5

Recursive Stochastic Models

Up to this point, except for short sections of Chapters 1 and 2, our models have been deterministic. The values of all of the parameters of the models and the form of the functions are known with certainty. Given some initial condition, these economies follow a prescribed path.

Models are approximations of reality and being approximations are at least (or, at best, only) partially false. The predictions that a model makes, even a very good and very complete model, will not coincide with what occurs. This failure of models to predict perfectly comes from two potential sources.

One source of this failure to predict perfectly is that these are variables that are not included in the model but that impact on the values of the variables included. Our (admittedly very simple) model of a deterministic Robinson Crusoe economy tells us how the decisions about how much to save and how much to work affect the amount of output in the economy but does not account for how the economy responds to the weather, to the physical and mental health of Robinson, or to the chance arrival of a book on tropical horticulture that gets washed up on a beach. Under this logic, if a model were sufficiently rich, it would be able to predict future outcomes with very great accuracy. However, our capacity to collect information and to construct, to test, and to solve models is limited, not the least by the capacity of our minds. Since we know we can't include everything in a model, one way to handle that which we cannot include is to allow the model to be stochastic. We simply let some part of the model, the value of some parameters in each period, for example, be determined by "nature," where nature embodies everything that is not in our model.

A second source may be that the universe is simply naturally stochastic and there are things that we cannot predict with absolute certainty even if we have full information about the current state of the universe. This idea is fairly well accepted in quantum physics, but is much more open to debate when more macro processes are studied. Many people are uncomfortable with the idea that their world could be inherently random. In fact, the whole area of mathematics know as "chaos theory" is based on adding extra dimensions to problems so that what looks like chaotic or random behavior in low dimensions is explained by a perfectly deterministic model in higher dimensions. The goal of chaos theory is to eliminate randomness.

Whatever its source, for an economist it is convenient to assume that the world is stochastic and that the way this randomness intrudes in our models can be described by probabilities. Adding a little randomness goes a long way in helping the predictions of our models better match the data that we observe.

5.1 PROBABILITY

Before discussing models with stochastic shocks, it is worth taking some time to briefly define exactly what is meant by probability and by a probability space. A probability space $(\Omega, \mathcal{F}, P)$ is comprised of three elements: 1) Ω, a set that contains all the states of nature that might occur, 2) $\mathcal{F}$, a collection of subsets of Ω, where each subset is called an event,[1] and 3) P, a probability measure over $\mathcal{F}$.

First consider what this means when Ω is a finite set of possible states of nature. For example, it might contain just two possible values for technology, A_1 and A_2. Then a natural way to define $\mathcal{F}$ is with four elements: the empty set, $[\ ]$, A_1, A_2, and the set $[A_1, A_2]$. A probability measure for these four sets is 0 for the empty set, some value $0 \le p_1 \le 1$ for A_1, $p_2 = 1 - p_1$ for A_2, and 1 for the set $[A_1, A_2]$. This says that either A_1 or A_2 will occur and, for a large enough sample, A_1 will occur with frequency p_1. For larger finite sets of possible states of nature, the structure is the same, but there are simply more elements to $\mathcal{F}$. If Ω were comprised of three elements, $A_1 = .9$, $A_2 = 1.05$, $A_3 = 1.10$, then, in addition to the sets given above, $\mathcal{F}$ would include A_3, $[A_1, A_3]$, $[A_2, A_3]$, and $[A_1, A_2, A_3]$. The event $[A_2, A_3]$ contains all possible technology levels greater than 1 and occurs with probability $p_2 + p_3$ (when the underlying events are independent and A_1 occurs with probability p_1, A_2 with p_2, A_3 with p_3 and $p_1 + p_2 + p_3 = 1$).

1. Strictly speaking, the set $\mathcal{F}$ of subsets of Ω needs to be a σ-field. A σ-field has the following properties: if subset A is a member of $\mathcal{F}$, then so is the complement of A (all elements of Ω excluding those in A). If subsets A and B are members of $\mathcal{F}$, so are the intersection, $A \cap B$, and union, $A \cup B$. In addition, a countably infinite union of sets in $\mathcal{F}$ is in $\mathcal{F}$.

It may seem like one goes to too much trouble with defining $\mathcal{F}$, the set of subsets of Ω, and then probabilities over this subset. In the finite case, with independent underlying events, one can frequently simply define the probability measure over the elements of Ω. Each underlying event has its probability, and the probability of any subset of these events is found by summing the probabilities of the events that make up the subset.

When the set of possible states of nature is continuous, then the definition is more useful. Consider a growth model where technology, A_t, can take on any value in the set [.9, 1.2], the closed continuous set of values between .9 and 1.2, that includes the end points. Suppose that the probability distribution is uniform, so that, in some sense, any value is equally as likely as any other inside the set. In this case, the probability that in some given period t, $A_t =$ 1.15565, for example, is zero. With a uniform distribution, or any continuous distribution, for that matter, the probability that technology has any specific value in any specific period is always zero.

It is in this case that defining subsets of [.9, 1.2] becomes useful. Imagine that we want to know the probability that technology will have a value in period t between .97 and 1.03. Since this is a uniform distribution, this probability can be calculated as $.06/.3 = .2$, or 20%. Although the probability of any one value occurring for A_t is always zero in this example, for any positive range of values, one can usually find a positive probability. Therefore, by defining probabilities over subsets of the states of nature, the definition encompasses situations with a continuous range of possible states of nature.

