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Linear time algorithms on circular-arc graphs *

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Abstract

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Circular-arc graphs are rich in combinatorial structures. Various characterization and optimization problems on circular-arc graphs have been studied. In this paper, we present an extremely simple O(n) algorithm which simultaneously solves the following three problems (the unweighted version) on circular-arc graphs: the maximum independent set, the minimum clique cover, and the minimum dominating set problems; whereas the best previous bounds for the latter two problems were O(n^2) and O(n^3), respectively. Our approach takes advantage of the underlying structure of circular-arc graphs that is amenable to greedy algorithms.

Keywords: Design of algorithms, combinatorial problems

1. Introduction

Circular-arc graphs are rich in combinatorial structures. Various characterization and optimization problems on circular-arc graphs have been studied (some of them are listed in the bibliography). In [5], Masuda and Nakajima gave an O(n) time algorithm for the (unweighted) maximum independent set problem (MIS) on circular-arc graphs (assuming a circular-arc model is given and the endpoints have already been sorted in the clockwise order). Golumbic and Hammer [2] considered both the graph model and the arc model and gave O($m \cdot \delta$) and O(n) algorithms, respectively, for the MIS problem. Independently, Manacher (private communication) produced simpler alternate algorithms for this problem.

In this paper, we present an extremely simple O(n) algorithm which simultaneously solves the following three problems (the unweighted version) on circular-arc graphs: the maximum independent set, the minimum clique cover, and the minimum dominating set problems; whereas the best previous bounds for the latter two problems were O(n^2) [3] and O(n^3) [1], respectively. Our

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approach takes advantage of the underlying structure of circular-arc graphs that is amenable to greedy algorithms. The basic idea is to define a function NEXT(i) and a greedy solution GD(i) for every arc i such that GD(NEXT(i)) is at least as good as GD(i). Starting with any arc j, apply the NEXT function to j repeatedly until some arc j^* appears twice in the list (namely, j^* is in a cycle). Then, GD(j^*) can be shown to be an optimal solution.

A circular-arc family F is a collection of arcs in a circle. Denote by $G=(V, E)$, a graph with a finite vertex set V and a set E of edges connecting vertices of V. A graph G is a *circular-arc graph* if there is a circular arc family F and a one-to-one mapping of the vertices of G and the arcs in F such that two vertices in G are adjacent if and only if their corresponding arcs in F overlap. For convenience, we shall consider arcs in the family F rather than vertices in its corresponding graph G. Let n be the number of arcs in F. A set of arcs is *independent* if no two of them overlap with each other. The *maximum independent set* (*MIS*) *problem* on an arc family F is to determine a maximum size (denoted by $\alpha(F)$) independent set in F. A *clique* in F is a set of mutually overlapping arcs. The *minimum clique cover* (*MQC*) problem is to determine a minimum size (denoted by $\theta(F)$) collection of cliques whose union is F. A set D in F is a *dominating set* if every arc in $F \setminus D$ overlaps with some arc in D. The *minimum dominating set* (*MDS*) problem is to determine a minimum size (denoted by $\tau(F)$) dominating set in F.

Without loss of generality, assume all arc endpoints are distinct and no arc covers the entire circle. Label the n arcs arbitrarily from 1 through n. Denote an arc i that begins at endpoint p and ends at endpoint q in the clockwise direction by (p, q). Define p to be the *head* of i (denoted by $h(i)$) and q to be the *tail* of i (denoted by $t(i)$).

The continuous part of the circle that begins with an endpoint c and ends with d in the clockwise direction is referred to as *segment* (c, d) of the circle. We consider the segment (c, d) as "open", namely, points c and d are not in (c, d). Define $[c, d)$ to be a segment containing c but not d. Similarly, one can define $(c, d]$ and $[c, d]$.

An arc (p, q) of F is also regarded as an open segment (p, q). A point on the circle is said to be *in* arc (p, q) if it is contained in segment (p, q).

