Notes of Linear Algebra

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Contents

Chapter 1

Background Knowledge

Definition 1.1 (Group)**.** A group is a set G with a binary low of composition

$$\mu : G \times G \to G$$

denoting as $\mu(x, y) = xy$.

- $(xy)\, z = x\,(yz)$
- There exists an element e called the identity s.t. $xe = ex = x$
- To each $x \in G$ there is an element x^{-1} s.t. $xx^{-1} = x^{-1}x = e$

Let G and H be two groups, then a mapping $\phi : G \to H$ is called a homomorphism if

$$\phi\,(xy) = \phi x \phi y \qquad x, y \in G$$

A group is called commutative or abelian if for each $x, y \in G$, $xy = yx$.

Definition 1.2 (field)**.** A field is a set K on which two binary lows of composition s.t.

- K is a commutative group with respect to addition.
- The set $K - \{0\}$ is a commutative group with respect to multiplication.
- Addition and multiplication are connected by the distributive low,

$$(\alpha + \beta)\,\gamma = \alpha\gamma + \beta\gamma$$

Chapter 2

Vector Space

2.1 Linear independence and basis

Definition 2.1 (linear independence)**.** A family of vectors $\{x_i\}_{i \in I}$ is called **linear independent** if the vectors x_i are linearly independent i.e.

$$\sum_{i \in I} \alpha_i x_i = 0 \implies \alpha_i = 0 \quad \text{for each } i$$

Definition 2.2 (system of generators)**.** A subset $S \subset E$ is called a system of generators of E if every vector $x \in E$ is a linear combination of vectors in S.

Proposition 2.1.

1. *Every finitely generated non-trivial vector space has a finite basis.*
2. *Suppose that $S = \{x_1, \ldots, x_m\}$ is a finite system of generators of E and that the subset $R \subset S$ by $R = \{x_1, \ldots, x_r\}\,(r \leq m)$ consists of linearly independent vectors. Then there exists a basis T of E s.t. $R \subset T \subset S$.*

Proof. Just need to notice that every basis is the system of generators, and it is a minimal one.

□

Theorem 2.1. *Let E be a non-trivial vector space. Suppose S is a system of generators and R is a family of linearly independent vectors in E s.t. $R \subset S$. Then there exists a basis T of E s.t. $R \subset T \subset S$.*

Proof. Consider the partially order defined between R and S, find some $X \subset E$ s.t.

- $R \subset X \subset S$

- the vectors in X are linearly independent.

We note this partially order as $\mathcal{P}(R,S)$.
Notice that for every chain $\{X_\alpha\} \subset \mathcal{P}(R,S)$ has a maximal element $A = \bigcup_\alpha X_\alpha$. It is obvious that $A \in \mathcal{P}(R,S)$ (Notice that $R \subset A \subset S$ and the property of a chain of the set that contains linearly independent vectors.)
So we prove that every chain $\{X_\alpha\} \subset \mathcal{P}(R,S)$ has a upper bound in $\mathcal{P}(R,S)$, so Zorn's Lemma implies that there exists a maximal element $T \in \mathcal{P}(R,S)$ s.t. vectors in T are linearly independent.
Then we just need to show that T generates E. Give $x \in E$, suppose that x is linearly independent to vectors in T. Notice that S generates E, so

$$x = \sum_{i \in I'} \alpha_i x_i \qquad \text{for some } x_i \in S$$

If x is linearly independent to vectors in T then exists some $i \in I'$ s.t. x_i is linearly independent to vectors in T and note this set as $\{x_j\}_{j\in J} \subset S$, consider the set $\{x_j\}_{j\in J} \cup T \supsetneq T$ which leads to a contradiction of the maxmality of T. So T is a basis of E.

□

Corollary 2.1. *1. Every system of generators of E contains a basis. In particular, every non-trivial vector space has a basis.*

2. Every family of linearly independent vectors of E can be extended to a basis.

2.2 Free vector space

Let X be an arbitrary set and consider all maps $f : X \to \mathbb{K}$ s.t. $f(x) \neq 0$ only for finitely many $x \in X$, denoting the set of these maps by $F(X)$, it is easy to show that $F(X)$ is a vector space.

Now give a basis of $F(X)$. For any $a \in X$, let f_a be:

$$f_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

Then $\{f_a\}_{a\in X}$ forms a basis of $F(X)$.

