CONVERGENCE

Xie zejian

xiezej@gmail.com

December 1, 2020

In the following paragraph, Ω is a space and τ is a topology on Ω, τ is a filter on Ω.

1 Filter

A **filter** is a non-empty collection $\mathcal{F}$ of subset in Ω s.t.

1. $A \in \mathcal{F}, A \subset \mathcal{B} \implies \mathcal{B} \in \mathcal{F}$
2. Closed under finite intersection.
3. $\emptyset \notin \mathcal{F}$

Note the definition of $\mathcal{F}$ is independent with topology τ.
A collection $\mathcal{B}$ of subset in Ω is a **base** for the fliter if

1. $\mathcal{B} \subset \mathcal{F}$
2. $\forall N \in \mathcal{F}, \exists W \in \mathcal{B} \ni W \subset N$

We say $\mathcal{B}$ generates $\mathcal{F}$, where

$$\mathcal{F} = \mathcal{B}^{\uparrow} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

For example,

1. Suppose A is any non-empty subset of Ω, all the subsets of Ω include A is a filter while $\{A\}$ is a base for it.
2. Suppose $x \in \Omega$ then

   $$\mathcal{N}(x) = \{\text{All neighbourhoods of } x\}$$

   is a filter on Ω, that is, **neighbourhood filter**, while each neighborhood base is a base for this filter. Note

   $$\tau(x) = \{X \in \tau : x \in X\}$$

   is a base for $\mathcal{N}(x)$ and thus $\mathcal{N}(x) = \tau(x)^{\uparrow}$.
3. Suppose Ω is infinite, the collection of all cofinite subsets(subset s with finite complement) is a filter on Ω, such filter is called **Frechet filter**.

To assert a collection is a base, we have

Theorem 1 Let $\mathcal{B}$ be a collection of nonempty subsets. Then $\mathcal{B}$ is a filter base, that is, $\mathcal{B}$ may generates a filter iff 1. The intersection of each finite family of sets in $\mathcal{B}$ inclueds a set in $\mathcal{B}$ 2. $\mathcal{B}$ is non-empty and $\notin \mathcal{B}$.
Proof

$$\mathcal{F} = \{X \in \mathcal{P}(\Omega) : \exists A \in \mathcal{B} \ni X \supset A\}$$

$\mathcal{F}$ is the filter generated by $\mathcal{B}$. ■

Let A be a collection of subsets of nonempty subsets, then construct A' by taking all finite intersection, if $\notin A'$, it's a base for some filter $\mathcal{F}$, we call $\mathcal{F}$ the filter generated by A.

Suppose $\mathcal{F}$ and G be filters on Ω. Then

$$X \in F \cap G \iff \exists P \in F \text{ and } Q \in G \ni X = P \cup G$$

$$X \in \{\text{finite intersection in } F \cup G\} \iff \exists P \in F \text{ and } Q \in G \ni X = P \cap Q$$

Suppose R is an order relation on Ω, then Ω is said to be **inductivelt ordered** by R if every totally ordered subset has an **supremum**.

Zorn's Lemma states that every inductively ordered set has a maximal element.

Theorem 2 The set of all filters on Ω is inductively ordered by inclusion.
Proof Suppose a collection A of filters is totally ordered, it's a base by theorem 1 since it's finite intersection is just a fliter in A with totally ordered. Then the supremum is just the fliter generates by A. ■

By Zorn's lemma, the set of all filters has maximal filters and we call such fliters **ultrafilters**.

Theorem 3 Let $\mathcal{F}$ be an ultrafilter on Ω, if A and B are subsets of Ω s.t. $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.
Proof If $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$, suppose $\mathcal{F}' = \{X : A \cup X \in \mathcal{F}\}$, and easy to verify $\mathcal{F}' \supset \mathcal{F}$, a contradiction. ■

To assert a filter is ultra, we have:

Theorem 4 Let A be a collection of subsets and $\mathcal{F}$ the filter generates by A. If

$$\forall X \subset \Omega, \text{either } X \in A \text{ or } X^c \in A$$

then A is an ultrafilter on Ω.
Proof Suppose $\mathcal{F}'$ is an ultrafilter include $\mathcal{F}$, we have $\mathcal{F}' \supset A$ clearly. Consider any $X \in \mathcal{F}'$, we claim that $X \in A$ since if $X^c \in A$ then $X^c \in \mathcal{F}'$ as $\mathcal{F}' \supset \mathcal{F} \supset A$ and $X \cap X^c = \emptyset \in \mathcal{F}'$ results in a contradiction. It follows that $A \supset \mathcal{F}'$ and thus $A = \mathcal{F}'$. ■

The kernel of ultrafilter is at most a singleton, if a filter has singleton kernel, it's ultra.

