homework 2

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2 Random vector and matrix

Exercise 2.1. Suppose $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then

1. $\mathbb{E}\,\mathbf{x}\mathbf{x}' = \Sigma + \boldsymbol{\mu}\boldsymbol{\mu}'$
2. $\mathbb{E}\,\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbb{E}\operatorname{tr}(\mathbf{x}\mathbf{x}'\mathbf{A}) = \operatorname{tr}(\boldsymbol{\Sigma}\mathbf{A}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$
3. Put $\boldsymbol{\mu} = \mu\mathbf{e}$, $\Sigma = \sigma^2\mathbf{I}$ and $A = \mathbf{I} - \mathbf{P}_\mathbf{e}$ where $\mathbf{P}_\mathbf{e} = \frac{\mathbf{e}\mathbf{e}'}{\mathbf{e}'\mathbf{e}}$, show $\mathbb{E}\,\frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\sigma^2} = \mathbf{e}'\mathbf{e} - 1$

Solution. 1. Note

$$\boldsymbol{\Sigma} = \mathbb{E}\,(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})' = \mathbb{E}\,\mathbf{x}\mathbf{x}' - \mathbb{E}\,\mathbf{x}\boldsymbol{\mu}' - \mathbb{E}\,\boldsymbol{\mu}\mathbf{x}' + \boldsymbol{\mu}\boldsymbol{\mu}' = \mathbb{E}\,\mathbf{x}\mathbf{x}' - \boldsymbol{\mu}\boldsymbol{\mu}'$$

then claim follows.

2. The first equality is clear as $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$. For the second, note

$$\mathbb{E}\,\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbb{E}\,(\mathbf{x}-\boldsymbol{\mu})'\mathbf{A}(\mathbf{x}-\boldsymbol{\mu}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$$

then

$$\begin{aligned}\mathbb{E}\,(\mathbf{x}-\boldsymbol{\mu})'\mathbf{A}(\mathbf{x}-\boldsymbol{\mu}) &= \mathbb{E}\operatorname{tr}\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})'\mathbf{A}\right] \\ &= \operatorname{tr}\left[\mathbb{E}\,(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})'\mathbf{A}\right] \\ &= \operatorname{tr}(\boldsymbol{\Sigma}\mathbf{A}) = \operatorname{tr}(\mathbf{A}\boldsymbol{\Sigma})\end{aligned}$$

3. By **2**, we have

$$\begin{aligned}\mathbb{E}\,\frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\sigma^2} &= \frac{1}{\sigma^2}\operatorname{tr}\left[\boldsymbol{\sigma}^2(\mathbf{I}-\mathbf{P}_\mathbf{e})\right] + \frac{1}{\sigma^2}\mu^2\mathbf{e}'(\mathbf{I}-\mathbf{P}_\mathbf{e})\mathbf{e} \\ &= \operatorname{tr}(\mathbf{I}-\mathbf{P}_\mathbf{e}) \quad (\text{as } \mathbf{e} \perp \mathbf{I}-\mathbf{P}_\mathbf{e}) \\ &= \operatorname{rank}(\mathbf{I}-\mathbf{P}_\mathbf{e}) = n-1 = \mathbf{e}'\mathbf{e} - 1\end{aligned}$$

Exercise 2.2. Suppose *r.v.* X and Y are *i.i.d.* with *p.d.f.*:

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

show that $X + Y$ and $\frac{X}{Y}$ are independent.

Solution. As X and Y are independent, the joint distribution is:

$$f(x, y) = f_X(x) f_Y(y) = e^{-x-y}$$

Let $U = X + Y$, $V = \frac{X}{Y}$ then $X = \frac{UV}{V+1}$, $Y = \frac{U}{V+1}$, thus the Jacobi determinant is

$$J(u, v) = \begin{vmatrix} \frac{v}{v+1} & \frac{u}{(1+v)^2} \\ \frac{1}{v+1} & \frac{-u}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^2}$$

and hence

$$f_{U,V}(u, v) = J(u, v) \cdot \exp -(\frac{uv + u}{v + 1}) = \frac{u}{(1 + v)^2} \exp -u$$

Then the marginal distribution of U and V are given by

$$f_U(u) = \int_0^\infty f_{U,V}(u, v) dv = ue^{-u}$$

$$f_V(v) = \int_0^\infty f_{U,V}(u, v) du = \frac{1}{(1 + v)^2}$$

So we observed that $f_{U,V} = f_U f_V$ and the claim follows.