$F(X)$ is called the **free vector space over** X.

2.3 Linear mappings

Definition 2.3 (linear mapping)**.** Suppose that E and F are vector spaces, and let $\varphi : E \to F$ be a set mapping s.t.

$$\varphi(x+y) = \varphi(x) + \varphi(y) \text{ for all } x, y \in E$$

and

$$\varphi(\alpha x) = \alpha\varphi(x) \text{ for all } \alpha \in \mathbb{K}, x \in E$$

Then we call the mapping φ satisfying above conditions linear mappings.
Moreover, if $F = \mathbb{K}$, then we called φ a **linear function** on E.

Corollary 2.2. *Linear mappings preserve linear relations.*

Proof. Suppose φ be a linear mappings, and let $u = \alpha x + \beta y \in E$, then

$$\varphi(u) = \varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$$

□

Let $\varphi : E \to F, \psi : F \to G$ be linear mappings, then the composition of them $\psi \circ \varphi : E \to G$ is defined by:

$$(\psi \circ \varphi)(x) = \psi(\varphi(x))$$

It is easy to show that $\psi \circ \varphi$ is still a linear mapping.

Proposition 2.2. *Suppose S is a system of generators of E and $\varphi_0 : S \to F$ where F is also a vector space. Then φ_0 can be extended in at most one way to linear mapping $\varphi : E \to F$.*
And the extension exists iff such an extension is that

$$\sum_i \alpha_i \varphi_0(x_i) = 0$$

whenever $\sum_i \alpha_i x_i = 0$.

Proof.

- $\Longrightarrow$: Suppose φ to be a linear mapping and it is the extension of φ_0, then $\varphi\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i \varphi(x_i)$ for each $x_i \in E$.
  And for each $x_i \in S$,

  $$\varphi\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i \varphi(x_i) = \sum_{i=1}^n \alpha_i \varphi_0(x_i)$$

  so $\varphi(0) = \varphi_0(0) = 0$.

- $\Longleftarrow$: For any $x \in E$, define there exists some $\{x_i\}_{i \in I} \subset S$ s.t. $x = \sum_{i \in I} \alpha_i x_i$. Define

  $$\varphi(x) = \sum_{i \in I} \alpha_i \varphi_0(x_i)$$

  It is obvious that φ is that linear mapping.

$\square$

Notice that if S is a basis of E, let φ_0 be a set map from S to E, then φ_0 can be extended in a unique way to a linear mapping $\varphi : E \to F$.

Proposition 2.3. *Let $\varphi : E \to F$ be a linear mapping and $\{x_\alpha\}$ be a basis of E. Then φ is a linear isomorphism iff the vectors $y_\alpha = \varphi(x_\alpha)$ form a basis for F.*

Proof.

- $\Longrightarrow$: As φ is a linear isomorphism, so for any $y \in F$, there exists a unique $x \in E$ s.t. $x = \varphi^{-1}(y)$. Notice that $\{x_\alpha\}$ is a basis, so $x = \sum_\alpha a_\alpha x_\alpha$ for some a_α, so $y = \varphi(x) = \varphi\left(\sum_\alpha a_\alpha x_\alpha\right) = \sum_\alpha a_\alpha \varphi(x_\alpha)$. That means $\{\varphi(x_\alpha)\}$ generates F. Then we need to prove the linear independence.
  Let $\sum_\alpha \lambda_\alpha x_\alpha = 0$, then $\lambda_\alpha = 0$ for each α. Then let $\sum_\alpha \gamma_\alpha \varphi(x_\alpha) = 0$, then

  $$\sum_\alpha \gamma_\alpha \varphi(x_\alpha) = \varphi\left(\sum_\alpha \gamma_\alpha x_\alpha\right) = 0$$

  so $\sum_\alpha \gamma_\alpha x_\alpha = 0$ which means $\gamma_\alpha = 0$ for each α. So $\{\varphi(x_\alpha)\}$ is a basis of F.

- $\Longleftarrow$: Let $\{y_\alpha = \varphi(x_\alpha)\}$ be a basis of F, then for each $y \in F$, there exists a unique components (λ_α) s.t. $\sum_\alpha \lambda_\alpha y_\alpha = y$. Then we have

  $$\sum_\alpha \lambda_\alpha \varphi(x_\alpha) = \varphi\left(\sum_\alpha \lambda_\alpha x_\alpha\right) = \varphi(x)$$

  for some unique $x \in E$.