Theorem 5 Every filter $\mathcal{F}$ is the intersection of all the ultrafilter which include $\mathcal{F}$.
Proof We claim that

$$\mathcal{F} = \cap\{\text{ultrafilter generates by } \{x\} : x \in \cap\mathcal{F}\}$$

■

Theorem 6 Let f be a mapping from Ω to Ω' and $\mathcal{B}$ a base for a fliter $\mathcal{F}$ on Ω. Then $f(\mathcal{B}) = \{f(X)\}_{X \in B}$ is also a base on Ω'. Moreover, if $\mathcal{F}$ is ultra then $f(\mathcal{B})$ also generates an ultrafilter.
Proof First assertion is straightforward and the second follows from $\mathcal{B}$ is collection of supset for some $\{x\}$, then $f(\mathcal{B})$ generates the fliter that generates by $\{f(x)\}$. ■

Theorem 7 In the same situation as previous theorem. If $\mathcal{B}'$ is a base on Ω', then $f^{-1}(\mathcal{B}')$ is a base on Ω iff every set in $\mathcal{B}'$ meets $f(\Omega)$

Proof We have

$$\Omega \in f^{-1}(\mathcal{B}') \implies f(\Omega) \subset X'$$

for some $X' \in f^{-1}(\mathcal{B}')$, by definition, $\implies$ is immediately. For $\impliedby$, suppose any finite family $X_i \in \mathcal{B}'$, then

$$\bigcap_{i=1} f^{-1}(X_i) = f^{-1}(\bigcap_i X_i) \in f^{-1}(\mathcal{B}')$$

Then the claim follows from theorem 1. ■

2 Limit

A point $x \in \Omega$ is said to be a **limit** or a **limit point** of the fliter $\mathcal{F}$ and $\mathcal{F}$ is said to **converge** to x, or $\mathcal{F} \to x$, if the neighborhood filter $\mathcal{N}(x) \subset \mathcal{F}$. For filter base $\mathcal{B}$, we define on the filter generated by $\mathcal{B}$, that is, if $\mathcal{N}(x) \subset \mathcal{B}^{\uparrow}$.

This implies a equivalent definition of finer topology:

$$\tau \supset \tau' \iff \mathcal{N}_\tau(x) \supset \mathcal{N}_{\tau'}(x) \iff \mathcal{F} \to a \implies \mathcal{F}' \to a$$

also, an equivalent definition of continuity as follows:

$f : (\Omega, \tau) \to (\Omega', \tau')$ is continous at x iff

$$\forall \mathcal{F} \to x, f(\mathcal{F}) \to f(x)$$

Proof By definition, $f(\mathcal{F}) \to f(x)$ if

$$\mathcal{N}(f(x)) \subset f(\mathcal{F})^{\uparrow}$$

That is, for any neighbourhood $N' \in \mathcal{N}(f(x))$, there exist some $A \in \mathcal{F}$ s.t. $f(A) \subset N'$, as $\mathcal{N}(x) \subset \mathcal{F}$ and f is continous at x, such A is always exists. Conversely, take $\mathcal{F} = \mathcal{N}(x)$ then the claim is follows ■

A point $x \in \Omega$ is said to be an **adherent point** of $\mathcal{F}$ if x is an adherent point of every set in $\mathcal{F}$. The **adherence** of $\mathcal{F}$, $\text{Adh}_\tau(\mathcal{F})$ or $\overline{\mathcal{F}}$ is the set of all adherent points, thus

$$\overline{\mathcal{F}} = \bigcap_{X \in F} \overline{X}$$

Define similarly on filter base $\mathcal{B}$ by the filter generated. By definition, we have

$$\overline{\mathcal{B}} = \bigcap_{X \in B} \overline{X}$$

Suppose A be a subset of Ω, then $x \in \overline{A}$ iff there is a filter $\mathcal{F}$ s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.

