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# MATRIX APPROCH TO GRS STATISTIC

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## 1 GRS for capm

Suppose we run time series regressions for each of the  $n$  risky assets

$$\mathbf{r}_t^e = \alpha + \beta r_{m,t}^e + \epsilon_t$$

where  $\alpha, \mathbf{r}_t^e, \beta, \nu_t$  are  $n \times 1$  vector and  $r_{m,t}^e$  is scalar.

By the discussion above,  $\alpha = \mathbf{0}$  when CAPM holds. Assume  $\{\nu_t\}_{t=1}^T$  i.i.d with  $\mathcal{N}(0, \Sigma)$ , we have  $\mathbf{r}_t^e \mid r_{m,t}^e \sim \mathcal{N}(\alpha + \beta r_{m,t}^e, \Sigma)$ .

Suppose

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1^{e'} \\ \mathbf{r}_2^{e'} \\ \mathbf{r}_3^{e'} \\ \vdots \\ \mathbf{r}_T^{e'} \end{bmatrix}, \mathbf{r}_m = \begin{bmatrix} r_{m,1}^e \\ r_{m,2}^e \\ r_{m,3}^e \\ \vdots \\ r_{m,T}^e \end{bmatrix}$$

The equation become

$$\mathbf{r}' = \alpha \mathbf{e}' - \beta \mathbf{r}_m' + \mathbf{E}$$

Now  $\mathbf{r}' \sim \mathcal{MN}_{n \times T}(\alpha \mathbf{e}' + \beta \mathbf{r}_m', \Sigma, \mathbf{I})$ , the p.d.f is (Wikipedia contributors 2019)

$$p(\mathbf{r}' | \beta \mathbf{r}_m', \Sigma, \mathbf{I}) = \frac{\exp(-\frac{1}{2} \text{Tr}[(\mathbf{r}' - \alpha \mathbf{e}' - \beta \mathbf{r}_m')' \Sigma^{-1} (\mathbf{r}' - \alpha \mathbf{e}' - \beta \mathbf{r}_m')])}{(2\pi)^{nT/2} T^{n/2} |\Sigma|^{T/2}}$$

thus the log likelihood function is

$$\log L = -\frac{1}{2} \text{Tr}[(\mathbf{r}' - \alpha \mathbf{e}' - \beta \mathbf{r}_m')' \Sigma^{-1} (\mathbf{r}' - \alpha \mathbf{e}' - \beta \mathbf{r}_m')] - \frac{nT}{2} \log 2\pi - \frac{n}{2} \log T - \frac{T}{2} \log |\Sigma|$$

FOC w.r.t  $\alpha$ , by chain rule(Petersen and Pedersen 2012)

$$\begin{aligned}
\partial \log L &= \text{Tr} \left( \frac{\partial \log L}{\partial (\mathbf{r}' - \boldsymbol{\alpha} \mathbf{e}' - \beta \mathbf{r}'_{\mathbf{m}})}' \partial (\mathbf{X} - \boldsymbol{\alpha} \mathbf{e}' - \beta \mathbf{r}'_{\mathbf{m}}) \right) \\
&= \text{Tr} \left( \frac{\partial \log L}{\partial (\mathbf{r}' - \boldsymbol{\alpha} \mathbf{e}' - \beta \mathbf{r}'_{\mathbf{m}})}' \partial \alpha (-\mathbf{e}') \right) \\
&= \text{Tr} \left( (-\mathbf{e}') \frac{\partial \log L}{\partial (\mathbf{r}' - \boldsymbol{\alpha} \mathbf{e}' - \beta \mathbf{r}'_{\mathbf{m}})}' \partial \alpha \right)
\end{aligned}$$

hence

$$\begin{aligned}
\frac{\partial \log L}{\partial \alpha} &= - \frac{\partial \log L}{\partial (\mathbf{r}' - \boldsymbol{\alpha} \mathbf{e}' - \beta \mathbf{r}'_{\mathbf{m}})} \mathbf{e} \\
&= -(\Sigma^- + \Sigma'^-)(\mathbf{r}' - \boldsymbol{\alpha} \mathbf{e}' - \beta \mathbf{r}'_{\mathbf{m}}) \mathbf{e} = 0
\end{aligned}$$

