# Hackathon

Group 2

11th April 2025

## 1 Model problem

We start with a four-dimensional data set for the electric field $E$ and the magnetic field $H$. They are both complex data of dimension $8 \times 121 \times 111 \times 126$ and can be interpreted as two functions

$$E_i : \mathbb{R}^3 \to \mathbb{C}^3, \quad H_i : \mathbb{R}^3 \to \mathbb{C}^3 \quad \{i = 1, \dots 8\}$$

which are only applied on a grid of grid-size 4mm. Especially the total field strength can be written as

$$E = \sum_{i=1}^{8} E_i, \quad H = \sum_{i=1}^{8} H_i$$

If we adapt the fields at each dipole through an amplitude $A_i$ and phase $\varphi$ we can write the custom fields as

$$\begin{aligned} \hat{H}_i &= A_i \cdot \exp(\mathrm{i} \cdot \varphi_i) \cdot H_i \\ \hat{E}_i &= A_i \cdot \exp(\mathrm{i} \cdot \varphi_i) \cdot E_i \end{aligned}$$

In the same way as before the total field strength of the adapted fields is then

$$\hat{E} = \sum_{i=1}^{8} \hat{E}_i, \quad \hat{H} = \sum_{i=1}^{8} \hat{H}_i$$

Now we can define the $B_1^+$-field as a function which takes a three-dimensional real input $x$:

$$\begin{aligned} B_1^+ &: \mathbb{R}^3 \to \mathbb{C} \\ & x \mapsto \mathrm{proj}_1(\hat{H}(x)) + \mathrm{i} \cdot \mathrm{proj}_2(\hat{H}(x)) \end{aligned}$$

Finally we define the Specific Absorption Rate (SAR) as

$$\mathrm{SAR} = \frac{|E(x)|^2 \cdot \sigma(x)}{\rho(x)}$$

Then we can define two cost functions as follows with $D$ as the human area and $|D|$ as the amount of voxels of the human area. The first one calculates the $B_1$-homogeneity

$$\mathrm{cost}_1(\varphi, A) = \frac{\mathrm{mean}(|B_1^+|)}{\mathrm{std}(|(B_1^+)|}$$

and based upon that the seconds cost function incorporates additionally the SAR

$$\mathrm{cost}_2(\varphi, A) = \frac{\mathrm{mean}(|B_1^+|)}{\mathrm{std}(|(B_1^+)|} + \lambda \cdot \frac{\min(|B_1^+|)}{\sqrt{\max(\mathrm{SAR})}}$$

## 2 Simplifying cost function 1

We start by writing the formula as a function of $H_i$.

$$
\begin{aligned}
&\text{mean}(|B_1^+|) = \frac{1}{|D|}\sum_{x\in D}|B_1^+(x)| \propto \sum_{x\in D}|B_1^+(x)| \\
&= \sum_{x\in D}\sqrt{\Re(B_1^+(x))^2 + \Im(B_1^+(x))^2} \\
&= \sum_{x\in D}\sqrt{\Re((\hat{H}(x))_1 + \mathrm{i}\cdot(\hat{H}(x))_2)^2 + \Im((\hat{H}(x))_1 + \mathrm{i}\cdot(\hat{H}(x))_2)^2} \\
&= \sum_{x\in D}\sqrt{\Re\left(\sum_{i=1}^{8}(\hat{H}_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(\hat{H}_i(x))_2\right)^2 + \Im\left(\sum_{i=1}^{8}(\hat{H}_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(\hat{H}_i(x))_2\right)^2} \\
&= \sum_{x\in D}\left(\Re\left(\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_2\right)^2\right. \\
&\quad \left. + \Im\left(\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_1 + \mathrm{i}\cdot\sum_{i=1}^{8}(A_i\cdot\exp(\mathrm{i}\cdot\varphi_i)\cdot(H_i(x))_2\right)^2\right)^{1/2}
\end{aligned}
$$

We can now write each element $(H_i(x))_j = (M_i^x)_j \cdot \exp(i\cdot(\phi_i^x)_j) \in \mathbb{C}, M_i^x \in [0,\infty)^3, \phi_i^x \in [0,2\pi)^3$ and insert it into our formula:

