

Notes of Infinite dimensional analysis

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Chapter 1

Odds and ends

1.1 Space of sequences

Definition 1.1. For $1 \leq p < \infty$, ℓ_p is defined to be the set of all sequences $x. = (x_1, x_2, \dots)$ for which $\|x\|_p < \infty$. Where

$$\|x\|_p = (\sum_1^{\infty} |x_i|^p)^{1/p}$$

is the ℓ_p **norm** of the sequences.

While ℓ_{∞} is defined as the set of all $\sup\{|x_n|\} \leq \infty$, such norm is called ℓ_{∞} **norm**, **supremum norm** or **uniform norm**.

All of these spaces are vector space. And sequence $\{\ell_i\}_{i=1}^{\infty}$ is increasing.

The space of all convergent sequence is denoted c and all sequences convergent to 0 is denoted c_0 . Finally, the collection of sequences with finite nonzero terms is φ . One can check that

$$\varphi \subset \ell_p \subset c_0 \subset \ell_{\infty} \subset \mathbb{R}^n$$

1.2 Spaces of functions

One can think \mathbb{R}^n as

$$\{f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}\} = \mathbb{R}^n = \mathbb{R}^{\{1, 2, \dots, n\}}$$

Replace $\{1, 2, \dots, n\}$ by an arbitrary X , then \mathbb{R}^X is all functions from X to \mathbb{R} .

For $1 \leq p < \infty$, $L_p(\mu)$ is defined to be the set of all μ measurable functions f for which $\|f\|_p < \infty$, where the L_p **norm** is defined as

$$\|f\|_p = \left(\int_{\Omega} |f|^p \right)^{1/p}$$

And the L_{∞} **norm**, or **essential supremum** is defined as

$$\|f\|_{\infty} = \text{ess sup } f = \sup\{t : \mu(\{x : |f(x)| \geq t\}) > 0\}$$

1.3 Ordinals

Suppose R is an order relation on Ω , then Ω is said to be **inductively ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Definition 1.2. A set X is **well ordered** by linear \preceq if every nonempty subset has a least element.

Definition 1.3. An **initial segment** of (X, \preceq) is any set of the form $I(x) = \{y \in X : y \preceq x\}$.

Definition 1.4. An **ideal** in a well ordered X is a subset A s.t. for all $a \in A$, $I(a) \subset A$.

Theorem 1.1 (Well Ordering Principle). *Every nonempty set can be well ordered.*

Proof. Let X nonempty, and let

$$\mathcal{X} = \{(A, \preceq_A) \text{ is well order} : A \subset X\}$$

all well ordered sets, and define \preceq on \mathcal{X} as $(B, \preceq_B) \preceq (A, \preceq_A)$ if B is an ideal in A and \preceq_A extends \preceq_B . Suppose every chain \mathcal{C} in \mathcal{X} , $(\cup \mathcal{C}, \cup \{\preceq_A : A \in \mathcal{C}\})$ clearly an upper bound of \mathcal{C} and well ordered. By Zorn's lemma, there is a maximal element of \mathcal{X} and it's actually X . \square

Kind of remarkable and useful well ordered set is exist:

Theorem 1.2. *There exist poset (Ω, \preceq) satisfy*

1. (Ω, \preceq) is well ordered.

2. Ω has a greast element ω_1
3. $I(x)$ is countable for $x < \omega_1$
4. $\{y \in \Omega : x \leq y \leq \omega_1\}$ is uncountable.
5. Every nonempty subset of Ω has a least upper bound.
6. A nonempty subset of $\Omega - \{\omega_1\}$ has greatst element iff it's countable. Every uncountable subset has least upper bound ω_1 .

Proof. Let (X, \preceq) be uncountable well ordered set, and let A

$$A = \{x \in X : I(x) \text{ is uncountable}\}$$

w.l.o.g we may assume A is nonempty. Then there is a first element and denoted by ω_1 . Then we show that $\Omega = I(\omega_1)$ enjoy all the properies.

1-4 is straightforward and 5 follows from all the upper bound is well ordered and thus least upper bound exist. For 6, suppose there is a countable $C \subset \Omega - \{\omega_1\}$, then $\bigcup_{i=1}^{\infty} I(x_i)$ is countable, so there is some $x < \omega_1 \notin \bigcup_{i=1}^{\infty} I(x_i)$, that is an upper bound. By 5, least upper bound is exist and belong to C . Conversely, if some subset C has some least upper bound $b < \omega_1$, then $C \subset I(b)$ and must countable. \square

The elements of Ω are called **ordinals** and ω_1 is called **first uncountable ordinal**. The elements of $\Omega_0 = \Omega - \{\omega_1\}$ is **countable ordinals**. We treat \mathbb{N} as a subset of Ω . Then the first element of $\Omega - \mathbb{N}$ is **first infinite ordinal**.

Theorem 1.3 (Interlacing Lemma). *Suppose sequence $\{x_n\}$ and $\{y_n\}$ in Ω_0 with $x_n \leq y_n \leq x_{n+1}$. Then they share the same least upper bound.*

Proof. Clearly since $x_n \leq y_n \leq x_{n+1}$. \square

Chapter 2

Topology

2.1 Topological spaces

Let Ω be as space

Definition 2.1. A class of subset τ of Ω is an **topology** if

1. \emptyset and Ω belongs to τ .
2. closed under arbitrary union.
3. closed under finite intersection.

(Ω, τ) called a **topological space** where Ω is called as **underlying set**. The sets in τ are called **open** while sets with complement in τ is **closed**. Both open and closed set is called **clopen**.

Definition 2.2. Countable intersection of open sets is \mathcal{G}_σ set and countable union of closed sets is \mathcal{F}_δ set.

Following is some examples of topological space.

Definition 2.3. (X, ρ) is a **semimetric space**, when ρ defined on $X \times X$ s.t. $\forall x, y, z \in X$:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

ρ is called a **semimetric**.

If $\rho(x, y) = 0 \iff x = y$, ρ become a **metric** and (X, ρ) become **metric space**. $B(a, r) = \{x \in E, d(x, a) < r\}$ is **r -ball** with center a .

U is **open** in (Ω, d) iff $\forall x \in U, \exists r_x > 0 \ni B_d(x, r_x) \subseteq U$. Let τ_d be the set of all open subsets of Ω , we call τ_d the **topology generated by d** . A Topological space is **metrizable** if there exist metric d generates it.

Suppose d is discrete, that is, $d(x, y) = 0$ iff $x = y$, otherwise, $d(x, y) = 1$. Then every subset is open hence $\tau_d = \mathcal{P}(\Omega)$ and called **discrete topology**. The zero semimetric, defined by $d(x, y) = 0$ for all $x, y \in \Omega$ generates $\tau_d = \{\emptyset, \Omega\}$ and called **trivial topology**.

Let $\Omega = \mathbb{R}^n$, $l^2 = \sqrt{\sum_1^n (x_i - y_i)^2}$ is called **Euclidean metric**. $l^1 = \sum_1^n |x_i - y_i|$ is called **taxi-cab metric** and $l^\infty = \sup\{|x_i - y_i|\}$ is called **sup norm metric**.

Note $d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n}d_{l^2}(x, y)$ and $d_{l^2}(x, y) \leq \sqrt{n}d_{l^\infty}(x, y) \leq \sqrt{n}d_{l^2}(x, y)$, then d_{l^∞} open $\iff d_{l^2}$ open $\iff d_{l^1}$ open. Hence $\tau_{d_{l^2}} = \tau_{d_{l^1}} = \tau_{d_{l^\infty}}$.

All topologies on Ω is poset with greatest element $\mathcal{P}(\Omega)$ and least $\{\emptyset, \Omega\}$. If $\tau' \subset \tau$, we say τ' **coarser** than τ while τ finer than τ' .

If τ can be form by taking union of families in some $\mathcal{B} \subset \tau$, we call \mathcal{B} the **base** for the topology τ .

