Normed space

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1 Space of sequences

For $1 \leq p < \infty$, ℓ_p is defined to be the set of all sequences $x. = (x_1, x_2, \cdots)$ for which $\|x\|_p < \infty$. Where

$$\|x\|_p = (\sum_1^\infty |x_i|^p)^{1/p}$$

is the ℓ_p **norm** of the sequences.

While ℓ_∞ is defined as the set of all $\sup\{|x_n|\} \leq \infty$, such norm is called ℓ_∞ **norm**, **supremum norm** or **uniform norm**.

All of these spaces are vector space. And sequence ℓ_i is increasing.

The space of all convergent sequence is denoted c and all sequences convergent to 0 is denoted c_0. Finally, the collection of sequences with finite nonzero terms is φ. One can check that

$$\varphi \subset \ell_p \subset c_0 \subset \ell_\infty \subset \mathbb{R}^n$$

2 Spaces of functions

One can think $\mathbb{R}^n$ as

$$\{f : \{1, 2, \cdots, n\} \to \mathbb{R}\} = \mathbb{R}^n = \mathbb{R}^{\{1,2,\cdots,n\}}$$

Replace $\{1, 2, \cdots, n\}$ by an arbitrary X, then $\mathbb{R}^X$ is all functions from X to $\mathbb{R}$.

For $1 \leq p < \infty$, $L_p(\mu)$ is defined to be the set of all μ measurable functions f for which $\|f\|_p < \infty$, where the L_p **norm** is defined as

$$\|f\|_p = (\int_\Omega |f|^p)^{1/p}$$

And the L_∞ **norm**, or **essential supremum** is defined as

$$\|f\|_\infty = \operatorname{ess\,sup} f = \sup\{t : \mu(\{x : |f(x)| \geq t\}) > 0\}$$

2.1 Existence of basis

Every non-zero vector space has a basis.

Proof Let $\mathcal{X}$ be the class of all independent subsets of space V. Then $(\mathcal{X}, \subset)$ is a poset. Forall chain $\mathcal{Y} \subset \mathcal{X}$, note $\cup\mathcal{Y} \in \mathcal{X}$ is a upper bound of $\mathcal{Y}$. Apply Zorn's lemma we can find a maximal element $B \in \mathcal{X}$ and $\langle B \rangle = V$, so B forms a basis of V.

2.2 Knaster-Tarski fixed point theorem

Let $(X, \succeq)$ be an inductive ordered set. Let $f : X \to X$ is monotone and assume there exist $x \in X$ s.t. $x \leq f(x)$. Then the set of all fixed point is nonempty and has a maximal fixed point.

Proof

3 Ordinals

A set X is **well ordered** by linear $\preceq$ if every nonempty subset has a least element. An **initial segement** of $(X, \preceq)$ is any set of the form $I(x) = \{y \in X : y \leq x\}$. An **ideal** in a well ordered X is a subset A s.t. forall $a \in A$, $I(a) \subset A$.

Theorem Every nonempty set can be well ordered.

Proof Let X nonempty, and let

$$\mathcal{X} = \{(A, \preceq_A) \text{ is well order} : A \subset X\}$$

all well ordered sets, and define $\preceq$ on $\mathcal{X}$ as $(B, \preceq_B) \preceq (A, \preceq_A)$ if B is an ideal in A and $\preceq_A$ extends $\preceq_B$. Suppose every chain $\mathcal{C}$ in $\mathcal{X}$, $(\cup\mathcal{C}, \cup\{\prec_A\colon A \in \mathcal{C}\})$ clearly an upper bound of $\mathcal{C}$ and well ordered. By Zorn's lemma, there is a maximal element of $\mathcal{X}$ and it's actually X. ■

Theorem There exist poset $(\Omega, \preceq)$ satisfy 1. $(\Omega, \preceq)$ is weel ordered. 2. Ω has a greast element ω_1 3. $I(x)$ is countable for $x < \omega_1$ 4. $\{y \in \Omega : x \leq y \leq \omega_1\}$ is uncountable. 5. Every nonempty subset of Ω has a least upper bound. 6. A nonempty subset of $\Omega - \{\omega_1\}$ has greatst element iff it's countable. Every uncountable subset has least upper bound ω_1.