Similarly, FOC w.r.t  $\beta$  and combine those results:

$$\begin{aligned}
(\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\beta} \mathbf{r}'_{\mathbf{m}}) \mathbf{r}_{\mathbf{m}} &= \mathbf{0} \\
(\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\beta} \mathbf{r}'_{\mathbf{m}}) \mathbf{e} &= \mathbf{0}
\end{aligned}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{r}'_{\mathbf{m}} \mathbf{r}_{\mathbf{m}} & \mathbf{r}'_{\mathbf{m}} \mathbf{e} \\ \mathbf{e}' \mathbf{r}_{\mathbf{m}} & \mathbf{e}' \mathbf{e} \end{bmatrix} \begin{bmatrix} \hat{\beta}' \\ \hat{\alpha}' \end{bmatrix} = \begin{bmatrix} \mathbf{r}'_{\mathbf{m}} \mathbf{r}' \\ \mathbf{e}' \mathbf{r}' \end{bmatrix}$$

Similarly to our deduction for mean-variance model, let  $a = \mathbf{r}'_{\mathbf{m}} \mathbf{r}_{\mathbf{m}}$ ,  $b = \mathbf{e}' \mathbf{e} = T$  and  $c = \mathbf{e}' \mathbf{r}_{\mathbf{m}} (c^2 < ab)$ , hence

$$\hat{\alpha} = \frac{\mathbf{r}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}}$$

and note

$$\hat{\beta} \mathbf{r}'_{\mathbf{m}} = \mathbf{r}' \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e} \mathbf{P}_{\mathbf{r}_{\mathbf{m}}} - (\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e} \mathbf{e}' \mathbf{P}_{\mathbf{r}_{\mathbf{m}}}}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}}$$

By assumption  $\mathbf{r}' \sim \mathcal{MN}_{n \times T}(\beta \mathbf{r}'_{\mathbf{m}}, \Sigma, \mathbf{I})$ , and  $\hat{\alpha} = \mathbf{r}'(c \mathbf{r}_{\mathbf{m}} - a \mathbf{e}) / (c^2 - ab)$ . By transformation of matrix normal distribution (Wikipedia contributors 2019)

$$\frac{[(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}]'[(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}]}{[\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}]^2} = \frac{1}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}}$$

we have

$$\hat{\alpha} \sim \mathcal{MN}(\mathbf{0}, \Sigma, \frac{1}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}}) \sim \mathcal{N}(0, \frac{1}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}} \Sigma)$$

which degenerated to multivariate normal distribution since  $\Sigma \otimes \frac{1}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}} = \frac{1}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}} \Sigma$ .

Then we may construct statistic  $J_0$  as

$$J_0 = \hat{\alpha}' \left( \frac{1}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}})\mathbf{e}} \Sigma \right)^- \hat{\alpha} \sim \chi_n^2$$

However,  $\Sigma$  is unknown so we should use  $\hat{\Sigma}$  instead of  $\Sigma$  and now  $J_0$  is just asymptotically chi-square distributed:

$$J_0 = \mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\hat{\alpha}'\hat{\Sigma}^-\hat{\alpha} \stackrel{A}{\sim} \chi_n^2$$

Return to the likelihood equation and FOC w.r.t  $\Sigma$ :

$$\frac{\partial \log L}{\partial \Sigma} = \frac{1}{2}(\Sigma^-(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_m)(\mathbf{r}' - \alpha\mathbf{e}' - \beta\mathbf{r}'_m)'\Sigma^-)' - \frac{T}{2}\Sigma'^- = 0$$

hence

$$\hat{\Sigma} = (\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_m)(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_m)'/T$$

note

$$\hat{\alpha}\mathbf{e}' + \hat{\beta}\mathbf{r}'_m = \mathbf{r}' \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{P}_{\mathbf{r}_m} + (\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}}$$

Thus

$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_m) = \mathbf{r}' \left( \mathbf{I} - \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{P}_{\mathbf{r}_m} + (\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}} \right)$$