An arc i (denoted by (a, b)) is said to be *contained in* another arc j (denoted by (c, d)) if segment (a, b) is contained in segment (c, d). An arc family F is said to be *proper* if no arc in F is contained in another. An arc in F is *minimal* if it does not contain any other arc of F. An arc in F is *maximal* if it is not contained in any other arc of F. Define the proper subfamily F' of F to be the collection of all minimal arcs in F and the family F'' of F to be the collection of all maximal arcs in F. It can be shown that solutions for the MIS and the MQC problem can be restricted to F' and the solutions for the MDS problem can be restricted to F''.

We divide our discussion as follows. In Section 2, a new $O(n)$ algorithm for the MIS problem is presented. The idea of this algorithm serves as a basis for solving the other two problems. We solve the MQC problem in Section 3 and the MDS problem in Section 4. Throughout this paper, we assume the endpoints of F are already sorted.

2. The maximum independent set (MIS) problem

Our approach for solving the MIS problem is based on a NEXT($\cdot$) function and greedy independent sets GD($\cdot$) defined below.

Definition. For each arc i, define NEXT(i) to be the arc j in F whose head is contained in $(t(i), t(j))$ and whose tail is first encountered in a clockwise traversal from $t(i)$. Define GD(i) to be the maximal independent set of the form $\{i_1(=i), i_2, \ldots, i_k\}$, where $i_t = \text{NEXT}(i_{t-1})$, $t = 2, \ldots, k$ (GD is short for GREEDY). Finally, define LAST(i) to be NEXT(i_k) (note that LAST(i) must overlap with i).

It is easy to verify that, for each arc i, NEXT(i) belongs to the proper subfamily F'.

Lemma 2.1. $|\text{GD}(i)|$ *is maximum among all independent sets containing* i.

Proof. Let GD(i) be $\{i(=i_1), i_2,\ldots,i_k\}$. Let P be an independent set containing i of maximum size. Since arcs $i_2, i_3,\ldots,i_k$ are minimal, each of the segments $[t(i_2), t(i_3))$, $[t(i_3), t(i_4)),\ldots,$ $[t(i_{k-1}), t(i_k))$, $[t(i_k), h(i_1))$ contains at most one tail of an arc in P. Hence, there are at most $k-1$ other arcs in P besides i and $|P| \leqslant k = |\mathrm{GD}(i)|$. □

Construct the directed graph $D=(V(D), E(D))$ with $V(D)=F$ and $(i, j)\in E(D)$ iff $j=\mathrm{NEXT}(i)$. Since each vertex of D has out-degree 1, D must contain a directed cycle and no two directed cycles can share a common vertex. We shall use the notions "node in D" and "arc in F" interchangeably. An arc i is called a *good arc* if $|\mathrm{GD}(i)| = \alpha(F)$.

Lemma 2.2. *Suppose $\alpha(F)>1$. Then D contains at least one directed cycle consisting of good arcs.*

Proof. Pick a node i in D that corresponds to a good arc of F. Since $\alpha(F)>1$, NEXT(i) is contained in GD(i). By Lemma 2.1, GD(NEXT(i)) is also an MIS. Define $i_1=i$ and $i_{s+1}=\mathrm{NEXT}(i_s)$ for $s\geqslant 1$. Then, the same argument implies that each i_s is a good arc. Since each vertex in D has out-degree 1, there must exist indices a and b such that $a<b$ and $i_a=i_b$. Hence, there exists a cycle C consisting of nodes of the form i_s, which is a desired directed cycle. □

Theorem 2.3. *Let C be any directed cycle in D. Each arc in C is a good arc.*

Proof. Let C and C^* be two disjoint cycles in D such that each node in C^* is a good arc in F. Since each node in $C\cup C^*$ is the NEXT arc of some other arc, we have $C\cup C^*\subseteq F'$.

