Final Quiz

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Lemma 1. *Process*

$$S_t = S_0 \exp\left\{\int_0^t \sigma_s dB_s - \int_0^t \frac{1}{2}\sigma_s^2 ds\right\}$$

is a martingale.

Proof. Let $X_t = \int_0^t \sigma_s dB_s - \int_0^t \frac{1}{2}\sigma_s^2 ds$, we have

$$\begin{cases} dX_t & = \sigma_t dB_t - \frac{1}{2}\sigma_t^2 dt \\ dX_t dX_t & = \sigma_t^2 dB_t dB_t = \sigma_t^2 dt \end{cases}$$

Let $f(x) = S_0 e^x$, by Ito-Doeblin formula:

$$\begin{aligned} dS_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t dX_t \\ &= S_0 e^{X_t} dX_t + \frac{1}{2}S_0 e^{X_t} dX_t dX_t \\ &= \sigma_t S_t dB_t \end{aligned}$$

thus S_t is martingale by theorem II.20 (Protter 2005). $\square$

Lemma 2. *For deterministic function* $\Delta(s)$*,*

$$I(t) := \int_0^t \Delta(s) dB_s \sim \mathcal{N}\left(0, \int_0^t \Delta^2(s) ds\right)$$

Proof. As $I(t)$ is martingale, we have $\mathbb{E}\, I(t) = \mathbb{E}\, I(0) = 0$, and by Ito's isometry:

$$\operatorname{Var} I(t) = \mathbb{E}\, I^2(t) = \int_0^t \Delta^2(s) ds$$

There is remain to show it's normally distributed, i.e. ,

$$\mathbb{E}\, e^{uI(t)} = \exp\left\{\frac{1}{2}u^2 \int_0^t \Delta^2(s) ds\right\}$$

that is

$$\mathbb{E} \exp\left\{\int_0^t u\Delta(s) dB_s - \frac{1}{2}\int_0^t [u\Delta(s)]^2 ds\right\} = 1$$

by lemma 1,

$$\exp\left\{\int_0^t u\Delta(s)dB_s - \frac{1}{2}\int_0^t [u\Delta(s)]^2\, ds\right\}$$

is martingale, and it's start with 1 clearly, this completes the proof. □

Exercise 1.

Solution. By Ito-Doeblin formula in integral form, take $f = x \mapsto \frac{1}{2}x^2$:

$$\begin{aligned}\frac{1}{2}B_t^2 = f(B_t) - f(B_0) &= \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \\ &= \int_0^t B_s dB_s + \frac{t}{2}\end{aligned}$$

thus

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}$$

Exercise 2.

Solution. By theorem II.39(Protter 2005), X_t is a Brownian motion.

$$\begin{aligned}\mathop{\mathbb{E}}_s \exp\left\{iuX_t + \frac{u^2t}{2}\right\} &= \mathop{\mathbb{E}}_s\left[\exp\{iu(X_t - X_s)\}\cdot\exp\left\{iuX_s + \frac{u^2t}{2}\right\}\right] \\ &= \exp\left\{iuX_s + \frac{u^2t}{2}\right\}\cdot\mathop{\mathbb{E}}_s[\exp\{iu(X_t - X_s)\}] \\ &= \exp\left\{iuX_s + \frac{u^2t}{2}\right\}\cdot\mathbb{E}[\exp\{iu(X_t - X_s)\}] \text{ (Stationary increments)} \\ &= \exp\left\{iuX_s + \frac{u^2t}{2}\right\}\cdot\mathbb{E}\left[\exp\left\{iu\mathfrak{z}\sqrt{t-s}\right\}\right] \\ &= \exp\left\{iuX_s + \frac{u^2t}{2}\right\}\exp\left\{-\frac{u^2(t-s)}{2}\right\} \text{ (MGF of normal distribution)} \\ &= \exp\left\{iuX_s + \frac{u^2s}{2}\right\}\end{aligned}$$

The integrability follows from X_t is Brownian motion:

$$\mathbb{E}\exp\left\{iuX_t + \frac{u^2t}{2}\right\} = \frac{u^2t}{2}\,\mathbb{E}\exp\{iuX_t\} = \frac{u^2t - u^2t}{2} = 0$$

this completes the proof.

