Mathematical Statistics and Data Analysis

Joint Distributions

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1.

a.

$$f_X(x)=\begin{cases}0.10+0.05+0.02+0.02=0.19 & x=1\\ 0.05+0.20+0.05+0.02=0.32 & x=2\\ 0.02+0.05+0.20+0.04=0.31 & x=3\\ 0.02+0.02+0.04+0.10=0.18 & x=4\end{cases}$$

$$f_Y(y)=\begin{cases}0.10+0.05+0.02+0.02=0.19 & y=1\\ 0.05+0.20+0.05+0.02=0.32 & y=2\\ 0.02+0.05+0.20+0.04=0.31 & y=3\\ 0.02+0.02+0.04+0.10=0.18 & y=4\end{cases}$$

b.

$$f_{X|Y=1}(x)=\begin{cases}\frac{f(1,1)}{P(Y=1)}=\frac{0.1}{0.19}=0.5263 & x=1\\ \frac{f(1,2)}{P(Y=1)}=\frac{0.05}{0.19}=0.2632 & x=2\\ \frac{f(1,3)}{P(Y=1)}=\frac{0.02}{0.19}=0.1053 & x=3\\ \frac{f(1,4)}{P(Y=1)}=\frac{0.02}{0.19}=0.1053 & x=4\end{cases}$$

$$f_{Y|X=1}(x)=\begin{cases}\frac{f(1,1)}{P(Y=1)}=\frac{0.1}{0.19}=0.5263 & y=1\\ \frac{f(1,2)}{P(Y=1)}=\frac{0.05}{0.19}=0.2632 & y=2\\ \frac{f(1,3)}{P(Y=1)}=\frac{0.02}{0.19}=0.1053 & y=3\\ \frac{f(1,4)}{P(Y=1)}=\frac{0.02}{0.19}=0.1053 & y=4\end{cases}$$

9.

a.

区域面积为

$$S=\int_{-1}^{1}\left(1-x^{2}\right)dx=\left(x-\frac{x^{3}}{3}\right)\Big|_{-1}^{1}=1-\frac{1}{3}-\left(-1-\frac{-1}{3}\right)=\frac{4}{3}$$

所以

$$f(x,y)=\frac{1}{4/3}=\frac{3}{4}$$

$$f_X(x)=\int_{0}^{1-x^{2}}\frac{3}{4}dy=\frac{3}{4}(1-x^{2})\quad -1\leq x\leq 1$$

$$f_Y(y)=\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\frac{3}{4}dx=\frac{3}{2}\sqrt{1-y}\quad 0\leq y\leq 1$$

b.

$$f_{X|Y}(x,y)=\frac{f(x,y)}{f_Y(y)}=\frac{1}{2\sqrt{1-y}}$$

$$f_{Y|X}(x,y)=\frac{f(x,y)}{f_X(x)}=\frac{1}{1-x^{2}}$$

10.

a.

$$f_Y(y)=\int_{0}^{\infty}xe^{-x(y+1)}dx=\frac{1}{1+y^{2}}$$

$$f_X(x)=\int_{0}^{\infty}xe^{-x(y+1)}dy=\frac{1}{e^{x}}$$

$f_X(x)f_Y(y)\neq f(x,y)$, 故 X 和 Y 不独立

b.

$$f_{X|Y}(x,y)=\frac{f(x,y)}{f_Y(y)}=xe^{-x(1+y)}\left(1+y^{2}\right)$$

$$f_{Y|X}(x,y)=\frac{f(x,y)}{f_X(x)}=xe^{x-x(1+y)}$$

15.

