Algorithms [illegible] Circular-Arc Graphs

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ABSTRACT

Consider a finite family of non-empty sets. Th[illegible]er-section graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. The intersection graph of a family of arcs on a circularly ordered set is called a circular-arc graph. In this paper we give a characterization of the circular-arc graphs and we describe efficient algorithms for recognizing two subclasses. Also, we describe efficient algorithms for finding a maximum independent set, a minimum covering by cliques and a maximum clique of a circular-arc graph.

1. INTRODUCTION

In this paper we consider only finite graphs G(V), with no parallel edges and no self-loops, where V is the set of the graph vertices. Two vertices of G connected by an edge are called *adjacent vertices*. A *subgraph* of G is a graph determined by a subset of V, two vertices of the subgraph being adjacent if and only if they are adjacent in G. A set of G vertices is called *independent* if no two of its elements are adjacent. A *maximum independent set* is one with the largest number of vertices of all independent sets. The number of vertices in a maximum independent set will be denoted by $\alpha(G)$. A *clique* is a maximal completely connected set of vertices; a *maximum clique* is one with a maximum number of elements. The number of vertices in a maximum clique will be denoted by $\beta(G)$. The set of vertices adjacent to a vertex v is denoted Γv. For a set A, $|A|$ is the number of its elements. For two sets A, B, A-B is

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the set of elements of A which are not in B. Throughout the paper, we will assume that the graph $G(V)$ has n vertices denoted $V = \{v_1,\ldots,v_n\}$.

The matrices we deal with in this paper are (0,1)-matrices. For a graph $G(V)$ and a family $A_1,\ldots,A_k$ of subsets of V, we will denote by $\mu(A_1,\ldots,A_k)$ the $k \times n$ matrix whose entry $\langle i,j \rangle$ is 1 if $v_j \in A_i$, and 0 if $v_j \notin A_i$.

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. The intersection graphs of families of sets with a defined topological pattern have applications in genetics, psychophysics, archeology and ecology. The paper [8] is a survey of problems and applications of the different intersection graphs. For example, the intersection graph of a family of intervals on a linearly ordered set is called an *interval graph* (see [1]-[3]).

The intersection graph of a family of arcs on a circularly ordered set is called a *circular-arc graph*. For example, the graph of Figure 1a is a circular-arc graph represented by the family of arcs $F = \{\bar{v}_1,\ldots,\bar{v}_8\}$ of Figure 1b. The problem of characterizing the circular-arc graphs first appeared in [7]. Klee discussed in [8] some problems related to this subject. Tucker [9] characterized the circular-arc graphs by means of their adjacency matrices, and asked for a recognition algorithm, yet unknown.

A graph is called a Δ *circular-arc graph* if it is the intersection graph of a family of arcs on a circle, so that for three arcs, if every pair intersects then the intersection of the three arcs is non-empty. A graph is called a θ *circular-arc graph* if it is the intersection graph of a family of arcs on a circle so that for every clique, the intersection of the arcs corresponding to the vertices of the clique is non-empty. Clearly, a θ circular-arc graph is also a Δ circular-arc graph. Consider the graph in Figure 1a. The set $\{v_1,v_2,v_3,v_4\}$ is a circuit without diagonals which can be represented only by the arcs $\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4$ as in Figure 1b. For representing the clique $\{v_5,v_6,v_7,v_8\}$ by four arcs with a non-empty intersection, it is necessary that the arc $\bigcap_{i=5}^{8} \bar{v}_i$ should intersect one of the arcs $\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4$. Hence, one of the vertices v_1,v_2,v_3,v_4 must be

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connected to all the vertices v_5,v_6,v_7,v_8. Thus the graph in Figure 1a is a Δ circular-arc graph which is not a θ circular-arc graph.

The purposes of this paper are to describe efficient algorithms for:

(i) Recognizing the Δ and θ circular-arc graphs and constructing the corresponding families of arcs.

(ii) Finding a maximum clique, a maximum independent set and a minimum covering by cliques of a circular-arc graph.

