3

Infinitely Lived Agents

A frequently used method of handling endogenous savings is to allow the agents to live forever and to make plans taking into account their future consumption stream. One can think of the agents as family dynasties, where those members of the dynasty who are alive today take into account the welfare of all members of the family, including those of generations not yet born. One of the main reasons for choosing this structure of infinitely lived agents is technical. Under conditions where certain structures of the economy (such as utility functions) don't change through time, we can use standard variational or recursive methods to solve for stationary states.

While it is obvious that people do not live forever, models with infinitely lived agents may be good approximations for the kinds of decisions that are made by young people. When the expected time of death is far enough into the future and future consumption is discounted ($\beta < 1$), the weight given to consumption after the expected date of death in utility can be small enough so that it has no effect on current economic decisions. If there is a bequest motive, then taking into account the consumption of one's spawn and that of their spawn soon gives an infinite horizon utility function. In addition, in models where agents live lots of periods, it is easier to think of the model as having potential for representing business cycles. Business cycles are normally thought to be less than ten years, and lives in a two-period-lived overlapping generations model are much too long to imagine that these models give a good representation of business cycles. The only obvious problem with infinitely lived agents is that there is little room for discussing retirement decisions, but overlapping generations models like those of Chapter 2 are often used for modeling retirement.

In this chapter, we find stationary states for a deterministic economy where individuals have perfect foresight using variational methods. In the next chapter we will use recursive methods to solve the same problem. Here, we first consider the case of a Robinson Crusoe type of economy where there is only one individual (or a social planner) who makes the savings decisions. Later in this chapter, we look for the stationary state of an economy comprised of many individuals who receive income from their work and from the earnings on their savings.

3.1 A ROBINSON CRUSOE ECONOMY WITH FIXED LABOR

Consider an economy with only one individual.[1] At time t, an infinitely lived individual wants to maximize a lifetime utility function of the form

$$\sum_{i=0}^{\infty} \beta^i u(c_{t+i}),$$

where $u()$ is a subutility function that does not change with time, $0 < \beta < 1$ is the factor by which future utility is discounted, and c_t is the consumption of the individual in period t. The function $u()$ is assumed to be increasing, continuous, concave, and with as many derivatives as are required.

The individual begins period t with a given amount of capital stock carried over from the previous period, k_t, will provide one unit of labor in every period, and decides how much to consume subject to the sequence of single-period budget constraints of the form

$$k_{t+1} = (1 - \delta)k_t + i_t$$

and

$$y_t = f(k_t) \geq c_t + i_t.$$

Here, the capital stock, k_t, is per worker capital by definition in a one-person economy and by the assumption that Robinson Crusoe provides one unit of labor in each period. Investment, i_t, is the only other use for production that is not consumed. As in the initial Solow model, δ is the depreciation rate.

3.1.1 Variational Methods

Variational methods are useful for finding the stationary state of a dynamic problem. They can also be used to find first-order conditions for dynamic

1. I have been using the convention of writing individual variables in the lowercase and aggregate in the uppercase. In the Robinson Crusoe economy, the individual is the aggregate, so there is room for ambiguity. I choose to use lowercase for a single-person economy.

models, as we will see later. The basic technique is to assume that the values for the endogenous variables for periods $s-1$ and $s+1$ are given and then maximize the objective equation for the values in period s. The resulting first-order conditions must hold in a stationary state along with the condition that the value for each of the variables is the same at times $s-1$, s, and $s+1$. These conditions are sufficient to find a stationary state.