Exercise 3.

Solution. Let $Y_t = X_0 + \sigma\int_0^t e^{\alpha s}dB_s$, we have

$$\begin{cases} dY_t &= \sigma e^{\alpha t}dB_t \\ dY_t dY_t &= \sigma^2 e^{2\alpha t}dt\end{cases}$$

Let $X_t = f(t, Y_t) = e^{-\alpha t}Y_t$, $f_t(t, Y_t) = -\alpha e^{-\alpha t}Y_t$, $f_x = e^{-\alpha t}$, $f_{xx} = 0$, Ito-Doeblin formula yields,

$$\begin{aligned}dX_t &= -\alpha e^{-\alpha t}Y_t dt + e^{-\alpha t}dY_t \\ &= -\alpha X_t dt + \sigma dB_t\end{aligned}$$

Exercise 4.

Solution. By lemma 2, take $\Delta(s) = s$, we have

$$\int_0^t s^2 ds = \frac{1}{3}$$

then the claim follows.

Exercise 5.

Solution. a. Let $f(t,x) = e^{\beta t}x$, by Ito-Doeblin formula:

$$\begin{aligned} d(e^{\beta t}R_t) = df(t,R_t) &= \beta e^{\beta t}R_t dt + e^{\beta t}\left[(\alpha - \beta R_t)\,dt + \sigma\sqrt{R_t}dB_t\right] \\ &= \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R_t}dB_t \end{aligned}$$

Integrate each sides:

$$e^{\beta t}R_t = R_0 + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma\int_0^t e^{\beta s}\sqrt{R_s}dB_s$$

As Integration w.r.t B_t is martingale and thus have zero expectation:

$$e^{\beta t}\,\mathbb{E}\,R_t = R_0 + \frac{\alpha}{\beta}(e^{\beta t} - 1)$$

that is

$$\mathbb{E}\,R_t = e^{-\beta t}r_0 + \frac{\alpha(1 - e^{-\beta t})}{\beta}$$

b. Let $X_t = e^{\beta t}R_t$, we already have

$$dX_t = \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R_t}dB_t = \alpha e^{\beta t}dt + \sigma e^{\frac{\beta t}{2}}\sqrt{X_t}dB_t$$

and

$$\mathbb{E}\,X_t = r_0 + \frac{\alpha}{\beta}(e^{\beta t} - 1)$$

Let $f(X) = x^2$, Ito-Doeblin formula yields:

$$d(X^2(t)) = 2\alpha e^{\beta t}X_t dt + 2\sigma e^{\frac{\beta t}{2}}X_t^{\frac{3}{2}}dB_t + \sigma^2 e^{\beta t}X_t dt$$

Integrate each sides, we have

$$X_t^2 = X_0^2 + (2\alpha + \sigma^2)\int_0^t e^{\beta s}X_s ds + 2\sigma\int_0^t e^{\frac{\beta s}{2}}X_s^{\frac{3}{2}}dW_s$$

Take expectation each sides:

$$\begin{aligned} \mathbb{E}\,X^2 &= X_0^2 + (2\alpha + \sigma^2)\int_0^t e^{\beta s}\,\mathbb{E}\,X_s ds \text{ (Fubini's theorem)} \\ &= r_0^2 + (2\alpha + \sigma^2)\int_0^t e^{\beta s}\left[r_0 + \frac{\alpha(e^{\beta s} - 1)}{\beta}\right]ds \end{aligned}$$

Then we can derivate $\operatorname{Var} R_t$ by

$$\begin{aligned} \operatorname{Var} R_t = \mathbb{E}\,R_t^2 - (\mathbb{E}\,R_t)^2 &= e^{-2\beta t}\,\mathbb{E}\,X_t^2 \\ &= \frac{\sigma^2 r_0}{\beta}(e^{-\beta t} - e^{-2\beta t}) + \frac{\alpha\sigma^2}{2\beta^2}(1 - 2e^{-\beta t} + e^{-2\beta t}) \end{aligned}$$

Reference

Protter, Philip E. 2005. *Stochastic Differential Equations.* Springer.