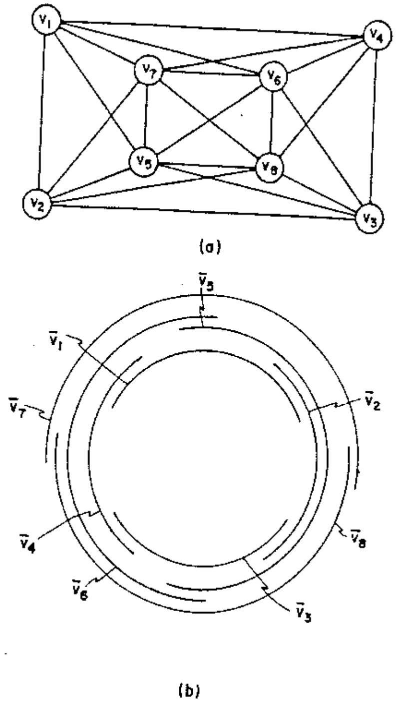


Fig. 1

A graph is called *chordal* if every simple circuit with more than three vertices has an edge connecting two non-consecutive vertices. Efficient recognition algorithms of these graphs are described in [3] and [5]. The number of cliques of a chordal graph is at most as the number of its vertices (see [3] and [4]). Let us denote an oriented edge from u to v, by $u \to v$. An orientation of a graph is called an *R-orientation* if it has no directed circuits and for every three vertices u,v,w, if $u \to v$ and $w \to v$, then either $u \to w$ or $w \to u$. In [3] and [5] it is proved that a graph is chordal if and only if it is R-orientable. The interval graphs are chordal (see [1] and [2]). We can obtain an R-orientation of an interval graph, in n^2 steps, as follows. Consider an interval graph G and its representing family of intervals F. Without loss of generality, we can assume that the intervals of F have no common endpoints. Then, for two adjacent vertices u,v of G we orient $u \to v$ if and only if the left endpoint of u appears on the left of the left endpoint of v. Clearly, this is an R-orientation of G. By the algorithms described in [4], based on the R-orientation, we can find a maximum clique, a maximum independent set, a minimum covering by cliques, and the set of cliques of an interval graph. For an interval graph, the intersection of the intervals corresponding to the vertices of a clique is a non-empty interval (see [1] or [2]).

Consider a matrix written on the lateral surface of a cylinder, so that the rows are generating lines. The matrix has a *circular 1's form* if the 1's in each column appear in a circular consecutive order. A matrix has the *circular 1's property* if by a permutation of the rows it can be transformed into a matrix with a circular 1's form. Tucker [9] described an efficient algorithm for constructing a circular 1's form of a matrix, if one exists. His algorithm takes at most m^3 steps, where m is the number of columns in the matrix.

Without loss of generality, we can assume that the families of arcs (on a circle) we deal with are chosen so that the arcs are open, no two arcs have a common endpoint, and none of the arcs covers the whole circle. By an arc $a = (e,f)$, we mean the arc beginning in e and continuing in clockwise direction until f; e will be called the left endpoint of a and f will be called the right endpoint of a. Consider a circular-arc graph G and its representing family of arcs F. We will assume that the union of the arcs of F covers the circle, for otherwise G is an interval graph. Thus we will consider only connected graphs. The corresponding arc in F of a vertex v of G will be denoted by $\bar{v}$.

2. CHARACTERIZATION OF THE CIRCULAR-ARC GRAPHS

Let G(V) be a circular-arc graph and F its family of representing arcs. Two arcs $\bar{v}_i, \bar{v}_j \in F$ are called *overlapping* if they intersect and no one is contained in another. Consider the set $S = \{s_1,\dots,s_r\}$ of all the arcs on the circle, such that every s_i, $1 \le i \le r$, satisfies:

(i) s_i does not contain endpoints of the arcs of F;

(ii) s_i is an arc of F or is the intersection of two overlapping arcs of F.

The set S will be called *the set of primitive arcs* for F. Clearly, every arc of F contains a primitive arc, and every two different primitive arcs have an empty intersection. For every $1 \le i \le r$, denote $V_i = \{v \mid v \in V,\ s_i \subseteq \bar{v}\}$.

