Contents

Definition 0.1.
- A set X is a **vector space** over $\mathbb{K}$ if there exists two mappings:

$$(x, y) \in X \times X \to x + y \in X$$
$$(\alpha, x) \in \mathbb{K} \times X \to \alpha x \in X$$

  there exists an element of X denoted as 0 s.t. $x + 0 = x$ for all $x \in X$, define $(-x)$ is a vector s.t. $x + (-x) = 0$.
- A **subspace** of a vector space X over $\mathbb{K}$ is any subset of X which is also a vector space over $\mathbb{K}$.
- Let Y and Z be two subspace of X then X is said to be the **direct sum** of Y, Z if any vector $x \in X$ can be written as

$$x = y + z \qquad y \in Y, z \in Z$$

  and such a decomposition is unique.
- A subspace B is called **subspace spanned by a subset** A of X consisting of all finite linear combinations of vectors of A, i.e., $x \in B$ of the form $x = \sum_{i \in I} \alpha_i a_i$ where the set I is finite and $\alpha_i \in \mathbb{K}$, $a_i \in A$, we said that

$$B = \text{span}A$$

- The **Hamel basis** in X is any family $\{e_i\}_{i \in I}$ of vectors $e_i \in X$ satisfying:
  - First, the family is linearly independent. It means that give any finite subfamily of $\{e_j\}_{j \in J}$ and any scalars $\alpha_j \in \mathbb{K}, j \in J$ s.t. $\sum_{j \in J} \alpha_j e_j = 0$ then $\alpha_j = 0, j \in J$.
  - Second, $\text{span}\{e_i\}_{i \in I} = X$.

Theorem 0.1. *Let $X \neq \{0\}$ be a vector space.*
- There exists a Hamel base of X - Let E, F be two Hamel bases of X. Then $cardE = cardF$.

Definition 0.2. A vector space X is finite-dimensional if there exists a finite Hamel basis of X, and its **dimension** denoted as $\text{dim}X$.
If E is a Hamel base of X, then $\text{dim}X = \text{card}E$

Definition 0.3 (norm)**.** Let X be a vector space over $\mathbb{K}$. A norm on X is a mapping $\|\cdot\| : X \to \mathbb{R}$ with: - $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$ - $\|\alpha x\| = \|\alpha\|\|x\|$ for all $\alpha \in \mathbb{K}, x \in X$ - $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

Definition 0.4 (distance in normed vector space)**.** Let $(X, \|\cdot\|)$ be a normed vector space, then the mapping $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ for all $x, y \in X$ is a **distance** on X.

Proof. First we need to show that $\big|\|x\| - \|y\|\big| \leq \|x - y\|$.
Assume that $\|x\| \geq \|y\|$, then consider $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$, so $\|x\| - \|y\| \leq \|x - y\|$, as they all non-negative, $\big|\|x\| - \|y\|\big| \leq \|x - y\|$ holds.
- $d(x, y) = \|x - y\| \geq 0$ - $d(x, y) = 0 \implies \|x - y\| = 0 \implies x = y$ - $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$ - $d(x, y) \leq d(x, z) + d(y, z)$, notice that $\|x - z\| + \|z - y\| \geq \|(x - z) + (z - y)\| = \|x + y\|$, so for any $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(y, z)$

So we find that $d(x, y) = \|x - y\|$ is truly a metric on X, so (X, d) is a metric topological space. It is also called the **norm topology** of X.

$\square$

Theorem 0.2. *Let X be a finite-dimensional vector space over $\mathbb{K}$, and let $(e_i)_{i=1}^n$ denote a basis of X :*

- *For each $p \in [1, \infty]$, the mapping $\|\cdot\|_p$ defined by:*

$$x = \sum_{i=1}^{n} x_i e_i \in X \to \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad \text{if } p \in [1, \infty)$$
$$x = \sum_{i=1}^{n} x_i e_i \in X \to \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \qquad \text{if } p = \infty$$

is a norm on X.

