Homework 1

Xie Zejian
11810105@mail.sustech.edu.cn

Department of Finance, SUSTech

Last compiled on 04:04, 24 , 2021

Contents

1 Simple Example

Suppose one buy a call option C which payoff $V_1 = (S_1 - K)^+$ at time 1, S_1 can be $uS_0 > K$ or $dS_0 < K$ determined in probability space $(\{H, T\}, 2^{\{H,T\}}, \mathbb{P})$. To replicate such option, we construct our portfolio by buying Δ_0 stock and investing remaining in risk-free asset at return r:

$$(V_0 - \Delta_0 S_0)(1 + r) + \Delta_0 S_1 = (S_1 - K)^+$$

solve:

$$V_0 = \frac{pV_1(H) + qV_1(T)}{1 + r}, \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

where $p = \frac{1+r-d}{u-d}, q = 1 - p$.

p and q can be then seen as probability assigned to $\mathbb{P}\{H\}$ and $\mathbb{P}\{T\}$. Then V_0 is just the discounted of expected value of such option, such measure $\mathbb{P}$ is called **risk-neutral**.

2 Single-period Model

Suppose $\mathbf{S_0} \in \mathbb{R}^N$ is price of N stocks at time 0, and $\mathbf{D} \in \mathbb{R}^{N\times n}$ is their price at time t for n states. For any portfolio $\boldsymbol{\theta} \in \mathbb{R}^N$, it cost $\mathbf{S_0'}\boldsymbol{\theta}$ and its value is $\mathbf{D'}\boldsymbol{\theta} \in \mathbb{R}^n$ for all n states.

An arbitrage is then defined as a portfolio $\theta, s.t.$ $\mathbf{S'}\boldsymbol{\theta}$ have different sign with $\mathbf{D'}\boldsymbol{\theta}$.

Definition 2.1 (State price)**.** A state price vector is $\psi \in \mathbb{R}^n_{++}$ $s.t.$ $\mathbf{S_0} = \mathbf{D}\boldsymbol{\psi}$.

To justify the name of state price, suppose we want to "bet" the state of market, *i.e.*, we would like earning $\mathbf{1}_{state=i}$, then our portfolio supposed to be $\mathbf{D'}\boldsymbol{\theta} = \mathbf{e_i}$, and it cost

$$\mathbf{S_0'}\boldsymbol{\theta} = \boldsymbol{\psi}'\mathbf{D'}\boldsymbol{\theta} = \boldsymbol{\psi}'\mathbf{e_i} = \psi_i$$

so the coordinate of $\boldsymbol{\psi}$ is the price of "betting" a state.

Theorem 2.1. *There is no arbitrage iff there is a state price vector.*

Proof.

Theorem 2.2 (Separating Hyperplane Theorem)**.** *Suppose M and K are closed convex cones in $\mathbb{R}^d$ that $M \cap K = \{\mathbf{0}\}$, if K isn't a liner space, then there is a nonzero linear f separated them, i.e., $f(x) < f(y)$ for any $x \in M$ and $y \in K - \{\mathbf{0}\}$.*

Theorem 2.3 (Riesz Representation Theorem)**.** *Any continuous linear function f on Hilbert space $\mathcal{H}$ can be written as $f(x) = \langle x, v \rangle$ for some $v \in \mathcal{H}$.*

Let $M = \left\{(-\mathbf{S_0'}\boldsymbol{\theta}, \mathbf{D'}\boldsymbol{\theta}) : \theta \in \mathbb{R}^N\right\}$ and $K = \mathbb{R}_+ \times \mathbb{R}_+^n$. Then there is no arbitrage iff $K \cap M = \{\mathbf{0}\}$.

$\Longrightarrow$, let f be the functional in theorem 2.2, note M is a linear space, f should be vanish on M, *i.e.*, $f(x) = 0, \forall x \in M$, otherwise, fix $f(y) > 0$ for $y \in K - \{0\}$, we can find $\lambda \in \mathbb{R}$ *s.t.* $\lambda f(x) = f(\lambda x) > f(y)$.

Then by theorem 2.3, we have $f(x) = x'v$ for some v, write $v = (\alpha, \phi)$ where $\alpha \in \mathbb{R}$ and $\phi \in \mathbb{R}^n$. Since $f(x) > 0$ for nonzero $x \in K$, α and ϕ should strictly positive, then

$$-\alpha \mathbf{S_0'}\boldsymbol{\theta} + \boldsymbol{\phi}'\mathbf{D'}\boldsymbol{\theta} = 0$$

which implies $-\alpha \mathbf{S_0} + \mathbf{D}\boldsymbol{\phi} = \mathbf{0}$ and thus $\frac{\phi}{\alpha}$ is a state price vector as required.

$\Longleftarrow$, Suppose $(-\mathbf{S_0'}\boldsymbol{\theta}, \mathbf{D'}\boldsymbol{\theta}) \in K$, then, $\boldsymbol{\psi}'\mathbf{D'}\boldsymbol{\theta} \leq 0$ and $\mathbf{D'}\boldsymbol{\theta} \geq 0$, note $\psi \gg 0$ force $\mathbf{D'}\boldsymbol{\theta} = \mathbf{0}$, thus $K \cap M = \{0\}$ as required.

□

Exercise 2.1. 1

Solution. Given above.

Exercise 2.2. 2

Solution. Setup:

$$\mathbf{S_0} = (1, S_0)', \mathbf{D} = \begin{bmatrix} 1+r & 1+r \\ uS_0 & dS_0 \end{bmatrix}$$

then the state price should be

$$\boldsymbol{\psi} = \mathbf{D^{-1}S_0} = [(-u+1+r)S_0, (d-1-r)S_0]' \gg 0$$

then claim follows.

Exercise 2.3. 3

Solution. No. Let $\psi = (\frac{1}{3}, \frac{1}{3})$, it's a state price vector.

Exercise 2.4. 4

Solution. Note column space of D is just $\{\lambda \cdot (1, 2, 3)' : \lambda \in \mathbb{R}\}$, which excluded $\overline{q}$, therefore there is no state price vector and thus arbitrage exists.