$\square$

2.4 Subspace and factor space

2.4.1 Subspace and Sum

Definition 2.4 (Subspace)**.** Let X be a vector space and let $A \subset X$ be a subset of X. Then A is called a subspace if A is also a vector space.

Let S be a non-empty subset of X and there exists a set, noting as X_S, is the linear combination of any vectors in S, X_S is truly a subspace which is called **the subspace generated by** S or **linear closure** of S.

Proposition 2.4. *Let A_1, A_2 be two subspaces of the vector space X and suppose that $A_1 \cap A_2 \neq \varnothing$ then $A_1 \cap A_2$ is still a subspace of X.*

Proof. Notice that if $x \in A_1 \cap A_2$, then $x \in A_1$ and $x \in A_2$, and A_1, A_2 are vector space thus provide the linearity of $A_1 \cap A_2$.

□

Definition 2.5 (sum of subspace)**.** Let A_1, A_2 be two subspaces of a vector space X, then $\{x = x_1 + x_2 : x_1 \in A_1, x_2 \in A_2\}$ is called the **sum of A_1 and A_2**, denote as $A_1 + A_2$. It is easy to determine that $A_1 + A_2$ is still a subspace of X.

Notice that the decomposition is not determined uniquely.
Let $x = x_1 + x_2 = x_1' + x_2'$, then $x_1 - x_1' = x_2 - x_2' = z \in A_1 \cap A_2$. Only if $A_1 \cap A_2 = \{0\}$, then $x = x_1 + x_2$ is uniquely determined. In this time, we called that sum as **direct sum** of A_1 and A_2, denote as $A_1 \oplus A_2$.

Proposition 2.5.
- *Let A_1, A_2 be subspaces of X and let S_1, S_2 be systems of generators of A_1 and A_2, then $S_1 \cup S_2$ generates $A_1 + A_2$.*
- *Suppose that $A_1 \cap A_2 = \{0\}$ and T_1, T_2 are basis of A_1, A_2, then $T_1 \cup T_2$ is the basis of $A_1 \oplus A_2$.*

Proof. Give any $x \in A_1 + A_2$, then $x = x_1 + x_2$ for some $x_1 \in A_1, x_2 \in A_2$. $x_1 = \sum_\alpha \lambda_\alpha x_\alpha$ for some $x_\alpha \in S_1$ and $x_2 = \sum_\beta \gamma_\beta x_\beta$ for some $x_\beta \in S_2$, so $x = \sum_\alpha \lambda_\alpha x_\alpha + \sum_\beta \gamma_\beta x_\beta$, notice that every $x_\alpha, x_\beta \in S_1 \cup S_2$, so $S_1 \cup S_2$ generates $A_1 + A_2$.

Now we need to prove that $T_1 \cup T_2$ is linearly independent.
Notice that $T_1 \subset A_1, T_2 \subset A_2$, $A_1 \cap A_2 = \{0\}$, so $T_1 \cap T_2 = \{0\}$. So consider $x \in A_1 \oplus A_2$, $x = \sum_\alpha \lambda_\alpha x_\alpha + \sum_\beta \gamma_\beta x_\beta = 0$, then $A_1 \ni x_1 = \sum_\alpha \lambda_\alpha x_\alpha = -\sum_\beta \gamma_\beta x_\beta = x_2 \in A_2$, so $x_1 = x_2 = 0$, then as the property of basis, $\lambda_\alpha = 0$ for all α and $\gamma_\beta = 0$ for all β.

□

Definition 2.6 (complementary subspace)**.** If A_1 is a subspace of X, and there exists a subspace A_2 s.t. $A_1 \oplus A_2 = E$, then A_2 is called the **complementary subspace** for A_1 in X.

Proposition 2.6 (existence of complementary subspace)**.** *If $A_1 \subset X$ is a subspace, then there exists a $A_2 \subset X$ a subspace s.t. $A_1 \oplus A_2 = X$*

Proof. According to the 2.1, suppose that $\{x_\alpha\}$ is a basis of A_1, then it is linearly independent and so can be extended to a basis of X, denote as $\{x_\gamma\}$. Notice that $\{x_\alpha\} \subset \{x_\gamma\}$ and let $\{x_\beta\} = \{x_\gamma\} - \{x_\alpha\}$. Then let A_2 be the subspace generated by $\{x_\beta\}$.

