Homework 2

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0.1 1

Suppose there are n risky asset $\mathbf{r}$ with expectation $\bar{\mathbf{r}}$ and variance $\mathbf{V}$ together with riskless asset r_f, the market portfolio is constructed by $r_m = \boldsymbol{\omega}'\mathbf{r}$. By CAPM, there is

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \boldsymbol{\beta}_{\mathbf{m}}(\bar{r}_m - r_f)$$

where

$$\boldsymbol{\beta} = \frac{\text{Cov}(\mathbf{r}, r_m)}{\text{Var}(r_m)}$$

0.1.1 a

By definition, $\sigma_m = \sqrt{\boldsymbol{\omega}'\mathbf{V}\boldsymbol{\omega}}$, thus

$$\frac{\partial \sigma_m}{\partial \boldsymbol{\omega}} = \frac{1}{2\sqrt{\boldsymbol{\omega}'\mathbf{V}\boldsymbol{\omega}}} \frac{\partial \boldsymbol{\omega}'\mathbf{V}\boldsymbol{\omega}}{\partial \boldsymbol{\omega}} = \frac{\mathbf{V}\boldsymbol{\omega}}{\sqrt{\boldsymbol{\omega}'\mathbf{V}\boldsymbol{\omega}}}$$

note $\text{Cov}(\mathbf{r}, r_m) = \text{Cov}(\mathbf{r}, \boldsymbol{\omega}'\mathbf{r}) = \mathbf{V}\boldsymbol{\omega}$ and write it as scalar form, we have

$$\frac{\partial \sigma_m}{\partial \omega_i} = \frac{\text{Cov}(r_i, r_m)}{\sigma_m}$$

0.1.2 b

Rewrite CAPM as

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \frac{\partial \sigma_m}{\partial \boldsymbol{\omega}} \frac{\bar{r}_m - r_f}{\sigma_m}$$

where $\frac{\bar{r}_m - r_f}{\sigma_m}$ is sharpe ratio of market portfolio and $\frac{\partial \sigma_m}{\partial \boldsymbol{\omega}}$ is the marginal risk of each asset. Thus the expected excess return is just r_f plus S_p reward for each one unit additional exposure risk.

0.1.3 c

$$\mathbf{r_t} = \boldsymbol{\alpha} + \boldsymbol{\beta} \boldsymbol{r}_{m,t} + \boldsymbol{\epsilon}_t$$

Take expectation both sides

$$\alpha = \mathrm{E}[\mathbf{r_t}] - \boldsymbol{\beta}\,\mathrm{E}[r_m] = r_f(\mathbf{e} - \boldsymbol{\beta})$$

0.1.4 d

Taking expectation both sides, if CAPM holds

$$\boldsymbol{\alpha} = \mathrm{E}[\mathbf{r} - \boldsymbol{r_f}\mathbf{e}] - \boldsymbol{\beta}\,\mathrm{E}[r_m - r_f] = \mathbf{0}$$

0.1.5 e

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f \mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f \mathbf{e})}$.By one-fund theorem, there is $\omega = c\omega_D$. Write $\omega_D = m\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f\mathbf{e})$ where $m = \frac{1}{\mathbf{e}'\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f \mathbf{e})}$, then we have

$$\bar{r} - r_f\mathbf{e} = \frac{1}{cm}\mathbf{V}\boldsymbol{\omega}$$

Note $\mathrm{Cov}(\mathbf{r}, \omega'\mathbf{r}) = \mathbf{V}\boldsymbol{\omega}$ and

$$\sigma^2 = \omega'\mathbf{V}\omega = cm\omega'(\bar{\mathbf{r}} - r_f\mathbf{e}) = cm\bar{r}_v - cmr_f$$

we have

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \frac{\bar{r}_v - r_f}{\sigma^2}\,\mathrm{Cov}(\mathbf{r}, r_v)$$

