theorem 1.1. *Let A be a countable set and $(A, \succsim)$ is a decision problem. Then, there is a utility function u representing $\succsim$.*

Proof. Let $A = \{a_1, a_2, \ldots\}$, then for each $i, j \in \mathbb{N}$:

$$h_{ij} = \begin{cases} 1 & a_i, a_j \in A \text{ and } a_i \succ a_j \\ 0 & otherwise \end{cases}$$

then $u(a_i) = \sum_{i=1}^{\infty} \frac{1}{2^j} h_{ij}$ and u represents $\succsim$.

$\square$

Definition 1.4 (Order dense and gap)**.** Let $(X, \succsim)$ be a decision problem. A set $B \subset A$ is order **dense** in A if for each $a_1, a_2 \in A$ with $a_2 \succ a_1$, there is $b \in B$ s.t. $a_2 \succsim b \succsim a_1$.

And (a_1, a_2) is a **gap** if for each $b \in A$, either $b \succsim a_2$ or $a_1 \succsim b$, such a_1, a_2 are **gap extremes**. Let A^* be the set of gap extremes.

Theorem 1.2. *Let $(A, \succsim)$ be a decision problem where $\succsim$ is antisymmetric. Then, $\succsim$ can be represented by a utility function iff there is a countable set $B \subset A$ is order dense in A.*

Proof. Let $B \subset A$ is a order dense subset in A. We say a is the first element in A if there is not $\overline{a} \in A$, $\overline{a} \neq a$ s.t. $\overline{a} \succsim a$. Last element is defined similarly.

$\square$

Remark. The equivalence also hold when $\succsim$ is not antisymmetric as there exist a utility function u' for $(A/\sim, \succsim_a)$ and we may define $u = u' \circ I$.

Remark. We may replace u by $f \circ u$ where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

1.2 Linear Utility

Definition 1.5. A **convex decision problem** is a decision problem $(X, \succsim)$ where X is convex in $\mathbb{R}^n$.

Let $(X, \succsim)$ be a convex decision problem. A utility function $\overline{u}$ representing $\succsim$ is **linear** if

$$\forall t \in [0,1], \overline{u}(tx + (1-t)y) = t\overline{u}(x) + (1-t)\overline{u}(y)$$

Remark. Ordinal utility only reveals the relative order of each pair, but linear utility also reveals how different they are.

Definition 1.6. Let $(X, \succsim)$ be a convex decision problem. We say that $\succsim$ is

- **independent** if for each triple $x, y, z \in X$ and $t \in (0,1], x \succsim y$ iff $tx + (1-t)z \succsim ty + (1-t)z$.
- **continuous** if for each triple $x \succ y \succ z$, there exist $t \in (0,1)$ with $y \sim tx + (1-t)z$.

Suppose $\succsim$ is linear, then it's continuous and independent clearly, for the converse:

Lemma 1.2. *Let $(X, \succsim)$ be a convex decision problem and $\succsim$ is independent. For $y \succ x$ and $s, t \in [0,1]$ where $s > t$, then*

$$sy + (1-s)x \succ ty + (1-t)x$$

Proof. By the independence of $\succsim$, we have

$$\frac{s-t}{1-t}y + \frac{1-s}{1-t}x \succ \frac{s-t}{1-t}x + \frac{1-s}{1-t}x = x$$

and note

$$sy + (1-s)x = ty + (1-t)(\frac{s-t}{1-t}x + \frac{1-s}{1-t}x) \succ ty + (1-t)x$$

$\square$

That implies if $\succsim$ is continuous and independent, the "t" in the definition is unique. Then we are ready for the main results:

Theorem 1.3. *Let $(X, \succsim)$ be a convex decision problem then $\succsim$ is independent and continuous iff there is a linear utility function $\overline{u}$ representing $\succsim$. And it's unique up to positive affine transformations.*

Proof. For $x_2 \succ x_1$, let $[x_1, x_2] := \{x \in X : x_2 \succ x \succ x_1\}$ and define $u : [x_1, x_2] \to \mathbb{R}$ by:

$$u(x) := \begin{cases} 0 & x \sim x_1 \\ 1 & x \sim x_2 \end{cases}$$

$\square$