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COMMUNICATION

ON THE COMPLEXITY OF RECOGNIZING PERFECTLY ORDERABLE GRAPHS*

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The question whether a polynomial time recognition algorithm for the class of perfectly orderable graphs exists was posed by Chvátal in 1981 when he introduced the notion of perfect orders. Since then several classes of perfectly orderable graphs have been identified. In this note we prove that recognizing perfectly orderable graphs is *NP*-complete.

1. Introduction and counterexample

One canonical way to colour the vertices of a graph by positive integers, so that adjacent vertices receive distinct colours is to enumerate the vertices of the graph as $v_1, \ldots, v_n$; then to scan the sequence from v_1 to v_n and to assign to each v_i the smallest colour available.

A graph G is called *perfectly ordered* if its ordering is such that for each induced subgraph H of G the above greedy procedure gives with the induced order on H an optimal colouring of H. A graph is called *perfectly orderable* if it admits a perfect order.

Chvátal [1] proved that a graph G is perfectly orderable if and only if G admits an acyclic orientation such that for no induced path $P = p_1p_2p_3p_4$ the edges p_1p_2 and p_3p_4 are oriented from p_1 to p_2 and from p_4 to p_3.

Comparability graphs form a subclass of perfectly orderable graphs. They can be characterized as follows: A graph G is a comparability graph iff it admits an acyclic orientation that is alternating on every induced path of three vertices (compare Ghouila-Houri [2] and Gilmore and Hoffman [3]). For comparability graphs a polynomial time recognition algorithm based on the following procedure works. First choose an edge and assign an orientation to it. Then look if the orientation of some other edges is now determined by the forbidden configuration (*Fig. 1*). If so, orient these edges the admissible way and look if now the orientation of further edges is determined and continue the process.

If a graph is not a comparability graph, it has an edge e such that the above process yields a directed cycle (possibly of length 2) if one chooses e as starting edge.

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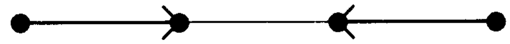

Fig. 1. Forbidden configuration.

A similar statement is true for the class of P_4-comparability graphs, that was introduced by Hoàng and Reed [4] as a subclass of perfectly orderable graphs that generalizes comparability graphs. A graph is a P_4-comparability graph if and only if it can be oriented acyclically such that every induced path of four vertices is oriented alternating.

The graph of *Fig. 2* shows that the analogous recognition algorithm for perfectly orderable graphs does not work.

The graph of *Fig. 2* is not perfectly orderable. Any orientation of edge *a* forces one of the edges *b* to have the depicted orientation. This forces edge *c* of the inner triangle to be oriented as shown. By the symmetries of the graph it follows that the inner triangle is forced to form a directed cycle. It is easy to check, that for no edge of the graph both possible orientations yield a directed cycle.

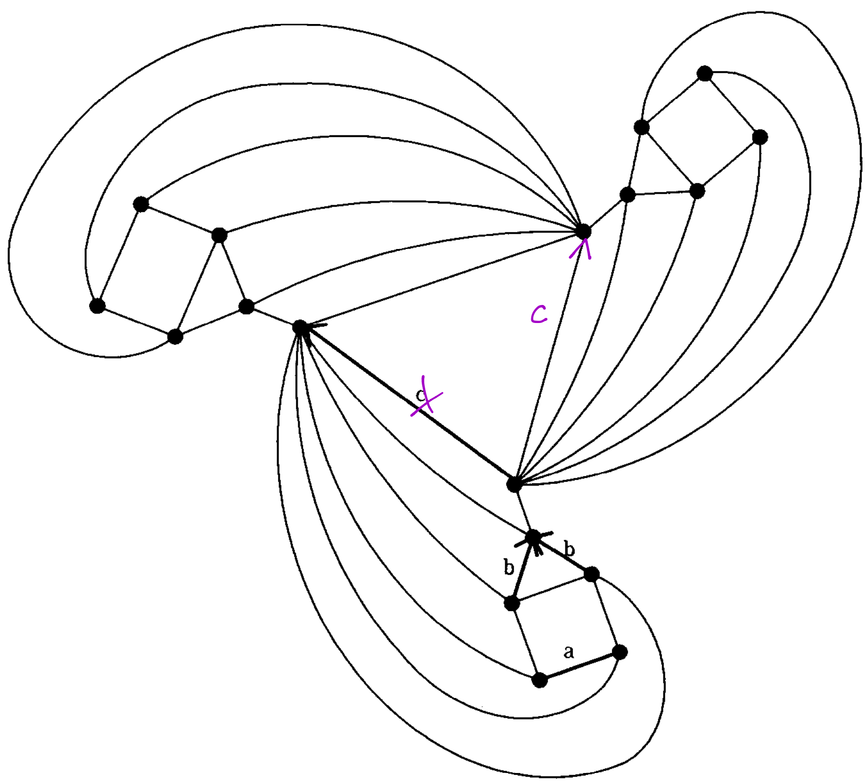

Fig. 2. A counterexample.

