FIN413 homework 1

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1

1

$$\min \frac{1}{2}\omega'\mathbf{V}\omega \quad s.t. \quad \mathbf{e}'\omega - \mathbf{1}, \omega'\overline{\mathbf{r}} = \overline{r}_p$$

By Lagrangian and FOC:

$$L = \frac{1}{2}\omega'\mathbf{V}\omega + \lambda(\overline{r}_p - \omega'\overline{\mathbf{r}}) + \gamma(\mathbf{1} - \omega'\mathbf{e})$$

$$\frac{\partial L}{\partial \omega} = \mathbf{V}\omega - \lambda\overline{\mathbf{r}} - \gamma\mathbf{e} = \mathbf{0} \implies \omega^* = \lambda\mathbf{V}^-\overline{\mathbf{r}} + \gamma\mathbf{V}^-\mathbf{e}$$

2

Substitute ω^* into $\overline{r}_p$ and $\mathbf{e}'\omega$:

$$\overline{r}_p = \overline{\mathbf{r}}'\omega^* = \lambda\overline{\mathbf{r}}'\mathbf{V}^-\overline{\mathbf{r}} + \gamma\overline{\mathbf{r}}'\mathbf{V}^-\mathbf{e}$$
$$\mathbf{1} = \mathbf{e}'\omega^* = \lambda\mathbf{e}'\mathbf{V}^-\overline{\mathbf{r}} + \gamma\mathbf{e}'\mathbf{V}^-\mathbf{e}$$

denoted $\delta = \mathbf{e}'\mathbf{V}^-\mathbf{e}, \alpha = \overline{\mathbf{r}}'\mathbf{V}^-\mathbf{e}, \xi = \overline{\mathbf{r}}'\mathbf{V}^-\overline{\mathbf{r}}$ where $\delta, \xi > 0$ since $\mathbf{V}$ is positive define. Thus we have a linear equations:

$$\begin{bmatrix} \xi & \alpha \\ \alpha & \delta \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} \overline{r}_p \\ 1 \end{bmatrix}$$

Note $\Delta = \delta\xi - \alpha^2 > 0$ since $(\alpha\bar{\mathbf{r}} - \xi\mathbf{e})'\mathbf{V}^-(\alpha\bar{\mathbf{r}} - \xi\mathbf{e}) = \xi(\delta\xi - \alpha^2) > 0$ and thus such equations is consistent.

solve and get ($\mathbf{Ax} = \mathbf{b} \implies \mathbf{x} = \mathbf{A}^-\mathbf{b}$)

$$\lambda = \frac{\delta\bar{r}_p - \alpha}{\delta\xi - \alpha^2}, \gamma = \frac{\xi - \alpha\bar{r}_p}{\delta\xi - \alpha^2}$$

hence

$$\begin{aligned}\omega^* &= \lambda\mathbf{V}^-\bar{\mathbf{r}} + \gamma\mathbf{V}^-\mathbf{e} \\ &= \frac{\delta\bar{r}_p - \alpha}{\delta\xi - \alpha^2}\mathbf{V}^-\bar{\mathbf{r}} + \frac{\xi - \alpha\bar{r}_p}{\delta\xi - \alpha^2}\mathbf{V}^-\mathbf{e} \\ &= a + b\bar{r}_p\end{aligned}$$

where $a = \frac{\xi\mathbf{V}^-\mathbf{e} - \alpha\mathbf{V}^-\bar{\mathbf{r}}}{\delta\xi - \alpha^2}$ and $b = \frac{-\alpha\mathbf{V}^-\mathbf{e} + \delta\mathbf{V}^-\bar{\mathbf{r}}}{\delta\xi - \alpha^2}$

3

The minimum variance is given by

$$\sigma_p^2 = \omega'^*\mathbf{V}\omega^* = \omega'^*(\lambda\bar{\mathbf{r}} + \gamma\mathbf{e}) = \lambda\bar{r}_p + \gamma = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$$

Thus the **global minimum variance**(GMV) portfolio is $\frac{1}{\delta}$ when $\bar{r}_p = \frac{\alpha}{\delta}$.