Let i, j be a pair of nodes in C^* and C, respectively, such that no head of any other node in $C^*\cup C$ falls in $(h(j), h(i))$. Denote GD(j) by $\{j(=j_1), j_2,\ldots,j_k\}$. By the definition of GD(j), the endpoints $t(j_3), t(j_4),\ldots,t(j_k), t(j_{k+1})$ are clockwise ordered and contained in $(t(j_2), t(j_1)]$. Since $C\cap C^*$ is empty, no tail of any arc in C can coincide with one in C^*. Now, each of the segments $(t(j_1), t(j_2)]$, $(t(j_2), t(j_3)]$, $(t(j_3), t(j_4)],\ldots,(t(j_k), t(j_{k+1})]$ contains at most one tail of arcs in GD(i). If $j_1=j_{k+1}$, then $|\mathrm{GD}(i)|\leqslant k \leqslant |\mathrm{GD}(j)|$. Otherwise, suppose $(t(j_{k+1}), t(j_1))$ contains the tail of an arc v in GD(i). Since $t(j_{k+1})$ is contained in j_1, we must have $t(j_{k+1})\in (h(j_1), t(v))$, which is contained in $(h(j_1), h(i_1))$. By the selection of i and j, NEXT(j_{k+1}) would have been i, contradictory to the fact that C and C^* are disjoint. Hence, $\alpha(F)=|\mathrm{GD}(i)|\leqslant k = |\mathrm{GD}(j)|$ and j is a good arc. □

Note that the size of a directed cycle could be greater than $\alpha(F)$. By the above analysis, to solve the MIS problem, it suffices to determine a node i in a directed cycle of D and find GD(i). The NEXT($\cdot$) function on every arc i in F can be calculated in O(n) time in a clockwise traversal. The set GD(i) for a specified arc i can also be calculated in O(n) time. The following GREEDY algorithm correctly determines a good arc in F. We set L[i] = 1 whenever arc i has been visited.

Algorithm GREEDY

1. Set $L[i]=0$, for each arc i in F;
   Arbitrarily select an arc i in F;
2. **While** $L[i]=0$ **do** $\{L[i]\leftarrow 1;\ i\leftarrow \mathrm{NEXT}(i)\}$;
3. Find GD(i);

3. The minimum clique cover (MQC) problem

We shall show that, by using a new set function $\mathrm{GD_q}(i)$ (a modification of GD(i)), the GREEDY algorithm can also be used to solve the MQC problem. There are two types of cliques in a circular-arc family. The first type of cliques contains three arcs which do not contain a common point of the circle. Such three arcs are referred to as a *singular triple*. The second type is defined below.

Definition. A clique is said to be *linear* if all arcs in the clique contain a common point of the circle. Denote by LQ(i) the linear clique consisting of i and all arcs which contain $t(i)$.

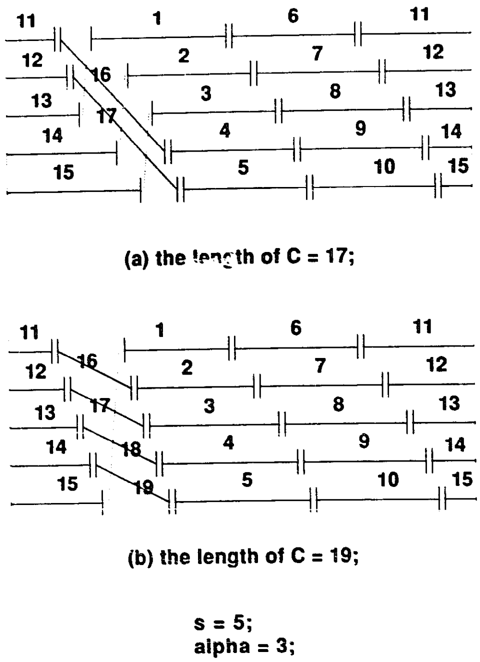

Fig. 1. The length of C.

Note that the cliques in an MQC are not required to be vertex disjoint. F is a clique itself if and only if $\alpha(F)=1$. We show that it suffices to use linear cliques in an MQC unless $\alpha(F)=1$. Hence, consider the case that $\alpha(F)>1$. Let C be a cycle in D. Define the *density of C at $t(i)$* to be the number of arcs (including i) in C which contain $t(i)$. Define the *density* s of the cycle C to be the maximum density among all tails of arcs in C.