Note that it is not always the case that in situations with a continuous range of states of nature, the probability of a single value is zero. Consider the range of states of nature for the exact daily rainfall in Buenos Aires. Over the last 100 years, the range has been from zero to slightly over 300 millimeters. However, there have been a great many days in which the rainfall was zero so that the probability of zero rainfall has a positive value. The probability of any other specific number is zero. The probability distribution is not continuous at zero, and the point zero contains a positive mass of probability.

5.2 A SIMPLE STOCHASTIC GROWTH MODEL

Imagine that the economy is much like that in the infinite horizon Robinson Crusoe model except that the production function in period t is

$$y_t = A^t f(k_t),$$

where the production function, $f(k_t)$, is an increasing, concave function of the capital stock (assuming that a constant 1 unit of labor is supplied) and A^t can take on two values (two states of nature, as they are called) where the level of technology given by

$$A^t = \begin{cases} A_1 \text{ with probability } p_1 \\ A_2 \text{ with probability } p_2 \end{cases}.$$

We assume that A_1 and A_2 are the only values that technology can take, so $p_1 + p_2 = 1$. The realization of A_i in any period is independent of the realizations that occurred in the past and those that will occur in the future. The probabilities p_1 and p_2 should be interpreted to mean that as the sample size increases, the fraction of periods in which we observe A_1 goes to p_1 and the fraction of periods in which we observe A_2 goes to p_2. To put some order on things, we assume that $A_1 > A_2$, or that the technology is more productive in state 1 than it is in state 2. As this model is written, there is no technological growth. There are two kinds of periods, one where, with the same capital, output is greater than in the other. One can think of a farming economy where state 1, in which the technology has the value A_1, is when good weather occurs and state 2, with the technology of value A_2, is when bad weather occurs. This is an appropriate example because we often speak of the states 1 and 2 as states of nature in which nature, with probabilities p_1 and p_2, chooses the state that occurs in any period t.

The rest of the model is similar, but not identical, to earlier growth models. The capital stock grows by the equation

$$k_{t+1} = A^t f(k_t) + (1-\delta)k_t - c_t.$$

At time 0, Robinson Crusoe wants to maximize an *expected* discounted utility function of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t).$$

The choice of consumption in each period will depend on both the capital stock of that period and the realization of technology in that period. Since future realizations of technology are not known, it is not possible to choose a complete consumption path. In fact, future consumption plans are represented by a kind of tree. Given some initial capital k_0, in period 0 there are two possible technology levels that could occur and two different amounts of production, represented by the ordered pair $[A_1 f(k_0), A_2 f(k_0)]$, with probabilities $[p_1, p_2]$. Depending on which state occurs in period 0, R.C. will choose some time 1 capital stocks of $[k_1^1, k_1^2]$. In period 1, production will be one of these four possibilities, $[A_1 f(k_1^1), A_2 f(k_1^1), A_1 f(k_1^2), A_2 f(k_1^2)]$, with probabilities $[p_1 p_1, p_1 p_2, p_2 p_1, p_2 p_2]$. These probabilities occur because there was a probability of p_1 that the capital would be k_1^1, and once that capital stock was chosen there was a probability p_1 that nature would choose the technology level A_1 in period 1. Notice that the four probabilities sum to one, $p_1 p_1 + p_1 p_2 = p_1$ and $p_2 p_1 + p_2 p_2 = p_2$, and $p_1 + p_2 = 1$. In period 1 there are four possible k_2^i that can be chosen that result in eight possible levels of output in period 2. In this way, the number of possible consumptions doubles in each period. One doesn't know which path will occur, but one can find the probability of each path.

Suppose that one can write the value of the maximum expected discounted utility given an initial capital stock of k_0, when the time 0 realization of technology is A_1, as

$$V(k_0, A_1) = \max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the budget constraint for $t = 0$,

$$k_1 = A_1 f(k_0) + (1-\delta)k_0 - c_0,$$

and those for $t \geq 1$,

$$k_{t+1} = A^t f(k_t) + (1-\delta)k_t - c_t,$$

and the independent realizations of $A^t = \left[A_1, A_2\right]$ with probabilities $\left[p_1, p_2\right]$. One can write an almost identical expression for $V(k_0, A_2)$, the maximum expected discounted utility for the same initial capital stock when the time 0 realization of technology is A_2, by simply replacing A_1 with A_2 in the first budget constraint.