Lemma 1: Let G(V) be a circular-arc graph and F its representing family of arcs. Then $\mu(V_1,\dots,V_r)$ has the circular 1's property.

Proof: Without loss of generality we can assume that the primitive arcs $s_1,\dots,s_r$ appear in a circular consecutive order. Hence, every arc $\bar{v}_j$ contains a circular consecutive sequence of primitive arcs. But $s_i \subseteq \bar{v}_j$ if and only if $v_j \in V_i$. Thus the 1's in the column j of $\mu(V_1,\dots,V_r)$ appear in a circular consecutive order. Therefore, $\mu(V_1,\dots,V_r)$ has a circular 1's form.

Q.E.D.

A family $A_1,\dots,A_k$ of completely connected sets of a graph G(V) is called a *covering system*, if it satisfies: $V = \bigcup_{i=1}^{k} A_i$; if $i \ne j$ then $A_i \not\subseteq A_j$; for every two adjacent vertices u,v there exists a set A_i containing them.

Theorem 1: A graph G(V) is a circular-arc graph if and only if it has a covering system $A_1,\dots,A_k$ such that $\mu(A_1,\dots,A_k)$ has the circular 1's property.

Proof: Assume that G(V) is a circular-arc graph and F is the representing family of arcs. Clearly, the family $V_1,\dots,V_k$ defined as above, is a covering system, and by Lemma 1, $\mu(V_1,\dots,V_k)$ has the circular 1's property.

Conversely, let $A_1,\ldots,A_k$ be a covering system of G, so that $\mu(A_1,\ldots,A_k)$ has the circular 1's property. Without loss of generality we can assume that the matrix has a circular 1's form. Denote k points consecutively in the clockwise direction on a circle, by $1,2,\ldots,k$. We construct the family F as follows. Let the 1's in a column i appear in a circular consecutive order in clockwise direction between the rows m and p, inclusively. If $m \neq 1$, then $\bar{v}_i = (m-1,p) \in F$ and if $m = 1$, then $\bar{v}_i = (k,p) \in F$. If the column i contains only 1's then $\bar{v}_i = (k,k) \in F$. Two vertices $v_i,v_j \in V$ are adjacent if and only if there exists an ℓ, $1 \leq \ell \leq k$, such that $v_i,v_j \in A_\ell$, hence if and only if $\bar{v}_i \cap \bar{v}_j \supseteq (\ell-1,\ell)$. Therefore, G is the intersection graph of F.

Q.E.D.

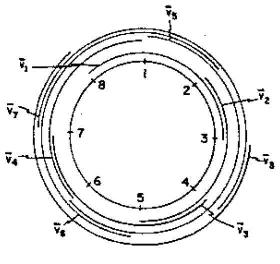
A covering system of the graph of Figure 1a is:

$A_1 = \{v_1,v_4,v_6,v_7\}$; $A_2 = \{v_1,v_5,v_6,v_7\}$; $A_3 = \{v_1,v_2,v_5,v_7\}$;
$A_4 = \{v_2,v_5,v_7,v_8\}$; $A_5 = \{v_2,v_3,v_5,v_8\}$; $A_6 = \{v_3,v_5,v_6,v_8\}$;
$A_7 = \{v_3,v_4,v_6,v_8\}$; $A_8 = \{v_4,v_6,v_7,v_8\}$.

A circular 1's form of $\mu(A_1,\ldots,A_8)$ is given in Figure 2a. In Figure 2b we see the representing family of arcs, constructed by the above method.

$$
\begin{array}{c|cccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
3 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
5 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
6 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
7 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
8 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}
$$

(a)



(b)

Fig. 2

3. RECOGNITION ALGORITHMS FOR THE Δ AND θ CIRCULAR-ARC GRAPHS

Consider a graph G(V), $V = \{v_1,\ldots,v_n\}$. For every vertex v_i, let G_i denote the subgraph defined by $\Gamma v_i \cup \{v_i\}$. Let $C_1^i,\ldots,C_{k_i}^i$ be all the cliques of G_i. We will denote the maximal elements of $\bigcup_{i=1}^{n}\{C_1^i,\ldots,C_{k_i}^i\}$ by $D_1,\ldots,D_k$.