Observe that $\{x_\alpha\} \cup \{x_\beta\}$ generates X, so $A_1 + A_2 = X$, then let $x \in A_1 \cap A_2$, so $x = \sum_\alpha \lambda_\alpha x_\alpha = \sum_\beta \omega_\beta x_\beta$ which means $\sum_\alpha \lambda_\alpha x_\alpha + \sum_\beta (-\omega_\beta) x_\beta = 0$. For vectors in $\{x_\alpha\}$ and $\{x_\beta\}$ are linearly independent, so $\lambda_\alpha = 0, \omega_\beta = 0$ for all α, β, then $A_1 \cap A_2 = \{0\}$ which means $X = A_1 \oplus A_2$.

$\square$

Corollary 2.3. *Let A_1 be a subspace of X and $\varphi_1 : A_1 \to F$ be a linear mapping. Then φ_1 may be extended to a linear mapping $\varphi : X \to F$.*

Proof. According to the above proposition, there exists a subspace $A_2 \subset X$ s.t. $A_1 \oplus A_2 = X$. Now define $\varphi_2 : A_2 \to F$ be a linear mapping. Then for any $x \in X$, notice that $x = x_1 + x_2$ where $x_1 \in A_1, x_2 \in A_2$, define

$$\varphi(x) = \varphi_1(x_1) + \beta\varphi_2(x_2) \qquad x = x_1 + x_2; \beta \in \mathbb{K}$$

It is easy to show that φ is a linear mapping as φ_1, φ_2 are.

$\square$

2.4.2 Factor Space

Definition 2.7 (factor space)**.** Suppose that X is a vector space and A_1 is a subspace of X. Two vectors $x, x' \in X$ is called **equivalent** mod A_1 if $x - x' \in A_1$. Then $x \sim x'$ is a equivalence relation, that is reflexive, symmetric and transitive.

Then we let X/A_1 denote the **set of equivalence classes**, X/A_1 is a vector space too and define a mapping:

$$\pi : X \to X/A_1$$

by letting $\pi x = \overline{x}, x \in X$ where $\overline{x}$ denotes the equivalence class containing x. Clearly, π is a surjective mapping.

Proof. Now prove the equivalent relation:

- let $x \sim x_1$, $x_1 \sim x_2$, which means $x - x_1 \in A_1$ and $x_1 - x_2 \in A_1$ then $x - x_2 = (x - x_1) + (x_1 - x_2) \in A_1$.
- Notice that $x - x = 0 \in A_1$ as A_1 is a subspace.
- Observe that $x - x_1 = (-1)(x_1 - x)$ which means the symmetry.

$\square$

Proposition 2.7. *There exists precisely one linear structure in X/A_1 s.t. π is a linear mapping.*

Proof. Assume that X/A_1 is made into a vector space s.t. π is a linear mapping. Then

$$\pi(x+y) = \pi(x) + \pi(y)$$

and $\pi(\lambda x) = \lambda\pi(x)$. It shows that we can use a linear mapping π to define the linear structure of X/A_1 and the linear structure of X/A_1 is determined by the linear structure of X, thus unique.

Now define the linear structure of X/A_1. Let $\overline{x}, \overline{y} \in X/A_1$ and $\overline{x} \neq \overline{y}$. Then there exists some $x, y \in X$ s.t. $\pi(x) = \overline{x}$ and $\pi(y) = \overline{y}$. Pick an arbitrary x and y, define:

$$\overline{x} + \overline{y} = \pi(x+y)$$

and

$$\lambda\overline{x} = \pi(\lambda x)$$

We only need to show that π is a linear mapping. Suppose that $x_1 - x_2 \in A_1$ and $y_1 - y_2 \in A_1$, notice that $(x_1+y_1)-(x_2+y_2) = (x_1-x_2)+(y_1-y_2) \in A_1$ as the property of subspace. Since the picking of x_1, x_2, y_1, y_2 is arbitrary, $\pi(x) = \overline{x}$, $\pi(x+y) = \overline{x} + \overline{y}$. Then π is a communicative group as above. Similarly, it is easy to show that $\pi(\lambda x) = \lambda\pi(x)$. Then π is linear, so it determines the linear structure of X/A_1.

□

Remark. The space discussed above like X/A_1 is called the factor space or quotient space and the linear mapping $\pi : X \to X/A_1$ is called the canonical projection of X onto A_1.

