Dynamic Stochastic Models

Homework 2 - Overlapping Generations

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Exercise 1

Find the equilibrium in a constant population economy where, in each period, the government imposes a small lump sum tax, τ, on each young person and gives that amount to each old person. Both the young and the old see these taxes as lump sum. Under what conditions can this tax improve welfare and under what conditions will it not?

The budget constraints are modified to look like this:

$$w_t h_t^h(t) = c_t^h(t) + l^h(t) + k^h(t+1) + \tau$$

So some of their total labor income will now have to go to pay the tax τ. Meanwhile, old people will be receiving that tax so the budget constraint in the next period is:

$$w_{t+1} h_t^h(t+1) + r_t l^h(t) + \text{rental}_{t+1} k^h(t+1) + \tau = c_t^h(t+1)$$

On the left we have their income in period $t+1$ of a member of generation t. We can see that they have more income than in the basic model which implies the old will consume more and the young will consume less. Intuitively, with a simple utility function increasing in consumption, we would expect welfare to improve if more old people can consume more than the forgone consumption of the young to play taxes. We can express the restriction in period t of generation t of the young in terms of loans l^h:

$$l^h(t) = w_t h_t^h(t) - c_t^h(t) - k^h(t+1) - \tau$$

We replace that in the second constraint, that of the second period of generation t.

$$w_{t+1} h_t^h(t+1) + r_t(w_t h_t^h(t) - c_t^h(t) - k^h(t+1) - \tau) + \text{rental}_{t+1} k^h(t+1) + \tau = c_t^h(t+1)$$

We can now impose the no-arbitrage condition, which says that, in this closed economy with perfect foresight, $\text{rental}_{t+1} = r_t$. That follows from the Inada conditions we assumed on the production function, by which if the equality doesn't hold either no one wants to borrow and holds no capital ($\text{rental}_{t+1} < r_t$) or no one lends and everybody wants capital ($\text{rental}_{t+1} > r_t$) . We then also divide through by r_t and get to this expression:

$$\frac{w_{t+1}h_t^h(t+1)}{r_t} + w_t h_t^h(t) - c_t^h(t) - k^h(t+1) - \tau + k^h(t+1) + \frac{\tau}{r_t} = \frac{c_t^h(t+1)}{r_t}$$

The terms next period k^h cancel out and we can rearrange to get:

$$\frac{w_{t+1}h_t^h(t+1)}{r_t} + w_t h_t^h(t) + \tau(\frac{1}{r_t} - 1) = \frac{c_t^h(t+1)}{r_t} + c_t^h(t)$$

That is the usual intertemporal constraint. The term $\tau(\frac{1}{r_t} - 1)$ only appears in this specification, so we can check whether intertempral consumption rises relative to the model without tax if that term is positive.

$$\tau(\frac{1}{r_t} - 1) \geq 0 \implies r_t \leq 1$$

So if $r_t \leq 1$, the tax improves welfare. We interpret that if the rate of return of the economy r is too low, lower than the return of just leaving the government to give your money back later (because τ and population are constant), then the best use of money is that lump sum transfer. If that was not the case, it would be better to take advantage of the high rate of return in the economy rather than pay tax.

Excercise 2

Solve the model of the class but where depreciation is equal to $\delta < 1$, for example, $\delta = .5$. Will output in the stationary state be higher or lower? Will the utility of households in the stationary state be higher or lower? Starting from some K_0 below the stationary state, will growth be faster or slower?

Now, for the capital that they saved in the fist period $k^h(t+1)$ in order to earn a rate rental_{t+1} later, now also a proportion of that saved capital materializes later, namely $(1-\delta)k^h(t+1)$, the proportion that didn't fully depreciate.

The constraints will be these:

$$w_t h_t^h(t) = c_t^h(t) + l^h(t) + k^h(t+1)$$

That one, that of the young, is the same as always. The old will face this constraint:

$$w_{t+1}h_t^h(t+1) + r_t l^h(t) + \text{rental}_{t+1}k^h(t+1) + (1-\delta)k^h(t+1) = c_t^h(t+1)$$

which means they sell some of the remaining capital to the young, which is an additional income for the old and means there will be more capital going around in the economy. Intuitively, we would expect this to improve output and household utility realtive to the model with full depreciation. There is more stuff! We can also express this as:

$$w_{t+1}h_t^h(t+1) + r_t l^h(t) + (1+\text{rental}_{t+1}-\delta)k^h(t+1) = c_t^h(t+1)$$

and there we have the term $(1 + \text{rental}_{t+1} - \delta)$ which would represent something like the effective return on capital, and tha is now equal to r, the rate on loans, through no arbitrage.For equilibrium in both lending and capital markets:

$$r_t = 1 + \text{rental}_{t+1} - \delta$$

And since $\text{rental}_{t+1} = F_K(K(t), H(t))$, (from perfect competition) it follows that:

$$r_t = F_K(K(t+1), H(t+1)) + (1-\delta)$$

To analyze the stationary state and dynamics, we will take the example economy with Cobb Douglass production and utility.