Notice that the value of expected utility is a function of two state variables, the amount of capital that is available in the period and the realization of the technology shock in that period. As shown in the previous chapter, this expression can be written recursively as

$$V(k_0, A^0) = \max_{c_0} \left[u(c_0) + \beta E_0 V\left(k_1, A^1\right)\right]$$

subject to the budget constraint

$$k_1 = A^0 f(k_0) + (1-\delta)k_0 - c_0.$$

There is a subtle change in how the value function is written; it is now written as a *function* of the time 0 realization of the technology shock. As this function is written, k_0 and A^0 are the state variables and c_0 is the control variable. The second part of the value function is written with the expectations term because given a choice for c_0 (and through the budget constraint of k_1), it will have a value of $V(k_1, A_1)$ with probability p_1, and a value of $V(k_1, A_2)$ with probability p_2. For any particular choice $\widehat{k}_1$ of the time 1 capital stock, the expectations expression is equal to

$$E_0 V\left(\widehat{k}_1, A^1\right) = p_1 V\left(\widehat{k}_1, A_1\right) + p_2 V\left(\widehat{k}_1, A_2\right).$$

For any initial time period t, the problem can be written as

$$V(k_t, A^t) = \max_{c_t} \left[u(c_t) + \beta E_t V\left(k_{t+1}, A^{t+1}\right) \right],$$

subject to the budget constraint

$$k_{t+1} = A^t f(k_t) + (1-\delta)k_t - c_t.$$

Of course, as we have seen earlier, there are other ways to define the object function and budget constraints so that we have different choices for state variables and control variables. When we rewrite the problem with k_{t+1} as the control variable, it becomes

$$V(k_t, A^t) = \max_{k_{t+1}} \left[u(A^t f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta E_t V\left(k_{t+1}, A^{t+1}\right) \right], \quad (5.1)$$

and the budget constraint (using the notation of Chapter 4) is

$$k_{t+1} = G(x_t, y_t) = k_{t+1}.$$

The solution to a stochastic recursive problem like that in equation 5.1 finds a function that gives the values of the control variables that maximizes the value function over the domain of the state variables. Since the state variables include both the results of previous choices of control variables and the results of nature's choices of the value for the stochastic state variables, we call the solution function a *plan* and write it (for the problem in equation 5.1) as

$$k_{t+1} = H(k_t, A^t).$$

The plan gives the optimizing choice of the control variables in every period as a function of the regular state variables and of the states of nature. A plan fulfills the condition that

$$V(k_t, A^t) = u(A^t f(k_t) + (1-\delta)k_t - H(k_t, A^t)) + \beta E_t V\left(H(k_t, A^t), A^{t+1}\right),$$

where no maximization is required because the plan chooses the maximizing values for the control variables.

5.3 A GENERAL VERSION

Using the notation of section 4.3, we can write the value function as

$$V(x_t, z_t) = \max_{\{y_s\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, y_s, z_s),$$

subject to the set of budget constraints given by

$$x_{s+1} = G(x_s, y_s, z_s),$$

for $s \geq t$, where x_t is the set of "regular" state variables and z_t is the set of state variables determined by nature, the stochastic state variables. The y_t are the control variables. Both the objective function, $F(x_s, y_s, z_s)$, and the budget constraints, $G(x_s, y_s, z_s)$, can be functions of the stochastic state variables.

This problem can be written recursively as a Bellman equation of the form

$$V(x_t, z_t) = \max_{y_t} \left[F(x_t, y_t, z_t) + \beta E_t V(x_{t+1}, z_{t+1}) \right], \tag{5.2}$$

subject to the budget constraint

$$x_{t+1} = G(x_t, y_t, z_t).$$

The solution (what was previously called a policy function) is a *plan* (a function) of the form

$$y_t = H(x_t, z_t),$$

where

$$V(x_t, z_t) = F(x_t, H(x_t, z_t), z_t) + \beta E_t V(G(x_t, H(x_t, z_t), z_t), z_{t+1})$$

holds for all values of the state variables (including the stochastic state variables).

The first-order conditions for the problem in equation 5.2, and its budget constraints, are

$$0 = F_y(x_t, y_t, z_t) + \beta E_t \left[V_x(G(x_t, y_t, z_t), z_{t+1}) G_y(x_t, y_t, z_t) \right],$$

where $X_q()$ is the partial derivative of function $X()$ with respect to variable q, where q can be a vector.

In addition, for interior solutions, we have the Benveniste-Scheinkman envelope theorem result,

$$V_x(x_t, z_t) = F_x(x_t, y_t, z_t) + \beta E_t \left[V_x(G(x_t, y_t, z_t), z_{t+1}) G_x(x_t, y_t, z_t) \right].$$

When one is able to choose the controls so that $G_x(x_t, y_t, z_t) = 0$, the above equation is

$$V_x(x_t, z_t) = F_x(x_t, y_t, z_t),$$

and the first-order conditions give the Euler equation (a stochastic Euler equation),

$$0 = F_y(x_t, y_t, z_t) + \beta E_t \left[F_x(G(x_t, y_t, z_t), y_{t+1}, z_{t+1}) G_y(x_t, y_t, z_t) \right].$$

Up to this point, the discussion of the general version has said nothing about the dimension of the stochastic variable z_t. In the example economy in the beginning of this chapter, the stochastic variable had only two possible realizations in any period. That the stochastic shock could take on so few values makes the exposition simple. In theory, there is no necessity that the dimension be small, and it is quite possible to describe a model in which the realization of the stochastic variable comes from a continuous distribution (one with an uncountable infinite set of possible realizations). In practice, the dimension of the state space and the variables in it do matter.

