Campi armonici

Se le sorgenti cambiano nel tempo sinusoidalmente, con pulsazione ω

$$\mathbf{J}(x, y, z, t) = \mathbf{J}(x, y, z)\cos(\omega t)$$

$$\rho(x, y, x, t) = \rho(x, y, z)\cos(\omega t + \alpha)$$

Allora, in un mezzo lineare, anche il campo e.m. varierà nel tempo con legge sinusoidale, con la medesima pulsazione ω

*Prendendo **l'onda progressiva**:*

$$E_x^+(z, t) = f^+\left(t - \frac{z}{v}\right)$$

In $z=0$

$$E_x^+(z=0,t) = f^+(t) = E_0^+\cos(\omega t)$$

Quindi, sostituendo a t (t-z/v), otteniamo

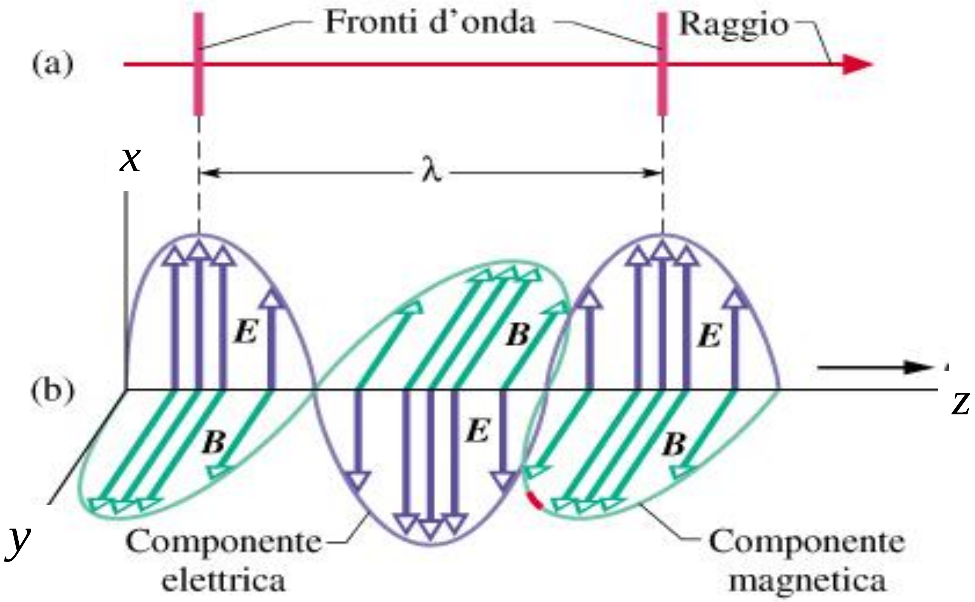
$$E_x^+(z,t) = f^+(t-z/v) = E_0^+\cos\left[\omega(t-z/v)\right]$$

$$= E_0^+\cos(\omega t - kz)$$

$$k = \omega / v$$

Fotografia in t=t0

$$\mathbf{E}(z,t) = \mathbf{u}_x E_0^{+} \cos(\omega t - kz)$$



Lunghezza d'onda

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega} v = \frac{v}{f}}$$

Dominio della frequenza

$$E_x^{\pm} = \mathrm{Re}\left\{E_0^{\pm} \exp\left[j(\omega t \mp kz)\right]\right\} =$$

$$= \mathrm{Re}\left\{E_0^{\pm} \exp\left(\mp jkz\right)\exp\left(j\omega t\right)\right\}$$

$$= \mathrm{Re}\left\{\hat{E}_x^{\pm} \exp\left(j\omega t\right)\right\}$$

$\hat{E}_x$ è detto FASORE

Se i campi sono armonici

$$\partial_t E_x^{\pm} = \partial_t \operatorname{Re}\left\{\hat{E}_x^{\pm} \exp(j\omega t)\right\} =$$

$$= \operatorname{Re}\left\{\hat{E}_x^{\pm} \partial_t \exp(j\omega t)\right\} =$$

$$= \operatorname{Re}\left\{\hat{E}_x^{\pm} j\omega \exp(j\omega t)\right\}$$

$$= \operatorname{Re}\left\{ \boxed{j\omega \hat{E}_x^{\pm}} \exp(j\omega t) \right\}$$

$$\partial_t^2 E_x^{\pm} = \partial_t\left(\partial_t E_x^{\pm}\right) = \; = \operatorname{Re}\left\{ \boxed{-\omega^2 \hat{E}_x^{\pm}} \exp(j\omega t) \right\}$$

Equazione d'onda per i fasori

$$\frac{\partial^2 \operatorname{Re}\left\{\hat{E}_x \exp(j\omega t)\right\}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \operatorname{Re}\left\{\hat{E}_x \exp(j\omega t)\right\}}{\partial t^2}$$

$$= -\omega^2 \mu\varepsilon \operatorname{Re}\left\{\hat{E}_x \exp(j\omega t)\right\}$$

Vera se:

$$\boxed{\frac{\partial^2 \hat{E}_x}{\partial z^2} = -\omega^2 \mu\varepsilon \hat{E}_x}$$

$$\hat{E}_x = \hat{E}_x^+ + \hat{E}_x^- = E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$$k^2 = \omega^2 \mu\varepsilon$$

$$\eta \hat{H}_y = E_0^+ e^{-jkz} - E_0^- e^{jkz}$$

Esercizio- Un’onda elettromagnetica piana armonica (o sinusoidale), di frequenza f=100 KHz, con il campo elettrico diretto lungo y, si propaga nel verso positivo dell’asse *x*, in un mezzo omogeneo $\mu_r = 1; \quad \varepsilon_r = 3.$

a 14 GHz $\rightarrow k_0 = w_0 \sqrt{\mu_0 \varepsilon_0} = 20.958 \ m^{-1}$

Calcolare:

1) v

*2) l’ampiezza del campo magnetico H, sapendo che l’ampiezza del campo elettrico E=*10 V/m

3) l’espressione nel tempo di E e di H, se all’istante $\bar{t}$*=7.5* μs *nel punto di ascissa* $\bar{x}$ *=57* m $\mathbf{E}(\bar{x},y,z,\bar{t})=\mathbf{u}_y 10$ V/m

1) $$v = \frac{1}{\sqrt{\mu\varepsilon}} \text{ m/s} = \frac{c_0}{\sqrt{\varepsilon_r}} \text{ m/s} = 1.73 \cdot 10^8 \text{ m/s}$$

$$k = 2\pi f \sqrt{\mu\varepsilon} = 20.958 \sqrt{\varepsilon_r}\, 10^{-4} \text{ m}^{-1}$$

Essendo l'onda sinusoidale e progressiva, il campo elettrico deve avere forma:

considero entrambe le situazioni
di seno e coseno

$$\mathbf{E}(x,y,z,t)=\mathbf{u}_y E\cos(\omega t - kx - \alpha) = \mathbf{u}_y[E_c\cos(\omega t - kx) + E_s\sin(\omega t - kx)]$$

