Notes of Probability and Stochastics

Xie Zejian Zhang Songxin

2021-01-26

Contents

Chapter 1

Measure and integrations

1.1 Measurable space

1.1.1 σ algebra

Definition 1.1. A nonempty system of subset of Ω is an algebra on Ω if it's

1. Closed under complement: $A^c \in \mathcal{A} \iff A \in \mathcal{A}$
2. Closed under finite union: $\cup_i A_i \in \mathcal{A} \impliedby \forall i, A_i \in \mathcal{A}$

it's an σ algebra on Ω if it's also closed under countable union.

Remark. $\mathcal{A}$ is an algebra auto implies $\varnothing \in \mathcal{A}$ and $\Omega \in \mathcal{A}$. So $\{\varnothing, \Omega\}$ is the minimum algebra on Ω and thus minimum σ algebra while the discrete algebra 2^Ω is maximum.

Let $\left\{\mathcal{A}_\gamma : \gamma \in \Gamma\right\}$ is a collection of σ algebra, then we have $\mathcal{A} = \cap_{\gamma \in \Gamma} \mathcal{A}_\gamma$ is also a σ algebra. Hence we can define the smallest σ algebra as intersection of all σ algebras contains $\mathcal{A}$, that called the σ algebra **generated** by $\mathcal{A}$ and denoted by $\sigma(\mathcal{A})$.

The smallest $\sigma-$algebra generated by the system of all open sets in a topological space (Ω, τ) is called **Borel σ algebra** on Ω and denoted by $\mathcal{B}(\Omega)$, its elements are called **Borel sets**.

1.1.2 π, λ, m systems

A collection of subsets $\mathcal{A}$ is called.

1. **m-system** if closed under monotone series, that is if $(A_n) \subset \mathcal{A}$ and $A_n \nearrow A$, then $A \in \mathcal{A}$.
2. $\pi-$**system** is closed under finite intersection
3. $\lambda-$ **system** if
    1. $\Omega \in \mathcal{A}$
    2. closed under proper difference:
    $$A - B \in \mathcal{A} \impliedby B \subset A \text{ and } B, A \in \mathcal{A}$$
    3. is an m-system.

Theorem 1.1. *Let $\mathcal{A}$ be a collection of subsets of Ω iff it's both a π system and λ system.*

Which can be proved as follows:

- $\implies$:
    1. $\Omega \in \mathcal{A}$
    2. $A - B = A \cap B^c \in \mathcal{A}$
    3. is an m-system
- $\impliedby$:
    1. $A^c = \Omega - A \in \mathcal{A}$
    2. $A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$
    3. hence $\mathcal{A}$ is an algebra and $\mathcal{A}$ is a m-system.

Similarly, for m, π, λ -system, they also has a minimum system generated by some collection $\mathcal{C}$.

Lemma 1.1. *Let $\mathcal{A}$ be an algebra, then*

1. *$m(\mathcal{A}) = \sigma(\mathcal{A})$*
2. *if $\mathcal{B}$ is an m class and $\mathcal{A} \subset \mathcal{B}$, then $\sigma(\mathcal{A}) = m(\mathcal{A}) \subset \mathcal{B}$*

Similarly, let $\mathcal{A}$ be a π class, then $\lambda(\mathcal{A}) = \sigma(\mathcal{A})$

Then we have **Monotone class theorem**:

Theorem 1.2. $\forall \mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(\Omega)$*, s.t.:*

1. *If $\mathcal{A}$ is a π-class, $\mathcal{B}$ is a $\lambda-$class, then $\sigma(\mathcal{A}) \subset \mathcal{B}$*
2. *If $\mathcal{A}$ is an algebra, $\mathcal{B}$ is a $m-$class, then $\sigma(\mathcal{A}) \subset \mathcal{B}$*

1.1.3 Measurable spaces

Definition 1.2 (Measurable Space)**.** Pair $(\Omega, \mathcal{A})$ where $\mathcal{A}$ is a σ-Algebra on Ω.