The most direct technique for a simple model like the one we are using here is to replace consumption in the lifetime utility function using the budget constraints and then maximize by choosing the sequence of optimizing capital stocks. Using the two budget constraints at equality, one can write consumption at time t as

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t.$$

The lifetime utility function can be written as

$$\sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+i+1} + (1-\delta)k_{t+i}). \tag{3.1}$$

Assuming that the capital stocks k_{s-1} and k_{s+1} are given, the first-order condition for some time $s \geq t$ capital stock is

$$\begin{aligned} 0 = {} & \beta^{s-t}u'(f(k_s) - k_{s+1} + (1-\delta)k_s)\left(f'(k_s) + (1-\delta)\right) \\ & - \beta^{s-t-1}u'(f(k_{s-1}) - k_s + (1-\delta)k_{s-1}). \end{aligned}$$

or that

$$f'(k_s) + (1-\delta) = \frac{u'(f(k_{s-1}) - k_s + (1-\delta)k_{s-1})}{\beta u'(f(k_s) - k_{s+1} + (1-\delta)k_s)}.$$

From this Euler equation (as the first-order condition is sometimes known), one can find the stationary state values for the capital stock. Assuming that $k_{s+1} = k_s = k_{s-1} = \bar{k}$, as they must in a stationary state, the first-order condition yields

$$f'(\bar{k}) = \frac{1}{\beta} - 1 + \delta.$$

Since the parameters β and δ are known, as is the function $f(\cdot)$, this equation can be solved for $\bar{k}$. The equation states that in a stationary state, the marginal product of capital is equal to the net real interest rate implicit in the discount factor plus depreciation. Once the stationary state capital stock is determined, the budget constraints can be used to determine the stationary state values of output and consumption.

Of course, the above maximization problem could just as easily have been handled using Lagrangian multipliers. In that case, the maximization problem for some period s would have been written as

$$\max_{k_s, c_s, i_s} \sum_{i=0}^{\infty} \beta^i \Big[u(c_{t+i}) - \lambda^1_{t+i} \left(k_{t+1+i} - (1-\delta)k_{t+i} - i_{t+i}\right)$$
$$-\lambda^2_{t+i} \left(f(k_{t+i}) - c_{t+i} - i_{t+i}\right)\Big].$$

The stationary state that results from solving the first-order conditions is exactly the same as that found by substitution.

The conditions given above are necessary conditions for an optimum. However, an extra condition is required: a boundary or limit condition called the *transversality condition*. This condition can best be seen in a general version. Later it is shown for the Robinson Crusoe model.

One can write the problem for Robinson Crusoe as the general problem

$$\max_{\{x_x\}_{s=t}^{\infty}} \sum_{i=0}^{\infty} \beta^i F\left(x_{t+i}, x_{t+1+i}\right).$$

The necessary Euler condition that was given above can be written as

$$0 = F_2\left(x_{s-1}, x_s\right) + \beta F_1\left(x_s, x_{s+1}\right),$$

where $F_j\,(,)$ is the partial derivative with respect to the jth component of $F\,(,)$. The transversality condition for this general problem is the limit condition

$$\lim_{s\to\infty} \beta^s F_1\left(x_s, x_{s+1}\right) x_s = 0.$$

This condition says that in an optimal path, the values of x_s for s far enough into the future have zero weight in the maximization problem.

For the problem given in equation 3.1, the transversality condition is

$$\lim_{i\to\infty} \beta^i u'\left(f(k_{t+i}) - k_{t+i+1} + (1-\delta)k_{t+i}\right)\left(f'(k_{t+i}) + (1-\delta)\right) k_{t+i} = 0.$$

This equation can be interpreted as saying that the utility gains from accumulating capital eventually grow slower than $1/\beta$. If this were not the case, it might be optimal to postpone consumption indefinitely and no stationary state equilibrium would exist.

Here is an example where the transversality condition does not hold. Consider an economy where $f(k_t) = .2k_t$ and $u(c_t) = c_t$. Both production and the utility function are linear. Let $\delta = .1$ and $\beta = .98$. Imagine, first, that an individual in this economy will die in period T and wants to maximize

$$\sum_{i=0}^{T} \beta^i u \left(f\left(k_{t+i}\right) - k_{t+1+i} + (1-\delta)\, k_{t+i}\right),$$

or, after putting in the specific equations,

$$\sum_{i=0}^{T} (.98)^i \left((1.1)k_{t+i} - k_{t+1+i}\right).$$

The maximum utility that can be achieved, given an initial capital of k_0, is the corner solution:

$$(.98 \cdot 1.1)^T\, k_0.$$

This utility is achieved by consuming nothing until period T, when all accumulated capital is consumed. This is optimal because the next period's utility rises more from any marginal savings than it is reduced because of the discount factor.