2. The theorem

Theorem. *To decide whether a graph admits a perfect order is NP-complete.*

Proof. We give a reduction of 3SAT to the problem of deciding whether a graph admits an acylic orientation such that for no induced path $P = p_1 p_2 p_3 p_4$ the edge $p_1 p_2$, is oriented from p_1 to p_2 and the edge $p_3 p_4$ is oriented from p_4 to p_3.

Given a 3SAT formula $\mathscr{C}$ with clauses $C_0 = c_{00}c_{01}c_{02}$, $C_1, \ldots, C_{n-1}$ and variables $v_0, \ldots, v_{n-1}$ we define a graph $G(C)$ such that $G(C)$ is perfectly orderable iff $\mathscr{C}$ is satisfiable.

First we define for every variable v_i a graph $G(v_i)$ (see *Fig. 3*) with vertex-set:

$$\{v(i, j, 1); j = 0, \ldots, 2n-1\}$$

$$\cup \{v(i, 2j, 2), v(i, 2j, 3);\ v_i \text{ occurs in clause } C_j\}$$

$$\cup \{v(2j+1, 2), v(i, 2j+1, 3);\ \neg v_i \text{ occurs in clause } C_j\};$$

and edge-set:

$$\{v(i, j, 1)v(i, j+1, 1); j = 0, \ldots, 2n-2\} \cup \{v(i, 2n-1, 1)v(i, 0, 1)\}$$

$$\cup \{v(i, j, 1)v(i, j, 2),\ v(i, j, 2)v(i, j, 3);\ v(i, j, 2), v(i, j, 3) \in V(G(v_i))\}.$$

For every clause $C_i = (c_{i0}, c_{i1}, c_{i2})$ we define a graph $G(C_i)$ (see *Fig. 4*) with

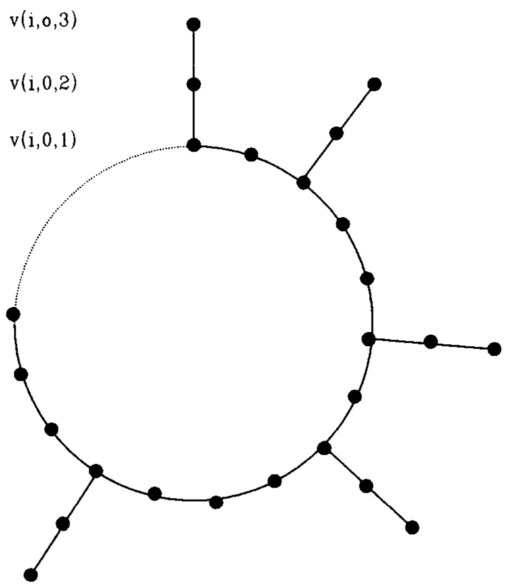

Fig. 3. $G(v_i)$.

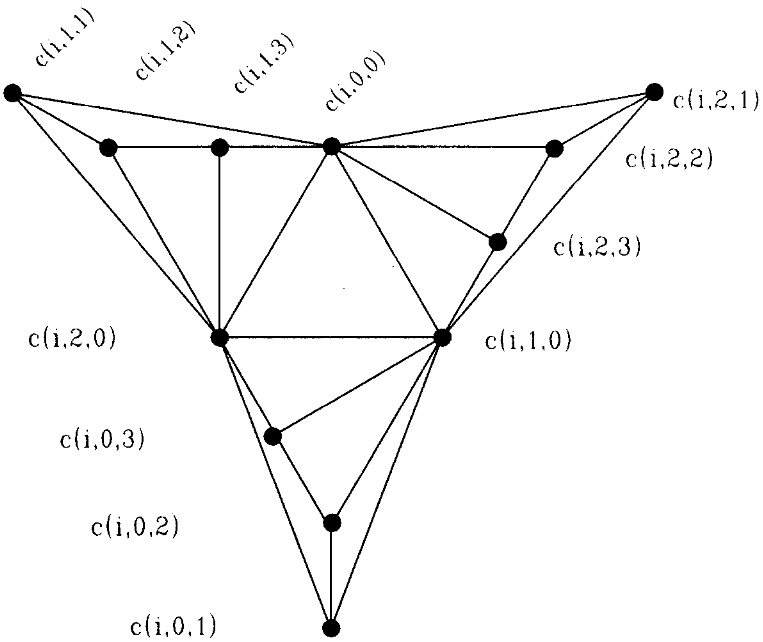

Fig. 4. $G(C_i)$.