We assume a Cobb-Douglas production function $F(K_t, H_t) = K_t^\alpha H_t^{1-\alpha}$ where $0 < \alpha < 1$, and logarithmic utility $U(c_t^h(t), c_t^h(t+1)) = \ln c_t^h(t) + \beta \ln c_t^h(t+1)$ where $\beta > 0$.

Defining $k_t = K_t/H_t$ as capital per worker, the marginal products are $F_K(K_t, H_t) = \alpha k_t^{\alpha-1}$ and $F_H(K_t, H_t) = (1-\alpha)k_t^\alpha$. Under perfect competition, the wage is $w_t = (1-\alpha)k_t^\alpha$ and the rental rate is $\text{rental}_t = \alpha k_t^{\alpha-1}$. The no-arbitrage condition becomes $r_t = \alpha k_{t+1}^{\alpha-1} + (1-\delta)$.

The household optimization problem is to maximize $\ln c_t^h(t) + \beta \ln c_t^h(t+1)$ subject to the budget constraints. From the first constraint, we have $c_t^h(t) = w_t h_t^h(t) - l^h(t) - k^h(t+1)$. From the second constraint, we have $c_t^h(t+1) = w_{t+1}h_t^h(t+1) + r_t l^h(t) + [\alpha k_{t+1}^{\alpha-1} + (1-\delta)]k^h(t+1)$.

The reduced form objective function becomes:

$$\max_{l^h(t), k^h(t+1)} \ln[w_t h_t^h(t) - l^h(t) - k^h(t+1)] + \beta \ln[w_{t+1}h_t^h(t+1) + r_t l^h(t) + [\alpha k_{t+1}^{\alpha-1} + (1-\delta)]k^h(t+1)]$$

Taking the first-order condition with respect to $l^h(t)$:

$$\frac{-1}{w_t h_t^h(t) - l^h(t) - k^h(t+1)} + \beta \frac{r_t}{w_{t+1} h_t^h(t+1) + r_t l^h(t) + [\alpha k_{t+1}^{\alpha-1} + (1-\delta)] k^h(t+1)} = 0$$

Taking the first-order condition with respect to $k^h(t+1)$:

$$\frac{-1}{w_t h_t^h(t) - l^h(t) - k^h(t+1)} + \beta \frac{\alpha k_{t+1}^{\alpha-1} + (1-\delta)}{w_{t+1} h_t^h(t+1) + r_t l^h(t) + [\alpha k_{t+1}^{\alpha-1} + (1-\delta)] k^h(t+1)} = 0$$

Equating these first-order conditions yields $\beta r_t = \beta[\alpha k_{t+1}^{\alpha-1} + (1-\delta)]$, which implies $r_t = \alpha k_{t+1}^{\alpha-1} + (1-\delta)$. This is exactly our no-arbitrage condition.

From the first-order conditions, we obtain the Euler equation:

$$\frac{1}{c_t^h(t)} = \frac{\beta r_t}{c_t^h(t+1)}$$

Therefore: $c_t^h(t+1) = \beta r_t c_t^h(t)$.

Stationary State

In the stationary state, all variables remain constant. We denote stationary state values with asterisks: $k_t = k_{t+1} = k^*$, $w_t = w_{t+1} = w^* = (1-\alpha)(k^*)^\alpha$, and $r_t = r^* = \alpha(k^*)^{\alpha-1} + (1-\delta)$.

Using simplified notation where c_1^h is consumption when young, c_2^h is consumption when old, l^h are loans, and k^h is capital investment (because they are stationary we skip the parenthesis with the period), the budget constraints in the stationary state are:

- Young: $w^* h = c_1^h + l^h + k^h$
- Old: $w^* h + r^* l^h + r^* k^h = c_2^h$

The Euler equation in the stationary state is $c_2^h = \beta r^* c_1^h$.

