Notes of Infinite dimensional analysis

Xie Zejian

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Chapter 1

Odds and ends

1.1 Space of sequences

Definition 1.1. For $1 \leq p < \infty$, ℓ_p is defined to be the set of all sequences $x. = (x_1, x_2, \cdots)$ for which $\|x\|_p < \infty$. Where

$$\|x\|_p = (\sum_1^\infty |x_i|^p)^{1/p}$$

is the ℓ_p **norm** of the sequences.

While ℓ_∞ is defined as the set of all $\sup\{|x_n|\} \leq \infty$, such norm is called ℓ_∞ **norm**, **supremum norm** or **uniform norm**.

All of these spaces are vector space. And sequence $\{\ell_i\}_{i=1}^\infty$ is increasing.

The space of all convergent sequence is denoted c and all sequences convergent to 0 is denoted c_0. Finally, the collection of sequences with finite nonzero terms is φ. One can check that

$$\varphi \subset \ell_p \subset c_0 \subset \ell_\infty \subset \mathbb{R}^n$$

1.2 Spaces of functions

One can think $\mathbb{R}^n$ as

$$\{f : \{1, 2, \cdots, n\} \to \mathbb{R}\} = \mathbb{R}^n = \mathbb{R}^{\{1,2,\cdots,n\}}$$

Replace $\{1, 2, \cdots, n\}$ by an arbitrary X, then $\mathbb{R}^X$ is all functions from X to $\mathbb{R}$.

For $1 \leq p < \infty$, $L_p(\mu)$ is defined to be the set of all μ measurable functions f for which $\|f\|_p < \infty$, where the L_p **norm** is defined as

$$\|f\|_p = (\int_\Omega |f|^p)^{1/p}$$

And the L_∞ **norm**, or **essential supremum** is defined as

$$\|f\|_\infty = \operatorname{ess\,sup} f = \sup\{t : \mu(\{x : |f(x)| \geq t\})0\}$$

1.3 Ordinals

Suppose R is an order relation on Ω, then Ω is said to be **inductively ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Definition 1.2. A set X is **well ordered** by linear $\preceq$ if every nonempty subset has a least element.

Definition 1.3. An **initial segement** of $(X, \preceq)$ is any set of the form $I(x) = \{y \in X : y \leq x\}$.

Definition 1.4. An **ideal** in a well ordered X is a subset A s.t. forall $a \in A$, $I(a) \subset A$.

Theorem 1.1 (Well Ordering Principle)**.** *Every nonempty set can be well ordered.*

Proof. Let X nonempty, and let

$$\mathcal{X} = \{(A, \preceq_A) \text{ is well order} : A \subset X\}$$

all well ordered sets, and define $\preceq$ on $\mathcal{X}$ as $(B, \preceq_B) \preceq (A, \preceq_A)$ if B is an ideal in A and $\preceq_A$ extends $\preceq_B$. Suppose every chain $\mathcal{C}$ in $\mathcal{X}$, $(\cup\mathcal{C}, \cup\{\prec_A : A \in \mathcal{C}\})$ clearly an upper bound of $\mathcal{C}$ and well ordered. By Zorn's lemma, there is a maximal element of $\mathcal{X}$ and it's actually X. $\square$

Kind of remarkable and useful well ordered set is exist:

Theorem 1.2. *There exist poset* $(\Omega, \preceq)$ *satisfy*

1. $(\Omega, \preceq)$ *is well ordered.*

2. *Ω has a greast element ω_1*
3. *$I(x)$ is countable for $x < \omega_1$*
4. *$\{y \in \Omega : x \leq y \leq \omega_1\}$ is uncountable.*
5. *Every nonempty subset of Ω has a least upper bound.*
6. *A nonempty subset of $\Omega - \{\omega_1\}$ has greatst element iff it's countable. Every uncountable subset has least upper bound ω_1.*

Proof. Let $(X, \preceq)$ be uncountable well ordered set, and let A

$$A = \{x \in X : I(x) \text{is uncountable}\}$$

w.l.o.g we may assume A is nonempty. Then there is a first element and denoted by ω_1. Then we show that $\Omega = I(\omega_1)$ enjoy all the properies.

1-4 is straightforward and 5 follows from all the upper bound is well ordered and thus least upper bound exist. For 6, suppose there is a countable $C \subset \Omega - \{\omega_1\}$, then $\bigcup_{i=1}^{\infty} I(x_i)$ is countable, so there is some $x < \omega_1 \notin \bigcup_{i=1}^{\infty} I(x_i)$, that is an upper bound. By 5, least upper bound is exist and belong to C. Conversely, if some subset C has some least upper bound $b < \omega_1$, then $C \subset I(b)$ and must countable. □

The elements of Ω are called **ordinals** and ω_1 is called **first uncountable ordinal**. The elements of $\Omega_0 = \Omega - \{\omega_1\}$ is **countable ordinals**. We treat $\mathbb{N}$ as a subset of Ω. Then the first element of $\Omega - \mathbb{N}$ is **first infinite ordinal**.

Theorem 1.3 (Interlacing Lemma)**.** *Suppose sequence $\{x_n\}$ and $\{y_n\}$ in Ω_0 with $x_n \leq y_n \leq x_{n+1}$. Then they share the same least upper bound.*

Proof. Clearly since $x_n \leq y_n \leq x_{n+1}$.

□

Chapter 2

Topology

2.1 Topological spaces

Let Ω be as space

Definition 2.1. A class of subset τ of Ω is an **topology** if

1. $\varnothing$ and Ω belongs to τ.
2. closed under arbitrary union.
3. closed under finite intersection.

(Ω, τ) called a **topological space** where Ω is called as **underlying set**. The sets in τ are called **open** while sets with complement in τ is **closed**. Both open and closed set is called **clopen**.

Definition 2.2. Countable intersection of open sets is $\mathscr{G}_\sigma$ set and countable union of closed sets is $\mathscr{F}_\delta$ set.

Definition 2.3. (X, ρ) is a **semimetric space**, when ρ defined on $X \times X$ s.t. $\forall x, y, z \in X$:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

ρ is called a **semimetric**.

If $\rho(x, y) = 0 \iff x = y$, ρ become a **metric** and (X, ρ) become **metric space**. $B(a, r) = \{x \in E, d(x, a) < r\}$ is r-ball with center a.

U is **open** in (Ω, d) iff $\forall x \in U, \exists r_x 0 \ni B_d(x, r_x) \subseteq U$. Let τ_d be the set of all open subsets of Ω, we call τ_d the **topology generated by** d. A Topological space is **metrizable** if there exist metric d generates it.

Suppose d is discrete, that is, $d(x,y) = 0$ iff $x = y$, otherwise, $d(x,y) = 1$. Then every subset is open hence $\tau_d = \mathcal{P}(\Omega)$ and called **discrete topology**. The zero semimetric, defined by $d(x,y) = 0$ for all $x, y \in \Omega$ generates $\tau_d = \{\varnothing, \Omega\}$ and called **trivial topology**.

Let $\Omega = \mathbb{R}^n$, $l^2 = \sqrt{\sum_1^n (x_i - y_i)^2}$ is called **Euclidean metric**. $l^1 = \sum_1^n |x_i - y_i|$ is called **taxi-cab metric** and $l^\infty = \sup\{|x_i - y_i|\}$ is called **sup norm metric**.

Note $d_{l^2}(x,y) \leq d_{l^1}(x,y) \leq \sqrt{n} d_{l^2}(x,y)$ and $d_{l^2}(x,y) \leq \sqrt{n} d_{l^\infty}(x,y) \leq \sqrt{n} d_{l^2}(x,y)$, then d_{l^∞} open $\iff$ d_{l^2} open $\iff$ d_{l^1} open. Hence $\tau_{d_{l^2}} = \tau_{d_{l^1}} = \tau_{d_{l^\infty}}$.

All topologies on Ω is poset with greatest element $\mathcal{P}(\Omega)$ and least $\{\varnothing, \Omega\}$. If $\tau' \subset \tau$, we say τ' **coarser** than τ while τ finer than τ'.

If τ can be form by taking union of families in some $\mathcal{B} \subset \tau$, we call $\mathcal{B}$ the **base** for the topology τ.

Theorem 2.1. *$\mathcal{B}$ is a base in (X, τ) iff $\forall U \in \tau, \forall x \in U, \exists W \in \mathcal{B} \ni x \in W \subset U$.*

Proof. $\implies$: Any U can be written as $U = \cup W_i$ and $x \in U \implies x \in W_i$ for some i and $W_i \in \mathcal{B}$. $\impliedby$: For any $U \in T$, consider arbitrary $x \in U$, then there exist W_x such that $x \in W_x \subset U$, thus we have $U = \cup_x W_x$.

$\square$

Let $\mathcal{S} \subset \tau$, suppose all topologies include $\mathcal{S}$. Then the intersection of all of them is again a topology, denoted as $\tau(S) = \cap \mathrm{T}$, then $\tau(\mathcal{S})$ is the smallest topology contains $\mathcal{S}$. We call it the topology **generated** by $\mathcal{S}$.

Theorem 2.2. *$\tau(\mathcal{S})$ is unions of families of finite intersections together with Ω, formally:*

$$\{\bigcup(\bigcap_1^N S_i)\} \cup \Omega$$

$\mathcal{S} \subset \tau$ is a **subbase** for τ if $\bigcup \mathcal{S} = \Omega$ then all finite intersections of $\mathcal{S}$ is a base. Note that if $\Omega \in \mathcal{S}$, $\mathcal{S}$ is the subbase of $\tau(\mathcal{S})$.

(Ω, τ) is **second countable** if τ has countable base. Clearly, a topology is second countable iff it has countable subbase.

For any subset X in (Ω, τ), then

$$\tau_X = \{X \cap V : V \in \tau\}$$

form a topology in X and we call (X, τ_X) a **subspace** or **relative topology**. Sets in τ_X are **relative open**. **Relative closed** sets of the form

$$X - (X \cap V) = X - V = X \cap V^c$$

2.2 Neighborhood

A subset V is called a **neighborhood** of a if there exists a open set $U \subset V$ contains a. Then we called $V' = V - \{a\}$ **punctured(deleted)** neighborhood. A **neighborhood base** is a collection of neighborhood $BN(a)$ s.t. for any neighborhood V of a, there exist a $W \in BN(a)$ and $W \subset V$. Clearly, all the neighborhoods is a neighborhood base and denoted as $\mathcal{N}(x)$, which is called **neighborhood system**.

Lemma 2.1. *A subset U is open iff it's a neighborhood for each of its points.*

Proof. $\Longrightarrow$ is trivial. $\Longleftarrow$ follows from $\cup_x G_x = U$ and unions of open set is still open. ■

□

This suggest a equivalent definition of finer topology:

Lemma 2.2. $\tau' \subset \tau \iff$ *τ' neighborhood is a τ neighborhood.*

Proof. $\Longrightarrow$ any open set G_x satisfy $x \in G_x \subset V$ in T' is still open in T, hence V is T neighborhood. $\Longleftarrow$ Consider any open set $G \in T'$, it's a T' neighborhood for each of its points implies it's a T neighborhood for each of its points and hence G is T open.

