7

Linear Quadratic Dynamic Programming

In the previous chapter, we solved for the first-order conditions of Hansen's model and then used log linearization techniques to convert the general problem into a linear approximation. The first-order conditions were found for a non linear dynamic optimization problem and only then was the problem linearized.

In this chapter, we show an alternative approach. Here we derive a quadratic approximation for the objective function of the infinite horizon dynamic optimization problem and, using linear budget constraints, find a policy function that gives the optimizing values for the control variables as a linear function of the state variables. Note that there are two problems facing the modeler: the budget constraints need to be linear and the objective function needs to be quadratic. The original budget constraints can frequently be converted into linear ones by appropriate changes of variables and/or by putting all of the nonlinearity into the objective function.

Since we are interested in the dynamics of a system, we need to have, at least, a quadratic approximation of the objective function. A linear (first-order) approximation would result in constants for the first-order conditions, and these do not produce interesting dynamics. Higher-order approximations of the objective function are likely to do a better job of capturing the dynamics, but solving them is more difficult.

The discounted quadratic objective function we are looking for is of the form

$$\sum_{t=0}^{\infty} \beta^t \left[x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t \right], \tag{7.1}$$

subject to the linear budget constraints

$$x_{t+1} = Ax_t + By_t,$$

where x_t is the $n \times 1$ vector of state variables, y_t is an $m \times 1$ vector of control variables, R and A are $n \times n$ matrices, Q is an $m \times m$ matrix, W is an $m \times n$, and B is an $n \times m$ matrix.

The method most commonly used for finding a quadratic approximation of the objective function is by taking second-order Taylor expansions.

7.1 TAYLOR APPROXIMATIONS OF THE OBJECTIVE FUNCTION

Brook Taylor (1685–1731) introduced the theorem that we use for approximating continuous and continuously differentiable functions by polynomials. What has become known as Taylor's Theorem is stated as follows:

Suppose that f is a function with domain D in $\mathbf{R}^p$ and range in $\mathbf{R}$, and suppose that f has continuous partial derivatives of order n in a neighborhood of every point on a line segment joining two points u, v in D. Then there exists a point $\tilde{u}$ on this line segment such that

$$\begin{aligned} f(v) = f(u) &+ \frac{1}{1!}Df(u)(v-u) + \frac{1}{2!}D^2 f(u)(v-u)^2 \\ &+ \cdots + \frac{1}{(n-1)!}D^{n-1}f(u)(v-u)^{n-1} + \frac{1}{n!}D^n f(\tilde{u})(v-u)^n. \end{aligned}$$

Because the error term in the polynomial,

$$\frac{1}{n!}D^n f(\tilde{u})(v-u)^n,$$

usually gets small quickly as n grows when the distance, $v - u$, is small, this polynomial is frequently used to approximate a function around a point with known value using only the first- or second-order expansion (up to the quadratic).

The *point of known value* that we use is that of the stationary state, which we calculate as in the previous chapter, using the Euler equations. Note that what we need is the value of the objective function at the stationary state. To get this value, we find the values of the state and control variables in the stationary state and plug these values into the objective function. In addition, we take the first and second derivatives of the objective function and, using the same values for the state and control variables, find the values of the derivatives. For example, if the discounted utility function is of the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t, h_t) = \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + A \ln(1 - h_t)\right],$$

the objective function is

$$\ln c_t + A \ln(1 - h_t),$$

the first derivative of the objective function is the vector,

$$\left[\begin{array}{cc} \frac{1}{c_t} & -\frac{A}{1-h_t} \end{array}\right],$$

and the second derivative is the matrix,

$$\begin{bmatrix} -\frac{1}{c_t^2} & 0 \\ 0 & \frac{A}{(1-h_t)^2} \end{bmatrix}.$$

The approximation of the objective function that we get from the Taylor expansion, when we evaluate the function and its first and second derivatives at the stationary state, $\bar{c}$ and $\bar{h}$, is

$$u(c_t, h_t) \approx \ln \bar{c} + A \ln(1 - \bar{h}) + \left[\begin{array}{cc} \frac{1}{\bar{c}} & -\frac{A}{1-\bar{h}} \end{array}\right] \begin{bmatrix} c_t - \bar{c} \\ h_t - \bar{h} \end{bmatrix}$$

$$+ \frac{1}{2} \left[\begin{array}{cc} c_t - \bar{c} & h_t - \bar{h} \end{array}\right] \begin{bmatrix} -\frac{1}{\bar{c}^2} & 0 \\ 0 & \frac{A}{\left(1-\bar{h}\right)^2} \end{bmatrix} \begin{bmatrix} c_t - \bar{c} \\ h_t - \bar{h} \end{bmatrix}.$$