Now take an infinitely lived person. Total discounted utility is found by letting $T \to \infty$. Utility is maximized by infinitely delaying consumption and having the discounted utility value of the capital stock go to infinity. The transversality condition is

$$\lim_{t \to \infty} \left(.98^t \cdot 1.1\right) (1.1)^{t-1}\, k_0$$

and this does not go to zero.

Stationary state allocations exist for this example economy: let $c_t = .1k_t$, for any value of k_t is one. However, no stationary state allocation is a utility-maximizing path.

The transversality condition removes the kinds of problems that the above example illustrates. Even when we do not explicitly do so in this book, one must check that the transversality conditions hold. They usually do in standard economies where the utility functions and production functions are concave and continuously differentiable and where the set of possible allocations in each period are convex and nonempty. (See Stokey, Lucas, and Prescott [83], section 4.5, for details and a theorem on the sufficiency of the Euler and transversality conditions.)

The main weakness of variational methods is that they do not provide an easy way of describing the dynamics of the model. To do that correctly, the values at both time $s - 1$ and time $s + 1$ are needed to find the solution. Therefore, the entire infinite sequence of capital needs to be solved simultaneously, a problem that is normally not one that we can handle in a tractable manner. Solving a stationary state dramatically reduces the dimensionality of the problem. Stationary states are tractable because the past and future values are going to be the same as the one we solve for. We reduce an infinite dimensional problem to one of just a single period.

3.2 A ROBINSON CRUSOE ECONOMY WITH VARIABLE LABOR

The first step in making the R.C. model a bit more realistic is to allow him to decide how to split his time between enjoying leisure and working.

3.2.1 The General Model

One can make the amount of labor that Robinson Crusoe supplies an individual choice as well. Assume that the utility function of the form

$$\sum_{i=0}^{\infty} \beta^i \overline{u}(c_{t+i}, l_{t+i}),$$

where $0 < l_{t+i} < 1$, is the amount of leisure that Robinson Crusoe chooses to consume in period $t+i$. Since there is a unit of time available each period, consuming l_{t+i} units of leisure in period $t+i$ means that one is using $h_{t+i} = 1 - l_{t+i}$ units of labor to produce goods. One can substitute this constraint into the utility function so that we think of utility as depending on consumption of goods and labor supplied,

$$\sum_{i=0}^{\infty} \beta^i u(c_{t+i}, h_{t+i}) \equiv \sum_{i=0}^{\infty} \beta^i \overline{u}(c_{t+i}, 1 - h_{t+i}) = \sum_{i=0}^{\infty} \beta^i \overline{u}(c_{t+i}, l_{t+i}),$$

where the partial derivatives are $u_c(c_{t+i}, h_{t+i}) > 0$ and $u_h(c_{t+i}, h_{t+i}) < 0$. The other budget constraints are the same capital accumulation equation as before,

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

and a new feasibility constraint where the production function includes the labor supply,

$$y_t = f(k_t, h_t) \geq c_t + i_t.$$

The Lagrangian is written as

$$\mathcal{L} = \sum_{i=0}^{\infty} \beta^i \Big[u(c_{t+i}, h_{t+i}) - \lambda^1_{t+i} \left(k_{t+1+i} - (1-\delta)k_{t+i} - i_{t+i} \right)$$
$$- \lambda^2_{t+i} \left(f(k_{t+i}, h_{t+i}) - c_{t+i} - i_{t+i} \right) \Big].$$

