Dynamic Stochastic Models

Homework 1 - Solow Model

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17 June, 2025

The Problem

We have to **solve the model in terms of growth rates**:

$$y_t = A_t k_t^{\theta} h_t^{1-\theta} K_t^{1-\theta}$$

(For simplicity, we omit the superscript j and work with a representative firm).

How would you get a log linear approximate version of this model?

Here, k is capital per worker and h represents workers. That model is Romer's learning by doing model (1987). Instead, we will consider only the case $K_t = k_t$, so no externalities are present. That is like the AK model from Romer (1986) but in a simple Solow context, without endogenous saving. We therfore get:

$$y_t = A_t k_t h_t^{1-\theta}$$

where the production function is linear in capital per worker. Our usual equation for the movement of capital is:

$$(1+\lambda)k_{t+1} = (1-\delta)k_t + i_t$$

Where $i_t = sy_t$ and $A_t = A_0(1+\alpha)^t$, but $\alpha = 0$ (so $A_t = A_0$, implying constant technology). Then we get the Solow first difference equation:

$$(1+\lambda)k_{t+1} = (1-\delta)k_t + sA_0 k_t h_t^{1-\theta}$$

Stationary state will be given by $k_t = k_{t+1} = k^*$:

$$(1+\lambda)k^* = (1-\delta)k^* + sA_0 k^* h^{1-\theta}$$

Solving for k^*:

$$k^*[(1+\lambda) - (1-\delta) - sA_0h^{1-\theta}] = 0$$

$$k^*[(\delta+\lambda) - sA_0h^{1-\theta}] = 0$$

This gives us two solutions:

1. $k^* = 0$ (trivial solution)

2. $(\delta+\lambda) = sA_0h^{1-\theta}$, which means any $k > 0$ can be a steady state if this condition holds

When $(\delta+\lambda) \neq sA_0h^{1-\theta}$, only $k^* = 0$ is a solution.

Stochastic Shocks

Using this stochastic process for technology:

$$A_t = (1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t$$

where ε_t is bounded both above and below and $|\gamma| < 1$.

The stochastic version of the Solow first difference equation becomes:

$$(1+\lambda)k_{t+1} = (1-\delta)k_t + sA_tk_th_t^{1-\theta}$$

Substituting the technology process:

$$(1+\lambda)k_{t+1} = (1-\delta)k_t + s[(1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t]k_th_t^{1-\theta}$$

This can be written as:

$$(1+\lambda)k_{t+1} = (1-\delta)k_t + s(1-\gamma)\bar{A}k_th_t^{1-\theta} + s\gamma A_{t-1}k_th_t^{1-\theta} + s\varepsilon_tk_th_t^{1-\theta}$$

Growth Rates

The growth rate for capital can be written as:

$$\gamma_t = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + sA_tk_th_t^{1-\theta}}{(1+\lambda)k_t} = \frac{1-\delta}{(1+\lambda)} + \frac{sA_th_t^{1-\theta}}{(1+\lambda)}$$

Substituting the stochastic technology process:

$$\gamma_t = \frac{1-\delta}{(1+\lambda)} + \frac{s[(1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t]h_t^{1-\theta}}{(1+\lambda)}$$

We first need to isolate the investment term:

$$\gamma_t - \frac{1-\delta}{1+\lambda} = \frac{s[(1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t]h_t^{1-\theta}}{(1+\lambda)}$$

Taking logs:

$$\ln\left[\gamma_t - \frac{1-\delta}{1+\lambda}\right] = \ln\frac{s}{1+\lambda} + \ln[(1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t] + (1-\theta)\ln h_t$$

Log Linear Approximation

We define $\tilde{k}_t = \ln(k_t) - \ln(\bar{k})$ where $\bar{k}$ is a stationary state value of capital (any, since there are many values if the condition for a steady state holds), so $\tilde{k}_t = \ln(k_t/\bar{k})$.

