
MAPPINGS

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(Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$
4. $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ always holds if f is injection while $ff^{-1}(A) = A$ always holds if g is surjection.
7. If f is bijection, $(f^{-1})^{-1}(A) = f(A)$ always hold.
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$

Proof see Terence tao *Analysis I*. ■

f is **continuous** if for every neighborhood V' of $f(x)$, there is a neighborhood V of x s.t. $f(V) \subset V'$.

f is continuous iff $f^{-1}(G')$ is open for every open subset G' of Ω' .

Proof \implies : For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood V s.t. $f(V) \subset G'$, and

$$x \in V \subset f^{-1}f(V) \subset f^{-1}(G')$$

\Leftarrow : For every neighborhood V' , there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $ff^{-1}(G') \subset G'$. ■

f is continuous iff

$$f^{-1}(A^\circ) \subset (f^{-1}(A))^\circ$$

for all $A \subset \Omega'$.

Proof \implies : $f^{-1}(A^\circ)$ is open and th claim follows from $f^{-1}(A^\circ) \subset f^{-1}(A)$. \Leftarrow : Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggests $f^{-1}(A)$ is open. ■

Suppose $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_2 \rightarrow \Omega_1$, $f \circ g$ is continuous if f and g are continuous.

Proof Suppose G_3 is open and the claims follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$. ■

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f , s.t f^{-1} is continuous and such f is called **homeomorphism**.

f is open if $f(G)$ is open for all open set $G \in \tau$ and is closed if $f(F)$ is closed for all closed set $f(F)^c \in \tau$.

Suppose f is bijection, then it's homeomorphism iff it's continous and either open or closed.

Proof By the continuity of f^{-1} , since $(f^{-1})^{-1}(G) = f(G)$ for all open set G .

$$f^{-1} \text{ is continous} \iff f(G) \text{ is open} \iff f \text{ is open.} \blacksquare$$

Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f continous.

Proof Suppose f is homeomorphism, T_0 is another topology where f is continous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1} ,

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0 . \blacksquare

Remark $\mathcal{P}(\otimes)$ let all f continous and $\{\emptyset, \Omega\}$ let all $g : \Omega' \rightarrow \Omega$ continous.

Suppose $\{(\Omega_i, \mathcal{T}_i)\}_{i \in I}$ a family of topological space and $(f_i)_{i \in I}$ w.r.t $\{(\Omega_i, \mathcal{T}_i)\}_{i \in I}$. Let A be the set of all the topologies s.t. f_i is continous for all i . We call $\cap A$ the **topology induced** on Ω by $(f_i)_{i \in I}$.

Theorem The topology generated by $(f_i)_{i \in I}$ is the topology $\tau(\mathcal{S})$ generated by

$$\mathcal{S} = \{X \in \mathcal{P}(\Omega) : \exists G_i \in \mathcal{T}_i \ni X = f_i^{-1}(G_i)\}$$

Proof Clearly $\mathcal{S} \subset \cap A$ and thus $\tau(\mathcal{S}) \subset \cap A$. On the other hand, $\tau(\mathcal{S}) \in A$ and thus $\tau(\mathcal{S}) \supset \cap A$. \blacksquare

Theorem g is (τ', τ) continous iff $f_i \circ g$ continous for each f_i . Where τ is $\tau(\mathcal{S})$ in above theorem.

Proof \implies is immediately. \Leftarrow , suppose $G \in \tau$, by above theorem, this implies

$$G = \cup_I \cap_F X = \cup_I \cap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continous and thus $g^{-1}(G) = \cup_I \cap_F g^{-1}f^{-1}(G) = \cup_I \cap_F (f \circ g)^{-1}(G)$. \blacksquare

The **subspace topology(relative topology)** τ_A in (Ω, τ) induced by subset $A \subset \Omega$ is just topology induced by identical mapping w.r.t (A, τ) , that is

$$\tau_A = \{A \cap U : U \in \tau\}$$

It's clear that a subset $B \in \tau_A$ is closed iff $B = A \cap F$ for some F closed in τ . In additionally,

$$\forall B \in \tau_A:$$

$$\text{Cl}_{\tau_A}(B) = A \cap \text{Cl}_{\tau}(B)$$

and

$$\text{Int}_{\tau}(B) \subset \text{Int}_{\tau_A}(B)$$

Proof Note

$$\text{Cl}_{\tau}(B) = \cap \{F : F \supset B \text{ and } F^c \in \tau\}$$

thus

$$A \cap \text{Cl}_{\tau}(B) = A \cap (\cap \{F : F \supset B \text{ and } F^c \in \tau\}) = \cap \{A \cap F : F \supset B \text{ and } F^c \in \tau\}$$

that is just $\text{Cl}_{\tau_A}(B)$. However

$$\text{Int}_{\tau}(B) = \cup \{G : G \subset B \text{ and } G \in \tau\}$$

thus

$$\text{Int}_{\tau_A}(B) \cap A \cap \text{Int}_{\tau}(B) = \cup \{A \cap G : G \subset B \text{ and } G \in \tau\} \subset \text{Int}_{\tau_A}(B)$$

The difference result from

$$A \cap G \text{ and } G \subset B \implies A \cap G \subset B$$

while

$$A \cap F \text{ and } F \supset B \iff A \cap F \supset B$$

If A is dense, suppose N is a neighborhood of $a \in A$ in τ_A , then \overline{N} is a neighborhood of a in τ .

Proof By definition, there is some $G \in \tau$ s.t.

$$a \in G \cap A \subset N$$

It's sufficient to show that $a \in G \subset \overline{N}$, that is $\forall x \in G, x \in \overline{N}$. Consider any neighborhood W of x in τ , then $W \cap G$ is also neighborhood, note A is dense, $x \in \overline{A}$, which follows $W \cap G \cap A \neq \emptyset \implies W \cap N \neq \emptyset$, thus $x \in \overline{N}$. ■

Suppose $A \cup B = \Omega$ and $M \in \tau_A$ and $M \in \tau_B$, then $M \in \tau$.

Proof By definition, there is $G_A, G_B \in \tau$ s.t.

$$M = A \cap G_A = B \cap G_B$$

It's sufficient to show that $M = G_A \cap G_B$:

$$G_A \cap G_B = (G_A \cap G_B) \cap (A \cup B) = (G_A \cap G_B \cap A) \cup (G_A \cap G_B \cap B) = M \cup M = M$$

Suppose (Ω, τ) is separable, then (A, τ_A) is also separable. **Proof** Suppose $\overline{D} = \Omega$ and D is countable, we claim that $\text{Cl}_{\tau_A}(D \cap A) = A$. Which follows from any neighborhood of N in τ_A for any $x \in A$,

$$N \cap D \cap A = (N \cap A) \cap D \neq \emptyset. \blacksquare$$

Let $((\Omega_i, \tau_i))_{i \in I}$ be family of topological spaces, let $\Omega = \prod_{i \in I} \Omega_i$ and π_i be Mappings from Ω to Ω_i and τ