The four first-order conditions (maximizing with respect to time s consumption, labor supply, capital, and investment, for some period $s \geq t$) are

$$\frac{\partial \mathcal{L}}{\partial c_s} = 0 = u_c(c_s, h_s) + \lambda^2_s,$$

$$\frac{\partial \mathcal{L}}{\partial h_s} = 0 = u_h(c_s, h_s) - \lambda^2_s f_h(k_s, h_s),$$

$$\frac{\partial \mathcal{L}}{\partial k_{s+1}} = 0 = -\lambda^1_s + \beta \lambda^1_{s+1}(1-\delta) - \beta \lambda^2_{s+1} f_k(k_{s+1}, h_{s+1}),$$

$$\frac{\partial \mathcal{L}}{\partial i_s} = 0 = \lambda^1_s + \lambda^2_s.$$

We close the system by adding the two budget constraints,

$$k_{s+1} = (1-\delta)k_s + i_s$$

and

$$f(k_s, h_s) = c_s + i_s.$$

These simplify to the conditions that

$$u_c(c_s, h_s) = \lambda^1_s = -\lambda^2_s,$$

$$\frac{u_h(c_s, h_s)}{u_c(c_s, h_s)} = -f_h(k_s, h_s),$$

$$\frac{u_c(c_s, h_s)}{u_c(c_{s+1}, h_{s+1})} = \beta \left[f_k(k_{s+1}, h_{s+1}) + (1-\delta) \right],$$

and the budget constraint

$$k_{s+1} = (1-\delta)k_s + f(k_s, h_s) - c_s.$$

In a stationary state, the third condition reduces to

$$\frac{1}{\beta} - (1-\delta) = f_k(k, h),$$

and the fourth to

$$\delta k = f(k, h) - c.$$

In a stationary state, the second condition is

$$\frac{u_h(c, h)}{u_c(c, h)} = -f_h(k, h).$$

We have three independent equations in three unknowns: c, h, and k, so a solution to the stationary state is normally available.

3.2.2 Solution for a Sample Economy

Consider a sample economy where the production function is Cobb-Douglas,

$$y_t = f(k_t, h_t) = k_t^{\theta} h_t^{1-\theta}.$$

In the literature, there are two subutility functions that are commonly used. One uses the log of consumption,

$$u(c_t, h_t) = \ln(c_t) + v(h_t),$$

and the other is the constant elasticity of substitution form,

$$u(c_t, h_t) = \frac{c_t^{1-\phi}}{1-\phi} v(h_t),$$

with $\phi > 0$ but $\phi \neq 1$, and where $v(h_t)$ is a concave function such that $v(h_t) \to -\infty$ as $h_t \to 1$. A function that is frequently used for the utility of labor is

$$v(h_t) = B \ln(1 - h_t),$$

for some positive constant B. For our example economy here, we use

$$u(c_t, h_t) = \ln(c_t) + B \ln(1 - h_t).$$

With these assumptions, the conditions for the stationary state are

$$\frac{1}{\beta} - (1 - \delta) = \theta \overline{k}^{\theta-1} \overline{h}^{1-\theta},$$

$$\delta \overline{k} = \overline{k}^{\theta} \overline{h}^{1-\theta} - c.$$

and

$$B\frac{c}{1-\bar{h}} = (1-\theta)\,\bar{k}^{\theta}\bar{h}^{-\theta}.$$

From the first of these three conditions, we get stationary state labor supply as a function of the stationary state capital,

$$\bar{h} = \left[\frac{1}{\beta\theta} - \frac{(1-\delta)}{\theta}\right]^{\frac{1}{1-\theta}} \bar{k} = G\bar{k},$$

where $G = \left[1/\beta\theta - (1-\delta)/\theta\right]^{\frac{1}{1-\theta}}$ is a constant. The first and second give the stationary state consumption as a function of the stationary state capital,

$$\bar{c} = \left[\frac{1}{\beta\theta} - \frac{(1-\delta)}{\theta} - \delta\right]\bar{k} = J\bar{k},$$

and J is the constant defined by this equation. The second and third give

$$B\frac{\bar{c}}{1-\bar{h}} = (1-\theta)\frac{\delta\bar{k}+\bar{c}}{\bar{h}}.$$

Combining these three gives the stationary state value of capital,

$$\bar{k} = \frac{(1-\theta)\,(\delta+J)}{G\,(BJ+(1-\theta)\,(\delta+J))}.$$

Given the stationary state value of capital, the stationary state values for the labor supply and consumption are immediate.

> **EXERCISE 3.1** Find the stationary state conditions for an example economy where the subutility function has the constant elasticity of substitution form given above.