vertex-set: $\{c(i, j, k); j=0, 1, 2; k=0, 1, 2, 3\}$, and edge-set:

$$\{c(i, j, 0)c(i, k, 0); 0 \leq j < k \leq 2\}$$
$$\cup \{c(i, j, 1)c(i, j, 2), c(i, j, 2)c(i, j, 3); j=0, 1, 2\}$$
$$\cup \{c(i, j, 1)c(i, k, 0); j \neq k; j, k=0, 1, 2\}$$
$$\cup \{c(i, j, 2)c(i, k, 0); k \equiv j+1 \bmod 3; j, k=0, 1, 2\}$$
$$\cup \{c(i, j, 3)c(i, k, 0); j \neq k, j, k=0, 1, 2\}.$$

Now we build $G(\mathscr{C})$. For $i, j=0, \ldots, n-1$ we identify the vertices $v(i, 2j, k)$ and $c(j, l, k)$ if $c_{jl}=v_i$; $k=1, 2, 3$ and $v(i, 2j+1, k)$ and $c(j, l, k)$ if $c_{jl}=\neg v_i$; $k=1, 2, 3$. Furthermore we introduce for $i=0, \ldots, n-1$; $k=0, 1, 2$ all edges $c(i, k, 0)x$ for every vertex x of $G(\mathscr{C}) - G(C_i)$. For $k=0, 1, 2, 3$ set $V_k :=$ $\{c(i, j, k); i=0, \ldots, n-1; j=0, 1, 2\}$. The elements of V_k are called vertices of type k.

Notice that a vertex $c(i, j, 0)$ is adjacent to every vertex of $G(\mathscr{C})$ except $c(i, j, k)$; $k=1, 2, 3$ and $c(i, l, 2)$ with $l \equiv j+1 \bmod 3$; $l \in \{0, 1, 2\}$.

Claim. If $G(\mathscr{C})$ is perfectly orderable then $\mathscr{C}$ is satisfiable.

Take an acyclic orientation of $G(\mathscr{C})$ with no forbidden configuration (see *Fig. 1*). Since the circuits in the graphs $G(v_i)$ are induced and have length >3 they are oriented alternating. For $i=0, \ldots, n-1$ we assign the value true to variable v_i if the vertex $v(i, 0, 1)$ (and thus each vertex $v(i, 2j, 1)$, $j = 0, \ldots, n-1$) is a source with respect to the edges of the circuit in $G(v_i)$, otherwise we assign the value false to variable v_i.

Since our orientation is acyclic, for every clause C_i at least one of the edges of the inner triangle in $G(C_i)$ is oriented from $c(i, k, 0)$ to $c(i, l, 0)$ with $l \equiv k - 1 \bmod 3$. (The inner triangle is the triangle of the three vertices of type 0 in $G(C_i)$.) Since $c(i, k, 0)c(i, l, 0)c(i, k, 3)c(i, k, 2)$ form an induced P_4, it follows that $c(i, k, 3)c(i, k, 2)$ is oriented from $c(i, k, 3)$ to $c(i, k, 2)$.

Then it follows that $c(i, k, 1)$ is a source in the circuit in $G(v_r)$, with r, s such that $c(i, k, 1) = v(r, s, 1)$. If s is even it follows that $c_{ik} = v_r$ and v_r was set true; if s is odd it follows that $c_{ik} = \neg v_r$ and v_r was set false. Thus our truth assignment defined satisfies $\mathscr{C}$. $\square$

Now we prove that from any truth assignment that satisfies $\mathscr{C}$ we can construct an acyclic orientation that proves the graph $G(\mathscr{C})$ to be perfectly orderable.

Given a truth assignment we first define an orientation for the graphs $G(v_i)$. We give the circuits in the $G(v_i)$'s an alternating orientation such that $v(i, 0, 1)$ is a source with respect to the circuit edges iff v_i has value true.