Meanwhile

$$\begin{aligned}\lambda &= 0 \\ \gamma &= \frac{1}{\delta} \\ \omega_{mv} &= \frac{\mathbf{V}^-\mathbf{e}}{\delta} = \frac{\mathbf{V}^-\mathbf{e}}{\mathbf{e}'\mathbf{V}^-\mathbf{e}}\end{aligned}$$

The minimum variance is given by

$$\sigma_p^2 = \omega'^*\mathbf{V}\omega^* = \omega'^*(\lambda\bar{\mathbf{r}} + \gamma\mathbf{e}) = \lambda\bar{r}_p + \gamma$$

by some algebra

$$\lambda\bar{r}_p + \gamma = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$$

4

The relationship between σ_p^2 and $\overline{r}_p$ is a parabola in $(\overline{r}_p, \sigma_p^2)$

5

Rewrite $\sigma_p^2 = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$ as

$$\frac{\sigma_p^2}{1/\delta} - \frac{(\overline{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 1$$

$$\begin{aligned} d\frac{\sigma_p^2}{1/\delta} - d\frac{(\overline{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = d1 &\implies d\sigma_p \frac{2\sigma_p}{1/\delta} - d\overline{r}_p \frac{2(\overline{r}_p - \frac{\alpha}{\delta})}{(\delta\xi - \alpha^2)/\delta^2} = 0 \\ &\implies \frac{d\overline{r}_p}{d\sigma_p} = \frac{\Delta\sigma_p}{\delta(\overline{r}_p - \frac{\alpha}{\delta})} \end{aligned}$$

Note when $\sigma_p \to \infty$:

$$\frac{\sigma_p^2}{1/\delta} - \frac{(\overline{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 0$$

hence

$$\lim_{\sigma_p \to \infty} \frac{\overline{r}_p - \frac{\alpha}{\delta}}{\sigma_p} = \sqrt{\Delta/\delta}$$

$$\lim_{\sigma_p \to \infty} \frac{\Delta\sigma_p}{\delta(\overline{r}_p - \frac{\alpha}{\delta})} = \frac{\Delta}{\delta}\sqrt{\frac{\delta}{\Delta}} = \sqrt{\frac{\Delta}{\delta}}$$

6

The **global minimum variance**(GMV) portfolio is $\frac{1}{\delta}$ when $\overline{r}_p = \frac{\alpha}{\delta}$ since $\sigma_p^2 = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$ meanwhile

$$\begin{aligned} \lambda &= \frac{\delta\overline{r}_p - \alpha}{\delta\xi - \alpha^2} = 0 \\ \gamma &= \frac{\xi - \alpha\overline{r}_p}{\delta\xi - \alpha^2} = \frac{\xi - \alpha\frac{\alpha}{\delta}}{\delta\xi - \alpha^2} = \frac{1}{\delta} \\ \omega_{mv} &= 0 + \frac{\mathbf{V}^{-}\mathbf{e}}{\delta} = \frac{\mathbf{V}^{-}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-}\mathbf{e}} \end{aligned}$$

2

Suppose covariance between portfolios p and q

$$\mathrm{Cov}(\mathbf{r_p}, \mathbf{r_q}) = \omega_p' \mathbf{V} \omega_q$$

Suppose q on the frontier, then

$$\omega_p' \mathbf{V} \omega_q = \omega_p'(\lambda \bar{\mathbf{r}} + \gamma \mathbf{e})$$

Moreover, if p is GMV portfolio, then

$$\omega_p'(\lambda \bar{\mathbf{r}} + \gamma \mathbf{e}) = \omega_p'(\frac{1}{\delta}\mathbf{e}) = \frac{1}{\delta}$$

since $\lambda = 0$ and $\gamma = \frac{1}{\delta}$.

3

Recall

$$\begin{bmatrix} \xi & \alpha \\ \alpha & \delta \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} \bar{r}_p \\ 1 \end{bmatrix}$$

hence we have $\lambda\alpha + \gamma\delta = 1$ clearly.

4

A portofolio ω is on the efficient froniter iff(if and only if) it's return $\bar{r}_p \geq r_{mv}$ and recall $\omega = a + b\bar{r}_p$, thus there is a bijection between ω_i and r_i. Suppose the return of ω_i is r_i,

$$\omega = \begin{bmatrix} \omega_1 & \cdots & \omega_n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Firstly, verify the convex combination is a legal portfolio:

$$\mathbf{e}'(\omega \mathbf{c}) = (\mathbf{e}'\omega)\mathbf{c} = \mathbf{e}'\mathbf{c} = 1$$

since $\forall \omega_i \mathbf{e}'\omega_{\mathbf{i}} = 1, \mathbf{e}'\mathbf{c} = 1$. Then it's sufficient to show that $\mathbf{c}'\mathbf{r} \geq r_{mv}$. It's clearly since

$$\mathbf{c}'\mathbf{r} = \sum c_i \omega_i \geq \sum c_i \min(r_i) = \min(r_i) \sum c_i = \min(r_i) \geq r_{mv}$$

