Business Analytics - Aplicaciones Computacionales en Negocios

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stochastic processes for optionality II

real vs financial options

it is often the case that a business needs to value a situation or contract with uncertain payoff

the payoff is a function of an underlying X_T which is unknown as seen from valuation date $t = 0$

examples:

a wind turbine, which is a real option on the future speed of wind

the right, but not obligation, to purchase 10% of YPF at 15 USD per share in July 2024. this is a financial option

in both cases the value of an uncertain payoff can be represented as its expected value, discounted by the fact that payment would occur in the future

$$V_0 = E[\frac{f(X_T)}{(1+r)^T}]$$

elements of our model?

trend? done

volatility? done

mean reversion? done

seasonality? done

correlations?

stochastic volatility?

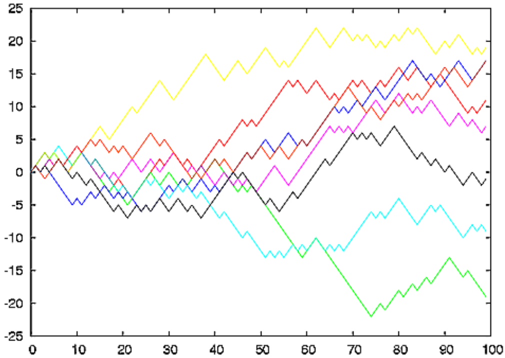
jumps?