- *For each $p \in [1, \infty]$, the space $(X, \|\cdot\|_p)$ is separable.*

Theorem 0.3 (Holder's and Minkowski's inequalities)**.**

- *Given a $p \in \mathbb{R}$ s.t. $p > 1$, let the real number q be defined by:*

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad \textit{hense } q > 1$$

and let $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ be two sequences of scalers satisfying

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \textit{ and } \sum_{i=1}^{\infty} |y_i|^q < \infty$$

Then the series $\sum_{i=1}^{\infty} |x_i y_i|$ converges and Holder's inequality holds:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$$

- *Give a real number $p \geq 1$ s.t.*

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \textit{ and } \sum_{i=1}^{\infty} |y_i|^p < \infty$$

Then $\sum_{i=1}^{\infty} |x_i + y_i|^p$ converges and Minkowski's inequality holds:

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

Proof.

1. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \qquad \text{for all } \alpha > 0,\ \beta > 0$$

To see this, note that the convexity of exponential function implies that

$$e^{\theta r + (1-\theta)s} \leq \theta e^r + (1-\theta)e^s$$

for all $\theta \in (0, 1)$ and $r, s \in \mathbb{R}$. Now let $\theta = \frac{1}{p}, r = p\text{Log}\alpha, s = q\text{Log}\beta$, the first inequality is proved.

2. Let $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ and $\|y\|_p = (\sum_{i=1}^{\infty} |y_i|^p)^{1/p}$. Let $\alpha = \frac{|x_i|}{\|x\|_p}$ and $\beta = \frac{|y_i|}{\|y\|_p}$. Then as shown above:

$$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} \leq \frac{|x_i|^p}{p(\|x\|_p)^p} + \frac{|y_i|^q}{q(\|y\|_q)^q}$$

for each $i \in \mathbb{N}, i \geq 1$. Then take sum of above inequality:

$$\sum_{i=1}^{n} \left(\frac{|x_i y_i|}{\|x\|_p \|y\|_q}\right) \leq \sum_{i=1}^{n} \left(\frac{|x_i|^p}{p(\|x\|_p)^p} + \frac{|y_i|^q}{q(\|y\|_q)^q}\right)$$

Notice that the right side of above:

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \implies \|x\|_p^p = \sum_{i=1}^{\infty} |x_i|^p$$

similar of $\|y\|_q$, so

$$\frac{\sum_{i=1}^{n} |x_i|^p}{p(\|x\|_p)^p} = \frac{\sum_{i=1}^{n} |x_i|^p}{p(\sum_{i=1}^{\infty} |x_i|^p)} \leq \frac{1}{p}$$

and the same as $\|y\|_q$, so the right side is less than $\frac{1}{p}+\frac{1}{q}=1$, so

$$\sum_{i=1}^{n}|x_iy_i| \le \|x\|_p\|y\|_q$$

holds for every $n\in\mathbb{N}$ and take the limit $n\to\infty$, the holder's inequality holds.

3. Notice that $\frac{1}{p}+\frac{1}{q}=1 \implies p+q=pq \implies p-1=\frac{p}{q}$.

$$\begin{aligned}\sum_{i=1}^{n}(|x_i|+|y_i|)^p &= \sum_{i=1}^{n}|x_i|(|x_i|+|y_i|)^{p-1}+\sum_{i=1}^{n}|y_i|(|x_i|+|y_i|)^{p-1}\\ &\le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/q}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/q}\\ &= \left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/q}\left(\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}\right)\end{aligned}$$

Notice that

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p}\le\left(\sum_{i=1}^{n}(|x_i|+|y_i|)^p\right)^{1/p}$$

so the Minkowski's inequality holds.

□

Proof. Now we prove that $\|x\|_p$ satisfies the triangle inequality.

$$\|x+y\|_p\le\|x\|_p+\|y\|_p$$

As shown above, when we prove Minkowski's inequality, before letting $n\to\infty$, we find that

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p}\le\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

which means $\|x+y\|_p\le\|x\|_p+\|y\|_p$ holds.