5.3.1 The Problem of Dimensionality

Logically, it should be possible to follow the same technique that we used in the deterministic case and begin with an initial guess for the value function, a function $V_0(x_t, z_t)$, and iterate on the equation

$$V_{j+1}(x_t, z_t) = \max_{y_t} \left[F(x_t, y_t, z_t) + \beta E_t V_j(G(x_t, y_t, z_t), z_{t+1}) \right]$$

to find approximations of the value function and the policy functions (the plans) that converge on the actual value function and plans. Indeed, it is possible to find the approximations numerically if the dimensions of x_t and z_t are not too big. For the calculations in the deterministic case, we used a reasonable dense discrete subset of the continuous domain of x_t and, when needed, found values between these points using a linear interpolation. If the dimension of the discrete domain of x_t (call it $dim\left(x_t\right)$) is not too large and the $V(x_t)$ function does not have too much curvature, this process gives useful results.

Once we add the stochastic state variables, the dimension of the optimization problem increases by the product of the dimension of these stochastic variables. If we add only one stochastic variable, the dimension of that variable is the discrete sampling that we do of its domain (call it $dim(z_t)$). The dimension of the calculations for the stochastic case is then $dim\left(x_t\right) dim(z_t)$, the product of the dimension of the discrete domain of the original state variable times the discrete domain of the stochastic variable. If one is using numerical techniques to calculate iterations of the value function, the number of points to be found in each iteration can become quite burdensome.

5.4 THE VALUE FUNCTION FOR THE SIMPLE ECONOMY

Using the model we built in the first section of this chapter, we describe how to find the value functions and the plans. The stochastic variable, A^t, has only two possible realizations, given by the ordered pair $[A_1, A_2]$ with constant probabilities $[p_1, p_2]$. We can write equation 5.1 as a pair of Bellman equations, one for each of the two possible time t realizations of A^t, as

$$V(k_t, A_1) = \max_{k_{t+1}} \left[u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta\left[p_1 V\left(k_{t+1}, A_1\right) + p_2 V\left(k_{t+1}, A_2\right)\right]\right]$$

and

$$V(k_t, A_2) = \max_{k_{t+1}} \left[u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta\left[p_1 V\left(k_{t+1}, A_1\right) + p_2 V\left(k_{t+1}, A_2\right)\right]\right]$$

where the expectations part of the value function, $E_t V\left(k_{t+1}, A^{t+1}\right)$, is replaced by its explicit representation, $p_1 V\left(k_{t+1}, A_1\right) + p_2 V\left(k_{t+1}, A_2\right)$. In this economy, for each choice of k_{t+1}, the expected value, $E_t V\left(k_{t+1}, A^{t+1}\right)$, is equal to the time $t+1$ value of discounted utility when nature chooses A_1 as the technology shock times the probability of this shock being chosen plus the value of discounted utility when nature chooses A_2 times the probability of nature choosing A_2.

The iteration process requires choosing starting functions for both $V_0(k_t, A_1)$ and $V_0(k_t, A_2)$. Given these initial functions, the functions from the first iteration, $V_1(k_t, A_1)$ and $V_1(k_t, A_2)$, are found by simultaneously calculating

$$V_1(k_t, A_1) = \max_{k_{t+1}} \left[u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta\left[p_1 V_0\left(k_{t+1}, A_1\right) + p_2 V_0\left(k_{t+1}, A_2\right)\right]\right]$$

and

$$V_1(k_t, A_2) = \max_{k_{t+1}} \left[u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta\left[p_1 V_0\left(k_{t+1}, A_1\right) + p_2 V_0\left(k_{t+1}, A_2\right)\right]\right]$$

over the discrete subset of values of k_t. To find the results of the next iterations, $V_2(k_t, A_1)$ and $V_2(k_t, A_2)$, we calculate

$$V_2(k_t, A_1) = \max_{k_{t+1}} \left[u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) \right.$$
$$\left. + \beta \left[p_1 V_1\left(k_{t+1}, A_1\right) + p_2 V_1\left(k_{t+1}, A_2\right) \right] \right]$$

and

$$V_2(k_t, A_2) = \max_{k_{t+1}} \left[u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) \right.$$
$$\left. + \beta \left[p_1 V_1\left(k_{t+1}, A_1\right) + p_2 V_1\left(k_{t+1}, A_2\right) \right] \right],$$

using the two $V_1(k_t, A^t)$ functions we found in the previous iteration. Repeated iterations result in a sequence of pairs of functions $\left\{V_j(k_t, A_1), V_j(k_t, A_2)\right\}_{j=0}^{\infty}$ that converge to the desired pair of value functions, $\left[V(k_t, A_1), V(k_t, A_2)\right]$.

