Factor model

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1 CAPM

1.1 Beta representation

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^{-}(\bar{\mathbf{r}}-r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\bar{\mathbf{r}}-r_f\mathbf{e})}$. Write $\omega_D = m\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f\mathbf{e})$ where $m = \frac{1}{\mathbf{e}'\mathbf{V}^{-}(\bar{\mathbf{r}}-r_f\mathbf{e})}$, then we have

$$\bar{r} - r_f\mathbf{e} = \frac{1}{m}\mathbf{V}\omega_D$$

Note $\operatorname{Cov}(\mathbf{r}, \omega'\mathbf{r}) = \mathbf{V}\omega$ and

$$\sigma_D^2 = \omega_D'\mathbf{V}\omega_D = m\omega_D'(\bar{\mathbf{r}} - r_f\mathbf{e}) = m\bar{r}_D - mr_f$$

we have

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \frac{\bar{r}_D - r_f}{\sigma_D^2}\operatorname{Cov}(\mathbf{r}, r_D)$$

Denote $\frac{\operatorname{Cov}(\mathbf{r}, r_D)}{\sigma_D^2} = \beta_D$, we have

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \beta_D(\bar{r}_D - r_f)$$

Similar results also holds for any portfolio $\bar{r}_p$ and the zero covariance portfolio $\bar{r}_q$ in the MVF:

$$\bar{\mathbf{r}} - \bar{r}_q\mathbf{e} = \beta_p(\bar{r}_p - \bar{r}_q)$$

It's clear in the view of every portfolio $\bar{r}_p$ is also a tangency portfolio by selecting proper r_f. One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega_p'\mathbf{V}\omega_q = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})(\bar{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

If the covariance is 0, we have

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Then

$$\begin{aligned}
\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} &= \overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e} \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\delta^2(\overline{r}_p - \alpha/\delta))(\overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta(\overline{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2))\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e} \\
&= \frac{(\delta^2\overline{r}_p\overline{\mathbf{r}} - \alpha\delta\overline{\mathbf{r}}) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e}}{\delta^2(\overline{r}_p - \alpha/\delta)} \\
&= \frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\delta(\overline{r}_p - \alpha/\delta)}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\beta_p &= \frac{\mathbf{V}\omega_{\mathbf{p}}}{\omega_p'\mathbf{V}\omega_p} \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\lambda_p\overline{\mathbf{r}} + \gamma\mathbf{e}) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{\xi\mathbf{e} - \alpha\overline{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha\mathbf{e} + \delta\overline{\mathbf{r}}}{\delta\xi - \alpha^2}\overline{r}_p) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\Delta})
\end{aligned}$$

Then it's remain to show that

$$(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) = \omega'\mathbf{V}\omega\Delta$$

It's clear since

$$\omega'\mathbf{V}\omega\Delta = \sigma_p^2\Delta = \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2$$

and

$$\begin{aligned}
(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) &= ((\overline{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\delta(\overline{r}_p - \alpha/\delta) \\
&= \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2
\end{aligned}$$

1.2 CAPM

In capital market equilibrium, the market portfolio is tangecy portfolio $\overline{r}_D = r_m$, then

$$\overline{\mathbf{r}} - \boldsymbol{r_f}\mathbf{e} = \boldsymbol{\beta}_{\mathbf{m}}(\overline{r}_m - r_f)$$

where

$$\beta_m = \begin{bmatrix} \frac{\text{Cov}(r_1,\overline{r}_m)}{\sigma_m^2} \\ \frac{\text{Cov}(r_2,\overline{r}_m)}{\sigma_m^2} \\ \cdots \\ \frac{\text{Cov}(r_n,\overline{r}_m)}{\sigma_m^2} \end{bmatrix}$$

this equation is called **Sharpe-Lintner CAPM**. $\overline{r}_m - r_f$ is called **market risk premium** and $\frac{\overline{r}_m - r_f}{\sigma_m}$ is called **market sharpe ratio**. Translate it from vector form, we get the **Security Market Line**:

$$r_i - r_f = \beta_{i,m}(\overline{r}_m - r_f)$$

1.3 Realized return

Now consider both r_i and r_m is random variable, let ϵ be a random vector with zero expection and zero covariance with r_i and r_m, then

$$r_i - r_f = \beta_{i,m}(r_m - r_f) + \epsilon_i$$

This is a regression equation, if one include an intercept, then the model

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \epsilon_i$$

is called **market model**, such α is called **Jensen's alpha**.