Notice that the Taylor approximation has a component (the part associated with the second derivative) that is, as this is written, a matrix in the controls. This is a part of the S matrix of the quadratic version of the objective function as written in equation 7.1. However, there is a difficulty that needs to be confronted. The rest of the Taylor expansion is a constant, the $\ln \bar{c} + A \ln(1 - \bar{h})$ part, and a linear component, the

$$\left[\begin{array}{cc} \frac{1}{\bar{c}} & -\frac{A}{1-\bar{h}} \end{array}\right] \begin{bmatrix} c_t - \bar{c} \\ h_t - \bar{h} \end{bmatrix}$$

part. We need to find a way to include these two components in the quadratic version of the objective function to be able to solve our model by linear dynamic programming techniques.

7.2 THE METHOD OF KYDLAND AND PRESCOTT

Kydland and Prescott [52] use a second-order Taylor series approximation as the discounted quadratic objective function for their model. To illustrate the technique they applied to include the constant and linear parts of the Taylor series, we use a Hansen model like the one discussed in Chapter 6.[1]

A general version of the problem is to maximize

$$\sum_{t=0}^{\infty} \beta^t F(x_t, y_t),$$

subject to the budget constraint

$$x_{t+1} = G(x_t, y_t) = Ax_t + By_t,$$

where x_t are the period t state variables and y_t are the period t control variables. Here we assume that the budget constraints, $G(x_t, y_t)$, are linear or have been made linear by putting all the nonlinear budget constraints into the objective function, $F(x_t, y_t)$.

The second-order Taylor expansion of the function $F(x_t, y_t)$ is

$$F(x_t, y_t) \approx F(\bar{x}, \bar{y}) + \begin{bmatrix} F_x(\bar{x}, \bar{y})' & F_y(\bar{x}, \bar{y})' \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}$$
$$+ \begin{bmatrix} (x_t - \bar{x})' & (y_t - \bar{y})' \end{bmatrix} \begin{bmatrix} \frac{F_{xx}(\bar{x},\bar{y})}{2} & \frac{F_{xy}(\bar{x},\bar{y})}{2} \\ \frac{F_{yx}(\bar{x},\bar{y})}{2} & \frac{F_{yy}(\bar{x},\bar{y})}{2} \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}.$$

To write this Taylor expansion in quadratic form, we define a vector z_t as

$$z_t = \begin{bmatrix} 1 \\ x_t \\ y_t \end{bmatrix}$$

and its value in the stationary state as

$$\bar{z} = \begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix}.$$

Since the vector x_t is of length k and the vector y_t is of length l, the vector z_t is of length $1 + k + l$. Consider the $(1 + k + l) \times (1 + k + l)$ matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}.$$

1. The actual model that Kydland and Prescott used is substantially more complicated. The techniques that they used are easier to explain in the basic Hansen model.

The matrix m_{11} is 1×1, m_{22} is $k \times k$, m_{33} is $l \times l$, and the rest of the matrices conform to make M square. The product

$$z_t' M z_t = m_{11} + (m_{12} + m_{21}')x_t + (m_{13} + m_{31}')y_t + x_t' m_{22} x_t + x_t'(m_{23} + m_{32}')y_t + y_t' m_{33} y_t. \tag{7.2}$$