If for $i = 0, \ldots, n-1$ and $j = 0, \ldots, 2n-1$ the verex $v(i, j, 1)$ is a sink in the circuit of $G(v_i)$ the edge $v(i, j, 2)v(i, j, 3)$ (if it exists) is oriented from $v(i, j, 2)$ to $v(i, j, 3)$. Otherwise it is oriented from $v(i, j, 3)$ to $v(i, j, 2)$. Notice that an edge $v(i, j, 2)v(i, j, 3)$ is an edge $c(k, l, 2)c(k, l, 3)$ if $c_{kl} = v_i$ and $j = 2k$ or if $c_{kl} = \neg v_i$ and $j = 2k + 1$.

Edges $c(i, j, 0)c(i, k, 0)$ with $k \equiv j + 1 \bmod 3$ are oriented from $c(i, j, 0)$ to $c(i, k, 0)$ if $c(i, k, 2)c(i, k, 3)$ is oriented from $c(i, k, 2)$ to $c(i, k, 3)$, which is exactly the case if $c_{ij} = v$ and v is set false or if $c_{ij} = \neg v$ and v is set true.

Since the formula $\mathscr{C}$ is satisfied we have so far no directed cycle in the graph $G(V_0)$. Thus we can exend our partial orientation to an acyclic orientation on $G(V_0)$ and fix this orientation.

Up to now we have not given an orientation to any edge xy where x is of type 0 and y is not of type 0. We orient all these edges from x to y.

The only edges that remain unoriented are the edges xy where x is of type 1 and y of type 2. We orient these edges from x to y.

Claim. Our orientation is acyclic.

Since every edge xy with x of type 0 and y not of type 0 is oriented from x to y no directed cycle K can contain both vertices of type 0 and vertices of other types. Since the orientation of $G(V_0)$ was chosen acyclic, K contains no vertex of type 0. Thus K is the circuit in $G(v_i)$ for some variable v_i. But these circuits were oriented alternating and we are done. $\square$

Claim. No induced path $P = p_1p_2p_3p_4$ is oriented such that p_1p_2 is oriented from p_1 and p_3p_4 is oriented from p_4 to p_3.

Assume for a contradiction that we have such a path P. Since $G(V_0)$ is a complete graph P cannot contain more than two vertices of type 0.

Case 1. P contains no vertex of type 0.

Then P lies completely in $G(v_i)$ for some variable v_i. Since edges xy with x of type 1 and y of type 2 are oriented from x to y, P cannot have an endpoint of type 2. Edges $v(i,j,2)v(i,j,3)$ are oriented from $v(i,j,3)$ to $v(i,j,2)$ only if $v(i,j,1)$ is a source in the circuit of $G(v_i)$. Thus P cannot have an endpoint of type 3. It follows that both endpoints are of type 1 and thus P lies completely in the circuit of $G(v_i)$ which is impossible since this circuit was oriented alternating.

Case 2. P contains exactly one vertex of type 0.

Since all edges xy with x of type 0 and y of different type are oriented from x to y the vertex of type 0 is without loss of generality p_1. Let $c(i,j,0)$ be this vertex. p_3 and p_4 are adjacent and both non-neighbours of p_1. Thus $\{p_3, p_4\} = \{c(i,j,1), c(i,j,2)\}$ or $\{p_3, p_4\} = \{c(i,j,2), c(i,j,3)\}$. p_2 is common neighbour of $p_1 = c(i,j,0)$ and p_3 which is not of type 0. Thus $p_3 = c(i,j,1)$ and $p_4 = c(i,j,2)$ which contradicts the fact that edges xy with x of type 1 and y of type 2 are oriented from x to y.

Case 3. P contains exactly two vertices of type 0.

Since $G(V_0)$ is complete and edges xy with x of type 0 and y different type are oriented from x to y, the vertices of type 0 must be without loss of generality p_1 and p_2. p_4 is a common non-neighbour of p_1 and p_2. Thus p_4 is $c(i,j,2)$ for some i and j and $\{p_1, p_2\} = \{c(i,j,0), c(i,k,0)\} k \equiv j-1 \bmod 3$. Thus P is either $c(i,j,0)c(i,k,0)c(i,j,3)c(i,j,2)$ or $c(i,j,0)c(i,k,0)c(i,j,1)c(i,j,2)$. But $c(i,j,1)c(i,j,2)$ is oriented from $c(i,j,1)$ to $c(i,j,2)$, and $c(i,j,0)c(i,k,0)$ was oriented from $c(i,k,0)$ to $c(i,j,0)$ if $c(i,j,2)c(i,j,3)$ is oriented from $c(i,j,2)$ to $c(i,j,3)$. $\square$

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