Theorem 2.1. \mathcal{B} is a base in (X, τ) iff $\forall U \in \tau, \forall x \in U, \exists W \in \mathcal{B} \ni x \in W \subset U$.

Proof. \implies : Any U can be written as $U = \cup W_i$ and $x \in U \implies x \in W_i$ for some i and $W_i \in \mathcal{B}$. \impliedby : For any $U \in \tau$, consider arbitrary $x \in U$, then there exist W_x such that $x \in W_x \subset U$, thus we have $U = \cup_x W_x$. \square

Let $\mathcal{S} \subset \tau$, suppose all topologies include \mathcal{S} . Then the intersection of all of them is again a topology, denoted as $\tau(\mathcal{S}) = \cap \mathcal{T}$, then $\tau(\mathcal{S})$ is the smallest topology contains \mathcal{S} . We call it the topology **generated** by \mathcal{S} .

Theorem 2.2. $\tau(\mathcal{S})$ is unions of families of finite intersections together with Ω , formally:

$$\{\bigcup \left(\bigcap_1^N S_i \right)\} \cup \Omega$$

$\mathcal{S} \subset \tau$ is a **subbase** for τ if all finite intersections of \mathcal{S} is a base. Note that if $\Omega \in \mathcal{S}$, \mathcal{S} is the subbase of $\tau(\mathcal{S})$. (Ω, τ) is **second countable** if τ has countable base. Clearly, a topology is second countable iff it has countable subbase.

For any subset X in (Ω, τ) , then

$$\tau_X = \{X \cap V : V \in \tau\}$$

form a topology in X and we call (X, τ_X) a **subspace** or **relative topology**. Sets in τ_X are **relative open**. **Relative closed** sets of the form

$$X - (X \cap V) = X - V = X \cap V^c$$

2.2 Neighborhood

A subset V is called a **neighborhood** of a if there exists a open set $U \subset V$ contains a . Then we called $V' = V - \{a\}$ **punctured(deleted)** neighborhood. A **neighborhood base** is a collection of neighborhood $BN(a)$ s.t. for any neighborhood V of a , there exist a $W \in BN(a)$ and $W \subset V$. Clearly, all the neighborhoods is a neighborhood base and denoted as $\mathcal{N}(x)$, which is called **neighborhood system**.

Lemma 2.1. *A subset U is open iff it's a neighborhood for each of its points.*

Proof. \Rightarrow is trivial. \Leftarrow follows from $\cup_x G_x = U$ and unions of open set is still open. ■ □

This suggest a equivalent definition of finiar topology:

Lemma 2.2. $\tau' \subset \tau \iff \tau' \text{ neighborhood is a } \tau \text{ neighborhood.}$

Proof. \Rightarrow any open set G_x satisfy $x \in G_x \subset V$ in T' is still open in T , hence V is T neighborhood. \Leftarrow Consider any open set $G \in T'$, it's a T' neighborhood for each of its points implies it's a T neighborhood for each of its points and hence G is T open. □

2.3 Closures

The **interior** of A is the union of all open sets which are included A , i.e., the largest open set included in A , we denote it A° . And the **closure** is the intersection of all closed sets which include A and thus the smallest closed set includes A , we denote it \overline{A} .

Lemma 2.3. *Following is some useful truth:*

1. $(A \cap B)^\circ = A^\circ \cap B^\circ$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. $A \subset \overline{B} \implies \overline{A} \subset \overline{B}$
4. $A^\circ \subset B \implies A^\circ \subset B^\circ$
5. $\overline{A^c} = (A^\circ)^c$

$$6. (\overline{A})^c = (A^c)^\circ$$

Proof. We only prove **5**, note $(A^\circ)^c$ is closed and

$$A^\circ \subset A \implies (A^c) \subset (A^\circ)^c$$

we have $\overline{A^c} \subset (A^\circ)^c$. On the other hand

$$\overline{A^c} \supset (A^\circ)^c \iff (\overline{A^c})^c \subset A^\circ \iff (\overline{A^c})^c \subset A \iff \overline{A^c} \supset A^c$$

□

The **frontier** of A is $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ$.

x is said to be an **interior point** of A if A is neighborhood of x .

x is said to be an **adherent point** if it's every neighborhood meets A , an ω **accumulation point** of A if every neighborhood of x contains **infinitely** many points of A and is a **condensation point** of A if every neighborhood of x contains **uncountable** many points of A .

x is a **cluster point** or **accumulation point** if every deleted neighborhood of x meets A and is **isolated point** if x is not cluster point. That is, $\{x\}$ is relative open in A . We denoted all the cluster points as A' and called **derived set**.

x is **frontier point** or **boundary point** if every neighborhood of x meets both A and A^c .

It's east to show that the points of A° are precisely all the interior points of A and \overline{A} are precisely all the adherent points. ∂A is precisely points of frontier. We claim that

$$\overline{A} = A^\circ \cup \partial A = A \cup A'$$

A subset A is called **perfect** if it's closed while point in A is cluster points in A , that is $A' = A = \overline{A}$.

2.4 Dense

A is said **dense** if $\overline{A} = \Omega$ and **nowhere dense** if $(\overline{A})^\circ = \emptyset$ (\mathbb{Q} is dense in \mathbb{R} while \mathbb{Z} is nowhere dense.) A is said to be **meagre** or **set of the first category** if it's countable union of nowhere dense. Sets which are not meagre is **set of the second category set**.

Space (Ω, τ) is **first countable** if every point of Ω has countable neighborhood base. The space is said **separable** if Ω has a countable dense subset.

Lemma 2.4. *Second countable space is separable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, by axiom of choice, we may take x_i in I , let $X = \{x_i\}_{i \in I} \subset \Omega$. Then we show that X is dense. For any $x \in \Omega$, it's neighborhood must contain some open G which is unions of \mathcal{B} and thus contains at least one element in X , that is, G meet X . Hence $\overline{X} = \Omega$. \square

Lemma 2.5. *Second countable space is first countable*

Proof. Suppose $\mathcal{B} = (B_i)_{i \in I}$ is a countable base, for each point $x \in \Omega$, one may take all the sets in \mathcal{B} which contains x as a neighborhood base. To verify it's neighborhood base, if there is a neighborhood N of x , then there is a open G contains x . By the definition of base, G is the union of sets of \mathcal{B} and those sets must at least one contains x and these sets is subset to G . \square

2.5 Mappings

Suppose (Ω, τ) and (Ω', τ') are two spaces and f is a mapping from Ω to Ω' in the following.

Lemma 2.6. *Following is some useful truth for mappings.*

1. $ff^{-1}(A) \subset A$
2. $f^{-1}f(A) \supset A$
3. $f^{-1}(U \cap N) = f^{-1}(U) \cap f^{-1}(N)$
4. $f^{-1}(U \cup N) = f^{-1}(U) \cup f^{-1}(N)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f^{-1}f(A) = A$ always holds if f is injection while $ff^{-1}(A) = A$ always holds if f is surjection.
7. If f is bijection, $(f^{-1})^{-1}(A) = f(A)$ always hold.
8. $(f \circ g)^{-1}(A) = g^{-1}f^{-1}(A)$
9. $f^{-1}(A) \subset f^{-1}(B) \iff A \subset B$
10. $f(A) \subset f(B) \iff A \subset B$

Definition 2.4. f is **continuous** at x if for every neighborhood N' of $f(x)$, there is a neighborhood N of x s.t. $f(N) \subset N'$. It's continuous if it's continuous at every points $x \in \Omega$.

Theorem 2.3. f is continuous iff

1. $f^{-1}(G')$ is open for every open subset G' of Ω' .
2. $f^{-1}(F')$ is closed for every closed subset F' of Ω' .
3. If $A \subset \Omega'$, then $f^{-1}(A^\circ) \subset (f^{-1}(A))^\circ$
4. If $A \subset \Omega$, then $f(\overline{A}) \subset \overline{f(A)}$

Proof. We only prove 1 and 3.