The transform matrix is symmetric and idempotent, thus

$$\begin{aligned} \text{rank}(\mathbf{I} - \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{P}_{\mathbf{r}_m} + (\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}}) &= \text{Tr}(\mathbf{I} - \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{P}_{\mathbf{r}_m} + (\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}}) \\ &= T - (1 + 1) \\ &= T - 2 \end{aligned}$$

By following lemma:

**Lemma** Suppose symmetric matrix  $p \times p$   $\mathbf{A}$ . It's idempotent of rank  $s$  iff there exist a  $p \times s$   $\mathbf{P} \ni \mathbf{P}\mathbf{P}' = \mathbf{A}$  and  $\mathbf{P}'\mathbf{P} = \mathbf{I}$ .

**Proof** Sufficiency is trivial. For necessity, since  $\mathbf{A}$  is symmetric and idempotent matrix, it can be spectral decomposed by  $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}'$ . Where the diagonal of  $\mathbf{\Lambda}$  is  $s$  1 and  $p - s$  0. Thus

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}' = (\mathbf{P}_1 \quad \mathbf{P}_2) \begin{pmatrix} \mathbf{I}_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}'_1 \\ \mathbf{P}'_2 \end{pmatrix} = \mathbf{P}_1\mathbf{P}'_1$$

Note

$$\mathbf{I}_p = \mathbf{Q}'\mathbf{Q} = \begin{pmatrix} \mathbf{P}'_1 \\ \mathbf{P}'_2 \end{pmatrix} (\mathbf{P}_1 \quad \mathbf{P}_2) = \begin{pmatrix} \mathbf{P}'_1\mathbf{P}_1 & \mathbf{P}'_1\mathbf{P}_2 \\ \mathbf{P}'_2\mathbf{P}_1 & \mathbf{P}'_2\mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}'_1\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}'_2\mathbf{P}_2 \end{pmatrix}$$

hence  $\mathbf{P}'_1\mathbf{P}_1 = \mathbf{I}_s$ . ■

We may find  $\mathbf{r}'\mathbf{P} \sim \mathcal{MN}_{n \times (T-2)}(\mathbf{0}, \Sigma, \mathbf{I})$ . Where  $\mathbb{E}[\mathbf{r}'\mathbf{P}] = \mathbb{E}[\mathbf{r}'\mathbf{P}\mathbf{P}'\mathbf{P}] = \mathbb{E}[\mathbf{r}'\mathbf{A}\mathbf{P}] = \mathbf{0}$  and thus (Wikipedia contributors 2020)

$$T\hat{\Sigma} = \mathbf{r}'\mathbf{A}\mathbf{r} = \mathbf{r}'\mathbf{P}\mathbf{P}'\mathbf{r} = \mathbf{r}'\mathbf{P}(\mathbf{r}'\mathbf{P})' \sim W_n(T-2, \Sigma)$$

**Theorem** Suppose  $\mathbf{A} \sim W_n(m, \Sigma)$  and  $\mathbf{x} \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$ , where  $m > n$ , then

$$\frac{m-n+1}{n} \mathbf{x}'\mathbf{A}^-\mathbf{x} \sim F_{n, m-n+1}$$

**Proof** Note

$$\mathbf{x}'\mathbf{A}^-\mathbf{x} = \frac{\mathbf{x}'\Sigma^-\mathbf{x}}{\mathbf{x}'\Sigma^-\mathbf{x}'/\mathbf{x}'\mathbf{A}^-\mathbf{x}}$$

where  $\mathbf{x}'\Sigma^-\mathbf{x} \sim \chi_n^2$  and  $\frac{\mathbf{x}'\Sigma^-\mathbf{x}}{\mathbf{x}'\mathbf{A}^-\mathbf{x}} \sim \chi_{m-n+1}^2$  (Gupta and Nagar 2018). Then this claim follows. ■

Thus, taking  $\mathbf{x} = \sqrt{\mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}}\hat{\boldsymbol{\alpha}}$  and apply above theorem, we have

$$J_1 = \frac{T-n-1}{n} \frac{1}{T} \mathbf{e}'(\mathbf{I} - \mathbf{P}_{\mathbf{r}_m})\mathbf{e}\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^-\hat{\boldsymbol{\alpha}} = \frac{T-n-1}{nT} J_0 \sim F_{n, T-n-1}$$

$J_1$  is the so called GRS statistic.

