Financial Modeling Analysis

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Single Period Model

2.1 Simple Example

Suppose one buy a call option C which payoff $V_1 = (S_1 - K)^+$ at time 1, S_1 can be $uS_0 > K$ or $dS_0 < K$ determined in probability space $(\{H, T\}, 2^{\{H,T\}}, \mathbb{P})$. To replicate such option, we construct our portfolio by buying Δ_0 stock and investing remaining in risk-free asset at return r:

$$(V_0 - \Delta_0 S_0)(1 + r) + \Delta_0 S_1 = (S_1 - K)^+$$

solve:

$$V_0 = \frac{pV_1(H) + qV_1(T)}{1 + r}, \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

where $p = \frac{1+r-d}{u-d}, q = 1 - p$.

p and q can be then seen as probability assigned to $\mathbb{P}\{H\}$ and $\mathbb{P}\{T\}$. Then V_0 is just the discounted of expected value of such option, such measure $\mathbb{P}$ is called **risk-neutral**.

2.2 State price

Suppose $\mathbf{S_0} \in \mathbb{R}^N$ is price of N stocks at time 0, and $\mathbf{D} \in \mathbb{R}^{N\times n}$ is their price at time t for n states. For any portfolio $\boldsymbol{\theta} \in \mathbb{R}^N$, it cost $\mathbf{S_0'}\boldsymbol{\theta}$ and its value is $\mathbf{D}'\boldsymbol{\theta} \in \mathbb{R}^n$ for all n states.

An arbitrage is then defined as a portfolio $\theta, s.t.$ $\mathbf{S}'\boldsymbol{\theta}$ have different sign with $\mathbf{D}'\boldsymbol{\theta}$.

Definition 2.1 (State price). A state price vector is $\boldsymbol{\psi} \in \mathbb{R}^n_{++}$ *s.t.* $\mathbf{S_0} = \mathbf{D}\boldsymbol{\psi}$.

To justify the name of state price, suppose we want to "bet" the state of market, *i.e.*, we would like earning $\mathbf{1}_{state=i}$, then our portfolio supposed to be $\mathbf{D}'\boldsymbol{\theta} = \mathbf{e_i}$, and it cost

$$\mathbf{S_0'}\boldsymbol{\theta} = \boldsymbol{\psi}'\mathbf{D}'\boldsymbol{\theta} = \boldsymbol{\psi}'\mathbf{e_i} = \psi_i$$

so the coordinate of $\boldsymbol{\psi}$ is the price of "betting" a state.

Theorem 2.1. *There is no arbitrage iff there is a state price vector.*

Theorem 2.2 (Separating Hyperplane Theorem). *Suppose M and K are closed convex cones in $\mathbb{R}^d$ that $M \cap K = \{\mathbf{0}\}$, if K isn't a liner space, then there is a nonzero linear f separated them, i.e., $f(x) < f(y)$ for any $x \in M$ and $y \in K - \{\mathbf{0}\}$.*

Theorem 2.3 (Riesz Representation Theorem). *Any continuous linear function f on Hilbert space $\mathcal{H}$ can be written as $f(x) = \langle x, v \rangle$ for some $v \in \mathcal{H}$.*

Proof. Let $M = \left\{(-\mathbf{S_0'}\boldsymbol{\theta}, \mathbf{D}'\boldsymbol{\theta}) : \theta \in \mathbb{R}^N\right\}$ and $K = \mathbb{R}_+ \times \mathbb{R}^n_+$. Then there is no arbitrage iff $K \cap M = \{\mathbf{0}\}$.

$\Longrightarrow$, let f be the functional in theorem 2.2, note M is a linear space, f should be vanish on M, *i.e.*, $f(x) = 0, \forall x \in M$, otherwise, fix $f(y) > 0$ for $y \in K - \{0\}$, we can find $\lambda \in \mathbb{R}$ *s.t.* $\lambda f(x) = f(\lambda x) > f(y)$.

Then by theorem 2.3, we have $f(x) = x'v$ for some v, write $v = (\alpha, \phi)$ where $\alpha \in \mathbb{R}$ and $\phi \in \mathbb{R}^n$. Since $f(x) > 0$ for nonzero $x \in K$, α and ϕ should strictly positive, then

$$-\alpha\mathbf{S_0'}\boldsymbol{\theta} + \boldsymbol{\phi}'\mathbf{D}'\boldsymbol{\theta} = 0$$

which implies $-\alpha\mathbf{S_0} + \mathbf{D}\boldsymbol{\phi} = \mathbf{0}$ and thus $\frac{\phi}{\alpha}$ is a state price vector as required.