Definition 1.3 (Products of measurable spaces)**.** Let $(E, \mathcal{E}), (F, \mathcal{F})$ be two measurable spaces. For $A \subset E, B \subset F$, $A \times B$ is the set of all pairs (x, y) : $x \in A, y \in B$. Note that $\mathcal{E} \times \mathcal{F}$ is also a σ-Algebra with all $A \times B$ where $A \in \mathcal{E}, B \in \mathcal{F}$ which is called *the product σ-Algebra*.

1.2 Measurable function

1.2.1 Mappings

Let $X : \Omega_1 \to \Omega_2$ be a mapping, $\forall B \subset \Omega_2$ and $\mathcal{G} \subset \mathcal{P}(\Omega_2)$, the **inverse image** of

- B is $X^{-1}(B) = \{\omega : \omega \in \Omega_1, X(\omega) \in B\} := \{X \in B\}$
- $\mathcal{G}$ is $X^{-1}(\mathcal{G}) = \left\{X^{-1}(B) : B \in \mathcal{G}\right\}$

There is some properties:

1. $X^{-1}\left(\Omega_2\right) = \Omega_1, X^{-1}(\varnothing) = \varnothing$
2. $X^{-1}\left(B^c\right) = \left[X^{-1}(B)\right]^c$
3. $$\begin{array}{l} X^{-1}\left(\cup_{\gamma \in \Gamma} B_\gamma\right) = \cup_{\gamma \in \Gamma} X^{-1}\left(B_\gamma\right) \text{ for } B_\gamma \subset \Omega_2, \gamma \in \Gamma \\ X^{-1}\left(\cap_{\gamma \in \Gamma} B_\gamma\right) = \cap_{\gamma \in \Gamma} X^{-1}\left(B_\gamma\right) \text{ for } B_\gamma \subset \Omega_2, \gamma \in \Gamma \end{array}$$
   where Γ may not countable.
4. $X^{-1}\left(B_1 - B_2\right) = X^{-1}\left(B_1\right) - X^{-1}\left(B_2\right) \forall B_1, B_2 \subset \Omega_2$
5. $B_1 \subset B_2 \subset \Omega_2 \implies X^{-1}(B_1) \subset X^{-1}(B_2)$
6. If $\mathcal{B}$ is a $\sigma-$algebra, $X^{-1}(\mathcal{B})$ is also a $\sigma-$algebra. It's easy to check $X^{-1}(\mathcal{B})$ is closed under complement and countable union. (From properties 2 and 3)
7. If $\mathcal{C}$ is nonempty, $X^{-1}(\sigma(\mathcal{C})) = \sigma\left(X^{-1}(\mathcal{C})\right)$

Remarks X^{-1} preserves all the set operations on Ω.

1.2.2 Measurable functions

Definition 1.4. For two measurable spaces $(\Omega_1, \mathcal{A})$, $(\Omega_1, \mathcal{B})$, $f : \Omega_1 \to \Omega_2$ is a **measurable mapping** if $f^{-1}(\mathcal{B}) \subset \mathcal{A}$, where

$$f^{-1}(\mathcal{B}) = \{f^{-1}(B) : B \in \mathcal{B}\}$$

It is a **measurable function** if $(\Omega_2, \mathcal{B}) = (\mathcal{R}^n, \mathcal{B}(\mathcal{R}^n))$, moreover, a **Borel function** if $(\Omega_1, \mathcal{A}) = (\mathcal{R}^m, \mathcal{B}(\mathcal{R}^m))$

Remark. If $\mathcal{B} = \sigma(\mathcal{C})$, the definition can be reduced to $X^{-1}(\mathcal{C}) \subset \mathcal{A}$ since

$$X^{-1}(\mathcal{B}) = X^{-1}(\sigma(\mathcal{C})) = \sigma\left(X^{-1}(\mathcal{C})\right) \subset \sigma(\mathcal{A}) = \mathcal{A}$$

Lemma 1.2. *Suppose $f : \mathcal{E} \to \mathcal{F}$ and $g : \mathcal{F} \to \mathcal{G}$ are measurable, then so is $f \circ g$.*

Proof. The same as how we proved composition of continuous function is continuous.