Definition 2.8. Let A_1 be a subspace of X, and suppose $\{x_\alpha\}$ is a family of vectors in X. Then x_α is called **linear dependent mod** A_1 if there are scalars λ_α not all zero s.t. $\sum_\alpha \lambda_\alpha x_\alpha \in A_1$.

A family of vectors is called linearly independent mod a subspace A_1 if they are not linearly dependent mod A_1.

Now consider the canonical projection $\pi : X \to X/A_1$, then $\{x_\alpha\}$ is linearly dependent mod A_1 iff the vectors $\pi(x_\alpha)$ are linearly dependent in X/A_1.

Proof.

- $\Longrightarrow$: Suppose $\{x_\alpha\}$ is linear dependent mod A_1, then $\sum_\alpha \lambda_\alpha x_\alpha \in A_1$ for not all zero λ_α, notice that the linearity of π,

$$\sum_\alpha \lambda_\alpha \pi(x_\alpha) = \pi\left(\sum_\alpha \lambda_\alpha x_\alpha\right)$$

Observe that $\sum_\alpha \lambda_\alpha x_\alpha = x \in A_1$, and only if $x \in A_1$, $\pi(x) = \overline{0}$ in X/A_1.

- $\Longleftarrow$: Omission.

□

Suppose that $\{x_\alpha\} \cup \{x_\beta\}$ is a basis of X and $\{x_\alpha\}$ generates A_1, then according to 2.6 there exists a A_2 generated by $\{x_\beta\}$ s.t. $A_1 \oplus A_2 = X$.

Proposition 2.8 (basis of a factor space)**.** *$\pi(x_\beta)$ for all β form a basis of X/A_1.*

Proof. First, we need to prove that $\pi(x_\beta)$ generates X/A_1.
Let $\overline{x} \in X/A_1$ be an arbitrary element. We only need to find a $x \in \pi^{-1}(\overline{x})$, notice that if $\overline{x}$ is non-trivial i.e. $\overline{x} \neq \overline{0}$, $x \notin A_1$, so there must exist some γ_β s.t. $x = \sum_\beta \gamma_\beta x_\beta$. Then

$$\pi\left(\sum_\beta \gamma_\beta x_\beta\right) = \pi(x) = \overline{x} = \sum_\beta \gamma_\beta \pi(x_\beta)$$

Second, we observe that $\{x_\beta\}$ is linearly independent mod A_1, so $\pi(x_\beta)$ are linearly independent in X/A_1.

□

2.5 Dimension

Recall 2.1, every system of generators contains a basis, so if the generators of the system is finite, there exists a finite base of the space.

Definition 2.9 (dim)**.** Consider a vector space X whose basis is the family of finite number of vectors i.e. $\{x_1, \ldots, x_n\}$ generates X and $\sum_{i=1}^n \alpha_i x_i = 0$ whenever $\alpha_i = 0$ for every i. Then denotes the **dim of** X as $\dim X = n$.

Proposition 2.9. *Suppose a vector space X has a basis of n vectors. Then every family of $(n+1)$ vectors is linearly dependent. That means n is the maximum number of linearly independent vectors in X and hence every basis of X consists of n vectors.*

Proof. We use mathematical induction to prove this proposition.

1. Let $n = 1$, let x_1 be a basis of X, then $y_1, y_2 \neq 0$ and $y_1, y_2 \in X$. Then $y_1 = \alpha x, y_2 = \beta x$. Now let $\gamma_1 y_1 + \gamma_2 y_2 = 0$, we can let $\gamma_1 = \alpha\beta, \gamma_2 = -\alpha\beta$ which means y_1, y_2 are linearly dependent.

2. Assume that the proposition holds for every vector space having basis of $r \leq n-1$ vectors by the induction.

3. Let X be a vector space and let $\{x_1, \ldots, x_n\}$ be the basis of X and $\{y_1, \ldots, y_{n+1}\}$ be an arbitrary family of vectors in X.
   Now consider the factor space $X/\operatorname{span} y_{n+1}$ and the canonical projection