Then $k_t = \bar{k}e^{\tilde{k}_t}$. The stochastic difference equation becomes:

$$(1+\lambda)\bar{k}e^{\tilde{k}_{t+1}} = (1-\delta)\bar{k}e^{\tilde{k}_t} + s[(1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t]\bar{k}e^{\tilde{k}_t}\bar{h}^{1-\theta}$$

In this AK-type model, capital enters with power 1, which is different from the simple model we used in class. Using the first-order Taylor approximation $e^{\tilde{k}_t} \approx 1 + \tilde{k}_t$ for small $\tilde{k}_t$:

$$(1+\lambda)\bar{k}(1+\tilde{k}_{t+1}) = (1-\delta)\bar{k}(1+\tilde{k}_t) + s[(1-\gamma)\bar{A} + \gamma A_{t-1} + \varepsilon_t]\bar{k}(1+\tilde{k}_t)\bar{h}^{1-\theta}$$

In stationary state: $(1+\lambda)\bar{k} = (1-\delta)\bar{k} + s\bar{A}\bar{k}\bar{h}^{1-\theta}$

This simplifies to: $(\delta+\lambda) = s\bar{A}\bar{h}^{1-\theta}$

Expanding the linearized equation and using the steady state condition:

$$(1+\lambda)\bar{k}\tilde{k}_{t+1} = (1-\delta)\bar{k}\tilde{k}_t + s\bar{A}\bar{k}\bar{h}^{1-\theta}\tilde{k}_t + s\gamma(A_{t-1} - \bar{A})\bar{k}\bar{h}^{1-\theta} + s\varepsilon_t\bar{k}\bar{h}^{1-\theta}$$

Since $s\bar{A}\bar{h}^{1-\theta} = (\delta+\lambda)$, we get:

$$(1+\lambda)\tilde{k}_{t+1} = [(1-\delta) + (\delta+\lambda)]\tilde{k}_t + s\gamma(A_{t-1} - \bar{A})\bar{h}^{1-\theta} + s\varepsilon_t\bar{h}^{1-\theta}$$

$$\tilde{k}_{t+1} = \frac{1+\lambda}{1+\lambda}\tilde{k}_t + \frac{s\gamma\bar{h}^{1-\theta}}{1+\lambda}(A_{t-1} - \bar{A}) + \frac{s\bar{h}^{1-\theta}}{1+\lambda}\varepsilon_t$$

$$\tilde{k}_{t+1} = \tilde{k}_t + \frac{s\gamma\bar{h}^{1-\theta}}{1+\lambda}(A_{t-1} - \bar{A}) + \frac{s\bar{h}^{1-\theta}}{1+\lambda}\varepsilon_t$$

This shows that in this AK model, capital follows a random walk with drift terms from technology shocks. The coefficient on $\tilde{k}_t$ is exactly 1, indicating that shocks have permanent effects on the capital stock.

Defining the technology deviation $\tilde{A}_t = A_t - \bar{A}$, which follows:

$$\tilde{A}_t = \gamma \tilde{A}_{t-1} + \varepsilon_t$$

Output dynamics: Since $y_t = A_t k_t h_t^{1-\theta}$, the log-linearized output is:

$$\tilde{y}_t = \tilde{k}_t + \tilde{A}_t$$

This AK-type model generates endogenous growth with stochastic technology shocks having permanent effects on both capital and output.

