Bachelor Thesis Marginal-Sampling

Michael Fedders

17.09.2020

Table of Contents

1. Normal-Gamma Prior

2. Laplacian noise

Small repetition

We are dealing with the case

$$\overline{y}_k = c + (h(x(t_k, \theta)) + \varepsilon_k$$

where

- $\overline{y}_k$ is the data
- c is the offset parameter
- h is the observation function
- ε_k is the noise

Normal-Gamma Prior

Environment

In contrast to last time we now assume the two priors from our likelihood are now a joint probability distribution, the so called normal-gamma distribution.
The two parameters which we want to marginalize are the offset parameter c and the precision λ, which is the same as $1/\sigma^2$.

prior-definition

The normal-gamma prior depends on 4 shape parameters, $\mu, \kappa, \alpha, \beta$ and has the following structure:

$$
\begin{align}
p(c, \lambda) &= f(c, \lambda \mid \mu, \kappa, \alpha, \beta) \tag{1} \\
&= \mathcal{N}(c \mid \mu, 1/(\kappa\lambda)) \cdot \Gamma(\lambda \mid \alpha, \beta). \tag{2}
\end{align}
$$

likelihood

As a result the likelihood is

$$p(D \mid \theta, c, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{N/2} \cdot \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{N}(\overline{y}_k - (c + h_k))^2\right) \quad (3)$$

Calculation

We can start with the usual setup:

$$p(D \mid \theta) = \int_{\mathbb{R}\times\mathbb{R}_+} p(D \mid \theta, c, \lambda) p(c, \lambda) \, d(c, \lambda) \tag{4}$$

Calculation

We can start with the usual setup:

$$p(D \mid \theta) = \int_{\mathbb{R}\times\mathbb{R}_+} p(D \mid \theta, c, \lambda) p(c, \lambda) \, d(c, \lambda) \tag{5}$$

$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\frac{\lambda}{2\pi}\right)^{N/2} \exp\left(-\frac{\lambda}{2} \sum_{k=1}^{N} (\overline{y}_k - (c + h_k))^2\right) \cdot \tag{6}$$

$$\frac{\beta^\alpha \sqrt{\kappa}}{\Gamma(\alpha)\sqrt{2\pi}} \lambda^{\alpha-1/2} \exp\left(-\frac{\lambda}{2}[\kappa(c-\mu)^2 + 2\beta]\right) \, dc \, d\lambda \tag{7}$$

Offset Integration

We pull out the constants and terms which only rely on λ to integrate over c.

$$\int_{\mathbb{R}} \exp\left(-\frac{\lambda}{2}\left(\left(\sum_{k=1}^{N}((\overline{y_k} - h_k) - c)^2\right) + \kappa(c-\mu)^2 + 2\beta\right)\right)\, dc \tag{8}$$

$$= \int_{\mathbb{R}} \exp\left(-\frac{\lambda}{2}\left((N+\kappa)c^2 - 2\left(\left(\sum_{k=1}^{N}\overline{y_k} - h_k\right) + \kappa\mu\right)c \right.\right. \tag{9}$$

$$\left.\left. + \left(\sum_{k=1}^{N}(\overline{y_k} - h_k)^2\right) + \kappa\mu^2 + 2\beta\right)\right) dc \tag{10}$$

Offset Integration

Now we can use the exponential integration formula:

$$\int_{\mathbb{R}} \exp(-a \cdot c^2 + b \cdot c - d)\, dc = \sqrt{\frac{\pi}{a}} \cdot \exp\left(\frac{b^2}{4a} - d\right)$$

Offset Integration

Now we can use the exponential integration formula:

$$\int_{\mathbb{R}} \exp(-a \cdot c^2 + b \cdot c - d)\, dc = \sqrt{\frac{\pi}{a}} \cdot \exp\left(\frac{b^2}{4a} - d\right)$$

and receive

$$\sqrt{\frac{2\pi}{\lambda(N+\kappa)}} \cdot \exp\left(\lambda \cdot \left(\frac{1}{2(N+\kappa)}\left(\left(\sum_{k=1}^{N} \overline{y_k} - h_k\right) + \kappa\mu\right)^2\right.\right. \tag{11}$$

$$\left.\left.- \frac{1}{2}\left(\left(\sum_{k=1}^{N} (\overline{y_k} - h_k)^2\right) + \kappa\mu^2 + 2\beta\right)\right)\right) \tag{12}$$

Offset Integration

In total we have the integral

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)(2\pi)^{\frac{N}{2}}} \cdot \sqrt{\frac{\kappa}{N+\kappa}} \int_{\mathbb{R}_+} \lambda^{\alpha+\frac{N}{2}-1} \cdot e^{-\lambda \cdot C} \, d\lambda,$$

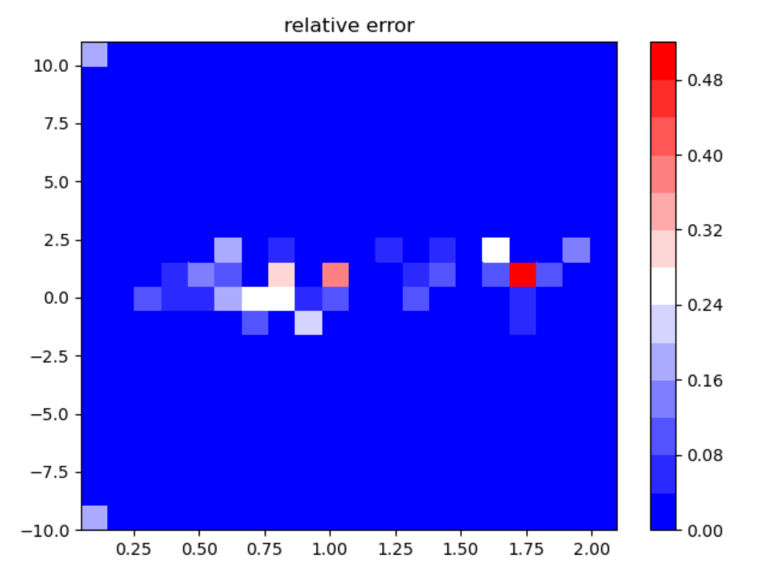
while C is a constant.