5.4.1 Calculating the Value Functions

Using the parameters from our standard example economy, $\delta = .1$, $\beta = .98$, $\theta = .36$, the production function, $f(k_t) = k_t^{\theta}$, and the subutility function, $\ln(c_t)$, we add the values of the stochastic state variable, $A_1 = 1.75$, $p_1 = .8$, $A_2 = .75$, and $p_2 = .2$. We choose constant initial value functions, $V_0(k_t, A_1) = 20$ and $V_0(k_t, A_2) = 20$. The first round of iterations results in calculations for $V_1(k_t, A^t)$ of

$$V_1(k_t, A_1 = 1.75) = \max_{k_{t+1}} \ln(1.75k_t^{.36} + .9k_t - k_{t+1}) + 19.6$$

and

$$V_1(k_t, A_2 = .75) = \max_{k_{t+1}} \ln(.75k_t^{.36} + .9k_t - k_{t+1}) + 19.6,$$

where the number 19.6 is simply 20 discounted by .98. The values of the two discounted next period value functions are the same in these two equations because we chose the same value for the initial guesses. In the next round, $V_1(k_t, A^t)$ will have different values for $A^t = A_1 = 1.75$ and $A^t = A_2 = .75$. The $V_2(k_t, A^t)$ functions are found maximizing

$$V_2(k_t, 1.75) = \max_{k_{t+1}} \left\{ \ln(1.75k_t^{.36} + .9k_t - k_{t+1}) \right.$$
$$\left. + \beta \left[.8V_1\left(k_{t+1}, 1.75\right) + .2V_1\left(k_{t+1}, .75\right) \right] \right\}$$

and

$$V_2(k_t, .75) = \max_{k_{t+1}} \left\{ \ln(.75k_t^{.36} + .9k_t - k_{t+1}) \right.$$
$$\left. + \beta \left[.8V_1\left(k_{t+1}, 1.75\right) + .2V_1\left(k_{t+1}, .75\right) \right] \right\}.$$

Continued iterations result in the value functions shown in Figure 5.1. In this figure, the two curves are shown every 50 iterations (the last is at iteration 250). The pair of policy functions that we have after 250 iterations is shown in Figure 5.2, where the lower one is for $A^t = .75$.

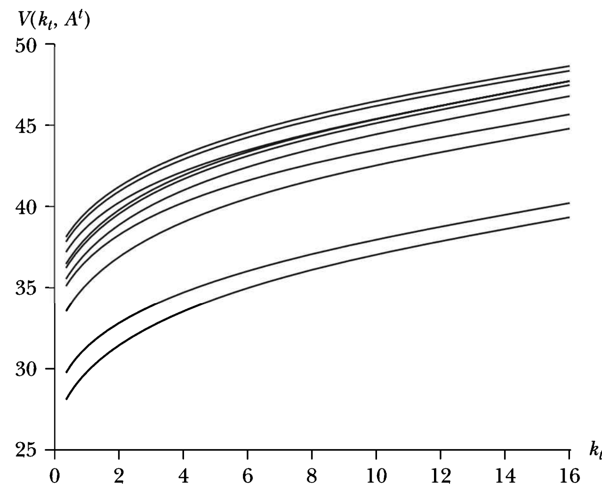

FIGURE 5.1 Iterations on the value function

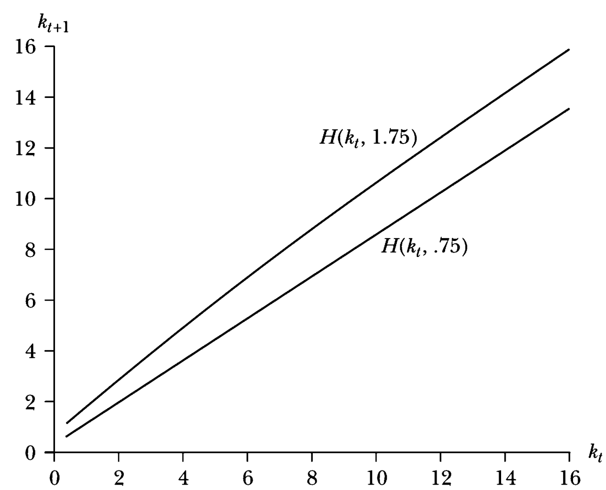

FIGURE 5.2 The plans

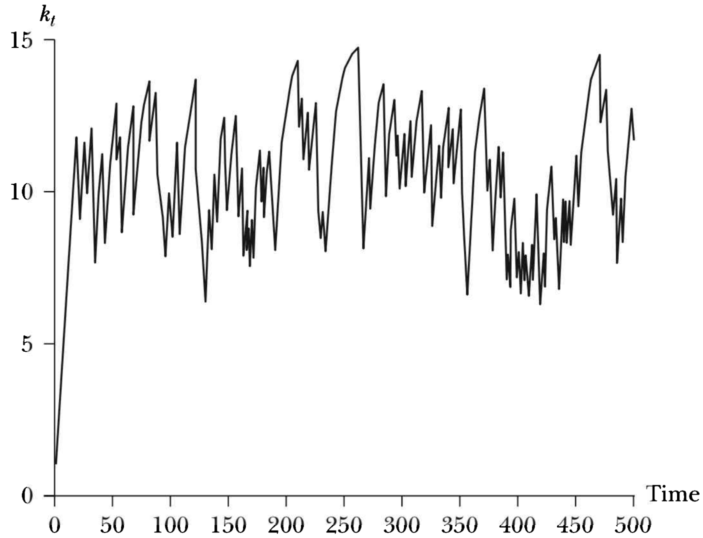

FIGURE 5.3 A simulated time path

One can generate simulations of the time paths of this economy using the plan and a uniform random number generation. Figure 5.3 shows a simulation of the time path of this economy beginning with $k_0 = 1$ and where, in each period, a uniform random number generation that produced numbers between 0 and 1 is used to determine whether nature has chosen state one or state two. State A_1 occurred in each period where the random number generator produced a value less than or equal to .8. Otherwise, state A_2 occurred.