Then we prove that $\|x\|_\infty$ is a norm.
- $\|x\|_\infty=\max_{1\le i\le n}|x_i|\ge 0$ - $\|\alpha x\|_\infty=\max_{1\le i\le n}|\alpha x_i|=|\alpha|\|x\|_\infty$ -

$$\begin{aligned}\|x+y\|_\infty &= \max_{1\le i\le n}|x_i+y_i|\le\max_{1\le i\le n}|x_i|+\max_{1\le i\le n}|y_i|\\ &=\|x\|_\infty+\|y\|_\infty\end{aligned}$$

□

Notice that when $p=2$, $\|x\|_2$ is the Euclidean distance between point $x\in\mathbb{R}^n$ and 0, and the distance generated by $\|x\|_2$, $d(x,y)=\|x-y\|_2$ is the Euclidean distance between x and y.

0.0.0.1 Space $\mathcal{C}(K;Y)$ with K compact

Theorem 0.4. *Let K be a compact topological space and let $(Y,\|\cdot\|_Y)$ be a normed vector space. Then $\mathcal{C}(K;Y)$ is a vector space with the norm $\|\cdot\|:\mathcal{C}(K;Y)\to\mathbb{R}$:*

$$\|f\|_{\mathcal{C}}=\sup_{x\in K}\|f(x)\|_Y$$

*for each $f\in\mathcal{C}(K;Y)$. It is called the **sup-norm** on $\mathcal{C}(K;Y)$.*

Proof. Notice that $(Y, \|\cdot\|)$ is a metric space and a compact subset in a metric space is bounded and closed. - $\sup \|f(x)\|_Y < \infty$ and $\sup \|f(x)\|_Y \geq 0$ - $\sup \|\alpha f(x)\|_Y = |\alpha| \sup \|f(x)\|_Y$ - $\sup \|f+g\|_Y \leq \sup \|f\|_Y + \sup \|g\|_Y$

□

Definition 0.5 (converge uniformly). A sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n \in \mathcal{C}(K;Y)$ is said to **converge uniformly** if $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{C}} = 0$. It means

$$\lim_{n\to\infty} \left(\sup_{x\in K} \|f_n(x) - f(x)\|_Y \right) = 0$$

Theorem 0.5. *Let X be any set and let $(Y, \|\cdot\|_Y)$ be a normed vector space. Then the set $\mathcal{B}(X;Y)$ of all bounded mappings $f : X \to Y$ i.e. $f(X) \subset Y$ is a bounded subset in Y is a vector space and the function $\|\cdot\|_{\mathcal{B}} : \mathcal{B}(X;Y) \to \mathbb{R}$ defined by:*

$$\|f\|_{\mathcal{B}} = \sup_{x\in X} \|f(x)\|_Y$$

is a norm on $\mathcal{B}(X;Y)$.

Proof. Notice that a bounded function f over $\mathbb{K}$, for any $\alpha \in \mathbb{K}$, αf is still bounded and for $f, g \in \mathcal{B}(X;Y)$, $f+g$ is still bounded.

It is easy to show that $\|f\|_{\mathcal{B}}$ is truly a norm on $\mathcal{B}(X;Y)$.

□

Definition 0.6 (local uniform convergence). Let X be a topological space and Y be a normed vector space. Then a sequence $(f_n)_{n=1}^{\infty}$ of mappings $f_n : X \to Y$ is said to converge locally uniformly to a mapping $f : X \to Y$ as $n \to \infty$ if given any $x_0 \in X$ there exists a neighborhood $V(x_0)$ of x_0 s.t.

$$\lim_{n\to\infty} \left(\sup_{x\in V(x_0)} \|f_n(x) - f(x)\|_Y \right) = 0$$

Notice that local uniform convergence and uniform convergence implies the pointwise convergence.

Theorem 0.6. *Let X is a topological space and Y be a normed vector space, let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous mapping from X to Y that converges locally uniformly to a $f : X \to Y$, then f is continuous on X.*

Moreover, if f_n continuous at x_0 and they locally uniformly convergence to f then f is continuous at x_0.