### 1.1 Interpretation of $J_1$

Consider a sub portfolio problem:

$$\min \frac{1}{2} \boldsymbol{\omega}' \mathbf{V} \boldsymbol{\omega} \quad s.t. \quad \mathbf{e}' \boldsymbol{\omega} - \mathbf{1}, \boldsymbol{\omega}' \bar{\mathbf{r}} = m$$

where  $r = \begin{bmatrix} r_{m,t}^e \\ \mathbf{r}_t^e \end{bmatrix}$ . Thus the MLE of  $\bar{\mathbf{r}}$

$$\hat{\mathbf{r}} = \mathbb{E}[\widehat{\begin{bmatrix} r_{m,t}^e \\ \mathbf{r}_t^e \end{bmatrix}}] = \frac{1}{T} \begin{bmatrix} \mathbf{r}_m' \mathbf{e} \\ \mathbf{r}' \mathbf{e} \end{bmatrix}$$

and  $\mathbf{V}$  follows by  $\hat{\sigma}^2 = \mathbf{r}_m' (\mathbf{I} - \mathbf{P}_1) \mathbf{r}_m = a - \frac{c^2}{b}$  and recall

$$\mathbf{r}_t^e = \boldsymbol{\alpha} + \beta r_{m,t}^e + \nu_t$$

then

$$\text{Cov}(\mathbf{r}_t^e, r_{m,t}^e) = \text{Cov}(\beta r_{m,t}^e, r_{m,t}^e) = \beta \hat{\sigma}^2$$

$$\text{Cov}(\mathbf{r}_t^e) = \text{Cov}(\beta r_{m,t}^e) + \text{Cov}(\nu_t) = \beta \sigma^2 \beta' + \Sigma$$

hence

$$\hat{\mathbf{V}} = \text{Cov}(\bar{\mathbf{r}}) = \begin{bmatrix} \hat{\sigma}^2 & \hat{\sigma}^2 \beta' \\ \hat{\sigma}^2 \beta & \Sigma + \beta \hat{\sigma}^2 \beta' \end{bmatrix}$$

Using the formula for a partitioned inverse:

$$\begin{aligned} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1} \mathbf{B} (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \mathbf{C} \mathbf{A}^{-1} & -\mathbf{A}^{-1} \mathbf{B} (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \mathbf{C} \mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} \mathbf{B} \mathbf{D}^{-1} \\ -\mathbf{D}^{-1} \mathbf{C} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{C} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} \mathbf{B} \mathbf{D}^{-1} \end{bmatrix} \end{aligned}$$

Thus

$$\hat{\mathbf{V}}^{-1} = \begin{bmatrix} \frac{1}{\hat{\sigma}^2} + \beta' \Sigma^{-1} \beta & -\beta' \Sigma^{-1} \\ -\Sigma^{-1} \beta & \Sigma^{-1} \end{bmatrix}$$

Note the maximal sharpe ratio is given by

$$S_p = \sqrt{\bar{\mathbf{r}}' \hat{\mathbf{V}}^{-1} \bar{\mathbf{r}}}$$

Directly algebra and the invariance property of MLE yields

$$\hat{\mathbf{r}}' \hat{\mathbf{V}}^{-1} \hat{\mathbf{r}} = \frac{\mathbf{r}'_{\mathbf{m}} \mathbf{e}^2}{T^2 \hat{\sigma}^2} + \frac{(\mathbf{r}'_{\mathbf{e}} - \hat{\beta} \mathbf{r}'_{\mathbf{m}} \mathbf{e})' \hat{\Sigma}^{-1} (\mathbf{r}'_{\mathbf{e}} - \hat{\beta} \mathbf{r}'_{\mathbf{m}} \mathbf{e})}{T^2}$$

