Notes of Infinite dimensional analysis

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Contents

Chapter 1

Odds and ends

1.1 Space of sequences

Definition 1.1. For $1 \leq p < \infty$, ℓ_p is defined to be the set of all sequences $x. = (x_1, x_2, \cdots)$ for which $\|x\|_p < \infty$. Where

$$\|x\|_p = (\sum_1^\infty |x_i|^p)^{1/p}$$

is the ℓ_p **norm** of the sequences.

While ℓ_∞ is defined as the set of all $\sup\{|x_n|\} \leq \infty$, such norm is called ℓ_∞ **norm**, **supremum norm** or **uniform norm**.

All of these spaces are vector space. And sequence $\{\ell_i\}_{i=1}^\infty$ is increasing.

The space of all convergent sequence is denoted c and all sequences convergent to 0 is denoted c_0. Finally, the collection of sequences with finite nonzero terms is φ. One can check that

$$\varphi \subset \ell_p \subset c_0 \subset \ell_\infty \subset \mathbb{R}^n$$

1.2 Spaces of functions

One can think $\mathbb{R}^n$ as

$$\{f : \{1, 2, \cdots, n\} \to \mathbb{R}\} = \mathbb{R}^n = \mathbb{R}^{\{1,2,\cdots,n\}}$$

Replace $\{1, 2, \cdots, n\}$ by an arbitrary X, then $\mathbb{R}^X$ is all functions from X to $\mathbb{R}$.

For $1 \leq p < \infty$, $L_p(\mu)$ is defined to be the set of all μ measurable functions f for which $\|f\|_p < \infty$, where the L_p **norm** is defined as

$$\|f\|_p = (\int_\Omega |f|^p)^{1/p}$$

And the L_∞ **norm**, or **essential supremum** is defined as

$$\|f\|_\infty = \operatorname{ess\,sup} f = \sup\{t : \mu(\{x : |f(x)| \geq t\})0\}$$

1.3 Ordinals

Suppose R is an order relation on Ω, then Ω is said to be **inductively ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Definition 1.2. A set X is **well ordered** by linear $\preceq$ if every nonempty subset has a least element.

Definition 1.3. An **initial segement** of $(X, \preceq)$ is any set of the form $I(x) = \{y \in X : y \leq x\}$.

Definition 1.4. An **ideal** in a well ordered X is a subset A s.t. forall $a \in A$, $I(a) \subset A$.

Theorem 1.1 (Well Ordering Principle)**.** *Every nonempty set can be well ordered.*

Proof. Let X nonempty, and let

$$\mathcal{X} = \{(A, \preceq_A) \text{ is well order} : A \subset X\}$$

all well ordered sets, and define $\preceq$ on $\mathcal{X}$ as $(B, \preceq_B) \preceq (A, \preceq_A)$ if B is an ideal in A and $\preceq_A$ extends $\preceq_B$. Suppose every chain $\mathcal{C}$ in $\mathcal{X}$, $(\cup\mathcal{C}, \cup\{\prec_A : A \in \mathcal{C}\})$ clearly an upper bound of $\mathcal{C}$ and well ordered. By Zorn's lemma, there is a maximal element of $\mathcal{X}$ and it's actually X. $\square$

Kind of remarkable and useful well ordered set is exist:

Theorem 1.2. *There exist poset* $(\Omega, \preceq)$ *satisfy*

1. $(\Omega, \preceq)$ *is well ordered.*

2. *Ω has a greast element ω_1*
3. *$I(x)$ is countable for $x < \omega_1$*
4. *$\{y \in \Omega : x \leq y \leq \omega_1\}$ is uncountable.*
5. *Every nonempty subset of Ω has a least upper bound.*
6. *A nonempty subset of $\Omega - \{\omega_1\}$ has greatst element iff it's countable. Every uncountable subset has least upper bound ω_1.*

Proof. Let $(X, \preceq)$ be uncountable well ordered set, and let A

$$A = \{x \in X : I(x) \text{is uncountable}\}$$

w.l.o.g we may assume A is nonempty. Then there is a first element and denoted by ω_1. Then we show that $\Omega = I(\omega_1)$ enjoy all the properies.

1-4 is straightforward and 5 follows from all the upper bound is well ordered and thus least upper bound exist. For 6, suppose there is a countable $C \subset \Omega - \{\omega_1\}$, then $\bigcup_{i=1}^{\infty} I(x_i)$ is countable, so there is some $x < \omega_1 \notin \bigcup_{i=1}^{\infty} I(x_i)$, that is an upper bound. By 5, least upper bound is exist and belong to C. Conversely, if some subset C has some least upper bound $b < \omega_1$, then $C \subset I(b)$ and must countable. □

The elements of Ω are called **ordinals** and ω_1 is called **first uncountable ordinal**. The elements of $\Omega_0 = \Omega - \{\omega_1\}$ is **countable ordinals**. We treat $\mathbb{N}$ as a subset of Ω. Then the first element of $\Omega - \mathbb{N}$ is **first infinite ordinal**.

Theorem 1.3 (Interlacing Lemma)**.** *Suppose sequence $\{x_n\}$ and $\{y_n\}$ in Ω_0 with $x_n \leq y_n \leq x_{n+1}$. Then they share the same least upper bound.*

Proof. Clearly since $x_n \leq y_n \leq x_{n+1}$.

□

Chapter 2

Topology

2.1 Topological spaces

Let Ω be as space

Definition 2.1. A class of subset τ of Ω is an **topology** if

1. $\emptyset$ and Ω belongs to τ.
2. closed under arbitrary union.
3. closed under finite intersection.

(Ω, τ) called a **topological space** where Ω is called as **uderlying set**. The sets in τ are called **open** while sets with complement in τ is **closed**. Both open and closed set is called **clopen**.

Definition 2.2. Countable intersection of open sets is $\mathcal{G}_\sigma$ set and countable union of closed sets is $\mathcal{F}_\delta$ set.

Following is some examples of topological space.

Definition 2.3. (X, ρ) is a **semimetric space**, when ρ defined on $X \times X$ s.t. $\forall x, y, z \in X$:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

ρ is called a **semimetric**.

If $\rho(x, y) = 0 \iff x = y$, ρ become a **metric** and (X, ρ) become **metric space**. $B(a, r) = \{x \in E, d(x, a) < r\}$ is r-**ball** with center a.

U is **open** in (Ω, d) iff $\forall x \in U, \exists r_x 0 \ni B_d(x, r_x) \subseteq U$. Let τ_d be the set of all open subsets of Ω, we call τ_d the **topology generated by** d. A Topological space is **metrizable** if there exist metric d generates it.

Suppose d is discrete, that is, $d(x, y) = 0$ iff $x = y$, otherwise, $d(x, y) = 1$. Then every subset is open hence $\tau_d = \mathcal{P}(\Omega)$ and called **discrete topology**. The zero semimetric, defined by $d(x, y) = 0$ for all $x, y \in \Omega$ generates $\tau_d = \{\emptyset, \Omega\}$ and called **trivial topology**.

Let $\Omega = \mathbb{R}^n$, $l^2 = \sqrt{\sum_1^n (x_i - y_i)^2}$ is called **Euclidean metric**. $l^1 = \sum_1^n |x_i - y_i|$ is called **texi-cab metric** and $l^\infty = \sup\{|x_i - y_i|\}$ is called **sup norm metric**.

Note $d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$ and $d_{l^2}(x, y) \leq \sqrt{n} d_{l^\infty}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$, then d_{l^∞} open $\iff$ d_{l^2} open $\iff$ d_{l^1} open. Hence $\tau_{d_{l^2}} = \tau_{d_{l^1}} = \tau_{d_{l^\infty}}$.

All topologies on Ω is poset with greatest element $\mathcal{P}(\Omega)$ and least $\{\emptyset, \Omega\}$. If $\tau' \subset \tau$, we say τ' **coarser** than τ while τ finer than τ'.

If τ can be form by taking union of families in some $\mathcal{B} \subset \tau$, we call $\mathcal{B}$ the **base** for the topology τ.

