Taller de Aprendizaje por Refuerzo: notes útiles sobre probabilidad*

April 2025

1 Conditional probabilities

The basic definition of the conditional probability of two events $\{S = s\}$ and $\{A = a\}$ is

$$P(\{S = s\}|\{A = a\}) = \frac{P(\{S = s\} \cap \{A = a\})}{P(\{A = a\})}$$

where the intersection means that $\{S = s\}$ and $\{A = a\}$ both happen. Even if probabilities are formally defined over sets, we often simplify notation

$$P(S = s|A = a) = \frac{P(S = s, A = a)}{P(A = a)} \quad (1)$$

To gain intuition, let us consider S and A to be states and actions. More specifically, the state $S \in \{\text{sick}, \text{healty}\}$ is a variable indicating if I am sick of a respiratory disease, and my action $A \in \{\text{masked}, \text{unmasked}\}$ indicate if I wore a mask or not.

Then $P(S = \text{sick}|A = \text{masked})$ is the probability that I am sick given that I wore the mask.

Intuitively, this probability could be estimated as the proportion of all individuals who became sick while using a mask

$$P(S = \text{sick}|A = \text{masked}) \simeq N(\text{masked,sick})/N(\text{masked}),$$

where $N(\text{masked,sick})$ represents the number of individuals wearing a mask and being sick, and $N(\text{masked})$ all those that wore a mask.

Then we can multiply and divide by the total population N to get

$$P(S = \text{sick}|A = \text{masked}) \simeq \left(\frac{N(\text{masked,sick})}{N}\right)\left(\frac{N}{N(\text{masked})}\right) \simeq P(S = \text{sick}, A = \text{masked})\frac{1}{P(A = \text{masked})}$$

*Estas notas son tomadas de las notas del curso ECE 2975 de la Universidad de Pittsburg, por Juan Bazerque

Again, this is just to gain intuition on the formula in (1).

2 Products of probabilities and independence

It is immediate from (1) that

$$P(S = s, A = a) = P(S = s|A = a)P(A = a). \tag{2}$$

Indeed, (2) and (1) are almost equivalent, just be aware of $P(A = a) = 0$ if you want to go from (2) to (1).

Now, (2) will be very useful for us in ECE2795. For instance, we read

$$P(S = \text{sick}, A = \text{masked}) = P(S = \text{sick}|A = \text{mask})P(A = \text{mask}). \tag{3}$$

as the following statement: *The probability of both being sick and wearing a mask equals the probability of wearing a mask times the probability of being sick if I wear a mask.*

I would say this is not true, but if we model the propagation of respiratory diseases such that the use of a mask does not change the probability of being sick, then

$$P(S = \text{sick}|A = \text{masked}) = P(S = \text{sick}),$$

and according to (3)

$$P(S = \text{sick}, A = \text{masked}) = P(S = \text{sick})P(A = \text{masked}).$$

Then we say that being sick and wearing a mask are independent.

We do not see many independence assumptions in ECE2975. Indeed we want the conditioning variable altering the probability of the conditioned variable, that is, $P(S = \text{sick}|A = \text{masked}) \neq P(S = \text{sick})$ because then observing one variable we obtain information about the other.

3 Law of total probability

An equality that we are going to use frequently is

$$P(S = s) = \sum_{a \in \mathcal{A}} P(S = s|A = a)P(A = a) \tag{4}$$

which is true if we use all possible values of a in the sum.

For instance

$$\begin{aligned} P(S = \text{sick}) = {} & P(S = \text{sick}|A = \text{masked})P(A = \text{mask}) && (5)\\ & + P(S = \text{sick}|A = \text{unmasked})P(A = \text{unmasked}) && (6) \end{aligned}$$

This is true because (proof)

$$\begin{aligned}
\sum_{a \in \mathcal{A}} P(S = s | A = a) P(A = a) &= \sum_{a \in \mathcal{A}} P(S = s, A = a) && (7)\\
&= \sum_{a \in \mathcal{A}} P(S = s, A = a) && (8)\\
&= \sum_{a \in \mathcal{A}} P(S = s, A = a) && (9)\\
&= P(\cup_{a \in \mathcal{A}} \{A = a\} \cap \{S = s\}) \quad = P(\Omega \cap \{S = s\}) && (10)\\
&= P(\{S = s\}) && (11)
\end{aligned}$$

Under our working example we read $P(\cup_{a \in \mathcal{A}} \{A = a\} \cap \{S = s\})$ as the proportion of all people that is sick and it is either masked or unmasked. So it is intuitive that the intersection is equal to is $P(\{S = \text{sick}\})$ [1]

[1] Ω represents the set of all people, so that $\Omega \cap \{S = \text{sick}\}$ is just the people that is sick.

4 Conditional probability laws

Notice that the conditional probabilities form themselves a probability law in terms of S, that is, $P(S = s|A = a) \geq 0$ for all s and $\sum_{s \in \mathcal{S}} P(S = s_i|A = a) = 1$ [2]

Now this is true if I consider a third variable, for instance if I got vaccinated. Consider $V \in \{\text{vaccinated, unvaccinated}\}$.