note  $(\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\beta} \mathbf{r}'_{\mathbf{m}} \mathbf{e}) = \mathbf{0}$ :

$$\hat{\alpha} \hat{\Sigma}^{-1} \hat{\alpha} = \hat{\mathbf{r}}' \hat{\mathbf{V}}^{-1} \hat{\mathbf{r}} - \frac{\mathbf{r}'_{\mathbf{m}} \mathbf{P}_1 \mathbf{r}_{\mathbf{m}}}{\mathbf{r}'_{\mathbf{m}} (\mathbf{I} - \mathbf{P}_1) \mathbf{r}_{\mathbf{m}}}$$

where  $\frac{\mathbf{r}'_{\mathbf{m}} \mathbf{P}_1 \mathbf{r}_{\mathbf{m}}}{\mathbf{r}'_{\mathbf{m}} (\mathbf{I} - \mathbf{P}_1) \mathbf{r}_{\mathbf{m}}}$  is the MLE of square market Sharpe ratio. Together with

$$\hat{\theta}_m^2 = \frac{\mathbf{r}'_{\mathbf{m}} \mathbf{P}_1 \mathbf{r}_{\mathbf{m}}}{\mathbf{r}'_{\mathbf{m}} (\mathbf{I} - \mathbf{P}_1) \mathbf{r}_{\mathbf{m}}} = \frac{\frac{c^2}{b}}{a - \frac{c^2}{b}} = \frac{c^2}{ab - c^2}$$

$$\frac{\mathbf{e}' (\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}}) \mathbf{e}}{T} = \frac{\mathbf{e}' (\mathbf{I} - \mathbf{P}_{\mathbf{r}_{\mathbf{m}}}) \mathbf{e}}{\mathbf{e}' \mathbf{e}} =$$

we finally have:

$$J_1 = \frac{T - n - 1}{n} \frac{\hat{\theta}_p^2 - \hat{\theta}_m^2}{1 + \hat{\theta}_m^2}$$

## 2 GRS for multi-factor model

Follow the similar fushion:

$$\mathbf{r}' = \alpha \mathbf{e}' + \mathbf{B} \mathbf{F}' + \mathbf{E}$$

where  $\mathbf{B}$  is  $n \times k$  and  $\mathbf{F}'$  is  $k \times t$

Now  $\mathbf{r}' \sim \mathcal{MN}_{n \times T}(\alpha \mathbf{e}' + \mathbf{B} \mathbf{F}', \Sigma, \mathbf{I})$ , the MLE satisfy:

$$(\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\mathbf{B}} \mathbf{F}') \mathbf{F} = \mathbf{0}$$

$$(\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\mathbf{B}} \mathbf{F}') \mathbf{e} = \mathbf{0}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{F}' \mathbf{F} & \mathbf{F}' \mathbf{e} \\ \mathbf{e}' \mathbf{F} & \mathbf{e}' \mathbf{e} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{B}}' \\ \hat{\alpha}' \end{bmatrix} = \begin{bmatrix} \mathbf{F}' \mathbf{r} \\ \mathbf{e}' \mathbf{r} \end{bmatrix}$$

Recall the inverse of partitioned matrix:

$$\begin{aligned} \begin{bmatrix} \hat{\mathbf{B}}' \\ \hat{\alpha}' \end{bmatrix} &= \begin{bmatrix} \mathbf{F}' \mathbf{F} & \mathbf{F}' \mathbf{e} \\ \mathbf{e}' \mathbf{F} & \mathbf{e}' \mathbf{e} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{F}' \mathbf{r} \\ \mathbf{e}' \mathbf{r} \end{bmatrix} \\ &= \frac{1}{\mathbf{e}' (\mathbf{I} - \mathbf{P}_{\mathbf{F}}) \mathbf{e}} \begin{bmatrix} \mathbf{e}' (\mathbf{I} - \mathbf{P}_{\mathbf{F}}) \mathbf{e} (\mathbf{F}' \mathbf{F})^{-1} + (\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}' \mathbf{e} \mathbf{e}' \mathbf{F} (\mathbf{F}' \mathbf{F})^{-1} & -(\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}' \mathbf{e} \\ -\mathbf{e}' \mathbf{F} (\mathbf{F}' \mathbf{F})^{-1} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{F}' \mathbf{r} \\ \mathbf{e}' \mathbf{r} \end{bmatrix} \end{aligned}$$