Substituting the Euler equation into the old agents' constraint: $w^* h + r^* l^h + r^* k^h = \beta r^* c_1^h$. From the young agents' constraint: $c_1^h = w^* h - l^h - k^h$. Substituting this expression:

$$w^* h + r^* l^h + r^* k^h = \beta r^* (w^* h - l^h - k^h)$$

$w^* h + r^* l^h + r^* k^h = \beta r^* w^* h - \beta r^* l^h - \beta r^* k^h$
$w^* h + r^* l^h (1+\beta) + r^* k^h (1+\beta) = \beta r^* w^* h$
$w^* h (1 - \beta r^*) = -r^* (1+\beta)(l^h + k^h)$

For the special case without intergenerational lending, we assume $l^h = 0$. Then:

$$w^* h (1 - \beta r^*) = -r^* (1+\beta) k^h$$

Solving for individual capital investment:

$$k^h = \frac{w^* h(\beta r^* - 1)}{r^*(1+\beta)}$$

In equilibrium, individual capital investment must equal the capital stock per capita: $k^h = k^*$. Therefore:

$$k^* = \frac{w^* h(\beta r^* - 1)}{r^*(1+\beta)}$$

Substituting $w^* = (1-\alpha)(k^*)^\alpha$ and $r^* = \alpha(k^*)^{\alpha-1} + (1-\delta)$:

$$k^* = \frac{(1-\alpha)(k^*)^\alpha h[\beta(\alpha(k^*)^{\alpha-1} + (1-\delta)) - 1]}{[\alpha(k^*)^{\alpha-1} + (1-\delta)](1+\beta)}$$

Multiplying both sides by $[\alpha(k^*)^{\alpha-1} + (1-\delta)](1+\beta)$ and dividing by k^*:

$$[\alpha(k^*)^{\alpha-1} + (1-\delta)](1+\beta) = (1-\alpha)(k^*)^{\alpha-1} h[\beta(\alpha(k^*)^{\alpha-1} + (1-\delta)) - 1]$$

This is the equilibrium condition that determines the steady-state capital stock k^*.

For comparison with the full depreciation case, when $\delta = 1$, the equilibrium condition becomes:

$$\alpha(k_1^*)^{\alpha-1}(1+\beta) = (1-\alpha)(k_1^*)^{\alpha-1} h[\beta\alpha(k_1^*)^{\alpha-1} - 1]$$

where k_1^* is the steady-state capital with full depreciation.

With partial depreciation ($\delta < 1$), the equilibrium condition is:

$$[\alpha(k_\delta^*)^{\alpha-1} + (1-\delta)](1+\beta) = (1-\alpha)(k_\delta^*)^{\alpha-1} h[\beta(\alpha(k_\delta^*)^{\alpha-1} + (1-\delta)) - 1]$$

where k_δ^* is the steady-state capital with partial depreciation.

The key difference is the additional term $(1-\delta) > 0$ which increases the effective return to capital. This higher return incentivizes greater saving, leading to the conclusion that $k_\delta^* > k_1^*$.

Since output per worker is $y^* = (k^*)^\alpha$, and $k_\delta^* > k_1^*$, we have $y_\delta^* = (k_\delta^*)^\alpha > (k_1^*)^\alpha = y_1^*$. Therefore, output in the stationary state will be higher with partial depreciation.

For household utility, consumption levels in the stationary state are:

$$c_1^h = w^* h - k^* = (1-\alpha)(k^*)^\alpha h - k^*$$

$$c_2^h = \beta r^* c_1^h = \beta[\alpha(k^*)^{\alpha-1} + (1-\delta)]c_1^h$$

Utility is $U^* = \ln c_1^h + \beta \ln c_2^h$. With $k_\delta^* > k_1^*$, we have higher wages $w_\delta^* = (1-\alpha)(k_\delta^*)^\alpha > (1-\alpha)(k_1^*)^\alpha = w_1^*$ and higher returns $r_\delta^* = \alpha(k_\delta^*)^{\alpha-1} + (1-\delta) > \alpha(k_1^*)^{\alpha-1} = r_1^*$. The net effect is positive because higher wages increase available income and higher returns increase the value of savings. Therefore, household utility in the stationary state will be higher with partial depreciation.

For the growth dynamics, starting from $K_0 < K^*$, the transition equation is:

$$K_{t+1} = k_t^h \cdot H_t = \frac{w_t h(\beta r_t - 1)}{r_t(1+\beta)} \cdot H_t$$

With partial depreciation, $r_t = \alpha k_t^{\alpha-1} + (1-\delta) > \alpha k_t^{\alpha-1}$, so for any $k_t < k^*$, the term $(\beta r_t - 1)$ is larger, implying greater saving and investment in each period. The larger gap $(k^* - k_t)$ creates stronger incentives to accumulate capital. Therefore, growth will be faster with partial depreciation.