1 \implies : For any $x \in f^{-1}(G')$, it's sufficient to show that $f^{-1}(G')$ is its neighborhood. By definition, there is a neighborhood N s.t. $f(N) \subset G'$, and

$$x \in N \subset f^{-1}f(N) \subset f^{-1}(G')$$

\Leftarrow : For every neighborhood N' , there is some open G' contain $f(x)$, and $f^{-1}(G')$ is neighborhood of x and $f^{-1}(G') \subset N'$.

3 \implies : $f^{-1}(A^\circ)$ is open and the claim follows from $f^{-1}(A^\circ) \subset f^{-1}(A)$. \Leftarrow : Suppose A is open, then $A^\circ = A$ and hence $f^{-1}(A) \subset (f^{-1}(A))^\circ$. Which suggests $f^{-1}(A)$ is open. \square

Lemma 2.7. Suppose $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_2 \rightarrow \Omega_3$, $f \circ g$ is continuous if f and g are continuous.

Proof. Suppose G_3 is open and the claim follows from $(f \circ g)^{-1}(G_3) = f^{-1}(g^{-1}(G_3))$. \square

Lemma 2.8. Suppose $f : (\Omega, \tau), (\Omega', \tau(\mathcal{S}))$, f is continuous iff $f^{-1}(S) \in \tau$ for any $S \in \mathcal{S}$.

(Ω, τ) and (Ω', τ') are said to be **homeomorphic** if there exist continuous bijection f , s.t f^{-1} is continuous and such f is called **homeomorphism**. In particular, f is an **embedding** if $f : (\Omega, \tau) \rightarrow (f(\Omega), \tau|_{f(\Omega)})$ is a homeomorphism.

f is **open** if $f(G)$ is open for all open set $G \in \tau$ and is **closed** if $f(F)$ is closed for all closed set $F^c \in \tau$.

Lemma 2.9. Suppose f is bijection, then it's homeomorphism iff it's continuous and either open or closed.

Proof. By the continuity of f^{-1} , since $(f^{-1})^{-1}(G) = f(G)$ for all open set G .

$$f^{-1} \text{ is continuous} \iff f(G) \text{ is open} \iff f \text{ is open}.$$

\square

Lemma 2.10. Suppose f is bijection, it's a homeomorphism iff τ' is the finest topology where f is continuous.

Proof. Suppose f is homeomorphism, T_0 is another topology where f is continuous. For any $G \in \tau_0$, $f^{-1}(G) \in \tau$ by the continuity of f^{-1} ,

$$G = (f^{-1})^{-1}(f^{-1}(G)) \in \tau'$$

That is τ' is finer than any τ_0 . \square

Note that $\mathcal{P}(\Omega)$ let all f continuous and $\{\emptyset, \Omega\}$ let all $g : \Omega' \rightarrow \Omega$ continuous.

b

2.6 Filter

Definition 2.5. A **filter** is a non-empty collection \mathcal{F} of subset in Ω s.t.

1. $A \in \mathcal{F}, A \subset B \implies B \in \mathcal{F}$
2. Closed under finite intersection.
3. $\emptyset \notin \mathcal{F}$

Note the definition of \mathcal{F} is independent with topology τ . A **free filter** is filter with $\ker \mathcal{F} = \bigcap_{F \in \mathcal{F}} F = \emptyset$. Not free filters are called **fixed**.

Filter can be formed by taking upward closure of a filter base.

Definition 2.6. A collection \mathcal{B} of subset in Ω is a **filter base** of or **prefilter** if

1. $\mathcal{B} \subset \mathcal{F}$
2. $\forall N \in \mathcal{F}, \exists W \in \mathcal{B} \ni W \subset N$

We say \mathcal{B} generates \mathcal{F} , where

$$\mathcal{F} = \mathcal{B}^\uparrow = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

For example,

- Suppose $x \in \Omega$ then

$$\mathcal{N}(x) = \{\text{All neighbourhoods of } x\}$$

is a filter on Ω , that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. Note

$$\tau(x) = \{X \in \tau : x \in X\}$$

is a base for $\mathcal{N}(x)$ and thus $\mathcal{N}(x) = \tau(x)^\uparrow$.

- Suppose Ω is infinite, the collection of all **cofinite** subsets(subset s with finite complement) is a filter on Ω , such filter is free and called **Frechet filter**.

To assert a collection is a base, we have

Theorem 2.4. *Let \mathcal{B} be a collection of nonempty subsets. Then \mathcal{B} is a filter base, that is, \mathcal{B} may generate a filter iff*

1. *The intersection of each finite family of sets in \mathcal{B} includes a set in \mathcal{B}*
2. *\mathcal{B} is non-empty and $\emptyset \notin \mathcal{B}$.*

Proof. We claim that

$$\mathcal{F} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

\mathcal{F} is the filter generated by \mathcal{B} . □

A family of subsets \mathcal{F} is said to have **finite intersection property** if intersection of every finite subfamily is nonempty.

Let \mathcal{A} be collection of subsets with finite intersection property, then collection of all finite intersection of \mathcal{A} is a base, we call the filter generated **filter generated by \mathcal{A}** . Formally

$$\mathcal{F} = \left\{ \bigcap_{A \in \mathcal{J}} A : \mathcal{J} \subset \mathcal{A} \text{ and } \mathcal{J} \text{ is finite} \right\}^\uparrow$$

A filter \mathcal{F} is **finer** than another \mathcal{G} if $\mathcal{F} \subset \mathcal{G}$. Clearly, the set of all filters on Ω is inductively ordered by inclusion. By Zorn's lemma, the set of all filters has maximal filters and we call such filters **ultrafilters**.

Lemma 2.11. *Every fixed ultrafilter of the form*

$$\mathcal{U}(x) = \{x\}^\uparrow$$

for any $x \in \Omega$. And every free ultrafilter contains no finite subsets.

To assert a filter is ultra, we have:

Theorem 2.5. *Let \mathcal{A} be a collection of subsets and \mathcal{F} the filter generated by \mathcal{A} . If*

$$\forall X \subset \Omega, \text{ either } X \in \mathcal{A} \text{ or } X^c \in \mathcal{A}$$

then \mathcal{A} is an ultrafilter on Ω .

Proof. Suppose \mathcal{F}' is an ultrafilter include \mathcal{F} , we have $\mathcal{F}' \supset A$ clearly. Consider any $X \in \mathcal{F}'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in \mathcal{F}'$ as $\mathcal{F}' \supset \mathcal{F} \supset A$ and $X \cap X^c = \emptyset \in \mathcal{F}'$ results in a contradiction. It follows that $A \supset \mathcal{F}'$ and thus $A = \mathcal{F}'$. \square

Theorem 2.6. *Every filter \mathcal{F} is the intersection of all the ultrafilter which include \mathcal{F} .*

Proof. We claim that

$$\mathcal{F} = \cap \{\text{ultrafilter generates by } \{x\} : x \in \cap \mathcal{F}\}$$

\square

Suppose mappings on a filter:

Theorem 2.7. *Let f be a mapping from Ω to Ω' and \mathcal{B} a base for a filter \mathcal{F} on Ω . Then $f(\mathcal{B}) = \{f(X)\}_{X \in \mathcal{B}}$ is also a base on Ω' . Moreover, if \mathcal{F} is ultra then $f(\mathcal{B})$ also generates an ultrafilter.*

Proof. First assertion is straightforward and the second follows from \mathcal{B} is collection of supset for some $\{x\}$, then $f(\mathcal{B})$ generates the filter that generates by $\{f(x)\}$. \square

Theorem 2.8. *In the same situation as previous theorem. If \mathcal{B}' is a base on Ω' , then $f^{-1}(\mathcal{B}')$ is a base on Ω iff every set in \mathcal{B}' meets $f(\Omega)$*

Proof. We have

$$\Omega \in f^{-1}(\mathcal{B}') \implies f(\Omega) \subset X'$$

for some $X' \in \mathcal{B}'$, by definition, \implies is immediately.

