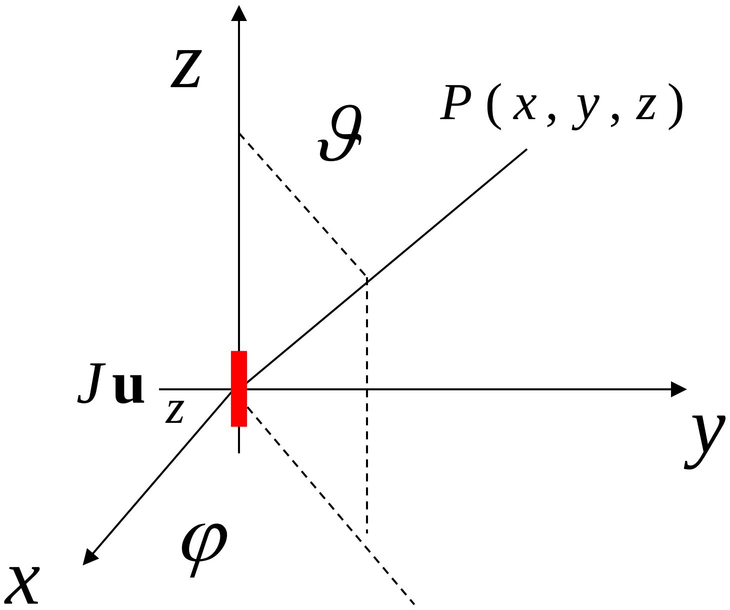
Campo di un dipolo elementare nell'origine

$$\nabla \times \hat{\mathbf{B}} = \mu \hat{\mathbf{J}} + j\omega\varepsilon\mu \; \hat{\mathbf{E}}$$

$$\nabla \times \hat{\mathbf{E}} = -j\omega\mu \; \hat{\mathbf{H}}$$



$$\nabla \cdot \hat{\mathbf{B}} = 0 \Rightarrow \hat{\mathbf{B}} = \nabla \times \hat{\mathbf{A}}$$

$$\nabla \times \hat{\mathbf{E}} = -j\omega \nabla \times \hat{\mathbf{A}} \Rightarrow \nabla \times \left(\hat{\mathbf{E}} + j\omega \hat{\mathbf{A}} \right) = 0$$

$$\mathbf{E} + j\omega \mathbf{A} = \nabla \phi$$

$$\nabla \times \nabla \times \hat{\mathbf{A}} = \mu \hat{\mathbf{J}} + j\omega\varepsilon\mu \left(\nabla \hat{\phi} - j\omega \hat{\mathbf{A}} \right) =$$

$$= \nabla\nabla \cdot \hat{\mathbf{A}} - \nabla^2 \hat{\mathbf{A}} = \mu \hat{\mathbf{J}} + j\omega\varepsilon\mu \nabla \hat{\phi} + k^2 \hat{\mathbf{A}}$$

$$\nabla \cdot \hat{\mathbf{A}} = j\omega\varepsilon\mu \hat{\phi}$$ Scelta di Lorentz

$$\nabla^2 \hat{\mathbf{A}} + k^2 \hat{\mathbf{A}} = -\mu \hat{\mathbf{J}}$$

$$\hat{\mathbf{J}} = \frac{I_0}{S} f(z) g(x,y) \mathbf{u}_z \qquad f(z) = \begin{cases} 1 & |z| < \dfrac{h}{2} \\ 0 & \text{altrove} \end{cases} \qquad g(x,y) = \begin{cases} 1 & \sqrt{x^2 + y^2} \leq R \\ 0 & \sqrt{x^2 + y^2} > R \end{cases}$$

Essendo R il raggio (piccolo) del filo

$$\hat{J}_x = \hat{J}_y = 0 \Rightarrow \hat{A}_x = \hat{A}_y = 0$$

Simmetria sferica $\quad \partial_\varphi = \partial_\vartheta = 0$

La proiezione dell'equazione di Helmoltz in *z* diventa:

$$\frac{1}{r^2} \partial_r (r^2 \partial_r) \hat{A}_z + k^2 \hat{A}_z = -\mu \frac{I_0 f(z)}{S} g(x,y)$$