Thus

$$\hat{\alpha} = \frac{\mathbf{r}' (\mathbf{I} - \mathbf{P}_{\mathbf{F}}) \mathbf{e}}{\mathbf{e}' (\mathbf{I} - \mathbf{P}_{\mathbf{F}}) \mathbf{e}} \sim \mathcal{N}(0, \frac{\Sigma}{\mathbf{e}' (\mathbf{I} - \mathbf{P}_{\mathbf{F}}) \mathbf{e}})$$

and note

$$\begin{aligned}\hat{\alpha}\mathbf{e}' + \hat{\mathbf{B}}\mathbf{F}' &= \mathbf{r}' \frac{(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{e}' + \mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{P}_F - (\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{e}'\mathbf{P}_F}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}} \\ &= \mathbf{r}' \frac{(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_F) + \mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{P}_F}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}}\end{aligned}$$

Return to the likelihood equation and FOC w.r.t  $\Sigma$ :

$$\hat{\Sigma} = (\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\mathbf{B}}\mathbf{F}')(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\mathbf{B}}\mathbf{F}')' / T$$

plug in  $\hat{\alpha}, \hat{\beta}$ :

$$\begin{aligned}(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\mathbf{B}}\mathbf{F}') &= \mathbf{r}' \left( \mathbf{I} - \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{P}_F + (\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}} \right) \\ \text{rank}(\mathbf{I} - \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{P}_F + (\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}}) &= \text{Tr}(\mathbf{I} - \frac{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{P}_F + (\mathbf{I} - \mathbf{P}_F)\mathbf{e}\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}}) \\ &= T - (k + 1) \\ &= T - k - 1\end{aligned}$$

thus

$$T\hat{\Sigma} \sim W_n(T - k - 1, \Sigma)$$

it's followed by

$$J_1 = \frac{T - n - k}{n} \frac{1}{T} \mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim F_{n, T-n-k}$$

Then we show that

$$\frac{\mathbf{e}'\mathbf{e}}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}} = 1 + \hat{\mathbf{f}}' \widehat{\text{Cov}(\mathbf{f})}^{-1} \hat{\mathbf{f}}$$

Where

$$\widehat{\text{Cov}(\mathbf{f})} = \frac{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1)\mathbf{F}}{\mathbf{e}'\mathbf{e}}, \hat{\mathbf{f}} = \frac{\mathbf{F}'\mathbf{e}}{\mathbf{e}'\mathbf{e}}$$

It's sufficient to show that

$$\mathbf{F}[\mathbf{F}'(\mathbf{I} - \mathbf{P}_1)\mathbf{F}]^{-1}\mathbf{F}'\mathbf{e} = \frac{\mathbf{P}_F\mathbf{e}\mathbf{e}'\mathbf{e}}{\mathbf{e}'(\mathbf{I} - \mathbf{P}_F)\mathbf{e}}$$

**Lemma** Given any 2 matrix  $A$  and  $B$ , we claim that

$$\mathbf{P}_A\mathbf{B}[\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B}]^{-1}\mathbf{B}'\mathbf{B} = \mathbf{A}[\mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{A}]^{-1}\mathbf{A}'\mathbf{B}$$

