Mathematical Statistics and Data Analysis

Joint Distributions

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记三个玩家赢得比赛的局数分别为 X_1, X_2, X_3, 则 X_1, X_2, X_3 满足多项分布, 其公式为:

$$p\left(y_1, y_2, \ldots, y_k\right)=\frac{n!}{y_1!y_2!\ldots y_k!} p_1^{y_1} p_2^{y_2} \ldots p_k^{y_k}$$

所以

$$p\left(x_1, x_2, x_3\right)=\frac{10!}{x_1!x_2!x_3!}\left(\frac{1}{3}\right)^{x_1}\left(\frac{1}{3}\right)^{x_2}\left(\frac{1}{3}\right)^{x_3}$$

补充题

依题意

$$X \sim B(3, \frac{1}{2})$$
$$Y=|X-(3-x)|=|2x-3|$$

	X=0	**X=1**	**X=2**	**X=3**
Y=0	0	0	0	0
Y=1	0	3/8	3/8	0
Y=2	0	0	0	0
Y=3	1/8	0	0	1/8

综上,

$$p(x,y)=\begin{cases} 3/8 & x=1,2 \quad y=1 \\ 1/8 & x=0,3 \quad y=0 \\ 0 & \text{其他} \end{cases}$$

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以 M 表示针的重点, 以 x 表示 M 与最近的一条平行线的距离, 记它们的夹角为 ϕ, 显然 x 与 ϕ 可以确定针的相对距离, 且有

$$
0 \leq x \leq \frac{D}{2}
$$
$$
0 \leq \phi \leq \pi
$$

为了使得针与平行线相交, 有

$$
x \leq \frac{L \sin \phi}{2}
$$

因为 X 和 Φ 均为均匀分布, 且彼此独立, 故

$$
p = P(X \leq \frac{l \sin \Phi}{2}) = \frac{1}{\pi} \int_0^\pi \frac{\frac{L \sin \phi}{2}}{\frac{D}{2}} = \frac{2L}{\pi D}
$$

在试验次数足够多时, 频率趋近于概率, 故而在大量试验后, 可由针相交的频率 r 估算 π

$$
\pi \approx \frac{2L}{rD}
$$

6.

$$
f_{XY}(x, y) = f_X(x|y) f_Y(y)
$$

对于 y 的密度函数, 其等于在每个 y 值对应的水平线长度比上椭圆面积

$$
f_Y(y) = \frac{2\sqrt{\frac{(a^2b^2 - a^2y^2)}{b^2}}}{\pi ab} = \frac{2\sqrt{b^2 - y^2}}{\pi b^2}
$$

同理有

$$
f_X(x) = \frac{2\sqrt{a^2 - x^2}}{\pi a^2}
$$

7.

联合密度为

$$
\frac{\partial F(x, y)}{\partial x \partial y} = \alpha \beta e^{\alpha(-x) - \beta y}
$$

对于边际密度

$$
\begin{aligned}
f_X(x) &= \int_0^{+\infty} f(x,y)dy \\
&= \int_0^{+\infty} \alpha\beta e^{-\alpha x-\beta y}dy \\
&= \alpha\beta e^{-\alpha x}\int_0^{+\infty} e^{-\beta y}dy \\
&= \alpha\beta e^{-\alpha x}\left(-\frac{e^{-\beta y}}{-\beta}\right)\Bigg|_0^{+\infty} \\
&= \alpha\beta e^{-\alpha x}\frac{e^{-0}}{\beta} \\
&= \alpha e^{-\alpha x}
\end{aligned}
$$

由对称性

$$
f_Y(y) = \beta e^{-\beta y}
$$

8.

a.

1

$$
\begin{aligned}
P(X>Y) &= \iint_{x>y} \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \int_y^1 \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \left(\frac{2}{7}(x+y)^3\right)\Bigg|_y^1 dy \\
&= \int_0^1 -2y^3 + \frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}dy \\
&= \left(-\frac{y^4}{2} + \frac{2y^3}{7} + \frac{3y^2}{7} + \frac{2y}{7}\right)\Bigg|_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

2.

$$
\begin{aligned}
P(X+Y \leq 1) &= \int_0^1 \int_0^{1-y} \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \left(\frac{2}{7}(x+y)^3 \right) \Bigg|_0^{1-y} dy \\
&= \int_0^1 -\frac{2}{7}y^3 + \frac{2}{7}dy \\
&= \left(-\frac{1}{14}y^4 + \frac{2}{7}y \right) \Bigg|_0^1 \\
&= \frac{3}{14}
\end{aligned}
$$

3.

$$
\begin{aligned}
P(X \leq \frac{1}{2} &= \int_0^1 \int_0^{1/2} \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \left(\frac{2}{7}(x+y)^3 \right) \Bigg|_0^{1/2} dy \\
&= \int_0^1 \frac{3}{7}y^2 + \frac{3}{14}y + \frac{1}{28}dy \\
&= \left(\frac{1}{7}y^3 + \frac{3}{28}y^2 + \frac{1}{28}y \right) \Bigg|_0^1 \\
&= \frac{2}{7}
\end{aligned}
$$

b

$$
\begin{aligned}
f_X(x) &= \int_0^1 \frac{6}{7}(x+y)^2 dy \\
&= \left(\frac{2}{7}(x+y)^3 \right) \Bigg|_0^1 \\
&= \frac{2}{7}(x+1)^3 - \frac{2}{7}x^3 \\
&= \frac{6}{7}x^2 + \frac{6}{7}x + \frac{2}{7}
\end{aligned}
$$

同理可得

$$
f_Y(y) = \frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}
$$

c

$$
f(y|X=x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{6}{7}(x+y)^2}{\frac{6}{7}x^2 + \frac{6}{7}x + \frac{2}{7}} = \frac{3(x+y)^2}{3x^2+3x+1}
$$

同理有

$$f(x|Y=y)=\frac{f(x,y)}{f_Y(y)}=\frac{\frac{6}{7}(x+y)^2}{\frac{6}{7}y^2+\frac{6}{7}y+\frac{2}{7}}=\frac{3(x+y)^2}{3y^2+3y+1}$$