Theorem 2.1. *$\mathcal{B}$ is a base in (X, τ) iff $\forall U \in \tau, \forall x \in U, \exists W \in \mathcal{B} \ni x \in W \subset U$.*

Proof. $\implies$: Any U can be written as $U = \cup W_i$ and $x \in U \implies x \in W_i$ for some i and $W_i \in \mathcal{B}$. $\impliedby$: For any $U \in T$, consider arbitary $x \in U$, then there exist W_x such that $x \in W_x \subset U$, thus we have $U = \cup_x W_x$. $\square$

Let $\mathcal{S} \subset \tau$, suppose all topologies include $\mathcal{S}$. Then the intersection of all of them is again a topology, denoted as $\tau(S) = \cap \mathrm{T}$, then $\tau(\mathcal{S})$ is the smallest topology contains $\mathcal{S}$. We call it the topology **generated** by $\mathcal{S}$.

Theorem 2.2. *$\tau(\mathcal{S})$ is unions of families of finite intersections together with Ω, formally:*

$$\{\bigcup (\bigcap_1^N S_i)\} \cup \Omega$$

$\mathcal{S} \subset \tau$ is a **subbase** for τ if all finite intersections of $\mathcal{S}$ is a base. Note that if $\Omega \in \mathcal{S}$, $\mathcal{S}$ is the subbase of $\tau(\mathcal{S})$. (Ω, τ) is **second countable** if τ has countable base. Clearly, a topology is second countable iff it has countable subbase.

For any subset X in (Ω, τ), then

$$\tau_X = \{X \cap V : V \in \tau\}$$

form a topology in X and we call (X, τ_X) a **subspace** or **relative topology**. Sets in τ_X are **relative open**. **Relative closed** sets of the form

$$X - (X \cap V) = X - V = X \cap V^c$$

2.2 Neighborhood

A subset V is called a **neighborhood** of a if there exists a open set $U \subset V$ contains a. Then we called $V' = V - \{a\}$ **punctured(deleted)** neighborhood. A **neighborhood base** is a collection of neighborhood $BN(a)$ s.t. for any neighborhood V of a, there exist a $W \in BN(a)$ and $W \subset V$. Clearly, all the neighborhoods is a neighborhood base and denoted as $\mathcal{N}(x)$, which is called **neighborhood system**.

Lemma 2.1. *A subset U is open iff it's a neighborhood for each of its points.*

Proof. $\Longrightarrow$ is trival. $\Longleftarrow$ follows from $\cup_x G_x = U$ and unions of open set is still open. ■ □

This suggest a equivalent definition of finear topology:

Lemma 2.2. $\tau' \subset \tau \iff$ *τ' neighborhood is a τ neighborhood.*

Proof. $\Longrightarrow$ any open set G_x satisfy $x \in G_x \subset V$ in T' is still open in T, hence V is T neighborhood. $\Longleftarrow$ Consider any open set $G \in T'$, it's a T' neighborhood for each of its points implies it's a T neighborhood for each of its points and hence G is T open. □

2.3 Closures

The **interior** of A is the union of all open sets which are included A, i.e., the largest open set included in A, we denote it A°. And the **closure** is the intersection of all closed sets which include A and thus the smallest closed set includes A, we denote it $\overline{A}$.

Lemma 2.3. *Following is some useful truth:*

1. $(A \cap B)^\circ = A^\circ \cap B^\circ$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. $A \subset \overline{B} \implies \overline{A} \subset \overline{B}$
4. $A^\circ \subset B \implies A^\circ \subset B^\circ$
5. $\overline{A^c} = (A^\circ)^c$

6. $(\overline{A})^c = (A^c)^\circ$

Proof. We only prove **5**, note $(A^\circ)^c$ is closed and

$$A^\circ \subset A \implies (A^c) \subset (A^\circ)^c$$

we have $\overline{A^c} \subset (A^\circ)^c$. On the other hand

$$\overline{A^c} \supset (A^\circ)^c \impliedby (\overline{A^c})^c \subset A^\circ \impliedby (\overline{A^c})^c \subset A \impliedby \overline{A^c} \supset A^c$$

$\square$

The **frontier** of A is $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ$.

x is said to be an **interior point** of A if A is neighborhood of x.

x is said to be an **adherent point** if it's every neighborhood meets A, an ω **accumulation point** of A if every neighborhood of x contains **infinitely** many points of A and is a **condensation point** of A if every neighborhood of x contains **uncountable** many points of A.

x is a **cluster point** or **accumulation point** if every deleted neighborhood of x meets A and is **isolated point** if x is not cluster point. That is, $\{x\}$ is relative open in A. We denoted all the cluster points as A' and called **derived set**.

x is **frontier point** or **boundary point** if every neighborhood of x meets both A and A^c.

It's east to show that the points of A° are precisely all the interior points of A and $\overline{A}$ are precisely all the adherent points. ∂A is precisely points of frontier. We claim that

$$\overline{A} = A^\circ \cup \partial A = A \cup A'$$

A subset A is called **perfect** if it's closed while point in A is cluster points in A, that is $A' = A = \overline{A}$.

2.4 Dense

A is said **dense** if $\overline{A} = \Omega$ and **nowhere dense** if $(\overline{A})^\circ = \emptyset$ ($\mathbb{Q}$ is dense in $\mathbb{R}$ while $\mathbb{Z}$ is nowhere dense.) A is said to be **meagre** or **set of the first category** if it's countable union of nowhere dense. Sets which are not meagre is **set of the second** category set.

Space (Ω, τ) is **first countable** if every point of Ω has countable neighborhood base. The space is said **separable** if Ω has a countable dense subset.

Lemma 2.4. *Second countable space is separable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, by axiom of choice, we may take x_i in I, let $X = \{x_i\}_{i \in I} \subset \Omega$. Then we show that X is dense. For any $x \in \Omega$, it's neighborhood must contain some open G which is unions of $\mathcal{B}$ and thus contains at least one element in X, that is, G meet X. Hence $\overline{X} = \Omega$. □

Lemma 2.5. *Second countable space is first countable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, for each point $x \in \Omega$, one may take all the sets in $\mathcal{B}$ which contains x as a neighborhood base. To verify it's neighborhood base, if there is a neighborhood N of x, then there is a open G contains x. By the definition of base, G is the union of sets of $\mathcal{B}$ and those sets must at least one contains x and these sets is subset to G. □

2.5 Mappings

Suppose (Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

Lemma 2.6. *Follwing is some useful truth for mappings.*

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap N) = f^{-1}(U) \cap f^{-1}(N)$
4. $f^{-1}(U \cup N) = f^{-1}(U) \cup f^{-1}(N)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ *always holds if* f *is injection while* $ff^{-1}(A) = A$ *always holds if* g *is surjection.*
7. *If* f *is bijection,* $(f^{-1})^{-1}(A) = f(A)$ *always hold.*
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$
9. $f^{-1}(A) \subset f^{-1}(B) \impliedby A \subset B$
10. $f(A) \subset f(B) \impliedby A \subset B$

Definition 2.4. f is **continuous** at x if for every neighborhood N' of $f(x)$, there is a neighborhood N of x s.t. $f(N) \subset N'$. It's continuous if it's continuous at every points $x \in \Omega$.

Theorem 2.3. f *is continuous iff*

1. $f^{-1}(G')$ *is open for every open subset* G' *of* Ω'*.*
2. $f^{-1}(F')$ *is closed for every closed subset* F' *of* Ω'*.*
3. *If* $A \subset \Omega'$*, then* $f^{-1}(A^\circ) \subset (f^{-1}(A))^\circ$
4. *If* $A \subset \Omega$*, then* $f(\overline{A} \subset \overline{f(A)})$

Proof. We only prove 1 and 3.

1 $\implies$: For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood N s.t. $f(N) \subset G'$, and

$$x \in N \subset f^{-1}f(N) \subset f^{-1}(G')$$

$\impliedby$: For every neighborhood N', there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $ff^{-1}(G') \subset G'$.