Notice that there is a constant term and two linear terms in equation 7.2 as well as the quadratic terms. By defining

$$m_{11} = F(\bar{x}, \bar{y}) - \bar{x}' F_x(\bar{x}, \bar{y}) - \bar{y}' F_y(\bar{x}, \bar{y}) + \frac{\bar{x}' F_{xx}(\bar{x}, \bar{y})\bar{x}}{2} + \bar{x}' F_{xy}(\bar{x}, \bar{y})\bar{y} + \frac{\bar{y}' F_{yy}(\bar{x}, \bar{y})\bar{y}}{2},$$

all of the constant components of the Taylor expansion are included in m_{11}. Defining

$$m_{12} = m_{21}' = \frac{F_x(\bar{x}, \bar{y})' - \bar{x}' F_{xx}(\bar{x}, \bar{y}) - \bar{y}' F_{yx}(\bar{x}, \bar{y})}{2}$$

and

$$m_{13} = m_{31}' = \frac{F_y(\bar{x}, \bar{y})' - \bar{x}' F_{xy}(\bar{x}, \bar{y}) - \bar{y}' F_{yy}(\bar{x}, \bar{y})}{2},$$

all the linear components of the Taylor expansion are included in M and we are making M a symmetric matrix. The quadratic components of the Taylor expansion are found in

$$m_{22} = \frac{F_{xx}(\bar{x}, \bar{y})}{2},$$

$$m_{23} = m_{32}' = \frac{F_{xy}(\bar{x}, \bar{y})}{2},$$

and

$$m_{33} = \frac{F_{yy}(\bar{x}, \bar{y})}{2}.$$

In this way, we have constructed a matrix M that is symmetric and gives the complete quadratic Taylor expansion as $z_t' M z_t$. The quadratic discounted dynamic programming problem to be solved is now

$$\sum_{t=0}^{\infty} \beta^t z_t' M z_t,$$

with $z_t' = [\,1 \quad x_t \quad y_t\,]$, subject to the budget constraints

$$\begin{bmatrix} 1 \\ x_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ x_t \end{bmatrix} + By_t.$$

7.2.1 An Example

A specific example of the problem to be solved is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, h_t),$$

subject to the budget constraint

$$c_t = f(k_t, h_t) + (1-\delta)k_t - k_{t+1}.$$

The budget constraint is not linear, given that the production function used is a Cobb-Douglas, so it can be substituted into the utility function to get a maximization problem of

$$\sum_{t=0}^{\infty} \beta^t u(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t),$$

where, in a deterministic version of the problem, k_t is the state variable and h_t and k_{t+1} are the controls. The linear budget constraint is simply

$$k_{t+1} = k_{t+1},$$

where the right-hand k_{t+1} is one of the controls in period t and the left-hand k_{t+1} is the state in period $t+1$.

The exact function $u(c_t, h_t)$ is

$$u(c_t, h_t) = \ln c_t + A \ln(1 - h_t),$$

or, after substituting in the budget constraint, is

$$u(k_t, k_{t+1}, h_t) = \ln\left(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}\right) + A \ln(1 - h_t),$$

where k_t is the period t state variable and k_{t+1} and h_t are the control variables. The production function, $f(k_t, h_t)$, is Cobb-Douglas, so

$$f(k_t, h_t) = k_t^{\theta} h_t^{1-\theta}.$$

The second-order Taylor expansion of the objective function is

$$
\begin{aligned}
u(k_t, k_{t+1}, h_t) \approx{} & \ln\left(f(\bar{k}, \bar{h}) - \delta\bar{k}\right) + A\ln(1-\bar{h}) \\
& + \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right]\left(k_t - \bar{k}\right) - \frac{1}{\bar{c}}\left(k_{t+1} - \bar{k}\right) \\
& + \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right]\left(h_t - \bar{h}\right) \\
& + \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(h_t - \bar{h}\right) \end{bmatrix}' \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(h_t - \bar{h}\right) \end{bmatrix},
\end{aligned}
$$

where the constant elements of the matrix are

$$a_{11} = -\frac{1}{2\bar{c}^2}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right]^2 - \frac{1}{2\bar{c}}\theta(1-\theta)\frac{\bar{y}}{\bar{k}^2},$$

$$a_{12} = a_{21} = \frac{1}{2\bar{c}^2}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right],$$

$$a_{13} = a_{31} = -\frac{1}{2\bar{c}^2}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right](1-\theta)\frac{\bar{y}}{\bar{h}} + \frac{1}{2\bar{c}}\theta(1-\theta)\frac{\bar{y}}{\bar{k}\bar{h}},$$

$$a_{22} = -\frac{1}{2\bar{c}^2},$$

$$a_{23} = a_{32} = \frac{1}{2\bar{c}^2}(1-\theta)\frac{\bar{y}}{\bar{h}},$$

and

$$a_{33} = -\frac{1}{2\bar{c}^2}\left[(1-\theta)\frac{\bar{y}}{\bar{h}}\right]^2 - \frac{1}{2\bar{c}}\theta(1-\theta)\frac{\bar{y}}{\bar{h}^2} - \frac{A}{2\left(1-\bar{h}\right)^2}.$$

