Mathematical Statistics and Data Analysis

Joint Distributions

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1.

a.

$$f_X(x) = egin{cases} 0.10 + 0.05 + 0.02 + 0.02 = 0.19 & x = 1 \ 0.05 + 0.20 + 0.05 + 0.02 = 0.32 & x = 2 \ 0.02 + 0.05 + 0.20 + 0.04 = 0.31 & x = 3 \ 0.02 + 0.02 + 0.04 + 0.10 = 0.18 & x = 4 \end{cases}$$
 $f_Y(y) = egin{cases} 0.10 + 0.05 + 0.02 + 0.02 = 0.19 & y = 1 \ 0.05 + 0.20 + 0.05 + 0.02 = 0.32 & y = 2 \ 0.02 + 0.05 + 0.20 + 0.04 = 0.31 & y = 3 \ 0.02 + 0.02 + 0.04 + 0.10 = 0.18 & y = 4 \end{cases}$

b.

$$f_{X|Y=1}(x) = egin{cases} rac{f(1,1)}{P(Y=1)} = rac{0.1}{0.19} = 0.5263 & x = 1 \ rac{f(1,2)}{P(Y=1)} = rac{0.05}{0.19} = 0.2632 & x = 2 \ rac{f(1,3)}{P(Y=1)} = rac{0.02}{0.19} = 0.1053 & x = 3 \ rac{f(1,4)}{P(Y=1)} = rac{0.02}{0.19} = 0.1053 & x = 4 \ \end{cases} \ f_{Y|X=1}(x) = egin{cases} rac{f(1,1)}{P(Y=1)} = rac{0.02}{0.19} = 0.5263 & y = 1 \ rac{f(1,2)}{P(Y=1)} = rac{0.05}{0.19} = 0.2632 & y = 2 \ rac{f(1,3)}{P(Y=1)} = rac{0.02}{0.19} = 0.1053 & y = 3 \ rac{f(1,4)}{P(Y=1)} = rac{0.02}{0.19} = 0.1053 & y = 4 \ \end{cases}$$

9.

a.

区域面积为

$$S = \int_{-1}^{1} \left(1 - x^2\right) dx = \left(x - rac{x^3}{3}
ight) \Big|_{-1}^{1} = 1 - rac{1}{3} - \left(-1 - rac{-1}{3}
ight) = rac{4}{3}$$

所以

$$f(x,y)=rac{1}{4/3}=rac{3}{4}$$
 $f_X(x)=\int_0^{1-x^2}rac{3}{4}dy=rac{3}{4}(1-x^2) \quad -1\leq x\leq 1$ $f_Y(y)=\int_{-\sqrt{1-y}}^{\sqrt{1-y}}rac{3}{4}dx=rac{3}{2}\sqrt{1-y} \quad 0\leq y\leq 1$

b.

$$f_{X|Y}(x,y) = rac{f(x,y)}{f_Y(y)} = rac{1}{2\sqrt{1-y}}$$
 $f_{Y|X}(x,y) = rac{f(x,y)}{f_X(x)} = rac{1}{1-x^2}$

10.

a.

$$f_Y(y) = \int_0^\infty x e^{-x(y+1)} dx = rac{1}{1+y^2} \ f_X(x) = \int_0^\infty x e^{-x(y+1)} dy = rac{1}{e^x}$$

 $f_X(x)f_Y(y) \neq f(x,y)$, 故 X 和 Y 不独立

b.

$$egin{align} f_{X|Y}(x,y) &= rac{f(x,y)}{f_Y(y)} = x e^{-x(1+y)} \left(1+y^2
ight) \ & f_{Y|X}(x,y) = rac{f(x,y)}{f_X(x)} = x e^{x-x(1+y)} \ \end{aligned}$$

15.

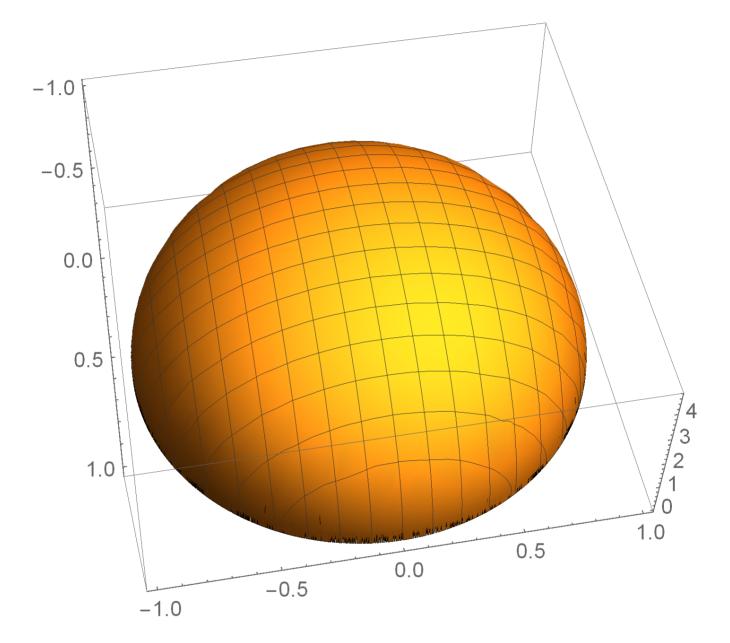
a.

$$\iint_{x^2+y^2\leqslant 1} c\sqrt{1-x^2-y^2} dx dy = c\iint_{x^2+y^2\leqslant 1} \sqrt{1-x^2-y^2} dx dy = \frac{2\pi}{3}c$$

所以

$$c = \frac{3}{2\pi}$$

b.



c.

$$P\left(X^2+Y^2\leqslant rac{1}{2}
ight)=\iint_{x^2+y^2\leq \left(rac{\sqrt{2}}{2}
ight)^2}\sqrt{1-\left(x^2+y^2
ight)}dxdy=1-rac{\sqrt{2}}{4}$$

d.