5

1

Substitute $\omega_0 = 1 - \mathbf{e}'\omega$, then

$$\omega'\overline{\mathbf{r}} + (1 - \mathbf{e}'\omega)r_f = \overline{r}_p \implies \omega'(\overline{\mathbf{r}} - r_f\mathbf{e}) + r_f = \overline{r}_p$$

The problem is

$$\min_{\omega,\omega_0} \frac{1}{2}\omega'\mathbf{V}\omega \quad s.t. \quad \omega'(\overline{\mathbf{r}} - r_f\mathbf{e}) + r_f = \overline{r}_p$$

by the Lagrangian

$$L = \frac{1}{2}\omega'\mathbf{V}\omega + \lambda(\overline{r}_p - \omega'(\overline{\mathbf{r}} - r_f\mathbf{e}) - r_f)$$

FOC with ω:

$$\frac{\partial L}{\partial \omega} = \mathbf{V}\omega - \lambda(\overline{\mathbf{r}} - r_f\mathbf{e}) = 0 \implies \omega^* = \lambda\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})$$

2

$$\overline{r}_p - \omega^{*\prime}(\overline{\mathbf{r}} - r_f\mathbf{e}) - r_f = 0 \implies \lambda = \frac{\overline{r}_p - r_f}{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})} = \frac{\overline{r}_p - r_f}{\xi - 2\alpha r_f + \delta r_f^2}$$

Substitute into ω^*:

$$\omega = \frac{\overline{r}_p - r_f}{\xi - 2\alpha r_f + \delta r_f^2}\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})$$

The tangency portofolio is

$$\omega_D = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})} = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}{\alpha - r_f\delta}$$

3

$$\sigma_p^2 = \omega'\mathbf{V}\omega = \lambda^2(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}\mathbf{V}\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e}) = \frac{(\overline{r}_p - r_f)^2}{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}$$

In geometry view, the frontier degenerate into two crossing line:

$$\overline{r}_p = r_f \pm \sqrt{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})} = r_f \pm \sqrt{\xi - 2\alpha r_f + \delta r_f^2}\sigma_p$$

4

$$\begin{aligned}
\mathrm{Cov}(\overline{r}_p, \overline{r}_q) &= \omega_p' \mathbf{V} \omega_q \\
&= (\lambda_p \mathbf{V}^- (\overline{\mathbf{r}} - r_f \mathbf{e}))' \mathbf{V} (\lambda_q \mathbf{V}^- (\overline{\mathbf{r}} - r_f \mathbf{e})) \\
&= \lambda_p \lambda_q (\overline{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^- (\overline{\mathbf{r}} - r_f \mathbf{e}) \\
&= \frac{(\overline{r}_p - r_f)(\overline{r}_q - r_f)}{(\overline{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^- (\overline{\mathbf{r}} - r_f \mathbf{e})} \\
&= \frac{(\overline{r}_p - r_f)(\overline{r}_q - r_f)}{\xi - 2\alpha r_f + \delta r_f^2}
\end{aligned}$$

5

The shrpe ratio is defined by

$$S_p = \frac{\overline{r}_p - r_f}{\sigma_p} = \frac{\omega' (\overline{\mathbf{r}} - r_f \mathbf{e})}{\sqrt{\omega' \mathbf{V} \omega}}$$

Let $\mathbf{r} = \overline{\mathbf{r}} - r_f \mathbf{e}$ and

$$\phi : w \mapsto \begin{bmatrix} \omega' \mathbf{r} \\ \omega' \mathbf{V} \omega \end{bmatrix}, \quad h(x, y) := \frac{x}{y^{1/2}}$$

Then $S_p = h \circ \phi(w)$, and thus

$$\begin{aligned}
\frac{\partial S_p}{\partial \omega} &= \frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial \omega} \\
&= \begin{bmatrix} \frac{1}{(\omega' \mathbf{V} \omega)^{1/2}} & -\frac{\omega' \mathbf{r}}{2(\omega' \mathbf{V} \omega)^{3/2}} \end{bmatrix} \begin{bmatrix} \mathbf{r}' \\ 2\omega' \mathbf{V} \end{bmatrix} \\
&= \frac{\omega' \mathbf{V} \omega \mathbf{r}' - \omega' \mathbf{r} \omega' \mathbf{V}}{(\omega' \mathbf{V} \omega)^{3/2}}
\end{aligned}$$

Setting to zero,

$$\omega' \mathbf{V} \omega \mathbf{r} - \omega' \mathbf{r} \mathbf{V} \omega = \mathbf{0} \implies \omega = \frac{\omega' \mathbf{V} \omega}{\omega' \mathbf{r}} \mathbf{V}^- \mathbf{r}$$