Simulation

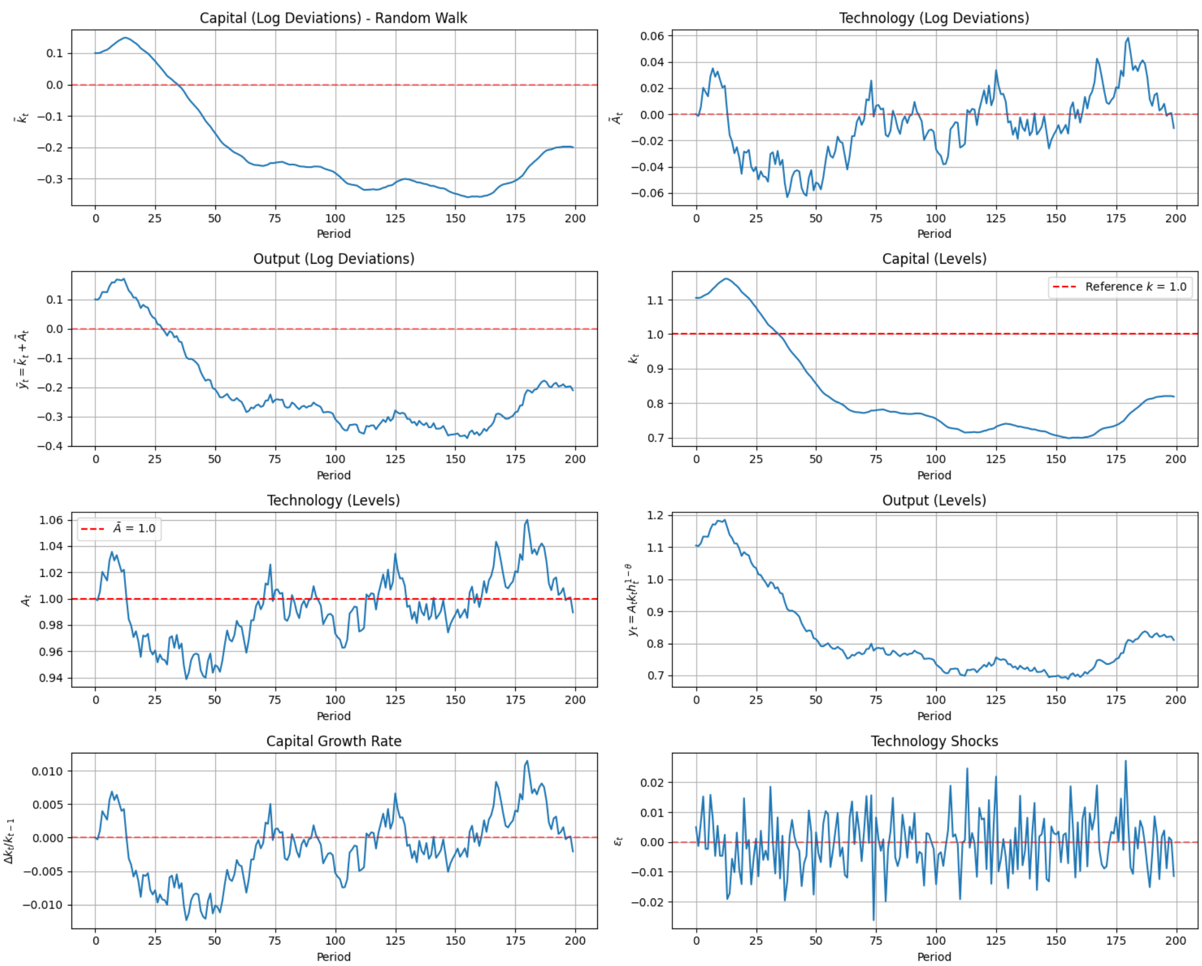

Figure 1: Smulation of the Solow model with $y = A_t k_t h_t^{1-\theta}$

Simulation Description

The simulation operates on the log-linearized system where $\tilde{k}_t = \ln(k_t/\bar{k})$ and $\tilde{A}_t = A_t - \bar{A}$:

$$\tilde{k}_{t+1} = \tilde{k}_t + \frac{s\gamma\bar{h}^{1-\theta}}{1+\lambda}\tilde{A}_t + \frac{s\bar{h}^{1-\theta}}{1+\lambda}\varepsilon_{t+1} \quad (1)$$

$$\tilde{A}_{t+1} = \gamma\tilde{A}_t + \varepsilon_{t+1} \quad (2)$$

$$\tilde{y}_t = \tilde{k}_t + \tilde{A}_t \quad (3)$$

where $\varepsilon_t \sim N(0, \sigma^2)$ are i.i.d. technology shocks.

Table 1: Simulation Parameters

Parameter	**Symbol**	**Value**
Labor share parameter	θ	0.33
Depreciation rate	δ	0.10
Savings rate	s	0.20
Population growth rate	λ	0.02
Technology persistence	γ	0.95
Technology shock std. dev.	σ	0.01
Workers (normalized)	h	1.0
Initial technology level	A_0	1.0
Reference capital level	$\bar{k}$	1.0
Simulation periods	T	200
Initial capital deviation	$\tilde{k}_0$	0.1

We chose those parameter values so as to satisfy the steady state condition $(\delta + \lambda) = s\bar{A}\bar{h}^{1-\theta}$ and because they were convenient. The technology persistence parameter $\gamma = 0.95$ ensures that technology shocks are highly persistent but stationary.