Il campo magnetico si ottiene considerando che:

$$\nabla \times \mathbf{E} = -\mu_0 \partial_t \mathbf{H}$$

$$\mathbf{u}_z\partial_x[E\cos(\omega t - kx - \alpha)]=\mathbf{u}_z\, k[E\sin(\omega t - kx - \alpha)]= -\mu_0\partial_t\mathbf{H}$$

Pertanto $\mathbf{H}=\dfrac{k}{\omega\mu_0}\mathbf{u}_z\,[E\cos(\omega t - kx - \alpha)]=$

$$\frac{\sqrt{\varepsilon_r}}{\eta_0}\mathbf{u}_z\,[E\cos(\omega t - kx - \alpha)]= \mathbf{u}_z\,[H\cos(\omega t - kx - \alpha)]$$

$$H = E\,\frac{\sqrt{\varepsilon_r}}{\eta_0}=10\,\text{V/m}\cdot 1.732\,/\,377\Omega \cong 0.046\ \text{A/m}$$

Quindi, l'espressione del campo magnetico è

$$\mathbf{H} = \frac{1}{\eta}\mathbf{u}_x \times \mathbf{E}$$

Ovvero, i vettori **E**, **H**,$\mathbf{u}_x$ *formano una terna ortogonale destrorsa*

2)

$$H = \frac{E}{\eta} = \frac{10\sqrt{3}}{377} = 0.046 \quad \text{A/m}$$

$$B = \mu_0 H = 4 \cdot \pi \cdot 10^{-7} \cdot 0.046 \quad \text{A/m} = 5.77 \cdot 10^{-8} \ \text{Wb/m}^2$$

3)

$$\mathbf{E}(\bar{x},y,z,\bar{t})=\mathbf{u}_y 10\ \mathrm{V/m}=\mathbf{u}_y 10\ \mathrm{V/m}\cos(\omega\bar{t}-k\bar{x}-\alpha)$$

Ne consegue che: $\omega\bar{t}-k\bar{x}-\alpha=2n\pi$

$\alpha=\omega\bar{t}-k\bar{x}-2n\pi$ *con n arbitrario*

Se prendiamo $n=0$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = -\mathbf{u}_x \partial_z E_y + \mathbf{u}_z \partial_x E_y$$

$$\partial_z E_y = 0$$

$$\mathbf{H} = \frac{\mathbf{u}_z}{-\mu_0} \int dt \left(\partial_x E_y \right) = \mathbf{u}_z \frac{\frac{2 \cdot \pi \cdot f}{c} \cdot \frac{1}{2 \cdot \pi \cdot f} E_0 \cos \left[2 \cdot \pi \cdot f \cdot \left(\frac{x_1}{c} - t_1 \right) \right]}{\mu_0}$$

$$= \mathbf{u}_z \frac{1}{\eta} E_0 \cos \left[2 \cdot \pi \cdot f \cdot \left(\frac{x_1}{c} - t_1 \right) \right]$$

Equazioni di Maxwell per i fasori

LE SORGENTI (e quindi i campi!!) variano nel tempo sinusoidalmente

$$\nabla \cdot \hat{\mathbf{H}} = 0 \qquad \nabla \times \hat{\mathbf{H}} = \hat{\mathbf{j}} + j\omega\varepsilon \, \overset{\frown}{\mathbf{E}}$$

$$\nabla \cdot \overset{\frown}{\mathbf{E}} = \hat{\rho} / \varepsilon \qquad \nabla \times \overset{\frown}{\mathbf{E}} = - j\omega\mu \, \hat{\mathbf{H}}$$

In assenza di sorgenti (le sorgenti ci sono eccome, sono sinusoidali e variano con pulsazione ω. Semplicemente non sono nella porzione di spazio che stiamo osservando!)

$$\nabla \times \hat{\mathbf{H}} = j\omega\varepsilon\,\overset{\frown}{\mathbf{E}}$$

$$\nabla \times \overset{\frown}{\mathbf{E}} = -\,j\omega\mu\,\hat{\mathbf{H}}$$

$$\partial_x = \partial_y = 0$$

$$\frac{\partial \hat{E}_x}{\partial z} = -\,j\omega\mu\,\hat{H}_y \qquad \frac{\partial \hat{E}_y}{\partial z} = j\omega\mu\,\hat{H}_x$$

$$\frac{\partial \hat{H}_y}{\partial z} = -\,j\omega\varepsilon\,\hat{E}_x \qquad \frac{\partial \hat{H}_x}{\partial z} = j\omega\varepsilon\,\hat{E}_y$$

$$\hat{E}_z = \hat{H}_z = 0$$

Equazione di Helmoltz monodimensionale

$$\frac{\partial^2 \hat{E}_x}{\partial z^2} = -\omega^2 \mu\varepsilon \hat{E}_x$$

$$\boxed{\hat{E}_x(z) = E_0^+ \exp(-jkz) + E_0^- \exp(+jkz)}$$

$k = \omega\sqrt{\mu\varepsilon}$ *NUMERO D'ONDA o COSTANTE DI FASE*

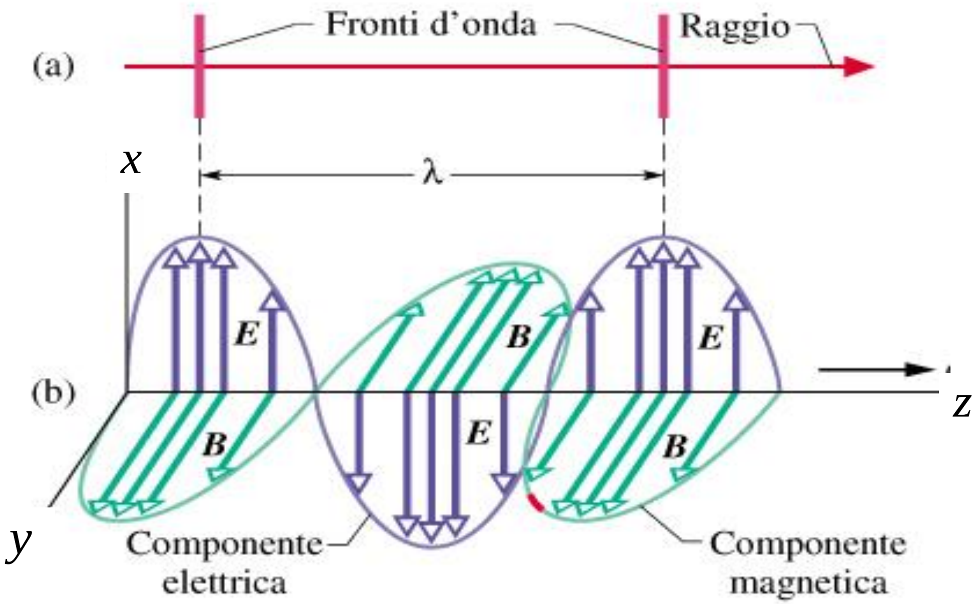
$$E_x(z,t) = \mathrm{Re}\left\{\hat{E}_x(z)\exp(j\omega t)\right\} =$$

$$\mathrm{Re}\left\{E_0^+ \exp\left[j(\omega t - kz)\right] + E_0^- \exp\left[j(\omega t + kz)\right]\right\} =$$