3.3 A COMPETITIVE ECONOMY

In the Robinson Crusoe economy, one person made consumption and production decisions for the whole economy. In a competitive economy, there are consumers who provide labor to the market and firms who hire this labor at the competitive wage, w_t. In this section, we assume that there is a continuum of identical agents of a unit mass and that all agents can provide up to one unit of labor to the market. All individuals are the same so that we can take the behavior of one agent as that of the whole economy since we simply integrate from 0 to 1 over identical agents.

As above, individual i gets utility out of consumption and disutility out of work, h_t^i, which is the amount of labor provided to the market by individual i. Each individual is endowed with one unit of time in each period, so the leisure of individual i in period t is equal to $l_t^i = 1 - h_t^i$. Subutility functions are of the form

$$u(c_t^i, h_t^i) = \overline{u}(c_t^i, 1 - h_t^i) = \overline{u}(c_t^i, l_t^i).$$

Firms produce goods each period according to a production function, $f(K_t, H_t)$, with the usual properties, where the uppercase letters K_t and H_t denote the society's aggregate amount of capital and labor applied to production. The firms pay a competitive wage, w_t, to workers and pay a competitive rent, r_t, for the use of capital, so the marginal product of labor is equal to the real wage and the marginal product of capital is equal to the rent paid for capital.

The general problem for an individual, i, at time 0 is to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i, h_t^i),$$

subject to the constraints from a competitive economy of

$$\begin{aligned} c_t^i &= w_t h_t^i + r_t k_t^i - I_t^i, \\ w_t &= f_h(K_t, H_t), \\ r_t &= f_k(K_t, H_t), \\ k_{t+1}^i &= (1 - \delta) k_t^i + I_t^i, \end{aligned}$$

where all variables are defined as above except I_t^i is the investment of individual i in period t. The individual's maximization problem can be written as the Lagrangian

$$\begin{aligned} \mathcal{L}^i = \sum_{t=0}^{\infty} \beta^t \Big[& u(c_t^i, h_t^i) - \lambda_t^1 \left(k_{t+1}^i - (1 - \delta) k_t^i - I_t^i \right) \\ & - \lambda_t^2 \left(f_h(K_t, H_t) h_t^i + f_k(K_t, H_t) k_t^i - c_t^i - I_t^i \right) \Big], \end{aligned}$$

subject to the aggregation rules that

$$H_t = \int_0^1 h_t^i di \quad \text{and} \quad K_t = \int_0^1 k_t^i di.$$

The use of capital letters for capital and labor in the production function reminds us that these are society-wide variables and are viewed as constants in the individual choice problem.

First-order conditions from the maximization problem of individual i in period s are

$$\frac{\partial \mathcal{L}^i}{\partial c_s^i} = 0 = u_c(c_s^i, h_s^i) + \lambda_s^2,$$

$$\frac{\partial \mathcal{L}^i}{\partial h_s^i} = 0 = u_h(c_s^i, h_s^i) - \lambda_s^2 f_h(K_s, H_s),$$

$$\frac{\partial \mathcal{L}^i}{\partial k_{s+1}^i} = 0 = -\lambda_s^1 + \beta\lambda_{s+1}^1(1-\delta) - \beta\lambda_{s+1}^2 f_k(K_{s+1}, H_{s+1}),$$

$$\frac{\partial \mathcal{L}^i}{\partial I_s^i} = 0 = \lambda_s^1 + \lambda_s^2.$$

These can be simplified to get

$$u_c(c_s^i, h_s^i) = \lambda_s^1 = -\lambda_s^2,$$

$$\frac{u_h(c_s^i, h_s^i)}{u_c(c_s^i, h_s^i)} = -f_h(K_s, H_s), \tag{3.2}$$

$$\frac{u_c(c_s^i, h_s^i)}{u_c(c_{s+1}^i, h_{s+1}^i)} = \beta\left[f_k(K_{s+1}, H_{s+1}) + (1-\delta)\right]. \tag{3.3}$$

To complete the system, we add the budget constraint

$$k_{t+1}^i = (1-\delta)k_t^i + f_h(K_t, H_t)h_t^i + f_k(K_t, H_t)k_t^i - c_t^i,$$

and the aggregation rules

$$H_t = \int_0^1 h_t^i di \quad \text{and} \quad K_t = \int_0^1 k_t^i di.$$

When all of the unit mass of individuals are identical, the aggregation rules simplify to $H_t = h_t^i$ and $K_t = k_t^i$.