To find the matrix M, we first define the four-element vector z_t as $z_t = [\,1 \quad k_t \quad k_{t+1} \quad h_t\,]'$. The 4×4 matrix M is

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & a_{11} & a_{12} & a_{13} \\ m_{31} & a_{21} & a_{22} & a_{23} \\ m_{41} & a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where the a_{ij}'s are defined as above and

$$m_{11} = \ln\left(f(\bar{k},\bar{h}) - \delta\bar{k}\right) + A\ln(1-\bar{h})$$
$$-\frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta) - 1\right]\bar{k} - \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right]\bar{h}$$
$$+\begin{bmatrix}\bar{k}\\ \bar{k}\\ \bar{h}\end{bmatrix}'\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\bar{k}\\ \bar{k}\\ \bar{h}\end{bmatrix},$$

$$m_{12} = m_{21} = \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right] - [\,\bar{k} \quad \bar{k} \quad \bar{h}\,]\begin{bmatrix}a_{11}\\ a_{21}\\ a_{31}\end{bmatrix},$$

$$m_{13} = m_{31} = -\frac{1}{\bar{c}} - [\,\bar{k} \quad \bar{k} \quad \bar{h}\,]\begin{bmatrix}a_{12}\\ a_{22}\\ a_{32}\end{bmatrix},$$

and

$$m_{14} = m_{41} = \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right] - [\,\bar{k} \quad \bar{k} \quad \bar{h}\,]\begin{bmatrix}a_{13}\\ a_{23}\\ a_{33}\end{bmatrix}.$$

The problem to be solved is to maximize

$$\sum_{t=0}^{\infty}\beta^t z_t' M z_t,$$

subject to the budget constraint

$$\begin{bmatrix}1\\ k_{t+1}\end{bmatrix} = A\begin{bmatrix}1\\ k_t\end{bmatrix} + B\begin{bmatrix}k_{t+1}\\ h_t\end{bmatrix},$$

where for this particular problem

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \text{ and } B = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

Notice that the first rows of A and B and the inclusion of 1 in the first row of x_t means that the first row of x_{t+1} will also be 1. Before we solve this particular problem, we show how to solve the general discounted linear quadratic dynamic programming problem.

7.2.2 Solving the Bellman Equation

Define the vector

$$z_t \equiv \begin{bmatrix} x_t \\ y_t \end{bmatrix}.$$

To keep things consistent with the development up to now, let the first element of x_t be the constant 1. Consider the more general discounted optimization problem in which one wants to maximize

$$\sum_{t=0}^{\infty} \beta^t z_t' M z_t,$$

subject to the linear budget constraint

$$x_{t+1} = Ax_t + By_t.$$

In keeping with the definition of the first element of x_t as equal to 1, the first row of A is made up of a 1 in the first position and zeros in all the others and the first row of B is all zeros. Written this way, the matrix multiplication guarantees that the first element of x_{t+1} will always be equal to 1.

The objective function is of the form

$$z_t' M z_t = [\, x_t' \quad y_t' \,] \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

where x_t is a $1 \times n$ vector, y_t is a $1 \times m$ vector, and z_t is therefore a $1 \times (n+m)$ vector. The matrix R is $n \times n$, Q is $m \times m$, and W is $m \times n$. R and Q are positive, symmetric, and semidefinite. Since $x_t' W' y_t = y_t' W x_t$,[2] this objective function can be written as

$$x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t. \tag{7.3}$$

What we are looking for is a value function for the model that can be expressed by a matrix P (positive, symmetric, and semidefinite), where $x_t' P x_t$ is the value of the solved discounted optimization problem given the state x_t. If such a matrix exists, the Bellman equation will be

$$x_t' P x_t = \max_{y_t} \left[z_t' M z_t + \beta x_{t+1}' P x_{t+1} \right],$$

2. A rule of transposition of matrices is that $(ABC)' = C'B'A'$. The products here result in scalars and the transpose of a scalar is equal to itself.

