Notes of GTM278

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Contents

1 Topology Background in Real Analysis

1.1 Meager Set

Definition 1.1. A subset E of a metric space X is said to be **dense in an open set** U if $U \subset \overline{E}$. E is defined to be **nowhere dense** if it is not dense in any open subset $U \subset X$. It means $\overline{E}$ does not contain any open set.

Definition 1.2 (first and second category)**.** A set E is said to be of **first category** in X if it is the union of a countable family of nowhere dense sets.

A set E is said to be a of **second category** in X if it is not the first category set.

Theorem 1.1 (Baire Category Theorem)**.** *A complete metric space X is not the union of a countable family of nowhere dense sets. That is, a complete metric space is of the second category.*

Proof. The proof of the Baire category theorem is to construct a sequence of balls and show that the center of the balls is a Cauchy sequence and find the limit of this sequence is not in X then result in a contradiction.

□

Theorem 1.2 (uniform boundedness theorem)**.** *Let $\mathcal{F}$ be a family of real-valued functions defined on a complete metric space X and suppose*

$$f^*(x) = \sup_{f \in \mathcal{F}} |f(x)| < \infty$$

for each $x \in X$.

Then there exists a nonempty open set $U \subset X$ and a constant M s.t. $|f(x)| \leq M$ for all $x \in U$ and $f \in \mathcal{F}$.

Proof. For each positive $i \in \mathbb{N}$, let

$$E_{i,f} = \{x; |f(x)| \le i\}, \ E_i = \bigcap_{f \in \mathcal{F}} E_{i,f}$$

Notice that $E_{i,f}$ is closed so is E_i and as the hypothesis, we find that for each $x \in X$, there is a M_x s.t. $f(x) \le M_x$ for all $f \in \mathcal{F}$, so

$$X = \bigcup_{i=1}^{\infty} E_i$$

And the Baire category theorem implies that there is some $E_M, M \in \mathbb{N}$ is not nowhere dense which means there is some open subset $U \subset E_M$ s.t. for all $x \in U$,and $f \in \mathcal{F}$, $|f(x)| \le M$.

□

1.2 Compactness in Metric Spaces

Lemma 1.1.
- *A convergent sequence in a metric space is Cauchy.*
- *A metric space which all the Cauchy sequence in it is convergence is **complete**.*
- *A metric space is a first countable space.*
- *A metric space is separable iff it is a second countable space.*

Proof. Give a sequence $(x_i) \to x$ in X, as X is a metric space, give any $\epsilon > 0$, there exists a $m \in \mathbb{N}$ s.t. for any $n_1, n_2 \ge m$, $d(x, x_{n_1}) \le \epsilon/2$, and $d(x, x_{n_2}) \le \epsilon/2$, so $d(x_{n_1}, x_{n_2}) \le d(x_{n_1}, x) + d(x, x_{n_2}) \le \epsilon$, so (x_i) is Cauchy.

□

Definition 1.3 (totally bounded)**.** If (X, d) is a metric space, a set $A \subset X$ is called totally bounded if for every $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ.

A set A is said to be bounded if there is $M \ge 0$ s.t. $d(x, y) \le M$ for all $x, y \in A$.

Notice that a totally bounded set is bounded but a bounded set may not be totally bounded.

Definition 1.4 (sequentially compact)**.** A set $A \subset X$ is said to be sequentially compact if every sequence in A has a subsequence that converges to a point $x \in A$.

Also, A is said to have **Bolzano-Weierstrass** property if every infinite subset of A has accumulation point in A.