Note the scale of ω is independent to S_p. If we assume $\mathbf{e}' \omega = 1$ additionally, then

$$\omega = \frac{\mathbf{V}^- \mathbf{r}}{\mathbf{e}' \mathbf{V}^- \mathbf{r}} = \frac{\mathbf{V}^- (\overline{\mathbf{r}} - r_f \mathbf{e})}{\mathbf{e}' \mathbf{V}^- (\overline{\mathbf{r}} - r_f \mathbf{e})} = \frac{\tilde{\omega}}{\mathbf{e}' \tilde{\omega}} = \omega_D$$

For now, we have

$$\omega = (1 - x) \omega_D + x \mathbf{e_i} \implies \frac{\partial \omega}{\partial x} = \mathbf{e_i} - \omega_D$$

where $\mathbf{e_i}$ is the vector all zero but the ith component is 1. Hence

$$\frac{\partial S_p}{\partial x} = \frac{\partial S_p}{\partial \omega}\frac{\partial \omega}{\partial x} = \frac{(\omega'\mathbf{V}\omega\mathbf{r'} - \omega'\mathbf{r}\omega'\mathbf{V})(\mathbf{e_i} - \omega_D)}{(\omega'\mathbf{V}\omega)^{3/2}}$$

When $x = 0$, $\omega = \omega_D$, note ω_D is the solution to $\frac{\partial S_p}{\partial \omega} = 0$, hence $\frac{\partial S_p}{\partial x} = 0$ also holds when $x = 0$, that is what we desired. Then consider

$$\begin{aligned}
\omega_D'\mathbf{V}\omega_D\mathbf{r} - \omega_D'\mathbf{r}\mathbf{V}\omega_D = \mathbf{0} &\implies \sigma_D^2\mathbf{r} - (r_D - r_f)\mathbf{V}\omega_{\mathbf{D}} = 0 \\
&\implies \sigma_D^2\mathbf{r} - (r_D - r_f)\mathrm{Cov}(\tilde{\mathbf{r}}, \tilde{r_D}) = 0 \\
&\implies \sigma_D^2(\bar{\mathbf{r}} - r_f\mathbf{e}) - (r_D - r_f)\mathrm{Cov}(\tilde{\mathbf{r}}, \tilde{r_D}) = 0 \\
&\implies \bar{\mathbf{r}} - r_f\mathbf{e} = \frac{\mathrm{Cov}(\tilde{\mathbf{r}}, \tilde{r_D})}{\sigma_D^2}(r_D - r_f)
\end{aligned}$$

Then note

$$\bar{\mathbf{r}} = \begin{bmatrix} E[\tilde{r_1}] \\ E[\tilde{r_2}] \\ \vdots \\ E[\tilde{r_n}] \end{bmatrix}, \mathrm{Cov}(\tilde{\mathbf{r}}, \tilde{r_D}) = \begin{bmatrix} \mathrm{Cov}(\tilde{r_1}, \tilde{r_D}) \\ \mathrm{Cov}(\tilde{r_2}, \tilde{r_D}) \\ \vdots \\ \mathrm{Cov}(\tilde{r_n}, \tilde{r_D}) \end{bmatrix}$$

thus we can split it into

$$E[\tilde{r_i}] - r_f = \frac{\mathrm{Cov}(\tilde{r_i}, \tilde{r_D})}{\sigma_D^2}E[\tilde{r_D} - r_f]$$

This is not CAPM since the tangency portfolio may not be the market portfolio.

6

1

It's increasing since

$$\frac{\partial U}{\partial W} = b\exp(-bW) > 0$$

and it's concave since

$$\frac{\partial^2 U}{\partial W^2} = -b^2\exp(-bW) < 0$$

2

$$E[U(W)] = E[U(r_p)] = E[-e^{-b(r_f+\omega'\mathbf{r})}]$$

where $\mathbf{r} = \tilde{\mathbf{r}} - r_f\mathbf{e} \sim N(\bar{\mathbf{r}} - r_f\mathbf{e}, \mathbf{V})$. It's sufficent to maximize

$$E[e^{(-b\omega)'\mathbf{r}}] = \exp\{(-b\omega)'E(\mathbf{r}) + b^2\omega'\mathbf{V}\omega/2\}$$

then

$$\frac{\partial(-b\omega)'E(\mathbf{r}) + b^2\omega'\mathbf{V}\omega/2}{\partial\omega} = -bE(\mathbf{r}) + b^2\mathbf{V}\omega = 0$$

hence

$$\omega = \frac{\mathbf{V}^{-}E(\mathbf{r})}{b} = \frac{\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f\mathbf{e})}{b} = \frac{\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f\mathbf{e})\mathbf{e}'\mathbf{V}(\bar{\mathbf{r}} - r_f\mathbf{e})}{b\mathbf{e}'\mathbf{V}(\bar{\mathbf{r}} - r_f\mathbf{e})} = \frac{\mathbf{e}'\mathbf{V}(\bar{\mathbf{r}} - r_f\mathbf{e})}{b}\omega_D$$