$\square$

1.3 Random Variable

A r.v. X is a measurable function from $(\Omega_1, \mathcal{A})$ to $(\mathcal{R}, \mathcal{B})$. It denoted by X is $\mathcal{A}$-measurable or $X \in \mathcal{A}$

(Another definition): A r.v. X is a measurable mapping from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{R}, \mathcal{B})$ such that

$$P(|X| = \infty) = P(\{\omega : |X(\omega)| = \infty\}) = 0$$

Lemma 1.3. *X is a r.v. from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$*

$$\iff X \le x = X^{-1}([-\infty], x) \in \mathcal{A} \quad \forall x \in \mathbb{R} \iff X \le x = X^{-1}([-\infty], x) \in \mathcal{A} \quad \forall x \in D$$

where D is a dense subset of $\mathbb{R}$, e.g. $\mathbb{Q}$. $\{X \le x\}$ above can be replaced by

$$\{X \le x\}, \quad \{X \ge x\}, \quad \{X < x\}, \quad \{X > x\}, \quad \{x < X < y\}$$

1.3.1 Construction of random variables

Lemma 1.4. *$\mathbf{X} = (X_1, \cdots, X_n)$ is a random vectors if X_k is a r.v. $\forall k$ iff $\mathbf{X}$ is a measurable function from $(\Omega, \mathcal{A})$ to $(\mathcal{R}^n, \mathcal{B}(\mathcal{R}^n))$.*

Proof. Note that

$$\{\mathbf{X} \in \prod I_n\} = \bigcap \{X_n \in I_n\} \in \mathcal{A}$$

where $I_k = (a_k, b_k], -\infty \leq a_k \leq b_k \leq \infty$ and follows from $\sigma(\{\prod I_n\}) = \mathcal{B}(\mathcal{R}^n)$. For the other direction, note

$$\{X_k \leq t\} = \{\mathbf{X} \in \prod_{i<k} \mathbb{R} \times \{-\infty, t\} \times \prod_{i>k} \mathbb{R}\} \in \mathcal{A}.\blacksquare$$

$\square$

Recall lemma 1.2 we have:

Theorem 1.3. *$\forall$ random n vectors $X = (X_{1:n})$ and Borel function f from $\mathcal{R}^n \to \mathcal{R}^m$, then $f(X)$ is a random m vectors.*

Remark. Note that continuous function are borel measurable since continuity leads to inverse image of an open set is still open. So if $X_{1:n}$ are r.v.'s, so are $\sum X_n$, $\sin(x)$, e^X, $\text{Poly}(X), \cdots$. That implies:

$\forall X, Y \in \mathcal{A}$, so are $aX+bY, X \vee Y = \max\{X, Y\}, X \wedge Y = \min\{X, Y\}, X^2, XY, X/Y, X^+ = \max(x, 0), X^- = -\min(x, 0), |X| = X^+ + X^-$

1.3.2 Limiting opts

Let (X_n) are r.v. on $(\Omega, \mathcal{A})$, then $\sup_{n\to\infty} X_n, \inf_{n\to\infty} X_n, \limsup_{n\to\infty} X_n, \liminf_{n\to\infty} X_n$ are r.v.'s. Moreover, if it exists, $\lim_{n\to\infty} X_n$ is r.v..

Proof. First two follows from, $\forall t \in \mathbb{R}$:

$$\{\sup_{n\to\infty} X_n \leq t\} = \bigcap_{n=1}^{\infty} \{X_n \leq t\} \in \mathcal{A}$$
$$\{\inf_{n\to\infty} X_n \geq t\} = \bigcap_{n=1}^{\infty} \{X_n \geq t\} \in \mathcal{A}$$

and the last two follows from $\limsup_{n\to\inf} = \inf_{k\to\infty} \sup_{m\geq k} X_m$ and $\liminf_{n\to\inf} = \sup_{k\to\infty} \inf_{m\geq k} X_m$.

$\square$

That implies

Lemma 1.5. *If $S = \sum_1^\infty X_n$ exists everywhere, then S is a r.v.*

Proof. Note $\sum_1^\infty X = \lim_{n\to\infty} \sum_n X_n$ is a r.v.

