

Notes of Infinite dimensional analysis

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Chapter 1

Odds and ends

1.1 Space of sequences

Definition 1.1. For $1 \leq p < \infty$, ℓ_p is defined to be the set of all sequences $x. = (x_1, x_2, \dots)$ for which $\|x\|_p < \infty$. Where

$$\|x\|_p = \left(\sum_1^{\infty} |x_i|^p \right)^{1/p}$$

is the ℓ_p **norm** of the sequences.

While ℓ_{∞} is defined as the set of all $\sup\{|x_n|\} \leq \infty$, such norm is called ℓ_{∞} **norm**, **supremum norm** or **uniform norm**.

All of these spaces are vector space. And sequence $\{\ell_i\}_{i=1}^{\infty}$ is increasing.

The space of all convergent sequence is denoted c and all sequences convergent to 0 is denoted c_0 . Finally, the collection of sequences with finite nonzero terms is φ . One can check that

$$\varphi \subset \ell_p \subset c_0 \subset \ell_{\infty} \subset \mathbb{R}^n$$

1.2 Spaces of functions

One can think \mathbb{R}^n as

$$\{f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}\} = \mathbb{R}^n = \mathbb{R}^{\{1, 2, \dots, n\}}$$

Replace $\{1, 2, \dots, n\}$ by an arbitrary X , then \mathbb{R}^X is all functions from X to \mathbb{R} .

For $1 \leq p < \infty$, $L_p(\mu)$ is defined to be the set of all μ measurable functions f for which $\|f\|_p < \infty$, where the L_p **norm** is defined as

$$\|f\|_p = \left(\int_{\Omega} |f|^p \right)^{1/p}$$

And the L_{∞} **norm**, or **essential supremum** is defined as

$$\|f\|_{\infty} = \text{ess sup } f = \sup \{t : \mu(\{x : |f(x)| \geq t\}) > 0\}$$

1.3 Ordinals

Suppose R is an order relation on Ω , then Ω is said to be **inductively ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Definition 1.2. A set X is **well ordered** by linear \preceq if every nonempty subset has a least element.

Definition 1.3. An **initial segment** of (X, \preceq) is any set of the form $I(x) = \{y \in X : y \preceq x\}$.

Definition 1.4. An **ideal** in a well ordered X is a subset A s.t. for all $a \in A$, $I(a) \subset A$.

Theorem 1.1 (Well Ordering Principle). *Every nonempty set can be well ordered.*

Proof. Let X nonempty, and let

$$\mathcal{X} = \{(A, \preceq_A) \text{ is well order} : A \subset X\}$$

all well ordered sets, and define \preceq on \mathcal{X} as $(B, \preceq_B) \preceq (A, \preceq_A)$ if B is an ideal in A and \preceq_A extends \preceq_B . Suppose every chain \mathcal{C} in \mathcal{X} , $(\cup \mathcal{C}, \cup \{\prec_A : A \in \mathcal{C}\})$ clearly an upper bound of \mathcal{C} and well ordered. By Zorn's lemma, there is a maximal element of \mathcal{X} and it's actually X . \square

Kind of remarkable and useful well ordered set is exist:

Theorem 1.2. *There exist poset (Ω, \preceq) satisfy*

1. (Ω, \preceq) is well ordered.
2. Ω has a greatest element ω_1
3. $I(x)$ is countable for $x < \omega_1$
4. $\{y \in \Omega : x \leq y \leq \omega_1\}$ is uncountable.

- 5. Every nonempty subset of Ω has a least upper bound.
- 6. A nonempty subset of $\Omega - \{\omega_1\}$ has greatest element iff it's countable. Every uncountable subset has least upper bound ω_1 .

Proof. Let (X, \leq) be uncountable well ordered set, and let A

$$A = \{x \in X : I(x) \text{ is uncountable}\}$$

w.l.o.g we may assume A is nonempty. Then there is a first element and denoted by ω_1 . Then we show that $\Omega = I(\omega_1)$ enjoy all the properties.

1-4 is straightforward and 5 follows from all the upper bound is well ordered and thus least upper bound exist. For 6, suppose there is a countable $C \subset \Omega - \{\omega_1\}$, then $\bigcup_{i=1}^{\infty} I(x_i)$ is countable, so there is some $x < \omega_1 \notin \bigcup_{i=1}^{\infty} I(x_i)$, that is an upper bound. By 5, least upper bound is exist and belong to C . Conversely, if some subset C has some least upper bound $b < \omega_1$, then $C \subset I(b)$ and must countable. \square

The elements of Ω are called **ordinals** and ω_1 is called **first uncountable ordinal**. The elements of $\Omega_0 = \Omega - \{\omega_1\}$ is **countable ordinals**. We treat \mathbb{N} as a subset of Ω . Then the first element of $\Omega - \mathbb{N}$ is **first infinite ordinal**.

Theorem 1.3 (Interlacing Lemma). *Suppose sequence $\{x_n\}$ and $\{y_n\}$ in Ω_0 with $x_n \leq y_n \leq x_{n+1}$. Then they share the same least upper bound.*

Proof. Clearly since $x_n \leq y_n \leq x_{n+1}$.

\square

Chapter 2

Topology

2.1 Topological spaces

Let Ω be as space

Definition 2.1. A class of subset τ of Ω is an **topology** if

1. \emptyset and Ω belongs to τ .
2. closed under arbitrary union.
3. closed under finite intersection.

(Ω, τ) called a **topological space** where Ω is called as **underlying set**. The sets in τ are called **open** while sets with complement in τ is **closed**. Both open and closed set is called **clopen**.

Definition 2.2. Countable intersection of open sets is \mathcal{G}_σ set and countable union of closed sets is \mathcal{F}_δ set.

Following is some examples of topological space.

Definition 2.3. (X, ρ) is a **semimetric space**, when ρ defined on $X \times X$ s.t. $\forall x, y, z \in X$:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

ρ is called a **semimetric**.

If $\rho(x, y) = 0 \iff x = y$, ρ become a **metric** and (X, ρ) become **metric space**. $B(a, r) = \{x \in E, d(x, a) < r\}$ is **r -ball** with center a .

U is **open** in (Ω, d) iff $\forall x \in U, \exists r_x 0 \ni B_d(x, r_x) \subseteq U$. Let τ_d be the set of all open subsets of Ω , we call τ_d the **topology generated by d** . A Topological space is **metrizable** if there exist metric d generates it.

Suppose d is discrete, that is, $d(x, y) = 0$ iff $x = y$, otherwise, $d(x, y) = 1$. Then every subset is open hence $\tau_d = \mathcal{P}(\Omega)$ and called **discrete topology**. The zero semimetric, defined by $d(x, y) = 0$ for all $x, y \in \Omega$ generates $\tau_d = \{\emptyset, \Omega\}$ and called **trivial topology**.

Let $\Omega = \mathbb{R}^n$, $l^2 = \sqrt{\sum_1^n (x_i - y_i)^2}$ is called **Euclidean metric**. $l^1 = \sum_1^n |x_i - y_i|$ is called **taxi-cab metric** and $l^\infty = \sup\{|x_i - y_i|\}$ is called **sup norm metric**.

Note $d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n}d_{l^2}(x, y)$ and $d_{l^2}(x, y) \leq \sqrt{n}d_{l^\infty}(x, y) \leq \sqrt{n}d_{l^2}(x, y)$, then d_{l^∞} open $\iff d_{l^2}$ open $\iff d_{l^1}$ open. Hence $\tau_{d_{l^2}} = \tau_{d_{l^1}} = \tau_{d_{l^\infty}}$.