5.5 MARKOV CHAINS

A somewhat richer random process can be obtained by the use of Markov chains. In a Markov chain, the probabilities for the realizations of the states of nature in period t are a function of the realization that occurred in period $t-1$ and only in period $t-1$. To be able to use our recursive methods, we want a Markov chain that is invariant with time; the probabilities depend on the realization that occurred in the previous period but not on which time period we are in.

There are three elements to a Markov chain stochastic process. The first is the set of realizations for the state of nature; in the example we have been using, it is the set of values that our A^t variable can take on. This set has a fixed, finite dimension, n, and $\{A^t\} = \{A_1, A_2, \ldots, A_n\}$. The dimension n and the values A_i are the same in every period. In the example economy above, $n = 2$ and $\{A^t\} = \{A_1 = 1.75, A_2 = .75\}$.

The second element of a Markov chain stochastic process is a matrix of transition probabilities, P, where element $p_{i,j}$ is the probability that state j will occur when the state of nature in the previous period was state i (which is the same as saying that A^{t-1} had value A_i). For, example, assume that A^t is the set from the earlier example, $A^t = \{1.75, .75\}$, and the probability matrix is

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} .90 & .10 \\ .40 & .60 \end{bmatrix}.$$

In this case, if the previous realization of the state of nature, A^{t-1}, had been $A_1 = 1.75$, then there is a 90% probability that the realization in period t will have the same value and a 10% probability that it will have the value .75. If the realization of A^{t-1} had been $A_2 = .75$, then in period t, there is a 60% probability that this same value will occur and a 40% probability that the realization will have the value 1.75. A Markov chain allows the stochastic process to, among other things, show more persistence than in the first example economy. The values in the example are chosen so that there is a higher probability that the economy will stay in the current state of nature than that the economy will move to the other state.

The third element of a Markov chain stochastic process is the initial state of nature, from period $t - 1$, that determines the row of the P matrix where we begin (or the probability distribution of the initial period's stochastic variable if this is different from a row of P). With the initial distribution given, the probability of any future outcome can be calculated.

The probabilities given by the matrix P are conditional probabilities in the sense that once the economy is in the state of nature A_1, the top row of the P matrix describes the probabilities for the state of nature in the next period. Once the economy is in the state of nature A_2, the second row of the P matrix describes the probabilities of ending up in the state of nature A_1 or A_2 in the next period. These are conditional probabilities since they tell how the economy will proceed once it is in a particular state. We may also be interested in the *unconditional probabilities* of the occurrence of the states of nature in this economy. These are the probabilities that the economy will be in state of nature A_1 or in state of nature A_2 when we know nothing about previous states. If we run the economy long enough, the unconditional probabilities will tell us how often we observe state of nature A_1,[2] independent of the initial conditions. Unique invariant unconditional probability distributions exist if

2. The frequency with which we observe state of nature A_2 is simply equal to one minus the frequency with which we observe state of nature A_1.

every element of the P matrix is positive, if every $p_{i,j} > 0$.[3] When every element of P is positive, there is a positive probability of moving from any state of nature to any other state of nature.

We can use an example to show how the convergence works. Suppose that p_0 is the initial probability distribution (the one for period 1). The unconditional probability distribution for period 2 is $p_0 P$, given the initial distribution. Multiplying p_0 by the transition probabilities gives the probability distribution for period 2. In period 3, the distribution is $p_0 P^2 = p_0 P P$. In any period $n + 1$, the probability distribution for the states of nature is $p_0 P^n$. The claim is that as $n \rightarrow \infty$, $p_0 P^n \rightarrow P^\infty$ independently of the initial probability p_0.

Using the 2×2 matrix P above, we calculate a sequence of P^n's, for $n = 1, 2, \ldots$. The first elements of this sequence are

$$P = \begin{bmatrix} .90 & .10 \\ .40 & .60 \end{bmatrix},$$

$$P^2 = \begin{bmatrix} .85 & .15 \\ .60 & .40 \end{bmatrix},$$

$$P^3 = \begin{bmatrix} .825 & .175 \\ .70 & .30 \end{bmatrix},$$

and in the limit,

$$P^\infty = \begin{bmatrix} .80 & .20 \\ .80 & .20 \end{bmatrix}.$$

Notice a special characteristic of the P^∞ matrix: all the rows of the matrix are identical. Every row gives the unique invariant unconditional probability distribution. This is because, independent of the initial state, the economy goes to this distribution. Therefore, if the economy starts in state 1 (the first row), it goes to the distribution $[\ .80 \quad .20\ ]$, and if it starts in state 2 (the second row), it also goes to the distribution $[\ .80 \quad .20\ ]$. Suppose that there is an initial probability distribution p_0. Let us choose $p_0 = [\ .36 \quad .64\ ]$, for example, but it could be any distribution. If we multiply this vector by the matrix P^∞, the result is

3. There are weaker conditions for having a unique invariant unconditional distribution. A discussion of these can be found in Ljungqvist and Sargent [54] or Breiman [16].

$$
\begin{aligned}
p_0 P^{\infty} &= [\,.36 \quad .64\,] \begin{bmatrix} .80 & .20 \\ .80 & .20 \end{bmatrix} \\
&= [\,.36 \times .80 + .64 \times .80 \quad .36 \times .20 + .64 \times .20\,] \\
&= [\,.80 \quad .20\,].
\end{aligned}
$$

For an economy that begins with initial probabilities for the two states given by p_0 and follows transition probabilities P long enough, the resulting limit probabilities for the two states are the unique invariant unconditional probability distribution.