a visit to the zoo of stochastic processes

a stochastic process is a sequence of random variables indexed by time

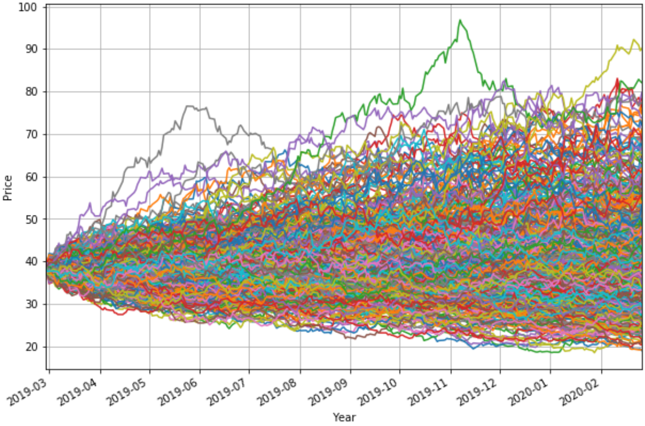
arithmetic random walk

$$S_{i+1} = S_i + \mu\Delta + \sigma\sqrt{\Delta}Z_{i+1}$$



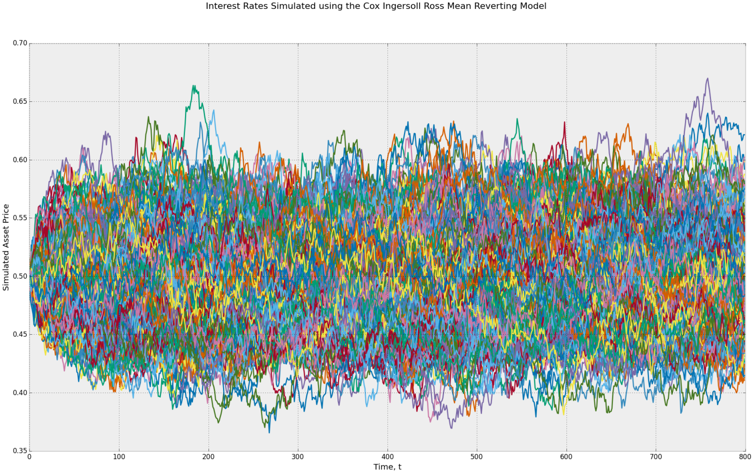
geometric random walk

$$S_{i+1} = S_i(1 + \mu\Delta + \sigma\sqrt{\Delta}Z_{i+1})$$

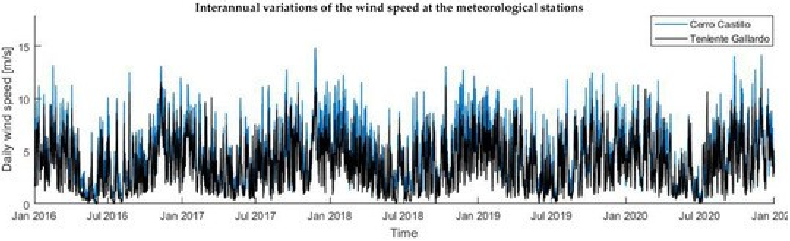


mean reverting process

$$S_{i+1} = S_i + \kappa(\theta - S_i)\Delta + S_i \sigma \sqrt{\Delta} Z_{i+1}$$



calibrating the seasonal drift



$$S_{i+1} = S_i + \kappa(\theta - S_i)\Delta + S_i\sigma\sqrt{\Delta}Z_{i+1}$$

$$\theta(x) = \theta_0 + Asin(\alpha x + \phi)$$

$$\theta_i = 3.75 + 1.25cos(2\pi i/365)?$$

short term correlated variables

it often happens that two or more quantities are correlated with each other in the short term

short term correlation because both feel short term fluctuations in a third variable

oil P_t and soybean S_t prices are sensitive to demand from Asia

$$P_{i+1} = P_i + ... + P_i \sigma Z_{i+1}$$

$$S_{i+1} = S_i + ... + S_i \sigma (\rho Z_{i+1} + \sqrt{1-\rho^2} W_{i+1})$$

where Z, W are independent random shocks

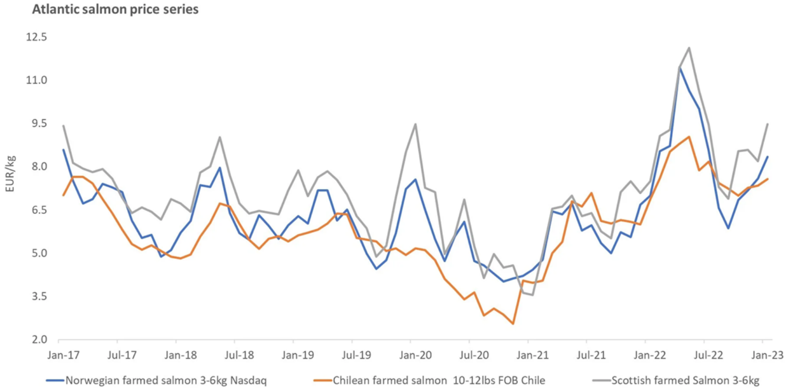
what is the volatility of oil and soybean returns? and the correlation of oil and soybeans returns?

can oil and soybeans drift far apart from each other?

long term correlated variables

it often happens that two or more quantities are correlated with each other as they evolve in time in the long term

prices of substitute goods, if one is too cheap, demand for the other will fall
Salmon from Chile and from Norway with prices C_t and N_t



long term correlated variables

Salmon from Chile and from Norway with prices C_t and N_t

define $X_t = (C_t + N_t)/2$

define $Y_t = C_t - N_t$

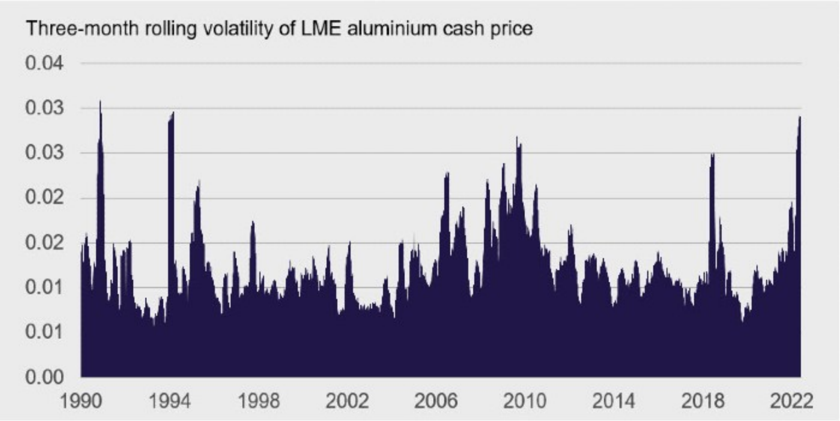
interpretation of X_t? what model do you choose for X_t?

