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# MULTIPLE INDEX MODELS AND THE BARRA RISK MODEL

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December 19, 2020

## 1 Index model

### 1.1 Single index model

Single index model extends CAPM by allowing  $\alpha$  in each asset and allow us to think about active investment. That is

$$r_i = \alpha_i + \beta_i r_m^e + \epsilon_i$$

That model also assume the covarince of  $\epsilon_i$  is 0 additionally.

Now we nolonger hold the market portfolio, instead, we should select assets by their **infomation ratio**. Infomation ratio of stock is given by

$$IR_i = \frac{\alpha_i}{\sigma_{\epsilon_i}}$$

Suppose we manage portfolio  $A$  constructed by  $\omega' \mathbf{r}^e + \omega_m r_m^e$ . The idea is we may choose  $\omega$  only for  $\alpha$  and invest some market index for  $\beta$ . By index model:

$$\mathbf{r}^e = \boldsymbol{\alpha} + \boldsymbol{\beta} r_m^e + \boldsymbol{\epsilon}$$

The optimize problem is:

$$\begin{aligned} \min \quad & \frac{1}{2} \sigma^2 = \frac{1}{2} [(\omega' \boldsymbol{\beta})^2 \sigma_m^2 + \boldsymbol{\omega}' \mathbf{V} \boldsymbol{\omega} + \omega_m^2 \sigma_m^2] \\ \text{s.t.} \quad & \boldsymbol{\omega}' (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \omega_{n+1} r_m^e = \bar{r}_p^e \end{aligned}$$

Where  $\mathbf{V} = \text{diag}(\sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \dots, \sigma_{\epsilon_n}^2)$ . To simplify, let

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} \boldsymbol{\omega} \\ \omega_m + \boldsymbol{\omega}' \boldsymbol{\beta} \end{bmatrix}, \mathbf{r} = \begin{bmatrix} \boldsymbol{\alpha} \\ \bar{r}_m^e \end{bmatrix}$$

and

$$\tilde{\mathbf{V}} = \text{diag}(\sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \dots, \sigma_{\epsilon_n}^2, \sigma_m^2) = \begin{bmatrix} \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \sigma_m^2 \end{bmatrix}$$

rewrite above optimize problem:

$$\min_{\tilde{\omega}, \lambda} \frac{1}{2} \tilde{\omega}' \tilde{\mathbf{V}} \tilde{\omega} + \lambda (\bar{r}_p^e - \tilde{\omega}' \mathbf{r})$$

FOC with  $\tilde{\omega}$ :

$$\tilde{\mathbf{V}} \tilde{\omega} - \lambda \mathbf{r} = 0 \implies \tilde{\omega} = \lambda \tilde{\mathbf{V}}^{-1} \mathbf{r}$$

Thus

$$\begin{aligned} \omega &= \lambda \mathbf{V}^{-1} \alpha \\ \omega_m + \omega' \beta &= \lambda \frac{\bar{r}_m^e}{\sigma_m^2} \end{aligned}$$

To get tangency portfolio, impose  $\omega' \mathbf{e} + \omega_m = 1$ , then

$$(\mathbf{e}' - \beta') \omega + \lambda \frac{\bar{r}_m^e}{\sigma_m^2} = 1 \implies \lambda = \frac{1}{(\mathbf{e}' - \beta') \mathbf{V}^{-1} \alpha + \frac{\bar{r}_m^e}{\sigma_m^2}}$$

The sharpe ratio of portfolio  $A$  is given by

$$SR_A^2 = \frac{(\tilde{\omega}' \mathbf{r})^2}{\tilde{\omega}' \tilde{\mathbf{V}} \tilde{\omega}} = \frac{\lambda^2 (\mathbf{r}' \tilde{\mathbf{V}}^{-1} \mathbf{r})^2}{\lambda^2 (\mathbf{r}' \tilde{\mathbf{V}}^{-1} \mathbf{r})} = \mathbf{r}' \tilde{\mathbf{V}}^{-1} \mathbf{r}$$

which can be decompositioned as

$$SR_A^2 = \alpha' \mathbf{V}^{-1} \alpha + \frac{(\bar{r}_m^e)^2}{\sigma_m^2} = IR_A^2 + \frac{(\bar{r}_m^e)^2}{\sigma_m^2} = \sum_{i=1}^n IR_i^2 + \frac{(\bar{r}_m^e)^2}{\sigma_m^2}$$

However, we can not know  $\alpha$  even in statistic, that motivate some extension of single index model.

## 2 Risk model

Risk model focus on the covarinace matrix of excess return

### 2.1 BARRA model

Now we model  $\alpha$  as linear with firm-level characteristics and estimate them by Fama-MacBeth regression.

$$r_t^e = \mathbf{X}_{t-1} \gamma_t + \beta_{t-1} \lambda_t + \epsilon_t$$