# Integration

Xie Zejian

## Contents



## Expectation and integration

### Integration

Let $f$ be Borel measurable on $(\Omega, \mathcal{A}, \mu)$. The **integral** of $f$ w.r.t $\mu$ is denoted by

$$\int f(\omega)\mu(d\omega) = \int f d\mu = \int f$$

1. If $f = \sum_1^n a_i I_{A_i}$ with $a_i \geq 0$,

$$\int f d\mu = \sum_1^n a_i \mu\left(A_i\right)$$

2. If $f \geq 0$,define

$$\int f d\mu = \lim_n \int f_n d\mu$$

   where $f_n$ are simple functions and $f_n \nearrow f$.

3. For any $f$, we have $f = f^+ - f^-$, define

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

4. $f$ is said to be integrable w.r.t. $\mu$ if $\int |f| d\mu < \infty$. We denote all integrable functions by $L^1$.

**Proposition**

1. If $f$ is positive and $a \leq f(x) \leq b$ and $\mu(\Omega) > \infty$, then

$$a\mu(\Omega) \leq \int f d\mu \leq b\mu(\Omega)$$

2. The integral of $f$ w.r.t $\mu$ over $A$ is defined by

$$\int_A f d\mu = \int f I_A d\mu = \int f(\omega) I_A(\omega) \mu(d\omega)$$

If $\mu(A) = 0$ and $f > 0$, then

$$\int_A f d\mu = 0$$

**MCT**

Suppose nonnegative $f_n \nearrow f$, then $\int f_n d\mu \nearrow \int f d\mu$.

**Proof** Note $\int f_n d\mu \leq \int f d\mu$, $\int f_n d\mu$ must converges to some $L \leq \int f$. Then we show $L \geq \int f$.

Let $s = \sum a_i \chi_{E_i}$ be simple function and $s \leq f$. Let $A_n = \{x : f_n(x) \geq cs(x)\}$ where $c \in (0, 1)$, then $A_n \nearrow X$. For each $n$

$$\begin{aligned}
\int f_n \geq \int_{A_n} f_n \geq c \int_{A_n} s \\
&= c \int_{A_n} \sum a_i \chi_{E_i} \\
&= c \sum a_i \mu(E_i \cap A_n) \\
&\nearrow c \int s
\end{aligned}$$

hence $L \geq c \int s \implies L \geq \int s \implies L = \lim L \geq \lim \int s_n = \int f$. $\blacksquare$

If $f$ and $g$ are intergrable or $f, g \geq 0$, then

note where $f_n$ is integrable not enough since MCT not hold.

$$\int f + g = \int f + \int g$$

Moreover, if $f_n > 0$ then

$$\int \sum_1^{\infty} f_n = \sum_1^{\infty} \int f$$

**Fatou's lemma**

If $f_n \geq 0$ then

$$\int \left(\liminf_n f_n\right) \leq \liminf_n \int f_n$$

**Proof** Suppose $g_n = \inf_{i \geq n} f_i$ and recall that $\lim g_n = \liminf f_n$. Clearly $g_n \leq f_i \forall i \geq n$ hence

$$\int g_n \leq \inf_{i \geq n} \int f_i$$

Take limit both side and note $g_n$ is increasing:

$$\lim \int g_n = \int \lim g_n = \int \liminf f_n \leq \liminf \int f_n$$

**Dominated convergence theorem**

Suppose $f_n(x) \to f(x) \forall x$, and there exists a nonnegative integrable $g$ s.t. $|f_n(x)| \leq g(x)$(then we get $f_n \in L^1$ immediately), then

$$\lim \int f_n d\mu = \int f d\mu$$

**Proof** Since $f_n + g \geq 0$

$$\int f + \int g = \int f + g \leq \liminf \int f_n + g = \liminf \int f_n + \int g$$

thus $\int f \leq \liminf_{n\to\infty} \int f_n$. Similarly, we can get $\int f \leq \liminf_{n\to\infty} \int f_n$ from $g - f_n \geq 0$. ■

## Properties of Lebesgue integrals

### Criteria for zero a.e.

Suppose $f$ is measurable and non-negative and $\int f d\mu = 0$. Then $f = 0$ a.e.