**Proof**

$$\begin{aligned}\mathbf{P}_A\mathbf{B}[\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B}]^{-1}\mathbf{B}'\mathbf{B} &= \mathbf{A}[\mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{A}]^{-1}\mathbf{A}'\mathbf{B} \\ \iff (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}[\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B}]^{-1}\mathbf{B}'\mathbf{B} &= [\mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{A}]^{-1}\mathbf{A}'\mathbf{B} \\ \iff \mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}[\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B}]^{-1}\mathbf{B}'\mathbf{B} &= \mathbf{A}'\mathbf{B} \\ \iff \mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{P}_A\mathbf{B}[\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B}]^{-1}\mathbf{B}'\mathbf{B} &= \mathbf{A}'\mathbf{B} \\ \iff \mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{P}_A\mathbf{B}[\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B}]^{-1} &= \mathbf{A}'\mathbf{B}[\mathbf{B}'\mathbf{B}]^{-1} \\ \iff \mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{P}_A\mathbf{B} &= \mathbf{A}'\mathbf{B}[\mathbf{B}'\mathbf{B}]^{-1}\mathbf{B}'(\mathbf{I} - \mathbf{P}_A)\mathbf{B} \\ \iff \mathbf{A}'(\mathbf{I} - \mathbf{P}_B)\mathbf{P}_A\mathbf{B} &= \mathbf{A}'\mathbf{P}_B(\mathbf{I} - \mathbf{P}_A)\mathbf{B}\end{aligned}$$

The claim follows from both sides equal to  $\mathbf{A}'(\mathbf{I} - \mathbf{P}_B\mathbf{P}_A)\mathbf{B}$ . ■

Thus,

$$J_1 = \frac{T - n - k}{n} \frac{\hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha}}{1 + \hat{\theta}_m^2} \sim F_{n, T-n-k}$$

where

$$\hat{\theta}_m^2 = \hat{\mathbf{f}}' \widehat{\text{Cov}(\mathbf{f})}^{-1} \hat{\mathbf{f}} = \frac{\mathbf{e}' \mathbf{F} [\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F}]^{-1} \mathbf{F}' \mathbf{e}}{\mathbf{e}' \mathbf{e}}$$

denoted the MLE of the maximal squared sharpe ration generated by the  $k$  risk factor  $\mathbf{F}'$ .

## 2.1 Interpretation of $J_1$

Consider a sub portfolio problem:

$$\min \frac{1}{2} \boldsymbol{\omega}' \mathbf{V} \boldsymbol{\omega} \quad s.t. \quad \mathbf{e}' \boldsymbol{\omega} - \mathbf{1}, \boldsymbol{\omega}' \bar{\mathbf{r}} = m$$

where  $\mathbf{r} = \begin{bmatrix} \mathbf{f}_t \\ \mathbf{r}_t^e \end{bmatrix}$ . Thus the MLE of  $\bar{\mathbf{r}}$

$$\hat{\mathbf{r}} = \frac{1}{T} \begin{bmatrix} \mathbf{F}' \mathbf{e} \\ \mathbf{r}' \mathbf{e} \end{bmatrix}$$

and  $\mathbf{V}$  is given by

$$\hat{\mathbf{V}} = \begin{bmatrix} \frac{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F}}{\mathbf{B}' \frac{\mathbf{e}' \mathbf{e}}{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F}}} & \frac{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F} \mathbf{B}'}{\mathbf{B}' \frac{\mathbf{e}' \mathbf{e}}{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F}}} \\ \hat{\Sigma} + \mathbf{B} \frac{\mathbf{e}' \mathbf{e}}{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F}} \mathbf{B}' \end{bmatrix}$$

Directly algebra and the invariance property of MLE yields

$$\hat{\theta}_p^2 = \hat{\mathbf{r}}' \hat{\mathbf{V}}^{-1} \hat{\mathbf{r}} = \frac{\mathbf{e}' \mathbf{F} [\frac{\mathbf{F}'(\mathbf{I} - \mathbf{P}_1) \mathbf{F}}{\mathbf{e}' \mathbf{e}}]^{-1} \mathbf{F}' \mathbf{e}}{T^2} + \frac{(\mathbf{r}' \mathbf{e} - \hat{\mathbf{B}} \mathbf{F}' \mathbf{e})' \hat{\Sigma}^{-1} (\mathbf{r}' \mathbf{e} - \hat{\mathbf{B}} \mathbf{F}' \mathbf{e})}{T^2}$$

that is

$$\hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} = \hat{\theta}_p^2 - \hat{\theta}_m^2$$

we finally have:

$$J_1 = \frac{T - n - k}{n} \frac{\hat{\theta}_p^2 - \hat{\theta}_m^2}{1 + \hat{\theta}_m^2}$$

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