Lemma 3.1. *If* $|C|>\alpha(F)>1$, *then* $\alpha(F)\cdot s<|C|<(\alpha(F)+1)\cdot s$.

Proof. Denote $|C|$ by r. Let i be an arc in C such that the density at $t(i)$ is s. Denote C by $\{i(=i^1), i^2,\ldots,i^r\}$, indexed by their clockwise tail order (see Fig. 1). Since the density at $t(i^1)$ is s, arc $i^2, i^3,\ldots,$ and i^s contain $t(i^1)$. Since $\alpha(F)>1$, i^{s+1} does not overlap with i^1. Hence $\mathrm{NEXT}(i^1)=i^{s+1}$. Since $\mathrm{NEXT}(i^2)\neq i^{s+1}$, $h(i^{s+1})$ must be in segment $(t(i^1), t(i^2))$. Therefore, the density at $t(i^2)$ is also s, and segment $(t(i^1), t(i^2))$ contains exactly one head $(h(i^{s+1}))$. By the same argument, $\mathrm{NEXT}(i^2)=i^{s+2}$. By induction, the density at the tail of any arc in C is s, and

$$\mathrm{NEXT}(i^m)=i^{m+s\ (\mathrm{mod}\ r)},$$

for $m=1, 2, 3,\ldots,r$. Then.

$$\mathrm{NEXT}^m(i^1)=i^{ms+1\,(\mathrm{mod}\ r)}.$$

Since $\mathrm{LAST}(i^1)\neq i^1$, $t(\mathrm{LAST}(i^1))$ is contained in i^1,

$$\mathrm{NEXT}(\mathrm{LAST}(i^1))\in\{i^2, i^3,\ldots,i^s\}.$$

Let $i^k=\mathrm{NEXT}(\mathrm{LAST}(i^1))$, where $2\leqslant k\leqslant s$. Then inductively, one can show that

$$\mathrm{NEXT}(\mathrm{LAST}(i^m))=i^{k+m-1},$$

for $1\leqslant m\leqslant s-k+1$, and $\mathrm{LAST}(i^m)=i^{k+m-s-1}$, for $s\geqslant m>s-k+1$. Thus, the arcs in C can be partitioned into the following sets:

$$\mathrm{GD}(i^1), \mathrm{GD}(i^2),\ldots,\mathrm{GD}(i^s),$$

$$\{\mathrm{LAST}(i^1), \mathrm{LAST}(i^2),\ldots,\mathrm{LAST}(i^{s-k+1})\}.$$

Therefore, $|C|=\alpha(F)\cdot s+(s-k+1)$. Since $2\leqslant k\leqslant s$, the lemma follows. □

Definition. For each arc i in a cycle of D, let $\mathrm{GD}(i)=\{i(=i_1), i_2,\ldots,i_{\alpha(F)}\}$, where $i_t=\mathrm{NEXT}(i_{t-1})$, $t=2,\ldots,\alpha(F)$. Define the greedy clique cover $\mathrm{GD_q}(i)$ to be

$$\{\mathrm{LQ}(i_1), \mathrm{LQ}(i_2),\ldots, \mathrm{LQ}(i_{\alpha(F)})\}$$

if $\mathrm{LAST}(i)=i$; and

$$\{\mathrm{LQ}(i_1), \mathrm{LQ}(i_2),\ldots,\mathrm{LQ}(i_{\alpha(F)+1})\},$$

otherwise.

Theorem 3.2. *Assume F is not a clique. Let i be any node in a cycle C of D. Then, $\mathrm{GD_q}(i)$ is an MQC for F.*

Proof. Denote C by $\{i(=i_1), i_2,\ldots,i_k\}$. Consider two cases:

Case (1): $\mathrm{LAST}(i)=i$. Then the tails of arcs in $\mathrm{GD}(i)=\{i(=i_1), i_2,\ldots,i_{\alpha(F)}\}$ divide the circle into $\alpha(F)$ segments. Suppose there is an arc j in $F\setminus\mathrm{GD}(i)$ that contains none of the tails of