1.4 Variance decomposition

Decomposition the variance as:

$$\text{Var}(r_i) = \overbrace{\underbrace{\beta_i \sigma_m^2}_{\text{Systematic risk}} + \underbrace{\text{Var}(\epsilon_i)}_{\text{Idiosyncratic risk}}}^{\text{total risk}}$$

The R^2 is just the proportion of systematic risk

$$R^2 = \frac{\beta_i^2 \sigma_m^2}{\beta_i^2 \sigma_m^2 + \sigma^2}$$

since Fraction of variance unexplained.

1.5 Testing CAPM

1.5.1 GRS test

see `GRS.pdf`

1.5.2 Cross-sectional regressions

By APT, we have

$$\text{E}[\mathbf{r_t}] - r_f \mathbf{e} = \mathbf{B}\boldsymbol{\lambda}$$

Take market excess return and some firm-level factor, that is

$$\mathbf{B}\boldsymbol{\lambda} = [\boldsymbol{\beta} \quad \mathbf{X}] \begin{bmatrix} \lambda \\ \gamma \end{bmatrix}$$

and the regression formula is

$$\mathrm{E}[\mathbf{r_t}] - r_f\mathbf{e} = \lambda_0 + \boldsymbol{\beta}\lambda + \mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$$

If CAPM holds, we have

$$\begin{cases} \lambda = \overline{r_m} - r_f \\ \lambda_0 = 0 \\ \boldsymbol{\gamma = 0} \\ R^2 \approx 1 \end{cases}$$

Where λ is the slope of **SML(Security Market Line)**. However, as we do not know $\overline{r_m} - r_f$, we should run a two pass cross-sectional regression. For example, we may test CAPM in such way:

Firstly, run a TS regression to estimate $\boldsymbol{\beta}$:

$$\mathbf{r_t^e} = \boldsymbol{\alpha} + \boldsymbol{\beta} \boldsymbol{r_{m,t}^e} + \boldsymbol{\epsilon_t}$$

Now we have a $\hat{\boldsymbol{\beta}}$. In the second pass, we run following regression:

$$\mathbf{r_t^e} = \boldsymbol{\lambda_0} + \hat{\boldsymbol{\beta}}\boldsymbol{\lambda}_t + \boldsymbol{\epsilon_t}$$

If CAPM holds, we have

$$\mathrm{E}[\boldsymbol{\lambda_0 = 0}], \mathrm{E}[\lambda_t] = \lambda = \mathrm{E}[r_m^e]$$

then we may use t test as usual.

1.6 Fama-MacBeth in depth

Writing the cross-sectional regression in a matrix form,

$$\mathbf{r} = \mathbf{X}\boldsymbol{\theta} + \boldsymbol{\epsilon}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{e} & \mathbf{x_2} & \cdots & \mathbf{x_m} \end{bmatrix}, \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix}$$

The OLS estimator of θ is given by

$$\hat{\boldsymbol{\theta}} = \mathbf{(X'X)^{-1}X'r_t}$$

Denoted $\mathbf{W' = (X'X)^{-1}X'}$ and suppose

$$\mathbf{W} = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \cdots & \mathbf{w_m} \end{bmatrix}$$

each $\mathbf{w_i}$ represent a portfolio and θ_i represents the excess return on corresponding portfolio. Note $\mathbf{W'X = I}$, we have

$$\mathbf{w_i'e} = \begin{cases} 0 & i \neq 1 \\ 1 & i = 1 \end{cases} \text{ and } \mathbf{w_i'x_j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

1.7 CAPM anomalies

However, past surveyresarch has found many $\mathbf{X}$ s.t. $\boldsymbol{\gamma} \neq \mathbf{0}$ significantly and even can't not reject $\lambda = 0$ in

$$\mathrm{E}[\mathbf{r_t}] - r_f \mathbf{e} = \lambda_0 + \boldsymbol{\beta}\lambda + \mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$$

That is known as **CAPM anomalies**.