Theorem 1.3. *If A is a subset of a metric space (X, d), the following are equivalent:*

- *A is compact.*
- *A is sequentially compact.*
- *A is complete and totally bounded.*
- *A has the Bolzano-Weierstrass property.*

Proof. We will give a proof from $1 \implies 2$:

- $1 \implies 2$: Let (x_i) be a sequence in A. Assume that (x_i) 's range is infinite, and suppose (x_i) has no convergent subsequence. Let E denotes the range of (x_i).
  Notice that every subsequence of (x_i) does not converge, so every point $x \in E$, there exists a r_x s.t. $B_r(x) \cap E = \{x\}$. Then as $\overline{E} = E \cup E^*$ where E^* denotes the set of accumulation point of E which is empty, so $\overline{E} = E \implies E$ is closed.
  A is compact and E is closed and $E \subset A$, so E is compact. However, E contains infinite points and every point is isolated, so the open cover $\{B_r(x) : r = r_x\}$ cant have a finite subcover that leads to a contradiction.

- $2 \implies 3$: First we need to show that if a subsequence of a Cauchy sequence converges, then the whole sequence converges.
  Let (x_i) be a Cauchy sequence and let $\left(x_{i(k)}\right)_{k=1}^{\infty}$ be a subsequence of (x_i) s.t. $\left(x_{i(k)}\right) \to x$ which means give a $\epsilon > 0$ there exists a $m(k) \in \mathbb{N}$ for all $k \geq m(k)$, $d\left(x_{i(k)}, x\right) \leq \epsilon/2$. Note that every subsequence of a Cauchy sequence is Cauchy, so there exists a $n(k) \in \mathbb{N}$ for all $k_1, k_2 \geq n(k)$, $d\left(x_{i(k_1)}, x_{i(k_2)}\right) \leq \epsilon/2$, pick $s = i(\max(m(k), n(k)))$, when $i \geq s$, $d(x_i, x) \leq \epsilon$.
  So A must be complete, if not there must be a Cauchy sequence (x_i) in A s.t. there exists a subsequence of (x_i) converges but (x_i) does not converge, which leads to a contradiction of the proposition above.

  About the totally bounded, suppose that A is not totally bounded and there exists a $\epsilon > 0$ s.t. A cannot be covered by finitely many balls of radius ϵ. Then we can choose a sequence in A as follows: Pick $x_1 \in A$, Then, since $A - B_\epsilon(x_1) \neq \varnothing$, we can choose $x_2 \in A - B_\epsilon(x_2)$. Note that $d(x_1, x_2) \geq \epsilon$, then similarly we choose

  $$x_i \in A - \bigcup_{j=1}^{i-1} B_\epsilon(x_j)$$

  Then as the cover cannot be finite, so (x_i) is a sequence in A with $d(x_i, x_j) \geq \epsilon$ when $i \neq j$ so clearly (x_i) does not have any convergent subsequence.

- $3 \implies 4$: Let $A \subset X$ be an infinite subset. Notice that A can be covered by a finite number of balls of radius 1, and there is a B_1 of those balls contains infinite points in A. Let x_1 be one of them. Similarly, there is a ball B_2 of radius $1/2$ s.t. $A \cap B_1 \cap B_2$ has infinitely many points, then pick $x_2 \neq x_1$ in it. Then we choose the ball B_i of radius $1/i$ and pick distinct x_k from:

  $$\bigcap_{i=1}^{k} A \cap B_i$$

  then the sequence (x_k) is Cauchy, then it converges as the completeness, then there is at least one accumulation point of A in A.

- $4 \implies 1$: Omission.

$\square$

Corollary 1.1 (Heine-Borel Theorem)**.** *A compact subset $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.*

Proof. First, compact means totally bounded thus bounded. And a compact subset of Hausdorff space is closed.

For the converse, if A is closed, it is complete. To show this, use the definition of Cauchy sequence and for any closed subset A, $A = \overline{A} = A \cup A^*$ where A^* denotes the set of the accumulation point of A.
Meanwhile, in $\mathbb{R}^n$, bounded means totally bounded. (So, when bounded means totally bounded? Why $\mathbb{R}^n$?).