Denote $\frac{\mathrm{Cov}(\mathbf{r}, r_v)}{\sigma^2} = \beta_v$, we have

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \beta_v(\bar{r}_D - r_f)$$

0.2 2

Denoted $\mu = \bar{\mathbf{r}}$ and $\Sigma = \mathbf{V}$. By Lagrangian

$$L = \boldsymbol{\omega'}\bar{\mathbf{r}} - \frac{1}{2\theta}\boldsymbol{\omega'}\mathbf{V}\boldsymbol{\omega} + \lambda_k(1 - \mathbf{e'}\boldsymbol{\omega})$$

The FOC w.r.t. ω is

$$\frac{\partial L}{\partial \boldsymbol{\omega}} = \bar{\mathbf{r}} - \frac{\mathbf{V}\boldsymbol{\omega}}{\theta} - \lambda_k\mathbf{e} = 0$$

0.2.1 b

Recall the optimal portfolio is achieved when $\frac{\partial L}{\partial \boldsymbol{\omega}} = 0$, denoted the optimal weight is ω_D, thus

$$\bar{\mathbf{r}} = \frac{\mathbf{V}\boldsymbol{\omega}_\mathbf{k}}{\theta_k} + \lambda_k\mathbf{e}$$

where $\mathbf{V}\boldsymbol{\omega}_\mathbf{k} = \mathrm{Cov}(\mathbf{r}, r_k)$.

0.2.2 c

Multiplying $W_k\theta_k$ both sides yield:

$$W_k\theta_k\bar{\mathbf{r}} - W_k\mathbf{V}\boldsymbol{\omega}_{\mathbf{k}} = W_k\theta_k\lambda_k\mathbf{e}$$

Suppose

$$\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix}, \boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\omega}_{\mathbf{1}}' \\ \boldsymbol{\omega}_{\mathbf{2}}' \\ \vdots \\ \boldsymbol{\omega}_{\mathbf{M}}' \end{bmatrix}$$

and stack $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ in similar way. Summing from 1 to M, we have

$$\mathbf{W}'\boldsymbol{\theta}\bar{\mathbf{r}} - \mathbf{V}\boldsymbol{\Omega}'\mathbf{W} = \mathbf{e}'\mathbf{W} \circ \boldsymbol{\theta} \circ \boldsymbol{\lambda}\mathbf{e}$$

Where $\circ$ is hadamard (elementwise) product. Note the wealth-weighted risk tolerance would be

$$\boldsymbol{\theta}_{\mathbf{M}} = \mathbf{W}'\boldsymbol{\theta}$$

and the covariance between $\mathbf{r}$ and market portfolio whould be (since market portfolio is $\sum W_k\boldsymbol{\omega}_{\mathbf{k}} = \boldsymbol{\Omega}'\mathbf{W}$)

$$\text{Cov}(\mathbf{r}, r_m) = \text{Cov}(\mathbf{r}, (\boldsymbol{\Omega}'\mathbf{W})'\mathbf{r}) = \mathbf{V}\boldsymbol{\Omega}'\mathbf{W}$$

thus

$$\bar{\mathbf{r}} = \frac{\mathbf{V}\boldsymbol{\Omega}'\mathbf{W}}{\boldsymbol{\theta}_{\mathbf{M}}} + \frac{\mathbf{e}'\mathbf{W} \circ \boldsymbol{\theta} \circ \boldsymbol{\lambda}\mathbf{e}}{\boldsymbol{\theta}_{\mathbf{M}}}$$

0.3 3

Similar results also holds for any portfolio $\overline{r}_p$ and the zero covariance portfolio $\overline{r}_q$ in the MVF:

$$\bar{\mathbf{r}} - \overline{r}_q\mathbf{e} = \beta_p(\overline{r}_p - \overline{r}_q)$$

It's clear in the view of every portfolio $\overline{r}_p$ is also a tangency portfolio by selecting proper r_f. One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega_p'\mathbf{V}\omega_q = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})(\overline{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