$$
\begin{aligned}
&= \sum_{x\in D}\left(\Re\left(\sum_{i=1}^{8}A_i(M_i^x)_1\cdot\exp(\mathrm{i}\cdot(\varphi_i + (\phi_i^x)_1) + A_i(M_i^x)_2\cdot\exp\Big(\mathrm{i}\cdot\Big(\varphi_i + (\phi_i^x)_2 + \frac{\pi}{2}\Big)\Big)\right)^2\right. \\
&\quad \left. + \Im\left(\sum_{i=1}^{8}A_i(M_i^x)_1\cdot\exp(\mathrm{i}\cdot(\varphi_i + (\phi_i^x)_1) + A_i(M_i^x)_2\cdot\exp\Big(\mathrm{i}\cdot\Big(\varphi_i + (\phi_i^x)_2 + \frac{\pi}{2}\Big)\Big)\right)^2\right)^{1/2} \\
&= \sum_{x\in D}\left(\left(\sum_{i=1}^{8}A_i((M_i^x)_1\cdot\cos(\varphi_i + (\phi_i^x)_1) - (M_i^x)_2\cdot\sin(\varphi_i + (\phi_i^x)_2))\right)^2\right. \\
&\quad \left. + \left(\sum_{i=1}^{8}A_i((M_i^x)_1\cdot\sin(\varphi_i + (\phi_i^x)_1) + (M_i^x)_2\cdot\cos(\varphi_i + (\phi_i^x)_2))\right)^2\right)^{1/2} \\
&= \sum_{x\in D}\left(\left(\sum_{i=1}^{8}\sum_{j=1}^{8}A_iA_j\cdot((M_i^x)_1\cdot\cos(\varphi_i + (\phi_i^x)_1) - (M_i^x)_2\cdot\sin(\varphi_i + (\phi_i^x)_2))\right.\right. \\
&\quad \left.\cdot((M_j^x)_1\cdot\cos(\varphi_j + (\phi_j^x)_1) - (M_j^x)_2\cdot\sin(\varphi_j + (\phi_j^x)_2))\right) \\
&+ \left(\sum_{i=1}^{8}\sum_{j=1}^{8}A_iA_j\cdot((M_i^x)_1\cdot\sin(\varphi_i + (\phi_i^x)_1) + (M_i^x)_2\cdot\cos(\varphi_i + (\phi_i^x)_2))\right. \\
&\quad \left.\left.\cdot((M_j^x)_1\cdot\sin(\varphi_j + (\phi_j^x)_1) + (M_j^x)_2\cdot\cos(\varphi_j + (\phi_j^x)_2))\right)\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{x\in D}\Bigg(\sum_{i=1}^{8}\sum_{j=1}^{8} A_iA_j\cdot\Big(&(M_i^x)_1(M_j^x)_1\cdot\ \ (\cos(\varphi_i+(\phi_i^x)_1)\cdot\cos(\varphi_j+(\phi_j^x)_1)\\
&\qquad\qquad + \sin(\varphi_i+(\phi_i^x)_1)\cdot\sin(\varphi_j+(\phi_j^x)_1)\\
&+ (M_i^x)_1(M_j^x)_2\cdot(\cos(\varphi_j+(\phi_j^x)_2)\cdot\sin(\varphi_i+(\phi_i^x)_1)\\
&\qquad\qquad - \cos(\varphi_i+(\phi_i^x)_1)\cdot\sin(\varphi_j+(\phi_j^x)_2)\\
&+ (M_i^x)_2(M_j^x)_1\cdot(\cos(\varphi_i+(\phi_i^x)_2)\cdot\sin(\varphi_j+(\phi_j^x)_1)\\
&\qquad\qquad - \cos(\varphi_j+(\phi_j^x)_1)\cdot\sin(\varphi_i+(\phi_i^x)_2))\\
&+ (M_i^x)_2(M_j^x)_2\cdot(\cos(\varphi_i+(\phi_i^x)_2)\cdot\cos(\varphi_j+(\phi_j^x)_2)\\
&\qquad\qquad + \sin(\varphi_i+(\phi_i^x)_2)\cdot\sin(\varphi_j+(\phi_j^x)_2)\Big)\Bigg)^{1/2}
\end{aligned}
$$

Now we can use the trigonometric equations

$$
\begin{aligned}
\sin(x)\cdot\sin(y) &= \frac{1}{2}\cdot(\cos(x-y)-\cos(x+y))\\
\cos(x)\cdot\cos(y) &= \frac{1}{2}\cdot(\cos(x-y)+\cos(x+y))\\
\sin(x)\cdot\cos(y) &= \frac{1}{2}\cdot(\sin(x-y)+\sin(x+y))
\end{aligned}
$$

to further simplify our equation:

$$
\begin{aligned}
= \sum_{x\in D}\Bigg(\sum_{i=1}^{8}\sum_{j=1}^{8} A_iA_j\cdot\Big(&(M_i^x)_1(M_j^x)_1\cdot\cos((\varphi_i-\varphi_j)+((\phi_i^x)_1-(\phi_j^x)_1))\\
&+ (M_i^x)_1(M_j^x)_2\cdot\sin((\varphi_i-\varphi_j)+((\phi_i^x)_1-(\phi_j^x)_2))\\
&+ (M_j^x)_1(M_i^x)_2\cdot\sin((\varphi_j-\varphi_i)+((\phi_j^x)_1-(\phi_i^x)_2))\\
&+ (M_i^x)_2(M_j^x)_2\cdot\cos((\varphi_i-\varphi_j)+((\phi_i^x)_2-(\phi_j^x)_2))\Big)\Bigg)^{1/2}
\end{aligned}
$$

