Random-Nikodym Theorem

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Contents

Signed measures 1

Signed measures

A finite set function $\mu : \mathcal{A} \to \mathbb{R}$ is called as a **signed measure** if it's σ additive.

Given a signed measure, its **total variance** is given by

$$|\mu|(E) = \sup_P \{\sum_j |\mu(E_j)|\}$$

for all partitions P of E. One can check total variance is also a finite measure.

Following proposition yield lots of signed measure:

> Let $(X, \mathcal{A}, \mu)$ be a measure space and $\omega \in L^1(\mu)$. Then define
>
> $$\nu(A) = \int_A \omega d\mu$$
>
> Such ν is a signed measure and $\mu \ll \nu$. For all measurable function $f : X \to [0, \infty]$, we have
>
> $$\int_A f d\nu = \int_A f \omega d\mu$$
>
> And its total variance is given by
>
> $$|\nu|(E) = \int_E |\omega| d\mu$$

Proof

ν is σ additive since lebesgue integral is σ additive over sets and absolute continuous follows from integral on an null sets is zero.

Suppose f is simple and $f = \sum a_i \chi_{E_i}$, then

$$\begin{aligned}
\int_A f dv &= \sum a_i \int_A \chi_{E_i} d\nu \\
&= \sum a_i \int_{A\cap E_i} d\nu \\
&= \sum a_i \nu(A\cap E_i) \\
&= \sum a_i \int_{A\cap E_i} \omega d\nu \\
&= \sum a_i \int_A \omega \chi_{E_i} d\nu \\
&= \int_A f\omega d\mu
\end{aligned}$$

If is measurable, then we can find $f_n \nearrow f$ and by MCT

$$\int_A f d\nu = \lim \int_A f_n d\nu = \lim \int_A f_n \omega d\mu = \int_A f\omega d\mu$$

For the total invariance, note $A = \{x \in E : \omega(x) \geq 0\}$ and $A = \{x \in E : \omega(x) < 0\}$ are a measurable partition of E, and

$$|\nu|(E) \geq |\nu(A)| + |\nu(B)| = \int_E |\omega| d\mu$$

On the other hand:

$$\begin{aligned}
\sum |\lambda(E_i)| &= \sum |\int_{E_i} \omega d\mu| \\
&\leq \sum \int_{E_i} |\omega| d\mu \\
&= \int_E |\omega| d\mu
\end{aligned}$$

Taking sup both sides, we get what we desired. ■

Let μ and ν be two measure, then ν is absolutely continuous w.r.t. μ if $\mu(A) = 0 \implies \nu(A) = 0$ and denoted as $v \ll \mu$.

Let μ and ν be two signed measure, then ν is absolutely continuous w.r.t. μ if $|\mu|(A) = (\mu^+ + \mu^-)(A) = 0 \implies \nu(A) = 0$ and denoted as $v \ll \mu$.

Let μ be a measure and ν be any kind measure, then ν is absolutely continuous w.r.t. μ if $\mu(A) = 0 \implies \nu(A) = 0$ and denoted as $v \ll \mu$.

Let μ be any kind of measure then μ is **concentrated** if $\forall E \in \mathcal{A}$, $\nu(A) = \nu(E \cap A)$.