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} rac{3}{2\pi} \sqrt{1-x^2-y^2} dy = rac{3}{4} ig(1-x^2ig)$$

同理可得

$$f_Y(y)=rac{3}{4}ig(1-y^2ig)$$

验证

$$f_X(x)f_Y(y) \neq f(x,y)$$

故X和Y不独立

e.

$$f_{X|Y}(x,y) = rac{f(x,y)}{f_Y(y)} = rac{2}{\pi} rac{\sqrt{1-x^2-y^2}}{1-y^2}$$

同理

$$f_{Y|X}(x,y) = rac{2}{\pi} rac{\sqrt{1-x^2-y^2}}{1-x^2}$$

19

a.

$$egin{aligned} f_{T_1}(t_1) &= lpha \cdot e^{-lpha t_1} \ f_{T_2}(t_2) &= eta \cdot e^{-eta t_2} \end{aligned}$$

所以联合分布为

$$egin{align*} f_{T_1,T_2}\left(t_1,t_2
ight) &= f_{T_1}\left(t_1
ight) \cdot f_{T_2}\left(t_2
ight) = lpha e^{-lpha t_1}eta e^{-eta t_2} \ P\left(T_1 > T_2
ight) &= \iint_{t_1 > t_2} f\left(t_1,t_2
ight) dt_1 dt_2 \ &= \iint_{t_1 > t_2} f_{T_1}\left(t_1
ight) \cdot f_{T_2}\left(t_2
ight) \ &= \int_0^{+\infty} \left(\int_{t_2}^{+\infty} lpha \cdot e^{-lpha t_1} \cdot eta \cdot e^{-eta t_2} dt_1
ight) dt_2 \ &= \int_0^{+\infty} lpha \cdot eta \cdot e^{-eta t_2} \cdot \left(-rac{e^{-lpha t_1}}{lpha}
ight)igg|_{t_2}^{+\infty} dt_2 = \int_0^{+\infty} eta \cdot e^{-(lpha + eta)t_2} dt_2 \ &= eta \cdot \left(-rac{e^{-(lpha + eta)t_2}}{lpha + eta}
ight)igg|_0^{+\infty} \ &= rac{eta}{lpha + eta} \end{split}$$

b.

$$egin{aligned} P\left(T_1>2T_2
ight) &= \int_{t_1>2t_2} f\left(t_1,t_2
ight) dt_1 dt_2 \ &= \iint_{t_1>2t_2} f_{T_1}\left(t_1
ight) \cdot f_{T_2}\left(t_2
ight) \ &= \int_0^{+\infty} \left(\int_{2t_2}^{+\infty} lpha \cdot e^{-lpha t_1} \cdot eta \cdot e^{-eta t_2} dt_1
ight) dt_2 \ &= \int_0^{+\infty} lpha \cdot eta \cdot e^{-eta t_2} \cdot \left(-rac{e^{-lpha t_1}}{lpha}
ight)igg|_{2t_2}^{+\infty} dt_2 = \int_0^{+\infty} eta \cdot e^{-(2lpha+eta)t_2} dt_2 \ &= \int_0^{+\infty} lpha \cdot eta \cdot e^{-eta t_2} \cdot \left(-rac{e^{-lpha t_1}}{lpha}
ight)igg|_{2t_2}^{+\infty} dt_2 = \int_0^{+\infty} eta \cdot e^{-(2lpha+eta)t_2} dt_2 \ &= eta \cdot \left(-rac{e^{-(2lpha+eta)t_2}}{2lpha+eta}
ight)igg|_0^{+\infty} \ &= rac{eta}{2lpha+eta} \end{aligned}$$

补充题

点P,Q服从均匀分布,记h为底边的高

$$f_{XY}(x,y)=egin{cases} rac{2}{h\cdot BC},(x,y)\in \Delta ABC\ 0,$$
其他 $f_{Z}(z)=egin{cases} rac{1}{BC},z\in (0,BC)\ 0,$ 其他

所以

$$f(x,y,z) = f_{XY}(x,y) f_Z(z)$$

PQ于线段AB相交等价于

$$P((x,y) \in \Delta ABQ, 0 < z < BC) = \frac{1}{2}$$

思考题

设三条线段长分别为 x, y, d - x - y

依题意

$$f(x) = rac{1}{d}$$
 $f_{Y|X}(y) = rac{1}{d-x}$

所以

$$f(x,y)=rac{1}{d(d-x)}$$
 $P($ 构成三角形 $)=\iint_{0< x<rac{d}{2},rac{d}{2}-x< y<rac{d}{2}}rac{1}{d(d-x)}dxdy=\int_{0}^{d/2}\int_{d/2-x}^{d/2}rac{1}{d(d-x)}dydx=ln2-1/2$