homework 1

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1.1 1

Lemma 1.1. *Suppose* $\mathbf{A}$ *with eigenvalues* λ_i *is symmetric, then*

$$
\begin{aligned}
eig(\mathbf{I} + c\mathbf{A}) &= 1 + c\lambda_i \\
eig(\mathbf{A} - c\mathbf{I}) &= \lambda_i - c
\end{aligned}
$$

Proof. Note

$$
|\mathbf{I} + c\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{I} + c\lambda_i\mathbf{I} - \lambda\mathbf{I}| = 0 \Rightarrow \lambda = 1 + c\lambda_i
$$

the other one can be proved similarly.

$\square$

Lemma 1.2. *Suppose* $\mathbf{A} \in \mathbb{R}^{n \times m}$ *and* $\mathbf{B} \in \mathbb{R}^{m \times n}$*, then*

$$eig(\mathbf{AB}) = eig(\mathbf{BA})$$

Proof. Suppose $\lambda_i \in \mathrm{eig}(\mathbf{AB})$, then

$$\mathbf{ABx} = \lambda_i \mathbf{x} \Rightarrow \mathbf{BABx} = \lambda_i \mathbf{Bx} \Rightarrow \lambda_i \in \mathrm{eig}(\mathbf{BA})$$

and the other direction follows from symmetry.

$\square$

Then we have following corollary:

Corollary 1.1. *Suppose* $\mathbf{x}$ *is a* $n-$*vector, then*

$$eig(\mathbf{xx'}) = \underbrace{0, 0, \ldots, 0}_{n-1}, \mathbf{x'x}$$

Note

$$\mathbf{A} = \mathbf{I} - \rho(-\mathbf{I} + \mathbf{ee'})$$

Where $\mathbf{e}$ is p all-one vector and $\mathbf{ee'}$ has one eigenvalue of p and $p-1$ eigenvalues of 0, then $\mathbf{A}$ has $p-1$ eigenvalues of $1-\rho$ and one $1+(p-1)\rho$ and thus

$$|\mathbf{A}| = (1-\rho)^{p-1}[1+(p-1)\rho]$$

1.2 2

Let $|A| = 0$, we have $(1-\rho) = 0$ or $1+(p-1)\rho = 0$ and thus

$$\rho = \begin{cases} 1 \\ -\frac{1}{p-1} \end{cases}$$

1.3 3

Already done in 1.

2 2

Note eigenvalues of $c\mathbf{A}$ is $c\lambda_i$. Plug $\rho = .5$ and $p = 3$, we find

$$\mathrm{eig}\left(\mathbf{A}\right) = 4, 1, 1$$

and the corresponding eigenvectors are:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

3 3

3.1 1

$$\begin{aligned}
\sum_{i=1}^{n} (x_i - \overline{x})^2 &= \left(\mathbf{x} - \overline{x}\mathbf{e}\right)' \left(\mathbf{x} - \overline{x}\mathbf{e}\right) \\
&= \left(\mathbf{x} - \frac{\mathbf{e'x}}{\mathbf{e'e}}\mathbf{e}\right)' \left(\mathbf{x} - \frac{\mathbf{e'x}}{\mathbf{e'e}}\mathbf{e}\right) \\
&= \mathbf{x'(I - P_e)'(I - P_e)x} \\
&= \mathbf{x'(I - P_e)x}
\end{aligned}$$

where $\mathbf{P_e} = \frac{\mathbf{ee'}}{\mathbf{e'e}}$ and thus

$$\mathbf{A} = \mathbf{I} - \mathbf{P_e}$$

3.2 2

It's symmetric and idempotent and thus a projection. In fact we have

Lemma 3.1. *If* $\mathbf{P}$ *is a projection matrix, so is* $\mathbf{I} - \mathbf{P}$

Proof. Symmetric follows from both $\mathbf{I}$ and $\mathbf{P}$ is symmetric and idempotent follows from

$$(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I^2} + \mathbf{P^2} - \mathbf{PI} - \mathbf{IP}) = \mathbf{I} - \mathbf{P}$$

□

3.3 3

Nonnegative define as $\sum_{i=1}^{n} (x_i - \overline{x})^2 \geq 0$ clearly.