Consider a Δ circular-arc graph G(V) and its representing family of arcs F. For every vertex v_i, denote:

$$F_i = \{\bar{v}_j^i \mid \bar{v}_j^i = \bar{v}_i \cap \bar{v}_j, v_j \in \Gamma v_i \cup \{v_i\}\}.$$

For two adjacent vertices $v_j,v_k \in \Gamma v_i$, we have $\bar{v}_i \cap \bar{v}_j \cap \bar{v}_k \neq \phi$, by the definition of the Δ circular-arc graphs, thus $\bar{v}_j^i \cap \bar{v}_k^i \neq \phi$. Therefore, G_i is the intersection graph of F_i, and F_i is a family of arcs which does not cover the whole circle. Hence, G_i is an interval graph. *Thus if G is a Δ circular-arc graph, then every G_i is an interval graph, and hence every G_i is chordal.*

Theorem 2: G is a Δ circular-arc graph if and only if $\mu(D_1,\ldots,D_k)$ has the circular 1's property.

Proof: Let G(V) be a Δ circular-arc graph, and F its representing family of arcs. Consider the set of primitive arcs $S = \{s_1,\ldots,s_r\}$. For every $1 \leq j \leq r$, denote $V_j = \{v \mid v \in V, s_j \subseteq \bar{v}\}$. Clearly, if $v_i \in V_j$, then V_j is a clique of G_i. On the other side, G_i is an interval graph, and the intersection of the arcs representing the vertices of a clique is non-empty and contains a primitive arc. Therefore, $V_1,\ldots,V_r$ are exactly all the maximal elements of $\bigcup_{i=1}^{n}\{C_1^i,\ldots,C_{k_i}^i\}$ and by Lemma 1, $\mu(V_1,\ldots,V_k)$ has the circular 1's property.

Conversely, consider a graph G such that $\mu(D_1,\ldots,D_k)$ has the circular 1's property. The family $D_1,\ldots,D_k$ is a covering system of G and we can construct to G a family of representing rcs F as in the proof of Theorem 1. Consider three vertices v_i, v_j, v_k, mutually adjacent. Hence $v_j, v_k \in G_i$ and there exists a clique of G_i which contains the three vertices. Thus there exist an ℓ, $1 \le \ell \le k$, such that $v_i, v_j, v_k \in D_\ell$. Therefore, by the construction of F, $\bar{v}_i \cap \bar{v}_j \cap \bar{v}_k \supseteq (\ell-1, \ell)$ on the circle of F. Thus, G is a Δ circular-arc graph.

Q.E.D.

By Theorem 2, the algorithm for recognizing whether a given graph G is a Δ circular-arc graph works as follows:

We check that every G_i, $1 \le i \le n$, is chordal. For every $1 \le i \le n$, we construct the set $\{C_1^i, \ldots, C_{k_i}^i\}$ of the cliques of G_i. Clearly, $k_i \le n$. Let $D_1, \ldots, D_k$ be the maximal elements of $\bigcup_{i=1}^{n} \{C_1^i, \ldots, C_{k_i}^i\}$. Then, G is a Δ circular-arc graph if and only if $\mu(D_1, \ldots, D_k)$ has the circular 1's property. A family F of representing arcs of G can be constructed as in the proof of Theorem 1. Since the number of steps required to test chordality is at most n^4, the above algorithm takes no more than n^5 steps.

Consider a graph G, and let $C_1, \ldots, C_k$ be its cliques.

Theorem 3: The graph G is a θ circular-arc graph if and only if $\mu(C_1, \ldots, C_k)$ has the circular 1's property.