Proof. Assume that f is continuous at x_0 which means give $\epsilon > 0$, there exists a neighborhood $V(x_0) \in \mathcal{N}_{x_0}$ s.t. for every $x \in V(x_0)$, $\|f(x_0) - f(x)\|_Y \leq \epsilon$.

Now suppose that $\epsilon > 0$ is given. As $(f_n) \to f$ locally uniformly. Then we can choose a $k \in \mathbb{N}$ s.t. for any $i \geq k$, we can find a neighborhood $V(x_0)$ s.t. for any $x \in V(x_0)$,

$$\sup_{x\in V(x_0)} \|f_i(x) - f(x)\|_Y \leq \epsilon/3$$

and as all $f_n : n \in \mathbb{N}$ is continuous at x_0, so we can find a neighborhood of x_0, $U(x_0) \in \mathcal{N}_{\S}$, s.t. for any $x \in U(x_0)$,

$$\sup_{x\in U(x_0)} \|f_i(x) - f_i(x_0)\|_Y \leq \epsilon/3$$

Then we consider the set $W(x_0) = U(x_0) \cap V(x_0) \in \mathcal{N}_{x_0}$, for any $x \in W(x_0)$:

$$\begin{aligned} \|f(x) - f(x_0)\|_Y &\leq \|f(x) - f_i(x)\|_Y + \|f_i(x) - f_i(x_0)\|_Y + \|f_i(x_0) - f(x_0)\|_Y \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

so if $(f_n) \to f$ locally uniformly, and f_n is continuous at x_0 for every n then f is continuous at x_0. Moreover, if f_n is continuous at every $x \in X$ i.e. continuous at X, then f is continuous at X.

□

0.0.0.2 ℓ^p space and L^p space

Definition 0.7 (ℓ^p space)**.** ℓ^p space is a normed vector space of all the infinite sequences $x = (x_i)_{i=1}^{\infty}$ of scalars $x_i \in \mathbb{K}$ that satisfy:

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \qquad \text{if } p \in [1, \infty)$$
$$\sup_{i \geq 1} |x_i| < \infty \qquad \text{if } p = \infty$$

For each $p \in [1, \infty]$, the set ℓ^p is a vector space with the norm $\|\cdot\|_p$:

$$x = (x_i) \in \ell^p \to \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \qquad \text{if } p \in [1, \infty)$$
$$x = (x_i) \in \ell^\infty \to \|x\|_\infty = \sup_{i \geq 1} |x_i| \qquad \text{if } p = \infty$$

is a norm on ℓ^p space.

Proof. Notice that from Minkowski's inequality, when $p \in [1, \infty)$ and $\sum_{i=1}^{\infty} |x_i|^p < \infty, \sum_{i=1}^{\infty} |y_i|^p < \infty$, $\sum_{i=1}^{\infty} |x_i + y_i|^p$ converges, and for a finite $\alpha \in \mathbb{K}$, $\sum_{i=1}^{\infty} \alpha |x_i|^p = \alpha \sum_{i=1}^{\infty} |x_i|^p$ also converges.

And with Minkowski's inequality, we can also easily to determine that $\|\cdot\|_p$ is a norm.

□

Theorem 0.7.
- *The normed vector space ℓ^p space is separable if $p \in [1, \infty)$*
- *The normed vector space ℓ^p space is not separable if $p = \infty$*

Proof. Let $\mathbb{K} = \mathbb{R}$, and $p \in [1, \infty)$, let

$$A = \bigcup_{n=1}^{\infty} \{(y_i) \in \ell^p; y_i \in \mathbb{Q} \text{ for } i \leq n, y_i = 0 \text{ for } i \geq n+1\}$$

Then we show $\overline{A} = \ell^p$, notice that ℓ^p is a metric space and we only need to show that for any $x \in \ell^p$ and any $\epsilon > 0$, there exists some $y \in A$ s.t. $\|x - y\|_p \leq \epsilon$.