$\pi : X \to X/\operatorname{span} y_{n+1}$. As $\{x_i : i = 1, \dots, n\}$ generates X and π is surjective, $\{\pi(x_i) : i = 1, \dots, n\}$ generates $X_1 = X/\operatorname{span} y_{n+1}$, so according to 2.1, it contains a basis of X_1 and as $y_{n+1} = \sum_{i=1}^n \alpha_i x_i$ for some not all zero α_i, $\{\overline{x_i} = \pi(x_i) : i = 1, \dots, n\}$ is linearly dependent, so $\dim X_1 \leq n - 1$, then by the hypothesis of induction, $\{\overline{y_i} = \pi(y_i) : i = 1, \dots, n\}$ are linearly independent.
so there exists:

$$\sum_{i=1}^{n} \gamma_i \overline{y_i} = 0 \text{ for non-trivial } \{\gamma_i\}$$

which means $\{y_i : i = 1, \dots, n\}$ are linearly dependent mod $\operatorname{span} y_{n+1}$ which means

$$\sum_{i=1}^{n} \gamma_i y_i = \lambda y_{n+1}$$

leads to the consult that $\{y_1, \dots, y_{n+1}\}$ are linearly dependent.

□

Give a vector space X and a subspace $A_1 \subset X$, then there exists a subspace $A_2 \subset X$ s.t. $A_1 \oplus A_2 = X$ by 2.6. Then let $\{x_\alpha\}$ be a basis of A_1 and $\{x_\beta\}$ be a basis of A_2, notice that $\{x_\alpha\} \cap \{x_\beta\} = \varnothing$ and $\{x_\alpha\} \cup \{x_\beta\}$ generates X. So we easily observe that $\dim X = \dim A_1 + \dim A_2$ if $A_1 \oplus A_2 = X$.

Then according to 2.8, let π be the canonical projection, $\{\overline{x_\beta} = \pi(x_\beta)\}$ forms a basis of X/A_1, so $\dim(X/A_1) = \operatorname{card}\{\overline{x_\beta}\} = \operatorname{card}\{x_\beta\} = \dim A_2$. So $\dim X = \dim A + \dim(X/A_1)$.

Proposition 2.10. *Let $A_1, A_2 \subset X$ be arbitrary subspace of X. Then*

$$dim\, A_1 + dim\, A_2 = dim(A_1 + A_2) + dim(A_1 \cap A_2)$$

Proof. Just let $\{x_\alpha\}$ be the basis of $A_1 \cap A_2$ and let $\{y_\beta\}, \{y_\gamma\}$ be the extending tail i.e. they don't intersect $\{x_\alpha\}$ and $\{x_\alpha\} \cup \{y_\beta\}$ is a basis of A_1 and $\{x_\alpha\} \cup \{y_\gamma\}$ is a basis of A_2.

Let $\operatorname{card}\{x_\alpha\} = \alpha, \operatorname{card}\{y_\beta\} = \beta, \operatorname{card}\{y_\gamma\} = \gamma$. Then $\dim A_1 = \alpha + \beta, \dim A_2 = \alpha + \gamma, \dim(A_1 \cap A_2) = \alpha$. Now we only need to show that $\{x_\alpha\} \cup \{y_\beta\} \cup \{y_\gamma\}$ generates $A_1 + A_2$. It is easy to show by the definition of generators of system. And notice that they are independent with each other. Thus $\{x_\alpha\} \cup \{y_\beta\} \cup \{y_\gamma\}$ is a basis of $A_1 + A_2$ which means $\dim(A_1 + A_2) = \operatorname{card}(\{x_\alpha\} + \{y_\beta\} + \{y_\gamma\}) = \alpha + \beta + \gamma$.

□

Chapter 3

Linear Mappings

3.1 Basic properties

Definition 3.1 (kernel and image). Suppose X, Y are vector spaces and $\varphi : E \to F$ be a linear mapping. Then the **kernel of** φ denoted as $\ker \varphi$ is the subset $K \subset X$ s.t. if $x \in K \implies \varphi(x) = 0$.

The **image space of** φ denoted as $\operatorname{Im} \varphi$ is the subset $I \subset Y$ s.t. $y \in I \implies$ there exists some $x \in X$ s.t. $\varphi(x) = y$.

Proposition 3.1. *1. Let $\varphi : X \to Y$ be a linear mapping, then $\ker \varphi$ is a vector space.*

2. The mapping $\varphi : X \to Y$ is injective iff $\ker \varphi = \{0\}$.

Proof. 1. Let $\varphi : X \to Y$ be a linear mapping, let $x_1, x_2 \in \ker \varphi$. Then

- $\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2) = 0$, so $x_1 + x_2 \in \ker \varphi$.
- $\varphi(\alpha x_1) = \alpha \varphi(x_1) = 0$, so $\alpha x_1 \in \ker \varphi$.