3 $\implies$: $f^{-1}(A^\circ)$ is open and th claim follows from $f^{-1}(A\circ) \subset f^{-1}(A)$. $\impliedby$: Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggets $f^{-1}(A)$ is open. □

Lemma 2.7. *Suppose $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$, $f \circ g$ is continuous if f and g are continuous.*

Proof. Suppose G_3 is open and the claims follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$. □

Lemma 2.8. *Suppose $f : (\Omega, \tau), (\Omega', \tau(\mathcal{S}))$, f is continous iff $f^{-1}(S) \in \tau$ for any $S \in \mathcal{S}$.*

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f, s.t f^{-1} is continuous and such f is called **homeomorphism**. In particular, f is an **embedding** if $f : (\Omega, \tau) \to (f(\Omega), \tau|f(\Omega))$ ia a homeomorphism.

f is **open** if $f(G)$ is open for all open set $G \in \tau$ and is **closed** if $f(F)$ is closed for all closed set $F^c \in \tau$.

Lemma 2.9. *Suppose f is bijection, then it's homeomorphism iff it's continuous and either open or closed.*

Proof. By the continuity of f^{-1}, since $(f^{-1})^{-1}(G) = f(G)$ for all open set G.

$$f^{-1} \text{ is continuous } \iff f(G) \text{ is open } \iff f \text{ is open }.$$

□

Lemma 2.10. *Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f continuous.*

Proof. Suppose f is homeomorphism, T_0 is another topology where f is continuous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1},

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0. □

Note that $\mathcal{P}(\Omega)$ let all f continuous and $\{\emptyset, \Omega\}$ let all $g : \Omega' \to \Omega$ continuous.

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2.6 Filter

Definition 2.5. A **filter** is a non-empty collection $\mathcal{F}$ of subset in Ω s.t.

1. $A \in \mathcal{F}, A \subset \mathcal{B} \implies \mathcal{B} \in \mathcal{F}$
2. Closed under finite intersection.
3. $\emptyset \notin \mathcal{F}$

Note the definition of $\mathcal{F}$ is independent with topology τ. A **free filter** is filter with $\ker \mathcal{F} = \bigcap_{F \in \mathcal{F}} F = \emptyset$. Not free filters are called **fixed**.

Filter can be formed by taking upward closure of a filter base.

Definition 2.6. A collection $\mathcal{B}$ of subset in Ω is a **fiter base** of or **prefilter** if

1. $\mathcal{B} \subset \mathcal{F}$
2. $\forall N \in \mathcal{F}, \exists W \in \mathcal{B} \ni W \subset N$

We say $\mathcal{B}$ generates $\mathcal{F}$, where

$$\mathcal{F} = \mathcal{B}^{\uparrow} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

For example,

- Suppose $x \in \Omega$ then

  $$\mathcal{N}(x) = \{\text{All neighbourhoods of } x\}$$

  is a filter on Ω, that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. Note

  $$\tau(x) = \{X \in \tau : x \in X\}$$

  is a base for $\mathcal{N}(x)$ and thus $\mathcal{N}(x) = \tau(x)^{\uparrow}$.

- Suppose Ω is infinite, the collection of all **cofinite** subsets(subset s with finite complement) is a filter on Ω, such filter is free and called **Frechet filter**.

To assert a collection is a base, we have

Theorem 2.4. *Let $\mathcal{B}$ be a collection of nonempty subsets. Then $\mathcal{B}$ is a filter base, that is, $\mathcal{B}$ may generates a filter iff*

1. *The intersection of each finite family of sets in $\mathcal{B}$ inclueds a set in $\mathcal{B}$*
2. *$\mathcal{B}$ is non-empty and $\emptyset \notin \mathcal{B}$.*

Proof. We claim that

$$\mathcal{F} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

$\mathcal{F}$ is the filter generated by $\mathcal{B}$. $\square$

A family of subsets $\mathcal{F}$ is said to have **finite intersection property** if intersection of every finite subfaimily is nonempty.

Let $\mathcal{A}$ be collection of subsets with finite intersection property, then collection of all finite intersection of $\mathcal{A}$ is a base, we call the filter generated **filter generated by** $\mathcal{A}$. Formally

$$\mathcal{F} = \{\bigcap_{A \in \mathcal{J}} A : \mathcal{J} \subset \mathcal{A} \text{ and } \mathcal{J} \text{ is finite}\}^{\uparrow}$$

A filter $\mathcal{F}$ is **finer** than another $\mathcal{G}$ if $\mathcal{F} \subset \mathcal{G}$. Clearly, the set of all filters on Ω is inductively ordered by inclusion. By Zorn's lemma, the set of all filters has maximal filters and we call such fliters **ultrafilters**.

Lemma 2.11. *Every fixed ultrafilter of the form*

$$\mathcal{U}(x) = \{x\}^{\uparrow}$$

for any $x \in \Omega$. And every free ultrafilter contains no finite subsets.

To assert a filter is ultra, we have:

Theorem 2.5. *Let A be a collection of subsets and $\mathcal{F}$ the filter generates by A. If*

$$\forall X \subset \Omega, \textit{either } X \in A \textit{ or } X^c \in A$$

then A is an ultrafilter on Ω.

Proof. Suppose $\mathcal{F}'$ is an ultrafilter include $\mathcal{F}$, we have $\mathcal{F}' \supset A$ clearly. Consider any $X \in \mathcal{F}'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in \mathcal{F}'$ as $\mathcal{F}' \supset \mathcal{F} \supset A$ and $X \cap X^c = \emptyset \in \mathcal{F}'$ results in a contradiction. It follows that $A \supset \mathcal{F}'$ and thus $A = \mathcal{F}'$. $\square$

Theorem 2.6. *Every filter $\mathcal{F}$ is the intersection of all the ultrafilter which include $\mathcal{F}$.*

Proof. We claim that

$$\mathcal{F} = \cap\{\text{ultrafilter generates by } \{x\} : x \in \cap\mathcal{F}\}$$

$\square$

Suppose mappings on a filter:

Theorem 2.7. *Let f be a mapping from Ω to Ω' and $\mathcal{B}$ a base for a fliter $\mathcal{F}$ on Ω. Then $f(\mathcal{B}) = \{f(X)\}_{X \in B}$ is also a base on Ω'. Moreover, if $\mathcal{F}$ is ultra then $f(\mathcal{B})$ also generates an ultrafilter.*

Proof. First assertion is straightforward and the second follows from $\mathcal{B}$ is collection of supset for some $\{x\}$, then $f(\mathcal{B})$ generates the fliter that generates by $\{f(x)\}$. $\square$

Theorem 2.8. *In the same situation as previous theorem. If $\mathcal{B}'$ is a base on Ω', then $f^{-1}(\mathcal{B}')$ is a base on Ω iff every set in $\mathcal{B}'$ meets $f(\Omega)$*

Proof. We have

$$\Omega \in f^{-1}(\mathcal{B}') \implies f(\Omega) \subset X'$$

for some $X' \in f^{-1}(\mathcal{B}')$, by definition, $\implies$ is immediately.

For $\Longleftarrow$, suppose any finite family $X_i \in \mathcal{B}'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}(\bigcap_i X_i) \in f^{-1}(\mathcal{B}')$$

Then the claim follows from theorem 2.4. $\square$

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the fliter $\mathcal{F}$ and $\mathcal{F}$ is said to **converge** to x, or $\mathcal{F} \to x$, if the neighborhood filter $\mathcal{N}(x) \subset \mathcal{F}$. For filter base $\mathcal{B}$, we define on the filter generated by $\mathcal{B}$, that is, if $\mathcal{N}(x) \subset \mathcal{B}^{\uparrow}$.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff \mathcal{N}_{\tau}(x) \supset \mathcal{N}_{\tau'}(x) \iff \mathcal{F} \to a \implies \mathcal{F}' \to a$$

also, an equivalent definition of continuity as follows:

Theorem 2.9. *$f : (\Omega, \tau) \to (\Omega', \tau')$ is continous at x iff*

$$\forall \mathcal{F} \to x, f(\mathcal{F}) \to f(x)$$

Proof. By definition, $f(\mathcal{F}) \to f(x)$ if

$$\mathcal{N}(f(x)) \subset f(\mathcal{F})^{\uparrow}$$

That is, for any neighbourhood $N' \in \mathcal{N}(f(x))$, there exist some $A \in \mathcal{F}$ s.t. $f(A) \subset N'$, as $\mathcal{N}(x) \subset \mathcal{F}$ and f is continous at x, such A is always exists. Conversely, take $\mathcal{F} = \mathcal{N}(x)$ then the claim is follows $\square$