Give any $x = (x_i) \in \ell^p$, there exists a $k \in \mathbb{N}$ s.t. $\sum_{i=k}^{\infty} |x_i|^p \leq \epsilon^p/2$, and there exists some $y \in A$ which means $y_i \in \mathbb{Q}$ for each i s.t. $\sum_{i=1}^{k-1} |x_i - y_i|^p \leq \epsilon^p/2$, then for these $x, y \in \ell^p$, we find that $\|x - y\|_p \leq \epsilon$.

Now give a proof of ℓ^∞ space is not separable.
Give a set

$$B = \{(x_i) \in \ell^\infty; x_i = 0 \text{ or } x_i = 1, i \geq 1\}$$

is an **uncountable set** since there is a one-to-one and onto mapping:

$$(x_i) \in B \to \sum_{i=1}^{\infty} \frac{1}{2^i} x_i$$

It is one-to-one obviously and onto $[0, 1]$ by the binary representation of a real number.
Now suppose there is a $C \subset \ell^\infty$ s.t. $\overline{C} = \ell^\infty$. Then give any $x \in B$, there exists a $y(x) \in C$ s.t. $\|y(x) - x\|_\infty < 1/2$ then the mapping $x \in B \to y(x) \in C$ is a injection since if $x_1, x_2 \in B$ with $x_1 \neq x_2$, then $\|x_1 - x_2\|_\infty = 1$, now let $y(x_1) = y(x_2) = y$, we find that $\|x_1 - x_2\|_\infty \leq \|x_1 - y\|_\infty + \|y - x_2\|_\infty$,

then we get the contradiction. So if $x_1 \neq x_2$, $y(x_1) \neq y(x_2)$, so this mapping must be one-to-one. It means $\operatorname{card} C \geq \operatorname{card} B$ so C is uncountable.

□

Definition 0.8 ($L^p(\Omega)$)**.** Let Ω is a open subset in $\mathbb{R}^n$ thus measurable. Remember that the $L^1(A)$ consists of all equivalence classes of real Lebesgue-measurable functions, i.e. those measurable functions $f : \Omega \to [-\infty, \infty]$ that satisfy:

$$\int_\Omega |f(x)|dx < \infty$$

Notice that a function $f : \Omega \to \overline{\mathbb{R}}$ is integrable iff $\int_\Omega |f(x)|dx < \infty$.

Now extend this definition. Let $p \in [1, \infty)$, we let $L^p(\Omega)$ denote the set formed by all equivalence classes of measurable functions $f : \Omega \to [-\infty, \infty]$ s.t. $f' = |f|^p \in L^1(\Omega)$ which means:

$$\int_\Omega |f(x)|^p dx < \infty \qquad \text{for some } \ p \in [1, \infty)$$

Theorem 0.8 (Holder and Minkowski's inequality for functions)**.**

- *Holder:*

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\int_\Omega |f(x)|^p dx < \infty \ \ \textit{and} \ \int_\Omega |g(x)|^q dx < \infty$$

$$\int_\Omega |f(x)g(x)|dx \leq \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \left(\int_\Omega |g(x)|^p dx \right)^{1/q}$$

- *Minkowski:*

$$\int_\Omega |f(x)|^p dx < \infty \ \ \textit{and} \ \int_\Omega |g(x)|^p dx < \infty$$

$$\left(\int_\Omega |f(x) + g(x)|^p dx \right)^{1/p} \leq \left(\int_\Omega |f(x)|^p dx \right)^{1/p} + \left(\int_\Omega |g(x)|^p dx \right)^{1/p}$$

Proof. Replace the sum to integral from the sequence Holder and Minkowski's inequality.

□

As we defined the space $L^p(\Omega)$ above, it is easy to verify that $L^p(\Omega)$ is a vector space and $\| \cdot \|_p : f \to \mathbb{R}$ defined by:

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \qquad p \in [1, \infty)$$

Now we define the space $L^\infty(\Omega)$.