2. Let φ be injective that means for each $y \in \operatorname{Im} \varphi$, $\varphi^{-1}(y) = x$ for some unique $x \in X$. So $\varphi^{-1}(0) = 0$ for only $0 \in X$.
For the converse, let $\ker \varphi = \{0\}$, give an arbitrary $y \in \operatorname{Im} \varphi$, suppose there exists $x_1, x_2 \in X$ s.t. $\varphi(x_1) = \varphi(x_2) = y$, then $\varphi(x_1 - x_2) = \varphi(x_1) - \varphi(x_2) = 0$, if $x_1 \neq x_2$, there leads to a contradiction about $\ker \varphi = \{0\}$. So φ is injective.

□

3.1.1 Induced Linear Mappings

Definition 3.2 (restriction of linear mapping). Suppose $\varphi : X \to Y$ is a linear mapping and $X_1 \subset X$, $Y_1 \subset Y$ are subspace s.t. $\varphi(x) \in Y_1$ when $x \in X_1$.

Then the linear mapping $\varphi_1 : X_1 \to Y_1$ defined by $\varphi_1(x) = \varphi(x), x \in X_1$ is called **the restriction of** φ to X_1.

Now we can find that $\varphi \circ i_{X_1} = i_{Y_1} \circ \varphi_1$ where $i_{X_1} : X_1 \to X$ is canonical injections, same as i_{Y_1}.

Equivalently, the diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \\ X_1 \arrow[u, "i_X"] \arrow[r, "\varphi_1"] & Y_1 \arrow[u, "i_Y"'] \end{tikzcd}$$

Let $\varphi : X \to Y$ be linear mapping and $\varphi_1 : X_1 \to Y_1$ be its restriction to subspace $X_1 \subset X, Y_1 \subset Y$. Then there exists precisely one linear mapping

$$\overline{\varphi} : X/X_1 \to Y/Y_1$$

s.t.

$$\overline{\varphi} \circ \pi_X = \pi_Y \circ \varphi$$

where π_X, π_Y are canonical projections on X, Y.

Notice that $\pi_Y(\varphi(x_1)) = \pi_Y(\varphi(x_2))$ whenever $\pi_X(x_1) = \pi_X(x_2)$. Because $\pi_X(x_1) = \pi_X(x_2)$ implies $\pi_X(x_1 - x_2) = \overline{0}$ so $x_1 - x_2 \in \ker \pi_X = X_1$. Then

$$\begin{aligned} \pi_Y \circ \varphi(x_2 - x_1) &= \pi_Y \circ \varphi(x) \qquad \text{for } x \in X_1 \\ &= \pi_Y(y) \qquad \text{for } y \in Y_1 \\ &= \overline{0} \end{aligned}$$

as the existence of the restriction φ_1.

Then we can assert that there exists a mapping s.t. $\overline{\varphi}(x)$ has only one value in Y/Y_1, thus a function. Then we need to show its linearity. Now let $\overline{x_1}, \overline{x_2} \in X/X_1$ and $x_1 \in \pi_X^{-1}(\overline{x_1})$ same as x_2.

$$\begin{aligned} \overline{\varphi}(\alpha\overline{x_1} + \beta\overline{x_2}) &= \overline{\varphi} \circ \pi_X(\alpha x_1 + \beta x_2) \\ &= \pi_Y \circ \varphi(\alpha x_1 + \beta x_2) \\ &= \alpha\pi_Y \circ \varphi(x_1) + \beta\pi_Y \circ \varphi(x_2) \\ &= \alpha\overline{\varphi}(\overline{x_1}) + \beta\overline{\varphi}(\overline{x_2}) \end{aligned}$$

which means the linearity.

Remark. The $\overline{\varphi}$ discussed above is called the **induced mapping in factor space** and the relation of $\overline{\varphi}$ is equivalent to the diagram:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi_X"] & Y \arrow[d, "\pi_Y"] \\ X/X_1 \arrow[r, "\overline{\varphi}"] & Y/Y_1 \end{tikzcd}$$

Notice that this diagram is commutative.

And the relation can be overwritten by $\overline{\varphi x} = \overline{\varphi x}$.

