MATRIX APPROCH TO GRS STATISTIC

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Suppose we run time series regressions for each of the n risky assets

$$\mathbf{r^e_t} = \alpha + \beta \mathbf{r^e_{m,t}} + \nu_\mathbf{t}$$

where $\alpha, \mathbf{r^e_t}, \beta, \nu_t$ are $n \times 1$ vector and $r^e_{m,t}$ is scalar.

By the discussion above, $\boldsymbol{\alpha} = \mathbf{0}$ when CAPM holds. Assume $\{\nu_t\}_{t=1}^T$ i.i.d with $\mathcal{N}(0, \Sigma)$, we have $\mathbf{r^e_t} \mid r^e_{m,t} \sim \mathcal{N}(\alpha + \beta r^e_{m,t}, \Sigma)$.

Suppose

$$\mathbf{r} = \begin{bmatrix} \mathbf{r^e_1}' \\ \mathbf{r^e_2}' \\ \mathbf{r^e_3}' \\ \vdots \\ \mathbf{r^e_T}' \end{bmatrix}, \mathbf{r_m} = \begin{bmatrix} r^e_{m,1} \\ r^e_{m,2} \\ r^e_{m,3} \\ \vdots \\ r^e_{m,T} \end{bmatrix}$$

Now $\mathbf{r'} \sim \mathcal{MN}_{n\times T}(\boldsymbol{\alpha}\mathbf{e'} + \boldsymbol{\beta}\mathbf{r'_m}, \boldsymbol{\Sigma}, \mathbf{I})$, the p.d.f is (Wikipedia contributors 2019)

$$p(\mathbf{r'}|\boldsymbol{\beta}\mathbf{r'_m}, \boldsymbol{\Sigma}, \mathbf{I}) = \frac{\exp(-\frac{1}{2}\operatorname{Tr}[(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})'\boldsymbol{\Sigma}^{-}(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})])}{(2\pi)^{nT/2}T^{n/2}|\boldsymbol{\Sigma}|^{T/2}}$$

thus the log likelihood function is

$$\log L = -\frac{1}{2}\operatorname{Tr}[(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})'\boldsymbol{\Sigma}^{-}(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})] - \frac{nT}{2}\log 2\pi - \frac{n}{2}\log T - \frac{T}{2}\log|\Sigma|$$

FOC w.r.t α, by chain rule(Petersen and Pedersen 2012)

$$\begin{aligned}
\partial \log L &= \operatorname{Tr}\left(\frac{\partial \log L}{\partial(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})}' \partial(\mathbf{X} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})\right) \\
&= \operatorname{Tr}\left(\frac{\partial \log L}{\partial(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})}' \partial\alpha(-\mathbf{e'})\right) \\
&= \operatorname{Tr}\left((-\mathbf{e'})\frac{\partial \log L}{\partial(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})}' \partial\alpha\right)
\end{aligned}$$

hence

$$\frac{\partial \log L}{\partial \alpha} = -\frac{\partial \log L}{\partial(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})}\mathbf{e}$$
$$= -(\Sigma^{-} + \Sigma'^{-})(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})\mathbf{e} = 0$$

Similarly, FOC w.r.t β and combine those results:

$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_{\mathbf{m}})\mathbf{r_m} = \mathbf{0}$$
$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_{\mathbf{m}})\mathbf{e} = \mathbf{0}$$

which leads to a linear equation

$$\begin{bmatrix} \mathbf{e'r_m} & \mathbf{r'_m r_m} \\ \mathbf{e'e} & \mathbf{r'_m e} \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{r'r_m} \\ \mathbf{r'e} \end{bmatrix}$$

Similarly to out deduction for mean-variance mdoel, let $a = \mathbf{r'_m r_m}$, $b = \mathbf{e'e} = T$ and $c = \mathbf{e'r_m}$($c^2 < ab$), hence

$$\begin{cases} \hat{\alpha} = c\mathbf{r'r_m} - a\mathbf{r'e}/(c^2 - ab) \\ \hat{\beta} = -b\mathbf{r'r_m} + c\mathbf{r'e}/(c^2 - ab) \end{cases}$$