All topologies on Ω is poset with greatest element $\mathcal{P}(\Omega)$ and least $\{\emptyset, \Omega\}$. If $\tau' \subset \tau$, we say τ' **coarser** than τ while τ finer than τ' .

If τ can be form by taking union of families in some $\mathcal{B} \subset \tau$, we call \mathcal{B} the **base** for the topology τ .

Theorem 2.1. \mathcal{B} is a base in (X, τ) iff $\forall U \in \tau, \forall x \in U, \exists W \in \mathcal{B} \ni x \in W \subset U$.

Proof. \implies : Any U can be written as $U = \cup W_i$ and $x \in U \implies x \in W_i$ for some i and $W_i \in \mathcal{B}$. \impliedby : For any $U \in \tau$, consider arbitrary $x \in U$, then there exist W_x such that $x \in W_x \subset U$, thus we have $U = \cup_x W_x$. \square

Let $\mathcal{S} \subset \tau$, suppose all topologies include \mathcal{S} . Then the intersection of all of them is again a topology, denoted as $\tau(\mathcal{S}) = \cap \mathcal{T}$, then $\tau(\mathcal{S})$ is the smallest topology contains \mathcal{S} . We call it the topology **generated** by \mathcal{S} .

Theorem 2.2. $\tau(\mathcal{S})$ is unions of families of finite intersections together with Ω , formally:

$$\{\bigcup \left(\bigcap_1^N S_i \right)\} \cup \Omega$$

$\mathcal{S} \subset \tau$ is a **subbase** for τ if all finite intersections of \mathcal{S} is a base. Note that if $\Omega \in \mathcal{S}$, \mathcal{S} is the subbase of $\tau(\mathcal{S})$. (Ω, τ) is **second countable** if τ has countable base. Clearly, a topology is second countable iff it has countable subbase.

For any subset X in (Ω, τ) , then

$$\tau_X = \{X \cap V : V \in \tau\}$$

form a topology in X and we call (X, τ_X) a **subspace** or **relative topology**. Sets in τ_X are **relative open**. **Relative closed** sets of the form

$$X - (X \cap V) = X - V = X \cap V^c$$

2.2 Neighborhood

A subset V is called a **neighborhood** of a if there exists a open set $U \subset V$ contains a . Then we called $V' = V - \{a\}$ **punctured(deleted)** neighborhood. A **neighborhood base** is a collection of neighborhood $BN(a)$ s.t. for any neighborhood V of a , there exist a $W \in BN(a)$ and $W \subset V$. Clearly, all the neighborhoods is a neighborhood base and denoted as $\mathcal{N}(x)$, which is called **neighborhood system**.

Lemma 2.1. *A subset U is open iff it's a neighborhood for each of its points.*

Proof. \Rightarrow is trivial. \Leftarrow follows from $\cup_x G_x = U$ and unions of open set is still open. ■ □

This suggest a equivalent definition of finear topology:

Lemma 2.2. $\tau' \subset \tau \iff \tau' \text{ neighborhood is a } \tau \text{ neighborhood.}$

Proof. \Rightarrow any open set G_x satisfy $x \in G_x \subset V$ in T' is still open in T , hence V is T neighborhood. \Leftarrow Consider any open set $G \in T'$, it's a T' neighborhood for each of its points implies it's a T neighborhood for each of its points and hence G is T open. □

2.3 Closures

The **interior** of A is the union of all open sets which are included A , i.e., the largest open set included in A , we denote it A° . And the **closure** is the intersection of all closed sets which include A and thus the smallest closed set includes A , we denote it \bar{A} .

Lemma 2.3. *Following is some useful truth:*

1. $(A \cap B)^\circ = A^\circ \cap B^\circ$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
3. $A \subset \bar{B} \implies \bar{A} \subset \bar{B}$
4. $A^\circ \subset B \implies A^\circ \subset B^\circ$
5. $\bar{A}^c = (A^\circ)^c$
6. $(\bar{A})^c = (A^c)^\circ$

Proof. We only prove **5**, note $(A^\circ)^c$ is closed and

$$A^\circ \subset A \implies (A^c) \subset (A^\circ)^c$$

we have $\bar{A}^c \subset (A^\circ)^c$. On the other hand

$$\bar{A}^c \supset (A^\circ)^c \iff (\bar{A}^c)^c \subset A^\circ \iff (\bar{A}^c)^c \subset A \iff \bar{A}^c \supset A^c$$

□

The **frontier** of A is $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ$.

x is said to be an **interior point** of A if A is neighborhood of x .

x is said to be an **adherent point** if it's every neighborhood meets A , an ω **accumulation point** of A if every neighborhood of x contains **infinitely** many points of A and is a **condensation point** of A if every neighborhood of x contains **uncountable** many points of A .

x is a **cluster point** or **accumulation point** if every deleted neighborhood of x meets A and is **isolated point** if x is not cluster point. That is, $\{x\}$ is relative open in A . We denoted all the cluster points as A' and called **derived set**.

x is **frontier point** or **boundary point** if every neighborhood of x meets both A and A^c .

It's east to show that the points of A° are precisely all the interior points of A and \overline{A} are precisely all the adherent points. ∂A is precisely points of frontier. We claim that

$$\overline{A} = A^\circ \cup \partial A = A \cup A'$$

A subset A is called **perfect** if it's closed while point in A is cluster points in A , that is $A' = A = \overline{A}$.

2.4 Dense

A is said **dense** if $\overline{A} = \Omega$ and **nowhere dense** if $(\overline{A})^\circ = \emptyset$ (\mathbb{Q} is dense in \mathbb{R} while \mathbb{Z} is nowhere dense.) A is said to be **meagre** or **set of the first category** if it's countable union of nowhere dense. Sets which are not meagre is **set of the second category** set.

Space (Ω, τ) is **first countable** if every point of Ω has countable neighborhood base. The space is said **separable** if Ω has a countable dense subset.

Lemma 2.4. *Second countable space is separable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, by axiom of choice, we may take x_i in I , let $X = \{x_i\}_{i \in I} \subset \Omega$. Then we show that X is dense. For any $x \in \Omega$, it's neighborhood must contain some open G which is unions of \mathcal{B} and thus contains at least one element in X , that is, G meet X . Hence $\overline{X} = \Omega$. □

Lemma 2.5. *Second countable space is first countable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, for each point $x \in \Omega$, one may take all the sets in \mathcal{B} which contains x as a neighborhood base. To verify it's neighborhood base, if there is a neighborhood N of x , then there is a open G contains x . By the definition of base, G is the union of sets of \mathcal{B} and those sets must at least one contains x and these sets is subset to G . \square

2.5 Mappings

Suppose (Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

Lemma 2.6. *Following is some useful truth for mappings.*

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap N) = f^{-1}(U) \cap f^{-1}(N)$
4. $f^{-1}(U \cup N) = f^{-1}(U) \cup f^{-1}(N)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ always holds if f is injection while $ff^{-1}(A) = A$ always holds if g is surjection.
7. If f is bijection, $(f^{-1})^{-1}(A) = f(A)$ always hold.
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$
9. $f^{-1}(A) \subset f^{-1}(B) \iff A \subset B$
10. $f(A) \subset f(B) \iff A \subset B$

Definition 2.4. f is **continuous** at x if for every neighborhood N' of $f(x)$, there is a neighborhood N of x s.t. $f(N) \subset N'$. It's continuous if it's continuous at every points $x \in \Omega$.

Theorem 2.3. f is continuous iff

1. $f^{-1}(G')$ is open for every open subset G' of Ω' .
2. $f^{-1}(F')$ is closed for every closed subset F' of Ω' .
3. If $A \subset \Omega'$, then $f^{-1}(A^\circ) \subseteq (f^{-1}(A))^\circ$
4. If $A \subset \Omega$, then $f(\overline{A}) \subset \overline{f(A)}$

Proof. We only prove 1 and 3.