$i_1, i_2, \ldots,$ and $i_{\alpha(F)}$. Then, j must be contained in a segment $(t(i_r), t(i_{r+1}))$ for some r in $\{1, \ldots, \alpha(F)\}$. Since $C \subseteq F'$, $h(j)$ must be contained in $(t(i_r), h(i_{r+1}))$. But then, $\text{NEXT}(i_r)$ would have been j instead of i_{r+1}, a contradiction. Hence, every arc in $F \setminus \text{GD}(i)$ must contain one of the tails of $i_1, i_2, \ldots,$ and $i_{\alpha(F)}$, and $\text{GD}_q(i) = \{\text{LQ}(i_1), \ldots, \text{LQ}(i_{\alpha(F)})\}$ is a clique cover for F. Since $|\text{GD}_q(i)| = \alpha(F) \leqslant \theta(F)$, $\text{GD}_q(i)$ is an MQC for F.

Case (2): $\text{LAST}(i) \neq i$. By an argument similar to the above, we have that every arc in $F \setminus \text{GD}(i)$ must pass through one of the tails of $i_1, i_2, \ldots,$ and $i_{\alpha(F)+1}$, and $\text{GD}_q(i) = \{\text{LQ}(i_1), \ldots, \text{LQ}(i_{\alpha(F)+1})\}$ is a clique cover for F, namely, $\theta(F) \leqslant \alpha(F) + 1$. Since $\theta(F) \geqslant \theta(C)$, it suffices for us to show that $\theta(C) > \alpha(F)$.

Denote the density of C by s. Any linear clique in C can contain at most s nodes. Since $|C| > \alpha(F) \cdot s$ by Lemma 3.1, any cover of C consisting of linear cliques must contain at least $\alpha(F) + 1$ cliques. If $\alpha(F) > 2$, C cannot contain any singular triple (every clique in C is linear) and $\theta(C)$ must be $\alpha(F) + 1$.

Hence, consider the case that $\alpha(F) = 2$. Suppose, on the contrary, $\theta(C) = 2$, and (Q_0, Q_1) is an MQC for C. Choose two arcs i and j in C such that no tail of any arc in C falls in segment $(t(i), t(j))$ and $i \in Q_0$, $j \in Q_1$ (see Fig. 2). Denote $\text{NEXT}(i)$ and $\text{NEXT}(j)$ by i_2 and j_2, respectively. Denote $\text{NEXT}(i_2)$ by i_3. Since C is proper, no tail of any arc in C can fall in segment $(t(i_2), t(j_2))$. The head of i_2 must be in segment $(t(i), t(j))$, hence $h(i_3)$ is in segment $(t(i_2), t(j_2))$. Since j_2 does not overlap with j, j_2 belongs to Q_0 and overlaps with i. Since i does not overlap with i_2, $h(i)$ must be in segment $(t(i_2), t(j_2))$. Therefore, both $h(i)$ and $h(i_3)$ are in segment $(t(i_2), t(j_2))$, and by assumption, $i_3 = \text{LAST}(i) \neq i$. Since C is a cycle in D, there must exist a unique arc i′ in C such that $\text{NEXT}(i') = i$. The tail of i' would then be in segment $(t(i_2), t(j_2))$, a contradiction. □

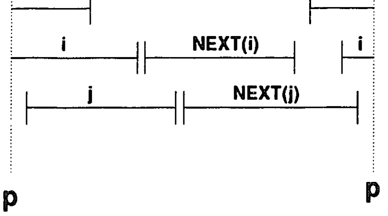

Fig. 2. Proof of Theorem 3.2.

4. The minimum dominating set (MDS) problem

Let F'' be the proper subfamily consisting of all maximal arcs in F. For each arc j which contains another arc i, an MDS K containing i can be changed to another dominating set $(K \setminus \{i\}) \cup \{j\}$ not containing i. Hence, it suffices to find an MDS whose arcs are contained in F''.