Una possibile soluzione… $\quad \hat{A}_z = C \dfrac{e^{-jkr}}{r}$

Nel dominio del tempo…..

$$C \in \Re$$

$$A_z(r,t) = C\ \mathrm{Re}\left(\frac{e^{-jkr}}{r} e^{j\omega t}\right) = \frac{C}{r}\cos(\omega t - kr)$$

La velocità di propagazione dell'onda si ottiene derivando l'argomento della funzione trigonometrica rispetto al tempo

$$\frac{d}{dt}(\omega t - kr) = \omega - k\frac{dr}{dt} = 0 \Rightarrow v = \frac{dr}{dt} = \omega / k$$

$$\partial_r \hat{A}_z = C\left(-jk\frac{e^{-jkr}}{r} - \frac{e^{-jkr}}{r^2} \right)$$

$$r^2 \partial_r \hat{A}_z = -Ce^{-jkr}\left(jkr + 1\right)$$

$$\partial_r \left(r^2 \partial_r \hat{A}_z\right) = -jk\left(-Ce^{-jkr}\left(jkr + 1\right)\right) - jkCe^{-jkr} =$$

$$= -k^2 rCe^{-jkr} \quad \boxed{\Rightarrow \frac{1}{r^2}\partial_r \left(r^2 \partial_r \hat{A}_z\right) = -k^2 \frac{Ce^{-jkr}}{r}}$$

Infatti, l'equazione omogenea

$$\frac{1}{r^2}\partial_r(r^2\partial_r)\hat{A}_z + k^2\hat{A}_z = 0 = \frac{1}{r}\partial_r^2(r\hat{A}_z) + k^2\hat{A}_z$$

essendo

$$\partial_r\left(r^2\partial_r\hat{A}_z\right) = 2r\partial_r\hat{A}_z + r^2\partial_r^2\hat{A}_z = r\partial_r^2(r\hat{A}_z)$$

E, quindi

$$\partial_r^2(r\hat{A}_z) + k^2(r\hat{A}_z) = 0$$

Cioè rA_z soddisfa l'equazione dell'onda 'piana'

$$\hat{A}_z = C\,\frac{e^{-jkr}}{r}$$

$$\int_{-l/2}^{+l/2} dl \int_S \mathbf{J} \cdot d\mathbf{S} = \int_{-l/2}^{+l/2} dl \int_S \frac{I_0}{S} \mathbf{u}_z \cdot \mathbf{u}_z d\mathbf{S} = \int_{-h/2}^{+h/2} dl\, I_0 = I_0 h$$

D'altra parte, nell'ipotesi che V sia piccolo:

$$\int_V \left(\nabla \cdot \nabla \hat{A}_z + k^2 \hat{A}_z \right) dV = \int_{\partial V} \nabla \hat{A}_z \cdot d\mathbf{S} = -\mu I_0 h$$

poiché $\qquad dV = r^2 \sin \vartheta \; dr \; d\vartheta \; d\varphi$

E, conseguentemente, $\int_V k^2 \hat{A}_z dV \rightarrow 0$ quando $\quad V \rightarrow 0$

$$\nabla \hat{A}_z = \partial_r \frac{Ce^{-jkr}}{r} \mathbf{u}_r = \frac{C\left(-jke^{-jkr} - 1/re^{-jkr}\right)}{r} \mathbf{u}_r$$

$$d\mathbf{S} = r^2 \sin\vartheta \, d\vartheta \, d\varphi \, \mathbf{u}_r$$

$$r \to 0$$

$$\nabla \hat{A}_z \cdot d\mathbf{S} = \frac{C\left(-jke^{-jkr} - 1/re^{-jkr}\right)}{r} r^2 \sin\vartheta \, d\vartheta \, d\varphi =$$