A point $x \in \Omega$ is said to be an **adherent point** of $\mathcal{F}$ if x is an adherent point of every set in $\mathcal{F}$. The **adherence** of $\mathcal{F}$, $\mathrm{Adh}_{\tau}(\mathcal{F})$ or $\overline{\mathcal{F}}$ is the set of all adherent points, thus

$$\overline{\mathcal{F}} = \bigcap_{X \in F} \overline{X}$$

Define similarly on filter base $\mathcal{B}$ by the filter generated. By definition, we have

$$\overline{\mathcal{B}} = \bigcap_{X \in B} \overline{X}$$

Lemma 2.12. *Suppose A be a subset of Ω, then $x \in \overline{A}$ iff there is a filter $\mathcal{F}$ s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.*

Theorem 2.10. *Suppose $BN(x)$ a neighbourhood base of x, then*

1. *$\mathcal{B}$ converges to x iff every set in $BN(x)$ includes a set in $\mathcal{B}$.*
2. *$x \in \overline{\mathcal{B}}$ iff every set in $BN(x)$ meets every set in $\mathcal{B}$.*

As consequence, we have

Corollary 2.1. *x is adherent to a filter $\mathcal{F}$ iff there is $\mathcal{F}' \supset \mathcal{F}$ and converges to x*

Proof. $\implies$ follows from taking $\mathcal{F} = BN(x)$. Conversely, $\forall N \in BN(x)$, we have $X' \subset N$ for some $X' \in \mathcal{F}'$, thus for any $X \in \mathcal{F}$, $N \cap X \subset X' \cap X \neq \emptyset$ as $X', X \in \mathcal{F}'$. $\square$

Corollary 2.2. *Every limit point of $\mathcal{F}$ is adherent to $\mathcal{F}$*

Proof. Clearly holds by applying theorem 2.10.1 and 2.10.2. □

Corollary 2.3. *Every adherent point of an ultra-filter is a limit point of it.*

Proof. Clearly as kernel of ultrafilter is a one point set. □

Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a **limit point** of f relative to $\mathcal{F}$ if $f(\mathcal{F}) \to x$.
2. an **adherent point** of f relative $\mathcal{F}$ if it's adherent point of $f(\mathcal{F})$.

Theorem 2.11. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$*

1. *x' is a limit point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, we have $f^{-1}(N') \in \mathcal{F}$.*
2. *x' is an adherent point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, it meets $f(X)$ for any $X \in \mathcal{F}$.*

Proof. x' is limit is equivalent to

$$\mathcal{N}(x') \subset f(\mathcal{F})^{\uparrow}$$

That is, there exist some $A = f(X) \subset N'$ for any N', followed by $X \subset f^{-1}f(X) \subset f^{-1}(N')$, then the claim follows from the definition of filter.

By theorem 2.10, x' is adherent to $f(\mathcal{F})$ iff

$$\forall N' \in BN(x'), \forall X \in \mathcal{F}, f(X) \cap N' \neq \emptyset$$

note for any $N' \in N'(x')$, there exist $N' \in BN(x') \ni N' \subset N'$, thus $f(X) \cap N' \neq \emptyset$ also holds. Conversely, making use of $BN(x') \subset N'(x')$. □

For example, suppose $f : (\mathbb{N}, \tau) \to (\Omega', \tau')$ and $\mathcal{F}$ the frechet filter on $\mathbb{N}$. Then x' is limit of f relative to $\mathcal{F}$ iff for all $N' \in N'(x')$, $f^{-1}(N') \in \mathcal{F} \iff f^{-1}(N')^c \subset [0, k] \iff f^{-1}(N') \supset \{n \in \mathbb{N} : n \geq k\}$ for some k, that is, $f(n) \in N'$ for any $n \geq k$.

Theorem 2.12. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ and let $\mathcal{F} = \mathcal{N}(x)$. By theorm 9, x' is limit of f relative to $\mathcal{N}(x)$ iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{N}(x) \iff N \subset f^{-1}(N') \iff f(N) \subset N'$ for some $N \in \mathcal{N}(x)$. That is, iff $x' = f(x)$, f is continous at x. Such limit points also called limit points of f at x.*

2.7 Net

$(D, \preceq)$ is called a **directed set** if every couple $\{x, y\}$ in which has an upper bound.

If $\{D_i\}_{i\in I}$ is family of directed set then $D = \prod_{i\in I} D_i$ is also directed under **product direction** defined by $(a_i)_{i\in I} \succeq (b_i)_{i\in I}$ for all $i \in I$.

Definition 2.7. Let $(D, \preceq)$ be a directed set, $\nu : D \to \Omega$ is called a **net** in Ω with domain D. The directed set is called **index set** of the net and members of D are **indexes**. We often write ν as $x.$ or $\{x_\alpha\}$.

Suppose A a subset of Ω, we say $x.$ **eventually in** A if there exist some $k \in D$ s.t. $x_n \in A$ for all $n \succeq k$. And we say ν is **frequently** in A if for all $n \in D$, there exist an $n' \succeq n$ s.t. $x_{n'} \in A$.

Lemma 2.13. *If $x.$ not frequently in A, then $x.$ eventually in A^c. Thus, for any $X \in \Omega$, $x.$ frequently in either X or X^c.*

Suppose $x \in \Omega$, then $x.$ is said **converge** to x, or $x. \to x$ if $x.$ eventually in N for all $N \in \mathcal{N}(x)$, i.e., $\mathcal{N}(x) \subset \mathcal{F}(x.)$. The point x is **adherent** to $x.$ if $x.$ frequently in N for all $N \in \mathcal{N}(x)$.

Theorem 2.13. *Suppose $A \in (\Omega, \tau)$, then $x \in \overline{A}$ iff it's the limit of some net in the set.*

Proof. $\Longleftarrow$ is clear. $\Longrightarrow$ follows from we may find a associated net taking value in A(since each neighborhood meets A) and such net converges to x. $\square$

As with sequence, if $x.$ is bounded, there is

$$\liminf x. = \sup \inf x. \preceq \limsup x. = \inf \sup x$$

Subnet generalizes subsequence.

Definition 2.8. Suppose D is directed, a subset B of D is called **cofinal** if for any $a \in D$, there exist $b \in B$ s.t. $a \preceq b$. A map $f : D \to A$ is **final** if $f(D)$ is cofinal of A.

Let $x.$ and $x.'$ are two nets in Ω with domains D and D' respectively. We say that $x.'$ is a **subnet** of $x.$ if there exists a final mapping $\varphi : D' \to D$ s.t. $x'_\alpha = x_{\varphi(\alpha)}$.

Theorem 2.14. *Let $\mathcal{A}$ be a collection of subsets that $x.$ is frequently in. If $\mathcal{A}$ is closed under finite intersection, then there exists a subnet $x'.$ of $x.$ and $x.'$ eventually in every member of $\mathcal{A}$*

Lemma 2.14. *Suppose $x.'$ is subnet of $x.$, we have*

1. $x. \to x \implies x.' \to x$
2. x *adherent to* $x.' \implies x$ *adherent to* $x.$.

Theorem 2.15. *A point x is adherent to $x.$ iff there is a subnet converges to x. While $x. \to x$ iff every subnet converges to x.*

Proof. $\implies$ is clear by lemma2.14. Conversely, suppose a is not adherent to x, there exist a neighborhood N that $x.$ not frequently in, i.e., exist k s.t. $x_n \notin N$ for any $n \geq k$, thus there is no subnet eventually in N.

For the second part, $\implies$ is also clear by lemma2.14 and $\impliedby$ comes from taking subnet as itself. $\square$

A net $x.$ is called **ultranet** or **universal net** if for all $X \in \Omega$, we have either $x.$ eventually in X or $x.$ eventually in X^c. Clearly, subnet of ultranet is ultra and

Lemma 2.15. *Every net has a ultra subnet.*

Proof. Consider collection of $\mathcal{Q}$ s.t. $x.$ is frequently in every member and closed under finite intersection. By Zorn's Lemma, there is a maximal $\mathcal{Q}_0$. By theorem 11, $x.$ has a subnet $x.'$ which eventually in every member of $\mathcal{Q}_0$. We claim that this subnet is ultra since, $\mathcal{Q}_0$ is maximal and thus either $X \in \mathcal{Q}_0$ or $X^c \in \mathcal{Q}_0$. $\square$

2.8 Nets and filters

Let

$$\mathcal{F}(x.) = \{X \in \mathcal{P}(\Omega) : x. \text{ is eventually in } X\}$$

Then $\mathcal{F}(x.)$ is a filter and we call it the **filter associated with the net** $x.$.