$\Longleftarrow$, Suppose $(-\mathbf{S_0'}\boldsymbol{\theta}, \mathbf{D}'\boldsymbol{\theta}) \in K$, then, $\boldsymbol{\psi}'\mathbf{D}'\boldsymbol{\theta} \leq 0$ and $\mathbf{D}'\boldsymbol{\theta}$, which contrast to $\psi \gg 0$. □

Exercise 2.1. 1

Solution. Given above.

Exercise 2.2. 2

Solution. Setup:

$$\mathbf{S_0} = (1, S_0)', \mathbf{D} = \begin{bmatrix} 1+r & 1+r \\ uS_0 & dS_0 \end{bmatrix}$$

then the state price should be

$$\boldsymbol{\psi} = \mathbf{D^{-1}S_0} = [(-u+1+r)S_0, (d-1-r)S_0]' \gg 0$$

then claim follows.

Exercise 2.3. 3

Solution. No. Let $\psi = (\frac{1}{3}, \frac{1}{3})$, it's a state price vector.

Exercise 2.4. 4

Solution. Note column space of D is just $\{\lambda \cdot (1,2,3)' : \lambda \in \mathbb{R}\}$, which excluded $\overline{q}$, therefore there is no state price vector and thus arbitrage exists.

2.2.1 Risk-neutral probability

If $\mathbf{p} > 0$ and $\mathbf{e}'\mathbf{p} = 1$, we can view $\mathbf{p} \in \mathbb{R}^n$ as a probability vector represent each state, as there is only n states, we can use it to represent probability measure. Then

$$\mathbb{E}\, S_T = \mathbf{Dp}$$

take $\mathbf{p} = \frac{\boldsymbol{\psi}}{\mathbf{e}'\boldsymbol{\psi}}$. Then

$$\mathbb{E}\, S_T = \frac{\mathbf{D}\boldsymbol{\psi}}{\mathbf{e}'\boldsymbol{\psi}} = \frac{\mathbf{S_0}}{\mathbf{e}'\boldsymbol{\psi}}$$

where $\mathbf{e}'\boldsymbol{\psi}$ is the discount on riskless borrowing. To confirm this, suppose the market allow positive riskless borrowing and we replicate it by investing a portfolio $\boldsymbol{\theta}$ for which

$$\mathbf{D}'\boldsymbol{\theta} = \mathbf{e}$$

and $\boldsymbol{\theta}$ cost $\mathbf{S_0}'\boldsymbol{\theta} = \boldsymbol{\psi}'\mathbf{D}'\boldsymbol{\theta} = \mathbf{e}'\boldsymbol{\psi}$. That is $\boldsymbol{\psi_0} = \mathbf{e}'\boldsymbol{\theta}$ is the riskless discount, i.e. $\frac{1}{\psi_0} = e^{rT}$.

If probability vector $\mathbf{p}$ also let $\mathbb{E}\, S_T$ have the same value, we said it's a risk-neutral probability measure.

A claim $C \in \mathbb{R}^n$ and it's said to be attainable or can be hedged if there is a portfolio θ that $\mathbf{D}'\boldsymbol{\theta} = C$.

Theorem 2.4. *With absence of arbitrage, the price of an attainable claim $C \in \mathbb{R}^n$ is $\mathbf{e}'\boldsymbol{\psi}\,\mathbb{E}\, C$ if $\mathbf{S_0} = \mathbf{e}'\boldsymbol{\psi}\,\mathbb{E}\, S_T$ for some probability measure $\boldsymbol{\psi}$.*

Proof. Suppose $\mathbf{D}'\boldsymbol{\theta} = \mathbf{C}$, then its price should be $\boldsymbol{\theta}'\mathbf{S_0}$

$$\mathbb{E}\, C = \mathbb{E}\, \mathbf{D}'\boldsymbol{\theta} = \frac{\boldsymbol{\theta}'\mathbf{D}\boldsymbol{\psi}}{\mathbf{e}'\boldsymbol{\psi}} = \boldsymbol{\theta}'\mathbf{S_0}$$

□

A market is said to be complete if every claim C is attainable.