Proof If $x \in \overline{A}$ then $\mathcal{F} = N(x) \cup \{A\}$ generates a fliter as required. Conversely,

$$N(x) \in \mathcal{F} \implies N \cap A \neq \forall N \in N(x)$$

Then the calim follows. ■

Theorem 8 Suppose $BN(x)$ a neighbourhood base of x, then

1. $\mathcal{B}$ converges to x iff every set in $BN(x)$ includes a set in $\mathcal{B}$.
2. $x \in \overline{\mathcal{B}}$ iff every set in $BN(x)$ meets every set in $\mathcal{B}$.

Proof Directly from definition. ■

As consequence, we have

Corollary 1 x is adherent to a filter $\mathcal{F}$ iff there is $\mathcal{F}' \supset \mathcal{F}$ and converges to x
Proof $\implies$ follows from taking $\mathcal{F} = BN(x)$. Conversely, $\forall N \in BN(x)$, we have $X' \subset N$ for some $X' \in \mathcal{F}'$, thus for any $X \in \mathcal{F}$, $N \cap X \subset X' \cap X \neq \emptyset$ as $X', X \in \mathcal{F}'$.
Corollary 2 Every limit point of $\mathcal{F}$ is adherent to $\mathcal{F}$
Proof Clearly holds by applying 1 and 21.
Corollary 3 Every adherent point of an ultra-filter is a limit point of it.
Proof Clearly as kernal of ultrafilter is a one point set. ■

Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$, a point $x' \in \Omega'$ is called

1. a **limit point** of f relative to $\mathcal{F}$ if $f(\mathcal{F}) \to x$.
2. an **adherent point** of f relative $\mathcal{F}$ if it's adherent point of $f(\mathcal{F})$.

Theorem 9

1. x' is a limit point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, we have $f^{-1}(N') \in \mathcal{F}$.
2. x' is an adherent point of f relative to $\mathcal{F}$ iff for any τ' neighbourhood $N' \in \mathcal{N}(x')$, it meets $f(X)$ for any $X \in \mathcal{F}$.

Proof x' is limit is equivalent to

$$\mathcal{N}(x') \subset f(\mathcal{F})^{\uparrow}$$

That is, there exist some $A = f(X) \subset N'$ for any N', followed by $X \subset f^{-1}f(X) \subset f^{-1}(N')$, then the claim follows from the definition of filter.
By theorem 8, x' is adherent to $f(\mathcal{F})$ iff

$$\forall N' \in BN(x'), \forall X \in \mathcal{F}, f(X) \cap N' \neq \emptyset$$

note for any $N' \in N'(x')$, there exist $N' \in BN(x') \ni N' \subset N'$, thus $f(X) \cap N' \neq \emptyset$ also holds. Conversely, making use of $BN(x') \subset N'(x')$. ■

For example, suppose $f : (\mathbb{N}, \tau) \to (\Omega', \tau')$ and $\mathcal{F}$ the frechet filter on $\mathbb{N}$. Then x' is limit of f relative to $\mathcal{F}$ iff for all $N' \in N'(x')$, $f^{-1}(N') \in \mathcal{F} \iff f^{-1}(N')^c \subset [0, k] \iff f^{-1}(N') \supset \{n \in \mathbb{N} : n \geq k\}$ for some k, that is, $f(n) \in N'$ for any $n \geq k$.

Theorem 10 Suppose $f : (\Omega, \tau) \to (\Omega', \tau')$ and let $\mathcal{F} = \mathcal{N}(x)$. By theorm 9, x' is limit of f relative to $\mathcal{N}(x)$ iff for all $N' \in \mathcal{N}(x')$, $f^{-1}(N') \in \mathcal{N}(x) \iff N \subset f^{-1}(N') \iff f(N) \subset N'$ for some $N \in \mathcal{N}(x)$. That is, iff $x' = f(x)$, f is continous at x. Such limit points also called limit points of f at x.
Proof Proved in statements. ■

3 Net

In the following paragraph, $(D, \preceq)$ is a ordered set. $x.(\nu)$ a net in Ω with domain D.

$(D, \preceq)$ is called a **directed set** if every couple $\{x, y\}$ in which has an upper bound. Let $(D, \preceq)$ be a directed set, $\nu : D \to \Omega$ is called a **net** in Ω with domain D. We often write ν as $x.$.

Suppose A a subset of Ω, we say $x.$ **eventually in** A if there exist some $k \in D$ s.t. $x_n \in A$ for all $n \succeq k$. And we say ν is **frequently** in A if for all $n \in D$, there exist an $n' \succeq n$ s.t. $x_{n'} \in A$.