1 \implies : For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood N s.t. $f(N) \subset G'$, and

$$x \in N \subset f^{-1}f(N) \subset f^{-1}(G')$$

\Leftarrow : For every neighborhood N' , there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $f f^{-1}(G') \subset G'$.

3 \implies : $f^{-1}(A^\circ)$ is open and th claim follows from $f^{-1}(A^\circ) \subset f^{-1}(A)$. \Leftarrow : Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggets $f^{-1}(A)$ is open. \square

Lemma 2.7. Suppose $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_2 \rightarrow \Omega_3$, $f \circ g$ is continuous if f and g are continuous.

Proof. Suppose G_3 is open and the claims follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$. \square

Lemma 2.8. Suppose $f : (\Omega, \tau), (\Omega', \tau(\mathcal{S}))$, f is continous iff $f^{-1}(S) \in \tau$ for any $S \in \mathcal{S}$.

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f , s.t f^{-1} is continuous and such f is called **homeomorphism**. In particular, f is an **embedding** if $f : (\Omega, \tau) \rightarrow (f(\Omega), \tau|_{f(\Omega)})$ ia a homeomorphism.

f is **open** if $f(G)$ is open for all open set $G \in \tau$ and is **closed** if $f(F)$ is closed for all closed set $F^c \in \tau$.

Lemma 2.9. Suppose f is bijection, then it's homeomorphism iff it's continuous and either open or closed.

Proof. By the continuity of f^{-1} , since $(f^{-1})^{-1}(G) = f(G)$ for all open set G .

$$f^{-1} \text{ is continuous } \iff f(G) \text{ is open } \iff f \text{ is open .}$$

\square

Lemma 2.10. Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f continuous.

Proof. Suppose f is homeomorphism, T_0 is another topology where f is continuous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1} ,

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0 . \square

Note that $\mathcal{P}(\Omega)$ let all f continuous and $\{\emptyset, \Omega\}$ let all $g : \Omega' \rightarrow \Omega$ continuous.

b

2.6 Filter

Definition 2.5. A **filter** is a non-empty collection \mathcal{F} of subset in Ω s.t.

1. $A \in \mathcal{F}, A \subset B \implies B \in \mathcal{F}$
2. Closed under finite intersection.
3. $\emptyset \notin \mathcal{F}$

Note the definition of \mathcal{F} is independent with topology τ . A **free filter** is filter with $\ker \mathcal{F} = \bigcap_{F \in \mathcal{F}} F = \emptyset$. Not free filters are called **fixed**.

Filter can be formed by taking upward closure of a filter base.

Definition 2.6. A collection \mathcal{B} of subset in Ω is a **filter base** of or **prefilter** if

1. $\mathcal{B} \subset \mathcal{F}$
2. $\forall N \in \mathcal{F}, \exists W \in \mathcal{B} \ni W \subset N$

We say \mathcal{B} generates \mathcal{F} , where

$$\mathcal{F} = \mathcal{B}^\uparrow = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

For example,

- Suppose $x \in \Omega$ then

$$\mathcal{N}(x) = \{\text{All neighbourhoods of } x\}$$

is a filter on Ω , that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. Note

$$\tau(x) = \{X \in \tau : x \in X\}$$

is a base for $\mathcal{N}(x)$ and thus $\mathcal{N}(x) = \tau(x)^\uparrow$.

- Suppose Ω is infinite, the collection of all **cofinite** subsets(subset s with finite complement) is a filter on Ω , such filter is free and called **Frechet filter**.

To assert a collection is a base, we have

Theorem 2.4. *Let \mathcal{B} be a collection of nonempty subsets. Then \mathcal{B} is a filter base, that is, \mathcal{B} may generates a filter iff*

1. *The intersection of each finite family of sets in \mathcal{B} includes a set in \mathcal{B}*
2. *\mathcal{B} is non-empty and $\emptyset \notin \mathcal{B}$.*

Proof. We claim that

$$\mathcal{F} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

\mathcal{F} is the filter generated by \mathcal{B} . □

A family of subsets \mathcal{F} is said to have **finite intersection property** if intersection of every finite subfamily is nonempty.

Let \mathcal{A} be collection of subsets with finite intersection property, then collection of all finite intersection of \mathcal{A} is a base, we call the filter generated **filter generated by \mathcal{A}** . Formally

$$\mathcal{F} = \left\{ \bigcap_{A \in \mathcal{J}} A : \mathcal{J} \subset \mathcal{A} \text{ and } \mathcal{J} \text{ is finite} \right\}^\uparrow$$

A filter \mathcal{F} is **finer** than another \mathcal{G} if $\mathcal{F} \subset \mathcal{G}$. Clearly, the set of all filters on Ω is inductively ordered by inclusion. By Zorn's lemma, the set of all filters has maximal filters and we call such filters **ultrafilters**.

Lemma 2.11. *Every fixed ultrafilter of the form*

$$\mathcal{U}(x) = \{x\}^\uparrow$$

for any $x \in \Omega$. And every free ultrafilter contains no finite subsets.

To assert a filter is ultra, we have:

Theorem 2.5. *Let A be a collection of subsets and \mathcal{F} the filter generated by A . If*

$$\forall X \subset \Omega, \text{ either } X \in A \text{ or } X^c \in A$$

then A is an ultrafilter on Ω .

Proof. Suppose \mathcal{F}' is an ultrafilter include \mathcal{F} , we have $\mathcal{F}' \supset A$ clearly. Consider any $X \in \mathcal{F}'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in \mathcal{F}'$ as $\mathcal{F}' \supset \mathcal{F} \supset A$ and $X \cap X^c = \emptyset \in \mathcal{F}'$ results in a contradiction. It follows that $A \supset \mathcal{F}'$ and thus $A = \mathcal{F}'$. □

Theorem 2.6. *Every filter \mathcal{F} is the intersection of all the ultrafilter which include \mathcal{F} .*

Proof. We claim that

$$\mathcal{F} = \cap \{\text{ultrafilter generates by } \{x\} : x \in \cap \mathcal{F}\}$$

□

Suppose mappings on a filter:

Theorem 2.7. *Let f be a mapping from Ω to Ω' and \mathcal{B} a base for a filter \mathcal{F} on Ω . Then $f(\mathcal{B}) = \{f(X)\}_{X \in \mathcal{B}}$ is also a base on Ω' . Moreover, if \mathcal{F} is ultra then $f(\mathcal{B})$ also generates an ultrafilter.*

Proof. First assertion is straightforward and the second follows from \mathcal{B} is collection of supset for some $\{x\}$, then $f(\mathcal{B})$ generates the filter that generates by $\{f(x)\}$. □

Theorem 2.8. *In the same situation as previous theorem. If \mathcal{B}' is a base on Ω' , then $f^{-1}(\mathcal{B}')$ is a base on Ω iff every set in \mathcal{B}' meets $f(\Omega)$*

Proof. We have

$$\Omega \in f^{-1}(\mathcal{B}') \implies f(\Omega) \subset X'$$

for some $X' \in \mathcal{B}'$, by definition, \implies is immediately.

