MULTIPLE INDEX MODELS AND THE BARRA RISK MODEL

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1 Index model

1.1 Single index model

Single index model extends CAPM by allowing α in each asset and allow us to think about active investment. That is

$$r_i = \alpha_i + \beta_i r_m^e + \epsilon_i$$

That model also assume the covarinace of ϵ_i is 0 additionaly.

Now we nolonger hold the market portfolio, instead, we should select assets by their **infomation ratio**. Infomation ratio of is given by

$$IR(r_i) = \frac{\alpha_i}{\sigma_{\epsilon_i}}$$

Suppose we manage portfolio A constructed by $\boldsymbol{\omega'}\mathbf{r^e} + \omega_m r_m^e$. The idea is we may choose ω only for α and invest some market index for β. By index model:

$$\mathbf{r^e} = \boldsymbol{\alpha} + \boldsymbol{\beta} r_m^e + \boldsymbol{\epsilon}$$

The optimize problem is:

$$\begin{aligned}
\min \quad & \frac{1}{2}\sigma^2 = \frac{1}{2}[(\boldsymbol{\omega'\beta})^2\sigma_m^2 + \boldsymbol{\omega'}\mathbf{V}\boldsymbol{\omega} + \omega_m^2\sigma_m^2] \\
s.t. \quad & \boldsymbol{\omega'(\alpha+\beta)} + \omega_{n+1} r_m^e = \overline{r}_p^e
\end{aligned}$$

Where $\mathbf{V} = diag\left(\sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \cdots, \sigma_{\epsilon_n}^2\right)$. To simplify, let

$$\widetilde{\boldsymbol{\omega}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{\omega_m} + \boldsymbol{\omega'\beta} \end{bmatrix}, \mathbf{r} = \begin{bmatrix} \boldsymbol{\alpha} \\ \overline{r}_m^e \end{bmatrix}$$

and

$$\widetilde{\mathbf{V}} = diag\left(\sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \cdots, \sigma_{\epsilon_n}^2, \sigma_m^2\right) = \begin{bmatrix} \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \sigma_m^2 \end{bmatrix}$$

rewrite above optimize problem:

$$\min_{\widetilde{\boldsymbol{\omega}},\boldsymbol{\lambda}} \frac{1}{2}\widetilde{\boldsymbol{\omega}}'\widetilde{\mathbf{V}}\widetilde{\boldsymbol{\omega}} + \lambda(\overline{r}_p^e - \widetilde{\boldsymbol{\omega}}'\mathbf{r})$$

FOC with $\widetilde{\boldsymbol{\omega}}$:

$$\widetilde{\mathbf{V}}\widetilde{\boldsymbol{\omega}} - \lambda\mathbf{r} = 0 \implies \widetilde{\boldsymbol{\omega}} = \lambda\widetilde{\mathbf{V}}^{-1}\mathbf{r}$$

Thus

$$\boldsymbol{\omega} = \lambda\mathbf{V}^{-1}\boldsymbol{\alpha}$$
$$\omega_m + \boldsymbol{\omega}'\boldsymbol{\beta} = \lambda\frac{\overline{r}_m^e}{\sigma_m^2}$$

To get tangency portfolio, impose $\boldsymbol{\omega}'\mathbf{e} + \boldsymbol{\omega}_\mathbf{m} = 1$, then

$$(\mathbf{e}' - \boldsymbol{\beta}')\boldsymbol{\omega} + \lambda\frac{\overline{r}_m^e}{\sigma_m^2} = 1 \implies \lambda = \frac{1}{(\mathbf{e}' - \boldsymbol{\beta}')\mathbf{V}^{-1}\boldsymbol{\alpha} + \frac{\overline{r}_m^e}{\sigma_m^2}}$$

The sharpe ratio of portfolio A is given by

$$SR_A^2 = \frac{(\widetilde{\boldsymbol{\omega}}'\mathbf{r})^2}{\widetilde{\boldsymbol{\omega}}'\widetilde{\mathbf{V}}\widetilde{\boldsymbol{\omega}}} = \frac{\lambda^2\left(\mathbf{r}'\widetilde{\mathbf{V}}^{-1}\mathbf{r}\right)^2}{\lambda^2(\mathbf{r}'\widetilde{\mathbf{V}}^{-1}\mathbf{r})} = \mathbf{r}'\widetilde{\mathbf{V}}^{-1}\mathbf{r}$$

which can be descompositioned as

$$SR_A^2 = \boldsymbol{\alpha}'\mathbf{V}^{-1}\boldsymbol{\alpha} + \frac{(\overline{r}_m^e)^2}{\sigma_m^2} = IR_A^2 + \frac{(\overline{r}_m^e)^2}{\sigma_m^2} = \sum_{i=1}^{n} IR_i^e + \frac{(\overline{r}_m^e)^2}{\sigma_m^2}$$