If the covariance is 0, we have

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Then

$$
\begin{aligned}
\overline{\mathbf{r}} - \overline{r}_q \mathbf{e} &= \overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e} \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\delta^2(\overline{r}_p - \alpha/\delta))(\overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta(\overline{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2))\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e} \\
&= \frac{(\delta^2\overline{r}_p\overline{\mathbf{r}} - \alpha\delta\overline{\mathbf{r}}) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e}}{\delta^2(\overline{r}_p - \alpha/\delta)} \\
&= \frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\delta(\overline{r}_p - \alpha/\delta)}
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\beta_p &= \frac{\mathbf{V}\omega_{\mathbf{p}}}{\omega_p'\mathbf{V}\omega_p} \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\lambda_p\overline{\mathbf{r}} + \gamma\mathbf{e}) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{\xi\mathbf{e} - \alpha\overline{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha\mathbf{e} + \delta\overline{\mathbf{r}}}{\delta\xi - \alpha^2}\overline{r}_p) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\Delta})
\end{aligned}
$$

Compare with the desire result, it's remain to show that

$$
(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) = \omega'\mathbf{V}\omega\Delta
$$

It's clear since

$$
\omega'\mathbf{V}\omega\Delta = \sigma_p^2\Delta = \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2
$$

and

$$
\begin{aligned}
(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) &= ((\overline{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\delta(\overline{r}_p - \alpha/\delta) \\
&= \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2
\end{aligned}
$$

We are done.

0.4 4

Suppose now there is multi-factor with k risk factor and

$$
\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon}_\mathbf{t}
$$

Where $\mathbf{r_t}, \boldsymbol{\epsilon}_\mathbf{t}$ is n vector while $\mathbf{f_t}$ is k vector and $\mathbf{B}$ is $n \times k$ matrix.
Taking expectation:

$$\mathbf{r_t} = \mathrm{E}[\mathbf{r}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

Note CAPM is just special case of APT when $k = 1$ and $\mathbf{f_t} = r_{m,t}$, which is the only factor affecting realized return. Thta is

$$\mathbf{r_t} = \boldsymbol{r_f}\mathbf{e} + \boldsymbol{\beta}\,\mathrm{E}[\boldsymbol{r_m} - \boldsymbol{r_f}] + \boldsymbol{\beta}(\boldsymbol{r_{m,t}} - \mathrm{E}[\boldsymbol{r_{m,t}}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

Assume there is non-arbitrage, that is, if one invest 0 and take no risk, then the expected return is 0. Formally, as $n \to \infty$,

$$\boldsymbol{\omega}'\,[\mathbf{e} \quad \mathbf{B}] = \mathbf{0} \implies \boldsymbol{\omega}'\,\mathrm{E}[\mathbf{r}] = \mathbf{0}$$

by Farkas lemma, that implies

$$[\mathbf{e} \quad \mathbf{B}]\begin{bmatrix}\lambda_0 \\ \boldsymbol{\lambda}\end{bmatrix} = \mathrm{E}[\mathbf{r}]$$

for some $\boldsymbol{\lambda} \geq \mathbf{0}$. Under CAPM, $\lambda_0 = r_f$ and $\mathrm{E}[r_m - r_f] = \lambda$ clearly. If one take no risk, that is $\mathbf{B} = \mathbf{0}$, s(he) get a λ_0 return, that implies $\lambda_0 = r_f$ immediately.

