FACTOR MODEL

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November 4, 2020

1 CAPM

1.1 Beta representation

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}}-r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}}-r_f\mathbf{e})}$. Write $\omega_D = m\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})$ where $m = \frac{1}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}}-r_f\mathbf{e})}$, then we have

$$\overline{r} - r_f\mathbf{e} = \frac{1}{m}\mathbf{V}\omega_D$$

Note $\mathrm{Cov}(\mathbf{r}, \omega'\mathbf{r}) = \mathbf{V}\omega$ and

$$\sigma_D^2 = \omega_D'\mathbf{V}\omega_D = m\omega_D'(\overline{\mathbf{r}} - r_f\mathbf{e}) = mr_D - mr_f$$

we have

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \frac{r_D - r_f}{\sigma_D^2}\,\mathrm{Cov}(\mathbf{r}, r_D)$$

Denote $\frac{\mathrm{Cov}(\mathbf{r},r_D)}{\sigma_D^2} = \beta_D$, we have

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \beta_D(r_D - r_f)$$

Similar results also holds for any portfolio $\overline{r}_p$ in the MVF:

$$\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} = \beta_p(\overline{r}_p - \overline{r}_q)$$

It's clear in the view of every portfolio $\overline{r}_p$ is also a tangency portfolio by selecting proper r_f. One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega_p'\mathbf{V}\omega_q = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})(\overline{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

If the covariance is 0, we have

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Then

$$\begin{aligned}
\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} &= \overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e} \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\delta^2(\overline{r}_p - \alpha/\delta))(\overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta(\overline{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2))\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e} \\
&= \frac{(\delta^2\overline{r}_p\overline{\mathbf{r}} - \alpha\delta\overline{\mathbf{r}}) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e}}{\delta^2(\overline{r}_p - \alpha/\delta)} \\
&= \frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\delta(\overline{r}_p - \alpha/\delta)}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\beta_p &= \frac{\mathbf{V}\omega_{\mathbf{p}}}{\omega_p'\mathbf{V}\omega_p} \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\lambda_p\overline{\mathbf{r}} + \gamma\mathbf{e}) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{\xi\mathbf{e} - \alpha\overline{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha\mathbf{e} + \delta\overline{\mathbf{r}}}{\delta\xi - \alpha^2}\overline{r}_p) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\Delta})
\end{aligned}$$

Then it's remain to show that

$$(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) = \omega'\mathbf{V}\omega\Delta$$

It's clear since

$$\omega'\mathbf{V}\omega\Delta = \sigma_p^2\Delta = \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2$$

and

$$\begin{aligned}
(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) &= ((\overline{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\delta(\overline{r}_p - \alpha/\delta) \\
&= \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2
\end{aligned}$$

1.2 CAPM

In capital market equilibrium, the market portfolio is tangecy portfolio $r_D = r_m$, then

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \beta_m(r_m - r_f)$$

where

$$\beta_m = \begin{bmatrix} \frac{\mathrm{Cov}(r_1, r_m)}{\sigma_m^2} \\ \frac{\mathrm{Cov}(r_2, r_m)}{\sigma_m^2} \\ \cdots \\ \frac{\mathrm{Cov}(r_n, r_m)}{\sigma_m^2} \end{bmatrix}$$

this equation is called **Sharpe-Lintner CAPM**. $r_m - r_f$ is called **market risk premium** and $\frac{r_m - r_f}{\sigma_m}$ is called **market sharpe ratio**. Translate it from vector form, we get the **Security Market Line**:

$$r_i - r_f = \beta_{i,m}(r_m - r_f)$$

1.2.1 Realized return

Now consider both r_i and r_m is random variable, let ϵ be a random vector with zero expection and zero covariance with r_i and r_m, then

$$r_i - r_f = \beta_{i,m}(r_m - r_f) + \epsilon_i$$

This is a regression equation, if one include an intercept, then the model

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \epsilon_i$$

is called **market model**, such α is called **Jensen's alpha**.

1.2.2 Variance decomposition

Decomposition the variance as:

$$\mathrm{Var}(r_i) = \overbrace{\underbrace{\beta_i \sigma_m^2}_{\text{Systematic risk}} + \underbrace{\mathrm{Var}(\epsilon_i)}_{\text{Idiosyncratic risk}}}^{\text{total risk}}$$

The R^2 is just the proportion of systematic risk

$$R^2 = \frac{\beta_i^2 \sigma_m^2}{\beta_i^2 \sigma_m^2 + \sigma^2}$$

since Fraction of variance unexplained.

1.2.3 Testing CAPM

Suppose we run time series regressions for each of the n risky assets

$$\mathbf{r_t^e} = \alpha + \beta r_{m,t}^e + \nu_t$$

where $\alpha, \mathbf{r_t^e}, \beta, \nu_t$ are $n \times 1$ vector and $r_{m,t}^e$ is scalar.