□

If $X = \lim_{n\to\infty} X_n$ holds **almost** everywhere, i.e., define Ω_0 is the set of all ω, such that $\lim_n X_n(\omega)$ exists, then $P(\Omega_0) = 1$, we say that X_n converges a.s. and write:

$$X_n \to X \quad a.s.$$

For a measurable function f, we may modify it at a null set into f' and it remain mesurable since for any open set G, $f'^{-1}(G)$ differ $f^{-1}(G)$ a at most null set, by the completion of lebesgue measure space, $f'-1(G)$ is measurable and thus f^{-1} measurable. Hence, for $f_n \to f$ a.s., we may ignore a null set and then $f_n \to f$ everywhere and thus f measurable.

1.3.3 Approximations of r.v. by simple r.v.'s

Definition 1.5. If $A \in \mathcal{A}$, the indicator function $\mathbf{1}_A$ is a r.v.

If $\Omega = \sum_1^n A_i$, where $A_i \in \mathcal{A}$, then $X = \sum_1^n a_i I_{A_i}$ is a r.v. and called **simple r.v.**

Any r.v. can be approximated by simple ones:

Theorem 1.4. *$\forall X \in \mathcal{A}$, $\exists 0 \le X_1 \le X_2 \cdots X_n$ s.t. $X_n(\omega) \nearrow X(\omega)$ everywhere.*

Proof. Suppose

$$X_n(\omega) = \sup\{\frac{j}{2^n} : j \in \mathbb{Z}, \frac{j}{2^n} \le \min(X(\omega), 2^n)\}$$

One can check X_n is simple r.v. and $X_n(\omega) \nearrow X(\omega)$ for all $\omega \in \Omega$.

□

1.3.4 σ algebra generated by r.v.

Let $\{X_\lambda, \lambda \in \Lambda\}$ is r.v.s on $(\Omega, \mathcal{A})$. Define

$$\sigma\left(X_\lambda, \lambda \in \Lambda\right) := \sigma\left(X_\lambda \in B, B \in \mathcal{B}, \lambda \in \Lambda\right) = \sigma\left(X_\lambda^{-1}(\mathcal{B}), \lambda \in \Lambda\right) = \sigma\left(\cup_{\lambda\in\Lambda} X_\lambda^{-1}(\mathcal{B})\right)$$

which is called σ algebra generated by $\{X_\lambda, \lambda \in \Lambda\}$, where Λ is a index set which can be uncountable.

For $\Lambda = \mathbb{N}^+$:

1.
$$\sigma(X_i) = \sigma\left(X_i^{-1}(\mathcal{B})\right) = X_i^{-1}(\mathcal{B}) = \{X_i \in \mathcal{B}\}$$
$$\sigma(X_1, \dots, X_n) = \sigma\left(\cup_{i=1}^n X_i^{-1}(\mathcal{B})\right) = \sigma\left(\cup_{i=1}^n \sigma(X_i)\right)$$

2.
$$\sigma(X_1) \subset \sigma(X_1, X_2) \subset \dots \dots \subset \sigma(X_1, \dots, X_n)$$
$$\sigma(X_1, X_2, \dots ..) \supset \sigma(X_2, X_3, \dots ..) \supset \dots \dots \supset \sigma(X_n, X_{n+1}, \dots ..)$$

3. $\bigcap_1^\infty \sigma(X_n, X_{n+1}, \cdots)$ is the tail σ algebra of $X_{1:}$

If $A_{1:n}$ are not mutually exclusive to each other, then we have

$$|\sigma(A_{1:n})| = 2^{2^n}$$

Which follows from for a partition $A_{1:n}$,

$$\sigma(A_1, \cdots, A_n) = \{\bigcup_{i \in J} A_i\}$$

where J is any subset of $\mathbb{N} \le n$ and $A_0 = \varnothing$. Hence for discrete r.v. Y, $\sigma(Y)$ can be generated from $A_i = \{Y = y_i\}$ forall y_i's. For continuous case, it's generated by all intervals.

1.3.5 Monotone classes of function

Definition 1.6 (monotone class)**.** $\mathcal{M}$ is called a monotone class if: - $1 \in \mathcal{M}$ - $f, g \in \mathcal{M}_b$ and $a, b \in \mathbb{R} \implies af + bg \in \mathcal{M}$ - $(f_n) \subset M_+, f_n \uparrow f \implies f \in \mathcal{M}$

where $\mathcal{M}_+$ is a subcollection consisting of positive functions in $\mathcal{M}$, and $\mathcal{M}_b$ for the bounded function in $\mathcal{M}$.