Definition 0.9 ($L^\infty(\Omega)$ space)**.**

- $L^\infty(\Omega)$ space denote the set of all measurable functions $f : \Omega \to [\infty, \infty]$ that satisfy:

$$\inf\{C \geq 0; |f| \leq C \ \text{ a.e. in } \ \Omega\} < \infty$$

- The norm $\| \cdot \|_\infty$ on $L^\infty(\Omega)$ is defined:

$$\|f\|_\infty = \inf\{C \geq 0; |f| \leq C \ \text{ a.e. in } \ \Omega\}$$

Definition 0.10 (essential supremum)**.** Give a measurable function $f : \Omega \to [-\infty, \infty]$, the extended real number

$$\inf\{C \geq 0; |f| \leq C \ \text{ a.e. in } \ \Omega\} \in [0, \infty]$$

is called the **essential supremum** of f.

Notice that $L^\infty(\Omega)$ space consists of all equivalence class of measurable functions whose essential supremum is finite.

Theorem 0.9. *Let Ω is a open subset of $\mathbb{R}^n$, define the space*

$$\mathcal{C}_c(\Omega) = \{g \in \mathcal{C}(\Omega);\ \textit{supp } g \textit{ is compact in } \Omega\}$$

Then, for each $p \in [1, \infty)$, the subspace $\mathcal{C}_c(\Omega)$ is dense in $L^p(\Omega)$.

Proof. To prove that $\mathcal{C}_c(\Omega)$ is a dense set, we need to show that for every $f \in L^p(\Omega)$, give any $\epsilon > 0$, we have some $g \in \mathcal{C}_c(\Omega)$ s.t. $\|f - g\|_p \leq \epsilon$.

1. There exists a measurable simple function $s = s(f, \epsilon)$ s.t.

   $$\mu(\{x \in \Omega; s(x) \neq 0\}) < \infty \text{ and } \|f - s\|_p \leq \epsilon/2$$

   to achieve this, assume that $f \geq 0$ then there exists a sequence of simple function with:

   $$0 \leq s_k \leq f \text{ for all } k \geq 1 \text{ and } (s_k) \nearrow f \text{ pointwise}$$

   Notice that $f \in L^p(\Omega)$, which means $\int_\Omega |f(x)|^p dx < \infty$. As $s_k \leq f$ holds for every $k \in \mathbb{N}$, $s_k \in L^p(\Omega)$. So $\mu(\{x \in \Omega; s_k(x) \neq 0\}) < \infty$ as the definition of the integral over a simple function.
   As $(s_k) \nearrow f$, notice that $|(f - s_k)|^p \leq |f|^p$ and $|f - s_k|^p \to 0$ when $k \to \infty$, using Lebesgue's dominated convergence theorem:

   $$\int_\Omega \lim_{k\to\infty} |f - s_k|^p = \lim_{k\to\infty} \int_\Omega |f - s_k|^p = 0$$

   so we can find some k s.t. $\int_\Omega |f - s_i|^p \leq (\epsilon/2)^p$ for all $i \geq k$, so there exists some $s = s(f, \epsilon)$ s.t. $\|f - s\|_p \leq \epsilon/2$.

2. Let $s = s(f, \epsilon)$ be the measurable simple function constructed in step 1. Then there exists a function $g = g(s, \epsilon) = g(f, \epsilon) \in \mathcal{C}_c(\Omega)$ s.t.

   $$\|s - g\|_p \leq \epsilon/2$$

   Since $\mu(\{x \in \Omega; s(x) \neq 0\}) < \infty$, Lusin's property implies that there exists a function $g \in \mathcal{C}_c(\Omega)$ that satisfies

   $$\sup_{x\in\Omega} |g(x)| \leq \|s\|_\infty$$

   $$\mu(\{x \in \Omega; g(x) \neq s(x)\}) \leq \left(\frac{\epsilon}{4\|s\|_\infty}\right)^p$$

   Then

   $$\|s - g\|_p = \left(\int_A |s - g|^p\right)^{1/p}$$

   Notice that $|s - g| \leq 2\|s\|_\infty$ as $\sup|g(x)| \leq \|s\|_\infty$, and A denotes the set $\{x \in \Omega; g(x) \neq s(x)\}$, so the integral above is less than $2\|s\|_\infty \cdot \mu A \leq \epsilon/2$.