E_0^+ e E_0^- reali

$$= E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz) =$$

$$= E_0^+ \cos\left[\omega\left(t - z/v\right)\right] + E_0^- \cos\left[\omega\left(t + z/v\right)\right]$$



Il campo magnetico

$$\frac{\partial \hat{E}_x}{\partial z} = -j\omega\mu \hat{H}_y \Rightarrow \hat{H}_y = \frac{1}{-j\omega\mu} \frac{\partial \hat{E}_x}{\partial z}$$

$$\hat{E}_x(z) = E_0^+ \exp(-jkz) + E_0^- \exp(+jkz)$$

$$\hat{H}_y = \boxed{\frac{-jk}{-j\omega\mu}} \left[E_0^+ \exp(-jkz) - E_0^- \exp(+jkz) \right] =$$

$$= \sqrt{\frac{\varepsilon}{\mu}} \left[E_0^+ \exp(-jkz) - E_0^- \exp(+jkz) \right]$$

Dal fasore di H a H(z,t)

$$H_y(z,t) = \mathrm{Re}\left\{\sqrt{\frac{\varepsilon}{\mu}}\left[E_0^+ \exp(-jkz) - E_0^- \exp(+jkz)\right]\exp(j\omega t)\right\} =$$

$$H_y(z,t) = \mathrm{Re}\left\{\sqrt{\frac{\varepsilon}{\mu}}\left[E_0^+ \exp\left[j(\omega t - kz)\right] - E_0^- \exp\left[j(\omega t + kz)\right]\right]\right\} =$$

$$H_y(z,t) = \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}\left[E_0^+ \cos(\omega t - kz) - E_0^- \cos(\omega t + kz)\right]$$

La quantità $\sqrt{\frac{\mu}{\varepsilon}} = \frac{j\omega\mu}{jk}$ *viene detta impedenza d'onda*

Usando una notazione vettoriale

$$\mathbf{E}(\mathbf{r},t) = \mathbf{u}_x E_x(z,t) = \mathbf{u}_x \left\{ E_0^+ \cos\left[\omega(t - z/v)\right] + E_0^- \cos\left[\omega(t + z/v)\right] \right\}$$

$$\mathbf{H}(\mathbf{r},t) = \mathbf{u}_y H_y(z,t) = \mathbf{u}_y \sqrt{\frac{\varepsilon}{\mu}} \left[E_0^+ \cos(\omega t - kz) - E_0^- \cos(\omega t + kz) \right]$$

$$\mathbf{H}^+(\mathbf{r},t) = \frac{1}{\eta} \mathbf{u}_z \times \mathbf{E}^+(\mathbf{r},t) = \frac{1}{\omega\mu_0} \mathbf{k} \times \mathbf{E}^+(\mathbf{r},t)$$

essendo

$$\mathbf{k} = k\mathbf{u}_z$$

$$kz = k\mathbf{u}_z \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{r}$$

Dove: $\mathbf{r} = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z$

Onda piana che si propaga in direzione ***k***

$$\mathbf{H}^\pm(\mathbf{r}) = \pm \frac{1}{\eta} \mathbf{u}_\mathbf{k} \times \mathbf{E}^\pm(\mathbf{r})$$

Propagazione di un'onda piana in direzione arbitraria **k** Si suppone che **E** e **H** non cambino nel piano ortogonale a **k**

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\eta}\mathbf{u}_{\mathbf{k}} \times \mathbf{E}(\mathbf{r})$$

$$\Uparrow$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{k} = k_x\mathbf{u}_x + k_y\mathbf{u}_y + k_z\mathbf{u}_z$$

Algebrizzazione eq Maxwell

$$\boxed{\mathbf{H}_0 = \frac{1}{\omega\mu}\mathbf{k} \times \mathbf{E}_0}$$

$$\mathbf{k} \times \mathbf{E}_0 = \omega\mu\,\mathbf{H}_0 \qquad \Rightarrow \mathbf{k} \times \mathbf{E}_0 \times \mathbf{k} = \omega^2\mu\varepsilon\,\mathbf{E}_0$$

$$-\mathbf{k} \times \mathbf{H}_0 = \omega\varepsilon\,\mathbf{E}_0 \qquad \Downarrow$$

$$\mathbf{k} \cdot \mathbf{H}_0 = 0 \qquad \left(k^2 - \omega^2\mu\varepsilon\right)\mathbf{E}_0 = 0$$

$$\mathbf{k} \cdot \mathbf{E}_0 = 0$$

E ed **H** ortogonali a **k**

Polarizzazione

Un insieme di onde piane che si propagano nella stessa direzione, ma con orientazioni e fasi arbitrarie dei campi, generano un'onda **non polarizzata**

Consideriamo invece un'onda piana armonica con due componenti

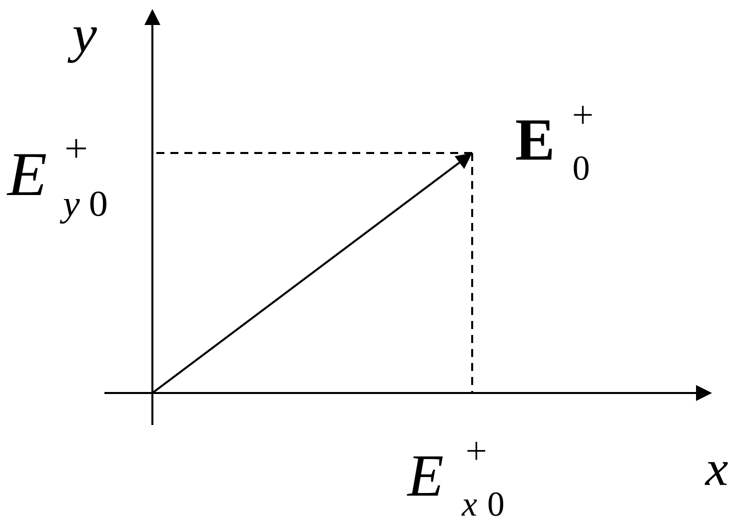
$$E_x(z,t) = E_{x0}^{+} \cos\left[\omega(t - z/v)\right]$$