Theorem 2.16. *Associated filter is the upward closure of the net's tail, that is*

$$\mathcal{F}(x.) = \{\{x_b : b \succeq a\} : a \in D\}$$

Motivated by the definition of filter that filter is closed under pairwise intersection, let $X \preceq Y \iff X \supset Y$, then any mapping $\nu : \mathcal{F} \to \Omega$ s.t. $\nu(X) \in X$ is a **net associated with the filter** $\mathcal{F}$.

By definition, we claim that $\mathcal{F}$ is the associated filter of every associated net and $x.$ is an associted net of the associated fiter.

Theorem 2.17. *Filter $\mathcal{F} \to x$ iff $x. \to x$ for any $x.$ associated with $\mathcal{F}$.*

Proof. Note

$$\forall N \in \mathcal{N}(x), x. \text{ eventually in } N \iff \mathcal{N}(x) \subset \mathcal{F}(x.)$$

Then is sufficient to show that $\mathcal{F}(x.) = \mathcal{F}$. It's follows from for any $X \in \mathcal{F}$, $x.$ eventually in X. □

Theorem 2.18.

$$x. \to x \iff \mathcal{F}(x.) \to x$$

Proof. Both side is equivalent to $\mathcal{N}(x) \subset \mathcal{F}(x.)$ □

Theorem 2.19. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, then f is continous at x iff $\forall x. \to x$, $f(x.) \to f(x)$.*

Proof. By theorem 2.18,2.17 and 2.12. □

By above theorems, we have

$$\text{Adh}(\mathcal{F}(x.)) = \text{Adh}(x.), \text{Lim}(\mathcal{F}(x.)) = \text{Lim}(x.)$$

and similarly results holds for any filter and one of associated nets.

Lemma 2.16. *If $x.$ is ultra then the associated filter $\mathcal{F}(x.)$ is also ultra and if $\mathcal{F}$ is ultra, every associated net is ultra.*

Proof. Directly from theorm 2.5. □

2.9 Separation

Definition 2.9. Space (Ω, τ) is said to be T_0 or **kolmogorov** if for every pair $(x, y) \in \Omega^2$, either there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ or $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Lemma 2.17. *τ isn't T_0 iff there exist pair (x, y), s.t:*

1. $\mathcal{N}(x) = \mathcal{N}(y)$.
2. $\overline{\{x\}} = \overline{\{y\}}$.

Proof. 1 If every $N \in \mathcal{N}(x)$ contains y, then $N \in \mathcal{N}(y) \implies \mathcal{N}(x) \subset \mathcal{N}(y)$, thus $\mathcal{N}(x) = \mathcal{N}(y)$.

2 If some point $a \in \overline{\{x\}}$, then every $N \in \mathcal{N}(a)$ also is neighborhood of x and thus neighborhood of y, hence $a \in \overline{\{y\}}$. □

Definition 2.10. Space (Ω, τ) is said to be T_1 or **Frechet** if for every pair $(x, y) \in \Omega^2$,there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ and $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Theorem 2.20. *Following statements are equivalent:*

1. *τ is T_1.*
2. *Singetons are closed.*
3. $\ker \mathcal{N}(x) = \{x\}$ *holds for any* $x \in \Omega$.

Proof. $1 \implies 2$ If there exist a singeton $\{x\}$ not closed, there is $y \in \overline{\{x\}}$, hence every neighborhood of y contains x, contradiction.

$2 \implies 3$ Suppose $\ker \mathcal{N}(x)$ contains y diifer x, that implies any neighborhood of x contains y and contradict 2.

$3 \implies 1$ is straightforward. □

Lemma 2.18. *Suppose (Ω, τ) with a finite base is T_1, then Ω is finite and τ is discrete.*

Definition 2.11. A topology (Ω, τ) is T_2, or **Hausdorff** or **separated** if every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $U \cap V = \emptyset$.

Theorem 2.21. *Following statements are equivalent:*

1. *τ is T_2.*
2. *Intersection of family of closed neighborhoods of x is x.*
3. *If a filter(net) converges to some point x, then $Adh(\mathcal{F}) = \{x\}$*
4. *Every net(filter) converges to at most one point.*

Proof. $1 \implies 2$ For any pair (x, y), by definition, there is $y \notin \overline{U}$, hence intersection of family of closed neighborhoods of x can only contains x.

$2 \implies 3$ follows from a point adherent to a filter converges to x must be in every closed neighborhood of x.

$3 \implies 4$ is clearly.

$4 \implies 1$ If there is a net $x.$ converges to both x and y, then $\mathcal{N}(x) \subset \mathcal{F}(x.)$ and $\mathcal{N}(y) \subset \mathcal{F}(x.)$, that is, U and V meets for any $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$. □

Definition 2.12. Space (Ω, τ) is said to be $T_{2.5}$ or **Completely Hausdorff** if for every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $\overline{U} \cap \overline{V} = \emptyset$.

Two nonempty sets are caled **separated by open sets** if they are included in disjoint open sets, and they are **separated by continous functions** if there is continos f taking values in $[0, 1]$ and assign 0 on one set and 1 on the other.

Space (Ω, τ) are said to be **regular** if every singeton and any closed A disjoint from it can be separated by open sets.

Definition 2.13. Space (Ω, τ) is said to be T_3 if it's T_1 and regular.

Space (Ω, τ) are said to **Completely regular** if every singeton and any closed A disjoint from it can be separated by continous function.

Definition 2.14. Space (Ω, τ) is said to be $T_{3.5}$ or **Tychonoff space** if it's T_1 and completely regular.

Theorem 2.22 (Tychonoff's Embedding Theorem)**.** *Space* (Ω, τ) *is* $T_{3.5}$ *iff it's homeomorphic to a subspace of* $([0,1]^n, \tau_{d_{l^1}})$.

Space (Ω, τ) is said to be **normal** if two disjoint closed subsets can be separated by open sets.

Definition 2.15. Space (Ω, τ) is said to be T_4 if it's normal and T_1.

Theorem 2.23 (Urysohn's Lemma)**.** *Following statements are equivalent:*

1. (Ω, τ) *is normal.*
2. *For any* $U \in \tau$ *and any closed* $A \subset U$*, there is a* $U' \in \tau$ *s.t.* $A \subset U'$ *and* $\overline{U'} \subset U$.
3. *Every two disjoint closed subsets can be separated by continous function.*

Proof. $1 \implies 2$ Apply normal property to A and U^c, there is a U' include A and V include U^c, as $U' \cap V = \emptyset \implies U' \subset V^c \implies \overline{U'} \subset V^c \subset U$.

$2 \implies 3$ Suppose A and B are two disjoint closed subset, apply 2 to A and $U_1 = B^c$ we have $A \subset U_0$ and $\overline{U_0} \subset U_1$. Apply again for $\overline{U_0}$ and U_1 to generates $U_0 \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$, repeat such process, that is, apply 2 to $\overline{U_{\frac{j}{2^k}}}$ and $U_{\frac{j+1}{2^k}}$ to generates $U_{\frac{2j+1}{2^{k+1}}}$. Finally, we construct a open strictly increasing squence U_r. where r is any dyadic rational in $[0,1]$, i.e., $r \in DR \cap [0,1]$.

Then define f as

$$f = \begin{cases} 1 & x \in B \\ \inf\{r : x \in U_r\} & x \in B^c \end{cases}$$

Then it's sufficient to show that f is continous. Note subspace $[0,1]$ of $\mathbb{R}$ can be generated by collection of $[0,s)$ and $(t,1]$ and

$$f^{-1}[0,s) = \bigcup_{r \in DR \cap [0,s)} U_r$$

$$f^{-1}(t,1] = \bigcup_{r \in DR \cap (t,1]} \overline{U_r}^c$$

Then the claim follows from lemma 2.8.