$$= -C \sin\vartheta \, d\vartheta \, d\varphi$$

$$\int_{\partial V} \nabla \hat{A}_z \cdot d\mathbf{S} == \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \left(-C \sin \vartheta\right) = -4\pi C$$

$$4\pi C = \mu I_0 h \Rightarrow \qquad \boxed{C = \frac{\mu I_0 h}{4\pi}}$$

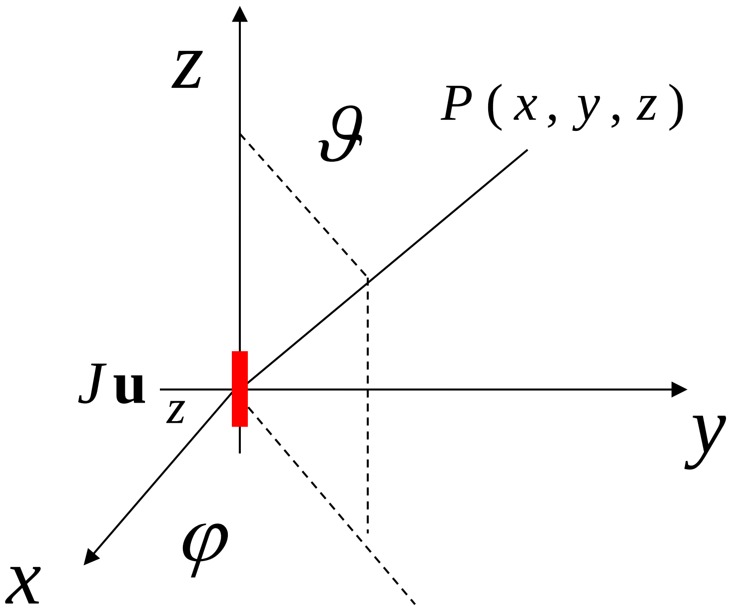
Espressione del potenziale

$$\hat{A}_z(k,r) = \frac{\mu I_0 h}{4\pi}\frac{e^{-jkr}}{r}$$

Nel dominio del tempo:

$$A_z(r,t) = \mathrm{Re}\left(\frac{\mu I_0 h}{4\pi}\frac{e^{-jkr}e^{j\omega t}}{r}\right) = \frac{\mu I_0 h}{4\pi}\frac{\cos(\omega t - kr)}{r}$$

Le componenti del potenziale in coordinate sferiche valgono allora,



$$\hat{A}_r = \hat{A}_z \cos\vartheta$$

$$\hat{A}_\vartheta = -\hat{A}_z \sin\vartheta$$

$$\hat{A}_\varphi = 0$$

$$\hat{\mathbf{B}} = \nabla \times (A_r \mathbf{u}_r + A_\vartheta \mathbf{u}_\vartheta)$$

Espressione del campo magnetico

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r^2 \sin\vartheta}\mathbf{u}_r & \frac{1}{r\sin\vartheta}\mathbf{u}_\vartheta & \frac{1}{r}\mathbf{u}_\varphi \\ \partial_r & \partial_\vartheta & \partial_\varphi \\ A_r & rA_\vartheta & r\sin\vartheta A_\varphi \end{vmatrix} =$$

$$= \frac{1}{r^2 \sin\vartheta}\mathbf{u}_r\left(-\partial_\varphi rA_\vartheta\right) + \frac{1}{r\sin\vartheta}\mathbf{u}_\vartheta \partial_\varphi A_r + \frac{1}{r}\mathbf{u}_\varphi\left(\partial_r\left(rA_\vartheta\right) - \partial_\vartheta A_r\right)$$

$$\hat{A}_z(k,r) = \frac{\mu I_0 h}{4\pi}\frac{e^{-jkr}}{r}$$

$$B_r = \frac{1}{r^2 \sin\vartheta}(-\partial_\varphi r(-\hat{A}_z \sin\vartheta)) = 0$$

