FACTOR MODEL

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1 CAPM

1.1 Beta representation

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}}-r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}}-r_f\mathbf{e})}$. Write $\omega_D = m\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})$ where $m = \frac{1}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}}-r_f\mathbf{e})}$, then we have

$$\overline{r} - r_f\mathbf{e} = \frac{1}{m}\mathbf{V}\omega_D$$

Note $\operatorname{Cov}(\mathbf{r}, \omega'\mathbf{r}) = \mathbf{V}\omega$ and

$$\sigma_D^2 = \omega_D'\mathbf{V}\omega_D = m\omega_D'(\overline{\mathbf{r}} - r_f\mathbf{e}) = mr_D - mr_f$$

we have

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \frac{r_D - r_f}{\sigma_D^2}\operatorname{Cov}(\mathbf{r}, r_D)$$

Denote $\frac{\operatorname{Cov}(\mathbf{r},r_D)}{\sigma_D^2} = \beta_D$, we have

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \beta_D(r_D - r_f)$$

Similar results also holds for any portfolio $\overline{r}_p$ in the MVF:

$$\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} = \beta_p(\overline{r}_p - \overline{r}_q)$$

It's clear in the view of every portfolio $\overline{r}_p$ is also a tangency portfolio by selecting proper r_f. One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega_p'\mathbf{V}\omega_q = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})(\overline{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

If the covariance is 0, we have

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Then

$$\begin{aligned}
\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} &= \overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e} \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\delta^2(\overline{r}_p - \alpha/\delta))(\overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta(\overline{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2))\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e} \\
&= \frac{(\delta^2\overline{r}_p\overline{\mathbf{r}} - \alpha\delta\overline{\mathbf{r}}) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e}}{\delta^2(\overline{r}_p - \alpha/\delta)} \\
&= \frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\delta(\overline{r}_p - \alpha/\delta)}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\beta_p &= \frac{\mathbf{V}\omega_{\mathbf{p}}}{\omega_p'\mathbf{V}\omega_p} \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\lambda_p\overline{\mathbf{r}} + \gamma\mathbf{e}) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{\xi\mathbf{e} - \alpha\overline{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha\mathbf{e} + \delta\overline{\mathbf{r}}}{\delta\xi - \alpha^2}\overline{r}_p) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\Delta})
\end{aligned}$$

Then it's remain to show that

$$(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) = \omega'\mathbf{V}\omega\Delta$$

It's clear since

$$\omega'\mathbf{V}\omega\Delta = \sigma_p^2\Delta = \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2$$

and

$$\begin{aligned}
(\overline{r}_p - \overline{r}_q)\delta(\overline{r}_p - \alpha/\delta) &= ((\overline{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\delta(\overline{r}_p - \alpha/\delta) \\
&= \frac{\Delta}{\delta} + \delta(\overline{r}_p - \frac{\alpha}{\delta})^2
\end{aligned}$$

1.2 CAPM

In capital market equilibrium, the market portfolio is tangecy portfolio $r_D = r_m$, then

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \beta_m(r_m - r_f)$$

where

$$\beta_m = \begin{bmatrix} \frac{\text{Cov}(r_1,r_m)}{\sigma_m^2} \\ \frac{\text{Cov}(r_2,r_m)}{\sigma_m^2} \\ \cdots \\ \frac{\text{Cov}(r_n,r_m)}{\sigma_m^2} \end{bmatrix}$$

this equation is called **Sharpe-Lintner CAPM**. $r_m - r_f$ is called **market risk premium** and $\frac{r_m - r_f}{\sigma_m}$ is called **market sharpe ratio**. Translate it from vector form, we get the **Security Market Line**:

$$r_i - r_f = \beta_{i,m}(r_m - r_f)$$

1.2.1 Realized return

Now consider both r_i and r_m is random variable, let ϵ be a random vector with zero expection and zero covariance with r_i and r_m, then

$$r_i - r_f = \beta_{i,m}(r_m - r_f) + \epsilon_i$$

This is a regression equation, if one include an intercept, then the model

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \epsilon_i$$

is called **market model**, such α is called **Jensen's alpha**.

1.2.2 Variance decomposition

Decomposition the variance as:

$$\text{Var}(r_i) = \overbrace{\underbrace{\beta_i \sigma_m^2}_{\text{Systematic risk}} + \underbrace{\text{Var}(\epsilon_i)}_{\text{Idiosyncratic risk}}}^{\text{total risk}}$$

The R^2 is just the proportion of systematic risk

$$R^2 = \frac{\beta_i^2 \sigma_m^2}{\beta_i^2 \sigma_m^2 + \sigma^2}$$

since Fraction of variance unexplained.

1.2.3 Testing CAPM

Suppose we run time series regressions for each of the n risky assets

$$\mathbf{r_t^e} = \alpha + \beta r_{m,t}^e + \nu_t$$

where $\alpha, \mathbf{r_t^e}, \beta, \nu_t$ are $n \times 1$ vector and $r_{m,t}^e$ is scalar.

