
FACTOR MODEL

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1 CAPM

1.1 Beta representation

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})}{\mathbf{e}' \mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})}$. Write $\omega_D = m \mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})$ where $m = \frac{1}{\mathbf{e}' \mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})}$, then we have

$$\bar{\mathbf{r}} - r_f \mathbf{e} = \frac{1}{m} \mathbf{V} \omega_D$$

Note $\text{Cov}(\mathbf{r}, \omega'_D \mathbf{r}) = \mathbf{V} \omega_D$ and

$$\sigma_D^2 = \omega'_D \mathbf{V} \omega_D = m \omega'_D (\bar{\mathbf{r}} - r_f \mathbf{e}) = m r_D - m r_f$$

we have

$$\bar{\mathbf{r}} - r_f \mathbf{e} = \frac{r_D - r_f}{\sigma_D^2} \text{Cov}(\mathbf{r}, r_D)$$

Denote $\frac{\text{Cov}(\mathbf{r}, r_D)}{\sigma_D^2} = \beta_D$, we have

$$\bar{\mathbf{r}} - r_f \mathbf{e} = \beta_D (r_D - r_f)$$

Similar results also holds for any portfolio \bar{r}_p in the MVF:

$$\bar{\mathbf{r}} - \bar{r}_q \mathbf{e} = \beta_p (\bar{r}_p - \bar{r}_q)$$

It's clear in the view of every portfolio \bar{r}_p is also a tangency portfolio by selecting proper r_f . One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega'_p \mathbf{V} \omega_q = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})(\bar{r}_q - \frac{\alpha}{\delta})}{\delta \xi - \alpha^2}$$

If the covariance is 0, we have

$$\bar{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)}$$

Then

$$\begin{aligned}\bar{\mathbf{r}} - \bar{r}_q \mathbf{e} &= \bar{\mathbf{r}} - \left(\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)} \right) \mathbf{e} \\ &= \frac{1}{\delta^2(\bar{r}_p - \alpha/\delta)} (\delta^2(\bar{r}_p - \alpha/\delta)) (\bar{\mathbf{r}} - \left(\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)} \right) \mathbf{e}) \\ &= \frac{1}{\delta^2(\bar{r}_p - \alpha/\delta)} (\bar{\mathbf{r}}(\delta^2(\bar{r}_p - \alpha/\delta)) - (\alpha\delta(\bar{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2)) \mathbf{e}) \\ &= \frac{1}{\delta^2(\bar{r}_p - \alpha/\delta)} (\bar{\mathbf{r}}(\delta^2(\bar{r}_p - \alpha/\delta)) - (\alpha\delta\bar{r}_p - \delta\xi) \mathbf{e}) \\ &= \frac{(\delta^2\bar{r}_p\bar{\mathbf{r}} - \alpha\delta\bar{\mathbf{r}}) - (\alpha\delta\bar{r}_p - \delta\xi) \mathbf{e}}{\delta^2(\bar{r}_p - \alpha/\delta)} \\ &= \frac{(\delta\bar{r}_p - \alpha)\bar{\mathbf{r}} - (\alpha\bar{r}_p - \xi) \mathbf{e}}{\delta(\bar{r}_p - \alpha/\delta)}\end{aligned}$$

On the other hand:

$$\begin{aligned}\beta_p &= \frac{\mathbf{V}\omega_p}{\omega_p' \mathbf{V}\omega_p} \\ &= \frac{1}{\omega_p' \mathbf{V}\omega_p} (\lambda_p \bar{\mathbf{r}} + \gamma \mathbf{e}) \\ &= \frac{1}{\omega_p' \mathbf{V}\omega_p} \left(\frac{\xi \mathbf{e} - \alpha \bar{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha \mathbf{e} + \delta \bar{\mathbf{r}}}{\delta\xi - \alpha^2} \bar{r}_p \right) \\ &= \frac{1}{\omega_p' \mathbf{V}\omega_p} \left(\frac{(\delta\bar{r}_p - \alpha)\bar{\mathbf{r}} - (\alpha\bar{r}_p - \xi) \mathbf{e}}{\Delta} \right)\end{aligned}$$