For \Leftarrow , suppose any finite family $X_i \in \mathcal{B}'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}\left(\bigcap_i X_i\right) \in f^{-1}(\mathcal{B}')$$

Then the claim follows from theorem 2.4. □

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the filter \mathcal{F} and \mathcal{F} is said to **converge** to x , or $\mathcal{F} \rightarrow x$, if the neighborhood filter $\mathcal{N}(x) \subset \mathcal{F}$. For filter base \mathcal{B} , we define on the filter generated by \mathcal{B} , that is, if $\mathcal{N}(x) \subset \mathcal{B}^\uparrow$.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff \mathcal{N}_\tau(x) \supset \mathcal{N}_{\tau'}(x) \iff \mathcal{F} \rightarrow a \implies \mathcal{F}' \rightarrow a$$

also, an equivalent definition of continuity as follows:

Theorem 2.9. *$f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continuous at x iff*

$$\forall \mathcal{F} \rightarrow x, f(\mathcal{F}) \rightarrow f(x)$$

Proof. By definition, $f(\mathcal{F}) \rightarrow f(x)$ if

$$\mathcal{N}(f(x)) \subset f(\mathcal{F})^\uparrow$$

That is, for any neighbourhood $N' \in \mathcal{N}(f(x))$, there exist some $A \in \mathcal{F}$ s.t. $f(A) \subset N'$, as $\mathcal{N}(x) \subset \mathcal{F}$ and f is continuous at x , such A is always exists. Conversely, take $\mathcal{F} = \mathcal{N}(x)$ then the claim is follows \square

A point $x \in \Omega$ is said to be an **adherent point** of \mathcal{F} if x is an adherent point of every set in \mathcal{F} . The **adherence** of \mathcal{F} , $\text{Adh}_\tau(\mathcal{F})$ or $\overline{\mathcal{F}}$ is the set of all adherent points, thus

$$\overline{\mathcal{F}} = \bigcap_{X \in \mathcal{F}} \overline{X}$$

Define similarly on filter base \mathcal{B} by the filter generated. By definition, we have

$$\overline{\mathcal{B}} = \bigcap_{X \in \mathcal{B}} \overline{X}$$

Lemma 2.12. Suppose A be a subset of Ω , then $x \in \overline{A}$ iff there is a filter \mathcal{F} s.t. $A \in \mathcal{F}$ and \mathcal{F} converges to x .

Theorem 2.10. Suppose $BN(x)$ a neighbourhood base of x , then

1. \mathcal{B} converges to x iff every set in $BN(x)$ includes a set in \mathcal{B} .
2. $x \in \overline{\mathcal{B}}$ iff every set in $BN(x)$ meets every set in \mathcal{B} .

As consequence, we have

Corollary 2.1. x is adherent to a filter \mathcal{F} iff there is $\mathcal{F}' \supset \mathcal{F}$ and converges to x

Proof. \Rightarrow follows from taking $\mathcal{F} = BN(x)$. Conversely, $\forall N \in BN(x)$, we have $X' \subset N$ for some $X' \in \mathcal{F}'$, thus for any $X \in \mathcal{F}$, $N \cap X \subset X' \cap X \neq \emptyset$ as $X', X \in \mathcal{F}'$. \square

Corollary 2.2. Every limit point of \mathcal{F} is adherent to \mathcal{F}

Proof. Clearly holds by applying theorem 2.10.1 and 2.10.2. \square

Corollary 2.3. Every adherent point of an ultra-filter is a limit point of it.

Proof. Clearly as kernel of ultrafilter is a one point set. \square

Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a **limit point** of f relative to \mathcal{F} if $f(\mathcal{F}) \rightarrow x$.
2. an **adherent point** of f relative \mathcal{F} if it's adherent point of $f(\mathcal{F})$.

Theorem 2.11. Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$

1. x' is a limit point of f relative to \mathcal{F} iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, we have $f^{-1}(N') \in \mathcal{F}$.
2. x' is an adherent point of f relative to \mathcal{F} iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, it meets $f(X)$ for any $X \in \mathcal{F}$.

Proof. x' is limit is equivalent to

$$\mathcal{N}(x') \subset f(\mathcal{F})^\uparrow$$

That is, there exist some $A = f(X) \subset N'$ for any $N' \in \mathcal{N}(x')$, followed by $X \subset f^{-1}f(X) \subset f^{-1}(N')$, then the claim follows from the definition of filter.

By theorem 2.10, x' is adherent to $f(\mathcal{F})$ iff

$$\forall N' \in \mathcal{N}(x'), \forall X \in \mathcal{F}, f(X) \cap N' \neq \emptyset$$

note for any $N' \in \mathcal{N}(x')$, there exist $N' \in \mathcal{N}(x') \ni N' \subset N'$, thus $f(X) \cap N' \neq \emptyset$ also holds. Conversely, making use of $\mathcal{N}(x') \subset \mathcal{N}(x')$. \square

For example, suppose $f : (\mathbb{N}, \tau) \rightarrow (\Omega', \tau')$ and \mathcal{F} the frechet filter on \mathbb{N} . Then x' is limit of f relative to \mathcal{F} iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{F} \iff f^{-1}(N')^c \subset [0, k] \iff f^{-1}(N') \supset \{n \in \mathbb{N} : n \geq k\}$ for some k , that is, $f(n) \in N'$ for any $n \geq k$.

Theorem 2.12. Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ and let $\mathcal{F} = \mathcal{N}(x)$. By theorem 9, x' is limit of f relative to $\mathcal{N}(x)$ iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{N}(x) \iff N \subset f^{-1}(N') \iff f(N) \subset N'$ for some $N \in \mathcal{N}(x)$. That is, iff $x' = f(x)$, f is continuous at x . Such limit points also called limit points of f at x .

2.7 Net

(D, \preceq) is called a **directed set** if every couple $\{x, y\}$ in which has an upper bound.

If $\{D_i\}_{i \in I}$ is family of directed set then $D = \prod_{i \in I} D_i$ is also directed under **product direction** defined by $(a_i)_{i \in I} \succeq (b_i)_{i \in I}$ for all $i \in I$.

Definition 2.7. Let (D, \preceq) be a directed set, $\nu : D \rightarrow \Omega$ is called a **net** in Ω with domain D . The directed set is called **index set** of the net and members of D are **indexes**. We often write ν as x . or $\{x_\alpha\}$.

Suppose A a subset of Ω , we say x . **eventually in** A if there exist some $k \in D$ s.t. $x_n \in A$ for all $n \succeq k$. And we say ν is **frequently** in A if for all $n \in D$, there exist an $n' \succeq n$ s.t. $x_{n'} \in A$.

Lemma 2.13. *If x . not frequently in A , then x . eventually in A^c . Thus, for any $X \in \Omega$, x . frequently in either X or X^c .*

Suppose $x \in \Omega$, then x . is said **converge** to x , or $x. \rightarrow x$ if x . eventually in N for all $N \in \mathcal{N}(x)$, i.e., $\mathcal{N}(x) \subset \mathcal{F}(x.)$. The point x is **adherent** to x . if x . frequently in N for all $N \in \mathcal{N}(x)$.

Theorem 2.13. *Suppose $A \in (\Omega, \tau)$, then $x \in \overline{A}$ iff it's the limit of some net in the set.*

Proof. \Leftarrow is clear. \Rightarrow follows from we may find a associated net taking value in A (since each neighborhood meets A) and such net converges to x . \square

As with sequence, if x . is bounded, there is

$$\liminf x. = \sup \inf x. \preceq \limsup x. = \inf \sup x$$

Subnet generalizes subsequence.

Definition 2.8. Suppose D is directed, a subset B of D is called **cofinal** if for any $a \in D$, there exist $b \in B$ s.t. $a \preceq b$. A map $f : D \rightarrow A$ is **final** if $f(D)$ is cofinal of A .

Let $x.$ and $x'.$ are two nets in Ω with domains D and D' respectively. We say that $x'.$ is a **subnet** of $x.$ if there exists a final mapping $\varphi : D' \rightarrow D$ s.t. $x'_\alpha = x_{\varphi(\alpha)}$.