The production function is homogeneous of degree one (constant returns to scale) and under conditions of perfect competition with free entry, firms do not make any profits. Given the aggregation rule, this implies that

$$f_h(K_t, H_t)H_t + f_k(K_t, H_t)K_t = f(K_t, H_t)$$

and that the budget constraint given above can be written as

$$K_{t+1} = (1-\delta)K_t + f(K_t, H_t) - C_t,$$

where aggregate consumption is

$$C_t = \int_0^1 c_t^i di.$$

When all the households are identical, the first-order conditions can be written as the aggregate conditions

$$\frac{u_h(C_s, H_s)}{u_c(C_s, H_s)} = -f_h(K_s, H_s),$$

$$\frac{u_c(C_s, H_s)}{u_c(C_{s+1}, H_{s+1})} = \beta \left[f_k(K_{s+1}, H_{s+1}) + (1-\delta) \right].$$

The result here is interesting. The conditions for the equilibrium in the competitive economy turn out to be an aggregate version of the same conditions of the Robinson Crusoe (R.C.) economy. When we find the stationary state equilibrium for this economy, it is the same stationary state that we found for the R.C. economy. Here, each individual supplies the same labor as the single individual did in the R.C. economy and each individual owns exactly the same amount of capital. Therefore, each individual has exactly the same income and faces the same marginal conditions as in the R.C. economy.

3.4 THE SECOND WELFARE THEOREM

The result that the equilibrium of a representative agent economy and that of a perfectly competitive one that is otherwise identical is not surprising. Mas-Colell, Whinston, and Green [59] write the second fundamental theorem of welfare economics as follows:

> The Second Fundamental Welfare Theorem. If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.* (p. 308)

This statement of the second fundamental welfare theorem holds for finite dimensional economies. Our economies have an infinite number of periods and, therefore, an infinite number of goods. The conditions for existence of a competitive equilibrium in infinite horizon economies are somewhat more complex than those for finite dimensional ones and some extra assumptions are required. See Stokey, Lucas, and Prescott [83], Chapter 16, for details. Here, we simply assume that a competitive equilibrium exists and are interested in its relationship to the social planner (Robinson Crusoe) economy.

The equilibrium that was found in the R. C. economy is Pareto optimal. It is the result of the "social planner" Robinson Crusoe finding production and consumption points that maximize the utility of the single individual in the economy, given his technological constraints.

The first fundamental welfare theorem tells us that any competitive equilibrium is necessarily Pareto optimal, so that the equilibrium found using a decentralized economy with factor and goods markets is also Pareto optimal. The second welfare theorem tells us that, since the production technologies and preferences are the same in the two economies, then with the right initial wealth conditions, the competitive economy can achieve an equilibrium that is identical to the social planner economy. It might achieve a different one if the initial wealth distribution were not correct. In our case, all individuals have an initial wealth that is identical to that of Robinson Crusoe, have identical preferences, and face the same technologies. Therefore, this initial distribution of wealth is sufficient to result in the same equilibrium as in the R. C. economy.

It is the second fundamental welfare theorem that permits us to use a representative agent economy to mimic a competitive economy. Since the second fundamental theorem is carefully worded, it should be clear that using a representative agent economy will not always give the appropriate results. If the economy is not perfectly competitive, if part of the economy has some monopoly power or if there are some external or internal restrictions that prevent some agents from behaving perfectly competitive, then the equilibrium found by the decentralized economy will not necessarily be achievable with a representative agent economy.

However, when the conditions are right, solving a representative agent economy is often technically much simpler than solving a decentralized economy. In this case, the second fundamental welfare theorem states that, with appropriate initial conditions, the solution of the representative agent economy is one for the decentralized economy.