$$E_y(z,t) = E_{y0}^{+} \cos\left[\omega(t - z/v) + \psi\right]$$

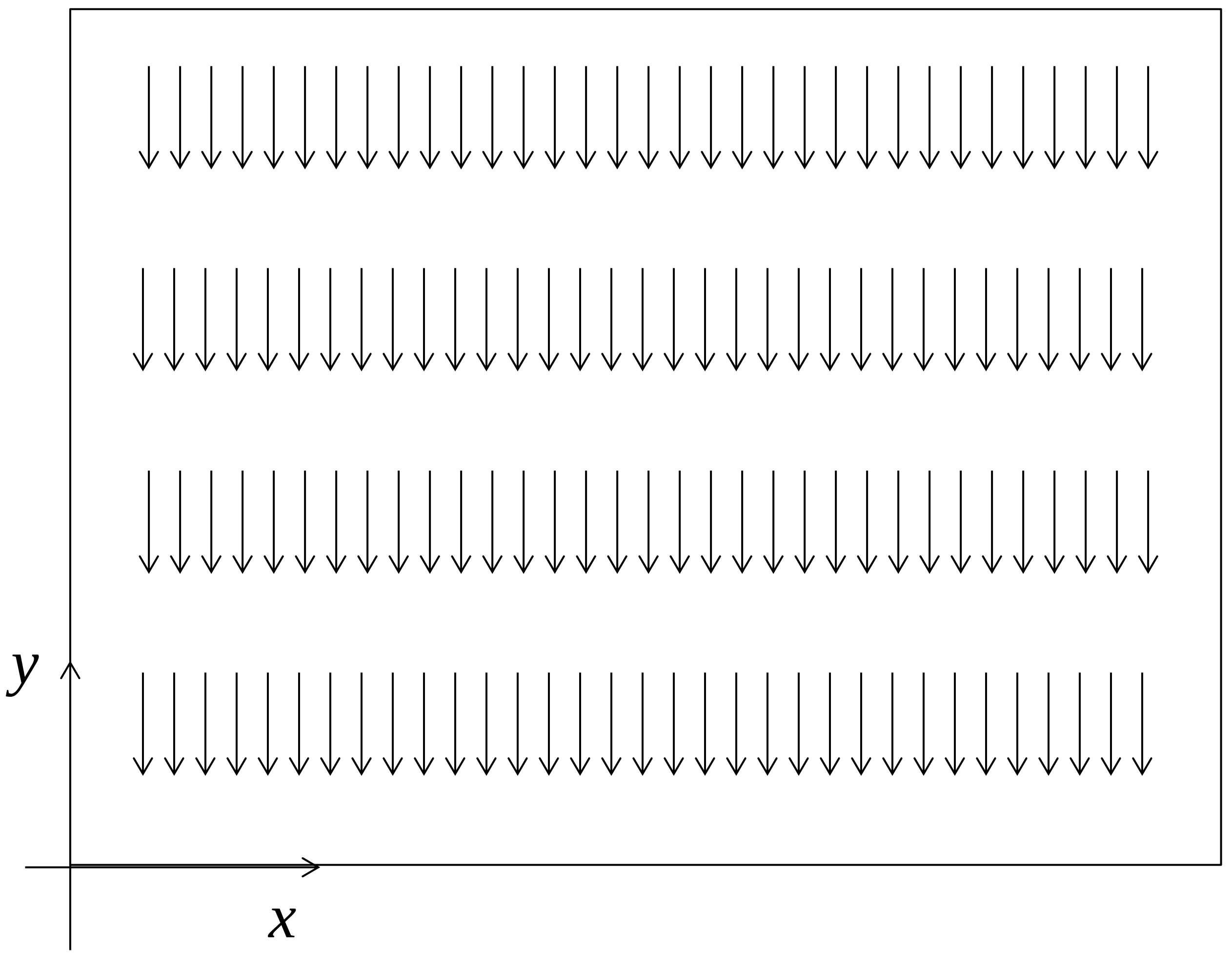
Polarizzazione lineare

$$\psi = 0$$

$$E_x(z,t) = E_{x0}^{+} \cos\left[\omega(t - z/v)\right]$$
$$E_y(z,t) = E_{y0}^{+} \cos\left[\omega(t - z/v)\right]$$
$$\Rightarrow \frac{E_x(z,t)}{E_y(z,t)} = \frac{E_{x0}^{+}}{E_{y0}^{+}}$$