In particular, this formulation allows us to write $|B_1^+|$ as the sum of all elements of a symmetric matrix. Indeed, assume that we define the matrix $F$ as

$$
\begin{aligned}
F_{i,j} =A_iA_j\cdot\Big(&(M_i^x)_1(M_j^x)_1\cdot\cos((\varphi_i-\varphi_j)+((\phi_i^x)_1-(\phi_j^x)_1))\\
&+ (M_i^x)_1(M_j^x)_2\cdot\sin((\varphi_i-\varphi_j)+((\phi_i^x)_1-(\phi_j^x)_2))\\
&+ (M_j^x)_1(M_i^x)_2\cdot\sin((\varphi_j-\varphi_i)+((\phi_j^x)_1-(\phi_i^x)_2))\\
&+ (M_i^x)_2(M_j^x)_2\cdot\cos((\varphi_i-\varphi_j)+((\phi_i^x)_2-(\phi_j^x)_2))\Big)
\end{aligned}
$$

Then we see that

1. $A_iA_j = A_jA_i$
2. 

$$
\begin{aligned}
&(M_i^x)_1(M_j^x)_1\cdot\cos((\varphi_i-\varphi_j)+((\phi_i^x)_1-(\phi_j^x)_1))\\
=&(M_j^x)_1(M_i^x)_1\cdot\cos(-((\varphi_j-\varphi_i)+((\phi_j^x)_1-(\phi_i^x)_1)))\\
=&(M_j^x)_1(M_i^x)_1\cdot\cos((\varphi_j-\varphi_i)+((\phi_j^x)_1-(\phi_i^x)_1))
\end{aligned}
$$

3.

$$
\begin{aligned}
&(M_i^x)_2(M_j^x)_2 \cdot \cos((\varphi_i - \varphi_j) + ((\phi_i^x)_2 - (\phi_j^x)_2)) \\
=&(M_j^x)_2(M_i^x)_2 \cdot \cos(-((\varphi_j - \varphi_i) + ((\phi_j^x)_2 - (\phi_i^x)_2))) \\
=&(M_j^x)_2(M_i^x)_2 \cdot \cos((\varphi_j - \varphi_i) + ((\phi_j^x)_2 - (\phi_i^x)_2))
\end{aligned}
$$

4.

$$
\begin{aligned}
&(M_i^x)_1(M_j^x)_2 \cdot \sin((\varphi_i - \varphi_j) + ((\phi_i^x)_1 - (\phi_j^x)_2)) \\
&+ (M_j^x)_1(M_i^x)_2 \cdot \sin((\varphi_j - \varphi_i) + ((\phi_j^x)_1 - (\phi_i^x)_2)) \\
=&(M_j^x)_1(M_i^x)_2 \cdot \sin((\varphi_j - \varphi_i) + ((\phi_j^x)_1 - (\phi_i^x)_2)) \\
&+ (M_i^x)_1(M_j^x)_2 \cdot \sin((\varphi_i - \varphi_j) + ((\phi_i^x)_1 - (\phi_j^x)_2))
\end{aligned}
$$

which exactly shows the symmetry. Consider now the standard deviation.

$$
\begin{aligned}
&\mathrm{std}(|B_1^+|) \\
=&\sqrt{\frac{1}{|D|} \cdot \sum_{x \in D} (|B_1^+(x)| - \mathrm{mean}(|B_1^+|))^2} \\
=&\frac{1}{\sqrt{|D|}} \cdot \sqrt{\sum_{x \in D} (|B_1^+(x)| - \mathrm{mean}(|B_1^+|))^2}
\end{aligned}
$$

This gives rise to an algorithm to calculate $\mathrm{cost}_1$:

1. Calculate all different combinations of $A_i \cdot A_j$ which corresponds to 36 multiplications
2. Calculate all different combinations of $\varphi_i - \varphi_j$ which corresponds to 28 calculations (trace is 0)
3. Calculate all different combinations of differences $(\phi_i^x)_1 - (\phi_j^x)_1, (\phi_i^x)_1 - (\phi_j^x)_2$ and $(\phi_i^x)_2 - (\phi_j^x)_2$ which corresponds to $28 + 64 + 28$ additions (again 0-trace for $(\phi_i^x)_1$ and $(\phi_i^x)_2$.
4. Calculate all different combinations of multiplications $(M_i^x)_1 \cdot (M_j^x)_1, (M_i^x)_1 \cdot (M_j^x)_2$ and $(M_j^x)_2 \cdot (M_j^x)_2$ which correspond to $36 + 64 + 36$ multiplications.