$$B_\vartheta = \frac{1}{r\sin\vartheta}\partial_\varphi \hat{A}_z \cos\vartheta = 0$$

$$B_\varphi = \frac{1}{r}\left(\partial_r\left(r(-\hat{A}_z \sin\vartheta)\right) - \partial_\vartheta(\hat{A}_z \cos\vartheta)\right) =$$

$$= \frac{1}{r}\left(-\hat{A}_z \sin\vartheta - r\sin\vartheta\,\partial_r \hat{A}_z + \hat{A}_z \sin\vartheta - \cos\vartheta\,\partial_\vartheta \hat{A}_z\right) =$$

$$= \frac{1}{r}\frac{\mu I_0 h}{4\pi}\left(-\frac{e^{-jkr}}{r}\sin\vartheta - r\sin\vartheta\,\frac{e^{-jkr}}{r}(-jk - \frac{1}{r}) + \frac{e^{-jkr}}{r}\sin\vartheta\right) =$$

$$H_\varphi = \frac{B_\varphi}{\mu} = \frac{I_0 h}{4\pi}\sin\vartheta\,\frac{e^{-jkr}}{r}\left(jk + \frac{1}{r}\right)$$

Condizioni statiche k=0

$$H_{\varphi}^{statico}(k=0)=\frac{I_0 h}{4\pi}\sin\vartheta\frac{1}{r^2}$$

Cfr I legge di Ampere-Laplace

$$h=dl$$

$$d\mathbf{H}=\frac{I_0}{4\pi}\frac{d\mathbf{l}\times\mathbf{r}}{r^3}=\frac{I_0}{4\pi}\frac{h}{r^2}\sin\vartheta\,\mathbf{u}_{\varphi}$$

Contributo dinamico k

$$H_{\varphi}^{dinamico} = \frac{B_{\varphi}}{\mu} = \frac{I_0 h}{4\pi} \sin\vartheta \; jk \, \frac{e^{-jkr}}{r}$$

$$\left| \frac{H_{\varphi}^{dinamico}}{H_{\varphi}^{statico}} \right| = kr$$

Allontanandosi dalla sorgente il contributo dinamico diventa dominante. Tale contributo è tanto maggiore quanto maggiore è la frequenza

Campo elettrico

$$\hat{\mathbf{E}} = \frac{1}{j\omega\varepsilon} \nabla \times \hat{\mathbf{H}}$$

$$\nabla \times \hat{\mathbf{H}} = \begin{vmatrix} \frac{1}{r^2 \sin\vartheta}\mathbf{u}_r & \frac{1}{r\sin\vartheta}\mathbf{u}_\vartheta & \frac{1}{r}\mathbf{u}_\varphi \\ \partial_r & \partial_\vartheta & \partial_\varphi \\ 0 & 0 & r\sin\vartheta \hat{H}_\varphi \end{vmatrix} =$$

$$= \frac{1}{r^2 \sin\vartheta}\mathbf{u}_r \left(\partial_\vartheta \left(r \sin\vartheta \hat{H}_\varphi\right)\right) - \frac{1}{r\sin\vartheta}\mathbf{u}_\vartheta \left(\partial_r \left(r\sin\vartheta \hat{H}_\varphi\right)\right)$$

$$H_\varphi = \frac{B_\varphi}{\mu} = \frac{I_0 h}{4\pi} \sin\vartheta \, \frac{e^{-jkr}}{r} \left(jk + \frac{1}{r} \right)$$

$$= \frac{1}{r^2 \sin\vartheta} \mathbf{u}_r \left(\partial_\vartheta \left(r \sin\vartheta \hat{H}_\varphi \right) \right) - \frac{1}{r \sin\vartheta} \mathbf{u}_\vartheta \left(\partial_r \left(r \sin\vartheta \hat{H}_\varphi \right) \right)$$

Pertanto..