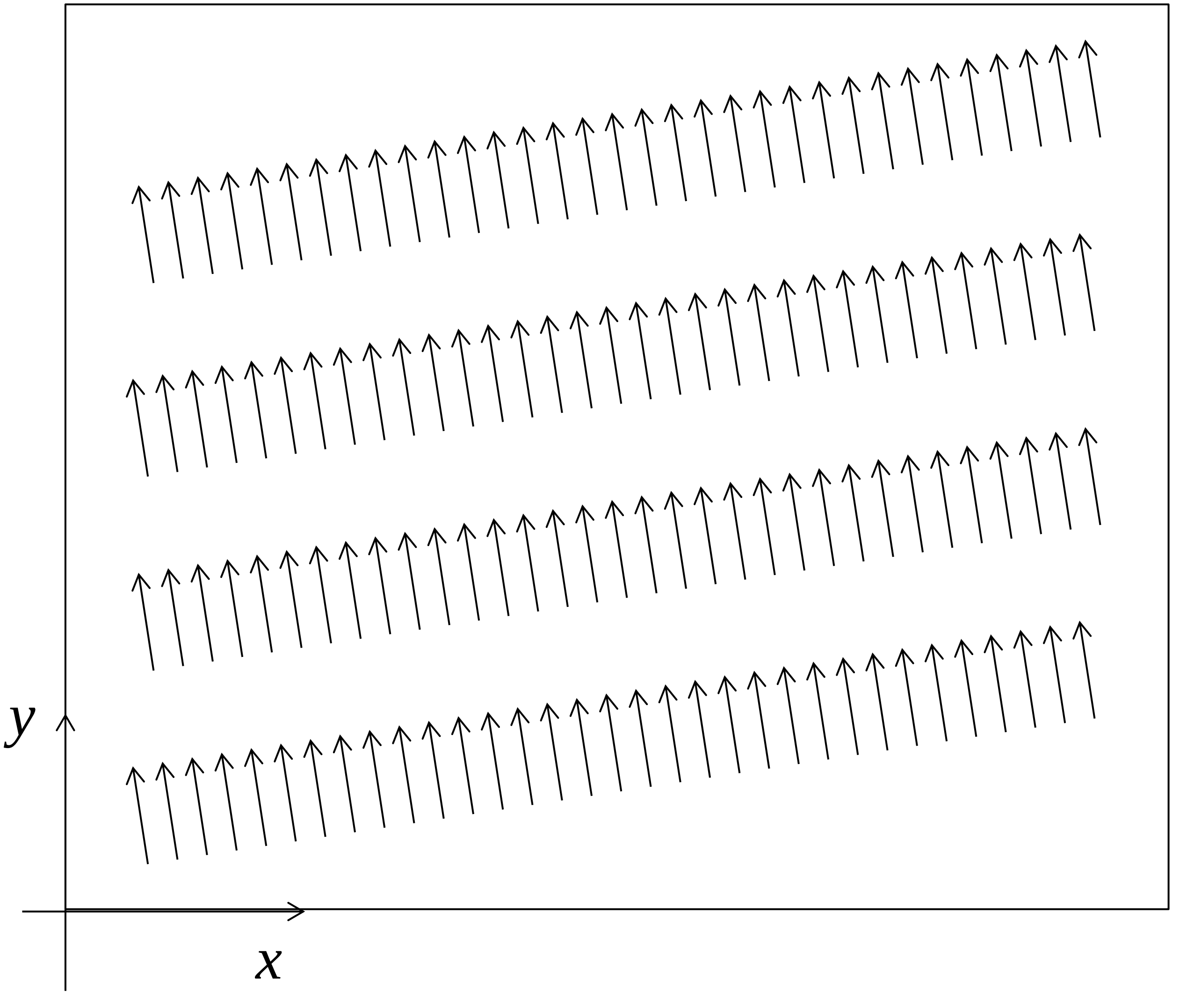


PORZIONE DI FRONTE D’ONDA PIANA POLARIZZATA LINEARMENTE



VISUALIZZAZIONE DEL CAMPO **E** *nei punti di applicazione dei vettori*

PORZIONE DI FRONTE D'ONDA PIANA POLARIZZATA LINEARMENTE



VISUALIZZAZIONE DEL CAMPO **E** *nei punti di applicazione dei vettori*

Polarizzazione circolare

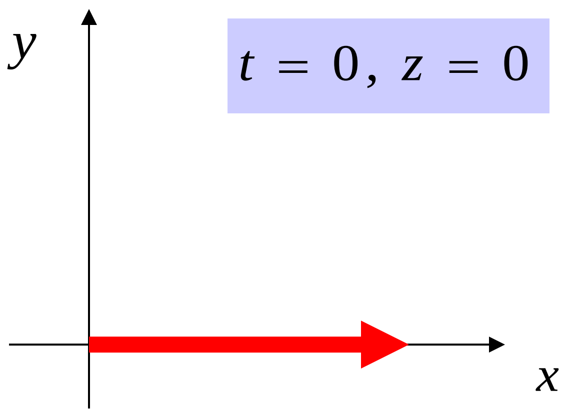
$$E^{+}_{x0} = E^{+}_{y0}$$

$$\psi = \frac{\pi}{2}$$

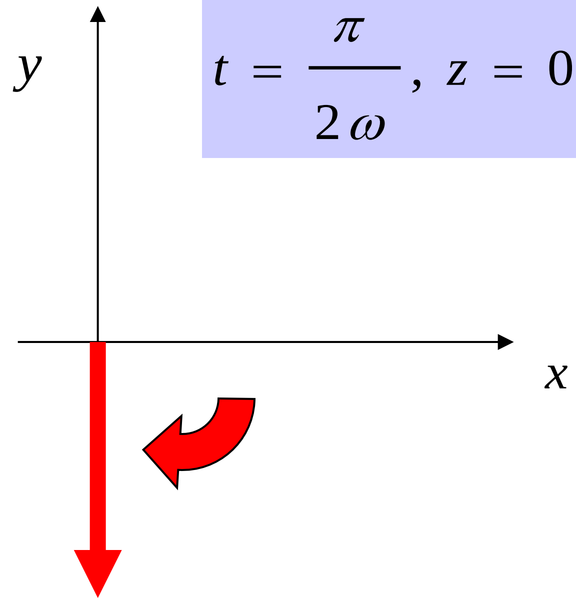
$$E_x(z,t) = E^{+}_{x0} \cos\left[\omega(t - z/v)\right]$$

$$E_y(z,t) = -E^{+}_{x0} \sin\left[\omega(t - z/v)\right]$$

$$\Rightarrow E_x^2(z,t) + E_y^2(z,t) = E^{+2}_{x0}$$

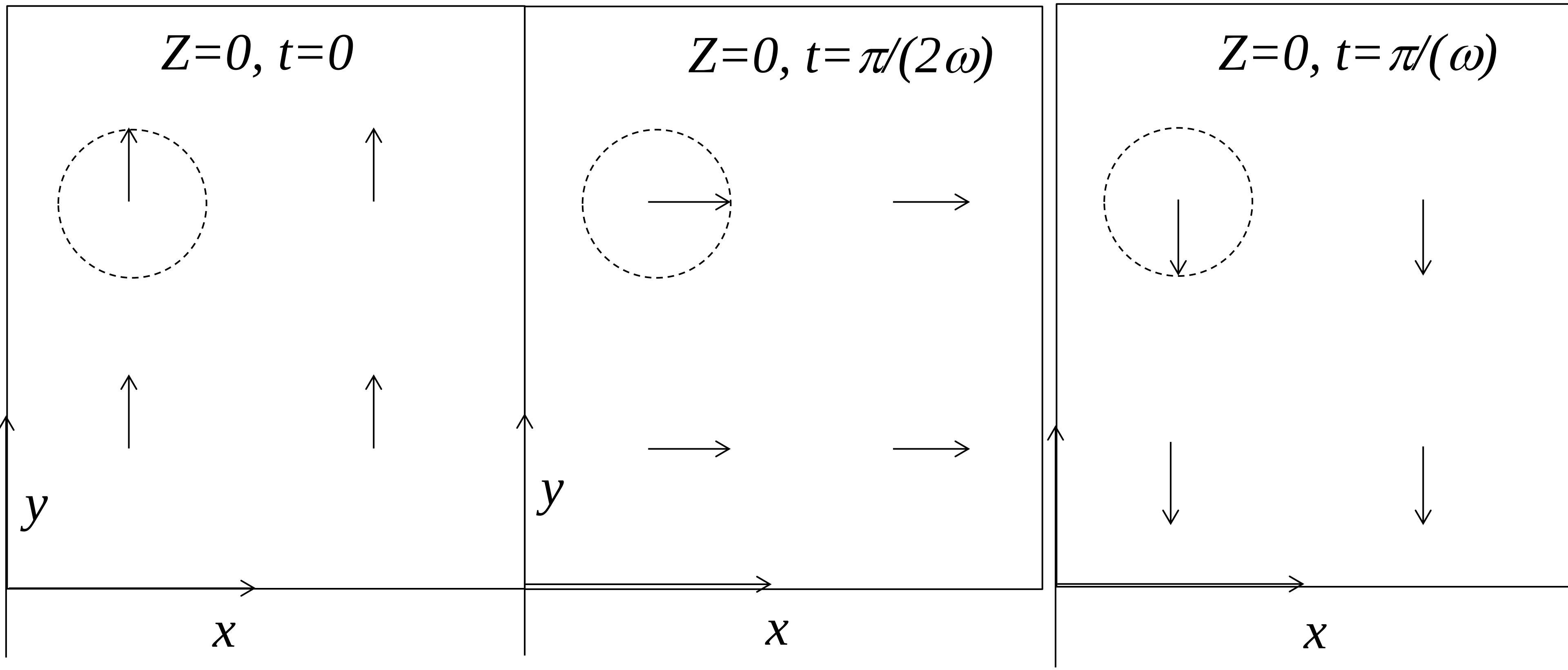


$t = 0,\ z = 0$



$$t = \frac{\pi}{2\omega},\ z = 0$$

PORZIONE DI FRONTE D'ONDA PIANA POLARIZZATA CIRCOLARMENTE



VISUALIZZAZIONE DEL CAMPO **E** *nei punti di applicazione dei vettori*

Polarizzazione ellittica

$E_x(z,t)=E_{x0}^{+}\cos(\omega t-kz)$
$E_y(z,t)=E_{y0}^{+}\cos(\omega t-kz+\psi)$