3

Larger b reduce the wealth invested in the tangency portfolio, that is why it measure the extent of risk aversion.

7

1

$$r_A - r_f = \beta(r_B - r_f) + \epsilon$$

where $\beta = \frac{\mathrm{Cov}(r_A - r_f, r_B - r_f)}{\sigma_B^2}$, since the intercept is 0, we have

$$\bar{r}_A - r_f = \beta(\bar{r}_B - r_f) \implies \beta = \frac{\bar{r}_A - r_f}{\bar{r}_B - r_f}$$

Thus

$$\begin{aligned}
\frac{\bar{r}_B - r_f}{\sigma_B} > \frac{\bar{r}_A - r_f}{\sigma_A} &\iff \frac{\bar{r}_A - r_f}{\bar{r}_B - r_f} < \frac{\sigma_A}{\sigma_B} \\
&\iff \beta < \frac{\sigma_A}{\sigma_B} \\
&\iff \frac{\mathrm{Cov}(r_A - r_f, r_B - r_f)}{\sigma_B^2} < \frac{\sigma_A}{\sigma_B} \\
&\iff \frac{\mathrm{Cov}(r_A, r_B)}{\sigma_B^2} < \frac{\sigma_A}{\sigma_B} \\
&\iff \mathrm{Cov}(r_A, r_B) < \sigma_A\sigma_B
\end{aligned}$$

which is clearly holds.

2

If $\alpha \neq 0$:

$$r_A - r_f = \alpha + \beta(r_B - r_f) + \epsilon$$

Denote $r = r_B - r_f$, $\mathrm{Var}(r) = \sigma_B^2, \mathrm{Var}(\epsilon) = \sigma^2$, then we have

$$\mathrm{Var}(r_A) = \mathrm{Var}(r_A - r_f) = \beta^2\sigma_B^2 + \sigma^2$$

Recall the maximum sharpe ratio is

$$S_p = \sqrt{(\bar{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}(\bar{\mathbf{r}} - r_f\mathbf{e})}$$

where

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \begin{bmatrix} r_A - r_f \\ r_B - r_f \end{bmatrix} = \begin{bmatrix} \beta r + \alpha \\ r \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \beta^2\sigma_B^2 + \sigma^2 & \beta\sigma_B^2 \\ \beta\sigma_B^2 & \sigma_B^2 \end{bmatrix}$$

Compute the maximun sharpe ratio directly:

$$\begin{aligned}
\begin{bmatrix} \beta r + \alpha \\ r \end{bmatrix}' \begin{bmatrix} \beta^2\sigma_B^2 + \sigma^2 & \beta\sigma_B^2 \\ \beta\sigma_B^2 & \sigma_B^2 \end{bmatrix}^{-} \begin{bmatrix} \beta r + \alpha \\ r \end{bmatrix} &= \begin{bmatrix} \beta r + \alpha \\ r \end{bmatrix}' \begin{bmatrix} \frac{1}{\sigma^2} & -\frac{\beta}{\sigma^2} \\ -\frac{\beta}{\sigma^2} & \frac{\beta^2\sigma_B^2+\sigma^2}{\sigma^2\sigma_B^2} \end{bmatrix} \begin{bmatrix} \beta r + \alpha \\ r \end{bmatrix} \\
&= r\left(\frac{r\left(\beta^2\sigma_B^2 + \sigma^2\right)}{\sigma^2\sigma_B^2} - \frac{\beta(\alpha + \beta r)}{\sigma^2}\right) + (\alpha + \beta r)\left(\frac{\alpha + \beta r}{\sigma^2} - \frac{\beta r}{\sigma^2}\right) \\
&= \frac{\alpha^2}{\sigma^2} + \frac{r^2}{\sigma_B^2}
\end{aligned}$$

$$\max_{\omega} S_p = \sqrt{\frac{\alpha^2}{\sigma^2} + \frac{(r_B - rf)^2}{\sigma_B^2}}$$