$\square$

Lemma 1.2 (Lebesgue number)**.** *Let (X, d) be a compact metric space, and let $\{V_i\}_{i \in I}$ be an open cover of X, then there exists some $\delta > 0$, called the **Lebesgue number** of the cover, s.t. for each $x \in X$ we have $B_\delta(x) \subset V_i$ for some $i \in I$.*

Proof. Assume that there is not any $\delta > 0$ satisfies.
Then for each n there exists some $x_n \in X$ s.t. $B_{1/n}(x_n) \cap V_i^c \neq \varnothing$ for each $i \in I$. If x is the limit point of some subsequence of (x_n), and $x \in X$, then $B_r(x) \ni x_i$ for some i for all $r > 0$ and also $B_r(x) \ni x_j$ where x_j in this subsequence and $j \geq i$. This means give $r > 0$, we can find $1/i \leq \epsilon \leq r/2$ s.t. $x \in B_\epsilon(x_i)$ for some i. Then $B_\epsilon(x_i) \subset B_r(x)$ which means $B_r(x)$ intersects V_i^c for all $i \in I$.
Notice that V_i^c is closed, so $\overline{V_i^c} = V_i^c$ and x is the accumulation point of all the V_i^c, so $x \in \bigcap_{i \in I} V_i^c = \left(\bigcup_{i \in I} V_i\right)^c = \varnothing$ which leads to a contradiction.

$\square$

Theorem 1.4 (Tychonoff product theorem)**.** *If $\{X_\alpha : \alpha \in A\}$ is a family of compact topological spaces and $X = \prod_{\alpha\in A} X_\alpha$ with the **product topology**, then X is compact.*

2 Continuous Function and Continuous Function Space

2.1 Continuous Function

Definition 2.1 (oscillation)**.** If $f : (X, d) \to (Y, \rho)$ is an arbitrary mapping, then the oscillation of f on a ball $B\left(x_0\right)$ is defined by:

$$\operatorname{osc}(f, B_r(x_0)) = \sup\left\{\rho(f(x), f(y)) : x, y \in B_r(x_0)\right\}$$

Notice that the oscillation is non-decreasing corresponding to r on each x_0.

Proposition 2.1. *A function $f : X \to Y$ is continuous at x_0 iff*

$$\lim_{r\to 0} osc(f, B_r(x_0)) = 0$$

Theorem 2.1. *Let $f : X \to Y$ be an arbitrary function. Then the set of points at which f is continuous is a G_δ set.*

Proof. Let

$$G_i = \left\{x \in X : \inf_{r>0} \operatorname{osc}(f, B_r(x)) < \frac{1}{i}\right\}$$

so the set that f is continuous is given by:

$$A = \bigcap_{i=1}^{\infty} G_i$$

Now we need to prove that G_i is open. Observe that $x \in G_i$ there exists $r > 0$ s.t. $\operatorname{osc}(f, B_r(x_0)) < 1/i$. Give $y \in B_r(x)$, there exists $t > 0$ s.t. $B_t(y) \subset B_r(x)$, so

$$\operatorname{osc}(f, B_y(t)) \leq \operatorname{osc}(f, B_r(x)) \leq 1/i$$

which means each point $y \in B_r(x)$ is an element of G_i, that is $B_r(x) \subset G_i$, as the arbitrary picking of x, G_i is thus a open set.

□

Theorem 2.2. *Let f be an arbitrary function defined on $[0, 1]$ and let*

$$E = \{x \in [0, 1] : \ f \ \textit{is continuous at} x\}$$

Then E cannot be the set of rational numbers in $[0, 1]$.

Proof. Observe that if E is the set of rational numbers, then the set of rational numbers in $[0, 1]$ is a G_δ set which implies that the irrational numbers in $[0, 1]$ is a F_σ set.
Notice that the rational numbers are the countable union of closed set (singletons). And since the rational numbers are dense in $[0, 1]$, so if the irrational number set is F_σ, then every closed set in this family cannot have any interiors which means the whole $[0, 1]$ is a F_σ set with a family of nowhere dense set, which is contrary with the Baire category theorem.