By assumption $\mathbf{r'} \sim \mathcal{MN}_{n\times T}(\boldsymbol{\beta}\mathbf{r'_m}, \mathbf{\Sigma}, \mathbf{I})$, and $\hat{\alpha} = \mathbf{r'}(c\mathbf{r_m} - a\mathbf{e})/(c^2 - ab)$. By transformation of matrix normal distribution(Wikipedia contributors 2019)

$$\frac{(c\mathbf{r_m} - a\mathbf{e})'(c\mathbf{r_m} - a\mathbf{e})}{(c^2 - ab)^2} = \frac{c^2a - 2ac^2 + a^2b}{(c^2 - ab)^2} = \frac{a}{ab - c^2}$$

we have

$$\hat{\alpha} \sim \mathcal{MN}(\mathbf{0}, \mathbf{\Sigma}, \frac{a}{ab - c^2}) \sim \mathcal{N}(0, \frac{a}{ab - c^2}\Sigma)$$

which degenerated to mutivariate normal distribution since $\Sigma \otimes \frac{a}{ab-c^2} = \frac{a}{ab-c^2}\Sigma$. For the same reason, $\hat{\beta} \sim \mathcal{N}(\boldsymbol{\beta}, \frac{b}{ab-c^2}\Sigma)$.

Then we may construct statistic J_0 as

$$J_0 = \hat{\alpha}'(\frac{a}{ab - c^2}\Sigma)^{-}\hat{\alpha} \sim \chi^2_n$$

However, Σ is unkonwn so we shound use $\hat{\Sigma}$ instead of Σ and now J_0 is just asymptotically chi-square distributed:

$$J_0 = \frac{ab - c^2}{a}\hat{\alpha}'\hat{\Sigma}^{-}\hat{\alpha} \overset{A}{\sim} \chi^2_n$$

Return to the likelihood equation and FOC w.r.t Σ:

$$\frac{\partial \log L}{\partial \Sigma} = \frac{1}{2}(\Sigma^{-}(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})(\mathbf{r'} - \boldsymbol{\alpha}\mathbf{e'} - \boldsymbol{\beta}\mathbf{r'_m})'\Sigma^{-})' - \frac{T}{2}\Sigma'^{-} = 0$$

hence

$$\hat{\Sigma} = (\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_{\mathbf{m}})(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}'_{\mathbf{m}})'/T$$

Where

$$(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r}_\mathbf{m}') = \mathbf{r'}(\mathbf{I} - \frac{c\mathbf{r_m}\mathbf{e'} - a\mathbf{ee'} - b\mathbf{r_m}\mathbf{r'_m} + c\mathbf{er'_m}}{c^2 - ab})$$

Easy to verify $\frac{c\mathbf{r_m}\mathbf{e'} - a\mathbf{ee'} - b\mathbf{r_m}\mathbf{r'_m} + c\mathbf{er'_m}}{c^2 - ab}$ is symmetric and idempotent, thus

$$\begin{aligned}
\text{rank}(\mathbf{I} - \frac{c\mathbf{r_m}\mathbf{e'} - a\mathbf{ee'} - b\mathbf{r_m}\mathbf{r'_m} + c\mathbf{er'_m}}{c^2 - ab}) &= \text{Tr}(\mathbf{I} - \frac{c\mathbf{r_m}\mathbf{e'} - a\mathbf{ee'} - b\mathbf{r_m}\mathbf{r'_m} + c\mathbf{er'_m}}{c^2 - ab}) \\
&= T - \frac{2c^2 - 2ab}{c^2 - ab} \\
&= T - 2
\end{aligned}$$

By following lemma:

Lemma Suppose symmetric matrix $p \times p$ $\mathbf{A}$. It's idempotent of rank s iff there exist a $p \times s$ $\mathbf{P}$ $\ni \mathbf{PP'} = \mathbf{A}$ and $\mathbf{P'P} = \mathbf{I}$.
Proof Sufficiency is trivial. For necessity, since $\mathbf{A}$ is symmetric and idempotent matrix, it can be spectral decompostioned by $\mathbf{A} = \mathbf{Q\Lambda Q'}$. Where the diagonal of $\mathbf{\Lambda}$ is s 1 and $p - s$ 0. Thus

$$\mathbf{A} = \mathbf{Q\Lambda Q'} = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \begin{pmatrix} \mathbf{I}_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1' \\ \mathbf{P}_2' \end{pmatrix} = \mathbf{P}_1\mathbf{P}_1'$$