Theorem 1.5 (Monotone class theorem for functions)**.** *Let $\mathcal{M}$ be a monotone class of functions on E. Suppose for some p-system $\mathcal{C}$ generating $\mathcal{E}$ and $1_A \in \mathcal{M}$ for every $A \in \mathcal{C}$. Then $\mathcal{M}$ includes all positive $\mathcal{E}$-measurable functions and all bounded $\mathcal{E}$-measurable functions.*

Proof. First we need to show that $1_A \in \mathcal{M}$ for every $A \in \mathcal{E}$. Let $\mathcal{D} = \{A \in \mathcal{E} : 1_A \in \mathcal{M}\}$. Now we check that $\mathcal{D}$ is a d-system: - $1_E = 1$, so $E \in \mathcal{D}$. - $B \subset A,\ A, B \in \mathcal{D}$. $1_{A-B} = 1_A - 1_B \in \mathcal{D}$ - $(A_n) \subset \mathcal{D}$ and $A_n \uparrow A$, then $1_{A_n} \uparrow 1_A$, so $1_A \in \mathcal{M}$, then $A \in \mathcal{D}$

By assuption, $\mathcal{C} \subset \mathcal{D}$, and $\sigma(\mathcal{C})$ is the smallest d-system by the theorem above, so $\mathcal{E} \subset \mathcal{D}$, so $1_A \in \mathcal{M}$ for every $A \in \mathcal{E}$.
As $1_A \in \mathcal{M}$ for every $A \in \mathcal{E}$, we can easily prove that all of the positive simple function is generated by the linear combination of 1_A s. And all positive $\mathcal{E}$-measurable functions is generated by a sequence of positive simple functions.

Then for general bounded $\mathcal{E}$-measurable function f, using $f = f^+ - f^-$ where $f^+, f^- \in \mathcal{M}$.

$\square$

Let $(E,\mathcal{E}), (F,\mathcal{F})$ be two mesurable spaces and f is a bijection $E \to F$. Then f is said to be a isomorphism of $(E,\mathcal{E})$ and $(F,\mathcal{F})$ if f is $\mathcal{E}$-measurable and f^{-1} is $\mathcal{F}$-measurable. These two spaces are called isomorphic if there exists an isomophism between them.
A measurable apce $(E,\mathcal{E})$ is said to be *standard* if it is isomorphic to $(F, \mathcal{B}_F)$ for some Borel subset $F \subset \mathbb{R}$.

1.4 Measure

Let Ω be a space and $\mathcal{A}$ a class, then function $\mu : \mathcal{A} \to R = [-\infty, \infty]$ is a **set function**.

It's

- 1. **finite** if $\forall A \in \mathcal{A}, \quad |\mu(A)| < \infty$
  2. **σ-finite** if $\exists A_n \subset \mathcal{A}, \quad s.t. \quad \cup_{i=1}^{\infty} A_i = \Omega \quad \forall n \quad |\mu(A_n)| < \infty$
  3. Σ **finite** if there exist finite (μ_n) s.t. $\mu = \sum_n \mu_n$.
- 1. **additive** $\iff \mu\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$
  2. $\sigma-$**additive**$\iff \mu\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

Remark. Finite implies σ finite and σ finite implies Σ finite.

μ is a **measure** on $\mathcal{A}$ if

1. $\forall A \in \mathcal{A} : \mu(A) \geq 0$
2. It's σ additive.

the triplet $(\Omega, \mathcal{A}, \mu)$ is a **measure space** when μ is a measure and $(\Omega, \mathcal{A})$ is a measurable space. Whose sets are called **measurable sets** or **$\mathcal{A}$-measurable**.A measure space is a **probability space** if $P(\Omega) = 1$.

Assume that $A_{1:n} \in \mathcal{A}$ and $A \in \mathcal{A}$ and μ is a measure.