Theorem 2.5. *The market in our setting is complete iff $N \geq n$ and $\mathbf{D}$ have full column rank.*

Proof. Completeness is precisely equivalent row space $\mathcal{C}(\mathbf{D}') = \mathbb{R}^n$ and then claim follows. □

In complete market, risk-neutral measure is unique.

2.3 Optimality and Asset Pricing

Suppose the market $(\mathbf{D}, \mathbf{S_0})$ is given, an agent is defined by an utility function $U : \mathbb{R}^n \to \mathbb{R}$ and an endowment $\boldsymbol{\varepsilon} \in \mathbb{R}^n_+$. Our optimal target is

$$\max_{\theta \in A} U(\boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta})$$

where

$$A = \{\mathbf{S_0}'\boldsymbol{\theta} \leq 0, \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta} \geq \mathbf{0}\}$$

and we assume there is $\boldsymbol{\theta_0}$ s.t. $\mathbf{D}'\boldsymbol{\theta_0} > 0$, that along with absence of arbitrage implies the optimal θ^* satisfy $\mathbf{S_0}'\boldsymbol{\theta} = 0$, otherwise we can invest some on $\boldsymbol{\theta_0}$ and get a better portfolio.

Note $A = \{\mathbf{S_0}'\boldsymbol{\theta} = 0, \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta} \geq \mathbf{0}\}$ is closed and bounded if there is no arbitrage and assume U is continuous, we have

Proposition 2.1. *The optimal problem has solution iff there is no arbitrage.*

Theorem 2.6. *If in optimal solution $\boldsymbol{\theta}^*$, $\mathbf{c}^* = \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta}^* \gg 0$, $\nabla U \gg 0$ at $\mathbf{c}^*$, There exist $\lambda > 0$ s.t. $\lambda \nabla U(\mathbf{c}^*)$ is a state price vector.*

Proof. Suppose $\boldsymbol{\theta}^*$ is solution, for any portfolio $\boldsymbol{\theta}$ s.t. $\mathbf{S_0}'\boldsymbol{\theta} = 0$, if we combine $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}$, utility will be

$$g(\alpha) = U[\varepsilon + \mathbf{D}'(\boldsymbol{\theta}^* + \alpha\boldsymbol{\theta})] = U(\mathbf{c}^* + \alpha\mathbf{D}'\boldsymbol{\theta})$$

where $\mathbf{c}^* = \boldsymbol{\varepsilon} + \mathbf{D}'\boldsymbol{\theta}^*$. As $\boldsymbol{\theta}^*$ is the solution, we have FOC on $\alpha = 0$:

$$g'(0) = [\nabla U(\mathbf{c}^*)]' \mathbf{D}'\boldsymbol{\theta} = [\mathbf{D}\nabla U(\mathbf{c}^*)]'\boldsymbol{\theta} = 0$$

that implies $\mathbf{D}\nabla U(\mathbf{c}^*) = \mu \mathbf{S}_0$ for some $\mu \in \mathbb{R}$. It's remaining to show that $\mu > 0$. Take $\boldsymbol{\theta_0}$ in assumption, we have

$$\mu \mathbf{S_0}'\boldsymbol{\theta_0} = [\nabla U(\mathbf{c}^*)]' \mathbf{D}'\boldsymbol{\theta_0} > 0$$

thus $\mu > 0$ as required. □

Since convex function automatically satisfy SOC, we have

Corollary 2.1. *If U is concave and strictly increasing, $\mathbf{c}^* \gg 0$, then $\boldsymbol{\theta}^*$ is the optimal solution iff $\lambda \nabla U(\mathbf{c}^*)$ is a state price vector for some $\lambda > 0$.*

2.3.1 Expected Utility Function

Now we consider a special case of utility:

$$U(\mathbf{c}) = \mathbb{E}\, u(c) = \mathbf{p}'\mathbf{u}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is concave and increasing and $\mathbf{u} = [u(c_1), u(c_2), \dots, u(c_n)]'$.

Then we have

$$\nabla U(\mathbf{c}) = \begin{bmatrix} p_1 u'(c_1) \\ p_2 u'(c_2) \\ \vdots \\ p_n u'(c_n) \end{bmatrix}$$

theorem 2.6 yields

$$\mathbf{S_0} = \mathbf{D}\boldsymbol{\psi} = \lambda \mathbf{D} \nabla U(\mathbf{c}^*) = \mathbb{E}\left[\lambda S_T \cdot u'(c)\right]$$

where

$$\boldsymbol{\psi} = \lambda \begin{bmatrix} p_1 u'(c_1) \\ p_2 u'(c_2) \\ \vdots \\ p_n u'(c_n) \end{bmatrix}$$

thus we can define risk-neutral measure $\mathbb{Q}$ by $\boldsymbol{\psi}, \mathbf{S_0} = \psi_0 \, \mathbb{E}^{\mathbb{Q}} S_T$.