Note

$$\mathbf{I}_p = \mathbf{Q}'\mathbf{Q} = \begin{pmatrix} \mathbf{P}_1' \\ \mathbf{P}_2' \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1'\mathbf{P}_1 & \mathbf{P}_1'\mathbf{P}_2 \\ \mathbf{P}_2'\mathbf{P}_1 & \mathbf{P}_2'\mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1'\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2'\mathbf{P}_2 \end{pmatrix}$$

hence$\mathbf{P_1'P_1} = \mathbf{I_s}$.■

We may find $\mathbf{r'P} \sim \mathcal{MN}_{n\times(T-2)}(\mathbf{0}, \Sigma, \mathbf{I})$. Where $\text{E}[\mathbf{r'P}] = \text{E}[\mathbf{r'PP'P}] = \text{E}[\mathbf{r'AP}] = \mathbf{0}$ and thus (Wikipedia contributors 2020)

$$T\hat{\Sigma} = \mathbf{r'Ar} = \mathbf{r'PP'r} = \mathbf{r'P(r'P)'} \sim W_n(T-2, \Sigma)$$

Theorem Suppose $\mathbf{A} \sim W_n(m, \Sigma)$ and $\mathbf{x} \sim \mathcal{N}_\setminus(\mathbf{0}, \Sigma)$, where $m > n$, then

$$\frac{m-n+1}{n}\mathbf{x'A^-x} \sim F_{n,m-n+1}$$

Proof Note

$$\mathbf{x'A^-x} = \frac{\mathbf{x'\Sigma^-x}}{\mathbf{x'\Sigma^-x'/x'A^-x}}$$

where $\mathbf{x'\Sigma^-x} \sim \chi^2_n$ and $\frac{\mathbf{x'\Sigma^-x}}{\mathbf{x'A^-x}} \sim \chi^2_{m-n+1}$(Gupta and Nagar 2018). Then this claim follows. ■

Thus, taking $\mathbf{x} = \sqrt{\frac{ab-c^2}{a}}\hat{\alpha}$ and apply above theorem, we have

$$J_1 = \frac{T-n-1}{n}\frac{ab-c^2}{a}\frac{1}{T}\hat{\alpha}'\hat{\Sigma}^-\hat{\alpha} = \frac{T-n-1}{nT}J_0 \sim F_{n,T-n-1}$$

J_1 is the so called GRS statistic.

0.1 Interpretion of J_1

Consider a sub portfolio problem:

$$\min \frac{1}{2}\boldsymbol{\omega'V\omega} \quad s.t. \quad \mathbf{e}'\omega - \mathbf{1}, \omega'\overline{\mathbf{r}} = m$$

where $r = \begin{bmatrix} r^e_{m,t} \\ \mathbf{r^e_t} \end{bmatrix}$. Thus the MLE of $\overline{\mathbf{r}}$

$$\hat{\mathbf{r}} = \mathrm{E}[\widehat{\begin{bmatrix} r^e_{m,t} \\ \mathbf{r^e_t} \end{bmatrix}}] = \frac{1}{T} \begin{bmatrix} \mathbf{r'_m e} \\ \mathbf{r'e} \end{bmatrix}$$

and $\mathbf{V}$ follows by $\hat{\sigma}^2 = \mathbf{r'_m(I - P_1)r_m} = a - \frac{c^2}{b}$ and recall

$$\mathbf{r^e_t = \boldsymbol{\alpha}} + \beta r^e_{m,t} + \nu_t$$

then

$$\mathrm{Cov}(\mathbf{r^e_t}, r^e_{m,t}) = \mathrm{Cov}(\beta r^e_{m,t}, r^e_{m,t}) = \beta\hat{\sigma}^2$$

$$\mathrm{Cov}(\mathbf{r^e_t}) = \mathrm{Cov}(\beta r^e_{m,t}) + \mathrm{Cov}(\nu_t) = \beta\sigma^2\beta' + \Sigma$$

hence

$$\hat{\mathbf{V}} = \mathrm{Cov}(\overline{\mathbf{r}}) = \begin{bmatrix} \hat{\sigma}^2 & \hat{\sigma}^2\beta' \\ \hat{\sigma}^2\beta & \Sigma + \beta\hat{\sigma}^2\beta' \end{bmatrix}$$