If there is an arc $i \in F''$ which overlaps with every other arc in F, then $\{i\}$ is a minimum dominating set and $\tau(F) = 1$ (we illustrate how to detect this in Algorithm $\text{NEXT}_d(i)$). Hence, consider the case that $\tau(F) > 1$. Below, we define $\text{NEXT}_d(\cdot)$ and $\text{GD}_d(\cdot)$ functions. For any subfamily $K \subseteq F$ and arc $i \in F$, define $N(i, K)$ to be the set of arcs in K overlapping with (but excluding) i.

Definition. For each arc i in F'', define the *first clockwise undominated* arc $U(i)$ of i to be the arc in $F \setminus N(i, F)$ whose tail is first encountered in a clockwise traversal from $t(i)$. Define $\text{NEXT}_d(i)$ to be the arc in $N(U(i), F'')$ whose tail is last encountered in a clockwise traversal from $t(U(i))$ (see Fig. 3).

By the definitions of $U(i)$ and $\text{NEXT}_d(i)$, it is easy to derive the following two lemmas:

Lemma 4.1. *Let v be any arc such that $t(v)$ is in segment $(t(\text{NEXT}_d(i)), t(i))$. The arc v does not overlap with $U(i)$.*

Lemma 4.2. *Let v be any arc with $t(v) \in$ segment $[t(i), t(\text{NEXT}_d(i))]$. Then v is dominated by either i or $\text{NEXT}_d(i)$.*

Definition. For each arc i in F'', define $\text{GD}_d(i)$ to be the minimal dominating set of the form

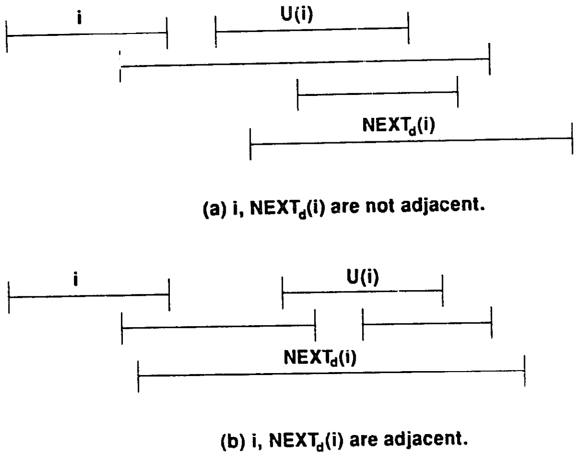

Fig. 3. The relative positions of arcs i and $\mathrm{NEXT_d}(i)$.

$\{i(=i_1), i_2, \ldots, i_r\}$, where $i_s = \mathrm{NEXT_d}(i_{s-1})$ for $s = 2, \ldots, r$.

Lemma 4.3. *Let $D \subseteq F''$ be a dominating set for F. For any i in D, $|\mathrm{GD_d}(i)| \leqslant |D|$.*

Proof. Let $\mathrm{GD_d}(i)$ be $\{i(=i_1), i_2, \ldots, i_r\}$. By Lemma 4.2, the endpoints $t(i_3), t(i_4), \ldots, t(i_r)$ are clockwise ordered and contained in $(t(i_2), t(i_1))$. By Lemma 4.1, for each $1 \leqslant x \leqslant r-1$, there must exist an arc $v \in D$ with $t(v) \in (t(i_s), t(i_{s+1})]$. Hence, besides arc i, there are at least $r-1$ other arcs in D, and $|D| \geqslant r$. □

Consider the directed graph $H = (V(H), E(H))$ with $V(H) = F''$ and $(i, j) \in E(H)$ iff $j = \mathrm{NEXT_d}(i)$. An arc $i \in F''$ is called a *good arc* if $|\mathrm{GD_d}(i)| = \tau(F)$. Similar to Lemma 2.2, one can argue that there is a directed cycle in D_d consisting of good arcs in F''. Furthermore, we have the following lemma:

Lemma 4.4. *Each node in a directed cycle C of H corresponds to a good arc in F''.*