Then it's remain to show that

$$(\bar{r}_p - \bar{r}_q)\delta(\bar{r}_p - \alpha/\delta) = \omega' \mathbf{V}\omega \Delta$$

It's clear since

$$\omega' \mathbf{V}\omega \Delta = \sigma_p^2 \Delta = \frac{\Delta}{\delta} + \delta(\bar{r}_p - \frac{\alpha}{\delta})^2$$

and

$$\begin{aligned}(\bar{r}_p - \bar{r}_q)\delta(\bar{r}_p - \alpha/\delta) &= \left((\bar{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)} \right) \delta(\bar{r}_p - \alpha/\delta) \\ &= \frac{\Delta}{\delta} + \delta(\bar{r}_p - \frac{\alpha}{\delta})^2\end{aligned}$$

1.2 CAPM

In capital market equilibrium, the market portfolio is tangency portfolio $r_D = r_m$, then

$$\bar{\mathbf{r}} - r_f \mathbf{e} = \beta_m (r_m - r_f)$$

where

$$\beta_m = \begin{bmatrix} \frac{\text{Cov}(r_1, r_m)}{\sigma_m^2} \\ \frac{\text{Cov}(r_2, r_m)}{\sigma_m^2} \\ \vdots \\ \frac{\text{Cov}(r_n, r_m)}{\sigma_m^2} \end{bmatrix}$$

this equation is called **Sharpe-Lintner CAPM**. $r_m - r_f$ is called **market risk premium** and $\frac{r_m - r_f}{\sigma_m}$ is called **market sharpe ratio**. Translate it from vector form, we get the **Security Market Line**:

$$r_i - r_f = \beta_{i,m}(r_m - r_f)$$

1.2.1 Realized return

Now consider both r_i and r_m is random variable, let ϵ be a random vector with zero expectation and zero covariance with r_i and r_m , then

$$r_i - r_f = \beta_{i,m}(r_m - r_f) + \epsilon_i$$

This is a regression equation, if one include an intercept, then the model

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \epsilon_i$$

is called **market model**, such α is called **Jensen's alpha**.

1.2.2 Variance decomposition

Decomposition the variance as:

$$\text{Var}(r_i) = \underbrace{\beta_i^2 \sigma_m^2}_{\text{Systematic risk}} + \underbrace{\text{Var}(\epsilon_i)}_{\text{Idiosyncratic risk}}$$

The R^2 is just the proportion of systematic risk

$$R^2 = \frac{\beta_i^2 \sigma_m^2}{\beta_i^2 \sigma_m^2 + \sigma^2}$$

since Fraction of variance unexplained.

1.2.3 Testing CAPM

Suppose we run time series regressions for each of the n risky assets

$$\mathbf{r}_t^e = \alpha + \beta r_{m,t}^e + \nu_t$$

where $\alpha, \mathbf{r}_t^e, \beta, \nu_t$ are $n \times 1$ vector and $r_{m,t}^e$ is scalar.

By the discussion above, $\alpha = \mathbf{0}$ when CAPM holds. Assume $\{\nu_t\}_{t=1}^T$ i.i.d with $N(0, \Sigma)$, we have $\mathbf{r}_t^e \mid r_{m,t}^e \sim N(\alpha + \beta r_{m,t}^e, \Sigma)$.

Hence the p.d.f is

$$f(\mathbf{r}_t^e) = (2\pi)^{-n/2} |\Sigma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e) \right\}$$

By the indepedence, from $t = 1$ to $t = T$, the joint p.d.f is $L = \prod_{t=1}^T f(\mathbf{r}_t^e) =$.

$$\log L = \sum_{t=1}^T \log f(\mathbf{r}_t^e) = -\frac{nT}{2} \ln(2\pi) - \frac{T}{2} \ln |\Sigma| - \frac{1}{2} \sum_{t=1}^T (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)$$