Set $\boldsymbol{\pi} = [\pi_i = \psi_i / p_i]$, then $\mathbb{E}\, \boldsymbol{\pi} = \psi_0$ and $\mathbf{S_0} = \mathbb{E}\, S_T \pi$

Remark. We use boldface to refer the vector representation of a random variable.

In fact, for any attainable claim C, we have

Theorem 2.7. *The price of C is given by $\mathbb{E}\,\pi\,\mathbb{E}^Q C$*

Proof. Suppose C can be attain by θ, then

$$S_0'\theta = [\mathbb{E}\, S_T \pi]'\theta = \mathbb{E}\,\pi S_T'\theta = \mathbb{E}\,\pi C = \psi_0 \overset{Q}{\mathbb{E}} C = \mathbb{E}\,\pi \overset{Q}{\mathbb{E}} C$$

□

2.3.2 Equilibrium

Definition 2.2. Equilibrium is a pair $(\boldsymbol{\theta_i})_{i \le m}, q$, where $\boldsymbol{\theta}_i$ maximize each one's utility and $\sum \boldsymbol{\theta}_i = \mathbf{0}$.

If each person invest $\boldsymbol{\theta}_i$, then we allocate $\mathbf{c}_i = \boldsymbol{\varepsilon}_i + \mathbf{S_T'}\boldsymbol{\theta_i}$ for each one, and therefore we can define Pareto optimal etc.

Theorem 2.8 (The First Welfare Theorem)**.** *Under complete market, equilibrium is Pareto optimal.*

2.4 Discrete Martingale

Under risk-neutral measure, for each time k, we have

$$\psi_0 \underset{k}{\mathbb{E}} S_{k+1} = S_k$$

thus if we define the discounted stock price as $\widetilde{S}_k = \psi_0^k S_k$(where each ψ_0 can be vary by time so it should be $\prod_{i=1}^k \psi_{0,k}$ but we abuse notation here), then $\widetilde{S}$ became a martingale. That implies the discounted claim $\widetilde{V}$ is also a martingale.

Now let (φ_n, ψ_n) be the amount of stock and bound at time n, the portfolio we holding value:

$$V_n = \varphi_n S_n + \psi_n B_n$$

since the portfolio is self-financing, the should equal to the value at the start of $n+1$(when the price is remain the same):

$$V_n = \varphi_{n+1} S_n + \psi_{n+1} B_n$$

Put them together and take discount, we have

$$\widetilde{V}_n = \varphi_{n+1}\widetilde{S}_n + \psi_{n+1} = \varphi_n \widetilde{S}_n + \psi_n$$

we have

$$\widetilde{V}_{n+1} - \widetilde{V}_n = \varphi_{n+1}(\widetilde{S}_{n+1} - \widetilde{S}_n)$$

By induction we have

$$\widetilde{V}_n = V_0 + \sum_{i=0}^{n-1} \varphi_{i+1}(\widetilde{S}_{i+1} - \widetilde{S}_i)$$

that is a martingale by invoking following lemma:

Lemma 2.1. *Suppose process $X = \{X_t\}_{t\in\mathbb{N}}$ is adapted to $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{N}}$ and $\{\varphi_i\}_{i\in\mathbb{N}^+}$ is $\mathbb{F}$-predictable. Then*

$$\left\{ Z_n \triangleq Z_0 + \sum_{i=0}^{n-1} \varphi_{i+1}(X_{i+1} - X_i) \right\}_{n\in\mathbb{N}}$$

is a martingale if so is X.

Now we turn to stocks market consist N stocks, construct sample space as Ω be all possible path $(\mathbb{R}^N)^T$. The absence of arbitrage give a risk-neutral measure for which

$$S_{t-1} = \psi_0 \underset{S_{t-1}}{\mathbb{E}} S_t$$

and as before, take discount $\widetilde{S}_t = \psi_0^t S_t$ we have a martingale. And we claim that absence of martingale is equivalent to existence of risk-neutral measure.