By the discussion above, $\alpha = \mathbf{0}$ when CAPM holds. Assume $\{\nu_t\}_{t=1}^T$ i.i.d with $N(0, \Sigma)$, we have $\mathbf{r_t^e} \mid r_{m,t}^e \sim N(\alpha + \beta r_{m,t}^e, \Sigma)$.

Hence the p.d.f is

$$f(\mathbf{r}_t^e) = (2\pi)^{-n/2} |\Sigma|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)' \Sigma^{-1} (\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e) \right\}$$

By the indepedence, from $t = 1$ to $t = T$, the joint p.d.f is $L = \prod_{t=1}^T f(\mathbf{r}_t^e) =$.

$$\log L = \sum_{t=1}^{T} \log f(\mathbf{r_t^e}) = -\frac{-nT}{2}\ln(2\pi) - \frac{T}{2}\ln|\Sigma| - \frac{1}{2}\sum_{t=1}^{T}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)'\Sigma^{-1}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)$$

MLE for (α, β, Σ) is found by maximize $\log L$, W.r.t. β, it's the same as minimize

$$\sum_{t=1}^{T}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)'\Sigma^{-1}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)$$

FOC with β and recall $\frac{\partial \mathbf{x'Ax}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}')\mathbf{x}$, $\frac{\partial \mathbf{f}}{\partial \mathbf{X}} = \mathbf{A}'\frac{\partial \mathbf{f}}{\partial \mathbf{AXB}}\mathbf{B}'$

$$\begin{aligned}
\frac{\partial \log L}{\partial \beta} &= \sum_{t=1}^{T}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)'\Sigma^{-1}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e) \\
&= \sum_{t=1}^{T}\frac{\partial(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)'\Sigma^{-1}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)}{\partial \beta} \\
&= \sum_{t=1}^{T}(\frac{\partial(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)'\Sigma^{-1}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)}{\partial \mathbf{r_t^e} - \alpha - \beta r_{m,t}^e}\frac{\partial \mathbf{r_t^e} - \alpha - \beta r_{m,t}^e}{\partial \beta}) \\
&= \sum_{t=1}^{T}(-2r_{m,t}^e\Sigma^{-1}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)) = 0
\end{aligned}$$

since Σ^{-1} is clearly invertible, $\sum_{t=1}^{T} r_{m,t}^e(\mathbf{r_t^e} - \hat{\alpha} - \hat{\beta} r_{m,t}^e) = \mathbf{0}$
Similarly, FOC w.r.t α, we get

$$\sum_{t=1}^{T}\mathbf{r_t^e} - \hat{\alpha} - \hat{\beta} r_{m,t}^e = \mathbf{0}$$

By the truth $\frac{\partial|\mathbf{X}|}{\mathbf{X}} = |\mathbf{X}|\mathbf{X}'^{-}$ and $\frac{\partial \mathbf{a'X^{-}b}}{\partial \mathbf{X}} = -\mathbf{X}'^{-}\mathbf{ab}'\mathbf{X}'^{-}$, we have

$$\frac{\partial \log L}{\partial \Sigma} = \sum_{t=1}^{T}(-\frac{\Sigma^{-}}{2} - + \frac{\Sigma^{-}}{2}(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)(\mathbf{r_t^e} - \alpha - \beta r_{m,t}^e)'\Sigma^{-})$$

thus have

$$\hat{\Sigma} = \sum_{t=1}^{T}(\mathbf{r_t^e} - \hat{\alpha} - \hat{\beta} r_{m,t}^e)(\mathbf{r_t^e} - \hat{\alpha} - \hat{\beta} r_{m,t}^e)'/T$$

Suppose

$$\mathbf{r} = \begin{bmatrix} \mathbf{r_1^e}' \\ \mathbf{r_2^e}' \\ \mathbf{r_3^e}' \\ \vdots \\ \mathbf{r_T^e}' \end{bmatrix}, \mathbf{r_m} = \begin{bmatrix} r_{m,1}^e \\ r_{m,2}^e \\ r_{m,3}^e \\ \vdots \\ r_{m,T}^e \end{bmatrix}$$

Now

$$\begin{aligned}
(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r_m}')\mathbf{r_m} = \mathbf{0} \\
(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r_m}')\mathbf{e} = \mathbf{0}
\end{aligned}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{e}'\mathbf{r_m} & \mathbf{r}'_\mathbf{m}\mathbf{r_m} \\ \mathbf{e}'\mathbf{e} & \mathbf{r}'_\mathbf{m}\mathbf{e} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \mathbf{r}'\mathbf{r_m} \\ \mathbf{r}'\mathbf{e} \end{bmatrix}$$

hence

$$\begin{cases} \alpha = \mathbf{r}'\mathbf{r_m}(\mathbf{r}'_\mathbf{m}\mathbf{e} - \mathbf{e}'\mathbf{e})/(\mathbf{r}'_\mathbf{m}\mathbf{e}\mathbf{e}'\mathbf{r_m} - \mathbf{e}'\mathbf{e}\mathbf{r}'_\mathbf{m}\mathbf{r_m}) \\ \beta = \mathbf{r}'\mathbf{e}(\mathbf{r}'_\mathbf{m}\mathbf{r_m} - \mathbf{e}'\mathbf{r_m}) \end{cases}$$