Homework 1

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Optimality and Asset Pricing

Suppose the market $(\mathbf{D}, \mathbf{S_0})$ is given, an agent is defined by an utility function $U : \mathbb{R}^n \to \mathbb{R}$ and an endowment $\boldsymbol{\varepsilon} \in \mathbb{R}^n_+$. Our optimal target is

$$\max_{\theta \in A} U(\boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta})$$

where

$$A = \{\mathbf{S_0}'\boldsymbol{\theta} \leq 0, \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta} \geq \mathbf{0}\}$$

and we assume there is $\boldsymbol{\theta_0}$ s.t. $\mathbf{D}'\boldsymbol{\theta_0} > 0$, that along with absence of arbitrage implies the optimal θ^* satisfy $\mathbf{S_0}'\boldsymbol{\theta} = 0$, otherwise we can invest some on $\boldsymbol{\theta_0}$ and get a better portfolio.

Note $A = \{\mathbf{S_0}'\boldsymbol{\theta} = 0, \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta} \geq \mathbf{0}\}$ is closed and bounded if there is no arbitrage and assume U is continuous, we have

Proposition 0.1. *The optimal problem has solution iff there is no arbitrage.*

Theorem 0.1. *If in optimal solution $\boldsymbol{\theta}^*$, $\mathbf{c}^* = \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta}^* \gg 0$, $\nabla U \gg 0$ at $\mathbf{c}^*$, There exist $\lambda > 0$ s.t. $\lambda \nabla U(\mathbf{c}^*)$ is a state price vector.*

Proof. Suppose $\boldsymbol{\theta}^*$ is solution, for any portfolio $\boldsymbol{\theta}$ s.t. $\mathbf{S_0}'\boldsymbol{\theta} = 0$, if we combine $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}$, utility will be

$$g(\alpha) = U[\varepsilon + \mathbf{D}'(\boldsymbol{\theta}^* + \alpha\boldsymbol{\theta})] = U(\mathbf{c}^* + \alpha\mathbf{D}'\boldsymbol{\theta})$$

where $\mathbf{c}^* = \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta}^*$. As $\boldsymbol{\theta}^*$ is the solution, we have FOC on $\alpha = 0$:

$$g'(0) = [\nabla U(\mathbf{c}^*)]' \mathbf{D}'\boldsymbol{\theta} = [\mathbf{D}\nabla U(\mathbf{c}^*)]'\boldsymbol{\theta} = 0$$

that implies $\mathbf{D}\nabla U(\mathbf{c}^*) = \mu \mathbf{S}_0$ for some $\mu \in \mathbb{R}$. It's remaining to show that $\mu > 0$. Take $\boldsymbol{\theta_0}$ in assumption, we have

$$\mu \mathbf{S_0}'\boldsymbol{\theta_0} = [\nabla U(\mathbf{c}^*)]' \mathbf{D}'\boldsymbol{\theta_0} > 0$$

thus $\mu > 0$ as required. $\square$

Since convex function automatically satisfy SOC, we have

Corollary 0.1. *If U is concave and strictly increasing, $\mathbf{c}^* \gg 0$, then $\boldsymbol{\theta}^*$ is the optimal solution iff $\lambda \nabla U(\mathbf{c}^*)$ is a state price vector for some $\lambda > 0$.*

Exercise 0.1. Given market

$$\mathbf{D} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \mathbf{S_0} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

and agent

$$U(\mathbf{x}) = \log x_1 + \log x_2, \boldsymbol{\varepsilon} = \begin{bmatrix} 1 \\ 100 \end{bmatrix}$$

Solution. All the state price vector is given by:

$$\boldsymbol{\psi} \in \left\{ \mathbf{D^{-1}S_0} = \frac{1}{10}\mathbf{S_0} \right\}$$

by 0.1 we have

$$\lambda \nabla U(\mathbf{c}^*) = \lambda \begin{bmatrix} \frac{1}{c_1} \\ \frac{1}{c_2} \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{1+10\theta_1} \\ \frac{1}{100+10\theta_2} \end{bmatrix} = \frac{1}{10}\mathbf{S_0}$$

also $4\theta_1 + 2\theta_2 = 0$, we have

$$\boldsymbol{\theta} = \begin{bmatrix} \frac{49}{20} \\ -\frac{49}{10} \end{bmatrix}, \lambda = 10.2$$

the risk-neutral probability is $\mathbf{p} = [\frac{2}{3}, \frac{1}{3}]'$ and thus the price of X is

$$\psi_0 \mathbf{p}'\mathbf{X} = \frac{3}{5} \cdot 15 = 9$$

Exercise 0.2. Given market

$$\mathbf{D} = \begin{bmatrix} 20 & 0 & 2 \\ 0 & 10 & 1 \end{bmatrix}, \mathbf{S_0} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

and agent

$$U(\mathbf{x}) = x_1^\alpha + x_2^\alpha + x_3^\alpha, \boldsymbol{\varepsilon} = \begin{bmatrix} 10 \\ 50 \\ 100 \end{bmatrix}$$

Solution. All the state price vector is given by

$$\boldsymbol{\psi} \in \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x \begin{bmatrix} 1 \\ 1 \\ -10 \end{bmatrix} : x \in (0, \frac{1}{5}) \right\}$$

and

$$\mathbf{c}^* = \begin{bmatrix} 10 + 20\theta_1 \\ 50 + 10\theta_2 \\ 100 + 2\theta_1 + \theta_2 \end{bmatrix}, \lambda \nabla U(\mathbf{c}^*) = \lambda\alpha \begin{bmatrix} c_1^{\alpha-1} \\ c_2^{\alpha-1} \\ c_3^{\alpha-1} \end{bmatrix}$$

Note the first two coordinate keep the same, thus $\boldsymbol{\theta} = (1, -2)'$. Therefore

$$\left(\frac{10}{3}\right)^{\alpha-1} = \frac{2-10x}{x} \implies x = \frac{2 \cdot 3^{\alpha-1}}{10^{\alpha-1} + 10 \cdot 3^{\alpha-1}}$$

and hence:

$$\boldsymbol{\psi} = (x, x, 2-10x)'$$

$$\lambda = \frac{x}{\alpha 30^{\alpha-1}} = \frac{\frac{2}{\alpha} \cdot 10^{1-\alpha}}{10^{\alpha-1} + 10 \cdot 3^{\alpha-1}}$$

The risk-neutral probability and the price is given by

$$\mathbf{p} = \frac{1}{2 \cdot 3^{\alpha-1} + 10^{\alpha-1}} \left[3^{\alpha-1}, 3^{\alpha-1}, 10^{\alpha-1}\right]',$$

$$\psi_0 \mathbf{X}'\mathbf{p} = \frac{25 \cdot 3^{\alpha-1} + 10 \cdot 10^{\alpha-1}}{2 \cdot 3^{\alpha-1} + 10^{\alpha-1}} \cdot (2-8x) = \frac{50 \cdot 3^{\alpha-1} + 20 \cdot 10^{\alpha-1}}{10^{\alpha-1} + 10 \cdot 3^{\alpha-1}}$$