$3 \implies 1$ By taking any disjoint open set A contains 0 and B contains 1 and looking $f^{-1}(A)$ and $f^{-1}(B)$. $\square$

Theorem 2.24 (Tietze's Extension Theorem)**.** *Let (Ω, τ) be normal, F any closed subset and I any bounded closed interval of $\mathbb{R}$. Then any continous $f : F \to I$ can be extended to $f' : \Omega \to I$ and remain continous.*

Proof. Suppose $I = [-1, 1]$, then $A = f^{-1}[-1, -\frac{1}{3}]$ and $f^{-1}[\frac{1}{3}, 1]$ are disjoint and closed. By Urysohn's Lemma, there is $g : \Omega \to [-\frac{1}{3}, \frac{1}{3}]$ s.t. $g(A) = \{-\frac{1}{3}\}$ and $g(B) = \frac{1}{3}$. Set $f_0 = f, g_0 = g, f_1 = f - g|_F$. Then we can show that $|f_1|$ is bounded by $\frac{2}{3}$.

Repeat such process, we have series of

$$\begin{aligned} f_n &: F \to [-(\frac{2}{3})^n, (\frac{2}{3})^n] \\ g_n &: E \to [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n] \\ f_{n+1} &= (f_n - g_n)|_F \end{aligned}$$

Then we show that $g = \sum_{i=0}^{\infty} g_i$ is the extension of f. That is g is continous and $f = g$ in F. Note for any x

$$|\sum_{i=m}^{n} g_i(x)| \leq \sum_{i=m}^{n} |g_i(x)| \leq \sum_{i=m}^{n} \frac{1}{3}(\frac{2}{3})^i \leq (\frac{2}{3})^m \to 0$$

Thus $\{\sum_{i=0}^{n} g_i\}_{n=0}^{\infty}$ converges uniformly by Cauchy's criterion, followed by g is continous. And $f = g$ on F follows from

$$|f(x) - \sum_{i=0}^{n} g_i(x)| = |f_0(x) - \sum_{i=0}^{n} g_i(x)| = |f_1(x) - \sum_{i=1}^{n} g_i(x)| = |f_{n+1}(x)| \leq (\frac{2}{3})^{n+1} \to 0$$

□

2.10 Compactness

A **cover** of a set K is collection of sets whose union includes K. A **subcover** is subcollection of a cover and also covers K. K is **compact** if every open cover has a finite subcover and called **relatively compact** if it's closure is compact. A topology (Ω, τ) is **compact** if Ω is compact

Compactness is a "topological" property. That is, subset compactness in a subspace iff it's also compact in full space.

Theorem 2.25. *Let (Ω, τ) be a space, following are equivalent.*

1. *(Ω, τ) is compact.*
2. *Every filter(net) has at least one adherent point.*
3. *Every ultrafilter(ultranet) converges.*
4. $\ker \mathcal{F} \neq \emptyset$ *For every collection $\mathcal{F}$ of closed sets having FIP.*

Proof. $4 \iff 1$ Taking contrapositive:

$$\neg \ker \mathcal{F} \neq \emptyset \equiv \ker \mathcal{F} = \emptyset \equiv \bigcup_{F \in \mathcal{F}} F^c = \Omega$$

and

$$\neg \forall \bigcap_{i}^{n} F_i = \emptyset \equiv \exists \bigcup_{i}^{n} F_i^c = \Omega$$

note that's pricesly the definition of compactness.

$1 \implies 2$ Suppose filter $\mathcal{F}$, then

$$\{\overline{F} : F \in \mathcal{F}\}$$

enjoy finite intersection property by definition, then $\overline{F}$ has at least one adherent point since $\ker\{\overline{F} : F \in \mathcal{F}\} = \overline{\mathcal{F}} \neq \emptyset$ by 4

$2 \implies 3$ Clearly by corollary 2.3.

$3 \implies 1$ Suppose $\mathcal{A}$ a family of closed subsets with finite intersection property. Then the filter generates by $\mathcal{A}$ has an ultrafilter with a limit point x. Note x is also adherent to $\mathcal{U}$ and thus adherent to $\mathcal{F}$, followed by $x \in A$ for any $A \in \mathcal{A}$, hence $\ker \mathcal{A} \supset \{x\}$. Then the claim follows from 4. □

Theorem 2.26. *Let (Ω, τ) be Hausdorff, then every compact subset and disjoint singelton can be separated by open sets. In particular, compact subset is closed.*

Proof. Suppose $F \subset \Omega$ is compact, for any $x \in \Omega$ not in F, by Hausdorff, there is $x \notin U_y$ and $y \notin V_y$. Then $\bigcup_{y \in F} U_y$ cover F, there is subcover $U = \bigcup_i^n U_{y_i}$ and $V = \bigcup_i^n V_{y_i}$ selected from the same family separated F and $\{x\}$. □

Theorem 2.27. *Closed subset is compact in compact topological space.*

Proof. Note any open cover of F plus F^c become a open cover of Ω. □

Theorem 2.28. *Every compact Hausdorff space is normal.*

Proof. Suppose A and B are closed and thus comapct by theorem 2.27. For any point $x \in A$, there exist disjoint $V_x \supset B$ and $x \in U_x$ by theorem 2.26. Note $\bigcup_{x \in A} U_x$ cover A, there exist subcover $U = \bigcup_i^n U_{x_i} \supset A$ and $V = \bigcap_i^n V_{x_i} \supset B$ separated A and B. □

Theorem 2.29. *Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ is continous, then $f(A)$ is comapct if A is compact.*

Proof. For any open cover of $f(A)$:

$$\cup G_i \supset f(A) \implies f^{-1}(\cup G_i) = \cup f^{-1}(G_i) \supset f^{-1}f(A) \supset A$$

Thus there exist subcover s.t.

$$\cup_1^n f^{-1}(G_i) = f^{-1}(\cup_1^n G_i) \supset A \implies \cup_1^n G_i \supset ff^{-1}(\cup_1^n G_i) \supset f(A)$$

which shows that $f(A)$ is compact. □

As consequence:

Corollary 2.4 (Extreme value theorem)**.** *A continous real valued function defined on a compact space achieves its maximum and minimum values.*

Theorem 2.30. *Let (Ω, τ) be compact and (Ω', τ') be housdorff and $f : (\Omega, \tau) \to (\Omega', \tau')$ is continous bijection. Then f is homeomorphism.*

Proof. It's sufficient to show that $f(F)$ is closed as lemma 2.9. Note F is closed and thus compact as theorem 2.27 then $f(F)$ is compact as theorem 2.29 and thus closed by theorem 2.26. □

A subset A of a topological space is **sequentially compact** if every sequence in A has a subsequence converging to an element of A. A topological space is sequentially compact if itself is a sequentially compact set.

2.11 Semicontinuous

$f : \Omega \to \mathbb{R}^*$ is

- **lower semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \leq c\}$ is closed.
- **upper semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \geq c\}$ is closed.

Clearly f is lower semicontinuous iff $-f$ is upper and vice versa. Also, f is continous iff it's both upper and lower semicontinuous.

Lemma 2.19. *Suppose* $\{f_i\}_{i\in I}$ *is family of lower(upper) semicontinuous function then* $\sup f_i(\inf f_i)$ *is lower(upper) semicontinuous.*

Proof. Note

$$\{x \in \Omega : \sup f_i(x) \leq c\} = \bigcap_{i\in I}\{x \in \Omega : f_i(x) \leq c\}$$

is closed. □

Lemma 2.20. $f : \Omega \to \mathbb{R}^*$ *is*

- ***lower semicontinuous*** *iff for any net*

$$x. \to x \implies \liminf f(x.) \geq f(x)$$

- ***upper semicontinuous*** *iff for any net*

$$x. \to x \implies \limsup f(x.) \leq f(x)$$

Proof. Suppos f is lower semicontinuous and $x. \to x$. For any $c < f(x)$, then $G = \{\omega \in \Omega : f(\omega)c\}$ is open and thus $x.$ eventually in, that is $x.c$ ecventually and thus $\liminf f(x.) \geq c$. This implies that $\liminf f(x.) \geq f(x)$.

Conversely, for any $c \in \mathbb{R}$, consider $F = \{\omega \in \Omega : f(\omega) \leq c\}$. Then we show that F is closed. Suppos $x.$ is nets in F and converges to some $x \in \Omega$. Then $c \geq \liminf f(x.) \geq f(x)$ thus x in F and thus F is closed. □

Then we can gennerlizes Weierstrass' Theorem in corollary 2.4.