Theorem 2.14. *Let \mathcal{A} be a collection of subsets that $x.$ is frequently in. If \mathcal{A} is closed under finite intersection, then there exists a subnet $x'.$ of $x.$ and $x'.$ eventually in every member of \mathcal{A}*

Lemma 2.14. *Suppose $x'.$ is subnet of $x.$, we have*

1. $x. \rightarrow x \Rightarrow x'.$ $\rightarrow x$
2. x adherent to $x'.$ $\Rightarrow x$ adherent to $x.$

Theorem 2.15. *A point x is adherent to x . iff there is a subnet converges to x . While $x. \rightarrow x$ iff every subnet converges to x .*

Proof. \Rightarrow is clear by lemma2.14. Conversely, suppose a is not adherent to x , there exist a neighborhood N that x . not frequently in, i.e., exist k s.t. $x_n \notin N$ for any $n \geq k$, thus there is no subnet eventually in N .

For the second part, \Rightarrow is also clear by lemma2.14 and \Leftarrow comes from taking subnet as itself. \square

A net $x.$ is called **ultranet** or **universal net** if for all $X \in \Omega$, we have either x . eventually in X or x . eventually in X^c . Clearly, subnet of ultranet is ultra and

Lemma 2.15. *Every net has a ultra subnet.*

Proof. Consider collection of \mathcal{Q} s.t. x is frequently in every member and closed under finite intersection. By Zorn's Lemma, there is a maximal \mathcal{Q}_0 . By theorem 11, x has a subnet x' which eventually in every member of \mathcal{Q}_0 . We claim that this subnet is ultra since, \mathcal{Q}_0 is maximal and thus either $X \in \mathcal{Q}_0$ or $X^c \in \mathcal{Q}_0$. \square

2.8 Nets and filters

Let

$$\mathcal{F}(x.) = \{X \in \mathcal{P}(\Omega) : x. \text{ is eventually in } X\}$$

Then $\mathcal{F}(x.)$ is a filter and we call it the **filter associated with the net** $x.$.

Theorem 2.16. *Associated filter is the upward closure of the net's tail, that is*

$$\mathcal{F}(x.) = \{\{x_b : b \succeq a\} : a \in D\}$$

Motivated by the definition of filter that filter is closed under pairwise intersection, let $X \preceq Y \iff X \supset Y$, then any mapping $\nu : \mathcal{F} \rightarrow \Omega$ s.t. $\nu(X) \in X$ is a **net associated with the filter** \mathcal{F} .

By definition, we claim that \mathcal{F} is the associated filter of every associated net and $x.$ is an associated net of the associated filter.

Theorem 2.17. *Filter $\mathcal{F} \rightarrow x$ iff $x. \rightarrow x$ for any $x.$ associated with \mathcal{F} .*

Proof. Note

$$\forall N \in \mathcal{N}(x), x. \text{ eventually in } N \iff \mathcal{N}(x) \subset \mathcal{F}(x.)$$

Then is sufficient to show that $\mathcal{F}(x.) = \mathcal{F}$. It's follows from for any $X \in \mathcal{F}$, $x.$ eventually in X . \square

Theorem 2.18.

$$x. \rightarrow x \iff \mathcal{F}(x.) \rightarrow x$$

Proof. Both side is equivalent to $\mathcal{N}(x) \subset \mathcal{F}(x.)$ \square

Theorem 2.19. *Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$, then f is continous at x iff $\forall x. \rightarrow x, f(x.) \rightarrow f(x)$.*

Proof. By theorem 2.18, 2.17 and 2.12. \square

By above theorems, we have

$$\text{Adh}(\mathcal{F}(x.)) = \text{Adh}(x.), \text{Lim}(\mathcal{F}(x.)) = \text{Lim}(x.)$$

and similarly results holds for any filter and one of associated nets.

Lemma 2.16. *If $x.$ is ultra then the associated filter $\mathcal{F}(x.)$ is also ultra and if \mathcal{F} is ultra, every associated net is ultra.*

Proof. Directly from theorem 2.5. □

2.9 Separation

Definition 2.9. Space (Ω, τ) is said to be T_0 or **kolmogorov** if for every pair $(x, y) \in \Omega^2$, either there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ or $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Lemma 2.17. τ isn't T_0 iff there exist pair (x, y) , s.t.:

1. $\mathcal{N}(x) = \mathcal{N}(y)$.
2. $\overline{\{x\}} = \overline{\{y\}}$.

Proof. 1 If every $N \in \mathcal{N}(x)$ contains y , then $N \in \mathcal{N}(y) \implies \mathcal{N}(x) \subset \mathcal{N}(y)$, thus $\mathcal{N}(x) = \mathcal{N}(y)$.

2 If some point $a \in \overline{\{x\}}$, then every $N \in \mathcal{N}(a)$ also is neighborhood of x and thus neighborhood of y , hence $a \in \overline{\{y\}}$. □

Definition 2.10. Space (Ω, τ) is said to be T_1 or **Frechet** if for every pair $(x, y) \in \Omega^2$, there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ and $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Theorem 2.20. *Following statements are equivalent:*

1. τ is T_1 .
2. Singetons are closed.
3. $\ker \mathcal{N}(x) = \{x\}$ holds for any $x \in \Omega$.

Proof. 1 \implies 2 If there exist a singeton $\{x\}$ not closed, there is $y \in \overline{\{x\}}$, hence every neighborhood of y contains x , contradiction.

2 \implies 3 Suppose $\ker \mathcal{N}(x)$ contains y diifer x , that implies any neighborhood of x contains y and contradict 2.

3 \implies 1 is straightforward. □

Lemma 2.18. *Suppose (Ω, τ) with a finite base is T_1 , then Ω is finite and τ is discrete.*

Definition 2.11. A topology (Ω, τ) is T_2 , or **Hausdorff** or **separated** if every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $U \cap V = \emptyset$.

Theorem 2.21. *Following statements are equivalent:*

1. τ is T_2 .
2. Intersection of family of closed neighborhoods of x is x .
3. If a filter(net) converges to some point x , then $\text{Adh}(\mathcal{F}) = \{x\}$
4. Every net(filter) converges to at most one point.

Proof. 1 \implies 2 For any pair (x, y) , by definition, there is $y \notin \overline{U}$, hence intersection of family of closed neighborhoods of x can only contains x .

2 \implies 3 follows from a point adherent to a filter converges to x must be in every closed neighborhood of x .

3 \implies 4 is clearly.

4 \implies 1 If there is a net $x.$ converges to both x and y , then $\mathcal{N}(x) \subset \mathcal{F}(x.)$ and $\mathcal{N}(y) \subset \mathcal{F}(x.)$, that is, U and V meets for any $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$. \square

Definition 2.12. Space (Ω, τ) is said to be $T_{2.5}$ or **Completely Hausdorff** if for every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $\overline{U} \cap \overline{V} = \emptyset$.

Two nonempty sets are called **separated by open sets** if they are included in disjoint open sets, and they are **separated by continuous functions** if there is continuous f taking values in $[0, 1]$ and assign 0 on one set and 1 on the other.

Space (Ω, τ) are said to be **regular** if every singleton and any closed A disjoint from it can be separated by open sets.

Definition 2.13. Space (Ω, τ) is said to be T_3 if it's T_1 and regular.

Space (Ω, τ) are said to be **Completely regular** if every singleton and any closed A disjoint from it can be separated by continuous function.

Definition 2.14. Space (Ω, τ) is said to be $T_{3.5}$ or **Tychonoff space** if it's T_1 and completely regular.