$$E_r = \frac{1}{j\omega\varepsilon} \frac{1}{r^2 \sin\vartheta} 2 \sin\vartheta \cos\vartheta \frac{I_0 h}{4\pi} e^{-jkr} \left(jk + \frac{1}{r} \right) \approx 0$$

a grande distanza

$$E_\vartheta = \frac{1}{j\omega\varepsilon} \frac{e^{-jkr}}{r} \frac{I_0 h}{4\pi} \left[jk \left(jk + \frac{1}{r} \right) + \frac{1}{r^2} \right] \sin\vartheta \approx - \frac{k^2}{j\omega\varepsilon} \frac{e^{-jkr}}{r} \frac{I_0 h}{4\pi} \sin\vartheta$$

Densità di potenza (1)

$$H_{\varphi} = jk \frac{I_0 h}{4\pi} \sin\vartheta \frac{e^{-jkr}}{r}$$

$$E_{\vartheta} = j\omega\mu \frac{I_0 h}{4\pi} \sin\vartheta \frac{e^{-jkr}}{r}$$

$$\frac{E_{\vartheta}}{H_{\varphi}} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

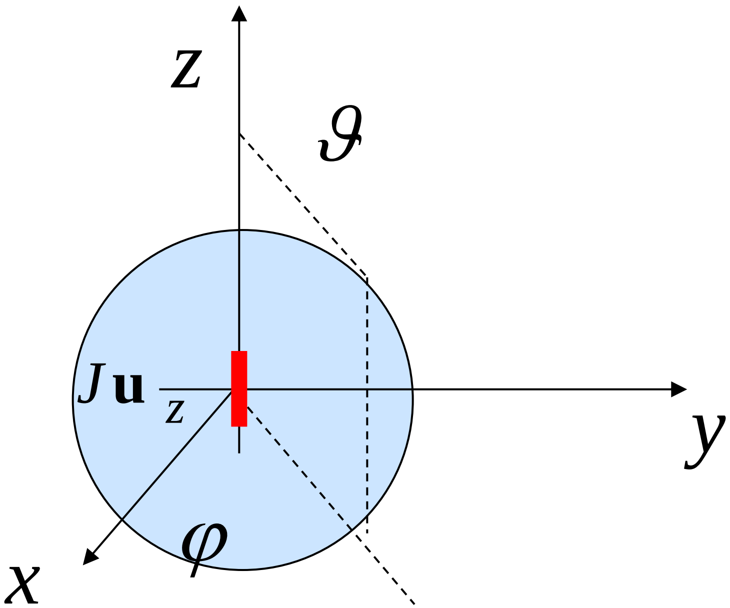
Densità di potenza (2)

$$E_\vartheta H_\varphi^* = j\omega\mu \frac{I_0 h}{4\pi}\sin\vartheta \frac{e^{-jkr}}{r}\left(-jk\frac{I_0^* h}{4\pi}\sin\vartheta \frac{e^{+jkr}}{r}\right) =$$

$$= \eta k^2 \left(\frac{|I_0| h}{4\pi}\right)^2 \sin^2\vartheta \frac{1}{r^2}$$

$$p(r,\vartheta,\phi) = \frac{1}{2}\mathrm{Re}\left(E_\vartheta H_\varphi^*\right) = \frac{\eta k^2}{2}\left(\frac{|I_0| h}{4\pi}\right)^2 \sin^2\vartheta \frac{1}{r^2}$$

Flusso di potenza attiva su una superficie sferica concentrica con l'origine 1



$$\overline{P} = \frac{1}{2}\mathrm{Re}\left(\int_S \hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \cdot d\mathbf{S}\right) =$$

$$\frac{1}{2}\mathrm{Re}\left(\int_S E_\vartheta H_\varphi^* \mathbf{u}_r \cdot \mathbf{u}_r \, dS\right) = \frac{1}{2}\eta k^2 \left(\frac{|I_0| h}{4\pi}\right)^2 \mathrm{Re}\left(\int_S \sin^2\vartheta \frac{1}{r^2} dS\right)$$