Proof: Assume that G is a θ circular-arc graph and F is the family of representing arcs. By the definition, for every clique C_i, $b_i = \bigcap_{v \in C_i} \bar{v} \neq \phi$. It is easy to see that $b_1, \ldots, b_k$ is the set of primitive arcs, and for every $1 \le i \le k$, $C_i = \{v \mid b_i \subseteq \bar{v}\}$. Thus by Lemma 1, $\mu(C_1, \ldots, C_k)$ has the consecutive 1's property.

Conversely, assume that $\mu(C_1, \ldots, C_k)$ has a circular 1's form. The family $C_1, \ldots, C_k$ is a covering system of G, and we can construct to G a family F of representing arcs as in the proof of Theorem 1. By the construction of F, for every i, $\bigcap_{v \in C_i} \bar{v} = (i-1, i)$. Therefore G is a θ circular-arc graph.

Q.E.D.

Let G be a θ circular-arc graph with n vertices and F its representing family of arcs. For every clique C of G, $\bigcap_{v \in C} \bar{v}$ is a primitive arc. The number of primitive arcs is at most n. *Thus the number of cliques of a θ circular-arc graph is at most n.* A subgraph of G with k vertices is also a θ circular-arc graph and thus it has at most k cliques.

Let G(V) be a given graph. The algorithm for recognizing if G is a θ circular-arc graph works as follows:

First, we must check that the number of its cliques is at most n. We do this by the algorithm described in [6]. For every $1 \le i \le n$, we construct the set P_i of all the cliques of the subraph G^i defined by the vertices $v_1, \ldots, v_i$. For $i = 1$, $P_1 = (\{v_1\})$. Assume that P_{i-1} was constructed. Find:

$$P_i' = \{\{v_i\} \cup (C \cap \Gamma v_i) \mid \text{for every } C \in P_{i-1}\}.$$

Then P_i is the set of maximal elements of $P_i' \cup P_{i-1}$. If in any stage i, the number of elements in P_i is more than i, then G^i is not a θ circular-arc graph, G cannot be either, and we stop. Assume that the process ends successfully. Then $P_n = \{C_1, \ldots, C_k\}$ is the set of cliques of G and $k \le n$. (This process requires at most n^3 steps.) Therefore, G is a θ circular-arc graph if and only if $\mu(C_1, \ldots, C_k)$ has the circular 1's property. This algorithm requires at most n^3 steps.

4. ALGORITHMS FOR A MAXIMUM INDEPENDENT SET AND A MINIMUM COVERING BY CLIQUES OF A CIRCULAR-ARC GRAPH

Consider a circular-arc graph G and its representing family of arcs F. Let us denote the endpoints of the representing arcs consecutively in the clockwise direction by $h_1, h_2, \ldots, h_{2n-1}, h_{2n}, h_1$. For every $1 \le i \le 2n$, denote $a_i = (h_i, h_{i+1})$ and $a_{2n} = (h_{2n}, h_1)$. Also, for every $1 \le i \le 2n$, denote $W_i = \{v \mid v \in V, a_i \subseteq \bar{v}\}$ and $U_i = V - W_i$. Let $K_i(U_i)$ be the subgraph of G defined by U_i. The set of arcs corresponding to the vertices of U_i does not cover the circle, since a_i is not covered. Thus every K_i is an interval graph. Let J be a maximum independent set of G. Hence for every two vertices $u, v \in J$, $\bar{u} \cap \bar{v} = \emptyset$. Clearly, J does not cover the whole circle, and there exists an a_i which intersects no arcs corresponding to vertices of J. Thus J is a maximum independent set of K_i. Therefore $\alpha(G) = \max_{1\le i\le 2n} \alpha(K_i)$. For every interval graph K_i we can find a maximum independent set J_i by the algorithm described in [4]. Then, a set with a maximum number of elements among $J_1,\ldots,J_{2n}$ is a maximum independent set of G. This algorithm requires at most n^4 steps.

