FACTOR MODEL

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0.1 CAPM

0.1.1 Beta representation

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^-(\bar{\mathbf{r}}-r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^-(\bar{\mathbf{r}}-r_f\mathbf{e})}$. Write $\omega_D = m\mathbf{V}^-(\bar{\mathbf{r}} - r_f\mathbf{e})$ where $m = \frac{1}{\mathbf{e}'\mathbf{V}^-(\bar{\mathbf{r}}-r_f\mathbf{e})}$, then we have

$$\bar{r} - r_f\mathbf{e} = \frac{1}{m}\mathbf{V}\omega_D$$

Note $\operatorname{Cov}(\mathbf{r}, \omega'\mathbf{r}) = \mathbf{V}\omega$ and

$$\sigma_D^2 = \omega_D'\mathbf{V}\omega_D = m\omega_D'(\bar{\mathbf{r}} - r_f\mathbf{e}) = m\bar{r}_D - mr_f$$

we have

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \frac{\bar{r}_D - r_f}{\sigma_D^2}\operatorname{Cov}(\mathbf{r}, r_D)$$

Denote $\frac{\operatorname{Cov}(\mathbf{r}, r_D)}{\sigma_D^2} = \beta_D$, we have

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \beta_D(\bar{r}_D - r_f)$$

Similar results also holds for any portfolio $\bar{r}_p$ and the zero covariance portfolio $\bar{r}_q$ in the MVF:

$$\bar{\mathbf{r}} - \bar{r}_q\mathbf{e} = \beta_p(\bar{r}_p - \bar{r}_q)$$

It's clear in the view of every portfolio $\bar{r}_p$ is also a tangency portfolio by selecting proper r_f. One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega_p'\mathbf{V}\omega_q = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})(\bar{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

If the covariance is 0, we have

$$\bar{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)}$$

Then

$$\begin{aligned}
\bar{\mathbf{r}} - \bar{r}_q\mathbf{e} &= \bar{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)})\mathbf{e} \\
&= \frac{1}{\delta^2(\bar{r}_p - \alpha/\delta)}(\delta^2(\bar{r}_p - \alpha/\delta))(\bar{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)})\mathbf{e}) \\
&= \frac{1}{\delta^2(\bar{r}_p - \alpha/\delta)}(\bar{\mathbf{r}}(\delta^2(\bar{r}_p - \alpha/\delta)) - (\alpha\delta(\bar{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2))\mathbf{e}) \\
&= \frac{1}{\delta^2(\bar{r}_p - \alpha/\delta)}(\bar{\mathbf{r}}(\delta^2(\bar{r}_p - \alpha/\delta)) - (\alpha\delta\bar{r}_p - \delta\xi)\mathbf{e} \\
&= \frac{(\delta^2\bar{r}_p\bar{\mathbf{r}} - \alpha\delta\bar{\mathbf{r}}) - (\alpha\delta\bar{r}_p - \delta\xi)\mathbf{e}}{\delta^2(\bar{r}_p - \alpha/\delta)} \\
&= \frac{(\delta\bar{r}_p - \alpha)\bar{\mathbf{r}} - (\alpha\bar{r}_p - \xi)\mathbf{e}}{\delta(\bar{r}_p - \alpha/\delta)}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\beta_p &= \frac{\mathbf{V}\omega_{\mathbf{p}}}{\omega_p'\mathbf{V}\omega_p} \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\lambda_p\bar{\mathbf{r}} + \gamma\mathbf{e}) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{\xi\mathbf{e} - \alpha\bar{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha\mathbf{e} + \delta\bar{\mathbf{r}}}{\delta\xi - \alpha^2}\bar{r}_p) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{(\delta\bar{r}_p - \alpha)\bar{\mathbf{r}} - (\alpha\bar{r}_p - \xi)\mathbf{e}}{\Delta})
\end{aligned}$$

Then it's remain to show that

$$(\bar{r}_p - \bar{r}_q)\delta(\bar{r}_p - \alpha/\delta) = \omega'\mathbf{V}\omega\Delta$$

It's clear since

$$\omega'\mathbf{V}\omega\Delta = \sigma_p^2\Delta = \frac{\Delta}{\delta} + \delta(\bar{r}_p - \frac{\alpha}{\delta})^2$$

and

$$\begin{aligned}
(\bar{r}_p - \bar{r}_q)\delta(\bar{r}_p - \alpha/\delta) &= ((\bar{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)})\delta(\bar{r}_p - \alpha/\delta) \\
&= \frac{\Delta}{\delta} + \delta(\bar{r}_p - \frac{\alpha}{\delta})^2
\end{aligned}$$

0.1.2 CAPM

In capital market equilibrium, the market portfolio is tangecy portfolio $\bar{r}_D = r_m$, then

$$\bar{\mathbf{r}} - \boldsymbol{r_f}\mathbf{e} = \boldsymbol{\beta}_{\mathbf{m}}(\bar{r}_m - r_f)$$

where

$$\beta_m = \begin{bmatrix} \frac{\mathrm{Cov}(r_1, \overline{r}_m)}{\sigma_m^2} \\ \frac{\mathrm{Cov}(r_2, \overline{r}_m)}{\sigma_m^2} \\ \cdots \\ \frac{\mathrm{Cov}(r_n, \overline{r}_m)}{\sigma_m^2} \end{bmatrix}$$

this equation is called **Sharpe-Lintner CAPM**. $\overline{r}_m - r_f$ is called **market risk premium** and $\frac{\overline{r}_m - r_f}{\sigma_m}$ is called **market sharpe ratio**. Translate it from vector form, we get the **Security Market Line**:

$$r_i - r_f = \beta_{i,m}(\overline{r}_m - r_f)$$

0.1.3 Realized return

Now consider both r_i and r_m is random variable, let ϵ be a random vector with zero expection and zero covariance with r_i and r_m, then

$$r_i - r_f = \beta_{i,m}(r_m - r_f) + \epsilon_i$$

This is a regression equation, if one include an intercept, then the model

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \epsilon_i$$

is called **market model**, such α is called **Jensen's alpha**.