When $\mathbf{f_t}$ is $\mathbf{r_t}$ or is $\mathbf{r_t} - \boldsymbol{r_f}\mathbf{e}$, regressing it on

$$\mathbf{f_t} = \mathrm{E}[\mathbf{f}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

we have $\mathbf{B} = \mathbf{I}$ and thus

$$\lambda_0\mathbf{e} + \boldsymbol{\lambda} = \mathrm{E}[\mathbf{f_t}] \implies \begin{cases}\mathrm{E}[\mathbf{f_t}] = \boldsymbol{r_f}\mathbf{e} + \boldsymbol{\lambda} & \text{return} \\ \mathrm{E}[\mathbf{f_t}] = \boldsymbol{\lambda} & \text{excess return}\end{cases}$$

0.4.1 a

$$\boldsymbol{\alpha} = \mathrm{E}[\mathbf{r}] - \mathbf{B}\,\mathrm{E}[\mathbf{f_t}] = \boldsymbol{r_f}\mathbf{e} + \mathbf{B}(\boldsymbol{\lambda} - \mathrm{E}[\mathbf{f_t}])$$

0.4.2 b

$$\boldsymbol{\alpha} = \mathbf{0}, \boldsymbol{\lambda} = \mathrm{E}[\mathbf{f_t}] - r_f\mathbf{e}$$

0.4.3 c

$$\boldsymbol{\alpha} = r_f\mathbf{e}, \boldsymbol{\lambda} = \mathrm{E}[\mathbf{f_t}]$$

0.5 5

```
mkt <- read.csv("mkt.csv")
mkt <- tibble(mkt)
bm <- read.csv("size_bm_25.csv",header=FALSE)
bm <- tibble(bm)
```

```
excess_m <- function(x)(x-mkt$RF)
cols=vector("character",25)
for (i in 2:26){
    cols[[i-1]] <- paste("V",i,sep="")
}
bm <- bm %>%
  mutate(across(cols,excess_m))
```

```
## Note: Using an external vector in selections is ambiguous.
## i Use ‘all_of(cols)‘ instead of ‘cols‘ to silence this message.
## i See <https://tidyselect.r-lib.org/reference/faq-external-vector.html>.
## This message is displayed once per session.
```

```
avgex <- bm %>%
  summarise(across(cols,mean))
```

```
tr_tibble <- function(df)(as_tibble(cbind(nms = names(df), t(df))))
avgex <- tr_tibble(avgex) %>% type.convert()
colnames(avgex) <- c("portfolio","excess return")
```

The sample average excess return is

portfolio	excess return
V2	0.5573475
V3	0.6907301
V4	0.9870240
V5	1.1743993
V6	1.3572828
V7	0.6181701
V8	0.9240850
V9	0.9861830
V10	1.0702680
V11	1.2497782
V12	0.7001386
V13	0.9054621
V14	0.9286137
V15	1.0108595
V16	1.1474122
V17	0.7129390
V18	0.7460813
V19	0.8588540
V20	0.9673198
V21	1.0227819
V22	0.6139741
V23	0.6263401
V24	0.6997967
V25	0.6500185
V26	0.9476617