Lemma If $x.$ not frequently in A, then $x.$ eventually in A^c. Thus, for any $X \in \Omega$, $x.$ frequently in either X or X^c.
Proof Clearly from definition. ■

A subset B of D is called **cofinal** if for any $a \in D$, there exist $b \in B$ s.t. $a \preceq b$. A map $f : D \to A$ is **final** if $f(D)$ is cofinal of A.

Let $x.$ and $x.'$ are two nets in Ω with domains D and D' respectively. We say that $x.'$ is a **subnet** of $x.$ if there exists a final mapping $\varphi : D' \to D$ s.t. $x'_\alpha = x_{\varphi(\alpha)}$.

Theorem 11 Let $\mathcal{A}$ be a collection of subsets that $x.$ is frequently in. If $\mathcal{A}$ is closed under finite intersection, then there exists a subnet $x'.$ of $x.$ and $x.'$ eventually in every member of $\mathcal{A}$
Proof (TODO). ■

Let

$$\mathcal{F}(x.) = \{X \in \mathcal{P}(\Omega) : x. \text{ is eventually in } X\}$$

Then $\mathcal{F}(x.)$ is a filter and we call it the **filter associated with the net** $x.$.

Motivated by the definition of filter that filter is closed under pairwise intersection, let $X \preceq Y \iff X \supset Y$, then any mapping $\nu : \mathcal{F} \to \Omega$ s.t. $\nu(X) \in X$ is a **net associated with the filter** $\mathcal{F}$.

By definition, we claim that $\mathcal{F}$ is the associated filter of every associated net and $x.$ is an associted net of the associated fiter.

Suppose $x \in \Omega$, then $x.$ is said **converge** to x, or $x. \to x$ if $x.$ eventually in N for all $N \in \mathcal{N}(x)$, i.e., $\mathcal{N}(x) \subset \mathcal{F}(x.)$. The point x is adherent to $x.$ if $x.$ frequently in N for all $N \in \mathcal{N}(x)$.

Suppose $x.'$ is subnet of $x.$, we have 1. $x. \to x \implies x.' \to x$ 2. x adherent to $x.' \implies x$ adherent to $x.$.
Proof Clearly from the definition. ■

Theorem 12 A point x is adherent to $x.$ iff there is a subnet converges to x.
Proof $\implies$ is clear by theorem 11. Conversely, suppose a is not adherent to x, there exist a neighborhood N that $x.$ not frequently in, i.e., exist k s.t. $x_n \notin N$ for any $n \geq k$, thus there is no subnet eventually in N.

Theorem 13 Filter $\mathcal{F} \to x$ iff $x. \to x$ for any $x.$ associated with $\mathcal{F}$.
Proof Note

$$\forall N \in \mathcal{N}(x), x. \text{ eventually in } N \iff \mathcal{N}(x) \subset \mathcal{F}(x.)$$

Then is sufficient to show that $\mathcal{F}(x.) = \mathcal{F}$. It's follows from for any $X \in \mathcal{F}$, $x.$ eventually in X.

Theorem 14

$$x. \to x \iff \mathcal{F}(x.) \to x$$

Proof Both side is equivalent to $\mathcal{N}(x) \subset \mathcal{F}(x.)$

Theorem 15 Suppose $f : (\Omega, \tau) \to (\Omega, \tau)$, then f is continous at x iff $\forall x. \to x$, $f(x.) \to f(x)$.
Proof By theorem 13,14 and the equivalent definition stated before. ■

A net $x.$ is called **ultranet** or **universal net** if for all $X \in \Omega$, we have either $x.$ eventually in X or $x.$ eventually in X^c. Clearly, subnet of ultranet is ultra and

Every net has a ultra subnet.
Proof Consider collection of $\mathcal{Q}$ s.t. $x.$ is frequently in every member and closed under finite intersection. By Zorn's Lemma, there is a maximal $\mathcal{Q}_0$. By theorem 11, $x.$ has a subnet $x.'$ which eventually in every member of $\mathcal{Q}_0$. We claim that this subnet is ultra since, $\mathcal{Q}_0$ is maximal and thus either $X \in \mathcal{Q}_0$ or $X^c \in \mathcal{Q}_0$. ■

If $x.$ is ultra then the associated filter $\mathcal{F}(x.)$ is also ultra and if $\mathcal{F}$ is ultra, every associated net is ultra.
Proof Directly from Theorem 4. ■