For \Leftarrow , suppose any finite family $X_i \in \mathcal{B}'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}\left(\bigcap_i X_i\right) \in f^{-1}(\mathcal{B}')$$

Then the claim follows from theorem 2.4. \square

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the filter \mathcal{F} and \mathcal{F} is said to **converge** to x , or $\mathcal{F} \rightarrow x$, if the neighborhood filter $\mathcal{N}(x) \subset \mathcal{F}$. For filter base \mathcal{B} , we define on the filter generated by \mathcal{B} , that is, if $\mathcal{N}(x) \subset \mathcal{B}^\uparrow$.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff \mathcal{N}_\tau(x) \supset \mathcal{N}_{\tau'}(x) \iff \mathcal{F} \rightarrow a \implies \mathcal{F}' \rightarrow a$$

also, an equivalent definition of continuity as follows:

Theorem 2.9. $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continous at x iff

$$\forall \mathcal{F} \rightarrow x, f(\mathcal{F}) \rightarrow f(x)$$

Proof. By definition, $f(\mathcal{F}) \rightarrow f(x)$ if

$$\mathcal{N}(f(x)) \subset f(\mathcal{F})^\uparrow$$

That is, for any neighbourhood $N' \in \mathcal{N}(f(x))$, there exist some $A \in \mathcal{F}$ s.t. $f(A) \subset N'$, as $\mathcal{N}(x) \subset \mathcal{F}$ and f is continous at x , such A is always exists. Conversely, take $\mathcal{F} = \mathcal{N}(x)$ then the claim is follows \square

A point $x \in \Omega$ is said to be an **adherent point** of \mathcal{F} if x is an adherent point of every set in \mathcal{F} . The **adherence** of \mathcal{F} , $\text{Adh}_\tau(\mathcal{F})$ or $\overline{\mathcal{F}}$ is the set of all adherent points, thus

$$\overline{\mathcal{F}} = \bigcap_{X \in \mathcal{F}} \overline{X}$$

Define similarly on filter base \mathcal{B} by the filter generated. By definition, we have

$$\overline{\mathcal{B}} = \bigcap_{X \in \mathcal{B}} \overline{X}$$

Lemma 2.12. Suppose A be a subset of Ω , then $x \in \overline{A}$ iff there is a filter \mathcal{F} s.t. $A \in \mathcal{F}$ and \mathcal{F} converges to x .

Theorem 2.10. Suppose $BN(x)$ a neighbourhood base of x , then

1. \mathcal{B} converges to x iff every set in $BN(x)$ includes a set in \mathcal{B} .
2. $x \in \overline{\mathcal{B}}$ iff every set in $BN(x)$ meets every set in \mathcal{B} .

As consequence, we have

Corollary 2.1. x is adherent to a filter \mathcal{F} iff there is $\mathcal{F}' \supset \mathcal{F}$ and converges to x

Proof. \implies follows from taking $\mathcal{F} = BN(x)$. Conversely, $\forall N \in BN(x)$, we have $X' \subset N$ for some $X' \in \mathcal{F}'$, thus for any $X \in \mathcal{F}$, $N \cap X \subset X' \cap X \neq \emptyset$ as $X', X \in \mathcal{F}'$. \square

Corollary 2.2. *Every limit point of \mathcal{F} is adherent to \mathcal{F}*

Proof. Clearly holds by applying theorem 2.10.1 and 2.10.2. \square

Corollary 2.3. *Every adherent point of an ultra-filter is a limit point of it.*

Proof. Clearly as kernel of ultrafilter is a one point set. \square

Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a **limit point** of f relative to \mathcal{F} if $f(\mathcal{F}) \rightarrow x$.
2. an **adherent point** of f relative \mathcal{F} if it's adherent point of $f(\mathcal{F})$.

Theorem 2.11. *Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$*

1. x' is a limit point of f relative to \mathcal{F} iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, we have $f^{-1}(N') \in \mathcal{F}$.
2. x' is an adherent point of f relative to \mathcal{F} iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, it meets $f(X)$ for any $X \in \mathcal{F}$.

Proof. x' is limit is equivalent to

$$\mathcal{N}(x') \subset f(\mathcal{F})^\uparrow$$

That is, there exist some $A = f(X) \subset N'$ for any N' , followed by $X \subset f^{-1}f(X) \subset f^{-1}(N')$, then the claim follows from the definition of filter.

By theorem 2.10, x' is adherent to $f(\mathcal{F})$ iff

$$\forall N' \in \mathcal{N}(x'), \forall X \in \mathcal{F}, f(X) \cap N' \neq \emptyset$$

note for any $N' \in \mathcal{N}(x')$, there exist $N' \in \mathcal{N}(x') \ni N' \subset N'$, thus $f(X) \cap N' \neq \emptyset$ also holds. Conversely, making use of $\mathcal{N}(x') \subset \mathcal{N}(x')$. \square

For example, suppose $f : (\mathbb{N}, \tau) \rightarrow (\Omega', \tau')$ and \mathcal{F} the frechet filter on \mathbb{N} . Then x' is limit of f relative to \mathcal{F} iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{F} \iff f^{-1}(N')^c \subset [0, k] \iff f^{-1}(N') \supset \{n \in \mathbb{N} : n \geq k\}$ for some k , that is, $f(n) \in N'$ for any $n \geq k$.

Theorem 2.12. *Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ and let $\mathcal{F} = \mathcal{N}(x)$. By theorem 9, x' is limit of f relative to $\mathcal{N}(x)$ iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{N}(x) \iff N \subset f^{-1}(N') \iff f(N) \subset N'$ for some $N \in \mathcal{N}(x)$. That is, iff $x' = f(x)$, f is continous at x . Such limit points also called limit points of f at x .*

2.7 Net

(D, \preceq) is called a **directed set** if every couple $\{x, y\}$ in which has an upper bound.

If $\{D_i\}_{i \in I}$ is family of directed set then $D = \prod_{i \in I} D_i$ is also directed under **product direction** defined by $(a_i)_{i \in I} \succeq (b_i)_{i \in I}$ for all $i \in I$.

Definition 2.7. Let (D, \preceq) be a directed set, $\nu : D \rightarrow \Omega$ is called a **net** in Ω with domain D . The directed set is called **index set** of the net and members of D are **indexes**. We often write ν as x . or $\{x_\alpha\}$.

Suppose A a subset of Ω , we say x . **eventually in** A if there exist some $k \in D$ s.t. $x_n \in A$ for all $n \succeq k$. And we say ν is **frequently** in A if for all $n \in D$, there exist an $n' \succeq n$ s.t. $x_{n'} \in A$.

Lemma 2.13. *If x . not frequently in A , then x . eventually in A^c . Thus, for any $X \in \Omega$, x . frequently in either X or X^c .*

Suppose $x \in \Omega$, then x . is said **converge** to x , or $x. \rightarrow x$ if x . eventually in N for all $N \in \mathcal{N}(x)$, i.e., $\mathcal{N}(x) \subset \mathcal{F}(x)$. The point x is **adherent** to x . if x . frequently in N for all $N \in \mathcal{N}(x)$.

Theorem 2.13. *Suppose $A \in (\Omega, \tau)$, then $x \in \overline{A}$ iff it's the limit of some net in the set.*

Proof. \Leftarrow is clear. \Rightarrow follows from we may find a associated net taking value in A (since each neighborhood meets A) and such net converges to x . \square

As with sequence, if x . is bounded, there is

$$\liminf x. = \sup \inf x. \preceq \limsup x. = \inf \sup x$$

Subnet generalizes subsequence.

Definition 2.8. Suppose D is directed, a subset B of D is called **cofinal** if for any $a \in D$, there exist $b \in B$ s.t. $a \preceq b$. A map $f : D \rightarrow A$ is **final** if $f(D)$ is cofinal of A .