3 Multivariate normal distribution

Exercise 3.1. Let $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} 9 & -3 & -3 \\ -3 & 5 & 1 \\ -3 & 1 & 5 \end{bmatrix}$$

Find conditional distribution of $x_1 | x_2, x_3$ and $(x_1, x_2) | x_3$.

Solution.

Lemma 3.1. *Suppose* $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, *and* $\mathbf{x}$ *is partitioned as* $\mathbf{x} = \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix}$ *and correspondingly:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu_1} \\ \boldsymbol{\mu_2} \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma_1} & \mathbf{B} \\ \mathbf{B'} & \boldsymbol{\Sigma_2} \end{bmatrix}$$

then the conditional $\mathbf{x_1} | \mathbf{x_2} = \mathbf{a} \sim \mathcal{N}(\overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\Sigma}})$ *for some* $\overline{\boldsymbol{\mu}}$ *and* $\overline{\boldsymbol{\Sigma}}$.

Proof. In the light of Schur components, note

$$\begin{bmatrix} \mathbf{I} & -\mathbf{B}\boldsymbol{\Sigma_2}^{-1} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma_1} & \mathbf{B} \\ \mathbf{B'} & \boldsymbol{\Sigma_2} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{B}\boldsymbol{\Sigma_2}^{-1} \\ 0 & \mathbf{I} \end{bmatrix}' = \begin{bmatrix} \boldsymbol{\Sigma_1} - \mathbf{B}\boldsymbol{\Sigma_2}^{-1}\mathbf{B'} & 0 \\ 0 & \boldsymbol{\Sigma_2} \end{bmatrix}$$

where $\overline{\boldsymbol{\Sigma}} := \boldsymbol{\Sigma_1} - \mathbf{B}\boldsymbol{\Sigma_2}^{-1}\mathbf{B'}$ is the Schur components of $\boldsymbol{\Sigma_1}$. Then define

$$\mathbf{y} = \begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix} := \begin{bmatrix} \mathbf{I} & -\mathbf{B}\boldsymbol{\Sigma_2}^{-1} \\ 0 & \mathbf{I} \end{bmatrix} \mathbf{x} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu_1} - \mathbf{B}\boldsymbol{\Sigma_2}\boldsymbol{\mu_2} \\ \boldsymbol{\mu_2} \end{bmatrix}, \begin{bmatrix} \overline{\boldsymbol{\Sigma}} & 0 \\ 0 & \boldsymbol{\Sigma_2} \end{bmatrix} \right)$$

Clearly, y_1 and y_2 are independent and thus we can find their joint distribution easily and that of $\mathbf{x}$ follows by simple transformation.

Note the Jacobian is just

$$\det\left(\begin{bmatrix}\mathbf{I} & -\mathbf{B\Sigma_2^{-1}} \\ 0 & \mathbf{I}\end{bmatrix}\right) = \det(\mathbf{I}) = 1$$

thus

$$f_{\mathbf{x_1}|\mathbf{x_2}} = \frac{f_{\mathbf{x}}}{f_{\mathbf{x_2}}} = \frac{f_{\mathbf{y_1}} \cdot f_{\mathbf{y_2}}}{f_{\mathbf{y_2}}} = f_{\mathbf{y_1}}(x_1 - \mathbf{B\Sigma_2^{-1}}x_2)$$

where $\mathbf{y_1} \sim \mathcal{N}(\boldsymbol{\mu_1 - \mathbf{B\Sigma_2^{-1}}\mu_2}, \overline{\mathbf{\Sigma}})$ and hence

$$\mathbf{x_1}|\mathbf{x_2} = \mathbf{a} \sim \mathcal{N}(\overline{\boldsymbol{\mu}}, \overline{\mathbf{\Sigma}})$$

where $\overline{\boldsymbol{\mu}} = \boldsymbol{\mu_1} + \mathbf{B\Sigma_2^{-1}(a} - \boldsymbol{\mu_2})$