As shown above, give $\epsilon > 0$ and $f \in L^p(\Omega)$ there is a $g(f, \epsilon)$ s.t. $\|f - g\|_p \leq \|f - s_k\|_p + \|s_k - g\|_p \leq \epsilon/2 + \epsilon/2 = \epsilon$. □

Theorem 0.10.
1. *$L^p(\Omega)$ is separable if $p \in [1, \infty)$*
2. *$L^\infty(\Omega)$ is not separable.*

Proof.
1. Let a $f \in L^p(\Omega)$ where $p \in [1, \infty)$ then there exists a $g = g(f, \epsilon) \in \mathcal{C}_c(\Omega)$ s.t.

   $$\|f - g\|_p \leq \epsilon/2$$

Since $K = \operatorname{supp} g$ is compact, there exists a bounded open set U s.t. $K \subset U \subset \Omega$. As U is bounded, $\overline{U}$ is bounded too, so g is uniformly continuous on $\overline{U}$, then there exists $\delta_0 > 0$ s.t.

$$|g(x) - g(y)| \leq \frac{\epsilon}{2(\mu(U))^{1/p}} = \epsilon'$$

for all $x, y \in \overline{U}$ s.t. $\|y - x\|_\infty < \delta_0$
As the compactness of K and the property of distance function, there exists $\delta_1 > 0$ s.t.

$$\{y \in \mathbb{R}^n; \|y - x\|_\infty < \delta_1\} \subset U \text{ for all } x \in K$$

Let $\delta \in \mathbb{Q}$ s.t. $0 < \delta \leq \min\{\delta_0, \delta_1\}$.
Let $(B_i)_{i \in I}$ denote the countable family formed by all open balls:

$$\left\{ y \in \mathbb{R}^n; \|x - y\|_\infty < \frac{\delta}{2} \text{ with } x_j = p_j \delta \text{ for some } p_j \in \mathbb{Z}, j \in [1, n] \right\}$$

Now pick the subfamily $(B_i)_{i \in I(K)}$ s.t. for any $i \in I(K)$, $B_i \cap K \neq \varnothing$. Then for each $i \in I(K)$, notice that $\delta/2$ makes sure that $\operatorname{diam}(B_i \cap K) \leq \delta \leq \delta_0$, so if $x \in K$, then $B_i \subset U$ and $|g(y_1) - g(y_2)| \leq \epsilon'$ for every $y_1, y_2 \in B_i$ since the property of uniform continuous. If $x \notin K$, then as its minimum is 0, we can also pick some α_i as blow:
we pick some $\alpha_i \in \mathbb{Q}$ s.t.

$$|g(y) - \alpha_i| \leq \epsilon' \text{ for all } y \in B_i$$

Now we can construct a simple function:

$$h = \sum_{i \in I(K)} \alpha_i \mathbf{1}_{B_i}$$

which satisfies that $|h(x) - g(x)| \leq \epsilon'$ for almost all $x \in U$ s.t.

$$\|h - g\|_p = \left(\int_U |h - g|^p \right)^{1/p} \leq (\mu(U))^{1/p} \cdot \frac{\epsilon}{2(\mu(U))^{1/p}} = \frac{\epsilon}{2}$$

Notice that $\|f - g\|_p + \|g - h\|_p \geq \|f - h\|_p$, so $\|f - h\|_p \leq \epsilon$ and as h is simple and $\alpha_i \in \mathbb{Q}$, so h is countable as $I(K)$ is always a finite subset of a countable set and $\mathbb{Q}$ is a countable set. So $L^p(\Omega)$ is separable.

$\square$