Let x . and $x'.$ are two nets in Ω with domains D and D' respectively. We say that $x'.$ is a **subnet** of x . if there exists a final mapping $\varphi : D' \rightarrow D$ s.t. $x'_\alpha = x_{\varphi(\alpha)}$.

Theorem 2.14. *Let \mathcal{A} be a collection of subsets that x . is frequently in. If \mathcal{A} is closed under finite intersection, then there exists a subnet $x'.$ of x . and $x'.$ eventually in every member of \mathcal{A}*

Lemma 2.14. *Suppose $x'.$ is subnet of x ., we have*

1. $x. \rightarrow x \implies x.' \rightarrow x$
2. $x \text{ adherent to } x.' \implies x \text{ adherent to } x..$

Theorem 2.15. *A point x is adherent to x . iff there is a subnet converges to x . While $x. \rightarrow x$ iff every subnet converges to x .*

Proof. \implies is clear by lemma2.14. Conversely, suppose a is not adherent to x , there exist a neighborhood N that x . not frequently in, i.e., exist k s.t. $x_n \notin N$ for any $n \geq k$, thus there is no subnet eventually in N .

For the second part, \implies is also clear by lemma2.14 and \Leftarrow comes from taking subnet as itself. \square

A net $x.$ is called **ultranet** or **universal net** if for all $X \in \Omega$, we have either $x.$ eventually in X or $x.$ eventually in X^c . Clearly, subnet of ultranet is ultra and

Lemma 2.15. *Every net has a ultra subnet.*

Proof. Consider collection of \mathcal{Q} s.t. $x.$ is frequently in every member and closed under finite intersection. By Zorn's Lemma, there is a maximal \mathcal{Q}_0 . By theorem 11, $x.$ has a subnet $x.'$ which eventually in every member of \mathcal{Q}_0 . We claim that this subnet is ultra since, \mathcal{Q}_0 is maximal and thus either $X \in \mathcal{Q}_0$ or $X^c \in \mathcal{Q}_0$. \square

2.8 Nets and filters

Let

$$\mathcal{F}(x.) = \{X \in \mathcal{P}(\Omega) : x. \text{ is eventually in } X\}$$

Then $\mathcal{F}(x.)$ is a filter and we call it the **filter associated with the net $x.$**

Theorem 2.16. *Associated filter is the upward closure of the net's tail, that is*

$$\mathcal{F}(x.) = \{\{x_b : b \succeq a\} : a \in D\}$$

Motivated by the definition of filter that filter is closed under pairwise intersection, let $X \preceq Y \iff X \supset Y$, then any mapping $\nu : \mathcal{F} \rightarrow \Omega$ s.t. $\nu(X) \in X$ is a **net associated with the filter \mathcal{F}** .

By definition, we claim that \mathcal{F} is the associated filter of every associated net and $x.$ is an associated net of the associated filter.

Theorem 2.17. *Filter $\mathcal{F} \rightarrow x$ iff $x. \rightarrow x$ for any $x.$ associated with \mathcal{F} .*

Proof. Note

$$\forall N \in \mathcal{N}(x), x. \text{ eventually in } N \iff \mathcal{N}(x) \subset \mathcal{F}(x.)$$

Then is sufficient to show that $\mathcal{F}(x.) = \mathcal{F}$. It's follows from for any $X \in \mathcal{F}$, $x.$ eventually in X . \square

Theorem 2.18.

$$x. \rightarrow x \iff \mathcal{F}(x.) \rightarrow x$$

Proof. Both side is equivalent to $\mathcal{N}(x) \subset \mathcal{F}(x.)$ \square

Theorem 2.19. Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$, then f is continous at x iff $\forall x. \rightarrow x, f(x.) \rightarrow f(x)$.

Proof. By theorem 2.18, 2.17 and 2.12. \square

By above theorems, we have

$$\text{Adh}(\mathcal{F}(x.)) = \text{Adh}(x.), \text{Lim}(\mathcal{F}(x.)) = \text{Lim}(x.)$$

and similarly results holds for any filter and one of associated nets.

Lemma 2.16. If $x.$ is ultra then the associated filter $\mathcal{F}(x.)$ is also ultra and if \mathcal{F} is ultra, every associated net is ultra.

Proof. Directly from theorem 2.5. \square

2.9 Separation

Definition 2.9. Space (Ω, τ) is said to be T_0 or **kolmogorov** if for every pair $(x, y) \in \Omega^2$, either there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ or $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Lemma 2.17. τ isn't T_0 iff there exist pair (x, y) , s.t:

1. $\mathcal{N}(x) = \mathcal{N}(y)$.
2. $\overline{\{x\}} = \overline{\{y\}}$.

Proof. 1 If every $N \in \mathcal{N}(x)$ contains y , then $N \in \mathcal{N}(y) \implies \mathcal{N}(x) \subset \mathcal{N}(y)$, thus $\mathcal{N}(x) = \mathcal{N}(y)$.

2 If some point $a \in \overline{\{x\}}$, then every $N \in \mathcal{N}(a)$ also is neighborhood of x and thus neighborhood of y , hence $a \in \overline{\{y\}}$. \square

Definition 2.10. Space (Ω, τ) is said to be T_1 or **Frechet** if for every pair $(x, y) \in \Omega^2$, there exist $N \in \mathcal{N}(x)$ s.t. $y \notin N$ and $N \in \mathcal{N}(y)$ s.t. $x \notin N$.

Theorem 2.20. *Following statements are equivalent:*

1. τ is T_1 .
2. Singetons are closed.
3. $\ker \mathcal{N}(x) = \{x\}$ holds for any $x \in \Omega$.

Proof. 1 \implies 2 If there exist a singeton $\{x\}$ not closed, there is $y \in \overline{\{x\}}$, hence every neighborhood of y contains x , contradiction.

2 \implies 3 Suppose $\ker \mathcal{N}(x)$ contains y diifer x , that implies any neighborhood of x contains y and contradict 2.

3 \implies 1 is straightforward. \square

Lemma 2.18. *Suppose (Ω, τ) with a finite base is T_1 , then Ω is finite and τ is discrete.*

Definition 2.11. A topology (Ω, τ) is T_2 , or **Hausdorff** or **separated** if every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $U \cap V = \emptyset$.

Theorem 2.21. *Following statements are equivalent:*

1. τ is T_2 .
2. Intersection of family of closed neighborhoods of x is x .
3. If a filter(net) converges to some point x , then $\text{Adh}(\mathcal{F}) = \{x\}$
4. Every net(filter) converges to at most one point.

Proof. 1 \implies 2 For any pair (x, y) , by definition, there is $y \notin \overline{U}$, hence intersection of family of closed neighborhoods of x can only contains x .

2 \implies 3 follows from a point adherent to a filter converges to x must be in every closed neighborhood of x .

3 \implies 4 is clearly.

4 \implies 1 If there is a net x . converges to both x and y , then $\mathcal{N}(x) \subset \mathcal{F}(x.)$ and $\mathcal{N}(y) \subset \mathcal{F}(x.)$, that is, U and V meets for any $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$. \square

Definition 2.12. Space (Ω, τ) is said to be $T_{2.5}$ or **Completely Hausdorff** if for every pair $(x, y) \in \Omega^2$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ s.t. $\overline{U} \cap \overline{V} = \emptyset$.

Two nonempty sets are caled **separated by open sets** if they are included in disjoint open sets, and they are **separated by continous functions** if there is continos f taking values in $[0, 1]$ and assign 0 on one set and 1 on the other.

Space (Ω, τ) are said to be **regular** if every singeton and any closed A disjoint from it can be separated by open sets.

Definition 2.13. Space (Ω, τ) is said to be T_3 if it's T_1 and regular.

Space (Ω, τ) are said to be **Completely regular** if every singleton and any closed A disjoint from it can be separated by continuous function.