$\square$

By above lemma and recall

$$f_{\mathbf{X}} = \frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k|\mathbf{\Sigma}|}}$$

we have, for $x_1|x_2, x_3$:

$$\overline{\boldsymbol{\mu}} = \frac{31 - x_2 - x_3}{2}, \overline{\mathbf{\Sigma}} = 6$$

and thus

$$f_{x_1|x_2,x_3}(x1) = \frac{1}{\sqrt{12\pi}}\exp\left(-\frac{(x - \frac{31-x_2-x_3}{2})}{12}\right)$$

And for $(x_1, x_2)|x_3$:

$$\overline{\boldsymbol{\mu}} = \begin{bmatrix}\frac{71-3x_3}{5} \\ \frac{13+x_3}{5}\end{bmatrix}, \overline{\mathbf{\Sigma}} = \begin{bmatrix}\frac{36}{5} & -\frac{12}{5} \\ -\frac{12}{5} & \frac{24}{5}\end{bmatrix}$$

thus

$$f_{(x_1,x_2)|x_3}(x_1, x_2) = \frac{\exp(-\frac{1}{2}\frac{x_1^2}{6} + \frac{x_1x_2}{6} + \frac{x_1x_3}{6} - \frac{31x_1}{6} + \frac{x_2^2}{4} - \frac{11x_2}{3} + \frac{x_3^2}{20} - \frac{71x_3}{30} + \frac{829}{20})}{24\pi/\sqrt{5}}$$

Exercise 3.2. Let $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \sigma^2\mathbf{I})$, and define

$$\overline{\mathbf{x}} = \frac{\mathbf{e'x}}{\mathbf{e'e}}, s^2 = \frac{1}{n-1}\mathbf{x'(I - P_e)x}$$

where n is length of $\mathbf{x}$. Then

1. $\overline{\mathbf{x}}$ and s^2 are independent.
2. $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$

Solution. **Independence**. Note $\overline{\mathbf{x}}$ is function of $\mathbf{e'x}$ while s^2 is so of $\mathbf{(I - P_e)x}$, so it's sufficient to show independence of $\mathbf{e'x}$ and $\mathbf{(I - P_e)x}$. To see that, suppose random vector:

$$\mathbf{y} := \begin{bmatrix}\mathbf{e'x} \\ \mathbf{(I - P_e)x}\end{bmatrix} = \begin{bmatrix}\mathbf{e'} \\ \mathbf{(I - P_e)}\end{bmatrix}\mathbf{x}$$

which is distributed as $\mathcal{N}_{n+1}$ with covariance:

$$\begin{bmatrix}\mathbf{e'} \\ \mathbf{(I - P_e)}\end{bmatrix}\sigma^2\mathbf{I}\begin{bmatrix}\mathbf{e'} \\ \mathbf{(I - P_e)}\end{bmatrix}' = \sigma^2\begin{bmatrix}\mathbf{e'e} & \mathbf{0} \\ \mathbf{0} & \mathbf{I - P_e}\end{bmatrix}$$

Thus $\mathbf{e'x}$ and $(\mathbf{I}-\mathbf{P_e})\mathbf{x}$ are independent and then claim follows.

Distribution. Note
$$\frac{(n-1)s^2}{\sigma^2}=\frac{\mathbf{x'(I-P_e)x}}{\sigma^2}=\mathbf{x'}\left(\frac{\mathbf{I-P_e}}{\sigma^2}\right)\mathbf{x}$$
which motivated us introduce following lemmas:

Lemma 3.2. *Suppose symmetric matrix $p\times p$ $\mathbf{A}$. It's idempotent of rank s iff there exist a $p\times s$ $\mathbf{P}$ s.t. $\mathbf{PP'}=\mathbf{A}$ and $\mathbf{P'P}=\mathbf{I}$.*

Proof. Sufficiency is trivial. For necessity, since $\mathbf{A}$ is symmetric and idempotent matrix, it can be spectral decompose by $\mathbf{A}=\mathbf{Q\Lambda Q'}$. Where the diagonal of $\mathbf{\Lambda}$ is s 1 and $p-s$ 0. Thus
$$\mathbf{A}=\mathbf{Q\Lambda Q'}=\begin{pmatrix}\mathbf{P_1} & \mathbf{P_2}\end{pmatrix}\begin{pmatrix}\mathbf{I}_s & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_1'\\ \mathbf{P}_2'\end{pmatrix}=\mathbf{P}_1\mathbf{P}_1'$$