Let $\varphi : X \to Y$ be a linear mapping and $X_1 = \ker \varphi$, $Y_1 = \{0\}$. Since $\varphi(x) = 0$ when $x \in X_1$, a linear mapping is **induced** by φ :

$$\overline{\varphi} : X/\ker \varphi \to Y/\{0\} = Y$$

s.t.

$$\overline{\varphi} \circ \pi = \varphi$$

where $\pi : X \to X/\ker \varphi$ is the canonical projection.

1. This mapping $\overline{\varphi}$ is injective. In fact if $\overline{\varphi} \circ \pi(x) = 0$, then $\varphi(x) = 0$ which means $x \in \ker \varphi$. Then $\pi(x) = \overline{0}$, so $\ker \overline{\varphi} = \left\{\overline{0}\right\}$, according to 3.1, $\overline{\varphi}$ is injective.

2. $\overline{\varphi}$ is a linear isomorphism between $X/\ker \varphi$ and $\operatorname{Im} \varphi$, i.e.

   $$\overline{\varphi} : X/\ker \varphi \xrightarrow{\simeq} \operatorname{Im} \varphi$$

   Notice that $\overline{\varphi}$ is injective and since $\operatorname{Im} \varphi$ it is surjective, thus one-to-one and onto.

Then every linear mapping $\varphi : X \to Y$ can be written as a composition of a surjective and injective linear mapping:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"] & Y \\ X/\ker \varphi \arrow[ur, "\overline{\varphi}"] & \end{tikzcd}$$

Now consider the linear mapping:

$$\varphi' : X_1/(X_1 \cap X_2) \xrightarrow{\simeq} (X_1 + X_2)/X_2$$

We need to show it is a isomorphism.
First we observe the canonical projection:

$$\pi : X_1 + X_2 \to (X_1 + X_2)/X_2$$

and $\pi\mid_{X_1}$ be the restriction on X_1. Notice that for $x \in X_1 + X_2$:

$$x = x_1 + x_2 \qquad x_1 \in X_1, x_2 \in X_2$$

then

$$\pi(x) = \pi(x_1 + x_2) = \pi(x_1) = \pi\mid_{X_1}(x_1)$$

So we find that $\pi\mid_{X_1}$ is surjective.
Define $\varphi = \pi\mid_{X_1} : X_1 \to (X_1 + X_2)/X_2$, then

$$\ker\varphi = \ker\pi \cap X_1 = X_1 \cap X_2$$

With the above discussion, we notice that $\varphi : X_1 \to (X_1 + X_2)/X_2$ and so

$$X_1/\ker\varphi \xrightarrow{\simeq} (X_1 + X_2)/X_2$$

Proposition 3.2. *Suppose that $\varphi : X \to Y$ and $\psi : X \to Z$ are linear mappings s.t. $\ker\varphi \subset \ker\psi$, then there exists a linear mapping $\omega : X \to Z$ s.t. $\omega \circ \varphi = \psi$.*

Proof. Notice that $\psi(x) = 0$ if $x \in \ker\varphi$, consider the induced linear mapping:

$$\overline{\psi} : X/\ker\varphi \to Z$$

s.t. $\overline{\psi} \circ \pi = \psi$ where $\pi : X \to X/\ker\varphi$ is the canonical projection. The existence of $\overline{\psi}$ is determined by the $\psi\mid_{\ker\varphi} : \ker\varphi \to \{0\}$.

Now let

$$\overline{\varphi} : X/\ker\varphi \xrightarrow{\simeq} \operatorname{Im}\varphi$$

be the linear isomorphism determined by φ and define $\overline{\psi}_1 : \operatorname{Im}\varphi \to Z$ by

$$\overline{\psi}_1 = \overline{\psi} \circ \overline{\varphi}^{-1}$$

Then let $\omega : X \to Z$ be a linear mapping which extends $\overline{\psi}_1$.
Notice that

$$\overline{\varphi}^{-1} \circ \varphi = \overline{\varphi}^{-1} \circ \overline{\varphi} \circ \pi = \pi$$

which means:

$$\omega \circ \varphi = \overline{\psi}_1 \circ \varphi = \overline{\psi} \circ \overline{\varphi}^{-1} \circ \varphi = \overline{\psi} \circ \pi = \psi$$

□

Remark. The result can be expressed in commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\psi"] & Y \arrow[dl, "\omega"] \\ Z & \end{tikzcd}$$