Definition 2.14. Space (Ω, τ) is said to be $T_{3.5}$ or **Tychonoff space** if it's T_1 and completely regular.

Theorem 2.22 (Tychonoff's Embedding Theorem). *Space (Ω, τ) is $T_{3.5}$ iff it's homeomorphic to a subspace of $([0, 1]^n, \tau_{d_{t1}})$.*

Space (Ω, τ) is said to be **normal** if two disjoint closed subsets can be separated by open sets.

Definition 2.15. Space (Ω, τ) is said to be T_4 if it's normal and T_1 .

Theorem 2.23 (Urysohn's Lemma). *Following statements are equivalent:*

1. (Ω, τ) is normal.
2. For any $U \in \tau$ and any closed $A \subset U$, there is a $U' \in \tau$ s.t. $A \subset U'$ and $\overline{U'} \subset U$.
3. Every two disjoint closed subsets can be separated by continuous function.

Proof. 1 \implies 2 Apply normal property to A and U^c , there is a U' include A and V include U^c , as $U' \cap V = \emptyset \implies U' \subset V^c \implies \overline{U'} \subset V^c \subset U$.

2 \implies 3 Suppose A and B are two disjoint closed subset, apply 2 to A and $U_1 = B^c$ we have $A \subset U_0$ and $\overline{U_0} \subset U_1$. Apply again for $\overline{U_0}$ and U_1 to generate $U_0 \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$, repeat such process, that is, apply 2 to $\overline{U_{\frac{j}{2^k}}}$ and $U_{\frac{j+1}{2^k}}$ to generate $U_{\frac{2j+1}{2^{k+1}}}$. Finally, we construct an open strictly increasing sequence U_r where r is any dyadic rational in $[0, 1]$, i.e., $r \in DR \cap [0, 1]$.

Then define f as

$$f = \begin{cases} 1 & x \in B \\ \inf\{r : x \in U_r\} & x \in B^c \end{cases}$$

Then it's sufficient to show that f is continuous. Note subspace $[0, 1]$ of \mathbb{R} can be generated by collection of $[0, s)$ and $(t, 1]$ and

$$\begin{aligned} f^{-1}[0, s) &= \bigcup_{r \in DR \cap [0, s)} U_r \\ f^{-1}(t, 1] &= \bigcup_{r \in DR \cap (t, 1]} \overline{U_r}^c \end{aligned}$$

Then the claim follows from lemma 2.8.

3 \implies 1 By taking any disjoint open set A contains 0 and B contains 1 and looking $f^{-1}(A)$ and $f^{-1}(B)$. \square

Theorem 2.24 (Tietze's Extension Theorem). *Let (Ω, τ) be normal, F any closed subset and I any bounded closed interval of \mathbb{R} . Then any continuous $f : F \rightarrow I$ can be extended to $f' : \Omega \rightarrow I$ and remain continuous.*

Proof. Suppose $I = [-1, 1]$, then $A = f^{-1}[-1, -\frac{1}{3}]$ and $f^{-1}[\frac{1}{3}, 1]$ are disjoint and closed. By Urysohn's Lemma, there is $g : \Omega \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ s.t. $g(A) = \{-\frac{1}{3}\}$ and $g(B) = \frac{1}{3}$. Set $f_0 = f, g_0 = g, f_1 = f - g|_F$. Then we can show that $|f_1|$ is bounded by $\frac{2}{3}$.

Repeat such process, we have series of

$$\begin{aligned} f_n : F &\rightarrow [-(\frac{2}{3})^n, (\frac{2}{3})^n] \\ g_n : E &\rightarrow [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n] \\ f_{n+1} &= (f_n - g_n)|_F \end{aligned}$$

Then we show that $g = \sum_{i=0}^{\infty} g_i$ is the extension of f . That is g is continuous and $f = g$ in F . Note for any x

$$|\sum_{i=m}^n g_i(x)| \leq \sum_{i=m}^n |g_i(x)| \leq \sum_{i=m}^n \frac{1}{3}(\frac{2}{3})^i \leq (\frac{2}{3})^m \rightarrow 0$$

Thus $\{\sum_{i=0}^n g_i\}_{n=0}^{\infty}$ converges uniformly by Cauchy's criterion, followed by g is continuous. And $f = g$ on F follows from

$$|f(x) - \sum_{i=0}^n g_i(x)| = |f_0(x) - \sum_{i=0}^n g_i(x)| = |f_1(x) - \sum_{i=1}^n g_i(x)| = |f_{n+1}(x)| \leq (\frac{2}{3})^{n+1} \rightarrow 0$$

□

2.10 Compactness

A **cover** of a set K is collection of sets whose union includes K . A **subcover** is subcollection of a cover and also covers K . K is **compact** if every open cover has a finite subcover and called **relatively compact** if its closure is compact.

A topology (Ω, τ) is **compact** if Ω is compact and **locally compact** if every point has a compact neighborhood. A subset $A \in \Omega$.

Compactness is a "topological" property. That is, subset compactness in a subspace iff it's also compact in full space.

Theorem 2.25. *Let (Ω, τ) be a space, following are equivalent.*

1. (Ω, τ) is compact.
2. Every filter(net) has at least one adherent point.
3. Every ultrafilter(ultranet) converges.
4. $\ker \mathcal{F} \neq \emptyset$ For every collection \mathcal{F} of closed sets having FIP.

Proof. 1 \implies 2 Suppose filter \mathcal{F} , then

$$\{\overline{F} : F \in \mathcal{F}\}$$

enjoy finite intersection property by definition, then \overline{F} has at least one adherent point since $\ker\{\overline{F} : F \in \mathcal{F}\} = \overline{\mathcal{F}} \neq \emptyset$ by theorem ??.

2 \implies 3 Clearly by corollary 2.3.

3 \implies 1 Suppose \mathcal{A} a family of closed subsets with finite intersection property. Then the filter generated by \mathcal{A} has an ultrafilter with a limit point x . Note x is also adherent to \mathcal{U} and thus adherent to \mathcal{F} , followed by $x \in A$ for any $A \in \mathcal{A}$, hence $\ker \mathcal{A} \supset \{x\}$. Then the claim follows from theorem ??.

4 \iff 1 Taking contrapositive:

$$\neg \ker \mathcal{F} \neq \emptyset \equiv \ker \mathcal{F} = \emptyset \equiv \bigcup_{F \in \mathcal{F}} F^c = \Omega$$

and

$$\neg \forall \bigcap_i^n F_i = \emptyset \equiv \exists \bigcup_i^n F_i^c = \Omega$$

note that's precisely the definition of compactness. \square

Theorem 2.26. *Let (Ω, τ) be Hausdorff, then every compact subset and disjoint singleton can be separated by open sets. In particular, compact subset is closed.*

Proof. Suppose $F \subset \Omega$ is compact, for any $x \in \Omega$ not in F , by Hausdorff, there is $x \notin U_y$ and $y \notin V_y$. Then $\bigcup_{y \in F} U_y$ cover F , there is subcover $U = \bigcup_i^n U_{y_i}$ and $V = \bigcup_i^n V_{y_i}$ selected from the same family separated F and $\{x\}$. \square

Theorem 2.27. *Closed subset is compact in compact topological space.*

Proof. Note any open cover of F plus F^c become a open cover of Ω . \square

Theorem 2.28. *Every compact Hausdorff space is normal.*

Proof. Suppose A and B are closed and thus compact by theorem 2.27. For any point $x \in A$, there exist disjoint $V_x \supset B$ and U_x by theorem 2.26. Note $\bigcup_{x \in A} U_x$ cover A , there exist subcover $U = \bigcup_i^n U_{x_i} \supset A$ and $V = \bigcap_i^n V_{x_i} \supset B$ separated A and B . \square

Theorem 2.29. *Suppose $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continuous, then $f(A)$ is compact if A is compact.*

Proof. For any open cover of $f(A)$:

$$\bigcup G_i \supset f(A) \implies f^{-1}(\bigcup G_i) = \bigcup f^{-1}(G_i) \supset f^{-1}f(A) \supset A$$

Thus there exist subcover s.t.

$$\bigcup_1^n f^{-1}(G_i) = f^{-1}(\bigcup_1^n G_i) \supset A \implies \bigcup_1^n G_i \supset f f^{-1}(\bigcup_1^n G_i) \supset f(A)$$

which shows that $f(A)$ is compact. \square

As consequence:

Corollary 2.4 (Extreme value theorem). *A continuous real valued function defined on a compact space achieves its maximum and minimum values.*

Theorem 2.30. *Let (Ω, τ) be compact and (Ω', τ') be hausdorff and $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continuous bijection. Then f is homeomorphism.*

Proof. It's sufficient to show that $f(F)$ is closed as lemma 2.9. Note F is closed and thus compact as theorem 2.27 then $f(F)$ is compact as theorem 2.29 and thus closed by theorem 2.26. \square

A subset A of a topological space is **sequentially compact** if every sequence in A has a subsequence converging to an element of A . A topological space is sequentially compact if itself is a sequentially compact set.