$\bar{E}_x=\cos(\omega t-kz)$
$\bar{E}_y=\cos(\omega t-kz)\cos\psi-\sin(\omega t-kz)\sin\psi=$

$\bar{E}_x\cos\psi-\sqrt{1-\bar{E}_x{}^2}\,\sin\psi$

$$\Longrightarrow \bar{E}_y{}^2+\bar{E}_x{}^2\cos^2\psi-2\bar{E}_x\bar{E}_y\cos\psi=\left(1-\bar{E}_x{}^2\right)\sin^2\psi$$

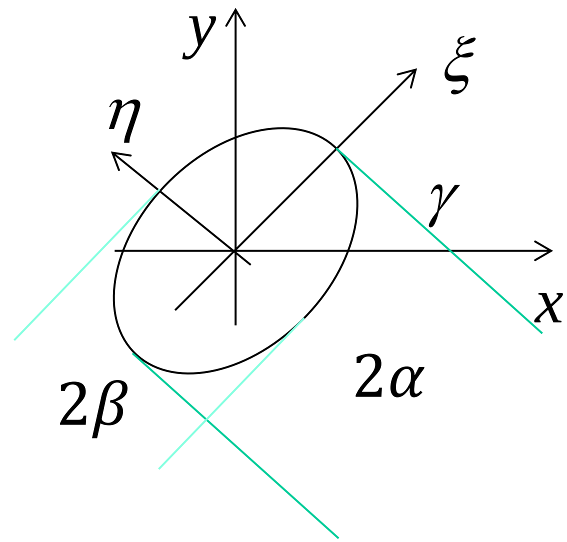
$$\Longrightarrow \bar{E}_y{}^2+\bar{E}_x{}^2-2\bar{E}_x\bar{E}_y\cos\psi=\sin^2\psi$$

Dividendo per $\sin^2\psi$

$$\breve{E}_y{}^2+\breve{E}_x{}^2-2\breve{E}_y\breve{E}_x\cos\psi=1$$

$\breve{E}=\bar{E}/\sin\psi$

$$\left(\frac{\xi}{\alpha}\right)^{2}+\left(\frac{\eta}{\beta}\right)^{2}=1$$



$$\xi(x,y) = x\cos\gamma + y\sin\gamma$$
$$\eta(x,y) = -x\sin\gamma + y\cos\gamma$$

$$\left(\frac{x\cos\gamma + y\sin\gamma}{\alpha}\right)^{2}+\left(\frac{-x\sin\gamma + y\cos\gamma}{\beta}\right)^{2}=1=$$
$$\left[\left(\frac{\cos\gamma}{\alpha}\right)^{2}+\left(\frac{\sin\gamma}{\beta}\right)^{2}\right]x^{2}+\left[\left(\frac{\sin\gamma}{\alpha}\right)^{2}+\left(\frac{\cos\gamma}{\beta}\right)^{2}\right]y^{2}+\left[\frac{1}{\alpha^{2}}-\frac{1}{\beta^{2}}\right]2xy\sin\gamma\cos\gamma$$

$$=\breve{E}_y^{\,2}+\breve{E}_x^{\,2}-2\breve{E}_y\breve{E}_x\cos\psi=1$$

Posto $x=E_x$ *e* $y=E_y$

$$\gamma=\frac{1}{2}\tan^{-1}\left[\frac{-2\cos\psi/(E_{x0}^{+}E_{y0}^{+})}{1/E_{x0}^{+2}-1/E_{y0}^{+2}}\right]$$

Rapporto Assiale (Axial Ratio)

$$AR = 20 \cdot \left| \log\left(\frac{\alpha}{\beta}\right) \right|$$

Teorema di Poynting

$$\int_{\partial V} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n} \; dS = \int_V \nabla \cdot \left(\mathbf{E} \times \mathbf{H} \right) dV =$$

$$= -\int_V \left(\mathbf{E} \cdot \nabla \times \mathbf{H} - \mathbf{H} \cdot \nabla \times \mathbf{E} \right) dV =$$

$$= -\int_V \left(\mathbf{E} \cdot (\mathbf{j} + \varepsilon \partial_t \mathbf{E}) - \mathbf{H} \cdot (-\mu_0 \partial_t \mathbf{H}) \right) dV =$$

$$= -\int_V \mathbf{E} \cdot \mathbf{j} \, dV - \frac{1}{2} \partial_t \left[\int_V \varepsilon_0 \left| \mathbf{E} \right|^2 + \int_V \mu_0 \left| \mathbf{H} \right|^2 \right]$$

Teorema di Poynting II

$$\int_V \nabla \cdot \mathbf{S}\, dV + \frac{1}{2}\partial_t \left[\int_V \varepsilon_0 \left|\mathbf{E}\right|^2 + \int_V \mu_0 \left|\mathbf{H}\right|^2 \right] = -\int_V \mathbf{E} \cdot \mathbf{j}\, dV$$

$$-\int_{\partial V} \mathbf{S} \cdot \mathbf{n}\, dS = \frac{1}{2}\partial_t \left[\int_V \varepsilon_0 \left|\mathbf{E}\right|^2 + \int_V \mu_0 \left|\mathbf{H}\right|^2 \right] + \int_V \mathbf{E} \cdot \mathbf{j}\, dV$$

dove $\int_V \mathbf{E} \cdot \mathbf{j}\, dV = \int_V \mathbf{E} \cdot nq\mathbf{v}\, dV = \int_V \mathbf{E} \cdot nq\mathbf{v}\, dV = \int_V \mathbf{F} \cdot \mathbf{v}\, dV$

n : numero di cariche per unità di volume, **F**: *forza agente sulle cariche nell'unità di volume,* **F · v** *: potenza dissipata per unità di volume per effetto Joule*

Teorema di Poynting: media temporale

$$\frac{1}{T}\int_0^T \int_{\partial V} \mathbf{S}\cdot\mathbf{n}\, dS\, dt + \frac{1}{T}\int_0^T \partial_t \left[\frac{1}{2}\int_V \left(\varepsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2\right) dV\right] dt +$$
$$+\frac{1}{T}\int_0^T \left[\int_V (\mathbf{E}\cdot\mathbf{j})\, dV\right] dt = 0$$

$$\omega = \frac{2\pi}{T}$$

$$\left|\mathbf{E}\right|^2 \propto \sin^2(\omega t + \phi)$$

$$\frac{1}{T}\int_0^T \frac{1}{2}\partial_t \left[\int_V \varepsilon_0 \left|\mathbf{E}\right|^2 + \int_V \mu_0 \left|\mathbf{H}\right|^2\right] = 0$$

