Homework 1

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Lemma 0.1. *σ algebra $\mathcal{A}$ can be seen as collection of numerical $\mathcal{A}$ measurable function, formerly,*

$$\mathcal{A} = \sigma\{f : f \text{ is } \mathcal{A} \text{ meaurable}\} = \sigma(\bigcup_f \sigma f)$$

Proof. Recall that f is $\mathcal{A}$ measurable (w.r.t borel $\mathcal{B}$) iff $\sigma f = f^{-1}(\mathcal{B}) \subset \mathcal{A}$, thus

$$\sigma(\bigcup_f \sigma f) \subset \mathcal{A}$$

On the other hand, for any $A \in \mathcal{A}$, $\mathbf{1}_A$ is measurable and hence:

$$\sigma(\bigcup_f \sigma f) \supset \bigcup_f \sigma(f) \supset \bigcup_{A \in \mathcal{A}} \sigma(\mathbf{1}_A) = \mathcal{A}$$

then claim follows. □

Hence we can write $V \in \mathcal{A}$ to mean V is $\mathcal{A}$ measurable without further comments.

Lemma 0.2. *$V \in \mathcal{A} \iff \{A \le r\} \in \mathcal{A}$ for any $r \in \mathbb{R}$.*

Proof. $\implies$ is immediately and $\impliedby$ follows from collection of $[-\infty, r]$ generates $\mathcal{B}$. □

For σ-algebras on stopping time, we have

Theorem 0.1. *$V \in \mathcal{F}_\tau$ iff $V\mathbf{1}_{\tau \le t} \in \mathcal{F}_t$ for any $t \in \overline{T}$.*

Proof.

$$\begin{aligned} V \in \mathcal{F}_\tau &\iff \{V > r\} \in \mathcal{F}_\tau \\ &\iff \{V > r\} \cap \{\tau \le t\} \in \mathcal{F}_t, \forall t \\ &\iff \{V\mathbf{1}_{\tau \le t} > r\} \in \mathcal{F}_t, \forall t \\ &\iff V\mathbf{1}_{\tau \le t} \in \mathcal{F}_t, \forall t \end{aligned}$$

□

Similarly, we abuse $\mathbb{F}$ following collection of process:

1. $X = \{X_t\}_{t \in T}$ is adapted to $\mathbb{F}$.
2. The path $t \mapsto X_t(\omega)$ is right continuous for every $\omega \in \Omega$ (RCLL or cadlag).

In such notations, we have the restatement of theorem 6 (Protter 2005):

Theorem 0.2. $\mathcal{F}_\tau = \{X_\tau : X \in \mathbb{F}\}$

Combine above theorems:

Theorem 0.3. *Let σ and τ be stopping times of $\mathbb{F}$, then*

1. *$\sigma \wedge \tau$ and $\sigma \vee \tau$ are stopping time of $\mathbb{F}$.*
2. *$\sigma \le \tau$ a.s. $\implies \mathcal{F}_\sigma \subset \mathcal{F}_\tau$*
3. *In general, $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$*
4. *If $V \in \mathcal{F}_\sigma$, then the following are in $\mathcal{F}_{\sigma\wedge\tau}$:*

$$V\mathbf{1}_{\sigma\le\tau}, V\mathbf{1}_{\sigma=\tau}, V\mathbf{1}_{\sigma<\tau}$$

Proof. **1** follows easily by noting

$$\{\tau \wedge \sigma \le t\} = \{\sigma \le t\} \cup \{\tau \le t\}$$

For **2**, suppose $V \in \mathcal{F}_\sigma$, then $X_t = V\mathbf{1}_{\sigma\le t}$ defines a process X adapted to $\mathbb{F}$ by theorem 0.1 and it's right continuous obviously, thus $X \in \mathbb{F}$. Then $X_\tau \overset{\text{a.s.}}{=\!=} V \in \mathcal{F}_\tau$ by theorem 0.2 and the claim follows.

To see **3**, we show **4** first. As $\sigma \wedge \tau$ is stopping time, use $V \in \mathcal{F}_\sigma$ in **2**, we have

$$X_{\sigma\wedge\tau} = V\mathbf{1}_{\sigma\le\sigma\wedge\tau} = V\mathbf{1}_{\sigma\le\tau} \in \mathcal{F}_{\sigma\wedge\tau}$$

take $V = 1$ we have $\{\sigma \le \tau\} \in \mathcal{F}_{\sigma\wedge\tau}$ and the others follows by symmetry and set operations.

Now let $H \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ and $V = \mathbf{1}_H$, then **4** implies $V = V\mathbf{1}_{\sigma<\tau} + V\mathbf{1}_{\tau<\sigma} \in \mathcal{F}_{\sigma\cap\tau}$ and that shows $F_\sigma \cap \mathcal{F}_\tau \subset \mathcal{F}_{\sigma\wedge\tau}$. The other direction follows easily by noting $\sigma \wedge \tau$ are dominated by both τ and σ. ☐

Exercise 0.1.