Note
$$\mathbf{I}_p=\mathbf{Q'Q}=\begin{pmatrix}\mathbf{P}_1'\\ \mathbf{P}_2'\end{pmatrix}\begin{pmatrix}\mathbf{P}_1 & \mathbf{P}_2\end{pmatrix}=\begin{pmatrix}\mathbf{P}_1'\mathbf{P}_1 & \mathbf{P}_1'\mathbf{P}_2\\ \mathbf{P}_2'\mathbf{P}_1 & \mathbf{P}_2'\mathbf{P}_2\end{pmatrix}=\begin{pmatrix}\mathbf{P}_1'\mathbf{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_2'\mathbf{P}_2\end{pmatrix}$$
hence $\mathbf{P_1'P_1}=\mathbf{I_s}$.

$\square$

Lemma 3.3. *If $\mathbf{Y}\sim N_p(\mu,\mathbf{I_p})$,if $\mathbf{A}$ is idempotent of rank s, then $\mathbf{Y'AY}\sim\chi_s^2(\frac{1}{2}\mu'\mathbf{A}\mu)$*

Proof. Since $\mathbf{A}=\mathbf{PP'}$ where $\mathbf{P}$ is $p\times s$. Thus
$$\mathbf{Y'AY}=\mathbf{Y'PP'Y}=\mathbf{X'X}$$
where $\mathbf{X}=\mathbf{P'Y}\sim N_s(\mathbf{P'}\mu,\mathbf{I})$. From previous result we have $\mathbf{X'X}\sim\chi_s^2(\lambda)$ where
$$\lambda\equiv\frac{1}{2}\left(\mathbf{P}_1'\boldsymbol{\mu}\right)'\mathbf{P}_1'\boldsymbol{\mu}=\frac{1}{2}\boldsymbol{\mu}'\mathbf{P}_1\mathbf{P}_1'\boldsymbol{\mu}=\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}.$$

$\square$

Lemma 3.4. *If $\mathbf{Y}\sim N_p(\mu,\mathbf{V})$, where we assume $\mathbf{V}$ is nonsingular, if $\mathbf{AV}$ is idempotent of rank s,then $\mathbf{Y'AY}\sim\chi_s^2(\frac{1}{2}\mu'\mathbf{A}\mu)$.*

Proof. We can construct $\mathbf{X}=\mathbf{V}^{-1/2}\mathbf{Y}\sim N_p(\mathbf{V}^{-1/2}\mu,\mathbf{I})$. Then
$$\mathbf{Y'AY}=\mathbf{X'V}^{1/2}\mathbf{AV}^{1/2}\mathbf{X}=\mathbf{X'BX}$$
Where $\mathbf{B}=\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}}$, it's idempotent since
$$\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}}\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}}=\sqrt{\mathbf{V}}\mathbf{AVA}\sqrt{\mathbf{V}}=\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}}$$
Note that $\mathbf{AVA}=\mathbf{A}$ since $\mathbf{AVAV}=\mathbf{AV}$ and $\mathbf{V}$ is nonsingular. then
$$rank(\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}})=tr(\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}})=tr(\mathbf{AV})=rank(\mathbf{AV})=s$$
From last result, we have $\mathbf{Y'AY}=\mathbf{X'BX}\sim\chi_s^2(\lambda)$, where
$$\lambda=\frac{1}{2}\left(\sqrt{\mathbf{V}}\boldsymbol{\mu}\right)'\mathbf{B}\left(\sqrt{\mathbf{V}}\boldsymbol{\mu}\right)=\frac{1}{2}\boldsymbol{\mu}'\sqrt{\mathbf{V}}\sqrt{\mathbf{V}}\mathbf{A}\sqrt{\mathbf{V}}\sqrt{\mathbf{V}}\boldsymbol{\mu}=\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$$

$\square$

Apply last lemma, note $\mathbf{AV} = \mathbf{I} - \mathbf{P_e}$ has rank $n-1$ and the noncentric parameter λ is

$$\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} = \frac{1}{2}\boldsymbol{\mu}'(\mathbf{I} - \mathbf{P_e})\boldsymbol{\mu}$$

thus

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}(\frac{1}{2}\boldsymbol{\mu}'(\mathbf{I} - \mathbf{P_e})\boldsymbol{\mu})$$