1. μ is continues from above, if $A_n \searrow A \implies \mu(A_n) \to \mu(A)$
2. μ is continues from below, if $A_n \nearrow A \implies \mu(A_n) \to \mu(A)$
3. μ is continues at A, if $A_n \to A \implies \mu(A_n) \to \mu(A)$

$\forall$ Measure μ is continues from below and may not continues from above. It will be continues from above if $\exists m < \infty, \mu(A_m) < \infty$.So finite measure μ are always continues.

1.4.1 Properties of measure

1.4.1.1 Semialgebras

Let μ be a nonnegative additive set function on a semialgebra $\mathcal{A}$. $\forall A, B \in \mathcal{A}$ and $\{A_n, B_n, n \geq 1\} \in \mathcal{A}$

1. (**Monotonicity**): $A \subset B \implies \mu(A) \leq \mu(B)$
2. (σ**-subadditivity**):
    1. $\sum_1^\infty A_n \subset A, \quad \implies \quad \sum_1^\infty \mu(A_n) \leq \mu(A)$
    2. Moreover, if μ is a measure, then

$$B \subset \sum_{n=1}^{\infty} B_n \implies \mu(B) \leq \sum_{n=1}^{\infty} \mu(B_n)$$

We can assert a nonnegative set function μ is a measure by:

1. μ is additive
2. μ is σ subadditive on $\mathcal{S}$

1.4.1.2 Algebras

Let μ be a measure on an algebra $\mathcal{A}$

$$A \subset \cup_1^\infty A_n \implies \mu(A) \leq \sum_1^\infty \mu(A_n)$$

Proof Note $A = A \cap (\cup A_n) = \cup(A \cap A_n)$, hence we can write A as union in $\mathcal{A}$ by take $B_n = A \cap A_n \in \mathcal{A}$.

$$A = \cup_1^\infty B_n$$

and then we can take $C_n = B_n - \cup_1^{n-1} B_i \in \mathcal{A}$ to write A as disjoint union:

$$A = \sum C_n$$

Then

$$\mu(\mathcal{A}) = \mu(\sum C_n) = \sum \mu(C_n) \leq \sum \mu(B_n) \leq \sum \mu(A_n)$$

as $C_n \subset B_n \subset A_n$. ■

1.4.1.3 σ algebras

Let μ be a measure on an σ algebra $\mathcal{A}$

1. Monotonicity
2. Boole's inequality(Countable Sub-Additivity)

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu\left(A_i\right)$$

3. Continuity from below
4. Continuity from above if μ is finite in A_i.

The sense of **4** follows from suppose $A_i \searrow A$, then $A_1 - A_i \nearrow A_1 - A$, then

$$\mu(A_1) - \mu(A) = \mu(A_1 - A) = \lim \mu(A_1 - A_i) = \mu(A_1) - \lim(A_i)$$

where $\mu(A_1)$ cannot be cancelled if $\mu(A_i) = \infty$.

Definition 1.7. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $N \subset \Omega$

1. N is a μ **null set** iff $\exists B \in \mathcal{A} \quad s.t. \quad \mu(B) = 0, \quad N \subset B$
2. This measure space is a **complete measure** space if $\forall$ μ null space N, $N \in \mathcal{A}$

Theorem 1.6. *Given any measure space* $(\Omega, \mathcal{A}, \mu)$ *, there exist a complete measure space* $(\Omega, \bar{\mathcal{A}}, \bar{\mu})$*, such that* $\mathcal{A} \subset \bar{\mathcal{A}}$ *and* $\bar{\mu}$ *is an extension of* μ*. This space is called completion of* $(\Omega, \mathcal{A}, \mu)$*.*

Proof. Take

$$\bar{\mathcal{A}} = \{A \cup N : A \in \mathcal{A}\}$$
$$\bar{\mathcal{B}} = \{A \Delta N : A \in \mathcal{A}\}$$

$\bar{\mathcal{A}} = \bar{\mathcal{B}}$ since $A \cup N = (A - B) \Delta (B \cap (A \cup N))$ and $A \Delta N = (A - B) \cup (B \cap (A \Delta N))$.