3.4 4

$$\begin{aligned}
\mathrm{rank}\left(\mathbf{I} - \mathbf{P_e}\right) &= \mathrm{tr}\left(\mathbf{I} - \mathbf{P_e}\right) \\
&= \mathrm{tr}\left(\mathbf{I}\right) - \mathrm{tr}\left(\mathbf{P_e}\right) \\
&= n - \mathrm{rank}\left(\mathbf{P_e}\right) \\
&= n - 1
\end{aligned}$$

4 4

By 1 we have find it's eigenvalues are $1+\rho$ and $1-\rho$, thus the corresponding eigenvectors is

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and thus

$$\mathbf{A} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1-\rho & \\ & 1+\rho \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Similarly,

$$\mathbf{A^{-1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{1-\rho} & \\ & \frac{1}{\rho+1} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and we find that $\frac{1}{\mathrm{eig}(\mathbf{A})} = \mathrm{eig}\left(\mathbf{A^{-1}}\right)$.

5 5

Note

$$\mathrm{eig}\left(\mathbf{A}\right) = \{6 \pm \sqrt{33}, 10\}$$

thus the maximum is $6+\sqrt{33}$ and minimum is $6-\sqrt{33}$.

6 6

6.1 1

Note $\frac{\partial \mathbf{X'AX}}{\partial \mathbf{x}} = (\mathbf{A}+\mathbf{A'})\mathbf{x}$ for general $\mathbf{A}$ and $\mathbf{x}$:

$$\frac{\partial \frac{\mathbf{x'Ax}}{\mathbf{x'x}}}{\partial \mathbf{x}} \propto 2\mathbf{Ax}(\mathbf{x'x}) - \mathbf{x'Ax}2\mathbf{x} = 0 \Rightarrow \mathbf{Ax} = \frac{\mathbf{x'Ax}}{\mathbf{x'x}}\mathbf{x}$$

that implies the extreme value occurs when $\mathbf{x}$ is the eigenvectors and the eigenvalues is the value of $\frac{\mathbf{x'Ax}}{\mathbf{x'x}}$. Thus

$$\max_{x \neq 0} \frac{\mathbf{x'Ax}}{\mathbf{x'x}} = \lambda_1$$
$$\min_{x \neq 0} \frac{\mathbf{x'Ax}}{\mathbf{x'x}} = \lambda_p$$

and holds when $\mathbf{x} = \mathbf{t_1}$ and $\mathbf{x} = \mathbf{t_p}$ respectively.

6.2 2

Lemma 6.1. *If* $\mathbf{A}$ *is positive definite then*

$$2\mathbf{x'Ax} = \mathbf{x'(A + A')x}$$

Proof. As $\mathbf{x'Ax}$ is a scalar, then $\mathbf{x'Ax = x'A'x}$ thus $\mathbf{x'(A - A')x = 0}$

$$\begin{aligned}\mathbf{X'(A + A')x} &= \mathbf{x'}[2\mathbf{A} - (\mathbf{A - A'})]\mathbf{x} \\ &= 2\mathbf{x'Ax}\end{aligned}$$

$\square$

By lemma 6.1:

$$\begin{aligned}\frac{\partial \frac{\mathbf{x'Ax}}{\mathbf{x'Bx}}}{\partial \mathbf{x}} &\propto 2\mathbf{Ax}(\mathbf{x'Bx}) - \mathbf{x'Ax(B + B')x} \\ &= 2\mathbf{Ax}(\mathbf{x'Bx}) - \mathbf{x'Ax}2\mathbf{Bx} \\ &\Rightarrow \mathbf{Ax} = \frac{\mathbf{x'Ax}}{\mathbf{x'Bx}}\mathbf{Bx}\end{aligned}$$

thus $\mathbf{x}$ is eigenvectors of $\mathbf{B^{-1}A}$ and correspond to $\frac{\mathbf{x'Ax}}{\mathbf{x'Bx}}$ and the claim follows easily.