By the discussion above, $\alpha = \mathbf{0}$ when CAPM holds. Assume $\{\nu_t\}_{t=1}^T$ i.i.d with $\mathcal{N}(0, \Sigma)$, we have $\mathbf{r_t^e} \mid r_{m,t}^e \sim \mathcal{N}(\alpha + \beta r_{m,t}^e, \Sigma)$.

Suppose

$$\mathbf{r} = \begin{bmatrix} \mathbf{r_1^e}' \\ \mathbf{r_2^e}' \\ \mathbf{r_3^e}' \\ \vdots \\ \mathbf{r_T^e}' \end{bmatrix}, \mathbf{r_m} = \begin{bmatrix} r_{m,1}^e \\ r_{m,2}^e \\ r_{m,3}^e \\ \vdots \\ r_{m,T}^e \end{bmatrix}$$

Now $\mathbf{r}' \sim \mathcal{MN}_{n\times T}(\alpha\mathbf{e}' + \beta\mathbf{r}'_{\mathbf{m}}, \mathbf{\Sigma}, \mathbf{I})$, the p.d.f is (Wikipedia contributors 2019)

$$p(\mathbf{r}'|\beta\mathbf{r}'_{\mathbf{m}}, \mathbf{\Sigma}, \mathbf{I}) = \frac{\exp(-\frac{1}{2}\operatorname{Tr}[(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})'\mathbf{\Sigma}^{-}(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})])}{(2\pi)^{nT/2}T^{n/2}|\mathbf{\Sigma}|^{T/2}}$$

thus the log likelihood function is

$$\log L = -\frac{1}{2}\operatorname{Tr}[(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})'\mathbf{\Sigma}^{-}(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})] - \frac{nT}{2}\log 2\pi - \frac{n}{2}\log T - \frac{T}{2}\log|\Sigma|$$

FOC w.r.t α, by chain rule(Petersen and Pedersen 2012)

$$\begin{aligned}
\partial \log L &= \operatorname{Tr}\left(\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})}' \partial(\mathbf{X} - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})\right) \\
&= \operatorname{Tr}\left(\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})}' \partial\alpha(-\mathbf{e}')\right) \\
&= \operatorname{Tr}\left((-\mathbf{e}')\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})}' \partial\alpha\right)
\end{aligned}$$

hence

$$\begin{aligned}
\frac{\partial \log L}{\partial \alpha} &= -\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})}\mathbf{e} \\
&= -(\Sigma^{-} + \Sigma'^{-})(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_{\mathbf{m}})\mathbf{e} = 0
\end{aligned}$$

Similarly, FOC w.r.t β and combine those results:

$$\begin{aligned}
(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_{\mathbf{m}})\mathbf{r}_{\mathbf{m}} = \mathbf{0} \\
(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_{\mathbf{m}})\mathbf{e} = \mathbf{0}
\end{aligned}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{e}'\mathbf{r}_{\mathbf{m}} & \mathbf{r}'_{\mathbf{m}}\mathbf{r}_{\mathbf{m}} \\ \mathbf{e}'\mathbf{e} & \mathbf{r}'_{\mathbf{m}}\mathbf{e} \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{r}'\mathbf{r}_{\mathbf{m}} \\ \mathbf{r}'\mathbf{e} \end{bmatrix}$$

Similarly to out deduction for mean-variance mdoel, let $a = \mathbf{r}'_{\mathbf{m}}\mathbf{r}_{\mathbf{m}}, b = \mathbf{e}'\mathbf{e} = T$ and $c = \mathbf{e}'\mathbf{r}_{\mathbf{m}}(c^2 < ab)$, hence

$$\begin{cases} \hat{\alpha} = c\mathbf{r}'\mathbf{r}_{\mathbf{m}} - a\mathbf{r}'\mathbf{e}/(c^2 - ab) \\ \hat{\beta} = -b\mathbf{r}'\mathbf{r}_{\mathbf{m}} + c\mathbf{r}'\mathbf{e}/(c^2 - ab) \end{cases}$$