□

2.3 Closures

The **interior** of A is the union of all open sets which are included A, i.e., the largest open set included in A, we denote it A°. And the **closure** is the intersection of all closed sets which include A and thus the smallest closed set includes A, we denote it $\overline{A}$.

Lemma 2.3. *Following is some useful truth:*

1. $(A \cap B)^\circ = A^\circ \cap B^\circ$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

3. $A \subset \overline{B} \implies \overline{A} \subset \overline{B}$
4. $A^\circ \subset B \implies A^\circ \subset B^\circ$
5. $\overline{A^c} = (A^\circ)^c$
6. $(\overline{A})^c = (A^c)^\circ$

Proof. We only prove **5**, note $(A^\circ)^c$ is closed and

$$A^\circ \subset A \implies (A^c) \subset (A^\circ)^c$$

we have $\overline{A^c} \subset (A^\circ)^c$. On the other hand

$$\overline{A^c} \supset (A^\circ)^c \impliedby (\overline{A^c})^c \subset A^\circ \impliedby (\overline{A^c})^c \subset A \impliedby \overline{A^c} \supset A^c$$

$\square$

The **frontier** of A is $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ$.

x is said to be an **interior point** of A if A is neighborhood of x.

x is said to be an **adherent point** if it's every neighborhood meets A, an ω **accumulation point** of A if every neighborhood of x contains **infinitely** many points of A and is a **condensation point** of A if every neighborhood of x contains **uncountable** many points of A.

x is a **cluster point** or **accumulation point** if every deleted neighborhood of x meets A and is **isolated point** if x is not cluster point. That is, $\{x\}$ is relative open in A. We denoted all the cluster points as A' and called **derived set**.

x is **frontier point** or **boundary point** if every neighborhood of x meets both A and A^c.

It's east to show that the points of A° are precisely all the interior points of A and $\overline{A}$ are precisely all the adherent points. ∂A is precisely points of frontier. We claim that

$$\overline{A} = A^\circ \cup \partial A = A \cup A'$$

A subset A is called **perfect** if it's closed while point in A is cluster points in A, that is $A' = A = \overline{A}$.

2.4 Dense

A is said **dense** if $\overline{A} = \Omega$ and **nowhere dense** if $(\overline{A})^\circ = \varnothing$ ($\mathbb{Q}$ is dense in $\mathbb{R}$ while $\mathbb{Z}$ is nowhere dense.) A is said to be **meagre** or **set of the first category** if it's countable union of nowhere dense. Sets which are not meagre is **set of the second** category set.

Space (Ω, τ) is **first countable** if every point of Ω has countable neighborhood base. The space is said **separable** if Ω has a countable dense subset.

Lemma 2.4. *Second countable space is separable*

Proof. Suppose $\mathcal{B} = (B_i)_{i\in I}$ is a countable base, by axiom of choice, we may take x_i in I, let $X = \{x_i\}_{i\in I} \subset \Omega$. Then we show that X is dense. For any $x \in \Omega$, it's neighborhood must contain some open G which is unions of $\mathcal{B}$ and thus contains at least one element in X, that is, G meet X. Hence $\overline{X} = \Omega$.

□

Lemma 2.5. *Second countable space is first countable*

Proof. Suppose $\mathcal{B} = (B_i)_{i\in I}$ is a countable base, for each point $x \in \Omega$, one may take all the sets in $\mathcal{B}$ which contains x as a neighborhood base. To verify it's neighborhood base, if there is a neighborhood N of x, then there is a open G contains x. By the definition of base, G is the union of sets of $\mathcal{B}$ and those sets must at least one contains x and these sets is subset to G.

□

2.5 Mappings

Suppose (Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

Lemma 2.6. *Following is some useful truth for mappings.*

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap N) = f^{-1}(U) \cap f^{-1}(N)$
4. $f^{-1}(U \cup N) = f^{-1}(U) \cup f^{-1}(N)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ *always holds if* f *is injection while* $ff^{-1}(A) = A$ *always holds if* g *is surjection.*
7. *If* f *is bijection,* $(f^{-1})^{-1}(A) = f(A)$ *always hold.*
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$
9. $f^{-1}(A) \subset f^{-1}(B) \impliedby A \subset B$

10. $f(A) \subset f(B) \impliedby A \subset B$

Definition 2.4. f is **continuous** at x if for every neighborhood N' of $f(x)$, there is a neighborhood N of x s.t. $f(N) \subset N'$. It's continuous if it's continuous at every points $x \in \Omega$.

Theorem 2.3. *f is continuous iff*

1. *$f^{-1}(G')$ is open for every open subset G' of Ω'.*
2. *$f^{-1}(F')$ is closed for every closed subset F' of Ω'.*
3. *If $A \subset \Omega'$, then $f^{-1}(A^\circ) \subset (f^{-1}(A))^\circ$*
4. *If $A \subset \Omega$, then $f(\overline{A} \subset \overline{f(A)})$*

Proof. We only prove 1 and 3.

1 $\implies$: For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood N s.t. $f(N) \subset G'$, and

$$x \in N \subset f^{-1}f(N) \subset f^{-1}(G')$$

$\impliedby$: For every neighborhood N', there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $ff^{-1}(G') \subset G'$.

3 $\implies$: $f^{-1}(A^\circ)$ is open and th claim follows from $f^{-1}(A\circ) \subset f^{-1}(A)$. $\impliedby$: Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggest $f^{-1}(A)$ is open.

□

Lemma 2.7 (Glueing Lemma)**.** *Let $X = A \cup B$ and A and B are both closed or both open, then $f : X \to Y$ is continuous iff it's restriction on A and B are both continuous.*

Proof. $\implies$ is trivial.

$\Leftarrow$ Suppose they are both open and U be any open set in Y. Note $f_{|A}^{-1}(U)$ is open in A and thus open in X, thus

$$f^{-1}(U) = (f^{-1}(U) \cap B) \cup (f^{-1}(U) \cap A) = f_{|A}^{-1}(U) + f_{|B}^{-1}(U)$$

is open.

□

Lemma 2.8. *Suppose $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$, $f \circ g$ is continuous if f and g are continuous.*

Proof. Suppose G_3 is open and the claims follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$.

$\square$

Lemma 2.9. *Suppose $f : (\Omega, \tau), (\Omega', \tau(\mathcal{S}))$, f is continuous iff $f^{-1}(S) \in \tau$ for any $S \in \mathcal{S}$.*

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f, s.t. f^{-1} is continuous and such f is called **homeomorphism**. In particular, f is an **embedding** if $f : (\Omega, \tau) \to (f(\Omega), \tau|f(\Omega))$ is a homeomorphism.

f is **open** if $f(G)$ is open for all open set $G \in \tau$ and is **closed** if $f(F)$ is closed for all closed set $F^c \in \tau$.

Lemma 2.10. *Suppose f is bijection, then it's homeomorphism iff it's continuous and either open or closed.*

Proof. By the continuity of f^{-1}, since $(f^{-1})^{-1}(G) = f(G)$ for all open set G.

$$f^{-1} \text{ is continuous } \iff f(G) \text{ is open } \iff f \text{ is open }.$$

$\square$

Lemma 2.11. *Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f continuous.*

Proof. Suppose f is homeomorphism, T_0 is another topology where f is continuous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1},

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0.

$\square$

Note that $\mathcal{P}(\Omega)$ let all f continuous and $\{\varnothing, \Omega\}$ let all $g : \Omega' \to \Omega$ continuous.

2.6 Semicontinuous

$f : \Omega \to \mathbb{R}^*$ is

- **lower semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \leq c\}$ is closed.

- **upper semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \geq c\}$ is closed.

Clearly f is lower semicontinuous iff $-f$ is upper and vice versa. Also, f is continuous iff it's both upper and lower semicontinuous.

Lemma 2.12. *Suppose $\{f_i\}_{i \in I}$ is family of lower(upper) semicontinuous function then $\sup f_i (\inf f_i)$ is lower(upper) semicontinuous.*

Proof. Note

$$\{x \in \Omega : \sup f_i(x) \leq c\} = \bigcap_{i \in I} \{x \in \Omega : f_i(x) \leq c\}$$

is closed.

□

Lemma 2.13. *$f : \Omega \to \mathbb{R}^*$ is*

- ***lower semicontinuous** iff for any net*

$$x. \to x \implies \liminf f(x.) \geq f(x)$$

- ***upper semicontinuous** iff for any net*

$$x. \to x \implies \limsup f(x.) \leq f(x)$$

Proof. Suppose f is lower semicontinuous and $x. \to x$. For any $c < f(x)$, then $G = \{\omega \in \Omega : f(\omega) c\}$ is open and thus $x.$ eventually in, that is $x.c$ eventually and thus $\liminf f(x.) \geq c$. This implies that $\liminf f(x.) \geq f(x)$.

Conversely, for any $c \in \mathbb{R}$, consider $F = \{\omega \in \Omega : f(\omega) \leq c\}$. Then we show that F is closed. Suppose $x.$ is nets in F and converges to some $x \in \Omega$. Then $c \geq \liminf f(x.) \geq f(x)$ thus x in F and thus F is closed.

□

Then we can generalizes Weierstrass' Theorem in corollary 2.5.

Theorem 2.4. *$f : \Omega \to \mathbb{R}^*$ on a compact set attains a minimum(maximum) value and set of minima(maxima) is compact if it's lower(upper) semicontinuous.*

Proof. Suppose X is compact and f is lower semicontinuous, then for every $c \in f(X)$, $F_c = \{x \in X : f(x) \leq c\}$ is closed and $\{F_c : c \in f(X)\}$ has FIP clearly. Note X is compact, $\ker\{F_c : c \in f(X)\}$ is nonempty by 2.28. That is just the set of minima and it's compact since it's closed.

□

2.7 Comparing topologies

We list some useful properties when comparing topologies, some of them has been mentioned before and proof omitted.

Lemma 2.14. *Suppose τ' and τ are two topologies on Ω, then the following are equivalent.*

1. $\tau' \subset \tau$
2. *Identity mapping $I : x \mapsto x$ from (Ω, τ) to (Ω', τ') is continuous.*
3. *τ' closed set is closed in τ.*
4. $x. \xrightarrow{\tau} x \implies x. \xrightarrow{\tau'} x$
5. $Cl_\tau(A) \subset Cl_{\tau'}(A)$

Lemma 2.15. *Suppose $\tau' \subset \tau$, then*

1. *Every τ compact set is τ' compact.*
2. *Every τ' continuous function is τ continuous.*
3. *Every τ dense set is τ' dense.*

2.8 Filter

Definition 2.5. A **filter** is a non-empty collection $\mathcal{F}$ of subset in Ω s.t.

1. $A \in \mathcal{F}, A \subset \mathcal{B} \implies \mathcal{B} \in \mathcal{F}$
2. Closed under finite intersection.
3. $\varnothing \notin \mathcal{F}$

Note the definition of $\mathcal{F}$ is independent with topology τ. A **free filter** is filter with $\ker \mathcal{F} = \bigcap_{F \in \mathcal{F}} F = \varnothing$. Not free filters are called **fixed**.