Flusso di potenza attiva su una superficie sferica concentrica con l'origine 2

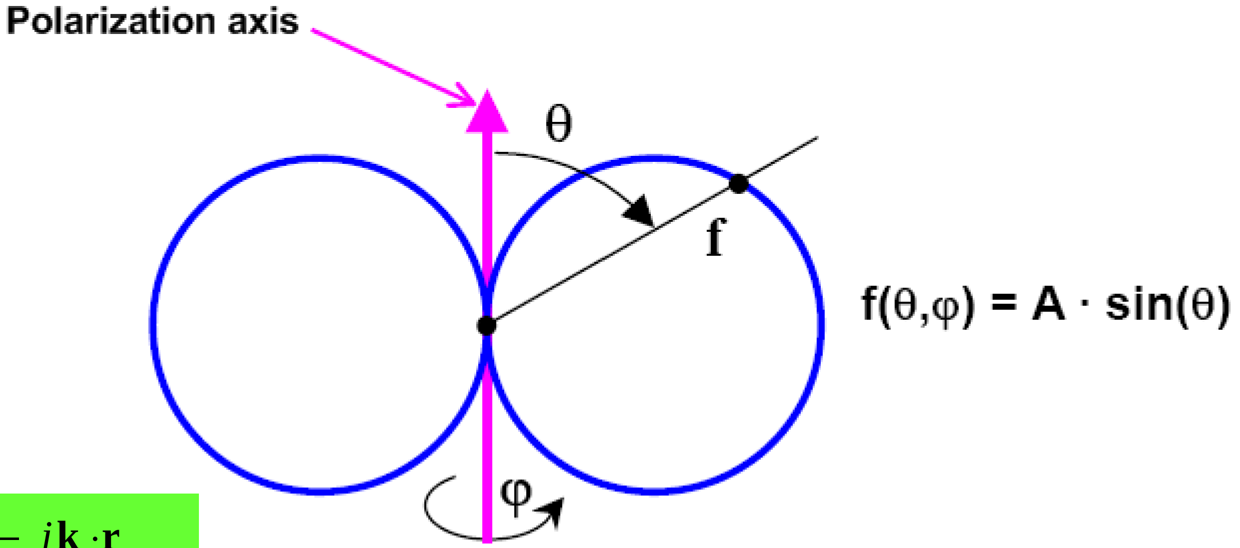
$$= \left(\frac{|I_0| h}{4\pi} \right)^2 \eta k^2 \frac{1}{2} \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \frac{1}{r^2} r^2 \sin^3 \vartheta = \left(\frac{I_0 h}{4\pi} \right)^2 \eta k^2 2\pi \frac{2}{3}$$

$$\overline{P} = \eta \frac{\pi}{3} I_0^2 \left(\frac{h}{\lambda} \right)^2 \quad [\mathrm{W}]$$

Direttività nella direzione (θ_0,ϕ_0) per un dipolo hertziano

$$D(\vartheta_0,\phi_0) = \frac{4\pi r^2 p(r,\vartheta_0,\phi_0)}{\overline{P}} = \frac{3}{2}\sin^2\vartheta_0$$

