Hackathon

Group 2

April 11, 2025

1 Model problem

We start with a four-dimensional data set for the electric field E and the magnetic field H. They are both complex data of dimension $8 \times 121 \times 111 \times 126$ and can be interpreted as two functions

$$E_i : \mathbb{R}^3 \to \mathbb{C}^3, \quad H_i : \mathbb{R}^3 \to \mathbb{C}^3 \quad \{i = 1, \ldots 8\}$$

which are only applied on a grid of grid-size 4mm. Especially the total field strength can be written as

$$E = \sum_{i=1}^{8} E_i, \quad H = \sum_{i=1}^{8} H_i$$

If we adapt the fields at each dipole through an amplitude A_i and phase φ we can write the custom fields as

$$\begin{aligned} \hat{H}_i &= A_i \cdot \exp(\mathrm{i} \cdot \varphi_i) \cdot H_i \\ \hat{E}_i &= A_i \cdot \exp(\mathrm{i} \cdot \varphi_i) \cdot E_i \end{aligned}$$

In the same way as before the total field strength of the adapted fields is then

$$\hat{E} = \sum_{i=1}^{8} \hat{E}_i, \quad \hat{H} = \sum_{i=1}^{8} \hat{H}_i$$

Now we can define the B_1^+-field as a function which takes a three-dimensional real input x:

$$\begin{aligned} B_1^+ &: \mathbb{R}^3 \to \mathbb{C} \\ &x \mapsto \mathrm{proj}_1(\hat{H}(x)) + \mathrm{i} \cdot \mathrm{proj}_2(\hat{H}(x)) \end{aligned}$$

Finally we define the Specific Absorption Rate (SAR) as

$$\mathrm{SAR} = \frac{|E(x)|^2 \cdot \sigma(x)}{\rho(x)}$$

Then we can define two cost functions as follows with D as the human area and $|D|$ as the amount of voxels of the human area. The first one calculates the B_1-homogeneity

$$\mathrm{cost}_1(\varphi, A) = \frac{\mathrm{mean}(|B_1^+|)}{\mathrm{std}(|(B_1^+)|}$$

and based upon that the seconds cost function incorporates additionally the SAR

$$\mathrm{cost}_2(\varphi, A) = \frac{\mathrm{mean}(|B_1^+|)}{\mathrm{std}(|(B_1^+)|} + \lambda \cdot \frac{\min(|B_1^+|)}{\sqrt{\max(\mathrm{SAR})}}$$

We start by writing the formula as a function of H_i.

$$\begin{aligned}
&\mathrm{mean}(|B_1^+|) = \frac{1}{|D|}\sum_{x\in D}|B_1^+(x)| \propto \sum_{x\in D}|B_1^+(x)| \\
&= \sum_{x\in D}\sqrt{\Re(B_1^+(x))^2 + \Im(B_1^+(x))^2} \\
&= \sum_{x\in D}\sqrt{\Re((\hat{H}(x))_1 + \mathrm{i}\cdot(\hat{H}(x))_2)^2 + \Im((\hat{H}(x))_1 + \mathrm{i}\cdot(\hat{H}(x))_2)^2} \\
&= \sum_{x\in D}\sqrt{\Re\left(\sum_{i=1}^{8}(\hat{H}_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(\hat{H}_i(x))_2\right)^2 + \Im\left(\sum_{i=1}^{8}(\hat{H}_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(\hat{H}_i(x))_2\right)^2} \\
&= \sum_{x\in D}\left(\Re\left(\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_2\right)^2\right. \\
&\quad \left. + \Im\left(\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_2\right)^2\right)^{1/2}
\end{aligned}$$

We can now write each element $(H_i(x))_j = (M_i^x)_j \cdot \exp(i\cdot(\phi_i^x)_j) \in \mathbb{C}, M_i^x \in [0,\infty)^3, \phi_i^x \in$

$[0, 2\pi)^3$ and insert it into our formula:

$$
\begin{aligned}
&= \sum_{x\in D} \Bigg(\Re\Bigg(\sum_{i=1}^{8} A_i (M_i^x)_1 \cdot \exp(\mathrm{i} \cdot (\varphi_i + (\phi_i^x)_1) + A_i (M_i^x)_2 \cdot \exp\Big(\mathrm{i} \cdot \Big(\varphi_i + (\phi_i^x)_2 + \frac{\pi}{2}\Big)\Big)\Bigg)^2 \\
&\quad + \Im\Bigg(\sum_{i=1}^{8} A_i (M_i^x)_1 \cdot \exp(\mathrm{i} \cdot (\varphi_i + (\phi_i^x)_1) + A_i (M_i^x)_2 \cdot \exp\Big(\mathrm{i} \cdot \Big(\varphi_i + (\phi_i^x)_2 + \frac{\pi}{2}\Big)\Big)\Bigg)^2 \Bigg)^{1/2} \\
&= \sum_{x\in D} \Bigg(\Bigg(\sum_{i=1}^{8} A_i ((M_i^x)_1 \cdot \cos(\varphi_i + (\phi_i^x)_1) - (M_i^x)_2 \cdot \sin(\varphi_i + (\phi_i^x)_2)) \Bigg)^2 \\
&\quad + \Bigg(\sum_{i=1}^{8} A_i ((M_i^x)_1 \cdot \sin(\varphi_i + (\phi_i^x)_1) + (M_i^x)_2 \cdot \cos(\varphi_i + (\phi_i^x)_2)) \Bigg)^2 \Bigg)^{1/2} \\
&= \sum_{x\in D} \Bigg(\Bigg(\sum_{i=1}^{8} \sum_{j=1}^{8} A_i A_j \cdot ((M_i^x)_1 \cdot \cos(\varphi_i + (\phi_i^x)_1) - (M_i^x)_2 \cdot \sin(\varphi_i + (\phi_i^x)_2)) \\
&\quad \cdot ((M_j^x)_1 \cdot \cos(\varphi_j + (\phi_j^x)_1) - (M_j^x)_2 \cdot \sin(\varphi_j + (\phi_j^x)_2)) \Bigg) \\
&+ \Bigg(\sum_{i=1}^{8} \sum_{j=1}^{8} A_i A_j \cdot ((M_i^x)_1 \cdot \sin(\varphi_i + (\phi_i^x)_1) + (M_i^x)_2 \cdot \cos(\varphi_i + (\phi_i^x)_2)) \\
&\quad \cdot ((M_j^x)_1 \cdot \sin(\varphi_j + (\phi_j^x)_1) + (M_j^x)_2 \cdot \cos(\varphi_j + (\phi_j^x)_2)) \Bigg) \Bigg)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{x\in D} \Bigg(\sum_{i=1}^{8} \sum_{j=1}^{8} A_i A_j \cdot \Big(&(M_i^x)_1 (M_j^x)_1 \cdot \ (\cos(\varphi_i + (\phi_i^x)_1) \cdot \cos(\varphi_j + (\phi_j^x)_1) \\
&\qquad + \sin(\varphi_i + (\phi_i^x)_1) \cdot \sin(\varphi_j + (\phi_j^x)_1) \\
&+ (M_i^x)_1 (M_j^x)_2 \cdot (\cos(\varphi_j + (\phi_j^x)_2) \cdot \sin(\varphi_i + (\phi_i^x)_1) \\
&\qquad - \cos(\varphi_i + (\phi_i^x)_1) \cdot \sin(\varphi_j + (\phi_j^x)_2) \\
&+ (M_i^x)_2 (M_j^x)_1 \cdot (\cos(\varphi_i + (\phi_i^x)_2) \cdot \sin(\varphi_j + (\phi_j^x)_1) \\
&\qquad - \cos(\varphi_j + (\phi_j^x)_1) \cdot \sin(\varphi_i + (\phi_i^x)_2)) \\
&+ (M_i^x)_2 (M_j^x)_2 \cdot (\cos(\varphi_i + (\phi_i^x)_2) \cdot \cos(\varphi_j + (\phi_j^x)_2) \\
&\qquad + \sin(\varphi_i + (\phi_i^x)_2) \cdot \sin(\varphi_j + (\phi_j^x)_2) \Big) \Bigg)^{1/2}
\end{aligned}
$$

Now we can use the trigonometric equations

$$
\begin{aligned}
\sin(x) \cdot \sin(y) &= \frac{1}{2} \cdot (\cos(x - y) - \cos(x + y)) \\
\cos(x) \cdot \cos(y) &= \frac{1}{2} \cdot (\cos(x - y) + \cos(x + y)) \\
\sin(x) \cdot \cos(y) &= \frac{1}{2} \cdot (\sin(x - y) + \sin(x + y))
\end{aligned}
$$

to further simplify our equation:

$$
\begin{aligned}
= \sum_{x\in D}\Bigg(\sum_{i=1}^{8}\sum_{j=1}^{8} A_i A_j \cdot \Big(& (M_i^x)_1 (M_j^x)_1 \cdot \cos((\varphi_i - \varphi_j) + ((\phi_i^x)_1 - (\phi_j^x)_1)) \\
& + (M_i^x)_1 (M_j^x)_2 \cdot \sin((\varphi_i - \varphi_j) + ((\phi_i^x)_1 - (\phi_j^x)_2)) \\
& + (M_j^x)_1 (M_i^x)_2 \cdot \sin((\varphi_j - \varphi_i) + ((\phi_j^x)_1 - (\phi_i^x)_2)) \\
& + (M_i^x)_2 (M_j^x)_2 \cdot \cos((\varphi_i - \varphi_j) + ((\phi_i^x)_2 - (\phi_j^x)_2))\Big)\Bigg)^{1/2}
\end{aligned}
$$