Using the formula for a partitioned inverse:

$$\begin{aligned} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & \left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1} & -\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix} \end{aligned}$$

Thus

$$\hat{\mathbf{V}}^- = \begin{bmatrix} \frac{1}{\hat{\sigma}^2} + \beta'\Sigma^-\beta & -\beta'\Sigma^- \\ -\Sigma^-\beta & \Sigma^- \end{bmatrix}$$

Note the maximal sharpe ratio is given by

$$S_p = \sqrt{\overline{\mathbf{r}}'\mathbf{V}^-\overline{\mathbf{r}}}$$

Directly algebra and the invariance property of MLE yields

$$\hat{\mathbf{r}}'\hat{\mathbf{V}}^-\hat{\mathbf{r}} = \frac{\mathbf{r'_m e}^2}{T^2\hat{\sigma}^2} + \frac{(\mathbf{r'e} - \hat{\beta}\mathbf{r'_m e})'\hat{\boldsymbol{\Sigma}}^-(\mathbf{r'e} - \hat{\beta}\mathbf{r'_m e})}{T^2}$$

note $(\mathbf{r}' - \hat{\alpha}\mathbf{e}' - \hat{\beta}\mathbf{r'_m})\mathbf{e} = \mathbf{0}$:

$$\hat{\alpha}\hat{\boldsymbol{\Sigma}}^-\hat{\alpha} = \hat{\mathbf{r}}'\hat{\mathbf{V}}^-\overline{\mathbf{r}} - \frac{\mathbf{r'_m P_1 r_m}}{\mathbf{r'_m(I - P_1)r_m}}$$

where $\frac{\mathbf{r'_m P_1 r_m}}{\mathbf{r'_m(I - P_1)r_m}}$ is the MLE of market Sharpe ratio. Together with

$$\hat{\theta_m}^2 = \frac{\mathbf{r'_m P_1 r_m}}{\mathbf{r'_m(I - P_1)r_m}} = \frac{\frac{c^2}{b}}{a - \frac{c^2}{b}} = \frac{c^2}{ab - c^2}$$

we finally have:

$$J_1 = \frac{T-n-1}{n}\frac{\hat{\theta_p}^2 - \hat{\theta_m}^2}{1+\hat{\theta_m}^2}$$

0.2 APT

Recall in the CAPM

$$\mathbf{r_t^e} = \boldsymbol{\alpha} + \beta r_{m,t}^e + \nu_t$$

Suppose now there is multi-factor with k risk factor and

$$\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f_t} + \boldsymbol{\epsilon}_\mathbf{t}$$

Where $\mathbf{r_t}, \boldsymbol{\epsilon}_\mathbf{t}$ is n vector while $\mathbf{f_t}$ is k vector and $\mathbf{B}$ is $n \times k$ matrix. Note CAPM is just special case of APT when $k = 1$. The only factor affecting realized return is market excess return.

Taking expectation:

$$\mathbf{r_t} = \mathrm{E}[\mathbf{r}] + \mathbf{B}(\mathbf{f_t} - \mathrm{E}[\mathbf{f_t}]) + \boldsymbol{\epsilon}_\mathbf{t}$$

Assume there is non-arbitrage, that is, if one invest 0 and take no risk, then the expected return is 0. Formally, as $n \to \infty$,

$$\boldsymbol{\omega}' \begin{bmatrix} e & \boldsymbol{B} \end{bmatrix} = \mathbf{0} \implies \boldsymbol{\omega}' \mathrm{E}[\mathbf{r}] = \mathbf{0}$$

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