2.11 Semicontinuous

$f : \Omega \rightarrow \mathbb{R}^*$ is

- **lower semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \leq c\}$ is closed.
- **upper semicontinuous** if for any $c \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \geq c\}$ is closed.

Clearly f is lower semicontinuous iff $-f$ is upper and vice versa. Also, f is continuous iff it's both upper and lower semicontinuous.

Lemma 2.19. *Suppose $\{f_i\}_{i \in I}$ is family of lower(upper) semicontinuous function then $\sup f_i(\inf f_i)$ is lower(upper) semicontinuous.*

Proof. Note

$$\{x \in \Omega : \sup f_i(x) \leq c\} = \bigcap_{i \in I} \{x \in \Omega : f_i(x) \leq c\}$$

is closed. □

Lemma 2.20. $f : \Omega \rightarrow \mathbb{R}^*$ is

- **lower semicontinuous** iff for any net

$$x. \rightarrow x \implies \liminf f(x.) \geq f(x)$$

- **upper semicontinuous** iff for any net

$$x. \rightarrow x \implies \limsup f(x.) \leq f(x)$$

Proof. Suppos f is lower semicontinuous and $x. \rightarrow x$. For any $c < f(x)$, then $G = \{\omega \in \Omega : f(\omega) > c\}$ is open and thus $x.$ eventually in, that is $x.c$ eventually and thus $\liminf f(x.) \geq c$. This implies that $\liminf f(x.) \geq f(x)$.

Conversely, for any $c \in \mathbb{R}$, consider $F = \{\omega \in \Omega : f(\omega) \leq c\}$. Then we show that F is closed. Suppos $x.$ is nets in F and converges to some $x \in \Omega$. Then $c \geq \liminf f(x.) \geq f(x)$ thus x in F and thus F is closed. □

Then we can gennerlizes Weierstrass' Theorem in corollary 2.4.

Theorem 2.31. $f : \Omega \rightarrow \mathbb{R}^*$ on a compact set attains a minimum(maximum) value and set of minima(maxima) is compact if it's lower(upper) semicontinuous.

Proof. Suppose X is compact and f is lower semicontinuous, then for every $c \in f(X)$, $F_c = \{x \in X : f(x) \leq c\}$ is closed and $\{F_c : c \in f(X)\}$ has FIP clearly. Note X is compact, $\ker\{F_c : c \in f(X)\}$ is nonempty by 2.25. That is just the set of minimas and it's compact since it's closed. □

2.12 Comparing topologies

We list some useful properties when comparing topologies, some of them have been mentioned before and proof omitted.

Lemma 2.21. *Suppose τ' and τ are two topologies on Ω , then the following are equivalent.*

1. $\tau' \subset \tau$
2. Identity mapping $I : x \mapsto x$ from (Ω, τ) to (Ω, τ') is continuous.
3. τ' closed set is closed in τ .
4. $x \xrightarrow{\tau} x \implies x \xrightarrow{\tau'} x$
5. $Cl_{\tau}(A) \subset Cl_{\tau'}(A)$

Lemma 2.22. *Suppose $\tau' \subset \tau$, then*

1. Every τ compact set is τ' compact.
2. Every τ' continuous function is τ continuous.
3. Every τ dense set is τ' dense.

2.13 Weak topology

Suppose $\{(Y_i, \tau_i)\}_{i \in I}$ a family of topological space and $f_i : X \rightarrow Y_{i \in I}$. Let \mathcal{F} be the set of all the topologies s.t. f_i is continuous for all i . We call $\cap \mathcal{F}$, i.e., the coarsest topology the **induced topology** or **weak topology** or **initial topology** on X by $\{f_i\}_{i \in I}$. The topology induced by $\{f_i\}_{i \in I}$ is generated by

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \tau_i\}$$

or

$$\mathcal{S} = \{f_i^{-1}(G_i) : G_i \in \mathcal{S}_i\}$$

where \mathcal{S}_i is a subbase for τ_i .

Lemma 2.23. *A net $x. \rightarrow x$ in the weak topology iff $f_i(x.) \rightarrow f_i(x)$ for each i .*

Proof. \implies is immediately. Conversely, noting sets of the form $\bigcap_1^n f_i^{-1}(V_i)$ consist a neighborhood base. \square

Theorem 2.32. *g is (τ', τ) continuous iff $f_i \circ g$ is continuous for each f_i . Where τ is $\tau(S)$ in above theorem.*

Proof. \implies is immediately. \Leftarrow , suppose $G \in \tau$, by above theorem, this implies

$$G = \cup_I \cap_F X = \cup_I \cap_F f_i^{-1}(G_i)$$

thus $g^{-1}(G)$ is open since $f \circ g^{-1}$ is continuous and thus $g^{-1}(G) = \cup_I \cap_F g^{-1}f^{-1}(G) = \cup_I \cap_F (f \circ g)^{-1}(G)$. \square

If the family \mathcal{F} consists of real function on X , the weak topology is denoted $\sigma(X, \mathcal{F})$. A subbase for $\sigma(X, \mathcal{F})$ consist of

$$U(f, x, \epsilon) = f^{-1}(B(f(x), \epsilon)) = \{y \in X : |f(y) - f(x)| < \epsilon\}$$

where $f \in \mathcal{F}, x \in X, \epsilon > 0$. \mathcal{F} is said **total** if $\forall f \in \mathcal{F}, f(x) = f(y) \implies x = y$. $\sigma(X, \mathcal{F})$ is Hausdorff iff \mathcal{F} is total.

Lemma 2.24. *Let A be a subset, then*

$$(A, \sigma(A, \mathcal{F}|_A)) = (A, \sigma(X, \mathcal{F})|_A)$$

Proof. Nets converges in $(A, \sigma(X, \mathcal{F})|_A)$ also converges in $(X, \sigma(X, \mathcal{F}))$, that is $\forall f, f_i(x.) \rightarrow x$. and thus the same as nets converges in $\sigma(A, \mathcal{F}|_A)$. That implies identical mapping is a homeomorphism since $x. \rightarrow x \iff I(x.) \rightarrow I(x)$. \square

The weak topology generated by $C(X)$ is also generated by $C_b(X)$ by noting for any $f \in C(X)$,

$$g(y) = \min\{f(x) + \epsilon, \max\{f(x) - \epsilon, f(y)\}\}$$

is boundeb by $B(f(x), \epsilon)$ and $U(g, x, \epsilon) = U(f, x, \epsilon)$.

Theorem 2.33. *$(X,)$ is completely regular iff $\tau = \sigma(X, C(X))$*

Suppose $\tau = \sigma(X, \mathcal{F})$ and is compelely regular, then we claim that $\mathcal{F} = C(X)$.