Theorem 2.22 (Tychonoff's Embedding Theorem). *Space (Ω, τ) is $T_{3.5}$ iff it's homeomorphic to a subspace of $([0, 1]^n, \tau_{d_1})$.*

Space (Ω, τ) is said to be **normal** if two disjoint closed subsets can be separated by open sets.

Definition 2.15. Space (Ω, τ) is said to be T_4 if it's normal and T_1 .

Theorem 2.23 (Urysohn's Lemma). *Following statements are equivalent:*

1. (Ω, τ) is normal.
2. For any $U \in \tau$ and any closed $A \subset U$, there is a $U' \in \tau$ s.t. $A \subset U'$ and $\overline{U'} \subset U$.
3. Every two disjoint closed subsets can be separated by continuous function.

Proof. $1 \implies 2$ Apply normal property to A and U^c , there is a U' include A and V include U^c , as $U' \cap V = \emptyset \implies U' \subset V^c \implies \overline{U'} \subset V^c \subset U$.

$2 \implies 3$ Suppose A and B are two disjoint closed subset, apply 2 to A and $U_1 = B^c$ we have $A \subset U_0$ and $\overline{U_0} \subset U_1$. Apply again for $\overline{U_0}$ and U_1 to generates $U_0 \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$, repeat such process, that is, apply 2 to $\overline{U_{\frac{j}{2^k}}}$ and $U_{\frac{j+1}{2^k}}$ to generates $U_{\frac{2j+1}{2^{k+1}}}$. Finally, we construct a open strictly increasing squence U_r . where r is any dyadic rational in $[0, 1]$, i.e., $r \in DR \cap [0, 1]$.

Then define f as

$$f = \begin{cases} 1 & x \in B \\ \inf\{r : x \in U_r\} & x \in B^c \end{cases}$$

Then it's sufficient to show that f is continous. Note subspace $[0, 1]$ of \mathbb{R} can be generated by collection of $[0, s)$ and $(t, 1]$ and

$$\begin{aligned} f^{-1}[0, s) &= \bigcup_{r \in DR \cap [0, s)} U_r \\ f^{-1}(t, 1] &= \bigcup_{r \in DR \cap (t, 1]} \overline{U_r}^c \end{aligned}$$

Then the claim follows from lemma 2.8.

$3 \implies 1$ By taking any disjoint open set A contains 0 and B contains 1 and looking $f^{-1}(A)$ and $f^{-1}(B)$. \square

Theorem 2.24 (Tietze's Extension Theorem). *Let (Ω, τ) be normal, F any closed subset and I any bounded closed interval of \mathbb{R} . Then any continous $f : F \rightarrow I$ can be extended to $f' : \Omega \rightarrow I$ and remain continous.*

Proof. Suppose $I = [-1, 1]$, then $A = f^{-1}[-1, -\frac{1}{3}]$ and $f^{-1}[\frac{1}{3}, 1]$ are disjoint and closed. By Urysohn's Lemma, there is $g : \Omega \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ s.t. $g(A) = \{-\frac{1}{3}\}$ and $g(B) = \frac{1}{3}$. Set $f_0 = f, g_0 = g, f_1 = f - g|_F$. Then we can show that $|f_1|$ is bounded by $\frac{2}{3}$.

Repeat such process, we have series of

$$\begin{aligned} f_n &: F \rightarrow [-\left(\frac{2}{3}\right)^n, \left(\frac{2}{3}\right)^n] \\ g_n &: E \rightarrow [-\frac{1}{3}\left(\frac{2}{3}\right)^n, \frac{1}{3}\left(\frac{2}{3}\right)^n] \\ f_{n+1} &= (f_n - g_n)|_F \end{aligned}$$

Then we show that $g = \sum_{i=0}^{\infty} g_i$ is the extension of f . That is g is continuous and $f = g$ in F . Note for any x

$$\left| \sum_{i=m}^n g_i(x) \right| \leq \sum_{i=m}^n |g_i(x)| \leq \sum_{i=m}^n \frac{1}{3} \left(\frac{2}{3}\right)^i \leq \left(\frac{2}{3}\right)^m \rightarrow 0$$

Thus $\{\sum_{i=0}^n g_i\}_{n=0}^{\infty}$ converges uniformly by Cauchy's criterion, followed by g is continuous. And $f = g$ on F follows from

$$\left| f(x) - \sum_{i=0}^n g_i(x) \right| = \left| f_0(x) - \sum_{i=0}^n g_i(x) \right| = \left| f_1(x) - \sum_{i=1}^n g_i(x) \right| = \left| f_{n+1}(x) \right| \leq \left(\frac{2}{3}\right)^{n+1} \rightarrow 0$$

□

2.10 Compatness

A **cover** of a set K is collection of sets whose union includes K . A **subcover** is subcollection of a cover and also covers K . K is **compact** if every open cover has a finite subcover and called **relatively compact** if it's closure is compact.

A topology (Ω, τ) is **compact** if Ω is compact and **locally compact** if every point has a compact neighborhood. A subset $A \in \Omega$.

Compatness is a “topological” property. That is, subset compactness in a subspace iff it's also compact in full space.

Theorem 2.25. *Let (Ω, τ) be a space, following are equivalent.*

1. (Ω, τ) is compact.
2. Every filter(net) has at least one adherent point.
3. Every ultrafilter(ultranet) converges.
4. $\ker \mathcal{F} \neq \emptyset$ For every collection \mathcal{F} of closed sets having FIP.

Proof. 1 \implies 2 Suppose filter \mathcal{F} , then

$$\{\overline{F} : F \in \mathcal{F}\}$$

enjoy finite intersection property by definition, then \overline{F} has at least one adherent point since $\ker\{\overline{F} : F \in \mathcal{F}\} = \overline{\mathcal{F}} \neq \emptyset$ by theorem ??.

2 \implies 3 Clearly by corollary 2.3.

3 \implies 1 Suppose \mathcal{A} a family of closed subsets with finite intersection property. Then the filter generated by \mathcal{A} has an ultrafilter with a limit point x . Note x is also adherent to \mathcal{U} and thus adherent to \mathcal{F} , followed by $x \in A$ for any $A \in \mathcal{A}$, hence $\ker \mathcal{A} \supset \{x\}$. Then the claim follows from theorem ??.

4 \iff 1 Taking contrapositive:

$$\neg \ker \mathcal{F} \neq \emptyset \equiv \ker \mathcal{F} = \emptyset \equiv \bigcup_{F \in \mathcal{F}} F^c = \Omega$$

and

$$\neg \forall \bigcap_i^n F_i = \emptyset \equiv \exists \bigcup_i^n F_i^c = \Omega$$

note that's precisely the definition of compactness. \square

Theorem 2.26. *Let (Ω, τ) be Hausdorff, then every compact subset and disjoint singleton can be separated by open sets. In particular, compact subset is closed.*

Proof. Suppose $F \subset \Omega$ is compact, for any $x \in \Omega$ not in F , by Hausdorff, there is $x \notin U_y$ and $y \notin V_y$. Then $\bigcup_{y \in F} U_y$ cover F , there is subcover $U = \bigcup_i^n U_{y_i}$ and $V = \bigcup_i^n V_{y_i}$ selected from the same family separated F and $\{x\}$. \square

Theorem 2.27. *Closed subset is compact in compact topological space.*

Proof. Note any open cover of F plus F^c become a open cover of Ω . \square

Theorem 2.28. *Every compact Hausdorff space is normal.*

Proof. Suppose A and B are closed and thus compact by theorem 2.27. For any point $x \in A$, there exist disjoint $V_x \supset B$ and $x \in U_x$ by theorem 2.26. Note $\bigcup_{x \in A} U_x$ cover A , there exist subcover $U = \bigcup_i^n U_{x_i} \supset A$ and $V = \bigcap_i^n V_{x_i} \supset B$ separated A and B . \square