Final steps

Together with the substitution of $\varphi(\lambda) = C \cdot \lambda$ and the knowledge about the gamma-function we conclude with the following form:

$$\frac{(\beta/C)^{\alpha}}{\Gamma(\alpha)(2\pi C)^{\frac{N}{2}}} \cdot \sqrt{\frac{\kappa}{N+\kappa}} \cdot \Gamma\left(\frac{N}{2} + \alpha\right)$$

Numerical Testing



Laplacian noise

Laplacian likelihood

We make the assumption that $\epsilon_k \sim \mathit{Laplace}(0, \sigma), \sigma \in (0, \infty)$, i.e. it has a Laplace distribution. The new likelihood has the following form:

$$p(D \mid \theta, c, \sigma) = \prod_{k=1}^{N} \mathit{Laplace}(\overline{y}_k \mid c + h_k, \sigma) \tag{13}$$

$$= \prod_{k=1}^{N} \frac{1}{2\sigma} \cdot \exp\left\{-\frac{|\overline{y}_k - c - h_k|}{\sigma}\right\} \tag{14}$$

Marginalisation Integral

The integral we receive is

$$\iint p(D \mid \theta, c, \sigma) p(c) p(\sigma) \, \mathrm{d}c \, \mathrm{d}\sigma \tag{15}$$

$$= \int_0^\infty \int_{-\infty}^\infty \prod_{k=1}^N \frac{1}{2\sigma} \cdot \exp\left\{ -\frac{|c - (\overline{y}_k - h_k)|}{\sigma} \right\} p(c) p(\sigma) \, \mathrm{d}c \, \mathrm{d}\sigma \tag{16}$$

Calculation

For calculation-reasons we renumber $\overline{y}_k$ and h_k so that $y_k - h_k$ are ordered from smallest to biggest, i.e. $\overline{y}_1 - h_1$ is the smallest number, $\overline{y}_N - h_N$ the biggest. Then we choose $b_0 = -\infty, b_i = \overline{y} - h_i (i = 1, \ldots, N), b_{N+1} = \infty$. Now we can split up the integral in the following parts:

$$\int_0^\infty \sum_{i=0}^{N} \int_{b_i}^{b_{i+1}} \frac{1}{2\sigma} \exp\left\{ -\frac{\sum_{k=1}^{N} |c - (\overline{y}_k - h_k)|}{\sigma} \right\} p(c) p(\sigma) \, \mathrm{d}c \, \mathrm{d}\sigma \tag{17}$$

Calculation

To remove the absolute value, we introduce the index $R_{k,i}$ which is defined like this:

$$r_{k,i} = \begin{cases} 1 & \text{if } k \leq i \\ -1 & \text{else} \end{cases}$$

Calculation

$$\int_0^\infty \frac{1}{2\sigma} \sum_{i=0}^{N} p(\sigma) \int_{b_i}^{b_{i+1}} \exp \underbrace{\left\{ -\frac{\sum_{k=1}^{N} r_{k,i}(c - (\overline{y}_k - h_k))}{\sigma} \right\}}_{=(*)} \mathrm{d}c \, \mathrm{d}\sigma \tag{18}$$

$$\text{with } (*) = \frac{-c(i-(N-i)) + \sum_{k=1}^{i} \overline{y}_k - h_k - \sum_{i+1}^{N} \overline{y}_k - h_K}{\sigma} \tag{19}$$

$$= \int_0^\infty \frac{1}{2\sigma} p(\sigma) \sum_{i=0}^{N} \exp \left\{ \frac{\sum_{k=1}^{i} \overline{y}_k - h_k - \sum_{i+1}^{N} \overline{y}_k - h_K}{\sigma} \right\} \tag{20}$$

$$\cdot \int_{b_i}^{b_{i+1}} e^{-c(2i-N)} \, \mathrm{d}c \, \mathrm{d}\sigma \tag{21}$$

$$\int_0^\infty \frac{1}{2\sigma} \sum_{i=1}^{N-1} \exp\left\{ \frac{\sum_{k=1}^{i} \overline{y}_k - h_k - \sum_{i+1}^{N} \overline{y}_k - h_K}{\sigma} \right\} \frac{1}{N-2i} \tag{22}$$

$$\cdot \left(e^{-b_{i+1}(2i-N)} - e^{-b_i(2i-N)} \right) \mathrm{d}\sigma \tag{23}$$

$$+ \int_0^\infty \frac{1}{2\sigma} \exp\left\{ \frac{-\sum_{k=1}^{N} \overline{y}_k - h_k}{\sigma} \right\} \underbrace{\int_{-\infty}^{b_1} e^{Nc}\, \mathrm{d}c}_{\frac{1}{N} e^{Nb_1}}\, \mathrm{d}\sigma \tag{24}$$

$$+ \int_0^\infty \frac{1}{2\sigma} \exp\left\{ \frac{\sum_{k=1}^{N} \overline{y}_k - h_k}{\sigma} \right\} \underbrace{\int_{b_N}^{\infty} e^{-Nc}\, \mathrm{d}c}_{\frac{1}{N} e^{-Nb_N}}\, \mathrm{d}\sigma \tag{25}$$