Theorem 2.31. $f : \Omega \to \mathbb{R}^*$ *on a compact set attains a minimum(maximum) value and set of minima(maxima) is compact if it's lower(upper) semicontinuous.*

Proof. Supoose X is compact and f is lower semicontinous, then for every $c \in f(X)$, $F_c = \{x \in X : f(x) \leq c\}$ is closed and $\{F_c : c \in f(X)\}$ has FIP clearly. Note X is compact, $\ker\{F_c : c \in f(X)\}$ is nonempty by 2.25. That is just the set of minimas and it's compact since it's closed. □

2.12 Comparing topologies

We list some useful properies when comparing topologies, some of them has been mentioned before and proof omitted.

Lemma 2.21. *Suppose τ' and τ are two tologies on Ω, then the following are equivalent.*

1. $\tau' \subset \tau$
2. *Identity mapping $I : x \mapsto x$ from (Ω, τ) to (Ω', τ') is continous.*
3. *τ' closed set is closed in τ.*
4. $x. \overset{\tau}{\to} x \implies x. \overset{\tau'}{\to} x$
5. $Cl_{\tau}(A) \subset Cl_{\tau'}(A)$

Lemma 2.22. *Suppose $\tau' \subset \tau$, then*

1. *Every τ compact set is τ' compact.*
2. *Every τ' continous function is τ continous.*
3. *Every τ dense set is τ' dense.*

2.13 Weak topology

Suppose $\{(Y_i, \tau_i)\}_{i\in I}$ a family of topological space and $f_i : X \to Y_{i_{i\in I}}$. Let $\mathcal{F}$ be the set of all the topologies s.t. f_i is continuous for all i. We call $\cap\mathcal{F}$, i.e., the corest topology the **induced topology** or **weak topology** or **initial topology** on X by $\{f_i\}_{i\in I}$. The topology induced by $\{f_i\}_{i\in I}$ is generated by

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \tau_i\}$$

or

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \mathcal{S}_i\}$$

where $\mathcal{S}_i$ is a subbase for τ_i.

Lemma 2.23. *A net $x. \to x$ in the weak topology iff $f_i(x.) \to f_i(x)$ for each i.*

Proof. $\implies$ is immediately. Conversely, noting sets of the form $\bigcap_1^n f_i^{-1}(V_i)$ consist a neighborhood base. ☐

Theorem 2.32. *g is (τ', τ) continuous iff $f_i \circ g$ continuous for each f_i. Where τ is $\tau(S)$ in above theorem.*

Proof. $\implies$ is immediately. $\impliedby$, suppose $G \in \tau$,by above theorem, this implies

$$G = \cup_I \cap_F X = \cup_I \cap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continuous and thus $g^{-1}(G) = \cup_I \cap_F g^{-1} f^{-1}(G) = \cup_I \cap_F (f \circ g)^{-1}(G)$. □

If the family $\mathcal{F}$ consists of real function on X, the weak topology is denoted $\sigma(X, \mathcal{F})$. A subbase for $\sigma(X, \mathcal{F})$ consist of

$$U(f, x, \epsilon) = f^{-1}(B(f(x), \epsilon)) = \{y \in X : |f(y) - f(x)| < \epsilon\}$$

where $f \in \mathcal{F}, x \in X, \epsilon > 0$. $\mathcal{F}$ is said **total** if $\forall f \in \mathcal{F}, f(x) = f(y) \implies x = y$. $\sigma(X, \mathcal{F})$ is Hausdorff iff $\mathcal{F}$ is total.

Lemma 2.24. *Let A be a subset, then*

$$(A, \sigma(A, \mathcal{F}|_A)) = (A, \sigma(X, \mathcal{F})|_A)$$

Proof. Nets converges in $(A, \sigma(X, \mathcal{F})|_A)$ also converges in $(X, \sigma(X, \mathcal{F}))$, that is $\forall f, f_i(x.) \to x$. and thus the same as nets converges in $\sigma(A, \mathcal{F}|_A)$. That implies identical mapping is a homeoporphism since $x. \to x \iff I(x.) \to I(x)$. □

The weak topology generated by $C(X)$ is also generated by $C_b(X)$ by noting for any $f \in C(X)$,

$$g(y) = \min\{f(x) + \epsilon, \max\{f(x) - \epsilon, f(y)\}\}$$

is boundeb by $B(f(x), \epsilon)$ and $U(g, x, \epsilon) = U(f, x, \epsilon)$.

Theorem 2.33. *(X,) is completely regular iff $\tau = \sigma(X, C(X))$*

Suppose $\tau = \sigma(X, \mathcal{F})$ and is compelely regurlar, then we claim that $\mathcal{F} = C(X)$.

2.14 Product topology

Let $((\Omega_i, \tau_i))_{i \in I}$ be family of topological spaces, let $\Omega = \prod_{i \in I} \Omega_i$ and π_i be projection mappings from Ω to Ω_i. The topology τ induced by $(\pi_i)_{i \in I}$ is called **product topology** on Ω and denoted by $\prod_{i \in I} \tau_i$. (Ω, τ) is called **topological product**. A subbase of this topology is all the sets of the form $\prod_{i \in I} X_i$ where $X_i = \Omega_i$ for all i but one is arbitary open set, or equally, sets of the form $\pi_i^{-1}(U_i)$ where $U_i \in \tau_i$.

Lemma 2.25. *Suppose $G \in \prod \tau_i$, then $\pi_i(G) = \Omega_i$ except a finite set in I.*

Proof. By definition,

$$G = \bigcup_I \bigcap_F (\prod_{i \in I} X_i)$$

where $X_i = \Omega_i$ for all i but one. Note there is a finitely intersection, that is

$$G = \bigcup_I (\prod_{i \in I} X_i)$$

where $X_i = \Omega_i$ for all i but finite exception. $\square$

Thus,$\{(x_i^\alpha)\}_{\{i \in I\}}$ in X converges to some $(x_i)_{i \in I}$ iff its every components converges to the components respectly. A function is called **jointly continuous** if it's continous w.r.t the product topology.

Theorem 2.34 (Closed Graph Theorem). *Function $f : (X, \tau) \to (Y, \tau)$ where Y is compact Hausdorff is continuous iff its graph $\mathrm{Gr}\, f$ is closed.*

Proof. $\implies$. For any net $(x., y.) \to (x, y)$, we show that $(x, y) \in \mathrm{Gr}\, f$. Note $f(x.) = y. \to y$, also, $f(x.) \to f(x)$ by continuity. It follows by $f(x) = y$ since Hausdorff and we finished.

$\impliedby$. Since Y is compact and Hausdorff, $f(x.)$ converges to precisely one point and denoted as y. As $\mathrm{Gr}\, f$ is closed, $y = f(x)$ and hence f is continuous. $\square$

Suppose A_i is subset of each i, then

$$\operatorname*{Cl}_\tau(\prod A_i) = \prod(\operatorname*{Cl}_{\tau_i}(A_i))$$

Thus we have an alternative definition of semicontinuous:

$f : X \to \mathbb{R}^*$ is

- lower semicontinuous iff its epigraph $\{(x, c) : c \geq f(x)\}$ is closed.
- upper semicontinuous iff its hypograph $\{(x, c) : c \leq f(x)\}$ is closed.

Theorem 2.35 (Tychonoff Product Theorem). *The product topology of a family of topologies $\tau = \prod_{i \in I} \tau_i$ is compact iff τ_i is compact for every $i \in I$.*

Proof. $\implies$ is clearly as projection is continuous.