Theorem 2.29. *Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continuous, then $f(A)$ is compact if A is compact.*

Proof. For any open cover of $f(A)$:

$$\bigcup G_i \supset f(A) \implies f^{-1}(\bigcup G_i) = \bigcup f^{-1}(G_i) \supset f^{-1}f(A) \supset A$$

Thus there exist subcover s.t.

$$\bigcup_1^n f^{-1}(G_i) = f^{-1}(\bigcup_1^n G_i) \supset A \implies \bigcup_1^n G_i \supset f f^{-1}(\bigcup_1^n G_i) \supset f(A)$$

which shows that $f(A)$ is compact. \square

As consequence:

Corollary 2.4 (Extreme value theorem). *A continuous real valued function defined on a compact space achieves its maximum and minimum values.*

Theorem 2.30. *Let (Ω, τ) be compact and (Ω', τ') be Hausdorff and $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continuous bijection. Then f is homeomorphism.*

Proof. It's sufficient to show that $f(F)$ is closed as lemma 2.9. Note F is closed and thus compact as theorem 2.27 then $f(F)$ is compact as theorem 2.29 and thus closed by theorem 2.26. \square

A subset A of a topological space is **sequentially compact** if every sequence in A has a subsequence converging to an element of A . A topological space is sequentially compact if itself is a sequentially compact set.

2.11 Semicontinuous

$f : \Omega \rightarrow \mathbb{R}^*$ is

- **lower semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \leq c\}$ is closed.
- **upper semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \geq c\}$ is closed.

Clearly f is lower semicontinuous iff $-f$ is upper and vice versa. Also, f is continuous iff it's both upper and lower semicontinuous.

Lemma 2.19. *Suppose $\{f_i\}_{i \in I}$ is family of lower(upper) semicontinuous function then $\sup f_i$ ($\inf f_i$) is lower(upper) semicontinuous.*

Proof. Note

$$\{x \in \Omega : \sup f_i(x) \leq c\} = \bigcap_{i \in I} \{x \in \Omega : f_i(x) \leq c\}$$

is closed. \square

Lemma 2.20. $f : \Omega \rightarrow \mathbb{R}^*$ is

- **lower semicontinuous** iff for any net

$$x. \rightarrow x \implies \liminf f(x.) \geq f(x)$$

- **upper semicontinuous** iff for any net

$$x. \rightarrow x \implies \limsup f(x.) \leq f(x)$$

Proof. Suppos f is lower semicontinuous and $x. \rightarrow x$. For any $c < f(x)$, then $G = \{\omega \in \Omega : f(\omega) < c\}$ is open and thus x eventually in, that is $x.c$ eventually and thus $\liminf f(x.) \geq c$. This implies that $\liminf f(x.) \geq f(x)$.

Conversely, for any $c \in \mathbb{R}$, consider $F = \{\omega \in \Omega : f(\omega) \leq c\}$. Then we show that F is closed. Suppos $x.$ is nets in F and converges to some $x \in \Omega$. Then $c \geq \liminf f(x.) \geq f(x)$ thus x in F and thus F is closed. \square

Then we can gennerlizes Weierstrass' Theorem in corollary 2.4.

Theorem 2.31. $f : \Omega \rightarrow \mathbb{R}^*$ on a compact set attains a minimum(maximum) value and set of minima(maxima) is compact if it's lower(upper) semicontinuous.

Proof. Suppose X is compact and f is lower semicontinuous, then for every $c \in f(X)$, $F_c = \{x \in X : f(x) \leq c\}$ is closed and $\{F_c : c \in f(X)\}$ has FIP clearly. Note X is compact, $\ker\{F_c : c \in f(X)\}$ is nonempty by 2.25. That is just the set of minimas and it's compact since it's closed. \square

2.12 Comparing topologies

We list some useful properies when comparing topologies, some of them has been mentioned before and proof omitted.

Lemma 2.21. Suppose τ' and τ are two tologies on Ω , then the following are equivalent.

1. $\tau' \subset \tau$
2. Identity mapping $I : x \mapsto x$ from (Ω, τ) to (Ω, τ') is continous.
3. τ' closed set is closed in τ .
4. $x. \xrightarrow{\tau} x \implies x. \xrightarrow{\tau'} x$
5. $Cl_\tau(A) \subset Cl_{\tau'}(A)$

Lemma 2.22. Suppose $\tau' \subset \tau$, then

1. Every τ compact set is τ' compact.
2. Every τ' continous function is τ continous.
3. Every τ dense set is τ' dense.

2.13 Weak topology

Suppose $\{(Y_i, \tau_i)\}_{i \in I}$ a family of topological space and $f_i : X \rightarrow Y_{i \in I}$. Let \mathcal{F} be the set of all the topologies s.t. f_i is continuous for all i . We call $\cap \mathcal{F}$, i.e., the corest topology the **induced topology** or **weak topology** or **initial topology** on X by $\{f_i\}_{i \in I}$. The topology induced by $\{f_i\}_{i \in I}$ is generated by

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \tau_i\}$$

or

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \mathcal{S}_i\}$$

where \mathcal{S}_i is a subbase for τ_i .

Lemma 2.23. *A net $x. \rightarrow x$ in the weak topology iff $f_i(x.) \rightarrow f_i(x)$ for each i .*

Proof. \Rightarrow is immediately. Conversely, noting sets of the form $\bigcap_1^n f_i^{-1}(V_i)$ consist a neighborhood base. \square

Theorem 2.32. *g is (τ', τ) continuous iff $f_i \circ g$ continuous for each f_i . Where τ is $\tau(S)$ in above theorem.*

Proof. \Rightarrow is immediately. \Leftarrow , suppose $G \in \tau$, by above theorem, this implies

$$G = \bigcup_I \bigcap_F X = \bigcup_I \bigcap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continuous and thus $g^{-1}(G) = \bigcup_I \bigcap_F g^{-1}f^{-1}(G) = \bigcup_I \bigcap_F (f \circ g)^{-1}(G)$. \square

If the family \mathcal{F} consists of real function on X , the weak topology is denoted $\sigma(X, \mathcal{F})$. A subbase for $\sigma(X, \mathcal{F})$ consist of

$$U(f, x, \epsilon) = f^{-1}(B(f(x), \epsilon)) = \{y \in X : |f(y) - f(x)| < \epsilon\}$$

where $f \in \mathcal{F}, x \in X, \epsilon > 0$. \mathcal{F} is said **total** if $\forall f \in \mathcal{F}, f(x) = f(y) \Rightarrow x = y$. $\sigma(X, \mathcal{F})$ is Hausdorff iff \mathcal{F} is total.

Lemma 2.24. *Let A be a subset, then*

$$(A, \sigma(A, \mathcal{F}|_A)) = (A, \sigma(X, \mathcal{F})|_A)$$

Proof. Nets converges in $(A, \sigma(X, \mathcal{F})|_A)$ also converges in $(X, \sigma(X, \mathcal{F}))$, that is $\forall f, f_i(x.) \rightarrow x$. and thus the same as nets converges in $\sigma(A, \mathcal{F}|_A)$. That implies identical mapping is a homeomorphism since $x. \rightarrow x \Leftrightarrow I(x.) \rightarrow I(x)$. \square

The weak topology generated by $C(X)$ is also generated by $C_b(X)$ by noting for any $f \in C(X)$,

$$g(y) = \min\{f(x) + \epsilon, \max\{f(x) - \epsilon, f(y)\}\}$$

is boundeb by $B(f(x), \epsilon)$ and $U(g, x, \epsilon) = U(f, x, \epsilon)$.