Filter can be formed by taking upward closure of a filter base.

Definition 2.6. A collection $\mathcal{B}$ of subset in Ω is a **fiter base** of or **prefilter** if

1. $\mathcal{B} \subset \mathcal{F}$
2. $\forall N \in \mathcal{F}, \exists W \in \mathcal{B} \ni W \subset N$

We say $\mathcal{B}$ generates $\mathcal{F}$, where

$$\mathcal{F} = \mathcal{B}^{\uparrow} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

For example,

- Suppose $x \in \Omega$ then

  $$\mathcal{N}(x) = \{\text{All neighbourhoods of } x\}$$

  is a filter on Ω, that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. Note

  $$\tau(x) = \{X \in \tau : x \in X\}$$

  is a base for $\mathcal{N}(x)$ and thus $\mathcal{N}(x) = \tau(x)^{\uparrow}$.

- Suppose Ω is infinite, the collection of all **cofinite** subsets(subset s with finite complement) is a filter on Ω, such filter is free and called **Frechet filter**.

To assert a collection is a base, we have

Theorem 2.5. *Let $\mathcal{B}$ be a collection of nonempty subsets. Then $\mathcal{B}$ is a filter base, that is, $\mathcal{B}$ may generates a filter iff*

1. *The intersection of each finite family of sets in $\mathcal{B}$ inclueds a set in $\mathcal{B}$*
2. *$\mathcal{B}$ is non-empty and $\varnothing \notin \mathcal{B}$.*

Proof. We claim that

$$\mathcal{F} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

$\mathcal{F}$ is the filter generated by $\mathcal{B}$. □

A family of subsets $\mathcal{F}$ is said to have **finite intersection property** if intersection of every finite subfaimily is nonempty.

Let $\mathcal{A}$ be collection of subsets with finite intersection property, then collection of all finite intersection of $\mathcal{A}$ is a base, we call the filter generated **filter generated by $\mathcal{A}$**. Formally

$$\mathcal{F} = \{\bigcap_{A \in \mathcal{I}} A : \mathcal{I} \subset \mathcal{A} \text{ and } \mathcal{I} \text{ is finite}\}^{\uparrow}$$

A filter $\mathcal{F}$ is **finer** than another $\mathcal{G}$ if $\mathcal{F} \subset \mathcal{G}$. Clearly, the set of all filters on Ω is inductively ordered by inclusion. By Zorn's lemma, the set of all filters has maximal filters and we call such fliters **ultrafilters**.

Lemma 2.16. *Every fixed ultrafilter of the form*

$$\mathcal{U}(x) = \{x\}^{\uparrow}$$

for any $x \in \Omega$. And every free ultrafilter contains no finite subsets.

To assert a filter is ultra, we have:

Theorem 2.6. *Let A be a collection of subsets and $\mathcal{F}$ the filter generates by A. If*

$$\forall X \subset \Omega, \textit{either } X \in A \textit{ or } X^c \in A$$

then A is an ultrafilter on Ω.

Proof. Suppose $\mathcal{F}'$ is an ultrafilter include $\mathcal{F}$, we have $\mathcal{F}' \supset A$ clearly. Consider any $X \in \mathcal{F}'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in \mathcal{F}'$ as $\mathcal{F}' \supset \mathcal{F} \supset A$ and $X \cap X^c = \varnothing \in \mathcal{F}'$ results in a contradiction. It follows that $A \supset \mathcal{F}'$ and thus $A = \mathcal{F}'$. □

Theorem 2.7. *Every filter $\mathcal{F}$ is the intersection of all the ultrafilter which include $\mathcal{F}$.*

Proof. We claim that

$$\mathcal{F} = \cap\{\text{ultrafilter generates by } \{x\} : x \in \cap\mathcal{F}\}$$

□

Suppose mappings on a filter:

Theorem 2.8. *Let f be a mapping from Ω to Ω' and $\mathcal{B}$ a base for a fliter $\mathcal{F}$ on Ω. Then $f(\mathcal{B}) = \{f(X)\}_{X \in B}$ is also a base on Ω'. Moreover, if $\mathcal{F}$ is ultra then $f(\mathcal{B})$ also generates an ultrafilter.*

Proof. First assertion is straightforward and the second follows from $\mathcal{B}$ is collection of supset for some $\{x\}$, then $f(\mathcal{B})$ generates the fliter that generates by $\{f(x)\}$. □

Theorem 2.9. *In the same situation as previous theorem. If $\mathcal{B}'$ is a base on Ω', then $f^{-1}(\mathcal{B}')$ is a base on Ω iff every set in $\mathcal{B}'$ meets $f(\Omega)$*

Proof. We have

$$\Omega \in f^{-1}(\mathcal{B}') \implies f(\Omega) \subset X'$$

for some $X' \in f^{-1}(\mathcal{B}')$, by definition, $\implies$ is immediately.

For $\impliedby$, suppose any finite family $X_i \in \mathcal{B}'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}(\bigcap_i X_i) \in f^{-1}(\mathcal{B}')$$

Then the claim follows from theorem 2.5. □

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the fliter $\mathcal{F}$ and $\mathcal{F}$ is said to **converge** to x, or $\mathcal{F} \to x$, if the neighborhood filter $\mathcal{N}(x) \subset \mathcal{F}$. For filter base $\mathcal{B}$, we define on the filter generated by $\mathcal{B}$, that is, if $\mathcal{N}(x) \subset \mathcal{B}^{\uparrow}$.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff \mathcal{N}_{\tau}(x) \supset \mathcal{N}_{\tau'}(x) \iff \mathcal{F} \to a \implies \mathcal{F}' \to a$$

also, an equivalent definition of continuity as follows:

Theorem 2.10. *$f : (\Omega, \tau) \to (\Omega', \tau')$ is continous at x iff*

$$\forall \mathcal{F} \to x, f(\mathcal{F}) \to f(x)$$

Proof. By definition, $f(\mathcal{F}) \to f(x)$ if

$$\mathcal{N}(f(x)) \subset f(\mathcal{F})^{\uparrow}$$

That is, for any neighbourhood $N' \in \mathcal{N}(f(x))$, there exist some $A \in \mathcal{F}$ s.t. $f(A) \subset N'$, as $\mathcal{N}(x) \subset \mathcal{F}$ and f is continous at x, such A is always exists. Conversely, take $\mathcal{F} = \mathcal{N}(x)$ then the claim is follows □

A point $x \in \Omega$ is said to be an **adherent point** of $\mathcal{F}$ if x is an adherent point of every set in $\mathcal{F}$. The **adherence** of $\mathcal{F}$, $\mathrm{Adh}_{\tau}(\mathcal{F})$ or $\overline{\mathcal{F}}$ is the set of all adherent points, thus

$$\overline{\mathcal{F}} = \bigcap_{X \in F} \overline{X}$$

Define similarly on filter base $\mathcal{B}$ by the filter generated. By definition, we have

$$\overline{\mathcal{B}} = \bigcap_{X \in B} \overline{X}$$

Lemma 2.17. *Suppose A be a subset of Ω, then $x \in \overline{A}$ iff there is a filter $\mathcal{F}$ s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.*

Theorem 2.11. *Suppose $BN(x)$ a neighbourhood base of x, then*

1. *$\mathcal{B}$ converges to x iff every set in $BN(x)$ includes a set in $\mathcal{B}$.*
2. *$x \in \overline{\mathcal{B}}$ iff every set in $BN(x)$ meets every set in $\mathcal{B}$.*

As consequence, we have

Corollary 2.1. *x is adherent to a filter $\mathcal{F}$ iff there is $\mathcal{F}' \supset \mathcal{F}$ and converges to x*

Proof. $\implies$ follows from taking $\mathcal{F} = BN(x)$. Conversely, $\forall N \in BN(x)$, we have $X' \subset N$ for some $X' \in \mathcal{F}'$, thus for any $X \in \mathcal{F}$, $N \cap X \subset X' \cap X \neq \varnothing$ as $X', X \in \mathcal{F}'$. □

Corollary 2.2. *Every limit point of $\mathcal{F}$ is adherent to $\mathcal{F}$*

Proof. Clearly holds by applying theorem 2.11.1 and 2.11.2. □

Corollary 2.3. *Every adherent point of an ultra-filter is a limit point of it.*

Proof. Clearly as kernel of ultrafilter is a one point set. □

Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a **limit point** of f relative to $\mathcal{F}$ if $f(\mathcal{F}) \to x$.
2. an **adherent point** of f relative $\mathcal{F}$ if it's adherent point of $f(\mathcal{F})$.

Theorem 2.12. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$*

1. *x' is a limit point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, we have $f^{-1}(N') \in \mathcal{F}$.*
2. *x' is an adherent point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, it meets $f(X)$ for any $X \in \mathcal{F}$.*

Proof. x' is limit is equivalent to

$$\mathcal{N}(x') \subset f(\mathcal{F})^{\uparrow}$$

That is, there exist some $A = f(X) \subset N'$ for any N', followed by $X \subset f^{-1}f(X) \subset f^{-1}(N')$, then the claim follows from the definition of filter.

By theorem 2.11, x' is adherent to $f(\mathcal{F})$ iff

$$\forall N' \in BN(x'), \forall X \in \mathcal{F}, f(X) \cap N' \neq \varnothing$$

note for any $N' \in N'(x')$, there exist $N' \in BN(x') \ni N' \subset N'$, thus $f(X) \cap N' \neq \varnothing$ also holds. Conversely, making use of $BN(x') \subset N'(x')$. □

For example, suppose $f : (\mathbb{N}, \tau) \to (\Omega', \tau')$ and $\mathcal{F}$ the frechet filter on $\mathbb{N}$. Then x' is limit of f relative to $\mathcal{F}$ iff for all $N' \in N'(x')$, $f^{-1}(N') \in \mathcal{F} \iff f^{-1}(N')^c \subset [0, k] \iff f^{-1}(N') \supset \{n \in \mathbb{N} : n \geq k\}$ for some k, that is, $f(n) \in N'$ for any $n \geq k$.

Theorem 2.13. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ and let $\mathcal{F} = \mathcal{N}(x)$. By theorm 9, x' is limit of f relative to $\mathcal{N}(x)$ iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{N}(x) \iff N \subset f^{-1}(N') \iff f(N) \subset N'$ for some $N \in \mathcal{N}(x)$. That is, iff $x' = f(x)$, f is continous at x. Such limit points also called limit points of f at x.*

2.9 Net

$(D, \preceq)$ is called a **directed set** if every couple $\{x, y\}$ in which has an upper bound.

If $\{D_i\}_{i\in I}$ is family of directed set then $D = \prod_{i\in I} D_i$ is also directed under **product direction** defined by $(a_i)_{i\in I} \succeq (b_i)_{i\in I}$ for all $i \in I$.

Definition 2.7. Let $(D, \preceq)$ be a directed set, $\nu : D \to \Omega$ is called a **net** in Ω with domain D. The directed set is called **index set** of the net and members of D are **indexes**. We often write ν as $x.$ or $\{x_\alpha\}$.

