Mathematical Statistics and Data Analysis

Joint Distributions

11810105 谢泽健

43.

U_1 和 U_2 的分布均为

$$f_{U_1}(u)=f_{U_2}(u)=1,\quad u\in[0,1]$$

由卷积公式

$$f_S(s)=\int_{-\infty}^{\infty}f_{U_1}(u)f_{U_2}(s-u)\mathrm{d}u$$

当 $0\le s\le 1$

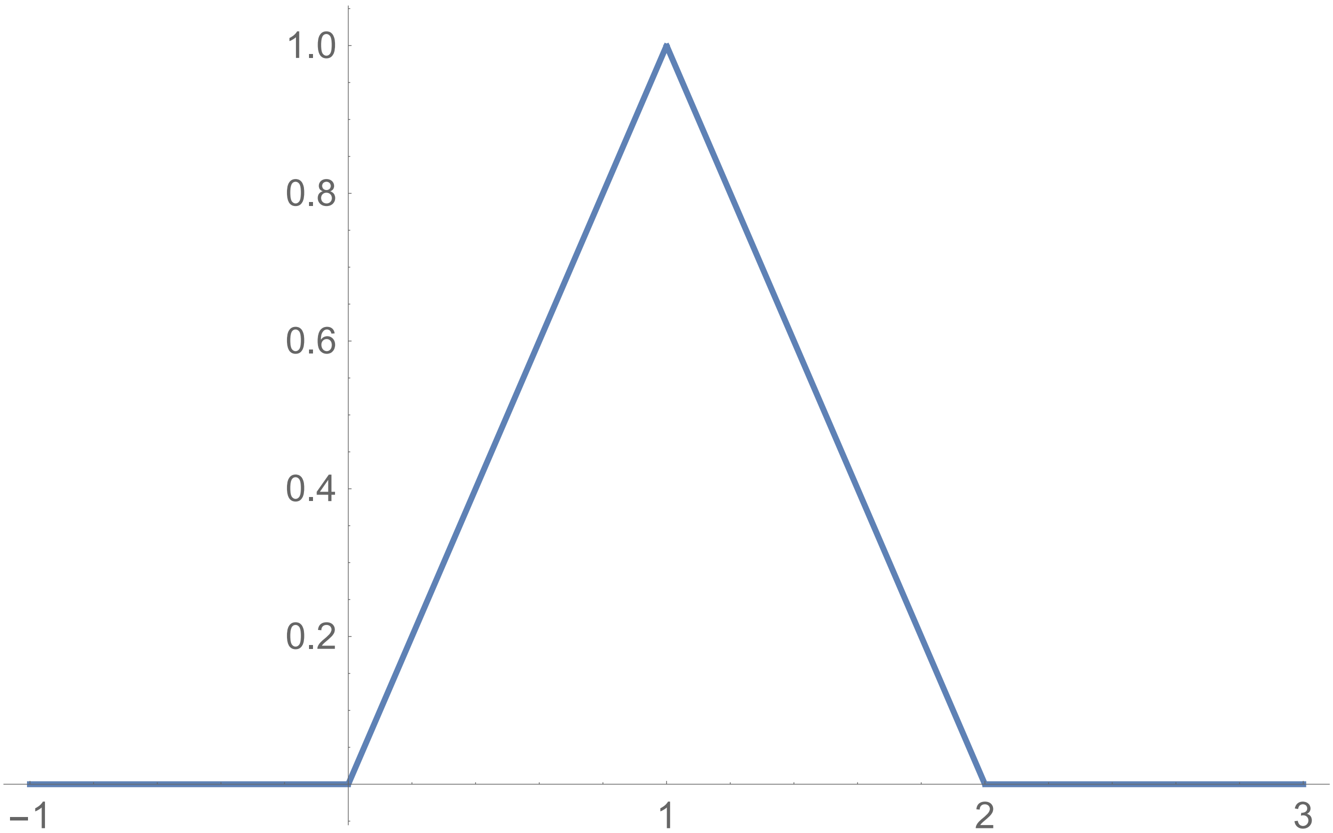
$$f_S(s)=\int_0^s f_{U_1}(u)f_{U_2}(s-u)\mathrm{d}u=\int_0^s 1\mathrm{d}u=s$$

当 $0\le s\le 1$

$$f_S(s)=\int_{s-1}^1 f_{U_1}(u)f_{U_2}(s-u)\mathrm{d}u=\int_{s-1}^1 1\mathrm{d}u=2-s$$

综上

$$f_S(s)=\begin{cases}s & s\in[0,1]\\ 2-s & s\in(1,2]\\ 0 & else\end{cases}$$



44.