3.4.1 An Example Where the Representative Agent Economy and the Decentralized Economy Are Not Equal

It is not difficult to find an example where distortions in the competitive economy cause the equilibrium to be different from that found in a representative agent economy. In the economy given here, the government applies a proportional tax on wage income and rebates the revenues to the households using a lump sum transfer. The tax on wage income changes the equilibrium so that it is no longer the same as that of the Robinson Crusoe economy.

Suppose that, in a simple decentralized economy, there exists a government that does nothing more than impose a wage tax and rebate the tax revenue by giving identical lump sum transfers to each family. Let the wage tax rate be t_w and the lump sum transfers in period t be T_t. The government makes no other use of this income, so its budget constraint is

$$t_w w_t H_t = T_t.$$

The general problem for an individual, i, at time 0 is to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i, h_t^i),$$

subject to the constraints from a competitive economy of

$$\begin{aligned} c_t^i &= \left(1 - t_w\right) w_t h_t^i + r_t k_t^i + T_t - I_t^i, \\ w_t &= f_h(K_t, H_t), \\ r_t &= f_k(K_t, H_t), \\ k_{t+1}^i &= (1-\delta) k_t^i + I_t^i. \end{aligned}$$

The individual's maximization problem can be written as the Lagrangian

$$\begin{aligned} \mathcal{L}^i = \sum_{t=0}^{\infty} \beta^t \Big[& u(c_t^i, h_t^i) - \lambda_t^1 \left(k_{t+1}^i - (1-\delta) k_t^i - I_t^i \right) \\ & - \lambda_t^2 \left(\left(1 - t_w\right) f_h(K_t, H_t) h_t^i + f_k(K_t, H_t) k_t^i + T_t - c_t^i - I_t^i \right) \Big], \end{aligned}$$

subject to the aggregation rules that

$$H_t = \int_0^1 h_t^i di \quad \text{and} \quad H_t = \int_0^1 k_t^i di.$$

The government's budget constraint is an equilibrium condition and needs to be applied to the economy after the individual optimization.

The first-order conditions are

$$\frac{\partial \mathcal{L}^i}{\partial c_s^i} = 0 = u_c(c_s^i, h_s^i) + \lambda_s^2,$$

$$\frac{\partial \mathcal{L}^i}{\partial h_s^i} = 0 = u_h(c_s^i, h_s^i) - \lambda_s^2 \left(1 - t_w\right) f_h(K_s, H_s),$$

$$\frac{\partial \mathcal{L}^i}{\partial k_{s+1}^i} = 0 = -\lambda_s^1 + \beta\lambda_{s+1}^1(1-\delta) - \beta\lambda_{s+1}^2 f_k(K_{s+1}, H_{s+1}),$$

$$\frac{\partial \mathcal{L}^i}{\partial I_s^i} = 0 = \lambda_s^1 + \lambda_s^2.$$

These simplify to

$$u_c(c_s^i, h_s) = \lambda_s^1 = -\lambda_s^2,$$

$$\frac{u_h(c_s^i, h_s^i)}{u_c(c_s^i, h_s)} = -\left(1 - t_w\right) f_h(K_s, H_s),$$

$$\frac{u_c(c_s^i, h_s^i)}{u_c(c_{s+1}^i, h_{s+1}^i)} = \beta\left[f_k(K_{s+1}, H_{s+1}) + (1-\delta)\right].$$

Now that the first-order conditions are determined, we add the aggregation (equilibrium) conditions. Production is homogeneous of degree 1, so

$$\begin{aligned} f(K_t, H_t) &= f_h(K_t, H_t)H_t + f_k(K_t, H_t)K_t \\ &= \left(1 - t_w\right) f_h(K_s, H_s)H_t + f_k(K_t, H_t)K_t + t_w f_h(K_s, H_s)H_t \\ &= \left(1 - t_w\right) f_h(K_s, H_s)H_t + f_k(K_t, H_t)K_t + T_t. \end{aligned}$$