and the covariance is

	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13
V2	150.63074	86.47086	86.78873	79.41490	90.34156	76.01681	68.70896	68.29165	70.35008	84.62038	70.03814	59.05967
V3	86.47086	97.58528	76.30361	74.45290	79.92993	68.16156	65.38479	62.39806	63.02097	71.64139	62.63035	52.15030
V4	86.78873	76.30361	81.49999	69.95241	77.51832	61.19466	61.02457	59.72094	61.37027	70.73557	58.32364	50.40413
V5	79.41490	74.45290	69.95241	70.02695	74.17501	58.18510	57.57761	56.66776	58.20423	66.83613	53.79209	46.77959
V6	90.34156	79.92993	77.51832	74.17501	86.94780	62.19920	62.62694	61.44475	64.17179	76.30892	57.13745	50.73329
V7	76.01681	68.16156	61.19466	58.18510	62.19920	64.16384	55.46588	52.05965	51.84410	57.79859	54.84699	46.50220
V8	68.70896	65.38479	61.02457	57.57761	62.62694	55.46588	56.55992	51.65494	51.92243	58.33013	51.51052	45.21039
V9	68.29165	62.39806	59.72094	56.66776	61.44475	52.05965	51.65494	52.95652	51.74589	58.19346	48.93026	43.79963
V10	70.35008	63.02097	61.37027	58.20423	64.17179	51.84410	51.92243	51.74589	55.34317	60.81178	48.62796	43.91260
V11	84.62038	71.64139	70.73557	66.83613	76.30892	57.79859	58.33013	58.19346	60.81178	76.15260	54.76607	49.83504
V12	70.03814	62.63035	58.32364	53.79209	57.13745	54.84699	51.51052	48.93026	48.62796	54.76607	55.30225	44.33731
V13	59.05967	52.15030	50.40413	46.77959	50.73329	46.50220	45.21039	43.79963	43.91260	49.83504	44.33731	42.09901
V14	58.63712	52.19900	51.71369	47.98878	52.75993	44.72083	45.13520	45.01177	45.77084	51.68157	43.23362	39.99389
V15	62.42918	56.14753	54.49249	51.79882	57.53963	46.31159	47.30590	47.18070	48.84710	55.80883	44.18206	41.21835
V16	79.93755	66.05180	67.68763	62.56791	72.25084	54.88984	55.99536	56.44626	58.55408	69.36803	52.11663	48.54386
V17	54.32008	46.93524	43.84823	40.44976	42.96432	44.58905	40.94049	38.54632	38.20638	42.98750	43.20325	37.49545
V18	52.02872	47.01696	46.10168	42.50614	46.34723	41.67427	41.37498	40.23297	40.84672	45.94876	40.56610	37.58139
V19	54.69244	49.53213	49.10935	45.47607	50.43757	42.20207	42.98515	42.68141	43.57715	50.00344	41.26761	38.63540
V20	61.79891	50.88982	51.37843	48.47789	54.36178	43.69742	44.16493	44.37532	46.42785	53.77507	42.11553	39.68433
V21	75.76631	65.23737	65.82717	61.94413	70.32552	54.08157	55.67554	56.30826	58.85006	69.13057	52.23777	48.80100
V22	42.16356	36.27345	35.05487	32.00067	34.13626	33.77287	32.03342	30.91240	30.70014	34.37299	33.86317	29.92462
V23	41.81086	35.74559	35.58220	32.79255	35.71130	32.67465	32.11141	31.73479	32.11233	36.43254	32.49378	30.39754
V24	44.55692	37.64006	37.66128	35.23789	39.09395	32.35006	33.19191	33.44417	34.95232	39.72444	32.45487	30.88884
V25	53.50955	45.49077	46.34158	43.32832	48.80886	38.36380	39.79229	40.47772	42.47763	49.05992	37.58496	36.05442
V26	72.21297	53.78162	58.77221	53.22177	61.40149	46.45758	46.71083	48.05715	51.16267	61.99170	44.89313	43.12918