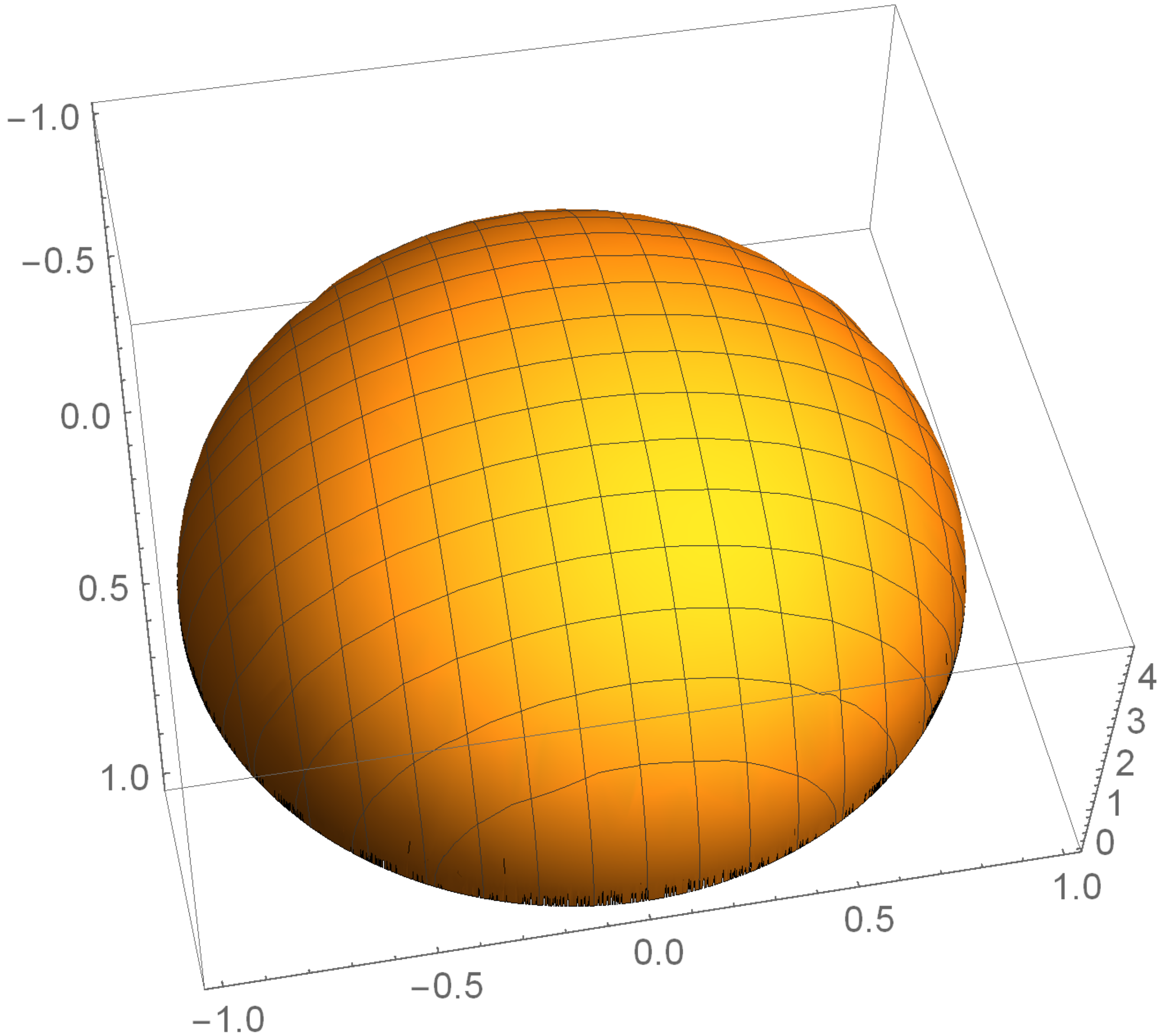
a.

$$\iint_{x^{2}+y^{2}\leqslant 1}c\sqrt{1-x^{2}-y^{2}}dxdy=c\iint_{x^{2}+y^{2}\leqslant 1}\sqrt{1-x^{2}-y^{2}}dxdy=\frac{2\pi}{3}c$$

所以

$$c = \frac{3}{2\pi}$$

b.



c.

$$P\left(X^2+Y^2 \leqslant \frac{1}{2}\right) = \iint_{x^2+y^2 \leq \left(\frac{\sqrt{2}}{2}\right)^2} \sqrt{1-\left(x^2+y^2\right)}dxdy = 1-\frac{\sqrt{2}}{4}$$

d.

