Homework 3

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Last compiled on 22:37, 15 December, 2021

Exercise 0.1.

Solution. Let $\tau_z = \inf\{s > 0 : Z_t \geq z\}$. Then τ_z is a stopping time and $\mathbb{P}\{\tau < \infty\} = 1$. Suppose coordinate map $\omega(t) = Z_t(\omega)$. Let $R = \inf\{s < t : Z_s \geq z\}$ and

$$Y_s(\omega) = \begin{cases} 1 & s \leq t, \omega(t-s) < z - y \\ 0, \text{otherwise} \end{cases}, Y_s'(\omega) = \begin{cases} 1 & s \leq t, \omega(t-s) > z - y \\ 0, \text{otherwise} \end{cases}$$

hence

$$Y_R \circ \theta_R(\omega) = \begin{cases} 1 & R \leq t, Z_t < z - y \\ 0, \text{otherwise} \end{cases}, Y_R' \circ \theta_R(\omega) = \begin{cases} 1 & R \leq t, Z_t > z - y \\ 0, \text{otherwise} \end{cases}$$

Strong Markov property implies that on $\{R < \infty\}$,

$$\mathop{\mathbb{E}}_0 (Y_R \circ \theta_R | \mathcal{F}_R) = \mathop{\mathbb{E}}_{Z_R} Y_R, \mathop{\mathbb{E}}_0 (Y_R' \circ \theta_R | \mathcal{F}_R) = \mathop{\mathbb{E}}_{Z_R} Y_R',$$

since $Z_R \geq z$ and Z is symmetric,

$$\mathop{\mathbb{E}}_a Y_s = \mathop{\mathbb{E}}_a \mathbf{1}_{Z_{t-s} < z-y} < \mathop{\mathbb{E}}_a \mathbf{1}_{Z_{t-s} > z+y} = \mathop{\mathbb{E}}_a Y_s'$$

By taking expectation,

$$\begin{aligned} \mathbb{P}\{\tau_z \leq t, Z_t < z - y\} &= \mathbb{E}\left(\mathop{\mathbb{E}}_{Z_R} Y_R : R < \infty\right) \\ &\leq \mathbb{E}\left(\mathop{\mathbb{E}}_{Z_R} Y_R' : R < \infty\right) \\ &= \mathbb{E}\left(\mathop{\mathbb{E}}_0 Y_R' \circ \theta_R | \mathcal{F}_R\right) \\ &= \mathop{\mathbb{P}}_0 \{\tau_z \leq t, Z_t > z + y\} \\ &= \mathbb{P}\{Z_t > z + y\} \end{aligned}$$

Exercise 0.2.

Solution. Define τ_z, R and $\omega(t)$ the same way in last exercise and let

$$Y_s(\omega) = \begin{cases} 1 & s \le t, \omega(t-s) > z \\ 0 & \text{otherwise} \end{cases}$$

Since $Z_R \ge z$ and Z is symmetric,

$$\mathop{\mathbb{E}}_{a} Y_s = \mathop{\mathbb{E}}_{a} \mathbf{1}_{\{Z_{t-s} \ge z\}} \ge \frac{1}{2}, \forall a \ge z, s < t$$

take expectation yield

$$\begin{aligned}
\mathop{\mathbb{P}}_{0} \{Z_t \ge z\} &= \mathop{\mathbb{P}}_{0} \{\tau_z \le t, Z_t \ge z\} \\
&= \mathbb{E}\left(\mathop{\mathbb{E}}_{0} (Y_R \circ \theta_R | \mathcal{F}_R) : R < \infty\right) \\
&= \mathbb{E}(\mathop{\mathbb{E}}_{Z_R Y_R} : R < \infty) \\
&\ge \mathbb{E}\left(\frac{1}{2} : R < \infty\right) = \frac{1}{2} \mathbb{P}\{S_t \ge z\}
\end{aligned}$$

Exercise 0.3.

Solution. By DCT,

$$\lim_{n\to\infty} \overset{\mathbb{Q}}{\mathbb{E}} (|X_n - X| \wedge 1) = \lim_{n\to\infty} \overset{\mathbb{P}}{\mathbb{E}} \left(|X_n - X| \wedge 1 \cdot \frac{d\mathbb{Q}}{d\mathbb{P}}\right) = 0$$

$|X_n - X| \wedge 1 \to 0 \in L^1(Q)$ implies $X_n \to X$ in $\mathbb{Q}$-probability.

Exercise 0.4.

Solution. B_t is continuous local martingale, then

$$[B, B]_t = \alpha^2 [X, X]_t + (1 - \alpha^2)[Y, Y]_t = t$$

as X and Y are independent, then by Levy theorem, B is a standard Brownian motion:

$$[X, B]_t = \alpha t, [Y, B]_t = \sqrt{1 - \alpha^2} t$$

Exercise 0.5.

Solution. Assume $f(0) = 0$, let $M_t = B_{f(t)}$ filtered by $\mathcal{G}_t = \mathcal{F}_{f(t)}$, where B is standard Brownian motion filtered by $\mathcal{F}_t$, then

$$\mathop{\mathbb{E}}_{\mathcal{G}_s} M_t = \mathop{\mathbb{E}}_{\mathcal{F}_{f(s)}} B_{f(t)} = B_{f(s)} = M_s$$

hence M_t is a martingale and continuous clearly. Finally,

$$[M, M]_t = [B, B]_{f(t)} = f(t)$$

If $f(0) > 0$, we can add a constant process A_t to $B_{f(t)}$ s.t.

$$2 A_t M_0 + A_t^2 = -B_0^2$$

to get desired result.

Exercise 0.6.

Solution. M is continuous as so is B, M is local martingale since B is locally square integrable local martingale.

$$[M, M]_t = \int_0^t H_s^2 ds = t$$

so M is Brownian motion by Levy's characterization theorem.

Exercise 0.7.

Solution. Note
$$[X^n, Z] = [H^n, Y \cdot Z] = H^n \cdot [Y, Z]$$
and
$$[X, Z] = [H, Y \cdot Z] = H \cdot [Y, Z]$$
and $[Y, Z]$ is a semimartingale. Then by continuity of Stochastic integral, $H^n \to H$ in ucp implies $H^n \cdot [Y, Z] \to H \cdot [Y, Z]$ and hence
$$[X^n, Z] \to [XZ]$$
in ucp