We can write the value function for an economy with a Markov chain stochastic process as

$$
V_{j+1}(x_t, z_t) = \max_{y_t} \left[F(x_t, y_t, z_t) + \beta E_t V_j(G(x_t, y_t, z_t), z_{t+1}) \mid z_t \right],
$$

where the addition of the expression, "$\mid z_t$" means that the expectations are taken *conditional on* the realization of z_t. For the growth economy, modified by using the Markov chain process described above, the value functions are

$$
\begin{aligned}
V(k_t, A_1) = \max_{k_{t+1}} \big[& u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) \\
& + \beta \left[p_{11} V\left(k_{t+1}, A_1\right) + p_{12} V\left(k_{t+1}, A_2\right) \right] \big]
\end{aligned}
$$

and

$$
\begin{aligned}
V(k_t, A_2) = \max_{k_{t+1}} \big[& u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) \\
& + \beta \left[p_{21} V\left(k_{t+1}, A_1\right) + p_{22} V\left(k_{t+1}, A_2\right) \right] \big],
\end{aligned}
$$

where the probabilities in each equation are those that are appropriate for the given realization of A^t.

As before, for specific economies, the value function is found by iterating, given some initial choice of the function: $V_0(k_t, A^t)$. For the example economy we have been using, but with the matrix P used in place of the constant probabilities, the $j+1$th iteration of the value function is

$$
\begin{aligned}
V_{j+1}(k_t, 1.75) = \max_{k_{t+1}} \Big\{ & \ln(1.75 k_t^{.36} + .9k_t - k_{t+1}) \\
& + \beta \left[.9 V_j\left(k_{t+1}, 1.75\right) + .1 V_j\left(k_{t+1}, .75\right) \right] \Big\}
\end{aligned}
$$

and

$$V_{j+1}(k_t, .75) = \max_{k_{t+1}} \Big\{ \ln(.75k_t^{.36} + .9k_t - k_{t+1}) \\ + \beta \left[.4V_j\left(k_{t+1}, 1.75\right) + .6V_j\left(k_{t+1}, .75\right)\right]\Big\}.$$

As before, these value functions converge to a $V(k_t, A^t)$ and the contingent plans that result are shown in Figure 5.4. A_1 is the state where technology is 1.75 and A_2 is the state where it is .75. These policy functions are similar to those shown in Figure 5.2. This should not be surprising since the unconditional distributions on the state variables are the same. However, they are not exactly the same because the conditional distributions are different. When we run some simulations of this economy, we notice that there is substantially more persistence in states than there was in the first stochastic version of the model. Compare the time path in Figure 5.5 to that of the simple stochastic model in Figure 5.3. Both of these time series were generated using the same realizations of a uniform random variable. The additional persistence is seen in both the continuation of the high technology state and the extended periods of convergence toward the higher stationary state and the longer periods in the low state.

One weakness of putting much of the persistence of the model into the Markov chain is that it rather avoids the *economic* problem of what generates

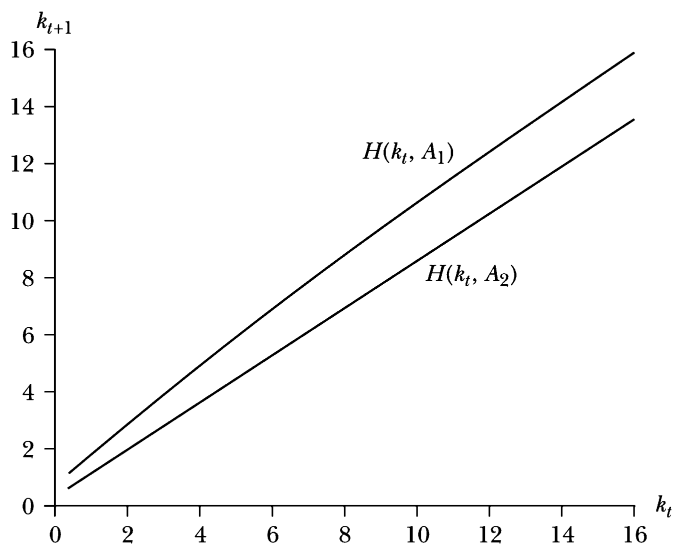

FIGURE 5.4 The plans with Markov chains

that persistence. The time series that result from these kinds of models can be made to display persistence, but this persistence is not explained in economic terms. If one is simply trying to replicate some time series, using larger-dimension Markov chains may do a good job of capturing the variation and the persistence. However, if one is using these models to try to understand what is causing the persistence, Markov chains do not really help.

EXERCISE 5.1 Not all conditional probabilities described by Markov chains result in unique invariant unconditional probability distributions. Consider a Markov chain with transition probabilities described by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find P^3, P^6, P^7, P^8,and P^9. What does this suggest about the nature of this process?