Solution. See theorem 0.3.2.

Exercise 0.2.

Solution. Let $\Omega = \mathbb{N}$, $\mathcal{F}_n = \sigma(\{1\}, \{2\}, \ldots \{n\})$.

Let $S(\omega) = \omega$, $T = 3$, then $\{S = n\} = \{n\} \in \mathcal{F}_n$ thus both are stopping time. However,

$$\{T - S = 1\} = \{2\} \notin \mathcal{F}_1$$

therefore $T - S$ isn't a stopping time.

Exercise 0.3.

Solution.

- $\sup_n \tau_n$: $\{\sup_n \tau_n \le t\} = \bigcap_n \{\tau_n \le t\} \in \mathcal{F}_t$
- $\inf_n \tau_n$: $\{\inf_n \tau_n < t\} = \bigcup_n \{\tau_n < t\} \in \mathcal{F}_t$
- $\limsup_{n\to\infty} = \inf_m \sup_{n\ge m} \tau_n$
- $\liminf_{n\to\infty} = \sup_m \inf_{n\ge m} \tau_n$

Exercise 0.4.

Solution. The first statement is clear by last exercise. By theorem 0.3.3 and monotonicity, we have

$$\mathcal{F}_\tau = \mathcal{F}_{\lim_{n\to\infty}\tau_n} = \lim_{n\to\infty} \mathcal{F}_{\wedge_{i\le n}\tau_i} = \lim_{n\to\infty} \bigcap_{i\le n} \mathcal{F}_i = \bigcap_n \mathcal{F}_n$$

Exercise 0.5.

Solution. Since $X \in L^p$, then $X \in L^1$

a. By Jensen's inequality

$$\mathbb{E}\,|M|^p = \mathbb{E}\left|\mathop{\mathbb{E}}_t X\right|^p \leq \mathbb{E}\mathop{\mathbb{E}}_t |X^p| = \mathbb{E}\,|X|^p < \infty$$

b. For $t \geq s \geq 0$,

$$\mathop{\mathbb{E}}_s M_t = \mathop{\mathbb{E}}_s \mathop{\mathbb{E}}_t X = \mathop{\mathbb{E}}_s X = M_s$$

thus M is a martingale. For the continuity, Jensen's inequality yields: $\forall p > 1$:

$$\mathbb{E}\,|M_t^n - M_t|^p = \mathbb{E}\left|\mathop{\mathbb{E}}_t (M_\infty^n - X)\right|^p \leq \mathbb{E}\mathop{\mathbb{E}}_t |M_\infty^n - X|^p = \mathbb{E}\,|M_\infty^n - X|$$

Take sup:

$$\sup_t \mathbb{E}\,|M_t^n - M_t| \leq \mathbb{E}\,|M_\infty^n - X| \to 0$$

For any $\varepsilon > 0$, we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_t |M_t^n - M_t| > \varepsilon\right\} &\leq \frac{1}{\varepsilon^p}\,\mathbb{E}\sup_t |M_t^n - M_t|^p && \text{(Chebychev's inequality)} \\
&\leq \frac{1}{\varepsilon^p}\left(\frac{p}{p-1}\right)^p \frac{\sup_t \mathbb{E}\,|M_t^n - M_t|^p}{\varepsilon^p} && \text{(Doob's inequality)} \\
&\to 0
\end{aligned}$$

Thus $M^n \to M$ uniformly in probability. As we can select a subsequence uniformly converges to M a.s. , then M is continuous a.s. since it's limit of uniformly continuous paths.

Exercise 0.6.

Solution.

$$\begin{aligned}
\mathbb{E}\,|N_t - \lambda t| &= \mathbb{E}\,(N_t - \lambda t) + 2\,\mathbb{E}\,(N_t - \lambda t)^- \\
&= 2\,\mathbb{E}\,(N_t - \lambda t)^- \\
&= 2\sum_{n=0}^{\lambda t} (\lambda t - n) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= 2 e^{-\lambda t} \frac{(\lambda t)^{\lambda t}}{(\lambda t - 1)!}
\end{aligned}$$

Exercise 0.7.

Solution. As Poison process has stationery increments, for $t \geq s \geq 0$,

$$\mathbb{E}\,(N_t - N_s)^2 = \mathbb{E}\,N_{t-s}^2 = \operatorname{Var} N_{t-s} + \overset{2}{\mathbb{E}}\,N_{t-s} = \lambda(t-s)\,[1 + \lambda(t-s)]$$

As $s \to t$, that tends to 0 and the claim follows.

Reference

Protter, Philip E. 2005. *Stochastic Differential Equations*. Springer.