Proof. Let C^*, C be two disjoint directed cycles in H such that each node in C^* corresponds to a good arc in F''. Let i, j be a pair of arcs in C^* and C, respectively, such that no tail of any arc in $C^* \cup C$ falls in $(t(i), t(j))$. Denote $\mathrm{GD_d}(j)$ by $\{j(=j_1), j_2, \ldots, j_r\}$. By Lemma 4.1, for each s, $1 \leqslant s \leqslant r-1$, there exists a distinct arc v in $\mathrm{GD_d}(i)$ such that $t(v) \in (t(j_s), t(j_{s+1})]$. By the selection of i, $t(i) \in (t(j_r), t(j))$. Hence, $\tau(F) = |\mathrm{GD_d}(i)| \geqslant r = |\mathrm{GD_d}(j)|$. □

Hence, the GREEDY algorithm in Section 2 can be used to find an MDS for F. We only need to describe an $O(n)$ time algorithm for computing $\mathrm{NEXT_d}(i)$ for all $i \in F''$. Denote F'' by $\{i_1, i_2, i_3, \ldots, i_r\}$, indexed by their clockwise tail order. In Algorithm $\mathrm{Next_d}$, we maintain three pointers i, u and j, which correspond to i_s, $U(i_s)$ and $\mathrm{NEXT_d}(i_s)$, respectively. The following lemma states that the tails of the $U(i_s)$'s are clockwise ordered and so are the tails of the $\mathrm{NEXT_d}(i_s)$'s. Therefore, in the execution of Algorithm $\mathrm{NEXT_d}$, the pointers i, u and j traverse the circle at most once, which takes $O(n)$ time.

Lemma 4.5. *For any $i, i' \in F''$, segment $(t(i), t(U(i))]$ is not contained in segment $(t(i'), t(U(i')))$, and segment $(t(i), t(\mathrm{NEXT_d}(i))]$ is not contained in segment $(t(i'), t(\mathrm{NEXT_d}(i')))$.*

Proof. If segment $(t(i), t(U(i))]$ is contained in segment $(t(i'), t(U(i')))$, then $U(i)$ does not overlap with i'. Hence, $U(i')$ should have been $U(i)$, a contradiction. The argument for the second part is similar. □

Algorithm $\mathrm{NEXT_d}$

1. Arrange arcs in F according to their clockwise tail orders. Construct a circular linked list P on arcs in F, where $t(P(i))$ is the first clockwise tail after $t(i)$ in F, and another linked list P' on arcs in F'', where $t(P'(i))$ is the first clockwise tail after $t(i)$ in F''.
2. Arbitrarily select an arc i^* in F''. If $N(i^*, F) = F \setminus \{i^*\}$, then stop /* $\{i^*\}$ is an MDS */. Otherwise, let arc $u^* = U(i^*)$ and $j^* = \mathrm{NEXT_d}(i^*)$. Set $i = P'(i^*)$, $u = u^*$ and $j = j^*$.
3. **While** $(i \neq i^*)$, **repeat** steps 4 to 9. /* find $U(i)$ and $\mathrm{NEXT_d}(i)$, for every arc i */
4. **If** $u = i$ **then** $u \leftarrow P(u)$ /* start searching for $U(i)$ with an arc $u \neq i$ */
5. **While** $(u \in N(i, F)$ and $u \neq i)$ **do** $u \leftarrow P(u)$ /* $U(i) = u$ */

6. **If** $i = u$ **then** stop; /* $\{i\}$ is a minimum dominating set */
7. **While** ($P'(j)$ contains $t(u)$) **do** $j \leftarrow P'(j)$;
8. $\text{NEXT}_d(i) = j$
9. $i \leftarrow P'(i)$;

Acknowledgment

The results in this paper appeared originally in [4] in 1988. It was unduly delayed by the review process of another journal before we finally decided to submit to IPL.

A lot has happened in the last three years. Recently, we learned that an equivalent form of Theorem 2.3 was discovered independently by D.T. Lee, M. Sarrafzadeh and Y.F. Wu, in "*Minimum Cuts for Circular-Arc Graphs*", to appear. Also, a referee pointed out that similar results on the MDS problem (though different approaches) have appeared in [6,7]. However, we should emphasize that the technique adopted in our paper is very simple and solves all three problems simultaneously.

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