In this case $P(S = \text{sick}|V = \text{vaccinated}) + P(S = \text{healty}|V = \text{vaccinated}) = 1$. Therefore I can define $\bar{P}(S = s) = P(S = \text{healty}|V = \text{vaccinated})$ and $\bar{P}(S = s)$ is a probability law in terms of s.

Similarly, $\bar{P}(A = a) := P(A = a|V = \text{vaccinated})$ is also a probability law.

But if all of the identities above are valid for any probability laws $P(S = s)$ and $P(A = a)$ then I can use $\bar{P}(S = s)$ $\bar{P}(A = a)$ instead. That means that all the identities in the previous sections are true if I substitute $P(S = s|V = \text{vaccinated})$ and $P(A = a|V = \text{vaccinated})$ for $P(S = s)$ and $P(A = a)$, in particular

$$P(S = s, A = a|V = \text{vaccinated}) = P(S = s|A = a, V = \text{vaccinated})P(A = a|V = \text{vaccinated})$$

and

$$P(S = s|V = \text{vaccinated}) = \sum a \in \mathcal{A} P(S = s|A = a, V = \text{vaccinated})P(A = a|V = \text{vaccinated})$$

5 Expectation

The expectation of the variable S is defined as

$$E[S] = \sum_{s \in \mathcal{S}} sP(S = s)$$

Notice that for this definition, s must be a number. For instance s could be coded as sick $\rightarrow s = 1$ and healthy $\rightarrow s = 0$. Formally, if s is not a number that we define a map $X : \mathcal{S} \rightarrow R$[3] and we name $X(s)$ a random variable, but it may be easier to start with s as a number and identifying $X(s) = s$.

Then $E[s]$ is the sum of all possible values of s that can occur weighted by the probability that they occur. If we think the probabilities as proportions, then we can approximate

$$E[S] \simeq \sum_{s \in \mathcal{S}} s \frac{N(s)}{N}$$

and the s becomes weighted by their frequency in which they occur.

[2] if we consider all possible values of S in the sum.

[3] More in general $X : \mathcal{S} \rightarrow C^N$

We can also define conditional expectations. Again, since $P(S = s|A = a)$ and $P(S = s|V = v)$ are probability laws with respect to s, then we can define

$$E[S|A = a] = \sum_{s \in \mathcal{S}} P(S = s|A = a)s$$

And again we compute the weighted average of s but this time we only consider the $N(A)$ cases in which $A = a$ occurs as our universe instead of considering all N cases.

6 Tower property of expectations

The law of total probability (**??**) leads to a corresponding identity involving expectations.

If we have two variables G and S'

$$E[G] = E_{S'}\Big[E_G[G|S']\Big] \tag{12}$$

This means computing the expectation $E[G]$ in two steps

First: $E_G[G|S'] = \int_g gp(g|S')dg = g(S')$

Then: $E_{S'}[g(S')] = \int_s g(s')ds'$

Notice that in the first step we condition on S', therefore, for each value of S' we obtain a different average that depends on S', which we called $g(S')$.

Intuitively, when we compute $E[G]$ we are averaging G with respect to all sources of randomness, including the randomness of all possible variables that may affect G. That general randomness is described by $p(g)$. The tower property suggest that we fix one of these variables S' affecting G and take the average with respect to all other sources randomness except S', that is $E_G[G|S']$. Then in a second step we average the result with respect to the previously omitted S'. After this second step all sources of randomness have been considered in the average, so we recover $E[G]$.

The formal proof of this property is a simple application of (**??**)

$$
\begin{aligned}
E_{S'}[g(S')] &= \int_{s'} g(s')p(s')ds' && (13)\\
&= \int_{s'} \left(\int_g gp(g|s')dg \right) p(s')ds' && (14)\\
&= \int_g g \left(\int_{s'} p(g|s')p(s')ds' \right) dg && (15)\\
&= \int_g gp(g)dg \qquad\qquad = E[G] && (16)
\end{aligned}
$$

If there is a third variable S involved in the randomness of S' and G, we may want to repeat all this but fixing S and keeping all expectations and probabilities conditioned on S.

$$E[G|S] = E_{S'}\left[\ E_G\left[G|S', S\right] \ \middle| S \right] \tag{17}$$

Now the two steps are

First: $E_G[G|S', S] = \int_g gp(g|S', S)dg = g(S', S)$

Then: $E_{S'}[g(S', S)|S] = \int_s p(s'|S)g(s', S)ds'$

The proof becomes

$$
\begin{aligned}
E_{S'}[g(S', S)|S] &= \int_{s'} g(s', S)p(s'|S)ds' && (18)\\
&= \int_{s'} \left(\int_g gp(g|s', S)dg \right) p(s'|S)ds' && (19)\\
&= \int_g g \left(\int_{s'} p(g|s', S)p(s'|S)ds' \right) dg && (20)\\
&= \int_g gp(g|S)dg && (21)\\
&= E[G|S] && (22)
\end{aligned}
$$