Mathematical Statistics

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Chap1

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68.

假设事件A, B, C分别为

$$A = a > 1 \tag{1}$$

$$B = b > 1 \tag{2}$$

$$C = a < 1 \tag{3}$$

容易验证这是一个反例

71.

$A \cap B$ 和 C:

A, B, C 相互独立,则:

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C) \tag{4}$$

于是

$$P((A \cap B) \cap C) = P(A)\,P(B)\,P(C) \tag{5}$$

注意到

$$P((A \cap B)) \geq P(A)\,P(B) \tag{6}$$

$$P((A \cap B) \cap C) \geq P((A \cap B))\,P(C) \tag{7}$$

即

$$P((A \cap B) \cap C) \geq P(A \cap B)\,P(C) \geq P(A)\,P(B)\,P(C) \tag{8}$$

由已知得,不等号均取等号,于是

$$P((A \cap B) \cap C) = P((A \cap B))\,P(C) \tag{9}$$

这表明 $A \cap B$ 和 C 是独立的.

$A \cup B$ 和 C:

要证明

$$P((A \cup B) \cap C) = P(A \cup B)\,P(C) \tag{10}$$

其中

$$P(A \cup B)\,P(C) = (P(A) + P(B) - P(A \cap B))\,P(C) \tag{11}$$

由于$A \cap B$ 和 C 独立:

$$P((A \cap B) \cap C) = P(A \cap B)\,P(C) = P(A)\,P(B)\,P(C) \tag{12}$$

所以

$$P(A \cap B) = P(A) P(B) \Rightarrow \text{A,B 独立} \tag{13}$$

同理可得

$$A, C \text{相互独立}, B, \text{C相互独立} \tag{14}$$

那么

$$P(A \cup B) P(C) = (P(A) + P(B) - P(A \cap B)) P(C) \tag{15}$$

$$= P(A) P(C) + P(B) P(C) - P(A \cap B) P(C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \tag{16}$$

另一方面

$$P((A \cup B) \cap C) = P((A \cap C) \cup (A \cap C)) \tag{17}$$

由容斥原理,展开即证

74.

记五个零件从左到右,从上到下的顺序损坏的事件为 A_1, A_2, A_3, A_4, A_5:

$$\begin{gathered}P((A_1 \cap A_2) \cup (A_3 \cap A_4) \cup A_5) = P(A_5) + P(A_1) \cdot P(A_2) + P(A_3) \cdot P(A_4) - P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) - \\ P(A_1) \cdot P(A_2) \cdot P(A_5) - P(A_3) \cdot P(A_4) \cdot P(A_5) + P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) \cdot P(A_5) = p + 2 p^2 - 2 p^3 - p^4 + p^5\end{gathered} \tag{18}$$

77.

扔 n 次从未命中靶心的概率为:

$$P(n) = (1 - 0.05)^n \tag{19}$$

求解

$$1 - (1 - 0.05)^n = 0.5 \tag{20}$$

得

$$n \simeq 13.5134 \simeq 14 \tag{21}$$

79.

a.

$$P(\mathrm{AA}) = P(\mathrm{aa}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \tag{22}$$

$$P(\mathrm{Aa}) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} \tag{23}$$

b.

没有疾病意味着他的基因型为 AA 或 Aa

$$P(\mathrm{Aa} \mid \mathrm{AA} \cup \mathrm{Aa}) = \frac{P(\mathrm{Aa})}{P(\mathrm{AA} \cup \mathrm{Aa})} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3} \tag{24}$$

c.

定义以下事件

$$\mathrm{Aa}_m = \{\text{父亲的基因型为Aa}\} \tag{25}$$

$$\mathrm{Aa}_f = \{\text{母亲的基因型为Aa}\} \tag{26}$$

$$\mathrm{Aa} = \{\text{第一代的基因型为Aa}\} \tag{27}$$

同理定义 $\mathrm{AA}_m, \mathrm{AA}_f, \mathrm{aa}, \mathrm{AA}$

由b知

$$P(\mathrm{Aa}_m) = \frac{2}{3},\ P(\mathrm{AA}_m) = \frac{1}{3} \tag{28}$$

由已知

$$P(\mathrm{Aa}_f) = p,\ P(\mathrm{AA}_f) = 1 - p \tag{29}$$

从而

$$P(\mathrm{aa}) = P(\mathrm{Aa}_m) \times P(\mathrm{Aa}_f) \times \frac{1}{4} = \frac{p}{6} \tag{30}$$

$$P(\mathrm{Aa}) = P(\mathrm{Aa}_m)\, P(\mathrm{Aa}_f) \times \frac{1}{2} + P(\mathrm{AA}_m)\, P(\mathrm{Aa}_f) \times \frac{1}{2} + P(\mathrm{Aa}_m)\, P(\mathrm{AA}_f) \times \frac{1}{2} = \frac{1}{3} + \frac{p}{6} \tag{31}$$