2.14 Product topology

Let $((\Omega_i, \tau_i))_{i \in I}$ be family of topological spaces, let $\Omega = \prod_{i \in I} \Omega_i$ and π_i be projection mappings from Ω to Ω_i . The topology τ induced by $(\pi_i)_{i \in I}$ is called **product topology** on Ω and denoted by $\prod_{i \in I} \tau_i$. (Ω, τ) is called **topological product**. A subbase of this topology is all the sets of the form $\prod_{i \in I} X_i$ where $X_i = \Omega_i$ for all i but one is arbitrary open set, or equally, sets of the form $\pi_i^{-1}(U_i)$ where $U_i \in \tau_i$.

Lemma 2.25. *Suppose $G \in \prod \tau_i$, then $\pi_i(G) = \Omega_i$ except a finite set in I .*

Proof. By definition,

$$G = \bigcup_I \bigcap_F \left(\prod_{i \in I} X_i \right)$$

where $X_i = \Omega_i$ for all i but one. Note there is a finitely intersection, that is

$$G = \bigcup_I \left(\prod_{i \in I} X_i \right)$$

where $X_i = \Omega_i$ for all i but finite exception. \square

Thus, $\{(x_i^\alpha)\}_{i \in I}$ in X converges to some $(x_i)_{i \in I}$ iff its every components converges to the components respectively. A function is called **jointly continuous** if it's continuous w.r.t the product topology.

Theorem 2.34 (Closed Graph Theorem). *Function $f : (X, \tau) \rightarrow (Y, \tau)$ where Y is compact Hausdorff is continuous iff its graph $\text{Gr } f$ is closed.*

Proof. \Rightarrow . For any net $(x., y.) \rightarrow (x, y)$, we show that $(x, y) \in \text{Gr } f$. Note $f(x.) = y. \rightarrow y$, also, $f(x.) \rightarrow f(x)$ by continuity. It follows by $f(x) = y$ since Hausdorff and we finished.

\Leftarrow . Since Y is compact and Hausdorff, $f(x.)$ converges to precisely one point and denoted as y . As $\text{Gr } f$ is closed, $y = f(x)$ and hence f is continuous. \square

Suppose A_i is subset of each i , then

$$\text{Cl}_\tau \left(\prod A_i \right) = \prod \left(\text{Cl}_{\tau_i} (A_i) \right)$$

Thus we have an alternative definition of semicontinuous:

$f : X \rightarrow \mathbb{R}^*$ is

- lower semicontinuous iff its epigraph $\{(x, c) : c \geq f(x)\}$ is closed.
- upper semicontinuous iff its hypograph $\{(x, c) : c \leq f(x)\}$ is closed.

Theorem 2.35 (Tychonoff Product Theorem). *The product topology of a family of topologies $\tau = \prod_{i \in I} \tau_i$ is compact iff τ_i is compact for every $i \in I$.*

Proof. \Rightarrow is clearly as projection is continuous.

\Leftarrow , suppose \mathcal{U} is ultrafilter in τ , then $\pi_i(\mathcal{U})$ is ultra base and thus coverges to some point, say x_i , then we claim that $\mathcal{U} \rightarrow x = (x_i)_{i \in I}$. Suppose V any neighborhood of x , there is

$$a \in \bigcap_{i \in J} \pi_i^{-1}(X_i) \subset V$$

where X_i is neighborhood of x_i and thus belong to $\pi_i(\mathcal{U})^\uparrow$, that implies there is $U \in \mathcal{U}$ s.t. $\pi_i(U) \subset X_i$, note $U \subset \pi_i^{-1}\pi_i(U) \subset \pi_i^{-1}(X_i)$, then $\pi_i^{-1}(X_i) \in \mathcal{U}$ and thus $V \in \mathcal{U}$. It followed by x is adherent to \mathcal{U} and thus $\mathcal{U} \rightarrow x$ as \mathcal{U} is ultra. \square

As consequence, we have

Theorem 2.36. *In the same notations, let K_i be compact for each i , G is open in τ and including $\prod_{i \in I} K_i$, then there exist basic open set sandwich by them.*

2.14.1 Coinduced topology

In the same notations, let K_i be compact for each i , G is open in τ and including $\prod_{i \in I} K_i$, then there exist basic open set sandwich by them.

Suppose $\{(\Omega_i, \mathcal{T}_i)\}_{i \in I}$ a family of topological space and $\{f_i : (\Omega_i, \mathcal{T}_i) \rightarrow (\Omega, \tau)\}_{i \in I}$. Let A be the set of all the topologies s.t. f_i is continuous for all i . We call the finest of A **topology coinduced** on Ω by $\{(f_i)\}_{i \in I}$.

Let R an equivalence relation on Ω , $\eta : \Omega \rightarrow \Omega/R$ the canonical surjection. The coinduced topology on Ω/R by η is denoted by τ/R and $(\Omega/R, \tau/R)$ is the quotient space w.r.t R .

2.15 Convergence

If \mathcal{F} is collection of functions on X , X can be seen as functions on \mathcal{F} by $e_x(f) = f(x)$ for each $x \in X$, such functions are called **evaluation functional**.

The product topology on \mathbb{R}^X is also called **topology of pointwise convergence** on X because a net $f. \rightarrow f$ iff $e_x(f.) \rightarrow e_x(f) \iff f.(x) \rightarrow f(x)$ for each $x \in X$.

There also exist induced topology $\sigma(\mathcal{F}, X)$ on \mathcal{F} , which is identical to the subspace $\mathbb{R}^X|_{\mathcal{F}}$ endowed the product topology. Formally

$$\sigma(\mathcal{F}, X) = \sigma(\mathbb{R}^X, X)|_{\mathcal{F}}$$

Lemma 2.26. *If \mathcal{F} is total, the function*

$$x \mapsto e_x : (X, \sigma(X, \mathcal{F})) \rightarrow (\mathbb{R}^{\mathcal{F}}, \sigma(\mathbb{R}^{\mathcal{F}}, \mathcal{F}))$$

is injective and thus an embedding.

Proof. It's remain to show the continuity.

$$\begin{aligned} x. \rightarrow x &\iff \forall f \in \mathcal{F}, f(x.) \rightarrow f(x) \\ &\iff \forall f \in \mathcal{F}, e_f(e_x.) \rightarrow e_f(e_x) \\ &\iff e_x. \rightarrow e_x \end{aligned}$$

□

By Tychonoff theorem 2.35, \mathcal{F} is compact iff $\forall x \in X$, $\{f(x)\}_{f \in \mathcal{F}}$ it's closed and pointwise bounded by borel theorem.

Definition 2.16. A net $f.$ converges uniformly to $f \in \mathbb{R}^X$ iff $|f.(x) - f(x)| < \epsilon$ eventually for each $x \in X$ after some f_α for any ϵ .

Theorem 2.37. *The uniform limit of a net is continuous.*

Proof. Suppose $f. \rightarrow f$ uniformly, then for any $x \in X$, for any $\alpha > \alpha_0$

$$|f_\alpha(x) - f(x)| < \epsilon$$

as f_α is continuous, for any $x. \rightarrow x$, for any $\lambda > \lambda_0$

$$|f_\alpha(x_\lambda) - f_\alpha(x)| < \epsilon$$

also, there is

$$|f_\alpha(x_\lambda) - f(x_\lambda)| < \epsilon$$

Hence, we have

$$|f(x_\lambda) - f(x)| < 3\epsilon$$

Thus, $f(x.) \rightarrow f$ and continuity follows.

□

Theorem 2.38 (Dini's Theorem). *If $f.$ converges monotonically to $f \in \mathbb{R}^X$ pointwise, then the net converges uniformly.*

If $f. \rightarrow f$, $f.(x) \rightarrow f(x)$ for each x and thus $f.(x)$ eventually in $B(f(x), \epsilon)$ after, say, f_{α_x} . Note

$$X = \bigcup_{x \in X} f^{-1}(B(f(x), \epsilon)) = \bigcup_{x \in J} f^{-1}(B(f(x), \epsilon))$$