interpretations of Y_t? what model do you choose for Y_t?

simulate X_t and Y_t, how do you then obtain local salmon paths?

is volatility constant?

Figure 1: Three-month rolling for volatility for aluminium reaches historical peaks



DATA: LME, CRU

stochastic volatility

discrete time stoch vol models with correlated variance process:

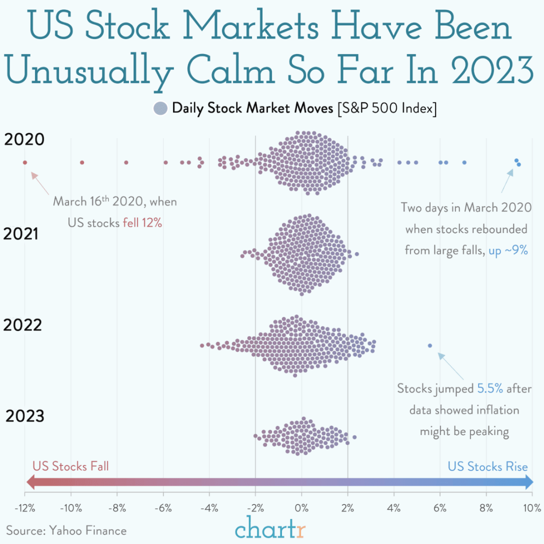
$$
\begin{aligned}
S_{i+1} &= S_i + S_i r\Delta + \sigma S_i \sqrt{V_i}\sqrt{\Delta} Z_{i+1}) \\
V_{i+1} &= \kappa(\theta - V_i) + \sqrt{V_i}\sqrt{\Delta}[\rho Z_{i+1} + \sqrt{1-\rho^2} W_{i+1}]
\end{aligned}
$$

$i = 0, 1, ..., N-1$. $Z_1, ..., Z_N$ and $W_1, ..., W_N$ are normal i.i.d components of $\{Z, W\}$. $\Delta = 1/252$.

ρ controls correlation btw asset price and variance process

should ρ be positive or negative? (stocks vs commodities?)

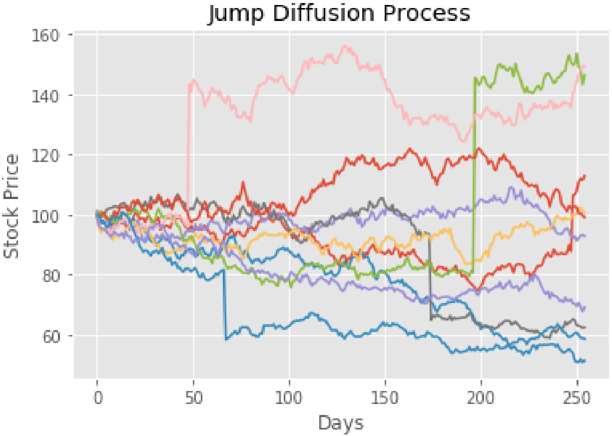
are asset prices continuous in time?



jump diffusion model

$$S_{i+1} = S_i(1 + \mu\Delta + \sigma\sqrt{\Delta}Z_{i+1} + (Y_{i+1} - 1)1_{U_{i+1}<\lambda})$$

where Y is lognormally distributed and U is a uniform r.v in $(0,1)$, and $\lambda << 1$
how frequent are jumps? how large?