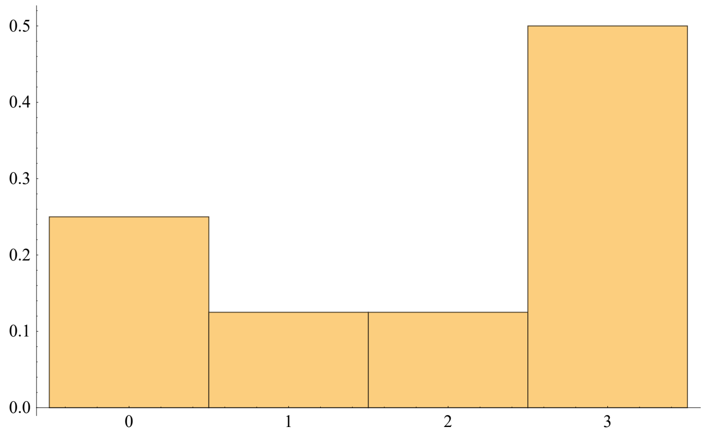
$$P(\mathrm{AA}) = 1 - P(\mathrm{aa}) - P(\mathrm{Aa}) = \frac{2}{3} - \frac{p}{3} \tag{32}$$

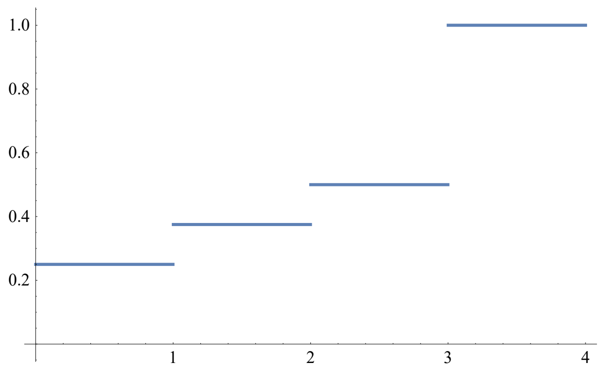
d.

$$P\left(\mathrm{Aa}_m \mid \mathrm{AA} \cup \mathrm{Aa}\right) = \frac{P\left(\mathrm{AA} \cup \mathrm{Aa} \mid \mathrm{Aa}_m\right) P(\mathrm{Aa}_m)}{P\left(\mathrm{AA} \cup \mathrm{Aa} \mid \mathrm{Aa}_m\right) P(\mathrm{Aa}_m) + P\left(\mathrm{AA} \cup \mathrm{Aa} \mid \mathrm{AA}_m\right) P(\mathrm{AA}_m)} \tag{33}$$

$$= \frac{\left(1 - p + \frac{3p}{4}\right) \times \frac{2}{3}}{\left(1 - p + \frac{3p}{4}\right) \times \frac{2}{3} + 1 \times \frac{1}{3}} = \frac{2}{p - 6} + 1 = \frac{p - 4}{p - 6} \tag{34}$$

1.





15.

A队在n局中获胜的局数X服从二项分布

$$X \approx \text{BinomialDistribution}[0.4,\ n] \tag{35}$$

在五局三胜制下

$$P(\text{Win}) = P(X \geq 3) = 1 - P(X = 1) - P(X = 2) = \frac{992}{3125} \simeq 0.31744 \tag{36}$$

在七局四胜制下

$$P(\text{Win}) = P(X \geq 4) = 1 - P(X = 1) - P(X = 2) - P(X = 3) = \frac{4528}{15\,625} \simeq 0.289792 \tag{37}$$

故应该选择五局三胜制

31.

a.

在十分钟内,泊松分布的参数为

$$\lambda = 2 \times \frac{1}{6} = \frac{1}{3} \tag{38}$$

Diane电话响起的次数X服从泊松分布

$$X \approx \text{PoissonDistribution}\left[\frac{1}{3}\right] \tag{39}$$

故电话响起的概率为

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{1}{e^{1/3}} = 0.283469 \tag{40}$$

b.

在k分钟内,Diane电话响起的次数X服从泊松分布

$$X \approx \text{PoissonDistribution}\left[\frac{k}{30}\right] \tag{41}$$

要使得电话响起的概率小于0.5,即

$$P(X \geq 1) = 1 - P(X = 0) < 0.5 \tag{42}$$

等价于求

$$P(X = 0) = e^{\frac{-k}{30}} > 0.5 \tag{43}$$

所以

$$k \leq 30\ln 2 \tag{44}$$

补充题

1.

$$P(X = x) = c\left(\frac{2}{3}\right)^x \tag{45}$$

由概率的定义:

$$\sum_{i=1}^{\infty} c\left(\frac{2}{3}\right)^x = 1 \tag{46}$$

注意到

$$c\sum_{x=1}^{\infty}\left(\frac{2}{3}\right)^x = 2\,c = 1 \tag{47}$$

所以

$$c = \frac{1}{2} \tag{48}$$

2.

$$P(X=k)=\frac{e^{-\lambda}\lambda^{k}}{k!} \tag{49}$$

$$P(X=k+1)=\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!} \tag{50}$$

$$\frac{P(X=k)}{P(X=k+1)}=\frac{\frac{e^{-\lambda}\lambda^{k}}{k!}}{\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}}=\frac{(k+1)!}{\lambda\, k!}=\frac{k+1}{\lambda} \tag{51}$$

当 $P(X=k)$ 到达最大值

$$\frac{k+1}{\lambda}>1 \text{ 且 } \frac{k}{\lambda}<1 \tag{52}$$

$$\lambda-1<k<\lambda \tag{53}$$

所以当

$$k=[\lambda] \text{ 时达到最大} \tag{54}$$