Teorema di Poynting: media temporale

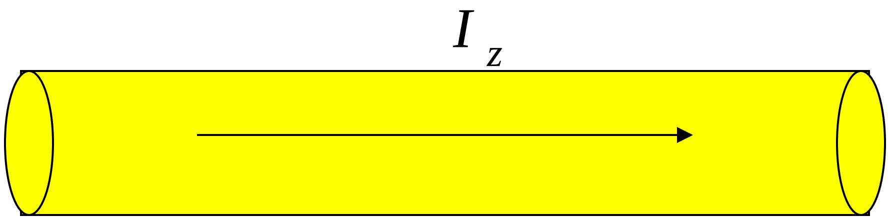
$$-\frac{1}{T}\int_0^T \int_{\partial V} \mathbf{S}\cdot\mathbf{n}\, dS\, dt = \frac{1}{T}\int_0^T \left[\int_V \ (\mathbf{E}\cdot\mathbf{j})\, dV\right] dt$$

$$\frac{1}{T}\int_0^T \left[\int_V \ (\mathbf{E}\cdot\mathbf{j})\, dV\right]\, dt = \frac{1}{T}\int_0^T \left[\int_V \ \mathrm{Re}\left(\hat{\mathbf{E}}e^{i\omega t}\right)\mathrm{Re}\left(\hat{\mathbf{j}}e^{i\omega t}\right) dV\right]\, dt =$$

$$= \frac{1}{T}\int_0^T \left[\int_V \ \left(\hat{\mathbf{E}}_R \cos(\omega t) - \hat{\mathbf{E}}_I \sin(\omega t)\right)(\hat{\boldsymbol{j}}_R \cos(\omega t) - \hat{\boldsymbol{j}}_I \sin(\omega t))\, dV\right]\, dt$$

$$= \frac{1}{2}\int_V \ \left(\hat{\mathbf{E}}_R\hat{\boldsymbol{j}}_R + \hat{\mathbf{E}}_I\hat{\boldsymbol{j}}_I\right) dV = \frac{1}{2}\mathrm{Re}\left[\int_V \ \hat{\mathbf{E}}\cdot\hat{\mathbf{j}}^*\, dV\ \right]$$

Perdite Ohmiche



$$\mathbf{E} = E_z \mathbf{u}_z = RI_z \mathbf{u}_z$$

$R =$ resistenza per unità di lunghezza

$$\mathbf{H} = H_\phi \mathbf{u}_\phi = \frac{I_z}{2\pi r} \mathbf{u}_\phi$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = RI_z \mathbf{u}_z \times H_\phi \mathbf{u}_\phi = -RI_z \frac{I_z}{2\pi r} \mathbf{u}_r$$

$$\int_{Cilindro} \mathbf{S} \cdot \mathbf{u}_r \, ds = -RI_z \frac{I_z}{2\pi r} 2\pi r = -RI_z^2 = -W$$

Teorema di Poynting per i fasori

$$\nabla \times \hat{\mathbf{H}} = \hat{\mathbf{j}} + j\omega\varepsilon\,\overset{\frown}{\mathbf{E}}$$

$$\nabla \times \overset{\frown}{\mathbf{E}} = -j\omega\mu\,\hat{\mathbf{H}}$$

$$\nabla \cdot \left(\overset{\frown}{\mathbf{E}} \times \hat{\mathbf{H}}^*\right) = \hat{\mathbf{H}}^* \cdot \nabla \times \overset{\frown}{\mathbf{E}} - \overset{\frown}{\mathbf{E}} \cdot \nabla \times \hat{\mathbf{H}}^* =$$

$$= \hat{\mathbf{H}}^* \cdot \left(-j\omega\mu\,\hat{\mathbf{H}}\right) - \overset{\frown}{\mathbf{E}} \cdot \left(\hat{\mathbf{j}}^* - j\omega\varepsilon\,\overset{\frown}{\mathbf{E}}^*\right) =$$

$$= -j\omega\left[\mu\left|\hat{\mathbf{H}}\right|^2 - \varepsilon\left|\overset{\frown}{\mathbf{E}}\right|^2\right] - \overset{\frown}{\mathbf{E}} \cdot \hat{\mathbf{j}}^* =$$

$$\int_{\partial V} \overset{\frown}{\mathbf{E}} \times \hat{\mathbf{H}}^* \cdot \mathbf{n}\,dS = -j\omega\int_V \left[\mu\left|\hat{\mathbf{H}}\right|^2 - \varepsilon\left|\overset{\frown}{\mathbf{E}}\right|^2\right] dV - \int_V \overset{\frown}{\mathbf{E}} \cdot \hat{\mathbf{j}}^*\,dV$$

$$\oint_{\partial V} \overset{\frown}{\mathbf{E}} \times \hat{\mathbf{H}}^{*} \cdot \mathbf{n}\, dS = -j\omega \int_{V} \left[\mu \left|\hat{\mathbf{H}}\right|^{2} - \varepsilon \left|\overset{\frown}{\mathbf{E}}\right|^{2} \right] dV - \int_{V} \overset{\frown}{\mathbf{E}} \cdot \hat{\mathbf{j}}^{*} dV$$

Perdite ohmiche:

$$\hat{\mathbf{j}} = \sigma\, \overset{\frown}{\mathbf{E}} \quad \Rightarrow \quad \int_{V} \overset{\frown}{\mathbf{E}} \cdot \hat{\mathbf{j}}^{*} dV = \int_{V} \sigma \left|\overset{\frown}{\mathbf{E}}\right|^{2} dV$$

$$\frac{1}{2} \operatorname{Re} \left\{ \oint_{\partial V} \overset{\frown}{\mathbf{E}} \times \hat{\mathbf{H}}^{*} \cdot \mathbf{n}\, dS \right\} = -\frac{1}{2} \int_{V} \sigma \left|\overset{\frown}{\mathbf{E}}\right|^{2} dV = \overline{P}_{joule}$$

Potenza media dissipata nel periodo

$$\overline{p} = \frac{1}{2} \operatorname{Re} \left\{ \overset{\frown}{\mathbf{E}} \times \hat{\mathbf{H}}^{*} \right\} \ \mathrm{W/m}^{2}$$