	V14	V15	V16	V17	V18	V19	V20	V21	V22	V23	V24	V25	V26
V2	58.63712	62.42918	79.93755	54.32008	52.02872	54.69244	61.79891	75.76631	42.16356	41.81086	44.55692	53.50955	72.21297
V3	52.19900	56.14753	66.05180	46.93524	47.01696	49.53213	50.88982	65.23737	36.27345	35.74559	37.64006	45.49077	53.78162
V4	51.71369	54.49249	67.68763	43.84823	46.10168	49.10935	51.37843	65.82717	35.05487	35.58220	37.66128	46.34158	58.77221
V5	47.98878	51.79882	62.56791	40.44976	42.50614	45.47607	48.47789	61.94413	32.00067	32.79255	35.23789	43.32832	53.22177
V6	52.75993	57.53963	72.25084	42.96432	46.34723	50.43757	54.36178	70.32552	34.13626	35.71130	39.09395	48.80886	61.40149
V7	44.72083	46.31159	54.88984	44.58905	41.67427	42.20207	43.69742	54.08157	33.77287	32.67465	32.35006	38.36380	46.45758
V8	45.13520	47.30590	55.99536	40.94049	41.37498	42.98515	44.16493	55.67554	32.03342	32.11141	33.19191	39.79229	46.71083
V9	45.01177	47.18070	56.44626	38.54632	40.23297	42.68141	44.37532	56.30826	30.91240	31.73479	33.44417	40.47772	48.05715
V10	45.77084	48.84710	58.55408	38.20638	40.84672	43.57715	46.42785	58.85006	30.70014	32.11233	34.95232	42.47763	51.16267
V11	51.68157	55.80883	69.36803	42.98750	45.94876	50.00344	53.77507	69.13057	34.37299	36.43254	39.72444	49.05992	61.99170
V12	43.23362	44.18206	52.11663	43.20325	40.56610	41.26761	42.11553	52.23777	33.86317	32.49378	32.45487	37.58496	44.89313
V13	39.99389	41.21835	48.54386	37.49545	37.58139	38.63540	39.68433	48.80100	29.92462	30.39754	30.88884	36.05442	43.12918
V14	42.67605	42.75635	51.04379	35.51934	37.66114	39.72452	41.11608	51.35307	29.30317	30.37829	32.24999	38.33921	45.61633
V15	42.75635	47.77187	54.53410	35.93945	38.76560	42.18031	44.55698	55.80907	29.67906	31.27963	34.19294	41.03020	48.91952
V16	51.04379	54.53410	72.73313	41.80683	45.63500	49.55489	53.70104	69.08960	34.78999	36.89980	40.56212	50.14898	62.57034
V17	35.51934	35.93945	41.80683	39.45499	34.91251	34.62054	35.11962	42.63225	30.52866	29.19665	28.31146	31.82247	37.88659
V18	37.66114	38.76560	45.63500	34.91251	37.20089	37.20630	37.49954	46.40549	28.93068	29.72703	30.57695	35.17855	41.36360
V19	39.72452	42.18031	49.55489	34.62054	37.20630	41.46900	41.08553	51.08557	29.19401	30.66653	32.69517	38.45979	45.38028
V20	41.11608	44.55698	53.70104	35.11962	37.49954	41.08553	46.60450	55.41905	29.66762	31.61682	34.51718	41.41315	50.25033
V21	51.35307	55.80907	69.08960	42.63225	46.40549	51.08557	55.41905	76.39623	35.99127	38.52103	42.58714	52.12067	65.13763
V22	29.30317	29.67906	34.78999	30.52866	28.93068	29.19401	29.66762	35.99127	28.76118	25.94853	25.29185	28.55290	33.49752
V23	30.37829	31.27963	36.89980	29.19665	29.72703	30.66653	31.61682	38.52103	25.94853	28.32659	27.52512	30.88724	35.95112
V24	32.24999	34.19294	40.56212	28.31146	30.57695	32.69517	34.51718	42.58714	25.29185	27.52512	32.04640	34.61740	39.56976
V25	38.33921	41.03020	50.14898	31.82247	35.17855	38.45979	41.41315	52.12067	28.55290	30.88724	34.61740	43.99416	48.75301
V26	45.61633	48.91952	62.57034	37.88659	41.36360	45.38028	50.25033	65.13763	33.49752	35.95112	39.56976	48.75301	73.62647

0.5.1 b

```r
#' enumerate(l=LETTERS)
reg.func <- function (y, m) {
        mod <- lm(y ~ m)
        mod <- summary(mod)
        mod$coefficients[1,]
    }
alphas <- apply(select(bm,cols), 2, reg.func, mkt$Mkt_rf) %>%
  as_tibble() %>%
  tr_tibble() %>%
  type.convert()

colnames(alphas) <- c("portfolio","estimated","std","t-statistic","pvalue")
```