Suppose A a subset of Ω, we say $x.$ **eventually in** A if there exist some $k \in D$ s.t. $x_n \in A$ for all $n \succeq k$. And we say ν is **frequently** in A if for all $n \in D$, there exist an $n' \succeq n$ s.t. $x_{n'} \in A$.

Lemma 2.18. *If $x.$ not frequently in A, then $x.$ eventually in A^c. Thus, for any $X \in \Omega$, $x.$ frequently in either X or X^c.*

Suppose $x \in \Omega$, then $x.$ is said **converge** to x, or $x. \to x$ if $x.$ eventually in N for all $N \in \mathcal{N}(x)$, i.e., $\mathcal{N}(x) \subset \mathcal{F}(x.)$. The point x is **adherent** to $x.$ if $x.$ frequently in N for all $N \in \mathcal{N}(x)$.

Theorem 2.14. *Suppose $A \in (\Omega, \tau)$, then $x \in \overline{A}$ iff it's the limit of some net in the set.*

Proof. $\Longleftarrow$ is clear. $\Longrightarrow$ follows from we may find a associated net taking value in A(since each neighborhood meets A) and such net converges to x. $\square$

As with sequence, if $x.$ is bounded, there is

$$\liminf x. = \sup \inf x. \preceq \limsup x. = \inf \sup x$$

Subnet generalizes subsequence.

Definition 2.8. Suppose D is directed, a subset B of D is called **cofinal** if for any $a \in D$, there exist $b \in B$ s.t. $a \preceq b$. A map $f : D \to A$ is **final** if $f(D)$ is cofinal of A.

Let $x.$ and $x.'$ are two nets in Ω with domains D and D' respectively. We say that $x.'$ is a **subnet** of $x.$ if there exists a final mapping $\varphi : D' \to D$ s.t. $x'_\alpha = x_{\varphi(\alpha)}$.

Theorem 2.15. *Let $\mathcal{A}$ be a collection of subsets that $x.$ is frequently in. If $\mathcal{A}$ is closed under finite intersection, then there exists a subnet $x'.$ of $x.$ and $x.'$ eventually in every member of $\mathcal{A}$*

Lemma 2.19. *Suppose $x.'$ is subnet of $x.$, we have*

1. $x. \to x \implies x.' \to x$
2. *x adherent to $x.'$ $\implies$ x adherent to $x.$.*

Theorem 2.16. *A point x is adherent to $x.$ iff there is a subnet converges to x. While $x. \to x$ iff every subnet converges to x.*

Proof. $\implies$ is clear by lemma2.19. Conversely, suppose a is not adherent to x, there exist a neighborhood N that $x.$ not frequently in, i.e., exist k s.t. $x_n \notin N$ for any $n \geq k$, thus there is no subnet eventually in N.

For the second part, $\implies$ is also clear by lemma2.19 and $\impliedby$ comes from taking subnet as itself. □

A net $x.$ is called **ultranet** or **universal net** if for all $X \in \Omega$, we have either $x.$ eventually in X or $x.$ eventually in X^c. Clearly, subnet of ultranet is ultra and

Lemma 2.20. *Every net has a ultra subnet.*

Proof. Consider collection of $\mathcal{Q}$ s.t. $x.$ is frequently in every member and closed under finite intersection. By Zorn's Lemma, there is a maximal $\mathcal{Q}_0$. By theorem 11, $x.$ has a subnet $x.'$ which eventually in every member of $\mathcal{Q}_0$. We claim that this subnet is ultra since, $\mathcal{Q}_0$ is maximal and thus either $X \in \mathcal{Q}_0$ or $X^c \in \mathcal{Q}_0$. □

2.10 Nets and filters

Let

$$\mathcal{F}(x.) = \{X \in \mathcal{P}(\Omega) : x. \text{ is eventually in } X\}$$

Then $\mathcal{F}(x.)$ is a filter and we call it the **filter associated with the net** $x.$.

Theorem 2.17. *Associated filter is the upward closure of the net's tail, that is*

$$\mathcal{F}(x.) = \{\{x_b : b \succeq a\} : a \in D\}$$

Motivated by the definition of filter that filter is closed under pairwise intersection, let $X \preceq Y \iff X \supset Y$, then any mapping $\nu : \mathcal{F} \to \Omega$ s.t. $\nu(X) \in X$ is a **net associated with the filter** $\mathcal{F}$.

By definition, we claim that $\mathcal{F}$ is the associated filter of every associated net and $x.$ is an associted net of the associated fiter.

Theorem 2.18. *Filter $\mathcal{F} \to x$ iff $x. \to x$ for any $x.$ associated with $\mathcal{F}$.*

Proof. Note

$$\forall N \in \mathcal{N}(x), x. \text{ eventually in } N \iff \mathcal{N}(x) \subset \mathcal{F}(x.)$$

Then is sufficient to show that $\mathcal{F}(x.) = \mathcal{F}$. It's follows from for any $X \in \mathcal{F}$, $x.$ eventually in X. □

Theorem 2.19.

$$x. \to x \iff \mathcal{F}(x.) \to x$$

Proof. Both side is equivalent to $\mathcal{N}(x) \subset \mathcal{F}(x.)$ □

Theorem 2.20. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, then f is continous at x iff $\forall x. \to x$, $f(x.) \to f(x)$.*

Proof. By theorem 2.19,2.18 and 2.13. □

By above theorems, we have

$$\text{Adh}(\mathcal{F}(x.)) = \text{Adh}(x.), \text{Lim}(\mathcal{F}(x.)) = \text{Lim}(x.)$$

and similarly results holds for any filter and one of associated nets.

Lemma 2.21. *If $x.$ is ultra then the associated filter $\mathcal{F}(x.)$ is also ultra and if $\mathcal{F}$ is ultra, every associated net is ultra.*

Proof. Directly from theorm 2.6. □

2.11 Convergence

If $\mathcal{F}$ is collection of functions on X, X can be seen as functions on $\mathcal{F}$ by $e_x(f) = f(x)$ for each $x \in X$, such functions are called **evaluation functional**.

The product topology on $\mathbb{R}^X$ is also called **topology of pointwise convergence** on X because a net $f. \to f$ iff $e_x(f.) \to e_x(f) \iff f.(x) \to f(x)$ for each $x \in X$.

There also exist induced topology $\sigma(\mathcal{F}, X)$ on $\mathcal{F}$, which is identical to the subspace $\mathbb{R}^X|_{\mathcal{F}}$ endowed the product topology. Formally

$$\sigma(\mathcal{F}, X) = \sigma(\mathbb{R}^X, X)|_{\mathcal{F}}$$

Lemma 2.22. *If $\mathcal{F}$ is total, the function*

$$x \mapsto e_x : (X, \sigma(X, \mathcal{F})) \to (\mathbb{R}^{\mathcal{F}}, \sigma(\mathbb{R}^{\mathcal{F}}, \mathcal{F}))$$

is injective and thus an embedding.

Proof. It's remain to show the continuity.

$$\begin{aligned} x. \to x &\iff \forall f \in \mathcal{F}, f(x.) \to f(x) \\ &\iff \forall f \in \mathcal{F}, e_f(e_{x.}) \to e_f(e_x) \\ &\iff e_{x.} \to e_x \end{aligned}$$

□

By Tychonoff theorem 2.44, $\mathcal{F}$ is compact iff $\forall x \in X$, $\{f(x)\}_{f \in \mathcal{F}}$ it's closed and pointwise bounded by borel theorem.

Definition 2.9. A net $f.$ converges uniformly to $f \in \mathbb{R}^X$ iff $|f.(x) - f(x)| < \epsilon$ eventually for each $x \in X$ after some f_α for any ϵ.

Theorem 2.21. *The uniform limit of a continuous net is continuous.*

Proof. Suppose $f. \to f$ uniformly, then for any $x \in X$, for any $\alpha > \alpha_0$

$$|f_\alpha(x) - f(x)| < \epsilon$$

as f_α is continuous, for any $x. \to x$, for any $\lambda > \lambda_0$

$$|f_\alpha(x_\lambda) - f_\alpha(x)| < \epsilon$$

also, there is

$$|f_\alpha(x_\lambda) - f(x_\lambda)| < \epsilon$$

Hence, we have

$$|f(x_\lambda) - f(x)| < 3\epsilon$$

Thus, $f(x.) \to f$ and continuity follows.

$\square$

Theorem 2.22 (Dini's Theorem)**.** *If continuous real function net $f.$ on a compact set converges monotonically to f pointwise, then the net converges to f uniformly.*

Proof. Let $g. = f. - f$, we have $g. \to 0$, $|g.|$ is decreasing as monotone. Then it's sufficient to show that $g. \to g$ uniformly. Note $|g.(x)| < \epsilon$ eventually for any $x \in X$ after, say, α_x. By continuity and compactness:

$$X = \bigcup_{x\in X} |g_{\alpha_x}|^{-1}(B(0,\epsilon)) = \bigcup_{x\in J} |g_{\alpha_x}|^{-1}(B(0,\epsilon))$$

Then we may pick $\alpha_0 \geq \alpha_x$ for all $x \in J$, and for any $\alpha \geq \alpha_0$ and any $x \in X$, suppose $x \in |g_{\alpha_{x_j}}|^{-1}(B(0,\epsilon))$

$$\epsilon > |g_{\alpha_{x_j}}(x)| > |g_\alpha(x)|$$

by monotone and thus $g. \to 0$ uniformly.

$\square$

2.12 Separation

Definition 2.10. Space (Ω, τ) is said to be T_0 or **kolmogorov** if for every pair $(x, y) \in \Omega^2$, either there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ or $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Lemma 2.23. *τ isn't T_0 iff there exist pair (x, y), s.t:*

1. *$\mathcal{N}(x) = \mathcal{N}(y)$.*
2. *$\overline{\{x\}} = \overline{\{y\}}$.*

Proof. 1 If every $N \in \mathcal{N}(x)$ contains y, then $N \in \mathcal{N}(y) \implies \mathcal{N}(x) \subset \mathcal{N}(y)$, thus $\mathcal{N}(x) = \mathcal{N}(y)$.

2 If some point $a \in \overline{\{x\}}$, then every $N \in \mathcal{N}(a)$ also is neighborhood of x and thus neighborhood of y, hence $a \in \overline{\{y\}}$. $\square$

Definition 2.11. Space (Ω, τ) is said to be T_1 or **Frechet** if for every pair $(x, y) \in \Omega^2$,there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ and $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Theorem 2.23. *Following statements are equivalent:*

1. τ *is* T_1.
2. *Singetons are closed.*
3. $\ker \mathcal{N}(x) = \{x\}$ *holds for any* $x \in \Omega$.

Proof. $1 \implies 2$ If there exist a singeton $\{x\}$ not closed, there is $y \in \overline{\{x\}}$, hence every neighborhood of y contains x, contradiction.

$2 \implies 3$ Suppose $\ker \mathcal{N}(x)$ contains y diifer x, that implies any neighborhood of x contains y and contradict 2.

$3 \implies 1$ is straightforward. $\square$

Lemma 2.24. *Suppose* (Ω, τ) *with a finite base is* T_1, *then* Ω *is finite and* τ *is discrete.*

Definition 2.12. A topology (Ω, τ) is T_2, or **Hausdorff** or **separated** if every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $U \cap V = \varnothing$.