□

Theorem 2.3. *A continuous functions carries a compact subset into a compact subset.*

Proof. Let X, Y be two topological space and $f : X \to Y$ is continuous, now we prove that if $K \subset X$ is compact, then $f(K) \subset Y$ is compact too.
Notice that $f\mid_K$ is surjective, so $f(f^{-1}(U)) = U$. Then consider a open cover $\mathcal{F}$ of $f(K)$, then the set $\mathcal{E} = \left\{f^{-1}(U) : U \in \mathcal{F}\right\}$ is a open cover of K, then there exists a finite open subcover $\{V_1, \dots, V_n : V_i \in \mathcal{E}\}$ s.t. $\bigcup_{i=1}^n V_i \supset K$ where $V_i, i = 1, \dots, n$ is $f^{-1}(U_i)$ for some $U_i \in \mathcal{F}$, so there exists some i s.t. $\bigcup_{i=1}^n f^{-1}(U_i) \supset K$, then

$$f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) = \bigcup_{i=1}^n f\left(f^{-1}(U_i)\right) = \bigcup_{i=1}^n U_i$$

Notice that

$$f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) \supset f(K)$$

so $f(K) \subset \bigcup_{i=1}^n U_i$.

□

Definition 2.2 (uniformly continuous)**.** A function $f : (X, d) \to (Y, \rho)$ is said to be uniformly continuous on X if for each $\epsilon > 0$, there exists $\delta > 0$ s.t. when $d(x, y) \leq \delta$, $\rho(f(x), f(y)) \leq \epsilon$ for all $x, y \in X$.

An equivalent formulation of uniform continuity can be stated in oscillation. For each $r > 0$, let

$$\omega_f(r) = \sup_{x \in X} \operatorname{osc}\left(f, B_r(x)\right)$$

The function ω_f is called the modulus of continuity of f. Observe that f is uniformly continuous if

$$\lim_{r \to 0} \omega_f(r) = 0$$

Proof. Give a $\epsilon > 0$, there exists a $\delta > 0$, when $r \leq \delta$, $\omega_f(r) \leq \epsilon$. Then

$$\sup_{x \in X} \operatorname{osc}(f, B_r(x)) \leq \epsilon$$

so when $d(x, y) \leq r \leq \delta$, $\sup_{x \in X} \rho(f(x), f(y)) \leq \epsilon$ which means uniform continuity.

□

Theorem 2.4. *Let $f : X \to Y$ be a continuous mapping. If X is compact, then f is uniformly continuous on X.*

Proof. From 2.2, we notice that if $\lim_{r \to 0} \omega_f(r) = 0$, then f is uniformly continuous.
Choose $\epsilon > 0$, the collection

$$\mathcal{F} = \left\{f^{-1}(B_{\epsilon/2}(y))) : y \in Y\right\}$$

is a open cover of X, then there exists a Lebesgue number $\delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_{\epsilon/2}(y))$ for all $x \in X$ follows from 1.2.
So $f(B_\delta(x)) \subset B_{\epsilon/2}(y)$ for some $y \in Y$ which means $\omega_f(\delta) \leq \epsilon$ for arbitrary ϵ, so f is uniformly continuous.

□

Theorem 2.5. *Let K be a compact topological space and let $(Y, \|\cdot\|_Y)$ be a normed vector space. Then $\mathcal{C}(K; Y)$ is a vector space with the norm $\|\cdot\| : \mathcal{C}(K; Y) \to \mathbb{R}$:*

$$\|f\|_{\mathcal{C}} = \sup_{x \in K} \|f(x)\|_Y$$

*for each $f \in \mathcal{C}(K; Y)$. It is called the **sup-norm** on $\mathcal{C}(K; Y)$.*

Proof. Notice that $(Y, \|\cdot\|)$ is a metric space and a compact subset in a metric space is bounded and closed. - $\sup \|f(x)\|_Y < \infty$ and $\sup \|f(x)\|_Y \geq 0$ - $\sup \|\alpha f(x)\|_Y = |\alpha| \sup \|f(x)\|_Y$ - $\sup \|f+g\|_Y \leq \sup \|f\|_Y + \sup \|g\|_Y$