The aggregate version of the individual budget constraint can be written as

$$K_{t+1} = (1-\delta)K_t + f(K_t, H_t) - C_t$$

and is the same as in the economy without the wage tax. When all the households are identical, the first-order conditions can be written as the aggregate conditions

$$\frac{u_h(C_s, H_s)}{u_c(C_s, H_s)} = -\left(1 - t_w\right) f_h(K_s, H_s),$$

$$\frac{u_c(C_s, H_s)}{u_c(C_{s+1}, H_{s+1})} = \beta\left[f_k(K_{s+1}, H_{s+1}) + (1-\delta)\right].$$

In a stationary state for the economy with a wage tax, the equations of the model are

$$\delta\overline{K} = f(\overline{K}, \overline{H}) - \overline{C},$$

$$1 = \beta\left[f_k(\overline{K}, \overline{H}) + (1-\delta)\right],$$

$$\frac{u_h(\overline{C}, \overline{H})}{u_c(\overline{C}, \overline{H})} = -\left(1 - t_w\right) f_h(\overline{K}, \overline{H}).$$

The second equation uses the fact that in a stationary state, $u_c(C_s, H_s) = u_c(C_{s+1}, H_{s+1}) = u_c(\overline{C}, \overline{H})$.

Compare the set of equations above to the equations for the stationary state of the same economy without the wage tax, when $t_w = 0$. The first two equations are the same, but the last one is simply

$$\frac{u_h(\overline{C}, \overline{H})}{u_c(\overline{C}, \overline{H})} = -f_h(\overline{K}, \overline{H}).$$

The stationary states will be different for different values of t_w. In the economy with a wage tax, the individuals pay a tax to the government through a percentage tax on wages and receive a lump sum transfer from the government for exactly the same amount as the tax they paid. These two amounts cancel out in the budget constraint. However, one of the first-order conditions is different, so the stationary states will, in general, be different.

Consider an example economy with a Cobb-Douglas production function,

$$f(K_t, H_t) = K_t^{\theta} H_t^{1-\theta},$$

and a subutility function,

$$u(c_t^i, h_t^i) = \ln\left(c_t^i\right) + A\ln\left(1 - h_t^i\right).$$

The parameters are $\beta = .98$, $\delta = .1$, $\theta = .36$, and $A = .5$. The stationary states for this economy with tax rates of $t_w = \{0, .1, .2\}$ are found by using a simple Matlab program and are shown in Table 3.1. In the table, $\overline{u}$ is the value of the subutility function in every period.

Table 3.1 shows how utility changes as the (distorting) wage tax rate increases. Stationary state economies where $t_w > 0$ are clearly not Pareto optimal, since they are dominated in every period by the economy where $t_w = 0$. The subutility value, $-.6953$, for $t_w = 0$ is greater than the subutility values for the higher values of t_w.

In the economy shown here, the distorting wage tax makes the competitive equilibrium different from the equilibrium found for a similar representative

Table 3.1 Stationary state values for example economies

t_w	0	.1	.2
$\overline{C}$	0.8387	0.8070	0.7705
$\overline{K}$	3.5770	3.4417	3.2862
$\overline{H}$	0.6461	0.6217	0.5936
$\overline{u}$	−.6953	−.7005	−.7109

agent economy. One needs to be careful when using representative agent economies and extremely careful when trying to use them for economies with distortions.

EXERCISE 3.2 Find the stationary state equilibria for an economy with a government that finances lump sum transfers to the public with a 10% tax on the income from capital. Compare this equilibrium to that of the same economy with no taxes. Compare the utility in this economy with that of an economy with a wage tax that raises the same amount of revenue for the government.

3.5 REPRISE

It is relatively easy to find stationary states for economies with single agents that live a long time and have utility that is comprised of additive discounted subutilities. These single agent economies can be thought of as representative agent economies for market economies where the individuals are identical. Variational methods are frequently good enough for finding stationary states, but they do not usually permit us to study the dynamic properties of the economy. If one is simply interested in comparing stationary states or finding the stationary state characteristics of an economy, variational methods are sufficient. As is often the case in economics, problems can arise when one wants to consider economies with many different types of agents. However, simple economies, such as the growth model given above, with groups of agents with different utility functions can frequently be solved using variational methods. One has to be careful during the aggregation procedures.

A classic text on variational methods is Hadley and Kemp [46].