Homework 2

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Theorem 0.1 (Lévy-Khintchine Formula)**.** *Let X be a Levy process in $\mathbb{R}^d$, there uniquely exist a triplet $(\mathbf{A}, \gamma, \nu)$ of*

$$\begin{cases} \mathbf{A} & \in \mathbb{R}^{d\times d} \geq 0 \\ \gamma & \in \mathbb{R}^d \\ \nu & \text{a Lévy measure on } \mathbb{R}^d \end{cases}$$

determine the process X, that is, $\mathbb{E}\, e^{i\mathbf{u}' X_t} = e^{-t\psi(\mathbf{u})}$, where

$$\psi(\mathbf{u}) = \frac{1}{2} \langle \mathbf{u}, \mathbf{A}\mathbf{u} \rangle - i \langle \mathbf{u}, \boldsymbol{\gamma} \rangle + \int_{\mathbb{R}^d} \left(1 + \mathbf{1}_{|x|\leq 1} i \langle \mathbf{u}, \mathbf{x} \rangle - e^{i\langle \mathbf{u}, \mathbf{x} \rangle}\right) \nu(d\mathbf{x})$$

If $\boldsymbol{\gamma_0} = \boldsymbol{\gamma} - \int_{|\mathbf{x}|\leq 1} d\nu$ is well-defined and finite,the we can rewrite above formula by $(\mathbf{A}, \gamma_0, \nu)_0$:

$$\psi(\mathbf{u}) = \frac{1}{2} \langle \mathbf{u}, \mathbf{A}\mathbf{u} \rangle - i \langle \mathbf{u}, \boldsymbol{\gamma_0} \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \mathbf{u}, \mathbf{x} \rangle}\right) \nu(d\mathbf{x})$$

If $\boldsymbol{\gamma_1} = \boldsymbol{\gamma} + \int_{|\mathbf{x}|> 1} d\nu$ is well-defined and finite,the we can rewrite above formula by $(\mathbf{A}, \gamma_1, \nu)_1$:

$$\psi(\mathbf{u}) = \frac{1}{2} \langle \mathbf{u}, \mathbf{A}\mathbf{u} \rangle - i \langle \mathbf{u}, \boldsymbol{\gamma_1} \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \mathbf{u}, \mathbf{x} \rangle} + i \langle \mathbf{u}, \mathbf{x} \rangle\right) \nu(d\mathbf{x})$$

Exercise 0.1 (11)**.**

Solution. Since $\mathbb{E}\, e^{iuZ_t} = e^{-t\psi(u)}$ and $\psi(u) = \int (1 - e^{iux})\nu(dx)$, Z_t is Lévy process with generating triplet $(0, 0, \nu)_0$ and $d = 1$ in theorem 0.1.

Exercise 0.2 (12)**.**

Solution. As compound Poisson process is Levy and thus for $t \geq s$

$$\mathop{\mathbb{E}}_s Z_t = \mathop{\mathbb{E}}_s (Z_t - Z_s + Z_s) = Z_s + \mathbb{E}\, Z_{t-s}$$

and Wald's equation yields

$$\mathbb{E}\, Z_t = \mathbb{E}\, N_t\, \mathbb{E}\, U_1 = \lambda t\, \mathbb{E}\, U_1$$

thus Z is integrable and

$$\mathop{\mathbb{E}}_s \left(Z_t - \lambda t\, \mathbb{E}\, U_1\right) = \mathop{\mathbb{E}}_s Z_t - \lambda t\, \mathbb{E}\, U_1 = Z_s - \lambda s\, \mathbb{E}\, U_1$$

completes the proof.

Exercise 0.3 (17)**.**

Solution. Note that the X is a cadlag and $[0, t]$ is a close set in $\mathbb{R}$, thus compact in $\mathbb{R}$. Fix some $\omega \in \Omega$, we can choose a subsequence $\{s_n\} \subset \mathbb{R}$ of the sequence whose jump is larger than ϵ s.t. $\lim_{n\to\infty} s_n = s$ for some $s \in [0, t]$ and for each n, $s_n \leq s_{n+1}$. By the assumption, there exists a $\delta \geq 0$ s.t. when $|k - s| \leq \delta$, $|X_k - X_{s-}| \leq \epsilon/3$, and $|X_{k-} - X_{s-}| \leq \epsilon/3$. And as the assumption, for some n s.t. $|s_n - s| \leq \delta$,

$$\left|X_{s_n} - X_{s-}\right| = \left|X_{s_n-} + \Delta X_{s_n} - X_{s-}\right| > 2\epsilon/3$$

which leads to a contradiction.

By the discussion above, we can just pick $\epsilon = 1/n$ for each n and note that the set $\{s \in [0, t] : |\Delta X_s| > 0\} = \bigcup_{n\in\mathbb{N}} \{s \in [0, t] : |\Delta X_s| > 1/n\}$ and each set of the right side is finite, thus the set of jumps is countable.

Exercise 0.4 (18)**.**

Solution. By corollary of theorem 36 and theorem 37 (Protter 2005), we have J^ε and $Z - J^\varepsilon$ are Lévy, the independency follows from noting

$$\psi_{J^\varepsilon} + \psi_{Z-J^\varepsilon} = \psi_Z$$

in theorem 0.1.

Exercise 0.5 (19)**.**

Solution. Let $\tau_n = \inf\{t > 0 : |X_t| > n\}$, it's series of stopping times since (n, ∞) is borel. Then let $\sigma_n = \tau_n \mathbf{1}_{X_0 \leq n}$, note

$$\{\sigma_n \leq t\} = \{\tau_n \leq t\} \cup \{X_0 > n\} \in \mathcal{F}_t$$

hence σ_n is stopping time. Then by the continuity of X, $\{\sigma_n\}$ justify that X is locally bounded.

Exercise 0.6 (24)**.**

Exercise 0.7 (25)**.**

Solution. Fix ε and t, note

$$\{|\Delta Z_t| > \varepsilon\} = \bigcup_n \bigcap_{n\geq m} \left\{\left|Z_t - Z_{t-\frac{1}{n}}\right| > \varepsilon\right\}$$

hence

$$\begin{aligned}
\mathbb{P}\{|\Delta Z_t| > \varepsilon\} &= \mathbb{P} \liminf_n \left\{\left|Z_t - Z_{t-\frac{1}{n}}\right| > \varepsilon\right\} \\
&\leq \liminf_n \mathbb{P}\left\{\left|Z_t - Z_{t-\frac{1}{n}}\right| > \varepsilon\right\} \\
&\leq \lim_{n\to\infty} \mathbb{P}\left\{\left|Z_t - Z_{t-\frac{1}{n}}\right| > \varepsilon\right\} = 0
\end{aligned}$$

Reference

Protter, Philip E. 2005. *Stochastic Differential Equations.* Springer.