U_1 和 U_2 的分布均为

$$p_U(u) = \begin{cases} \frac{1}{3} & u = 0 \\ \frac{1}{3} & u = 1 \\ \frac{1}{3} & u = 2 \end{cases}$$

由卷积公式

$$P(Z = z_l) = \sum_{i=1}^{\infty} P(X = x_i) P(Y = z_l - x_i)$$

当 $0 \le z \le 2$

$$p_Z(z) = \sum_{i=0}^{z} P(X = i) P(Y = z - i) = \sum_{i=0}^{z} \frac{1}{9} = \frac{z+1}{9}$$

当 $2 < z \le 4$

$$p_Z(z) = \sum_{i=z-2}^{2} P(X = i) P(Y = z - i) = \sum_{i=z-2}^{2} \frac{1}{9} = \frac{5-z}{9}$$

综上

Z	0	1	2	3	4
P	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

51.

令

$$\begin{cases} xy = z \\ y = y \end{cases}$$

Jacobi式为

$$\frac{\partial(x,y)}{\partial(z,y)} = \begin{vmatrix} \frac{1}{y} & \frac{-z}{y^2} \\ 0 & 1 \end{vmatrix} = \left|\frac{1}{y}\right|$$

于是

$$F_Z(z) = \int_{-\infty}^{z} \left(\int_{-\infty}^{+\infty} f(\frac{u}{y}, y) \left|\frac{1}{y}\right| \mathbf{d}y \right) \mathbf{d}u$$

所以

$$f_Z(z) = \int_{-\infty}^{+\infty} f(\frac{z}{y}, y) | \left|\frac{1}{y}\right| | \mathbf{d}y$$

52.

令 X 和 Y 均为 均匀变量, 它们的联合分布为

$$f_{XY}(x,y) = 1$$

令

$$\begin{cases} \frac{x}{y} = z \\ y = y \end{cases}$$

Jacobi式为

$$\frac{\partial(x,y)}{\partial(z,y)} = \begin{vmatrix} y & u \\ 0 & 1 \end{vmatrix} = |y|$$

于是

$$F_Z(z) = \int_{-\infty}^{z} \left(\int_{-\infty}^{+\infty} f(uy, y) \, |y| \, \mathbf{d}y \right) \mathbf{d}u$$

所以

当 $z > 1$

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_{XY}(zy, y)\,|y|\,\mathbf{d}y \\
&= \int_0^{\frac{1}{z}} |y|\,\mathbf{d}y \\
&= \frac{1}{2z^2}
\end{aligned}
$$

当 $0 \le z \le 1$

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_{XY}(zy, y)\,|y|\,\mathbf{d}y \\
&= \int_0^{1} |y|\,\mathbf{d}y \\
&= \frac{1}{2}
\end{aligned}
$$

综上

$$
f_Z(z) = \begin{cases} \frac{1}{2z^2} & z \in (1, \infty) \\ \frac{1}{2} & z \in [0, 1] \\ 0 & else \end{cases}
$$

57.

Y_1 和 Y_2 的联合分布为

$$
f_{Y_1 Y_2}(y_1, y_2) = \frac{e^{-y_1^2 + y_2 y_1 - \frac{y_2^2}{2}}}{2\pi}
$$

用已知得

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

取矩阵的逆得到

$$
\begin{pmatrix} \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} & -\frac{a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \\ -\frac{a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
$$

Jacobi式为

$$
\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} & -\frac{a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \\ -\frac{a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \end{vmatrix} = \left| \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \right|
$$

X_1 和 X_2 的联合分布为

$$f_{X_1X_2}(x_1,x_2)=\frac{\exp\left(-\frac{\left(a_{21}^2+2a_{22}a_{21}+2a_{22}^2\right)x_1^2+(-2a_{11}a_{21}-2a_{12}a_{21}-2a_{11}a_{22}-4a_{12}a_{22})x_2x_1+\left(a_{11}^2+2a_{12}a_{11}+2a_{12}^2\right)x_2^2}{2(a_{12}a_{21}-a_{11}a_{22})^2}\right)}{2\pi}$$

若 X_1 和 X_2 均为标准正态随机变量,

$$f_{X_1X_2}(x_1,x_2)=\frac{e^{\frac{1}{2}\left(-x^2-y^2\right)}}{2\pi}$$

解得

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

70.

$$\begin{aligned}
F(x,y) &= \mathrm{P}\left(X_n\leqslant y\right)-\mathrm{P}\left(X_1>x,X_n\leqslant y\right)\\
&=[F(y)]^n-\mathrm{P}\left(x<X_1\leqslant y,x<X_2\leqslant y,\cdots,x<X_n\leqslant y\right)\\
&=[F(y)]^n-\prod_{i=1}^{k}\mathrm{P}\left(x<X_i\leqslant y\right)\\
&=[F(y)]^n-[F(y)-F(x)]^k
\end{aligned}$$