Suppose $f$ is integrable and $\int_A f = 0$ for all measurable $A$. Then $f = 0$ a.e.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $\int_a^x f = 0$ for all $x$, then $f = 0$ a.e.

**Proof** For any interval $I = [c, d]$,

$$\int_i f = \int_a^d f - \int_a^c f = 0$$

Then the integral is 0 for finite disjoint union of intervals from additivity. Note open sets $G$ can be written as countable union of disjoint open intervals $G = \sum_1^\infty I_i = \lim \sum I_n \implies$

$$\int_G f = \int f\chi_G = \int f \sum_1^\infty \chi_{I_i} = \int \lim f \sum \chi_{I_i} = \lim \int f \sum \chi_{I_i} = 0$$

If $G_n \searrow H$, then

$$\int_H f = \int f\chi_H = \int \lim f\chi_{G_n} = \lim \int f\chi_{G_n} = \lim \int_{G_n} f = 0$$

where we apply DMT twice and take domiated function $g = |f|$.

Finally, for any borel measurable set $E$, there is $G_\delta \supset E$ and $m(G_\delta - E) = 0$, then

$$\int_E f = \int f\chi_E = \int f\chi_{G_\delta} = \int_{G_\delta} f = 0$$

Recall proposition **2**, we are done. ■

(**Absolute integrability**). $\int f$ is finite iff $\int |f|$ is finite.

**(Linearity)** If $f, g, a, b \geq 0$ or $f, g \in L^1$

$$\int (af + bg) = a \int f + b \int g$$

**($\sigma$ additivity over sets)** If $A = \sum_{i=1}^\infty A_i$, then

$$\int_A f = \sum_{i=1}^\infty \int_{A_i} f$$

**(Positivity)** If $f \geq 0$ a.s., then $\int f \geq 0$

**(Monotonicity)** If $f_1 \leq f \leq f_2$ a.s., then $\int f_1 \leq \int f \leq \int f_2$

**(Mean value theorem)** If $a \leq f \leq b$ a.s., then

$$a\mu(A) \leq \int_A f \leq b\mu(A)$$

**(Modulus inequality)**: $|\int f| \leq \int |f|$

**(Fatou's) inequality** If $f_n \geq 0$ a.s., then

$$\int \left(\liminf_n f_n\right) \leq \liminf_n \int f_n$$

**(Dominated Convergence Theorem)** If $f_n \to f$ a.s., $|f_n| \leq g$ a.s. for all n and $\int g < \infty$, then

$$\lim_n \int f_n = \int f = \int \lim_n f_n$$

**(Monotone Convergence Theorem)** If $0 \leq f_n \nearrow f$, then

$$\lim_n \int f_n = \int f = \int \lim_n f_n$$

**(Integration term by term)** If $\sum_{i=1}^{\infty} \int |f_n| < \infty$, then

$$\sum_{i=1}^{\infty} |f_n| < \infty, \ a.s.$$

and

$$\int \left(\sum_{i=1}^{\infty} f_n\right) = \sum_{i=1}^{\infty} \int f_n$$

**An approximation result**

Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable, then $\forall \epsilon > 0$, there exists a continuous $g$ with compact support and

$$\int |f - g| < \epsilon$$

**Proof** Note $\int f\chi_{[-n,n]} \nearrow \int f$, hence we may assume $f$ is bounded.

If $f = \chi_A$, there exist $F \subset A \subset G$ and $m(G - F) < \epsilon$, take $\delta = d(K, G^c)$, let

$$g(x) = (1 - \frac{d(x, F)}{\delta})$$

Then $g$ has compact support $\overline{G}$ and $\int |g - \chi_A| \leq \int \chi_G - \chi_F = m(G - F) < \epsilon$.

If $f = \sum a_i \chi_{A_i}$ is simple with bounded $A_i$. Then we may take $g = \sum a_i g_i$ with compact support is $\overline{\cup G_i}$.

If $f$ is non-negaive, there exist $\int s_n \nearrow \int f$, then we can pick $s$ s.t.

$$\int |f - s| < \epsilon/2$$

and we can pick

$$\int |s - g| < \epsilon/2$$

hence we find $\int |f - g| < \epsilon$. ■

## Riemann Integration

Suppose an interval $I$ and $f : I \to \mathbb{R}$ is riemann integrable