MLE for (α, β, Σ) is found by maximize $\log L$, W.r.t. β , it's the same as minimize

$$\sum_{t=1}^T (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)$$

FOC with β and recall $\frac{\partial \mathbf{x}' \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}') \mathbf{x}$, $\frac{\partial \mathbf{f}}{\partial \mathbf{X}} = \mathbf{A}' \frac{\partial \mathbf{f}}{\partial \mathbf{A} \mathbf{X} \mathbf{B}} \mathbf{B}'$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} &= \sum_{t=1}^T (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e) \\ &= \sum_{t=1}^T \frac{\partial (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)}{\partial \beta} \\ &= \sum_{t=1}^T \left(\frac{\partial (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)}{\partial \mathbf{r}_t^e - \alpha - \beta r_{m,t}^e} \frac{\partial \mathbf{r}_t^e - \alpha - \beta r_{m,t}^e}{\partial \beta} \right) \\ &= \sum_{t=1}^T (-2r_{m,t}^e \Sigma^{-1} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)) = 0 \end{aligned}$$

since Σ^{-1} is clearly invertible, $\sum_{t=1}^T r_{m,t}^e (\mathbf{r}_t^e - \hat{\alpha} - \hat{\beta} r_{m,t}^e) = \mathbf{0}$

Similarly, FOC w.r.t α , we get

$$\sum_{t=1}^T \mathbf{r}_t^e - \hat{\alpha} - \hat{\beta} r_{m,t}^e = \mathbf{0}$$

By the truth $\frac{\partial |\mathbf{X}|}{\partial \mathbf{X}} = |\mathbf{X}| \mathbf{X}'^{-1}$ and $\frac{\partial \mathbf{a}' \mathbf{X}^{-1} \mathbf{b}}{\partial \mathbf{X}} = -\mathbf{X}'^{-1} \mathbf{a} \mathbf{b}' \mathbf{X}'^{-1}$, we have

$$\frac{\partial \log L}{\partial \Sigma} = \sum_{t=1}^T \left(-\frac{\Sigma^{-1}}{2} + \frac{\Sigma^{-1}}{2} (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e) (\mathbf{r}_t^e - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} \right)$$

thus have

$$\hat{\Sigma} = \sum_{t=1}^T (\mathbf{r}_t^e - \hat{\alpha} - \hat{\beta} r_{m,t}^e) (\mathbf{r}_t^e - \hat{\alpha} - \hat{\beta} r_{m,t}^e)' / T$$

Suppose

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1^{e'} \\ \mathbf{r}_2^{e'} \\ \mathbf{r}_3^{e'} \\ \vdots \\ \mathbf{r}_T^{e'} \end{bmatrix}, \mathbf{r}_m = \begin{bmatrix} r_{m,1}^e \\ r_{m,2}^e \\ r_{m,3}^e \\ \vdots \\ r_{m,T}^e \end{bmatrix}$$

Now

$$\begin{aligned} (\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\beta} \mathbf{r}_m') \mathbf{r}_m &= \mathbf{0} \\ (\mathbf{r}' - \hat{\alpha} \mathbf{e}' - \hat{\beta} \mathbf{r}_m') \mathbf{e} &= \mathbf{0} \end{aligned}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{e}'\mathbf{r}_m & \mathbf{r}_m'\mathbf{r}_m \\ \mathbf{e}'\mathbf{e} & \mathbf{r}_m'\mathbf{e} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \mathbf{r}_m'\mathbf{r}_m \\ \mathbf{r}_m'\mathbf{e} \end{bmatrix}$$

hence

$$\begin{cases} \alpha = \mathbf{r}_m'\mathbf{r}_m(\mathbf{r}_m'\mathbf{e} - \mathbf{e}'\mathbf{e})/(\mathbf{r}_m'\mathbf{e}\mathbf{e}'\mathbf{r}_m - \mathbf{e}'\mathbf{e}\mathbf{r}_m'\mathbf{r}_m) \\ \beta = \mathbf{r}_m'\mathbf{e}(\mathbf{r}_m'\mathbf{r}_m - \mathbf{e}'\mathbf{r}_m) \end{cases}$$