$\impliedby$, suppose $\mathcal{U}$ is ultrafilter in τ, then $\pi_i(\mathcal{U})$ is ultra base and thus coverges to some point, say x_i, then we claim that $\mathcal{U} \to x = (x_i)_{i \in I}$. Suppose V any neighborhood of x, there is

$$a \in \bigcap_{i \in J} \pi_i^{-1}(X_i) \subset V$$

where X_i is neighborhood of x_i and thus belong to $\pi_i(\mathcal{U})^\uparrow$, that implies there is $U \in \mathcal{U}$ s.t. $\pi_i(U) \subset X_i$, note $U \subset \pi_i^{-1}\pi_i(U) \subset \pi_i^{-1}(X_i)$, then $\pi_i^{-1}(X_i) \in \mathcal{U}$ and thus $V \in \mathcal{U}$. It followed by x is adherent to $\mathcal{U}$ and thus $\mathcal{U} \to x$ as $\mathcal{U}$ is ultra. $\square$

As consequence, we have

Theorem 2.36. *In the same notations, let K_i be compact for each i, G is open in τ and including $\prod_{i\in I} K_i$, then there exist basic open set sandwich by them.*

2.14.1 Coinduced topology

In the same notations, let K_i be compact for each i, G is open in τ and including $\prod_{i\in I} K_i$, then there exist basic open set sandwich by them.

Suppose $\{(\Omega_i, \mathcal{T}_i)\}_{i\in I}$ a family of topological space and $\{f_i : (\Omega_i, \mathcal{T}_i) \to (\Omega, \tau)\}_{i\in I}$. Let A be the set of all the topologies s.t. f_i is continuous for all i. We call the finest of A **topology coinduced** on Ω by $\{(f_i)\}_{i\in I}$.

Let R an equivalence relation on Ω, $\eta : \Omega \to \Omega/R$ the canonical surjection. The coinduced topology on Ω/R by η is denoted by τ/R and $(\Omega/R, \tau/R)$ is the quotient space w.r.t R.

2.15 Convergence

If $\mathcal{F}$ is collection of functions on X, X can be seen as functions on $\mathcal{F}$ by $e_x(f) = f(x)$ for each $x \in X$, such functions are called **evaluation functional**.

The product topology on $\mathbb{R}^X$ is also called **topology of pointwise convergence** on X because a net $f. \to f$ iff $e_x(f.) \to e_x(f) \iff f.(x) \to f(x)$ for each $x \in X$.

There also exist induced topology $\sigma(\mathcal{F}, X)$ on $\mathcal{F}$, which is identical to the subspace $\mathbb{R}^X|_{\mathcal{F}}$ endowed the product topology. Formally

$$\sigma(\mathcal{F}, X) = \sigma(\mathbb{R}^X, X)|_{\mathcal{F}}$$

Lemma 2.26. *If $\mathcal{F}$ is total, the function*

$$x \mapsto e_x : (X, \sigma(X, \mathcal{F})) \to (\mathbb{R}^{\mathcal{F}}, \sigma(\mathbb{R}^{\mathcal{F}}, \mathcal{F}))$$

is injective and thus an embedding.

Proof. It's remain to show the continuity.

$$\begin{aligned} x. \to x &\iff \forall f \in \mathcal{F}, f(x.) \to f(x) \\ &\iff \forall f \in \mathcal{F}, e_f(e_{x.}) \to e_f(e_x) \\ &\iff e_{x.} \to e_x \end{aligned}$$

□

By Tychonoff theorem 2.35, $\mathscr{F}$ is compact iff $\forall x \in X$, $\{f(x)\}_{f\in\mathscr{F}}$ it's closed and pointwise bounded by borel theorem.

Definition 2.16. A net $f.$ converges uniformly to $f \in \mathbb{R}^X$ iff $|f.(x) - f(x)| < \epsilon$ eventually for each $x \in X$ after some f_α for any ϵ.

Theorem 2.37. *The uniform limit of a continuous net is continuous.*

Proof. Suppose $f. \to f$ uniformly, then for any $x \in X$, for any $\alpha > \alpha_0$

$$|f_\alpha(x) - f(x)| < \epsilon$$

as f_α is continuous, for any $x. \to x$, for any $\lambda > \lambda_0$

$$|f_\alpha(x_\lambda) - f_\alpha(x)| < \epsilon$$

also, there is

$$|f_\alpha(x_\lambda) - f(x_\lambda)| < \epsilon$$

Hence, we have

$$|f(x_\lambda) - f(x)| < 3\epsilon$$

Thus, $f(x.) \to f$ and continuity follows.

□

Theorem 2.38 (Dini's Theorem)**.** *If continuous real function net $f.$ on a compact set converges monotonically to f pointwise, then the net converges to f uniformly.*

Proof. Let $g. = f. - f$, we have $g. \to 0$, $|g.|$ is decreasing as monotone. Then it's sufficient to show that $g. \to g$ uniformly. Note $|g.(x)| < \epsilon$ eventually for any $x \in X$ after, say, α_x. By continuity and compactness:

$$X = \bigcup_{x\in X} |g_{\alpha_x}|^{-1}(B(0,\epsilon)) = \bigcup_{x\in J} |g_{\alpha_x}|^{-1}(B(0,\epsilon))$$

Then we may pick $\alpha_0 \geq \alpha_x$ for all $x \in J$, and for any $\alpha \geq \alpha_0$ and any $x \in X$, suppose $x \in |g_{\alpha_{x_j}}|^{-1}(B(0,\epsilon))$

$$\epsilon > |g_{\alpha_{x_j}}(x)| > |g_\alpha(x)|$$

by monotone and thus $g. \to 0$ uniformly.

□

2.16 Locally compact spaces

Definition 2.17. A topological space is **locally compact** if every point has a compact neighborhood.

Theorem 2.39 (Compact neighborhood base)**.** *Suppose (Ω, τ) is locally compact and Hausdorff, then every neighborhood of x includes a compact neighborhood. Consequently, that imply the existence of a compact neighborhood base.*

Proof. Begin by open neighborhood G and compact K for x s.t. $A = K - G \neq \emptyset$. For any $y \in A$, there is $U_y \cap W_y = \emptyset$ by Hausdorff, where $y \in U_y$ and $x \in W_y \subset K$. Note A is also compact and then there exist:

$$U = \bigcup_{i=1}^{k} U_{y_i} \supset A$$

Respectively, consider $W = \bigcap_{i=1}^{k} W_{y_i}$, and we claim that $\overline{W}$ is compact and included in G. Compactness is clear as $\overline{V} \subset K$. By theorem 2.26, $\overline{W} \cap U = \emptyset$. Consequently,

$$\overline{V} \cap G^c = \overline{V} \cap W \cap G^c = \overline{V} \cap A \subset \overline{V} \cap U = \emptyset$$

hence $\overline{V} \subset G$.

□

Corollary 2.5. *Suppose G is open and F is closed in a locally compact Hausdorff space, then $G \cap A$ is locally compact. That implies every closed and open set is locally compact.*

Proof. Note

$$G = (G \cap F^c) + (G \cap F)$$

and G and $G \cap F^c$ is open, by theorem 2.39, they are locally compact. Thus their difference is also locally compact.

□

Corollary 2.6. *If K is compact in a locally compact Hausdorff space and G is an open set including K, then there is an open V with compact closure s.t.*

$$K \subset V \subset \overline{V} \subset G$$

Proof. For any $x \in K$, by theorem 2.39, we have

$$x \in V_x \subset \overline{V_x} \subset G$$

then note

$$K \subset \bigcup_{i=1}^{k} V_{x_i} = V$$

we claim that V is desired. Since

$$\overline{V} = \overline{\bigcup_{i=1}^{k} V_{x_i}} = \bigcup_{i=1}^{k} \overline{V_{x_i}}$$

is compact and included in G.

□

Definition 2.18. A **Compactification** of a Hausdorff space X is a compact Hausdorff space $\hat{X}$ s.t. X is a homeomorphic to a dense subset of $\hat{X}$

For short, we treat X as an actual dense subset of $\hat{X}$ and τ a subspace of $\hat{\tau}$.

Theorem 2.40. *X is locally compact iff X is open of $\hat{X}$.*

Proof. $\Longleftarrow$ comes from corollary 2.5.

$\Longrightarrow$ Suppose $(\hat{X}, \hat{\tau})$ is compactification of Hausdorff (X, τ). For any $x \in X$, we may pick $x \in G \subset K$, where G is open and K is compact in τ. Consider $W \in \hat{\tau}$ where $W \cap X = G$, we have

$$W = W \cap \hat{X} = W \cap \overline{X} \subset \overline{W \cap X} = \overline{G} \subset K \subset X$$

that implies $x \in X^\circ \implies X^\circ = X$, i.e. X is open.

□