Then we can show that $\bar{\mathcal{A}}$ is a σ algebra. Let $E_i = A_i \cup N_i \in \bar{\mathcal{A}}$, then

$$\bigcup_1^\infty E_i = \bigcup_1^\infty A_i \cup \bigcup_1^\infty N_i$$

and note $\bigcup_1^\infty A_i \in \mathcal{A}$ and $\mu(\bigcup_1^\infty N_i) \leq \mu(\cup_1^\infty B_i) \leq \cup_1^\infty \mu(B_i) = 0$. Thus $\bar{\mathcal{A}}$ is closed by countable union. As for complements, note $E^c = A^c \cap N^c = (A^c \cap N^c \cap B^c) \cup (A^c \cap N^c \cap B) = (A^c \cap B^c) \cup (A^c \cap N^c \cap B) \in \bar{\mathcal{A}}$.

Finally we define a measure $\bar{\mu}$ on $\bar{\mathcal{A}}$ by

$$\bar{\mu}(A \cup N) = \mu(A)$$

We should prove it's well defined. Suppose $A_1 \cup N_1 = A_2 \cup N_2 \in \bar{\mathcal{A}}$, note $A \Delta B \Delta C = A \Delta (B \Delta C)$ and $A \Delta B = B \Delta A$.

$$\begin{aligned} (A_1 \Delta A_2) \Delta (N_1 \Delta N_2) &= (A_1 \Delta A_2 \Delta N_1) \Delta N_2 \\ &= (A_1 \Delta N_1) \Delta (A_2 \Delta N_2) \\ &= \varnothing \end{aligned}$$

Hence $A_1 \Delta A_2 = N_1 \Delta N_2$, note $N_1 \Delta N_2 \subset N_1 \cup N_2 \subset B_1 \cup B_2$, hence $\mu(A_1 \Delta A_2) = 0$ and thus $\mu(A_1 - A_2) = \mu(A_2 - A_1) = 0$. Therefore

$$\mu(A_1) = \mu(A_1 - A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2)$$
$$\mu(A_2) = \mu(A_2 - A_1) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2)$$

$\bar{\mu}$ is do well defined. μ^* is auto σ additive since so is μ and is easy to check that all μ^* null set is μ null set.

□

1.4.2 Specification of measures

Theorem 1.7. *Let $(E, \mathcal{E})$ be a measurable space. Let μ, ν be measures on it with $\mu(E) = \nu(E) < \infty$. If μ, ν agree on a p-system generating $\mathcal{E}$, then μ, ν are identical. >*

Proof. Let $\mathcal{C}$ be the p-system generating $\mathcal{E}$ and $\mu(A) = \nu(A)$ for every $A \in \mathcal{C}$. Consider $\mathcal{D} = \{A \in \mathcal{E} : \mu(A) = \nu(A)\}$ which satisfies $\mathcal{C} \subset \mathcal{D} \subset \mathcal{E}$. Then we need to prove that $\mathcal{D}$ is a d-system: - $E \in \mathcal{D}$ by the assuption. - Let $A, B \in \mathcal{D}$ and $B \subset A$. Then $\mu(A - B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A - B)$, so $A - B \in \mathcal{D}$ - Let $(A_n) \uparrow A$ and $(A_n) \subset \mathcal{D}$, then $\mu(A_n) \uparrow \mu(A)$, $\nu(A_n) \uparrow \nu(A)$, since $\mu(A_n) = \nu(A_n)$ for every n, so $\mu(A) = \nu(A)$.
So $\mathcal{D}$ is a d-system.

So $\mathcal{D} = \mathcal{E}$, so for every $A \in \mathcal{E}$, $\mu(A) = \nu(A)$.

□

Let $(E, \mathcal{E})$ be a measurable space. Suppose that the sigleton $\{x\} \in \mathcal{E}$ for every $x \in E$.
Let μ be a measure on $(E, \mathcal{E})$. A point x is said to be an atom if $\mu(\{x\}) > 0$, the measure is said to be *diffuse* if it has no atoms. It is said to be *purely atomic* if the set D of its atoms is countable and $\mu(E - D) = 0$.

Theorem 1.8. *Let μ be a Σ-finite measure on $(E, \mathcal{E})$. Then $\mu = \nu + \lambda$ where λ is a diffuse measure and ν is purely atomic.*