portfolio	estimated	std	t-statistic	pvalue
V2	-0.5024840	0.2632399	-1.9088445	0.0565470
V3	-0.2249165	0.1943875	-1.1570519	0.2475070
V4	0.0950023	0.1594663	0.5957518	0.5514659
V5	0.3495979	0.1485509	2.3533885	0.0187817
V6	0.4606041	0.1732289	2.6589329	0.0079548
V7	-0.2042884	0.1296606	-1.5755630	0.1154194
V8	0.1271945	0.1110494	1.1453867	0.2523028
V9	0.2081186	0.1041763	1.9977541	0.0459938
V10	0.2827259	0.1101840	2.5659425	0.0104236
V11	0.3540071	0.1413512	2.5044503	0.0124107
V12	-0.1094236	0.0992222	-1.1028141	0.2703536
V13	0.1739685	0.0719276	2.4186619	0.0157422
V14	0.1982596	0.0763027	2.5983293	0.0094954
V15	0.2574860	0.0917229	2.8072156	0.0050871
V16	0.2524158	0.1298447	1.9439820	0.0521570
V17	0.0027993	0.0683197	0.0409736	0.9673245
V18	0.0441499	0.0574084	0.7690488	0.4420325
V19	0.1336827	0.0719256	1.8586239	0.0633524
V20	0.2170200	0.0872584	2.4870976	0.0130291
V21	0.0997658	0.1304937	0.7645261	0.4447208
V22	-0.0075611	0.0469347	-0.1610989	0.8720456
V23	0.0086301	0.0458169	0.1883606	0.8506294
V24	0.0700453	0.0680191	1.0297873	0.3033404
V25	-0.0707608	0.0891685	-0.7935629	0.4276242
V26	0.0949580	0.1498063	0.6338717	0.5262988

The number of $\hat{\alpha}$ differ zero significantly is

```r
alphas %>%
  filter(pvalue<.05) %>%
  summarise(n())
```

```
## # A tibble: 1 x 1
##   `n()`
##   <int>
## 1     9
```

0.5.2 c

```r
r=as.vector(avgex$`excess return`/100)
r <- append(r,mean(mkt$Mkt_rf)/100)
covmat <- cov(select(bm,cols) %>%
      mutate(m=mkt$Mkt_rf)) / 10000
```

```
p1 <- mvFrontier(r,covmat,wmin=0,wmax=1.0,n=26)
```

