MATRIX APPROCH FOR GRS

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Suppose

$$\mathbf{r} = \begin{bmatrix} \mathbf{r_1^e}' \\ \mathbf{r_2^e}' \\ \mathbf{r_3^e}' \\ \vdots \\ \mathbf{r_T^e}' \end{bmatrix}, \mathbf{r_m} = \begin{bmatrix} r_{m,1}^e \\ r_{m,2}^e \\ r_{m,3}^e \\ \vdots \\ r_{m,T}^e \end{bmatrix}$$

Now $\mathbf{r}' \sim \mathcal{MN}_{n\times T}(\alpha\mathbf{e}' + \beta\mathbf{r}_\mathbf{m}', \mathbf{\Sigma}, \mathbf{I})$, the p.d.f is (Wikipedia contributors 2019)

$$p(\mathbf{r}'|\beta\mathbf{r}_\mathbf{m}', \mathbf{\Sigma}, \mathbf{I}) = \frac{\exp(-\frac{1}{2}\operatorname{Tr}[(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')'\mathbf{\Sigma}^-(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')])}{(2\pi)^{nT/2}T^{n/2}|\mathbf{\Sigma}|^{T/2}}$$

thus the log likelihood function is

$$\log L = -\frac{1}{2}\operatorname{Tr}[(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')'\mathbf{\Sigma}^-(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')] - \frac{nT}{2}\log 2\pi - \frac{n}{2}\log T - \frac{T}{2}\log|\Sigma|$$

FOC w.r.t α, by chain rule(Petersen and Pedersen 2012)

$$\begin{aligned} \partial \log L &= \operatorname{Tr}\left(\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')}'\partial(\mathbf{X} - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')\right) \\ &= \operatorname{Tr}\left(\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')}'\partial\alpha(-\mathbf{e}')\right) \\ &= \operatorname{Tr}\left((-\mathbf{e}')\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')}'\partial\alpha\right) \end{aligned}$$

hence

$$\begin{aligned} \frac{\partial \log L}{\partial\alpha} &= -\frac{\partial \log L}{\partial(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')}\mathbf{e} \\ &= -(\Sigma^- + \Sigma'^-)(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}_\mathbf{m}')\mathbf{e} = 0 \end{aligned}$$

Similarly, FOC w.r.t β and combine those results:

$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_\mathbf{m})\mathbf{r}_\mathbf{m} = \mathbf{0}$$
$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_\mathbf{m})\mathbf{e} = \mathbf{0}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{e}'\mathbf{r}_\mathbf{m} & \mathbf{r}'_\mathbf{m}\mathbf{r}_\mathbf{m} \\ \mathbf{e}'\mathbf{e} & \mathbf{r}'_\mathbf{m}\mathbf{e} \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{r}'\mathbf{r}_\mathbf{m} \\ \mathbf{r}'\mathbf{e} \end{bmatrix}$$

Similarly to out deduction for mean-variance mdoel, let $a = \mathbf{r}'_\mathbf{m}\mathbf{r}_\mathbf{m}$, $b = \mathbf{e}'\mathbf{e} = T$ and $c = \mathbf{e}'\mathbf{r}_\mathbf{m}(c^2 < ab)$, hence

$$\begin{cases} \hat{\alpha} = c\mathbf{r}'\mathbf{r}_\mathbf{m} - a\mathbf{r}'\mathbf{e}/(c^2 - ab) \\ \hat{\beta} = -b\mathbf{r}'\mathbf{r}_\mathbf{m} + c\mathbf{r}'\mathbf{e}/(c^2 - ab) \end{cases}$$

By assumption $\mathbf{r}' \sim \mathcal{MN}_{n\times T}(\beta\mathbf{r}'_\mathbf{m}, \mathbf{\Sigma}, \mathbf{I})$, and $\alpha = \mathbf{r}'(c\mathbf{r}_\mathbf{m} - a\mathbf{e})/(c^2 - ab)$
By transformation of matrix normal distribution(Wikipedia contributors 2019)

$$\frac{(c\mathbf{r}_\mathbf{m} - a\mathbf{e})'(c\mathbf{r}_\mathbf{m} - a\mathbf{e})}{(c^2 - ab)^2} = \frac{c^2a - 2ac^2 + a^2b}{(c^2 - ab)^2} = \frac{a}{ab - c^2}$$

we have

$$\hat{\alpha} \sim \mathcal{MN}(\mathbf{0}, \mathbf{\Sigma}, \frac{a}{ab - c^2})$$

