ht quotes in verbatim environments

Mathematical Statistics and Data Analysis

Joint Distributions

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43.

U_1 和U_2 的分布均为

$$f_{U_1}(u) = f_{U_2}(u) = 1, \quad u \in [0, 1]$$

由卷积公式

$$f_S(s) = \int_{-\infty}^{\infty} f_{U_1}(u) f_{U_2}(s-u) \mathrm{d}u$$

当$0 \leq s \leq 1$

$$f_S(s) = \int_0^s f_{U_1}(u) f_{U_2}(s-u) \mathrm{d}u = \int_0^s 1 \mathrm{d}u = s$$

当$0 \leq s \leq 1$

$$f_S(s) = \int_{s-1}^1 f_{U_1}(u) f_{U_2}(s-u) \mathrm{d}u = \int_{s-1}^1 1 \mathrm{d}u = 2 - s$$

综上

$$f_S(s) = \begin{cases} s & s \in [0, 1] \\ 2 - s & s \in (1, 2] \\ 0 & else \end{cases}$$

Figure 1:

44.

U_1 和U_2 的分布均为

$$p_U(u) = \begin{cases} \frac{1}{3} & u = 0 \\ \frac{1}{3} & u = 1 \\ \frac{1}{3} & u = 2 \end{cases}$$

由卷积公式

$$P\left(Z = z_l\right) = \sum_{i=1}^{\infty} P\left(X = x_i\right) P\left(Y = z_l - x_i\right)$$

当$0 \leq z \leq 2$

$$p_Z(z) = \sum_{i=0}^{z} P\left(X = i\right) P\left(Y = z - i\right) = \sum_{i=0}^{z} \frac{1}{9} = \frac{z+1}{9}$$

当$2 < z \leq 4$

$$p_Z(z) = \sum_{i=z-2}^{2} P\left(X = i\right) P\left(Y = z - i\right) = \sum_{i=z-2}^{2} \frac{1}{9} = \frac{5-z}{9}$$

综上

Z	0	1	2	3	4
P	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

51.

令

$$\begin{cases} xy = z \\ y_2 = y \end{cases}$$

Jacobi式为

$$\frac{\partial(x,y)}{\partial(z,y)} = \begin{vmatrix} \frac{1}{y} & \frac{-z}{y^2} \\ 0 & 1 \end{vmatrix} = \left|\frac{1}{y}\right|$$

于是

$$F_Z(z) = \int_{-\infty}^{z} \left(\int_{-\infty}^{+\infty} f(\frac{u}{y}, y) \left|\frac{1}{y}\right| \mathbf{d}y \right) \mathbf{d}u$$

所以

$$f_Z(z) = \int_{-\infty}^{+\infty} f(\frac{z}{y}, y)| \left|\frac{1}{y}\right| |\mathbf{d}y$$

52.

令X 和Y 均为均匀变量, 它们的联合分布为

$$f_{XY}(x,y) = 1$$

令

$$\begin{cases} \frac{x}{y} = z \\ y = y \end{cases}$$

Jacobi式为

$$\frac{\partial(x,y)}{\partial(z,y)} = \begin{vmatrix} y & u \\ 0 & 1 \end{vmatrix} = |y|$$

于是

$$F_Z(z) = \int_{-\infty}^{z} \left(\int_{-\infty}^{+\infty} f(uy, y) \, |y| \, \mathbf{d}y \right) \mathbf{d}u$$

所以

当$z > 1$

$$
\begin{aligned}
f_Z(z) = &\int_{-\infty}^{+\infty} f_{XY}(zy, y)\,|y|\,\mathbf{d}y \\
= &\int_0^{\frac{1}{z}} |y|\,\mathbf{d}y \\
= &\frac{1}{2z^2}
\end{aligned}
$$

当$0 \le z \le 1$

$$
\begin{aligned}
f_Z(z) = &\int_{-\infty}^{+\infty} f_{XY}(zy, y)\,|y|\,\mathbf{d}y \\
= &\int_0^{1} |y|\,\mathbf{d}y \\
= &\frac{1}{2}
\end{aligned}
$$

综上

$$
f_Z(z) = \begin{cases} \frac{1}{2z^2} & z \in (1, \infty) \\ \frac{1}{2} & z \in [0, 1] \\ 0 & else \end{cases}
$$

57.

Y_1 和Y_2 的联合分布为

$$
f_{Y_1Y_2}(y_1, y_2) = \frac{e^{-y_1^2 + y_2 y_1 - \frac{y_2^2}{2}}}{2\pi}
$$

用已知得

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

取矩阵的逆得到

$$\begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & -\frac{a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ -\frac{a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Jacobi式为

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & -\frac{a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ -\frac{a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{vmatrix} = \left| \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \right|$$

X_1 和X_2 的联合分布为

$$f_{X_1X_2}(x_1, x_2) = \frac{\exp\left(-\frac{\left(a_{21}^2+2a_{22}a_{21}+2a_{22}^2\right)x_1^2+(-2a_{11}a_{21}-2a_{12}a_{21}-2a_{11}a_{22}-4a_{12}a_{22})x_2x_1+\left(a_{11}^2+2a_{12}a_{11}+2a_{12}^2\right)x_2^2}{2(a_{12}a_{21}-a_{11}a_{22})^2}\right)}{2\pi}$$

若X_1 和X_2 均为标准正态随机变量,

$$f_{X_1X_2}(x_1, x_2) = \frac{e^{\frac{1}{2}\left(-x^2-y^2\right)}}{2\pi}$$

解得

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

70.

$$\begin{aligned} F(x, y) &= \mathrm{P}\left(X_n \leqslant y\right) - \mathrm{P}\left(X_1 > x, X_n \leqslant y\right) \\ &= [F(y)]^n - \mathrm{P}\left(x < X_1 \leqslant y, x < X_2 \leqslant y, \cdots, x < X_n \leqslant y\right) \\ &= [F(y)]^n - \prod_{i=1}^{k} \mathrm{P}\left(x < X_i \leqslant y\right) \\ &= [F(y)]^n - [F(y) - F(x)]^k \end{aligned}$$