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Here is my theorem.
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4 Inequality

4.0.1 Young's inequality

Let f be a continues and strictly increasing function with $f(0) = 0$, then we have

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$$

Take $f(x) = x^{p-1}$, then $f^{-1}(x) = x^{q-1}$ if $(p-1)(q-1) = 1 \iff \frac{1}{p} + \frac{1}{q} = 1$. Hence we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

4.0.2 Holder's inequality

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{\ltimes}$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum |a_i b_i| = |\mathbf{a}|'|\mathbf{b}| \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q$$

4.0.3 Minkowski's inequality

For $\mathbf{a}, \mathbf{b} \in \mathbb{F}^n$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{a} + \mathbf{b}\|_p \leq \|\mathbf{a}\|_p + \|\mathbf{b}\|_p$$

4.1 Normed Vector spaces

Let $(X, +, \cdot)$ be a vector space over $\mathbb{F}$. A norm on X is a function from $X \to \mathbb{R} \geq \not\vdash$ satisfy:

1. $\|x\| \geq 0$ and $=$ occurs iff $x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|cx\| = |c|\|x\|$

A vector space with a norm is **normed vector space**.

Let **c** is $n \times 1$ and $\mathbf{X} = [\mathbf{x_1} \quad \mathbf{x_2} \quad \cdots \quad \mathbf{x_n}]$ is $n \times n$ where x_i is n vector. Then

$$\begin{aligned} \|\mathbf{Xc}\| &= \|\sum c_i \mathbf{x_i}\| \\ &\leq \sum \|c_i \mathbf{x_i}\| \\ &= \sum |c_i| \|\mathbf{x_i}\| \\ &= \|\mathbf{X}\| |\mathbf{c}| \end{aligned}$$

where

$$\|\mathbf{X}\| = [\|\mathbf{x_1}\| \quad \|\mathbf{x_2}\| \quad \cdots \quad \|\mathbf{x_n}\|], |\mathbf{c}| = \begin{bmatrix} |c_1| \\ |c_2| \\ \vdots \\ |c_n| \end{bmatrix}$$

Let $(X, \|\cdot\|)$ be a normed space, define $d(x, y) = \|x - y\|$, one can check d is a metric and is called as induced metric of the form. Then we can talk about convergence in this space. Clearly, the norm is a continuous function and $+$ and $\cdot$ are also continuous.

If $x_n \to x$ in $\|\cdot\|_1 \implies x_n \to x$ in $\|\cdot\|_2$, we say $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$. If they are stonger than each other, we say they are equivalent.

All norm on finite dimensional space are equivalent.

Proof It's sufficient to show that every norm is equivalet to $\|\cdot\|_2$:

$$\|\mathbf{x}\| = \|\mathbf{Ex}\| \leq \|\mathbf{E}\| |\mathbf{x}| \leq \|\mathbf{x}\|_2 \|(\|\mathbf{E}\|')\|_2 = c\|\mathbf{x}\|_2$$

where

$$\mathbf{E} = [\mathbf{e_1} \quad \mathbf{e_2} \quad \cdots \quad \mathbf{e_n}] = \mathbf{I}$$

This state that $\|\cdot\|$ stronger than any norm. On the other hand, consider

$$\alpha = \inf\{\|\mathbf{x}\| : \|\mathbf{x}\|_2 = 1\}$$

It's positive since $\{\|\mathbf{x}\|_2 = 1\}$ is compact. Then we have

$$\alpha \leq \|\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\| = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_2} \implies \|\mathbf{x}\| \geq \alpha \|\mathbf{x}\|_2$$

For any abstract space X, $x \in X$ can be presented as linear combinations of bias, say $x = \sum a_i e_i$, then $x \mapsto (a_1, \cdots, a_n)$ is isomorph from X to $\mathbb{R}^\ltimes$. And any norm iduced a norm on $\mathbb{R}^\ltimes$ by

$$\|x\| = \widetilde{\|(a_1, \cdots, a_n)\|}$$

Hence all norm is equivalent.

4.1.1 Separability

A subset E of (X, d) is a **dense set** if its closure is X:

$$\overline{E} = X$$

A metric space is called **separable** if it has a countable dense subset.

4.1.2 Completeness

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Every metric space has a completion