Theorem 2.33. *$(X,)$ is completely regular iff $\tau = \sigma(X, C(X))$*

Suppose $\tau = \sigma(X, \mathcal{F})$ and is compelely regurlar, then we claim that $\mathcal{F} = C(X)$.

2.14 Product topology

Let $((\Omega_i, \tau_i))_{i \in I}$ be family of topological spaces, let $\Omega = \prod_{i \in I} \Omega_i$ and π_i be projection mappings from Ω to Ω_i . The topology τ induced by $(\pi_i)_{i \in I}$ is called **product topology** on Ω and denoted by $\prod_{i \in I} \tau_i$. (Ω, τ) is called **topological product**. A subbase of this topology is all the sets of the form $\prod_{i \in I} X_i$ where $X_i = \Omega_i$ for all i but one is arbitrary open set, or equally, sets of the form $\pi_i^{-1}(U_i)$ where $U_i \in \tau_i$.

Lemma 2.25. *Suppose $G \in \prod \tau_i$, then $\pi_i(G) = \Omega_i$ except a finite set in I .*

Proof. By definition,

$$G = \bigcup_I \bigcap_F \left(\prod_{i \in I} X_i \right)$$

where $X_i = \Omega_i$ for all i but one. Note there is a finitely intersection, that is

$$G = \bigcup_I \left(\prod_{i \in I} X_i \right)$$

where $X_i = \Omega_i$ for all i but finite exception. □

Thus, $\{(x_i^\alpha)\}_{i \in I}$ in X converges to some $(x_i)_{i \in I}$ iff its every components converges to the components respectively. A function is called **jointly continuous** if it's continuous w.r.t the product topology.

Theorem 2.34 (Closed Graph Theorem). *Function $f : (X, \tau) \rightarrow (Y, \tau)$ where Y is compact Hausdorff is continuous iff its graph $\text{Gr } f$ is closed.*

Proof. \Rightarrow . For any net $(x., y.) \rightarrow (x, y)$, we show that $(x, y) \in \text{Gr } f$. Note $f(x.) = y. \rightarrow y$, also, $f(x.) \rightarrow f(x)$ by continuity. It follows by $f(x) = y$ since Hausdorff and we finished.

\Leftarrow . Since Y is compact and Hausdorff, $f(x.)$ converges to precisely one point and denoted as y . As $\text{Gr } f$ is closed, $y = f(x)$ and hence f is continuous. □

Suppose A_i is subset of each i , then

$$\text{Cl}_\tau \left(\prod A_i \right) = \prod (\text{Cl}_{\tau_i} A_i)$$

Thus we have an alternative definition of semicontinuous:

$f : X \rightarrow \mathbb{R}^*$ is

- lower semicontinuous iff its epigraph $\{(x, c) : c \geq f(x)\}$ is closed.
- upper semicontinuous iff its hypograph $\{(x, c) : c \leq f(x)\}$ is closed.

Theorem 2.35 (Tychonoff Product Theorem). *The product topology of a family of topologies $\tau = \prod_{i \in I} \tau_i$ is compact iff τ_i is compact for every $i \in I$.*

Proof. \Rightarrow is clearly as projection is continuous.

\Leftarrow , suppose \mathcal{U} is ultrafilter in τ , then $\pi_i(\mathcal{U})$ is ultra base and thus coverges to some point, say x_i , then we claim that $\mathcal{U} \rightarrow x = (x_i)_{i \in I}$. Suppose V any neighborhood of x , there is

$$a \in \bigcap_{i \in J} \pi_i^{-1}(X_i) \subset V$$

where X_i is neighborhood of x_i and thus belong to $\pi_i(\mathcal{U})^\dagger$, that implies there is $U \in \mathcal{U}$ s.t. $\pi_i(U) \subset X_i$, note $U \subset \pi_i^{-1}\pi_i(U) \subset \pi_i^{-1}(X_i)$, then $\pi_i^{-1}(X_i) \in \mathcal{U}$ and thus $V \in \mathcal{U}$. It followed by x is adherent to \mathcal{U} and thus $\mathcal{U} \rightarrow x$ as \mathcal{U} is ultra. \square

As consequence, we have

Theorem 2.36. *In the same notations, let K_i be compact for each i , G is open in τ and including $\prod_{i \in I} K_i$, then there exist basic open set sandwich by them.*

2.14.1 Coinduced topology

In the same notations, let K_i be compact for each i , G is open in τ and including $\prod_{i \in I} K_i$, then there exist basic open set sandwich by them.

Suppose $\{(\Omega_i, \mathcal{T}_i)\}_{i \in I}$ a family of topological space and $\{f_i : (\Omega_i, \mathcal{T}_i) \rightarrow (\Omega, \tau)\}_{i \in I}$. Let A be the set of all the topologies s.t. f_i is continuous for all i . We call the finest of A **topology coinduced** on Ω by $\{(f_i)\}_{i \in I}$.

Let R an equivalence relation on Ω , $\eta : \Omega \rightarrow \Omega/R$ the canonical surjection. The coinduced topology on Ω/R by η is denoted by τ/R and $(\Omega/R, \tau/R)$ is the quotient space w.r.t R .

2.15 Convergence

The product topology on \mathbb{R}^X is also called **topology of pointwise convergence** on X because a net $f. \rightarrow f$ iff $f.(x) \rightarrow f(x)$ for each $x \in X$.

If \mathcal{F} is collection of functions on X , X can be seen as functions on \mathcal{F} by $e_x(f) = f(x)$ for each $x \in X$, such functions are called **evaluation functional**. There also exist induced topology $\sigma(\mathcal{F}, X)$ on \mathcal{F} , which is identical to the subspace $\mathbb{R}^X|_{\mathcal{F}}$ endowed the product topology. Formally

$$\sigma(\mathcal{F}, X) = \sigma(\mathbb{R}^X, X)|_{\mathcal{F}}$$

Lemma 2.26. *If \mathcal{F} is total, the function*

$$x \mapsto e_x : (X, \sigma(X, \mathcal{F})) \rightarrow (\mathbb{R}^{\mathcal{F}}, \sigma(\mathbb{R}^{\mathcal{F}}, \mathcal{F}))$$

is injective and thus an embedding.

Proof.

$$\begin{aligned} x. \rightarrow x &\iff \forall f \in \mathcal{F}, f(x.) \rightarrow f(x) \\ &\iff \forall f \in \mathcal{F}, e_f(e_{x.}) \rightarrow e_f(e_x) \\ &\iff e_{x.} \rightarrow e_x \end{aligned}$$

□

By Tychonoff theorem 2.35, \mathcal{F} is compact iff it's closed and pointwise bounded by borel theorem.

Definition 2.16. A net $f.$ converges uniformly to $f \in \mathbb{R}^X$ iff $|f.(x) - f(x)| < \epsilon$ eventually for each $x \in X$ at the same time.

Theorem 2.37. *The uniform limit of a net is continous.*

::: {theorem # name="Dini's Theorem"}

If $f.$ converges monotonically to $f \in \mathbb{R}^X$ pointwise, then the net converges uniformly.

:::

If $f. \rightarrow f$, $f.(x) \rightarrow f(x)$ for each x and thus $f.(x)$ eventually in $B(f(x), \epsilon)$ after, say, f_{α_x} . Note

$$X = \bigcup_{x \in X} f^{-1}(B(f(x), \epsilon)) = \bigcup_{x \in J} f^{-1}(B(f(x), \epsilon))$$