Theorem 2.24. *Following statements are equivalent:*

1. τ *is* T_2.
2. *Intersection of family of closed neighborhoods of* x *is* x.
3. *If a filter(net) converges to some point* x, *then* $Adh(\mathcal{F}) = \{x\}$
4. *Every net(filter) converges to at most one point.*

Proof. $1 \implies 2$ For any pair (x, y), by definition, there is $y \notin \overline{U}$, hence intersection of family of closed neighborhoods of x can only contains x.

$2 \implies 3$ follows from a point adherent to a filter converges to x must be in every closed neighborhood of x.

$3 \implies 4$ is clearly.

$4 \implies 1$ If there is a net $x.$ converges to both x and y, then $\mathcal{N}(x) \subset \mathcal{F}(x.)$ and $\mathcal{N}(y) \subset \mathcal{F}(x.)$, that is, U and V meets for any $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$. $\square$

Definition 2.13. Space (Ω, τ) is said to be $T_{2.5}$ or **Completely Hausdorff** if for every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $\overline{U} \cap \overline{V} = \varnothing$.

Two nonempty sets are caled **separated by open sets** if they are included in disjoint open sets, and they are **separated by continous functions** if there is continos f taking values in $[0, 1]$ and assign 0 on one set and 1 on the other.

Space (Ω, τ) are said to be **regular** if every singeton and any closed A disjoint from it can be separated by open sets.

Definition 2.14. Space (Ω, τ) is said to be T_3 if it's T_1 and regular.

Space (Ω, τ) are said to **Completely regular** if every singeton and any closed A disjoint from it can be separated by continous function.

Definition 2.15. Space (Ω, τ) is said to be $T_{3.5}$ or **Tychonoff space** if it's T_1 and completely regular.

Theorem 2.25 (Tychonoff's Embedding Theorem)**.** *Space (Ω, τ) is $T_{3.5}$ iff it's homeomorphic to a subspace of $([0,1]^n, \tau_{d_{l^1}})$.*

Space (Ω, τ) is said to be **normal** if two disjoint closed subsets can be separated by open sets.

Definition 2.16. Space (Ω, τ) is said to be T_4 if it's normal and T_1.

Theorem 2.26 (Urysohn's Lemma)**.** *Following statements are equivalent:*

1. *(Ω, τ) is normal.*
2. *For any $U \in \tau$ and any closed $A \subset U$, there is a $U' \in \tau$ s.t. $A \subset U'$ and $\overline{U'} \subset U$.*
3. *Every two disjoint closed subsets can be separated by continous function.*

Proof. $1 \implies 2$ Apply normal property to A and U^c, there is a U' include A and V include U^c, as $U' \cap V = \varnothing \implies U' \subset V^c \implies \overline{U'} \subset V^c \subset U$.

$2 \implies 3$ Suppose A and B are two disjoint closed subset, apply 2 to A and $U_1 = B^c$ we have $A \subset U_0$ and $\overline{U_0} \subset U_1$. Apply again for $\overline{U_0}$ and U_1 to generates $U_0 \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$, repeat such process, that is, apply 2 to $\overline{U_{\frac{j}{2^k}}}$ and $U_{\frac{j+1}{2^k}}$ to generates $U_{\frac{2j+1}{2^{k+1}}}$. Finally, we construct a open strictly increasing squence U_r. where r is any dyadic rational in $[0,1]$, i.e., $r \in DR \cap [0,1]$.

Then define f as

$$f = \begin{cases} 1 & x \in B \\ \inf\{r : x \in U_r\} & x \in B^c \end{cases}$$

Then it's sufficient to show that f is continous. Note subspace $[0,1]$ of $\mathbb{R}$ can be generated by collection of $[0,s)$ and $(t,1]$ and

$$f^{-1}[0,s) = \bigcup_{r \in DR \cap [0,s)} U_r$$
$$f^{-1}(t,1] = \bigcup_{r \in DR \cap (t,1]} \overline{U_r}^c$$

Then the claim follows from lemma 2.9.

$3 \implies 1$ By taking any disjoint open set A contains 0 and B contains 1 and looking $f^{-1}(A)$ and $f^{-1}(B)$. ☐

Theorem 2.27 (Tietze's Extension Theorem)**.** *Let (Ω, τ) be normal, F any closed subset and I any bounded closed interval of $\mathbb{R}$. Then any continous $f : F \to I$ can be extended to $f' : \Omega \to I$ and remain continous.*

Proof. Suppose $I = [-1, 1]$, then $A = f^{-1}[-1, -\frac{1}{3}]$ and $f^{-1}[\frac{1}{3}, 1]$ are disjoint and closed. By Urysohn's Lemma, there is $g : \Omega \to [-\frac{1}{3}, \frac{1}{3}]$ s.t. $g(A) = \{-\frac{1}{3}\}$ and $g(B) = \frac{1}{3}$. Set $f_0 = f, g_0 = g, f_1 = f - g|_F$. Then we can show that $|f_1|$ is bounded by $\frac{2}{3}$.

Repeat such process, we have series of

$$\begin{aligned} f_n &: F \to [-(\frac{2}{3})^n, (\frac{2}{3})^n] \\ g_n &: E \to [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n] \\ f_{n+1} &= (f_n - g_n)|_F \end{aligned}$$

Then we show that $g = \sum_{i=0}^{\infty} g_i$ is the extension of f. That is g is continous and $f = g$ in F. Note for any x

$$|\sum_{i=m}^{n} g_i(x)| \leq \sum_{i=m}^{n} |g_i(x)| \leq \sum_{i=m}^{n} \frac{1}{3}(\frac{2}{3})^i \leq (\frac{2}{3})^m \to 0$$

Thus $\{\sum_{i=0}^{n} g_i\}_{n=0}^{\infty}$ converges uniformly by Cauchy's criterion, followed by g is continous. And $f = g$ on F follows from

$$|f(x) - \sum_{i=0}^{n} g_i(x)| = |f_0(x) - \sum_{i=0}^{n} g_i(x)| = |f_1(x) - \sum_{i=1}^{n} g_i(x)| = |f_{n+1}(x)| \leq (\frac{2}{3})^{n+1} \to 0$$

□

2.13 Compactness

Definition 2.17. A **cover** of a set K is collection of sets whose union includes K. A **subcover** is subcollection of a cover and also covers K.

Definition 2.18. K is **compact** if every open cover has a finite subcover and called **relatively compact** if it's closure is compact. A topology (Ω, τ) is **compact** if Ω is compact.

Compactness is a "topological" property. That is, subset compactness in a subspace iff it's also compact in full space.

Theorem 2.28. *Let (Ω, τ) be a space, TFAE:*

1. *(Ω, τ) is compact.*
2. *Every filter(net) has at least one adherent point.*
3. *Every ultrafilter(ultranet) converges.*
4. *$\ker \mathcal{F} \neq \varnothing$ For every collection $\mathcal{F}$ of closed sets having FIP.*

Proof. $4 \iff 1$ Taking contrapositive:

$$\neg \ker \mathcal{F} \neq \varnothing \equiv \ker \mathcal{F} = \varnothing \equiv \bigcup_{F \in \mathcal{F}} F^c = \Omega$$

And

$$\neg \forall \bigcap_i^n F_i = \varnothing \equiv \exists \bigcup_i^n F_i^c = \Omega$$

note that's precisely the definition of compactness.

$1 \implies 2$ Suppose filter $\mathcal{F}$, then

$$\{\overline{F} : F \in \mathcal{F}\}$$

Enjoy finite intersection property by definition, then $\overline{F}$ has at least one adherent point since $\ker\{\overline{F} : F \in \mathcal{F}\} = \overline{\mathcal{F}} \neq \varnothing$ by 4

$2 \implies 3$ Clearly by corollary 2.3.

$3 \implies 1$ Suppose $\mathcal{A}$ a family of closed subsets with finite intersection property. Then the filter generates by $\mathcal{A}$ has an ultrafilter with a limit point x. Note x is also adherent to $\mathcal{U}$ and thus adherent to $\mathcal{F}$, followed by $x \in A$ for any $A \in \mathcal{A}$, hence $\ker \mathcal{A} \supset \{x\}$. Then the claim follows from 4.

□

Theorem 2.29. *Let (Ω, τ) be Hausdorff, then every compact subset and disjoint singleton can be separated by open sets. In particular, compact subset is closed.*

Proof. Suppose $F \subset \Omega$ is compact, for any $x \in \Omega$ not in F, by Hausdorff, there is $x \notin U_y$ and $y \notin V_y$. Then $\bigcup_{y \in F} U_y$ cover F, there is subcover $U = \bigcup_i^n U_{y_i}$ and $V = \bigcup_i^n V_{y_i}$ selected from the same family separated F and $\{x\}$.

□

Theorem 2.30. *Closed subset is compact in compact topological space.*

Proof. Note any open cover of F plus F^c become a open cover of Ω.

□

Theorem 2.31. *Every compact Hausdorff space is normal.*

Proof. Suppose A and B are closed and thus compact by theorem 2.30. For any point $x \in A$, there exist disjoint $V_x \supset B$ and $x \in U_x$ by theorem 2.29. Note $\bigcup_{x \in A} U_x$ cover A, there exist subcover $U = \bigcup_i^n U_{x_i} \supset A$ and $V = \bigcap_i^n V_{x_i} \supset B$ separated A and B.

□

Theorem 2.32. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ is continuous, then $f(A)$ is compact if A is compact.*

Proof. For any open cover of $f(A)$:

$$\cup G_i \supset f(A) \implies f^{-1}(\cup G_i) = \cup f^{-1}(G_i) \supset f^{-1} f(A) \supset A$$

Thus there exist subcover s.t.

$$\cup_1^n f^{-1}(G_i) = f^{-1}(\cup_1^n G_i) \supset A \implies \cup_1^n G_i \supset f f^{-1}(\cup_1^n G_i) \supset f(A)$$

Which shows that $f(A)$ is compact.

□

Corollary 2.4. *Let X be compact and Y be Hausdorff and $f : X \to Y$ is continuous bijection, then f is closed.*

Proof. Note F is closed and thus compact as theorem 2.30 then $f(F)$ is compact as theorem 2.32 and thus closed by theorem 2.29.

□

As consequence:

Corollary 2.5 (Extreme value theorem)**.** *A continuous real valued function defined on a compact space achieves its maximum and minimum values.*

Theorem 2.33. *Let X be compact and Y be Hausdorff and $f : X \to Y$ is continuous bijection. Then f is homeomorphism.*

Proof. By lemma 2.10 and corollary 2.4.

□

2.13.1 Sequentially compact

A subset A of a topological space is **sequentially compact** if every sequence in A has a subsequence converging to an element of A. A topological space is sequentially compact if itself is a sequentially compact set.

Example 2.1. The open interval $(0, 1)$ is not sequentially compact because $\{\frac{1}{n}\}$ has no convergent subsequence.

2.14 Locally compact spaces

Definition 2.19. A topological space is **locally compact** if every point has a compact neighborhood.