0.1.4 Variance decomposition

Decomposition the variance as:

$$\mathrm{Var}(r_i) = \overbrace{\underbrace{\beta_i \sigma_m^2}_{\text{Systematic risk}} + \underbrace{\mathrm{Var}(\epsilon_i)}_{\text{Idiosyncratic risk}}}^{\text{total risk}}$$

The R^2 is just the proportion of systematic risk

$$R^2 = \frac{\beta_i^2 \sigma_m^2}{\beta_i^2 \sigma_m^2 + \sigma^2}$$

since Fraction of variance unexplained.

0.1.5 Testing CAPM

see `GRS.pdf`

0.2 Multi-factor model

0.2.1 APT

Recall in the CAPM

$$\mathbf{r_t^e} = \boldsymbol{\alpha} + \beta r_{m,t}^e + \nu_t$$

Suppose now there is multi-factor with k risk factor and

$$\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon_t}$$

Where $\mathbf{r_t}, \boldsymbol{\epsilon_t}$ is n vector while $\mathbf{f_t}$ is k vector and $\mathbf{B}$ is $n \times k$ matrix.

Taking expectation:

$$\mathbf{r_t} = \mathrm{E}[\mathbf{r}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon_t}$$

Note CAPM is just special case of APT when $k = 1$ and $\mathbf{f_t} = r_{m,t}$, which is the only factor affecting realized return. Thta is

$$\mathbf{r_t} = \boldsymbol{r_f}\mathbf{e} + \boldsymbol{\beta}\,\mathrm{E}[\boldsymbol{r_m} - \boldsymbol{r_f}] + \boldsymbol{\beta}(\boldsymbol{r_{m,t}} - \mathrm{E}[\boldsymbol{r_{m,t}}]) + \boldsymbol{\epsilon_t}$$

Assume there is non-arbitrage, that is, if one invest 0 and take no risk, then the expected return is 0. Formally, as $n \to \infty$,

$$\boldsymbol{\omega}'\,[\mathbf{e} \quad \mathbf{B}] = \mathbf{0} \implies \boldsymbol{\omega}'\,\mathrm{E}[\mathbf{r}] = \mathbf{0}$$

by Farkas lemma(Farkas 1902), that implies

$$[\mathbf{e} \quad \mathbf{B}] \begin{bmatrix} \lambda_0 \\ \boldsymbol{\lambda} \end{bmatrix} = \mathrm{E}[\mathbf{r}]$$

for some $\boldsymbol{\lambda} \geq \mathbf{0}$. Under CAPM, $\lambda_0 = r_f$ and $\mathrm{E}[r_m - r_f] = \lambda$ clearly. If one take no risk, that is $\mathbf{B} = \mathbf{0}$, s(he) get a λ_0 return, that implies $\lambda_0 = r_f$ immediately.

When factors are returns When $\mathbf{f_t}$ is $\mathbf{r_t}$, regressing it on

$$\mathbf{f_t} = \mathrm{E}[\mathbf{f}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon_t}$$

we have $\mathbf{B} = \mathbf{I}$ and thus

$$\lambda_0\mathbf{e} + \boldsymbol{\lambda} = \mathrm{E}[\mathbf{f_t}] \implies \mathrm{E}[\mathbf{f_t}] = \boldsymbol{r_f}\mathbf{e} + \boldsymbol{\lambda}$$

When factors are excess returns If $\mathbf{f_t}$ is excess return for some tradable assets on time t, e.g.

$$f_{k,t} = r_{a,t} - r_{b,t}$$

where $r_{a,t}$ and $r_{b,t}$ is some componets in $\mathbf{r_t}$, that is

$$f_{k,t} = [0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad \cdots \quad -1 \quad \cdots \quad 0]\,\mathbf{r_t}$$

where the ath componets is 1 while bth is -1. If each componets is excess return, we may represent $\mathbf{f_t} = \mathbf{C}\mathbf{r_t}$ where each row of $\mathbf{C}$ of such form(All zero but a 1 and a -1). Recall

$$\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon_t}$$

It follows that

$$\mathbf{f_t} = \mathbf{C}\mathbf{r_t} = \mathbf{C}\boldsymbol{\alpha} + \mathbf{C}\mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon}$$

thus $\mathbf{C}\mathbf{B} = \mathbf{I}$. Recall

$$r_f\mathbf{e} + \mathbf{B}\boldsymbol{\lambda} = \mathrm{E}[\mathbf{r_t}]$$

thus

$$\mathrm{E}[\mathbf{f_t}] = \lambda$$

as $\mathbf{C}\mathbf{e} = \mathbf{0}$ clearly.

0.2.2 ICAPM

Farkas, Julius. 1902. "Theorie Der Einfachen Ungleichungen." *Journal Für Die Reine Und Angewandte Mathematik* 1902 (124): 1–27.