Homework 4

Xie Zejian
11810105@mail.sustech.edu.cn

Department of Finance, SUSTech

Last compiled on 18:02, 04 January, 2022

Exercise 0.1.

Solution. Let σ_n be a sequence of random partition tending to identity. By theorem 22 (Protter 2005),

$$
\begin{aligned}
[M, A] &= M_0 A_0 + \lim_{n\to\infty} \sum_i \left(M^{T^n_{i+1}} - M^{T^n_i}\right)\left(A^{T^n_{i+1}} - A^{T^n_i}\right) \\
&\leq 0 + \lim_{n\to\infty} \sup_i \left|M^{T^n_{i+1}} - M^{T^n_i}\right| \sum_i \left|A^{T^n_{i+1}} - A^{T^n_i}\right| = 0
\end{aligned}
$$

since M is continuous and $\sum_i \left|A^{T^n_{i+1}} - A^{T^n_i}\right| < \infty$ by hypothesis. Similarly,

$$
\begin{aligned}
[A, A] &= A_0^2 + \lim_{n\to\infty} \sum_i \left(A^{T^n_{i+1}} - A^{T^n_i}\right)\left(A^{T^n_{i+1}} - A^{T^n_i}\right) \\
&\leq 0 + \lim_{n\to\infty} \sup_i \left|A^{T^n_{i+1}} - A^{T^n_i}\right| \sum_i \left|A^{T^n_{i+1}} - A^{T^n_i}\right| = 0
\end{aligned}
$$

Thus we conclude that

$$[X, X] = [M, M] + 2[M, A] + [A, A] = [M, M]$$

Exercise 0.2.

Solution. X^2 is $\mathbb{P}$ semimartingale and hence $\mathbb{Q}$ semimartingale by theorem 2 (Protter 2005). By corollary of theorem 15 (Protter 2005), $(X\_ \cdot X)^{\mathbb{Q}}$ is indistinguishable from $(X\_ \cdot X)^{\mathbb{P}}$, then by definition,

$$[X, X]^{\mathbb{P}} = X^2 - (X\_ \cdot X)^{\mathbb{P}} = X^2 - (X\_ \cdot X)^{\mathbb{Q}} = [X, X]^{\mathbb{Q}}$$

Exercise 0.3.

Solution. By Ito's formula and given SDE,

$$d(\exp^{\alpha t} X_t) = \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t}(-\alpha X_t dt + \sigma dB_t) = \sigma e^{\alpha t} dX_t$$

integrate yields

$$X_t = e^{-\alpha t}\left(X_0 + \sigma \int_0^t e^{as} dB_s\right)$$

Exercise 0.4.

Solution. By the law of iterated logarithm,

$$\limsup_{t\to\infty} \frac{B_t}{t} = 0 \text{ a.s.}$$

In particular, for a.s. ω, there exists $t_0(\omega)$ s.t. $t > t_0(\omega)$ implies $\frac{B_t}{t} < \frac{1}{2} - \varepsilon$ for any $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then

$$\lim_{t\to\infty} \mathcal{E}(B_t) = \lim_{t\to\infty} \exp\left\{t\left(\frac{B_t}{t} - \frac{1}{2}\right)\right\} \le \lim_{t\to\infty} e^{-\varepsilon t} = 0, \text{ a.s.}$$

Exercise 0.5.

Solution. $\mathcal{E}(X)^{-1} = \mathcal{E}(-X + [X,X])$ by corollary of theorem 38 (Protter 2005). This implies that $\mathcal{E}(X)^{-1}$ is the solution of SDE:

$$\mathcal{E}(X)_t^{-1} = 1 + \int_0^t \mathcal{E}(X)_{s-}^{-1} d(-X_s + [X,X]_s)$$

as desired.

Reference

Protter, Philip E. 2005. *Stochastic Differential Equations.* Springer.