Bachelor Thesis Marginal-Sampling

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Normal-Gamma Prior

Environment

In contrast to last time we now assume the two priors from our likelihood are now a joint probability distribution, the so called normal-gamma distribution.
The two parameters which we wanna marginalize are the offset parameter c and the precision λ, which is the same as $1/\sigma^2$.

prior-definition

The normal-gamma prior depends on 4 shape parameters, $\mu, \kappa, \alpha, \beta$ and has the following structure:

$$p(c, \lambda) = f(c, \lambda \mid \mu, \kappa, \alpha, \beta) \tag{1}$$

$$= \mathcal{N}(c \mid \mu, 1/(\kappa\lambda)) \cdot \Gamma(\lambda \mid \alpha, \beta). \tag{2}$$

likelihood

As a result we also get a different likelihood:

$$p(D \mid \theta, c, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{N/2} \cdot \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{N}(\overline{y}_k - (c + h_k))^2\right) \quad (3)$$

Calculation

We can start with the usual setup:

$$p(D \mid \theta) = \int_{\mathbb{R}\times\mathbb{R}_+} p(D \mid \theta, c, \lambda) p(c, \lambda) \, d(c, \lambda) \tag{4}$$

Calculation

We can start with the usual setup:

$$p(D \mid \theta) = \int_{\mathbb{R}\times\mathbb{R}_+} p(D \mid \theta, c, \lambda) p(c, \lambda)\, d(c, \lambda) \tag{5}$$

$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\frac{\lambda}{2\pi}\right)^{N/2} \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{N}(\bar{y}_k - (c + h_k))^2\right) \cdot \tag{6}$$

$$\frac{\beta^\alpha \sqrt{\kappa}}{\Gamma(\alpha)\sqrt{2\pi}} \lambda^{\alpha-1/2} \exp\left(-\frac{\lambda}{2}[\kappa(c-\mu)^2 + 2\beta]\right) dc\, d\lambda \tag{7}$$

Offset Integration

We pull out the constants and terms which only rely on λ to integrate over c.

$$\int_{\mathbb{R}} \exp\left(-\frac{\lambda}{2}\left(\left(\sum_{k=1}^{N}((\overline{y_k} - h_k) - c)^2\right) + \kappa(c-\mu)^2 + 2\beta\right)\right)\, dc \tag{8}$$

$$= \int_{\mathbb{R}} \exp\left(-\frac{\lambda}{2}\left((N+\kappa)c^2 - 2\left(\left(\sum_{k=1}^{N}\overline{y_k} - h_k\right) + \kappa\mu\right)c\right.\right. \tag{9}$$

$$\left.\left. + \left(\sum_{k=1}^{N}(\overline{y_k} - h_k)^2\right) + \kappa\mu^2 + 2\beta\right)\right) dc \tag{10}$$

Offset Integration

Now we can use the exponential integration formula:

$$\int_{\mathbb{R}} \exp(-a \cdot c^2 + b \cdot c - d)\, dc = \sqrt{\frac{\pi}{a}} \cdot \exp\left(\frac{b^2}{4a} - d\right)$$

Offset Integration

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and receive

$$\sqrt{\frac{2\pi}{\lambda(N+\kappa)}} \cdot \exp\left(\lambda \cdot \left(\frac{1}{2(N+\kappa)}\left(\left(\sum_{k=1}^{N} \overline{y_k} - h_k\right) + \kappa\mu\right)^2 \right.\right. \tag{11}$$

$$\left.\left. - \frac{1}{2}\left(\left(\sum_{k=1}^{N} (\overline{y_k} - h_k)^2\right) + \kappa\mu^2 + 2\beta\right)\right)\right) \tag{12}$$

Offset Integration

In total we have the integral

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)(2\pi)^{\frac{N}{2}}} \cdot \sqrt{\frac{\kappa}{N+\kappa}} \int_{\mathbb{R}_+} \lambda^{\alpha+\frac{N}{2}-1} \cdot e^{-\lambda \cdot C} \, d\lambda,$$

while C is a constant.

Final steps

Together with the substitution of $\varphi(\lambda) = C \cdot \lambda$ and the knowledge about the gamma-function we conclude with the following form:

$$\frac{(\beta/C)^{\alpha}}{\Gamma(\alpha)(2\pi C)^{\frac{N}{2}}} \cdot \sqrt{\frac{\kappa}{N+\kappa}} \cdot \Gamma\left(\frac{N}{2} + \alpha\right)$$

Numerical Testing

The relative error

Log-Normal Prior

Log-Normal likelihood

We make the assumption that $\varepsilon_k \sim e^{\sigma \cdot Z}$, i.e. it has a log-normal distribution. The new likelihood has the following form:

$$p(D \mid \theta, c, \sigma^2) = \prod_{k=1}^{N} \frac{1}{\overline{y}_k - c - h_k} \mathcal{N}\big(\ln\big(\overline{y}_k - c - h_k\big) \mid 0, \sigma^2\big) \tag{13}$$

$$= \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\Bigg(-\frac{1}{2\sigma^2}\big(\ln^2(y_k - c - h_k) \tag{14}$$

$$+ 2\sigma^2 \cdot \ln(y_k - c - h_k)\big)\Bigg) \tag{15}$$

Arising problem

Problem: We can't substitute here as every $\ln(\overline{y}_k - c - h_k)$ part has a different substraction and the logarithm can't be summed up.

Laplacian Prior