1. CAPM anomalies rise because of mispricing.
2. CAPM anomalies rise because of other sources of risks. nonzero $\boldsymbol{\alpha}$ contains missing risk factors.

2 Multi-factors model

2.1 APT

Recall in the CAPM

$$\mathbf{r_t^e} = \boldsymbol{\alpha} + \beta r_{m,t}^e + \nu_t$$

Suppose now there is multi-factor with k risk factor and

$$\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon}_\mathbf{t}$$

Where $\mathbf{r_t}, \boldsymbol{\epsilon}_\mathbf{t}$ is n vector while $\mathbf{f_t}$ is k vector and $\mathbf{B}$ is $n \times k$ matrix.
Taking expectation:

$$\mathbf{r_t} = \mathrm{E}[\mathbf{r}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

Note CAPM is just special case of APT when $k = 1$ and $\mathbf{f_t} = r_{m,t}$, which is the only factor affecting realized return. Thta is

$$\mathbf{r_t} = \boldsymbol{r_f}\mathbf{e} + \boldsymbol{\beta}\,\mathrm{E}[\boldsymbol{r_m} - \boldsymbol{r_f}] + \boldsymbol{\beta}(\boldsymbol{r_{m,t}} - \mathrm{E}[\boldsymbol{r_{m,t}}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

Assume there is non-arbitrage, that is, if one invest 0 and take no risk, then the expected return is 0. Formally, as $n \to \infty$,

$$\boldsymbol{\omega}' \begin{bmatrix} \mathbf{e} & \mathbf{B} \end{bmatrix} = \mathbf{0} \implies \boldsymbol{\omega}' \,\mathrm{E}[\mathbf{r}] = \mathbf{0}$$

by Farkas lemma(**???**), that implies

$$\begin{bmatrix} \mathbf{e} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \boldsymbol{\lambda} \end{bmatrix} = \mathrm{E}[\mathbf{r}]$$

for some $\boldsymbol{\lambda} \geq \mathbf{0}$. Under CAPM, $\lambda_0 = r_f$ and $\mathrm{E}[r_m - r_f] = \lambda$ clearly. If one take no risk, that is $\mathbf{B} = \mathbf{0}$, s(he) get a λ_0 return, that implies $\lambda_0 = r_f$ immediately.

2.1.1 When factors are returns

When $\mathbf{f_t}$ is $\mathbf{r_t}$, regressing it on

$$\mathbf{f_t} = \mathrm{E}[\mathbf{f}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

we have $\mathbf{B} = \mathbf{I}$ and thus

$$\lambda_0 \mathbf{e} + \boldsymbol{\lambda} = \mathrm{E}[\mathbf{f_t}] \implies \mathrm{E}[\mathbf{f_t}] = \boldsymbol{r_f}\mathbf{e} + \boldsymbol{\lambda}$$

2.1.2 When factors are excess returns

If $\mathbf{f_t}$ is excess return for some tradable assets on time t, e.g.

$$f_{k,t} = r_{a,t} - r_{b,t}$$

where $r_{a,t}$ and $r_{b,t}$ is some componets in $\mathbf{r_t}$, that is

$$f_{k,t} = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & -1 & \cdots & 0 \end{bmatrix} \mathbf{r_t}$$

where the ath componets is 1 while bth is -1. If each componets is excess return, we may represent $\mathbf{f_t} = \mathbf{C}\mathbf{r_t}$ where each row of $\mathbf{C}$ of such form(All zero but a 1 and a -1). Recall

$$\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon}_\mathbf{t}$$

It follows that

$$\mathbf{f_t} = \mathbf{C}\mathbf{r_t} = \mathbf{C}\boldsymbol{\alpha} + \mathbf{C}\mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon}$$

thus $\mathbf{CB} = \mathbf{I}$. Recall

$$r_f\mathbf{e} + \mathbf{B}\boldsymbol{\lambda} = \mathrm{E}[\mathbf{r_t}]$$

thus

$$\mathrm{E}[\mathbf{f_t}] = \lambda$$

as $\mathbf{Ce} = \mathbf{0}$ clearly.

2.2 ICAPM

SKIP

2.3 Test Multi-factor model

see `GRS.pdf`