Let the number of cliques in a minimum covering by cliques of a graph H be denoted by $\xi(H)$. Every K_i, $1 \le i \le 2n$, is an interval graph, and thus (see [4]) $\alpha(K_i) = \xi(K_i)$. W_i is a completely connected set and if we add it to a minimum covering by cliques of K_i we obtain a covering by completely connected sets of G. Hence

$$\xi(G) \le \min_{1\le i\le 2n} \xi(K_i)+1 = \min_{1\le i\le 2n} \alpha(K_i)+1 \le \alpha(G)+1.$$

But $\alpha(G) \le \xi(G)$. *Thus in a circular-arc graph G, $\alpha(G) \le \xi(G) \le \alpha(G)+1$.*

Consider a circular-arc graph G for which $\alpha(G) = \xi(G)$. There exists an r, $1 \le r \le 2n$, such that $\alpha(K_r) = \alpha(G)$. Clearly, if $v \in U_r$, then $\bar v \cap a_r = \phi$ ($a_r = (h_r,h_{r+1})$). Consider a minimum covering by cliques $C_1,\ldots,C_{\xi(G)}$ of G, and denote $C_i' = C_i - W_r$, for every $1 \le i \le \xi(G)$. Clearly $C_1',\ldots,C'_{\xi(G)}$ is a covering by completely connected sets of K_r and $\xi(G) = \alpha(G) = \alpha(K_r) = \xi(K_r)$. Therefore, every C_i', $1 \le i \le \xi(G)$, is non-empty and $C_1',\ldots,C'_{\xi(G)}$ form a minimum covering by completely connected sets of K_r. For every $1 \le i \le \xi(G)$, denote $t_i = \bigcap_{v\in C_i'} \bar v$. Clearly, $i \ne j$ implies $t_i \cap t_j = \phi$. Assume that $t_1,\ldots,t_{\xi(G)}$ appear in a circular consecutive order and $t_1, t_{\xi(G)}$ are the neighbors of a_r: t_1 is the neighbor of h_r and $t_{\xi(G)}$ is the neighbor of h_{r+1}. Let $\bar u_1 = (x_1,y_1)$, $\bar u_2 = (x_2,y_2)$ be the arcs corresponding to the vertices of U_r such that (x_1,h_r) contains no left endpoints of arcs $\bar v$, $v \in U_r$, and (h_{r+1},y_2) contains no right endpoints of arcs $\bar v$, $v \in U_r$. Then, x_1 is the left endpoint of t_1 and y_2 is

the right endpoint of $t_{\xi(G)}$, since otherwise u_1 or u_2 is not covered by $C_1',\ldots,C'_{\xi(G)}$. Let us assume that there exists a vertex $v \in W_r$ ($a_r \subseteq \bar v$) such that $\bar v$ does not intersect all the arcs corresponding to the vertices of C_1' and also it does not intersect all the arcs corresponding to the vertices of $C'_{\xi(G)}$. Therefore, $v \notin C_1$ and $v \notin C_{\xi(G)}$. Clearly $\bar v \subset (x_1,y_2)$. For some j, $1 < j < \xi(G)$, C_j contains v and thus $\bar v$ intersects every arc $\bar u$, $u \in C_j'$. Therefore, every arc $\bar u$, $u \in C_j'$, contains x_1 or y_2 and hence $C_j' \subseteq C_1' \cup C'_{\xi(G)}$, contradicting the fact that $C_1',\ldots,C'_{\xi(G)}$ form a minimum covering by completely connected sets of K_r. Therefore, for every $v \in W_r$, $\bar v$ intersects all the arcs $\bar u$, $u \in C_1'$, or $\bar v$ intersects all the arcs $\bar w$, $w \in C'_{\xi(G)}$. For an arc a, denote $V_a = \{v | v\in U_r, a \subseteq \bar v\}$. For two arcs a,b, let K(a,b) be the subgraph of K_r defined by $U_r - (V_a \cup V_b)$. *Thus if $\alpha(G) = \xi(G)$, then there exist two arcs $t_1 = (x_1,t^1)$, $t_2 = (t^2,y_2)$, $t^1 \in (x_1,h_r)$, $t^2 \in (h_{r+1},y_2)$, such that $\xi(K(t_1,t_2)) \le \xi(K_r)-2$, and for every $v \in W_r$, $\bar v$ intersects all the arcs $\bar u$, $u \in V_{t_1}$, or $\bar v$ intersects all the arcs, $\bar w$, $w \in V_{t_2}$.*

The algorithm for finding a minimum covering by cliques of a circular-arc graph G works as follows.