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}dy = \frac{3}{4}\left(1-x^2\right)$$

同理可得

$$f_Y(y)=\frac{3}{4}\left(1-y^2\right)$$

验证

$$f_X(x)f_Y(y)\neq f(x,y)$$

故 X 和 Y 不独立

e.

$$f_{X|Y}(x,y)=\frac{f(x,y)}{f_Y(y)}=\frac{2}{\pi}\frac{\sqrt{1-x^2-y^2}}{1-y^2}$$

同理

$$f_{Y|X}(x,y)=\frac{2}{\pi}\frac{\sqrt{1-x^2-y^2}}{1-x^2}$$

19

a.

$$f_{T_1}(t_1)=\alpha\cdot e^{-\alpha t_1}$$

$$f_{T_2}(t_2)=\beta\cdot e^{-\beta t_2}$$

所以联合分布为

$$f_{T_1,T_2}\left(t_1,t_2\right)=f_{T_1}\left(t_1\right)\cdot f_{T_2}\left(t_2\right)=\alpha e^{-\alpha t_1}\beta e^{-\beta t_2}$$

$$\begin{aligned}
P\left(T_1>T_2\right)=&\iint_{t_1>t_2}f\left(t_1,t_2\right)dt_1dt_2\\
=&\iint_{t_1>t_2}f_{T_1}\left(t_1\right)\cdot f_{T_2}\left(t_2\right)\\
=&\int_0^{+\infty}\left(\int_{t_2}^{+\infty}\alpha\cdot e^{-\alpha t_1}\cdot\beta\cdot e^{-\beta t_2}dt_1\right)dt_2\\
=&\int_0^{+\infty}\alpha\cdot\beta\cdot e^{-\beta t_2}\cdot\left(-\frac{e^{-\alpha t_1}}{\alpha}\right)\Bigg|_{t_2}^{+\infty}dt_2=\int_0^{+\infty}\beta\cdot e^{-(\alpha+\beta)t_2}dt_2\\
=&\beta\cdot\left(-\frac{e^{-(\alpha+\beta)t_2}}{\alpha+\beta}\right)\Bigg|_0^{+\infty}\\
=&\frac{\beta}{\alpha+\beta}
\end{aligned}$$

b.

$$
\begin{aligned}
P\left(T_1>2T_2\right)=&\int_{t_1>2t_2} f\left(t_1, t_2\right) dt_1 dt_2\\
=&\iint_{t_1>2t_2} f_{T_1}\left(t_1\right)\cdot f_{T_2}\left(t_2\right)\\
=&\int_0^{+\infty}\left(\int_{2t_2}^{+\infty}\alpha\cdot e^{-\alpha t_1}\cdot\beta\cdot e^{-\beta t_2} dt_1\right) dt_2\\
=&\int_0^{+\infty}\alpha\cdot\beta\cdot e^{-\beta t_2}\cdot\left.\left(-\frac{e^{-\alpha t_1}}{\alpha}\right)\right|_{2t_2}^{+\infty} dt_2=\int_0^{+\infty}\beta\cdot e^{-(2\alpha+\beta) t_2} dt_2\\
=&\int_0^{+\infty}\alpha\cdot\beta\cdot e^{-\beta t_2}\cdot\left.\left(-\frac{e^{-\alpha t_1}}{\alpha}\right)\right|_{2t_2}^{+\infty} dt_2=\int_0^{+\infty}\beta\cdot e^{-(2\alpha+\beta) t_2} dt_2\\
=&\beta\cdot\left.\left(-\frac{e^{-(2\alpha+\beta) t_2}}{2\alpha+\beta}\right)\right|_0^{+\infty}\\
=&\frac{\beta}{2\alpha+\beta}
\end{aligned}
$$

补充题

点P, Q服从均匀分布, 记 h 为底边的高

$$
f_{XY}(x, y)=\begin{cases}\frac{2}{h\cdot BC},(x, y)\in\Delta ABC\\ 0, \text{其他}\end{cases}
$$

$$
f_Z(z)=\begin{cases}\frac{1}{BC}, z\in(0, BC)\\ 0, \quad\text{其他}\end{cases}
$$

所以

$$
f(x, y, z)=f_{XY}(x, y) f_Z(z)
$$

PQ于线段AB相交等价于

$$
P((x, y)\in\Delta ABQ, 0<z<BC)=\frac{1}{2}
$$

思考题

设三条线段长分别为 $x, y, d-x-y$

依题意

$$
\begin{gathered}
f(x)=\frac{1}{d}\\
f_{Y|X}(y)=\frac{1}{d-x}
\end{gathered}
$$

所以

$$f(x,y)=\frac{1}{d(d-x)}$$

$$P(\text{构成三角形})=\iint_{0<x<\frac{d}{2},\frac{d}{2}-x<y<\frac{d}{2}}\frac{1}{d(d-x)}dxdy=\int_{0}^{d/2}\int_{d/2-x}^{d/2}\frac{1}{d(d-x)}dydx=ln2-1/2$$