pick a stochastic model

the stochastic processes we saw in this and previous classes can be formulated as continuous time models. but the math is more complicated

we work with models in discrete time, $i = 1, ..., N$

we select a model with dynamics that resembles actual empirical data

careful interpretation of model parameters often lead to well defined parameter ranges prior to running any simulation. be careful and consistent with the choice of units...

say you like a model

$S_{i+1} = f(a, b, c, S_i, Z_{i+1})$

but still need to calibrate precise parameter values, can be done by simulation

structure of the simulation

initialize parameter values a, b, c and initial condition S_0

set data structure to store M paths of N time steps: $S(path, time)$

loop over M paths
$S(path, 0) = S_0$
  loop over N time steps
    sample Z
    $S(path, step+1) = f(a, b, c, S(path, step), Z)$
  end
end

estimator for expected terminal S: $\hat{E} = \frac{1}{M}\sum_{i=1}^{M} S(i, N)$
estimator for variance of terminal S: $\hat{V} = \frac{1}{M}\sum_{i=1}^{M} S(i, N)^2 - \hat{E}^2$

do more time steps contribute to smaller estimation error?

carrying less data

initialize parameter values a, b, c and initial condition S_0

if we only care about statistics for terminal S, no need to store full path

loop over M paths
$S(old) = S_0$
  loop over N time steps
    sample Z
    $S(new) = f(a, b, c, S(old), Z)$
    $S(old) = S(new)$
  end
$\hat{E} = \hat{E} + S(new)$
$\hat{V} = \hat{V} + S(new)^2$
end

estimator for expected terminal S: $\hat{E} = \frac{1}{M}\hat{E}$
estimator for variance of terminal S: $\hat{V} = \frac{1}{M}\hat{V} - \hat{E}^2$

estimation error? how many paths?

are M paths enough for a simulation?

we have:
estimator for expected terminal S: $\hat{E} = \frac{1}{M}\hat{E}$
estimator for variance of terminal S: $\hat{V} = \frac{1}{M}\hat{V} - \hat{E}^2$

for E, true expectation of $S(N)$ that is our goal, we know that

$$E \approx \hat{E} + \frac{\sqrt{\hat{V}}}{\sqrt{M}}$$

error size depends on intrinsic variability of $S(N)$ and on number of paths M (do more time steps reduce estimation error?)

set desired error d based on business / econ considerations. say, we want error to be less than 3 dollars

run simulation until $\frac{\sqrt{\hat{V}}}{\sqrt{M}} < d$

calibrating a model

to take into account when calibrating a model:

match empirical local properties. for example, relate mean and standard deviation of short term fluctuations to model parameters (μ, σ)

conjecture set of parameter values, generate many paths and compare simulated estimates for range, mean reversion, skewness, kurtosis, to those based on historical data

iterate over different choices of parameter values until simulated dynamics resembles local and global aspects of empirical data

now, you have a model properly calibrated and implemented as a simulation. you can now run it to compute statistics for the payoffs associated to your business situation

simulation applied to a payoff

initialize parameter a, b, c at their calibrated values and initial condition S_0

set data structure to store M paths of N time steps: $S(path, time)$

loop over M paths
$S(path, 0) = S_0$
  loop over N time steps
    sample Z
    $S(path, step + 1) = f(a, b, c, S(path, step), Z)$
  end
end
estimator for expected payoff : $\hat{E} = \frac{1}{M}\sum_{i=1}^{M} payoff(S(i))$
estimator for payoff variance : $\hat{V} = \frac{1}{M}\sum_{i=1}^{M} payoff(S(i))^2 - \hat{E}^2$
error in payoff estimator: $\frac{\sqrt{\hat{V}}}{\sqrt{M}}$

payoff could be profits obtained in a path, or number of times the process S exceeded some threshold, how many times a wind turbine was turned off, etc.