Find a K_r such that $\xi(K_r) = \alpha(K_r) = \alpha(G)$. Let $\bar u_1 = (x_1,y_1)$, $\bar u_2 = (x_2,y_2)$ be the arcs corresponding to vertices of U_r such that (x_1,h_r) contains no left endpoints of arcs $\bar v$, $v \in U_r$, and (h_{r+1},y_2) contains no right endpoints of arcs $\bar v$, $v \in U_r$. Let A be the set of all the arcs a, $a = (x_1,y)$, such that y is a right endpoint of an arc of F and $y \in (x_1,h_r)$. Similarly, let B be the set of arcs b, $b = (x,y_2)$, such that x is the left endpoint of an arc of F and $x \in (h_{r+1},y_2)$. Clearly $|A|,|B| \le n$. For every arc $a \in A \cup B$, let

$$W_r^a = \{v | v\in W_r, \bar v \text{ intersects every } \bar u, u\in V_a\}.$$

Let $X = \{\langle a,b\rangle | a\in A, b\in B, W_r^a \cup W_r^b = W_r\}$.

Clearly $|X| \leq n^2$. If $X = \phi$, then by the previous remark $\xi(G) = \alpha(G)+1$. Let us assume that $X \neq \phi$. For every $\langle a,b\rangle \in X$, find a minimum covering by cliques of $K(a,b)$. If for some $\langle a,b\rangle \in X$, $\xi(K(a,b)) \leq \xi(K_r)-2$, then the minimum covering by cliques of $K(a,b)$ together with $V_a \cup W_r^a$ and $V_b \cup W_r^b$ form a minimum covering, with $\xi(K_r) = \alpha(G)$ completely connected sets of G and $\xi(G) = \alpha(G)$. If, for every $\langle a,b\rangle \in X$, $\xi(K(a,b)) > \xi(K_r)-2$, then, by the previous remark, $\xi(G) = \alpha(G)+1$. If $\xi(G) = \alpha(G)+1$, then a minimum covering by completely connected sets of G can be obtained by adding W_r to a minimum covering by cliques of K_r.

The above algorithm requires at most n^5 steps.

5. AN ALGORITHM FOR A MAXIMUM CLIQUE OF A CIRCULAR-ARC GRAPH

Consider a circular-arc graph $G(V)$ and its representing family of arcs F. Every vertex v_i is represented by an arc $\bar{v}_i = (e_i, f_i)$. Let

$$X_i = \{v | v \in V \text{ and } e_i \in \bar{v}\} \cup \{v_i\}$$
$$Y_i = \{v | v \in V - X_i \text{ and } f_i \in \bar{v}\}.$$

Consider the subgraph M_i defined by $X_i \cup Y_i$. X_i and Y_i are completely connected sets. Thus the complement M_i' of M_i is a bipartite graph. Therefore, we can obtain a maximum clique of M_i by applying to M_i' the algorithm for finding a maximum independent set, described in [10].

Let C be a clique of G. There exists a vertex $v_i \in C$ such that for any other vertex v of C, $\bar{v} \not\subseteq \bar{v}_i$. Hence, for every $v \in C$ such that $v \neq v_i$, there exists $e_i \in \bar{v}$ or $f_i \in \bar{v}$. Therefore, C is a clique of M_i. Thus a maximum clique of the circular-arc graph G can be obtained as follows:

for every v_i, $1 \leq i \leq n$, construct the subgraph M_i;
for every $1 \leq i \leq n$, find a maximum clique C_i of M_i;

a clique with a maximum number of vertices among $C_1,\ldots,C_n$ is a maximum clique of G.

This algorithm required at most n^3 steps.

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Supported by the Rome Air Development Center under Contract F30602-72-C-0281.

Paper received January 28, 1974.