Mathematical Statistics

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Chap2

2019.10.14

33.

$$F(x) = 1 - e^{-\alpha x^{\beta}} \tag{1}$$

$$F'(x) = \alpha \beta x^{\beta-1} e^{\alpha(-x^{\beta})} > 0 \tag{2}$$

这表明 $F(x)$ 单调递增, 又当 $x \to \infty$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} 1 - e^{-\alpha x^{\beta}} = 1 \tag{3}$$

且有

$$F(0) = 0 \tag{4}$$

又 $F(x)$ 在定义域上处处可导, 这满足了连续性.

所以 F 是 CDF, 它的密度函数为

$$f(x) = F'(x) = \alpha \beta x^{\beta-1} e^{\alpha(-x^{\beta})} \tag{5}$$

40.

在 x 的其余取值

$$f(x) = 0 \tag{6}$$

因此

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 cx^2\, dx = \frac{c}{3} \tag{7}$$

a.

$$\frac{c}{3} = 1 \Rightarrow c = 3 \tag{8}$$

b.

当 $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \int_0^x f(u)\, du = x^3 \tag{9}$$

所以 CDF $F(x)$ 为

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{10}$$

c.

$$P(0.1 \le X \le 0.5) = F(0.5) - F(0.1) = 0.125 - 0.001 = 0.124 \tag{11}$$

45.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda x} \tag{12}$$

a.

$$P(X \le 10) = F(10) = 1 - \frac{1}{e} = 0.632121 \tag{13}$$

b.

$$P(5 \le X \le 15) = F(15) - F(5) = \left(1 - e^{-1.5}\right) - \left(1 - e^{-0.5}\right) = 0.3834 \tag{14}$$

c.

$$P(X > t) = 1 - P(X \le t) = 1 - F(t) = e^{-\lambda x} \tag{15}$$

$$e^{-\lambda x} = 0.01 \tag{16}$$

解得

$$x = -10 \ln 0.01 = 46.0517 \tag{17}$$

52.

a.

$$X \approx N\left(70,\ 3^2\right) \tag{18}$$

a.

$$P(X > 72) = P\left(Z > \frac{72 - 70}{3}\right) = P\left(Z > \frac{2}{3}\right) \tag{19}$$

其中 Z 是标准正态分布

$$Z \approx N(0,\ 1) \tag{20}$$

所以

$$P(X > 72) = 1 - P(X \le 72) = 1 - P\left(Z \le \frac{2}{3}\right) = 1 - 0.747507 = 0.252493 \tag{21}$$

b.

用厘米表示:

$$\text{UnitConvert[\{Quantity[70, "Inches"], Quantity[3, "Inches"]\}, "Centimeters"]} = \left\{\frac{889}{5}\text{ cm},\ \frac{381}{50}\text{ cm}\right\} \tag{22}$$

$$X \approx N\left(\frac{889}{5},\ \left(\frac{381}{50}\right)^2\right) \tag{23}$$

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$$\text{UnitConvert[\{Quantity[70, "Inches"], Quantity[3, "Inches"]\}, "Meters"]} = \left\{\frac{889}{500}\text{ m},\ \frac{381}{5000}\text{ m}\right\} \tag{24}$$

$$X \approx N\left(\frac{889}{500},\ \left(\frac{381}{5000}\right)^2\right) \tag{25}$$

53.

$$X \approx N\left(5,\ 10^2\right) \tag{26}$$

a.

$$P(X > 10) = 1 - P(X \le 10) = 1 - \Phi\left(\frac{10 - 5}{10}\right) = 0.308538 \tag{27}$$

b.

$$P(-20 < X < 15) = P(X < 15) - P(X < -20) = \Phi\left(\frac{15-5}{10}\right) - \Phi\left(\frac{-20-5}{10}\right) = 0.835135 \tag{28}$$

c.

$$P(X > x) = 1 - P(X \le x) = 1 - \Phi\left(\frac{x-5}{10}\right) = 0.05 \tag{29}$$

解得

$$x = 21.4485 \tag{30}$$

54.

注意到

$$P(Y \le y) = P(-y \le X \le y) = P(X \le y) - P(X \le -y) \tag{31}$$

由对称性

$$P(X \le -y) = P(X \ge y) = 1 - P(X \le y)$$

所以 Y 的CDF有

$$F_Y(y) = 2\,F_X(y) - 1 \tag{32}$$

从而

$$f_Y(y) = 2\,f_X(y) = \frac{\sqrt{\frac{2}{\pi}}\; e^{-\frac{y^2}{2\sigma^2}}}{\sigma} \tag{33}$$

59.

定义随机变量 X, 它满足

$$X = U^2 \tag{34}$$

那么

$$P(X \le x) = P\left(U^2 \le x\right) = P\left(-\sqrt{x} \le U \le \sqrt{x}\right) = P\left(U \le \sqrt{x}\right) - P\left(U \le -\sqrt{x}\right) \tag{35}$$

由对称性

$$P\left(U \le -\sqrt{x}\right) = P\left(U \ge \sqrt{x}\right) = 1 - P\left(U \le \sqrt{x}\right) \tag{36}$$

所以

$$F_X(x) = 2\,F_U\left(\sqrt{x}\right) - 1 = \sqrt{x} \tag{37}$$

$$f_{U^2}(u) = \frac{1}{2\sqrt{u}}\ (0 \le u \le 1) \tag{38}$$

64.

$$P(Y \le y) = P(aX + b \le y) \tag{39}$$

当 $a > 0$

$$P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right) \tag{40}$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \tag{41}$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \tag{42}$$

当 $a < 0$

$$P(Y \le y) = P(aX + b \le y) = P\left(X \ge \frac{y-b}{a}\right) = 1 - P\left(X \le \frac{y-b}{a}\right) \tag{43}$$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \tag{44}$$

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) \tag{45}$$

综上

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \tag{46}$$

补充题1

$$p(y) = \begin{cases} \frac{1}{5} & y = 0 \\ \frac{1}{6} + \frac{1}{15} = \frac{7}{30} & y = 1 \\ \frac{1}{5} + \frac{11}{30} = \frac{17}{30} & y = 4 \\ 0 & \text{Anything else} \end{cases} \tag{47}$$

补充题2

$$P(Y \le y) = P(\text{Sin}\, X \le y) = 1 - P(\text{ArcSin}(y) < x < \pi - \text{ArcSin}(y)) \tag{48}$$

$$= 1 - (F_X(\pi - \text{ArcSin}(y)) - F_X(\text{ArcSin}(y))) \tag{49}$$

$$F_Y(y) = 1 - F_X(\pi - \text{ArcSin}(y)) + F_X(\text{ArcSin}(y)) \tag{50}$$

$$f_Y(y) = -f_X(\pi - \text{ArcSin}(y))\left(-\frac{1}{\sqrt{1-y^2}}\right) + f_X(\text{ArcSin}(y))\left(\frac{1}{\sqrt{1-y^2}}\right) = \tag{51}$$

$$\frac{f_X(\pi - \text{ArcSin}(y)) + f_X(\text{ArcSin}(y))}{\sqrt{1-y^2}} = \frac{2\pi}{\pi^2\sqrt{1-y^2}} = \frac{2}{\pi\sqrt{1-y^2}}\ (0 < y \le 1) \tag{52}$$