Parte immaginaria

$$\mathrm{Im}\left\{\int_{\partial V} \overset{\frown}{\mathbf{E}} \times \hat{\mathbf{H}}^{*} \cdot \mathbf{n}\, dS\right\} = 4\omega \int_{V} \frac{1}{2}\left[\frac{1}{2}\varepsilon\left|\overset{\frown}{\mathbf{E}}\right|^{2} - \frac{1}{2}\mu\left|\hat{\mathbf{H}}\right|^{2}\right] dV =$$

$$= 4\omega\left(\overline{U}_{E} - \overline{U}_{H}\right)$$

Potenza media in un'onda piana polarizzata linearmente

$$\hat{\mathbf{E}}(\mathbf{r}) = \mathbf{u}_x \left(E_0^+ \exp(-jkz) + E_0^- \exp(+jkz) \right)$$

$$\hat{\mathbf{H}}(\mathbf{r}) = \frac{1}{\eta} \mathbf{u}_y \left(E_0^+ \exp(-jkz) - E_0^- \exp(+jkz) \right)$$

$$\hat{\mathbf{E}} \times \hat{\mathbf{H}}^* = \frac{1}{\eta} \mathbf{u}_z \left(E_0^+ \exp(-jkz) + E_0^- \exp(+jkz) \right) \left(E_0^{+*} \exp(+jkz) - E_0^{-*} \exp(-jkz) \right) =$$

$$= \frac{1}{\eta} \mathbf{u}_z \left[\left(\left| E_0^+ \right|^2 - \left| E_0^- \right|^2 \right) + \left(- E_0^+ E_0^{-*} \exp(-2jkz) + E_0^- E_0^{+*} \exp(+2jkz) \right) \right] =$$

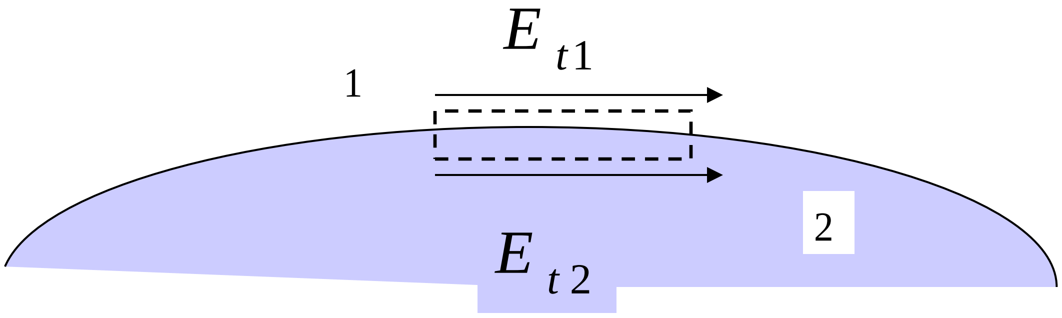
$$= \frac{1}{\eta} \mathbf{u}_z \left[\left(\left| E_0^+ \right|^2 - \left| E_0^- \right|^2 \right) + 2j \,\mathrm{Im} \left(E_0^- E_0^{+*} \exp(+2jkz) \right) \right]$$

Potenza media in un'onda piana polarizzata linearmente II

$$\frac{1}{2}\operatorname{Re}\left(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^{*}\right)=\frac{1}{2\eta}\mathbf{u}_z\left(\left|E_0^{+}\right|^2-\left|E_0^{-}\right|^2\right)$$

$$\operatorname{Im}\left(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^{*}\right)=\frac{1}{\eta}\mathbf{u}_z\,2\operatorname{Im}\left(E_0^{-}E_0^{+*}\exp(+2jkz)\right)$$

Condizioni di continuità dei campi: E tangenziale



$$\oint \mathbf{E} \cdot d\mathbf{l} = \left(E_{t1} - E_{t2}\right)\Delta l = -\partial_t \Phi_B = 0$$

$$\Rightarrow E_{t1} = E_{t2}$$

Condizioni di continuità dei campi: H tangenziale

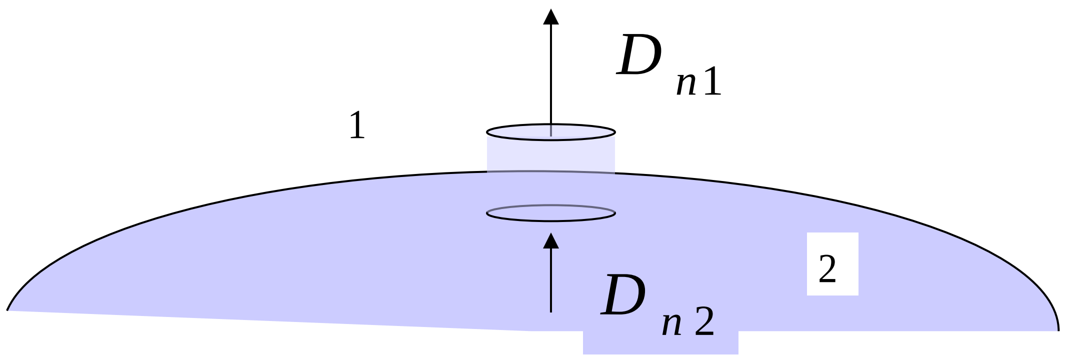
Analogamente,

$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(H_{t1} - H_{t2}\right)\Delta l = j_s \Delta l + \partial_t \Phi_D$$

$$\Phi_D = 0 \quad \Rightarrow H_{t1} - H_{t2} = j_s$$

$$\Phi_D = 0 \;\; j_s = 0 \qquad H_{t1} = H_{t2}$$

Condizioni di continuità dei campi: componenti normali



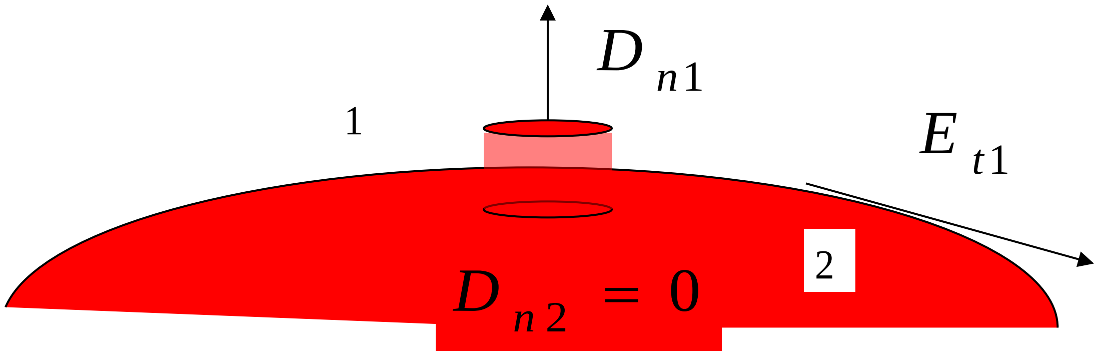
$$\left(D_{n1} - D_{n2}\right)\Delta S = \sigma \Delta S \quad \Rightarrow \quad D_{n1} - D_{n2} = \sigma$$

$$\text{Se} \quad \sigma = 0 \qquad D_{n1} = D_{n2} \Rightarrow \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

$$\textit{Analogamente,} \qquad B_{n1} = B_{n2} \Rightarrow \mu_1 H_{n1} = \mu_2 H_{n2}$$

Condizioni al contorno sulla superficie di un conduttore

$$\mathbf{E} = 0 \quad \text{nel conduttore ideale}$$



$$D_{n1} = \sigma \qquad B_{n1} = 0 = \mu_1 H_{n1} = 0$$

$$E_{t1} = 0 \qquad H_{t1} = j_s$$