$\square$

Definition 2.3 (converge uniformly)**.** A sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n \in \mathcal{C}(K;Y)$ is said to **converge uniformly** if $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{C}} = 0$. It means

$$\lim_{n\to\infty} \left(\sup_{x\in K} \|f_n(x) - f(x)\|_Y \right) = 0$$

Theorem 2.6. *Let X be any set and let $(Y, \|\cdot\|_Y)$ be a normed vector space. Then the set $\mathcal{B}(X;Y)$ of all bounded mappings $f : X \to Y$ i.e. $f(X) \subset Y$ is a bounded subset in Y is a vector space and the function $\|\cdot\|_{\mathcal{B}} : \mathcal{B}(X;Y) \to \mathbb{R}$ defined by:*

$$\|f\|_{\mathcal{B}} = \sup_{x\in X} \|f(x)\|_Y$$

is a norm on $\mathcal{B}(X;Y)$.

Proof. Notice that a bounded function f over $\mathbb{K}$, for any $\alpha \in \mathbb{K}$, αf is still bounded and for $f, g \in \mathcal{B}(X;Y)$, $f+g$ is still bounded.

It is easy to show that $\|f\|_{\mathcal{B}}$ is truly a norm on $\mathcal{B}(X;Y)$.

$\square$

Definition 2.4 (local uniform convergence)**.** Let X be a topological space and Y be a normed vector space. Then a sequence $(f_n)_{n=1}^{\infty}$ of mappings $f_n : X \to Y$ is said to converge locally uniformly to a mapping $f : X \to Y$ as $n \to \infty$ if given any $x_0 \in X$ there exists a neighborhood $V(x_0)$ of x_0 s.t.

$$\lim_{n\to\infty} \left(\sup_{x\in V(x_0)} \|f_n(x) - f(x)\|_Y \right) = 0$$

Notice that local uniform convergence and uniform convergence implies the pointwise convergence.

Theorem 2.7. *Let X is a topological space and Y be a normed vector space, let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous mapping from X to Y that converges locally uniformly to a $f : X \to Y$, then f is continuous on X.*

Moreover, if f_n continuous at x_0 and they locally uniformly convergence to f then f is continuous at x_0.

Proof. Assume that f is continuous at x_0 which means give $\epsilon > 0$, there exists a neighborhood $V(x_0) \in \mathcal{N}_{x_0}$ s.t. for every $x \in V(x_0)$, $\|f(x_0) - f(x)\|_Y \leq \epsilon$.

Now suppose that $\epsilon > 0$ is given. As $(f_n) \to f$ locally uniformly. Then we can choose a $k \in \mathbb{N}$ s.t. for any $i \geq k$, we can find a neighborhood $V(x_0)$ s.t. for any $x \in V(x_0)$,

$$\sup_{x\in V(x_0)} \|f_i(x) - f(x)\|_Y \leq \epsilon/3$$

and as all $f_n : n \in \mathbb{N}$ is continuous at x_0, so we can find a neighborhood of x_0, $U(x_0) \in \mathcal{N}_{\S}$, s.t. for any $x \in U(x_0)$,

$$\sup_{x\in U(x_0)} \|f_i(x) - f_i(x_0)\|_Y \leq \epsilon/3$$

Then we consider the set $W(x_0) = U(x_0) \cap V(x_0) \in \mathcal{N}_{x_0}$, for any $x \in W(x_0)$:

$$\begin{aligned} \|f(x) - f(x_0)\|_Y &\leq \|f(x) - f_i(x)\|_Y + \|f_i(x) - f_i(x_0)\|_Y + \|f_i(x_0) - f(x_0)\|_Y \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

so if $(f_n) \to f$ locally uniformly, and f_n is continuous at x_0 for every n then f is continuous at x_0. Moreover, if f_n is continuous at every $x \in X$ i.e. continuous at X, then f is continuous at X.

$\square$