Definition 2.20. Subset $A \subset X$ is said **precompact** if $\overline{A}$ is compact.

Theorem 2.34 (Compact neighborhood base)**.** *Let X be Hausdorff, TFAE*

1. *X is locally compact.*
2. *Every $x \in X$ has a precompact neighborhood.*
3. *X has a basis of precompact open sets, i.e., there exist $x \in K^\circ \subset K \subset N$.*

Proof. It's clear that $3 \Rightarrow 2 \Rightarrow 1$ even without Hausdorff, so we show that $1 \Rightarrow 3$.

Begin by open G and compact K neighborhood for x s.t. $A := K - G \neq \varnothing$. For any $y \in A$, there is $U_y \cap W_y = \varnothing$ by Hausdorff, where $y \in U_y$ and $x \in W_y \subset K$. Note A is also compact and then there exist:

$$U = \bigcup_{i=1}^{k} U_{y_i} \supset A$$

Respectively, consider $W = \bigcap_{i=1}^{k} W_{y_i}$, and we claim that $\overline{W}$ is compact and included in G. Compactness is clear as $\overline{V} \subset K$. By theorem 2.29, $\overline{W} \cap U = \varnothing$. Consequently,

$$\overline{W} \cap G^c = \overline{W} \cap K \cap G^c = \overline{W} \cap A \subset \overline{W} \cap U = \varnothing$$

hence $\overline{W} \subset G$.

$\square$

Consequently, that imply the existence of a compact neighborhood base.

Corollary 2.6. *Suppose G is open and F is closed in a locally compact Hausdorff space, then $G \cap F$ is locally compact. That implies every closed and open set is locally compact.*

Proof. Let $x \in G \cap F$, and $N \cap G \cap F$ be neighborhood of x in the subspace, by theorem 2.34, there exist K s.t.

$$x \in K^\circ \subset K \subset N \cap G$$

Then $F \cap K$ is compact as it's closed in compact Hausdorff subspace K.

□

Corollary 2.7. *If K is compact in a locally compact Hausdorff space and G is an open set including K, then there is an open V with compact closure s.t.*

$$K \subset V \subset \overline{V} \subset G$$

Proof. For any $x \in K$, by theorem 2.34, we have

$$x \in V_x \subset \overline{V_x} \subset G$$

then note

$$K \subset \bigcup_{i=1}^{k} V_{x_i} = V$$

we claim that V is desired. Since

$$\overline{V} = \overline{\bigcup_{i=1}^{k} V_{x_i}} = \bigcup_{i=1}^{k} \overline{V_{x_i}}$$

is compact and included in G.

□

2.14.1 Compactification

Locally compact Hausdorff space is very close to a compact Hausdorff space

Definition 2.21. A **Compactification** of a space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

Definition 2.22. Let (X, τ) be a space and define $\hat{X} = X \cup \{\infty\}$, with topology $\hat{\tau}$ consisting of sets that:

1. $G \in \tau$.
2. $\infty \in G$ and $\hat{X} - G = X - G \subset X$ is compact.

Theorem 2.35. *If X is Hausdorff and noncompact, then $\hat{X}$ is a compactification.*

Proof. Firstly we show that $\hat{X}$ is a space. By definition, $\varnothing$ and $\hat{X}$ are open clearly. To show it's closed under countable intersection, it suffices to show that $U_1 \cap U_2$ is open when U_1 and U_2 are so. We classify cases by whether ∞ occurs.

1. If $\infty \notin U_1 \cup U_2$, $U_1 \cap U_2 \in \hat{\tau}$ as $U_1 \cap U_2 \in \tau$.
2. If $\infty \in U_1$ and $\infty \notin U_2$, then $X - U_1$ is compact, as X is Hausdorff, $X - U_1$ is closed in X and thus $X - (X - U_1) = U_1 - \{\infty\}$ is open in X, it follows that $U_1 \cap U_2 = (U_1 - \{\infty\}) \cap U_2$ and the same as 1.
3. If $\infty \in U_1 \cap U_2$, then
$$X - (U_1 \cap U_2) = (X - U_1) \cup (X - U_2)$$
is compact as it's union of compact sets and thus $U_1 \cap U_2$ is open.

Now we turn to show closed under union. Suppose $\bigcup_{i \in I} U_i$ is a collection of open sets. If none contain ∞, $\bigcup_{i \in I} U_i$ is open clearly as it's open in X. If $\infty \in U_j, \forall j \in J$ for some $J \subset I$. Then

$$X - \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i)$$

is closed subset of any compact Hausdorff space $X - U_j$ and thus compact. It follows that $\bigcap_{i \in I} U_i$ is open.

Next, we show that $\iota : X \to \hat{X}$ is an embedding. It's injective and open clearly and it suffices to show it continuity by lemma 2.10. For open sets G in $\hat{X}$:

$$\iota^{-1}(G) = \begin{cases} G & \infty \notin G \\ G - \{\infty\} & \infty \in G \end{cases}$$

is also open as $G - \{\infty\} = X - (X - G)$ is open have shown above.

To see $\iota(X)$ is dense, it suffices to see $\{\infty\}$ is not open and that follows from definition of $\hat{X}$.

Finally, we show that $\hat{X}$ is compact. Let $\mathcal{G}$ be open cover, then there is some $G \in \mathcal{G}$ contains ∞. Note remaining of $\mathcal{G}$ still cover $X - G$ and thus have a finite cover then claim follows easily,

$\square$

Lemma 2.25. *If noncompact X is Hausdorff and locally compact, $\hat{X}$ is also Hausdorff.*

Proof. Let x_1 and x_2 in $\hat{X}$. If neither is ∞, we have desired disjoint neighborhood immediately. If $x_2 = \infty$, let $x_1 \in U \subset K$ then U and $V = \hat{X} - K$ are what we desired.

□

Lemma 2.26. *$\hat{X}$ is not Hausdorff if there is no subset G and K of X s.t. $G \subset K$.*

Proof. Suppose $\hat{X}$ is Hausdorff, then there is $\infty \in U$ s.t. $K = X - U$ is compact and disjoint to some V open in X, note

$$\begin{aligned} U \cap V = \varnothing &\Rightarrow (U - \{\infty\}) \cap V = \varnothing \\ &\Rightarrow (X - K) \cap V = \varnothing \\ &\Rightarrow V \subset K \end{aligned}$$

□

Example 2.2. $\hat{\mathbb{Q}}$ is non Hausdorff as any open sets G of the form $(a, b) \cap \mathbb{Q}$, if it's contained in a compact subset K, then $\overline{G}$ would be compact, which contradict to $[a, b] \cap \mathbb{Q}$ is not compact.

Theorem 2.36. *X is locally compact iff X is open of $\hat{X}$.*

Proof. $\Longleftarrow$ comes from corollary 2.6.

$\Longrightarrow$ Suppose $(\hat{X}, \hat{\tau})$ is compactification of Hausdorff (X, τ). For any $x \in X$, we may pick $x \in G \subset K$, where G is open and K is compact in τ. Consider $W \in \hat{\tau}$ where $W \cap X = G$, we have

$$W = W \cap \hat{X} = W \cap \overline{X} \subset \overline{W \cap X} = \overline{G} \subset K \subset X$$

that implies $x \in X^\circ \implies X^\circ = X$, i.e. X is open.

□

Lemma 2.27. *Let X be a locally compact Hausdorff space and $f : X \to Y$ a compactification, then f is open.*

Proof. As f is an embedding, we can pretend $X \subset Y$ and f is just inclusion. Then it suffices to show that X is open and that follows from theorem 2.36.

□

Theorem 2.37 (Universal property of compactification)**.** *Let X be a locally compact Hausdorff space and $f : X \hookrightarrow Y$ be a compactification. Then there is a unique quotient map $q : Y \to \hat{X}$ s.t. $q \circ f = \iota$.*

$$\begin{tikzcd} Y \arrow[rr, dashed, "\exists! q"] & & \hat{X} \\ & X \arrow[ul, "f"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

Let X be locally compact and Hausdorff and let $f : X \hookrightarrow Y$ be a compactification. Then there is a unique quotient map q: $Y \to \hat{X}$ s.t. $q \circ f = \iota$.

2.15 Weak topology

Suppose $\{(Y_i, \tau_i)\}_{i\in I}$ a family of topological space and $f_i : X \to Y_{i_{i\in I}}$. Let $\mathcal{F}$ be the set of all the topologies s.t. f_i is continuous for all i. We call $\cap\mathcal{F}$, i.e., the coarsest topology the **induced topology** or **weak topology** or **initial topology** on X by $\{f_i\}_{i\in I}$. The topology induced by $\{f_i\}_{i\in I}$ is generated by

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \tau_i\}$$

or

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \mathcal{S}_i\}$$

where $\mathcal{S}_i$ is a subbase for τ_i.

Lemma 2.28. *A net $x. \to x$ in the weak topology iff $f_i(x.) \to f_i(x)$ for each i.*

Proof. $\implies$ is immediately. Conversely, noting sets of the form $\bigcap_1^n f_i^{-1}(V_i)$ consist a neighborhood base.

□

Theorem 2.38. *g is (τ', τ) continuous iff $f_i \circ g$ continuous for each f_i. Where τ is $\tau(S)$ in above .theorem.*

Proof. $\implies$ is immediately. $\impliedby$, suppose $G \in \tau$,by above .theorem, this implies

$$G = \cup_I \cap_F X = \cup_I \cap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continuous and thus $g^{-1}(G) = \cup_I \cap_F g^{-1} f^{-1}(G) = \cup_I \cap_F (f \circ g)^{-1}(G)$.

□

If the family $\mathcal{F}$ consists of real function on X, the weak topology is denoted $\sigma(X, \mathcal{F})$. A subbase for $\sigma(X, \mathcal{F})$ consist of

$$U(f, x, \epsilon) = f^{-1}(B(f(x), \epsilon)) = \{y \in X : |f(y) - f(x)| < \epsilon\}$$

where $f \in \mathcal{F}, x \in X, \epsilon > 0$. $\mathcal{F}$ is said **total** if $\forall f \in \mathcal{F}, f(x) = f(y) \implies x = y$. $\sigma(X, \mathcal{F})$ is Hausdorff iff $\mathcal{F}$ is total.

Lemma 2.29. *Let A be a subset, then*

$$(A, \sigma(A, \mathcal{F}|_A)) = (A, \sigma(X, \mathcal{F})|_A)$$

Proof. Nets converges in $(A, \sigma(X, \mathcal{F})|_A)$ also converges in $(X, \sigma(X, \mathcal{F}))$, that is $\forall f, f_i(x.) \to x.$ and thus the same as nets converges in $\sigma(A, \mathcal{F}|_A)$. That implies identical mapping is a homeomorphism since $x. \to x \iff I(x.) \to I(x)$.

$\square$

The weak topology generated by $C(X)$ is also generated by $C_b(X)$ by noting for any $f \in C(X)$,

$$g(y) = \min\{f(x) + \epsilon, \max\{f(x) - \epsilon, f(y)\}\}$$

is bounded by $B(f(x), \epsilon)$ and $U(g, x, \epsilon) = U(f, x, \epsilon)$.