By assumption $\mathbf{r}' \sim \mathcal{MN}_{n\times T}(\beta\mathbf{r}'_{\mathbf{m}}, \mathbf{\Sigma}, \mathbf{I})$, and $\alpha = \mathbf{r}'(c\mathbf{r}_{\mathbf{m}} - a\mathbf{e})/(c^2 - ab)$
By transformation of matrix normal distribution(Wikipedia contributors 2019)

$$\frac{(c\mathbf{r}_{\mathbf{m}} - a\mathbf{e})'(c\mathbf{r}_{\mathbf{m}} - a\mathbf{e})}{(c^2 - ab)^2} = \frac{c^2a - 2ac^2 + a^2b}{(c^2 - ab)^2} = \frac{a}{ab - c^2}$$

we have

$$\hat{\alpha} \sim \mathcal{MN}(\mathbf{0}, \mathbf{\Sigma}, \frac{a}{ab - c^2})$$

which degenerated to $\mathcal{N}(0, \frac{\mathbf{a}}{\mathbf{ab - c^2}}\mathbf{\Sigma})$ since $\Sigma \otimes \frac{a}{ab-c^2} = \frac{a}{ab-c^2}\Sigma$.(Wikipedia contributors 2019) For the same reason, $\hat{\beta} \sim \mathcal{N}(\beta, \frac{b}{ab-c^2}\Sigma)$

FOC w.r.t Σ(Petersen and Pedersen 2012):

$$\frac{\partial \log L}{\partial \Sigma} = \frac{1}{2}(\Sigma^{-}(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_{\mathbf{m}}')(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_{\mathbf{m}}')'\Sigma^{-})' - \frac{T}{2}\Sigma'^{-} = 0$$

hence

$$\hat{\Sigma} = (\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}_{\mathbf{m}}')(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}_{\mathbf{m}}')'/T$$

Where

$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}_{\mathbf{m}}') = \mathbf{r}'(\mathbf{I} - \frac{c\mathbf{r_m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r_m}\mathbf{r}_{\mathbf{m}}' + c\mathbf{e}\mathbf{r}_{\mathbf{m}}'}{c^2 - ab})$$

Easy to verify $\frac{c\mathbf{r_m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r_m}\mathbf{r}_{\mathbf{m}}' + c\mathbf{e}\mathbf{r}_{\mathbf{m}}'}{c^2 - ab}$ is symmetric and idempotent, thus

$$\begin{aligned}
\mathrm{rank}(\mathbf{I} - \frac{c\mathbf{r_m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r_m}\mathbf{r}_{\mathbf{m}}' + c\mathbf{e}\mathbf{r}_{\mathbf{m}}'}{c^2 - ab}) &= \mathrm{Tr}(\mathbf{I} - \frac{c\mathbf{r_m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r_m}\mathbf{r}_{\mathbf{m}}' + c\mathbf{e}\mathbf{r}_{\mathbf{m}}'}{c^2 - ab}) \\
&= T - \frac{2c^2 - 2ab}{c^2 - ab} \\
&= T - 2
\end{aligned}$$

By following lemma:

> Suppose symmetric matrix $p \times p$ $\mathbf{A}$. It's idempotent of rank s iff there exist a $p \times s$ $\mathbf{P} \ni \mathbf{P}\mathbf{P}' = \mathbf{A}$ and $\mathbf{P}'\mathbf{P} = \mathbf{I}$.

Proof Sufficiency is trivial. For necessity, since $\mathbf{A}$ is symmetric and idempotent matrix, it can be spectral decompostioned by $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}'$. Where the diagonal of $\mathbf{\Lambda}$ is s 1 and $p - s$ 0. Thus

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}' = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \begin{pmatrix} \mathbf{I}_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1' \\ \mathbf{P}_2' \end{pmatrix} = \mathbf{P}_1\mathbf{P}_1'$$

Note

$$\mathbf{I}_p = \mathbf{Q}'\mathbf{Q} = \begin{pmatrix} \mathbf{P}_1' \\ \mathbf{P}_2' \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1'\mathbf{P}_1 & \mathbf{P}_1'\mathbf{P}_2 \\ \mathbf{P}_2'\mathbf{P}_1 & \mathbf{P}_2'\mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1'\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2'\mathbf{P}_2 \end{pmatrix}$$

hence$\mathbf{P_1'P_1} = \mathbf{I_s}$.■

We may find $\mathbf{r}'\mathbf{P} \sim \mathcal{MN}_{n\times(T-2)}(\mathbf{0}, \Sigma, \mathbf{I})$. Where $\mathrm{E}[\mathbf{r}'\mathbf{P}] = \mathrm{E}[\mathbf{r}'\mathbf{P}\mathbf{P}'\mathbf{P}] = \mathrm{E}[\mathbf{r}'\mathbf{A}\mathbf{P}] = \mathbf{0}$ and thus (Wikipedia contributors 2020)

$$T\hat{\Sigma} = \mathbf{r}'\mathbf{A}\mathbf{r} = \mathbf{r}'\mathbf{P}\mathbf{P}'\mathbf{r} = \mathbf{r}'\mathbf{P}(\mathbf{r}'\mathbf{P})' \sim W_n(T - 2, \Sigma)$$

Petersen, Kaare Brandt, and Michael Syskind Pedersen. 2012. "The Matrix Cookbook, Nov 2012." *URL Http://Www2. Imm. Dtu. Dk/Pubdb/P. Php* 3274: 14.

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