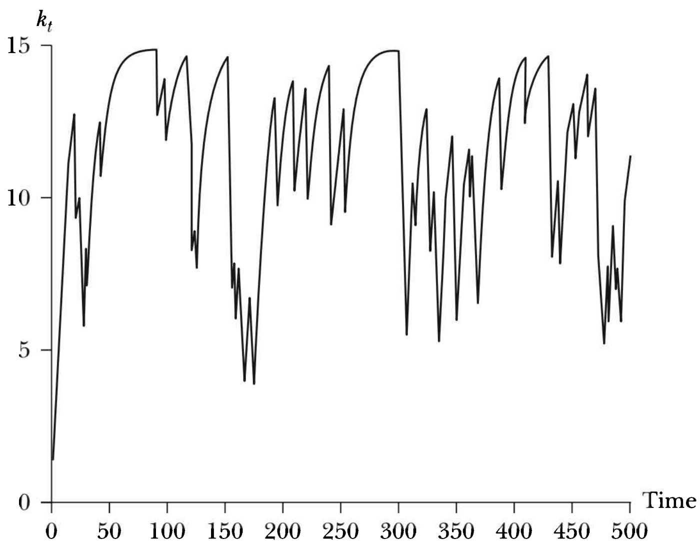

FIGURE 5.5 A simulation with Markov chains

EXERCISE 5.2 Find the unique invariant unconditional probability distribution for the Markov chain with transition probabilities described by

$$P = \begin{bmatrix} .7 & .2 & .1 \\ .3 & .4 & .3 \\ .1 & .1 & .8 \end{bmatrix}.$$

A doubling technique might be helpful. Define $Q = PP$. Then define $Q = QQ$ and repeatedly solve for Q. This process produces the sequence P^2, P^4, P^8, P^{16}, etc., and normally converges quite rapidly.

5.6 REPRISE

Adding finite dimensioned stochastic shocks to a recursive model is fairly simple. Logically little changes, the calculations are just a bit more complicated. The realization in each period of the stochastic shock is treated as a new state variable; the state of nature and its value helps determine the policy that the optimizing agent will follow. To emphasize the fact that the policy function needs to respond to the state of nature as well as the other state variables, and the realizations of future states of nature are yet unknown, we refer to the policy function as a plan.

In the Bellman equation with stochastic states of nature, the distribution (probabilities) of these states of nature show up as weights for determining the expectation of the next period's value of the value function. We use the same technique as before for finding an approximation of the value function, only this time we find a value function for each of the values that the stochastic variable can have. Once the value functions (and the plans) are known, we can start with some initial values for the state variables and, using a random number generator from our computer package, we can simulate time paths for the economy.

More persistence (as well as other characteristics) can be added to the model by using Markov chains for the stochastic processes. In a Markov chain, the probabilities of transition to the next state of nature depend on the current state of nature. While this permits us to find time paths for our economy that can better approximate those of the real world, the method is heavy on probability theory and light on economic theory. The persistence comes through the Markov chains and not through our ability to produce an economic model that captures why the persistence occurs.

The bibliographic references of Chapter 4 are also relevant for this chapter, especially Ljungqvist and Sargent [54], which deals extensively with Markov chains.

5.7 MATLAB CODE

The following program gives the Matlab code for finding the value functions and plans of the Markov chain example in the text.

The Main Program

The main program first sets the parameters and then does a loop that iterates on the two value functions. The value function is found for values for capital beginning at .4 and going to 16 in steps of .4. It draws the current pair of value functions every 50 iterations. At the end is the plot command that draws the final plans.

```
global vlast1 vlast2 beta delta theta k0 kt At p1 p2
hold off
hold all
vlast1=20*ones(1,40);
vlast2=vlast1;
k0=0.4:0.4:16;
kt11=k0;
kt12=k0;
beta=.98;
delta=.1;
theta=.36;
A1=1.75;
p11=.9;
p12=1-p11;
p21=.4;
p22=1-p21;
A2=.75;
numits=250;
for k=1:numits
   for j=1:40
      kt=k0(j);
      At=A1;
      p1=p11;
      p2=p12;
      %we are using a bounded function minimization routine here
      z=fminbnd(@valfunsto,.41,15.99);
      v1(j)=-valfunsto(z);
      kt11(j)=z;
      At=A2;
```

```
      p1=p21;
      p2=p22;
      z=fminbnd(@valfunsto,.41,15.99);
      v2(j)=-valfunsto(z);
      kt12(j)=z;
   end
   if k/50==round(k/50)
      plot(k0,v1,k0,v2)
      drawnow
   end
   vlast1=v1;
   vlast2=v2;
end
hold off
%optional plotting program for the plans
%plot(k0,kt11,k0,kt12)
```

Subroutine valfunsto

Note that interpolation of the previous value function is linear.

```
function val=valfunsto(x)
global vlast1 vlast2 beta delta theta k0 kt At p1 p2
k=x;
g1=interp1(k0,vlast1,k,'linear');
g2=interp1(k0,vlast2,k,'linear');
kk=At*kt^theta-k+(1-delta)*kt;
val=log(kk)+beta*(p1*g1+p2*g2);
val=-val;
```