Theorem 2.39. *(X,) is completely regular iff $\tau = \sigma(X, C(X))$*

Suppose $\tau = \sigma(X, \mathcal{F})$ and is completely regular, then we claim that $\mathcal{F} = C(X)$.

2.16 Product topology

Theorem 2.40 (Universal property of the Cartesian product)**.** *Let X, Y and Z be any space and given $f_X : Z \to X$ and $f_Y : Z \to Y$, there exist unique function $f : Z \to X \times Y$ s.t.*

$$f_X = \pi_X \circ f \textit{ and } f_Y = \pi_Y \circ f$$

and f is just (f_X, f_Y).

$$\begin{tikzcd}
 & & X \\
Z \arrow[r, dashed, "\exists! f"'] \arrow[urr, bend left, "f_X"] \arrow[drr, bend right, "f_Y"'] & X \times Y \arrow[ur, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
 & & Y
\end{tikzcd}$$

Lemma 2.30. *Suppose $\varphi : X \times Y \to Z$ is continuous, for each $x \in X$, define $\hat{\varphi} : Y \to Z$ by $\hat{\varphi}_x(y) = \varphi(x, y)$, then φ_x is continuous.*

Proof. Note $\hat{\varphi}_x$ is composition by $Y \xrightarrow{i_x} X \times Y \xrightarrow{\varphi} Z$, so it suffices to show that i_x is continuous. And that is just the product of constant map $Y \to X$ and identity map $Y \to Y$. Then the claim follows as both is continuous.

□

Also, φ is continuous if $\hat{\varphi}$ is continuous as φ is composition by

$$X \times Y \xrightarrow{\hat{\varphi}\times i} \mathcal{C}(Y, Z) \times Y \xrightarrow{eval} Z$$

Where we should use the truth that product of continuous function is continuous:

Theorem 2.41. *Let $f : X \to Y$ and $f' : X' \to Y'$ be continuous. Then the product $f \times f' : X \times X' \to Y \times Y'$ is also continuous.*

Proof. Clearly as the factor $X \times X' \to Y$ is the composition $X \times X' \xrightarrow{\pi_X} X \xrightarrow{f} Y$

□

Let $((\Omega_i, \tau_i))_{i\in I}$ be family of topological spaces, let $\Omega = \prod_{i\in I} \Omega_i$ and π_i be projection mappings from Ω to Ω_i. The topology τ induced by $(\pi_i)_{i\in I}$ is called **product topology** on Ω and denoted by $\prod_{i\in I} \tau_i$. (Ω, τ) is called **topological product**.

A subbase of this topology is all the sets of the form $\pi_i^{-1}(U_i) = \prod_{i\in I} X_i$ where $X_j = \Omega_j$ for all $i \neq j$ and $X_i = U_i$.

Lemma 2.31. *Suppose $G \in \prod \tau_i$, then $\pi_i(G) = \Omega_i$ except a finite set in I.*

Proof. By definition,

$$G = \bigcup_I \bigcap_F (\prod_{i\in I} X_i)$$

where $X_i = \Omega_i$ for all i but one. Note there is a finitely intersection, that is

$$G = \bigcup_I (\prod_{i\in I} X_i)$$

where $X_i = \Omega_i$ for all i but finite exception. And the claim is easily follows.

□

The product topology satisfy similar universal property if I is finite, that is

Theorem 2.42. *Given any space Z and $\{f_i : Z \to \Omega_i\}_{i\in I}$, there exist unique continuous $f : Z \to \prod_{i\in I}\Omega_i$ s.t. $\forall i \in I, \pi_i \circ f = f_i$.*

Proof. Existence is clear as we may define f by $f(z)_i = f_i(z)$ and $\pi_i \circ f = f_\alpha$ suggests the uniqueness. Then it suffices to show that continuity. Note the product topology has subbasis $\pi_i^{-1}(U_i)$ and

$$f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i) = f_i^{-1}(U_i)$$

is open as f_i is continuous.

□

We call the topology generated by $\{\prod_{i\in I} U_i\}$ **box topology** and it's finer than product topology unless I is finite and can't enjoy universal property. But they still share following property.

Lemma 2.32. *Let $A_i \subset \Omega_i$ for each $i \in I$, then*

$$\prod_{i\in I}\overline{A_i} = \overline{\prod_{i\in I} A_i}$$

in both product and box topology.

Proof. $\subset$: Let $(x_i)_{\{i\in I\}} \in \prod_{i\in I}\overline{A_i}$, and $U = \prod_{i\in I} U_i$ be a open neighborhood of which, then U_i is neighborhood of x_i and thus U_i meet A_i in, say, y_i, then we may find $(y_i) \in U \cap \prod_{i\in I} A_i$ and thus $(x_i) \in \overline{\prod_{i\in I} A_i}$.

$\supset$: Note product closed set is closed as

$$\left(\prod_{i\in I} F_i\right)^c = \bigcup_{i\in I}\prod_{i=I} X_i$$

Where $X_j = \Omega_j$ for $j \neq i$ and $X_i = F_i^c$, that is open clearly. And the claim follows as closure is minimum.

□

Lemma 2.33. *Ω_i is Hausdorff for each i iff so is $\prod_{i\in I}\Omega_j$ in both product and box topology.*

Proof. $\Rightarrow$: Pick any different (x_i) and (x_i') in $\prod_{i\in I}\Omega_i$ and suppose $x_\ell \neq x_\ell'$ for particular ℓ and they can be separated by U_ℓ and U_ℓ'. Then (x_i) and $(x_i)'$ can be separated by $\pi_\ell^{-1}(U_i)$ and $\pi_\ell^{-1}(U_i')$ and thus Hausdorff. For box topology, it's Hausdorff clearly as it's finer than product topology.

$\Leftarrow$: Note Hausdorff property is hereditary and we may treat factor Ω_ℓ as subspace by define embedding

$$f_\ell(x)_j : \Omega_\ell \to \prod_{i \in I} \Omega_i = \begin{cases} x & j = \ell \\ y_j & j \neq \ell \end{cases}$$

where y_j is any fixed point for each j. It's continuous and injective certainly, to see it's embedding, it suffices to show that it's open. Suppose any open $U_\ell \subset \Omega_\ell$, then

$$f_\ell(U_\ell) = \pi_\ell^{-1}(U_\ell) \cap f_\ell(\Omega_\ell)$$

is open in subspace $f_\ell(\Omega_\ell)$.

$\square$

Thus,$\{(x_i^\alpha)\}_{\{i \in I\}}$ in X converges to some $(x_i)_{i \in I}$ iff its every components converges to the components respectably. A function is called **jointly continuous** if it's continuous w.r.t. the product topology.

Theorem 2.43 (Closed Graph Theorem)**.** *Function $f : (X, \tau) \to (Y, \tau)$ where Y is compact Hausdorff is continuous iff its graph $\mathrm{Gr}\, f$ is closed.*

Proof. $\implies$. For any net $(x., y.) \to (x, y)$, we show that $(x, y) \in \mathrm{Gr}\, f$. Note $f(x.) = y. \to y$, also, $f(x.) \to f(x)$ by continuity. It follows by $f(x) = y$ since Hausdorff and we finished.

$\impliedby$. Since Y is compact and Hausdorff, $f(x.)$ converges to precisely one point and denoted as y. As $\mathrm{Gr}\, f$ is closed, $y = f(x)$ and hence f is continuous.

$\square$

Suppose A_i is subset of each i, then

$$\underset{\tau}{\mathrm{Cl}}(\prod A_i) = \prod(\underset{\tau_i}{\mathrm{Cl}}(A_i))$$

Thus we have an alternative definition of semicontinuous:

$f : X \to \mathbb{R}^*$ is

- lower semicontinuous iff its epigraph $\{(x, c) : c \geq f(x)\}$ is closed.
- upper semicontinuous iff its hypograph $\{(x, c) : c \leq f(x)\}$ is closed.

Theorem 2.44 (Tychonoff Product Theorem)**.** *The product topology of a family of topologies $\tau = \prod_{i \in I} \tau_i$ is compact iff τ_i is compact for every $i \in I$.*

Proof. $\Longrightarrow$ is clearly as projection is continuous.

$\Longleftarrow$, suppose $\mathcal{U}$ is ultrafilter in τ, then $\pi_i(\mathcal{U})$ is ultra base and thus converges to some point, say x_i, then we claim that $\mathcal{U} \to x = (x_i)_{i\in I}$. Suppose V any neighborhood of x, there is

$$a \in \bigcap_{i\in J} \pi_i^{-1}(X_i) \subset V$$

where X_i is neighborhood of x_i and thus belong to $\pi_i(\mathcal{U})^{\uparrow}$, that implies there is $U \in \mathcal{U}$ s.t. $\pi_i(U) \subset X_i$, note $U \subset \pi_i^{-1}\pi_i(U) \subset \pi_i^{-1}(X_i)$, then $\pi_i^{-1}(X_i) \in \mathcal{U}$ and thus $V \in \mathcal{U}$. It followed by x is adherent to $\mathcal{U}$ and thus $\mathcal{U} \to x$ as $\mathcal{U}$ is ultra.

□

As consequence, we have

Theorem 2.45. *In the same notations, let K_i be compact for each i, G is open in τ and including $\prod_{i\in I} K_i$, then there exist basic open set sandwich by them.*

2.17 coinduced topology

If we turn all of the arrows around in the diagram of product, that is,

Theorem 2.46. *Given space Z and f_X and f_Y, there is a unique map from $X \coprod Y$ to Z:*

$$\begin{tikzcd}
X \arrow[dr, "\iota_X"'] \arrow[drr, bend left, "f_X"] & & \\
& X \coprod Y \arrow[r, dashed, "\exists! f"'] & Z \\
Y \arrow[ur, "\iota_Y"] \arrow[urr, bend right, "f_Y"'] & &
\end{tikzcd}$$

The coproduct of $\{X_i\}_{i\in I}$ is given by

$$\coprod_{i\in I} X_i = \bigcup_{i\in I} (X_i \times \{i\})$$

Clearly, there are nature inclusions $\iota_{X_i} : X_i \hookrightarrow \coprod_{i\in I} X_i = x_i \mapsto (x_i, i)$. We topologize the coproduct by giving it the finest topology s.t. all ι_{X_i} are continuous.

Proof. Suppose $V \subset Z$ is open, then is open in $\coprod_{i\in I} X_i$ if each $\iota_i^{-1} f^{-1}(V)$ is open. Note

$$\left(f \circ \iota_i\right)^{-1}(V) = f_i^{-1}(V)$$

is open as each f_i is continuous.

$\square$

Lemma 2.34. *Let X_i be a space for $i \in I$, then $\coprod_{i\in I} X_i$ is Hausdorff iff all X_i are Hausdorff.*

Proof. $\Rightarrow$ is trivial as X_i embeds as a subset. For $\Rightarrow$, suppose $x \neq y$ in $\coprod_{i\in I} X_i$, if x and y come from different X_i, we simple select X_i and X_j they live, otherwise, X_i is Hausdorff and guarantee a disjoint neighborhood.