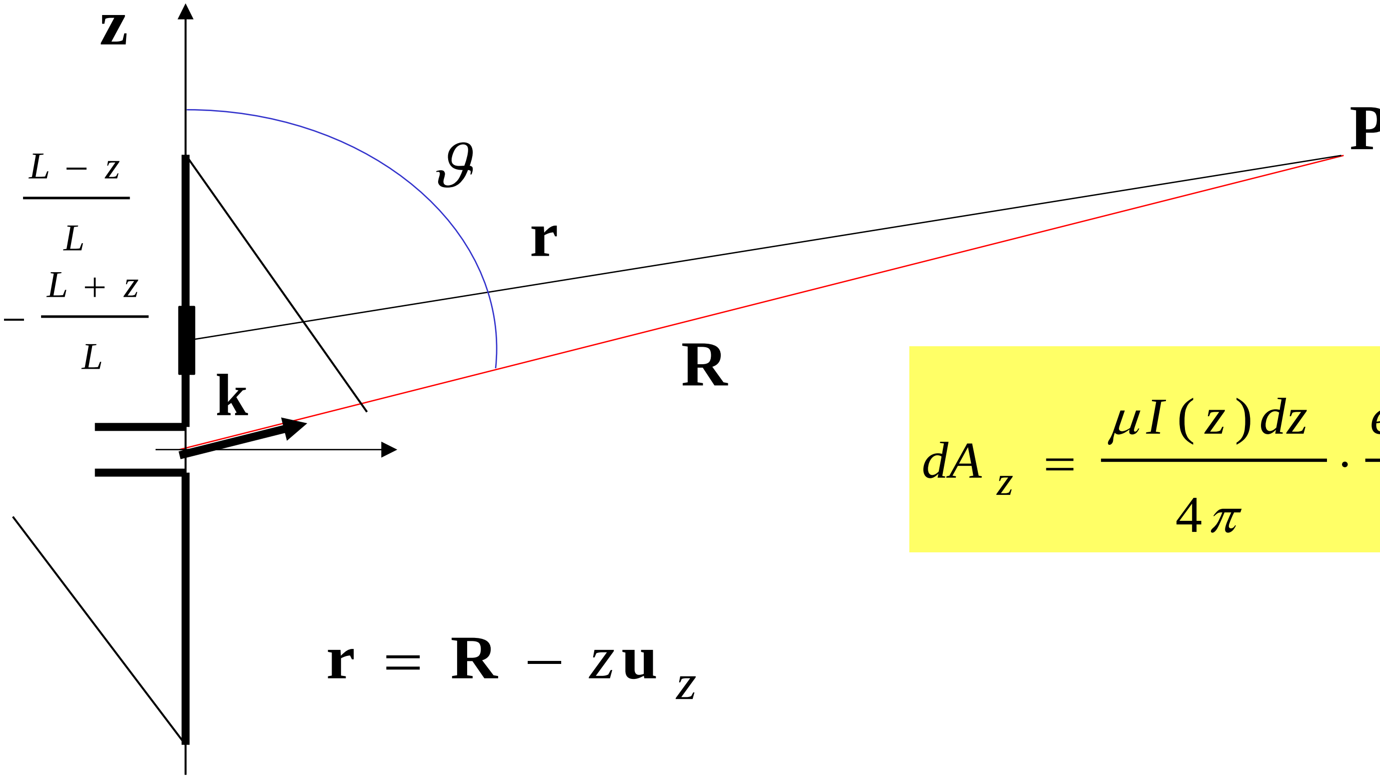
Diagramma di Radiazione



$$E_{\vartheta} = f(\vartheta,\phi)\frac{je^{-j\mathbf{k}\cdot\mathbf{r}}}{r}$$

$$f(\vartheta,\phi) = \omega\mu\frac{I_0 h}{4\pi}\sin\vartheta$$

Antenna Filiforme



$$I(z) = I_0 \begin{cases} \dfrac{L - z}{L} \\ -\dfrac{L + z}{L} \end{cases}$$

$$dA_z = \frac{\mu I(z)\,dz}{4\pi} \cdot \frac{e^{-j\mathbf{k}\cdot\mathbf{r}}}{r}$$

$$\mathbf{r} = \mathbf{R} - z\mathbf{u}_z$$

$$r \approx R$$

$$\mathbf{k}\cdot\mathbf{r} = \mathbf{k}\cdot(\mathbf{R} - z\mathbf{u}_z) = k\cdot\sqrt{(R\cos\vartheta - z)^2 + (R\sin\vartheta)^2} \approx kR - z\cos\vartheta$$

$$dA_z = \frac{\mu I dz}{4\pi} \cdot \frac{e^{-jkR}}{R} e^{jkz\cos\vartheta} \Rightarrow A_z = \frac{\mu}{4\pi}\frac{e^{-jkR}}{R}\int_{-L}^{+L} I(z)\,dz \cdot e^{jkz\cos\vartheta}$$