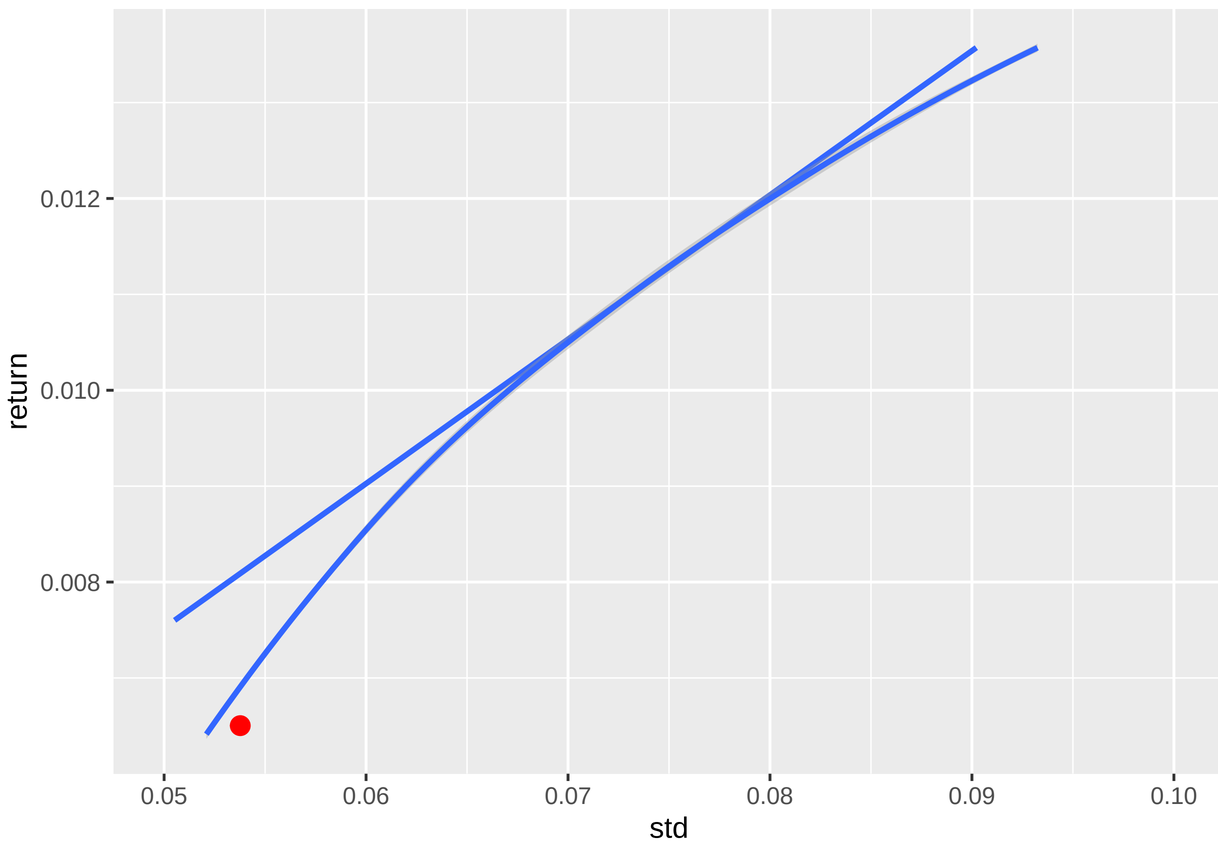
1

```
## Loading required namespace: quadprog
```

```
p2 <- mvFrontier(r,covmat,wmin=0,wmax=1.0,n=26,rf = 0)
```

```
ggplot()+
  geom_smooth(mapping=aes(x=p2$volatility,y=p2$return))+
  geom_smooth(mapping=aes(x=p1$volatility,y=p1$return))+
  geom_point(aes(sqrt(covmat[26,26]),r[26]),color="red",size=3)+
  xlim(.05,.1)+
  xlab("std")+
  ylab("return")
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```



The red point is the market portfolio, which is not on the frontier.

```
sample.sr <- function(x) {
    mu <- mean(x)
    sg <- sd(x)
    return(mu / sg)
}
```

```
res.func <- function (y, m) {
        mod <- lm(y ~ m)
        summ <- summary(mod)
        mod$residuals
}

T_=dim(bm)[1]
N=25

e.hat <- apply(select(bm,cols), 2, res.func, mkt$Mkt_rf) %>%
  as.matrix()

sig.hat <- t(e.hat) %*% e.hat / T_

alpha.hat <- alphas$estimated

mkt.sr <- sample.sr(mkt$Mkt_rf)

W=t(alpha.hat) %*% solve(sig.hat) %*% alpha.hat

GRS=(T_-N-1)/N*W[1,1]/(1+mkt.sr^2)

GRS
```

2

```
## [1] 3.50551
```

```
pf(GRS,df1 = N,df2 = T_-N-1)
```

```
## [1] 1
```

The GRS statistic is 3.5055097 and the p-value is 0.

3 The ex-post tangency portfolio's sharpe ratio is

```
ep.sr <- sqrt(W[1,1]+mkt.sr^2)
ep.sr
```

```
## [1] 0.3143707
```

and the market sharpe ratio is 0.1209335, thus the GRS is given by

```
GRS_=(T_-N-1)/N*(ep.sr^2-mkt.sr^2)/(1+mkt.sr^2)
GRS_
```

```
## [1] 3.50551
```

they are equal.

4 Yes, since the p-value is just 0.

5 The test in b not jointly test all alpha equal to zero.