$\square$

2.17.1 Quotient

Suppose $q : X \to Y$ is any subjective function, we define $\sim$ by $x \sim y$ if $q(x) = q(y)$, then $X/\sim \to Y$ is bijection and we can treat q as function that $X/\sim \to Y$. And that gives the universal property of the quotient.

Definition 2.23. A surjection $q : X \to Y$ is a **quotient map** if $V \subset Y$ is open iff $q^{-1}(V)$ is open in X.

Theorem 2.47 (Universal property of quotient)**.** *Let $q : X \to Y$ be a quotient map and $f : X \to Z$ is continuous and constant on the fiber of q, then there exist a unique continuous $g : Y \to Z$.*

$$\begin{tikzcd} X \arrow[dr, "q"'] \arrow[drr, "f"] & & \\ & Y \arrow[r, dashed, "\exists! g"'] & Z \end{tikzcd}$$

Proof. Clearly g must be defined by $g = f \circ q^{-1}$ and it remains to show that g is continuous. Let $G \subset Z$ is open then $g^{-1}(G) \subset Y$ is open iff $q^{-1}(g^{-1}(G)) = (g \circ q)^{-1}(G) = f^{-1}(G)$ is open, and that follows from f is continuous.

$\square$

Lemma 2.35. *Let $q : X \to Y$ be a continuous open surjection, then it's quotient map. The same is true if q is closed instead of open.*

Proof. Open case follows easily. For the other, for $V \subset Y$ s.t. $q^{-1}(V) \subset X$ is open, then $q^{-1}(V^c)$ is closed and thus $q(q^{-1}(V^c)) = V^c$ is close as surjection.

$\square$

However, the converse is not true.

Definition 2.24. Let $q : X \to Y$ be a continuous surjection. We say $U \subset X$ is **saturated** w.r.t. q if $U = q^{-1}(V)$ for some $V \subset Y$, i.e., $q^{-1}(q(U)) = U$.

Lemma 2.36. *Let $q : X \to Y$ be a continuous surjection, then it's a quotient map iff it takes saturated open sets to open sets.*

Proof. Suppose $q^{-1}(V) \subset X$ is open, then it's a saturated open sets, thus $q(q^{-1}(V)) = V$ is open. And the other implication follows from definition of continuity and quotient map.

□

Suppose $\{(\Omega_i, \mathcal{T}_i)\}_{i\in I}$ a family of topological space and $\{f_i : (\Omega_i, \mathcal{T}_i) \to (\Omega, \tau)\}_{i\in I}$. Let A be the set of all the topologies s.t. f_i is continuous for all i. We call the finest of A **topology coinduced** on Ω by $\{(f_i)\}_{i\in I}$.

Let R an equivalence relation on Ω, $\eta : \Omega \to \Omega/R$ the canonical surjection. The coinduced topology on Ω/R by η is denoted by τ/R and $(\Omega/R, \tau/R)$ is the quotient space w.r.t. R.

2.18 Connection

Definition 2.25. Two subset A and B are said to be **separated** if

$$A \cap \overline{B} = B \cap \overline{A} = \varnothing$$

Clearly, if disjoint A and B are both open or closed, they are separated.

Definition 2.26. Two nonempty separated subset A and B are called a **separation** if $A \cup B = X$.

Lemma 2.37. *Separation are both clopen.*

Proof. Suppose A and B is a separation, then

$$\overline{A} = \overline{A} \cap X = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = \overline{A} \cap A = A$$

thus A and B are closed, that implies A and B are open.

□

Definition 2.27. Space X is said to be **connected** if the only clopen set is X and $\varnothing$. Not connected space is said to be disconnection. Subset A is said to be *connected* or *disconnected* according to the connectedness of their subspace (A, τ_A)

Note separation are clopen, thus X is disconnected iff there exist a separation in X.

Theorem 2.48. *Suppose A is connected in X, then every set B s.t. $A \subset B \subset \overline{A}$ is connected.*

Proof. Suppose B is disconnected and separated by X and Y, then

$$A = (A \cap X) \cup (A \cap Y)$$

also construct a separation, as A is connected, we have, say $A \cap X = \varnothing$ and thus $A \subset Y$. It follows that

$$X \subset B \subset \overline{A} \subset \overline{Y}$$

whence contradict to $X \cap \overline{Y} = \varnothing$.

□

Theorem 2.49. *Suppose $\{A_i\}_{i\in I}$ is a family of connected subsets, then $A = \bigcup_{i\in I} A_i$ is connected if* $\ker\{A_i\}_{i\in I} \neq \varnothing$.

Proof. Suppose A is disconnected and separated by X and Y, then

$$A_i = A_i \cap A = (A_i \cap X) \cup (A_i \cap Y)$$

also construct a separation, as A_i is connected, we have $A_i \cap X = \varnothing$ or $A_i \cap Y = \varnothing$, suppose $I_X + I_Y = I$ and $A_i \cap X = \varnothing$ for $i \in I_X$ and $A_i \cap Y = \varnothing$ for $i \in I_Y$. Note $A_i \cap X = \varnothing \Rightarrow A_i \cap Y = A_i$ and thus

$$\begin{aligned}
\varnothing = X \cap Y &\supset (X \cap \bigcap_{i\in I_Y} A_i) \cap (Y \cap \bigcap_{i\in I_X} A_i) \\
&= \left(\bigcap_{i\in I_Y} A_i\right) \cap \left(\bigcap_{i\in I_X} A_i\right) \\
&= \ker\{A_i\}_{i\in I}
\end{aligned}$$

A contradiction.

□

Theorem 2.50. *Suppose $f : X \to Y$ is continuous, then f bring connected set subset $A \subset X$ to connected subset of Y.*

Proof. Suppose $f(A)$ is disconnected and separated by two open set, say, $f(A)\cap U$ and $f(A)\cap V$, where U, V are open in Y. That implies $f(A)\subset U\cup V$, note

$$A\subset f^{-1}f(A)\subset f^{-1}(U\cup V)=f^{-1}(U)\cup f^{-1}(V)$$

thus A is separated by $A\cap f^{-1}(U)$ and $A\cap f^{-1}(V)$, say, $A\cap f^{-1}(U)=\varnothing$, then

$$A\subset f^{-1}(V)\Rightarrow f(A)\subset V\Rightarrow f(A)\cap U=\varnothing$$

A contradiction.

□

Theorem 2.51. *Suppose each of family $\{X_i\}_{i\in I}$ is nonempty, then their product topology $\prod_{i\in I} X_i$ is connected iff each X_i is closed.*

Proof. $\Rightarrow$ follows from π_i is continuous and theorem 2.50(uses each X_i is nonempty).

$\Leftarrow$ Firstly, we should prove that in finite case, i.e., when I is finite. By induction, it suffices to show that $X_1\times X_2$ is connected. Pick fixed $z\in X_2$ we have the embedding $f(x): X_1\to X_1\times X_2=(x,z)$ and thus $D=f(X_1)$ is connected as theorem 2.50. Then for each $x\in X$, define embedding $g_x(y)=(x,y)$, let $D_x=g_x(X_2)\cup C$, it's connected as theorem 2.49, then $X_1\times X_2=\bigcup_{x\in X_1} D_x$ is connected for the same reason.

Now we are ready for the general case. Pick some $(z_i)_{i\in I}\in\prod_{i\in I} X_i$, for each finite collection $S_j\subset I$, let

$$F_{S_j}=\bigcap_{i\notin S_j}\pi_i^{-1}(z_i)\subset\prod_{i\in I} X_i$$

Clearly $F_{S_j}\cong\prod_{i\in S_j} X_i$, so it follows that F_{S_j} is connected and $(z_i)\in F_{S_j}$ for each S_j, so it follows that

$$F=\bigcup_{j\in J} F_{S_j}$$

is connected. Then it remains to show that F is dense in $\prod_{i\in I} X_i$ as lemma **??**. Recall any nonempty basis element of the form $\bigcap_{i\in S_j}\pi_i^{-1}(U_i)$ for some S_j and thus meet $F_{S_j}(X\times\cdots\times X\times U\times\cdots\times U\times X\times\cdots\times X$ and $z\times\cdots\times z\times X\times\cdots\times X\times z\times\cdots\times z)$, that implies F must be dense.

□

Definition 2.28. $A\subset X$ is said **path-connected** if every distinction singleton a and b has a **path** $f:[0,1]\to A$ s.t. $f(a)=0$ and $f(b)=1$.

Lemma 2.38. *Path-connected implies connected.*

Proof. Pick any $a_0 \in A$, for each other $b \in A$, there exit a path f_b, then $f_b(I)$ is connected. Then

$$A = \bigcup_{b \in A} f_b(I)$$

is connected as theorem 2.49.

□

Path-connected is quite similar to connected.

Theorem 2.52.
1. *Image of path-connected spaces are path-connected.*
2. *Overlapping unions of path-connected spaces are path-connected.*
3. *Product is path-connected iff every factor is path-connected.*

Proof. We only prove part 3. $\Rightarrow$ is trivial. To achieve $\Leftarrow$, for any pair (x_i) and (y_i), there exist path f_i for each $i \in I$, and then we get a continuous path $f = (f_i)$ by the universal property.

□

The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 2.29. Let $x \in X$, **connected component** of x is defined as:

$$C_x = \bigcup \{C | C \text{ is connected and } x \in C\}$$

Similarly, the **path-component** is

$$PC_x = \bigcup \{C | C \text{ is path-connected and } x \in C\}$$

Example 2.3. Suppose $\mathbb{Q}$ equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are singletons, so $C_x = \{x\}$. Such a space is said **totally disconnected**

In the light of connected component is maximum, each component C_x is closed as $\overline{C_x}$ is connected.

Definition 2.30. Let X be a space, it's **locally connected** if any neighborhood U of any x contains a connected neighborhood. And we define **locally path connected** in a similar way.

Theorem 2.53. *Let X be a space. TFAE:*

1. *X is locally connected.*
2. *X has a basis consisting of connected open sets.*
3. *For every open set $G \subset X$, any component $C \subset U$ is open in X.*

Proof. $1 \Rightarrow 3$. For any open $G \subset X$ and any $C \subset G$, for any $x \in C$, there exist connected neighborhood $x \in U \subset G$, as C is component, we have $U \subset C$ and thus C is open.

$3 \Rightarrow 1$. Let G be a open neighborhood of x, then the component C_x is the desired neighborhood.

$3 \Leftrightarrow 2$. $3 \Rightarrow 2$ is clear, for the converse, note $2 \Rightarrow 1$ and thus implies 3.

□

The property of path-connected is even better.

Theorem 2.54. *Let X be a space, TFAE:*

1. *X is locally path-connected.*
2. *X has a basis consisting of path-connected open sets.*
3. *For every open $G \subset X$, the path-component of G are open in X.*
4. *For every open set $G \subset X$, every component of G is path-connected and thus a path-component.*

Proof. We only show that $1 \Leftrightarrow 4$. Suppose X is locally path-connected, and let $P \subset C \subset G \subset X$, where P, C, G are path-component, component and open set respectly. Then P is open.

□