which degenerated to $\mathcal{N}(0, \frac{\mathbf{a}}{\mathbf{ab - c^2}}\mathbf{\Sigma})$ since $\Sigma \otimes \frac{a}{ab-c^2} = \frac{a}{ab-c^2}\Sigma$.(Wikipedia contributors 2019) For the same reason, $\hat{\beta} \sim \mathcal{N}(\beta, \frac{b}{ab-c^2}\Sigma)$
FOC w.r.t Σ(Petersen and Pedersen 2012):

$$\frac{\partial \log L}{\partial \Sigma} = \frac{1}{2}(\Sigma^{-}(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_\mathbf{m})(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_\mathbf{m})'\Sigma^{-})' - \frac{T}{2}\Sigma'^{-} = 0$$

hence

$$\hat{\Sigma} = (\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_\mathbf{m})(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_\mathbf{m})'/T$$

Where

$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_\mathbf{m}) = \mathbf{r}'(\mathbf{I} - \frac{c\mathbf{r}_\mathbf{m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r}_\mathbf{m}\mathbf{r}'_\mathbf{m} + c\mathbf{e}\mathbf{r}'_\mathbf{m}}{c^2 - ab})$$

Easy to verify $\frac{c\mathbf{r}_\mathbf{m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r}_\mathbf{m}\mathbf{r}'_\mathbf{m} + c\mathbf{e}\mathbf{r}'_\mathbf{m}}{c^2 - ab}$ is symmetric and idempotent, thus

$$\begin{aligned} \operatorname{rank}(\mathbf{I} - \frac{c\mathbf{r}_\mathbf{m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r}_\mathbf{m}\mathbf{r}'_\mathbf{m} + c\mathbf{e}\mathbf{r}'_\mathbf{m}}{c^2 - ab}) &= \operatorname{Tr}(\mathbf{I} - \frac{c\mathbf{r}_\mathbf{m}\mathbf{e}' - a\mathbf{e}\mathbf{e}' - b\mathbf{r}_\mathbf{m}\mathbf{r}'_\mathbf{m} + c\mathbf{e}\mathbf{r}'_\mathbf{m}}{c^2 - ab}) \\ &= T - \frac{2c^2 - 2ab}{c^2 - ab} \\ &= T - 2 \end{aligned}$$

By following lemma:

> Suppose symmetric matrix $p \times p$ $\mathbf{A}$. It's idempotent of rank s iff there exist a $p \times s$ $\mathbf{P} \ni \mathbf{P}\mathbf{P}' = \mathbf{A}$ and $\mathbf{P}'\mathbf{P} = \mathbf{I}$.

Proof Sufficiency is trivial. For necessity, since $\mathbf{A}$ is symmetric and idempotent matrix, it can be spectral decompostioned by $\mathbf{A} = \mathbf{Q\Lambda Q'}$. Where the diagonal of $\mathbf{\Lambda}$ is s 1 and $p - s$ 0. Thus

$$\mathbf{A} = \mathbf{Q\Lambda Q'} = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \begin{pmatrix} \mathbf{I}_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1' \\ \mathbf{P}_2' \end{pmatrix} = \mathbf{P}_1\mathbf{P}_1'$$

Note

$$\mathbf{I}_p = \mathbf{Q'Q} = \begin{pmatrix} \mathbf{P}_1' \\ \mathbf{P}_2' \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1'\mathbf{P}_1 & \mathbf{P}_1'\mathbf{P}_2 \\ \mathbf{P}_2'\mathbf{P}_1 & \mathbf{P}_2'\mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1'\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2'\mathbf{P}_2 \end{pmatrix}$$

hence$\mathbf{P_1'P_1} = \mathbf{I_s}$.■

We may find $\mathbf{r'P} \sim \mathcal{MN}_{n\times(T-2)}(\mathbf{0}, \Sigma, \mathbf{I})$. Where $\mathrm{E}[\mathbf{r'P}] = \mathrm{E}[\mathbf{r'PP'P}] = \mathrm{E}[\mathbf{r'AP}] = \mathbf{0}$ and thus (Wikipedia contributors 2020)

$$T\hat{\Sigma} = \mathbf{r'Ar} = \mathbf{r'PP'r} = \mathbf{r'P}(\mathbf{r'P})' \sim W_n(T-2, \Sigma)$$

Petersen, Kaare Brandt, and Michael Syskind Pedersen. 2012. "The Matrix Cookbook, Nov 2012." *URL Http://Www2. Imm. Dtu. Dk/Pubdb/P. Php* 3274: 14.

Wikipedia contributors. 2019. "Matrix Normal Distribution — Wikipedia, the Free Encyclopedia." `https://en.wikipedia.org/w/index.php?title=Matrix_normal_distribution&oldid=902125596`.

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