subject to the linear budget constraints

$$x_{t+1} = Ax_t + By_t.$$

Using the version of the objective function in equation 7.3 and substituting in the budget constraint for x_{t+1}, the Bellman equation can be written as

$$x_t' P x_t = \max_{y_t} \left[x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t + \beta \left(A x_t + B y_t \right)' P \left(A x_t + B y_t \right) \right].$$

The first-order conditions[3] for the problem are

$$\left[Q + \beta B' P B \right] y_t = - \left[W + \beta B' P A \right] x_t,$$

which gives the policy function (matrix), F, where

$$y_t = F x_t = - \left[Q + \beta B' P B \right]^{-1} \left[W + \beta B' P A \right] x_t. \tag{7.4}$$

P is still undefined. Substitute this policy function into the Bellman equation in place of y_t and, after a fair amount of matrix algebra, one arrives at the expression

$$P = R + \beta A' P A - \left(\beta A' P B + W' \right) \left[Q + \beta B' P B \right]^{-1} \left(\beta B' P A + W \right)$$

The matrix P can be found from an initial guess for P, for example, P_0 equals the identity matrix, and iterating on the matrix Ricotti equation,

$$P_{k+1} = R + \beta A' P_k A - \left(\beta A' P_k B + W' \right) \left[Q + \beta B' P_k B \right]^{-1} \left(\beta B' P_k A + W \right). \tag{7.5}$$

The sequence of $\left\{ P_k \right\}$, $k \to \infty$, converges to the desired P. Once P is approximated, the policy function, F, is found using equation 7.4. The matrix F gives a linear approximation of the optimal plan in the neighborhood of the stationary state.

7.2.3 Calibrating the Example Economy

As above for the Hansen economy, we use an economy with $\beta = .99$, $\delta = .025$, $\theta = .36$, and $A = 1.72$. The stationary state that is associated with these parameters has $\bar{h} = .3335$, $\bar{k} = 12.6695$, $\bar{y} = 1.2353$, and $\bar{c} = .9186$. Using these

3. To get the first-order conditions, we used the rules for matrix differentiation given in Ljungqvist and Sargent [54], p. 71. These definitions are that $\frac{\partial x' A x}{\partial x} = \left(A + A' \right) x$, $\frac{\partial y' B x}{\partial y} = B x$, and $\frac{\partial y' B x}{\partial x} = B' y$.

values, the matrix a is given by

$$a = \begin{bmatrix} -0.6056 & 0.5986 & -1.3823 \\ 0.5986 & -0.5926 & 1.4048 \\ -1.3823 & 1.4048 & -6.6590 \end{bmatrix},$$

and the matrix M is therefore

$$M = \begin{bmatrix} -1.6374 & 1.0996 & -1.0886 & 1.9361 \\ 1.0996 & -0.6056 & 0.5986 & -1.3823 \\ -1.0886 & 0.5986 & -0.5926 & 1.4048 \\ 1.9361 & -1.3823 & 1.4048 & -6.6590 \end{bmatrix}.$$

The matrices R, Q, and W come from the matrix M, where $M = \begin{bmatrix} R & W' \\ W & Q \end{bmatrix}$, so

$$R = \begin{bmatrix} -1.6374 & 1.0996 \\ 1.0996 & -0.6056 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.5926 & 1.4048 \\ 1.4048 & -6.6590 \end{bmatrix},$$

and

$$W = \begin{bmatrix} -1.0886 & 0.5986 \\ 1.9361 & -1.3823 \end{bmatrix}.$$

We begin iterating in equation 7.5 with the identity matrix,

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and get

$$P_1 = \begin{bmatrix} -.7515 & .9987 \\ .9987 & -0.4545 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} -1.6909 & .8247 \\ .8247 & -0.1924 \end{bmatrix}.$$

After 200 iterations, the values in P have settled down to

$$P = \begin{bmatrix} -96.3615 & .8779 \\ .8779 & -0.0259 \end{bmatrix}.$$

The policy function that is found using this P is

$$F = \begin{bmatrix} 0.5869 & 0.9537 \\ 0.4146 & -0.0064 \end{bmatrix}.$$

Recall that the control variables are $y_t = [\, k_{t+1} \quad h_t \,]'$ and the state variables are $x_t = [\, 1 \quad k_t \,]'$. Checking to see that this linear version of the model has been calculated correctly, we put in the value of the stationary state for k_t and use

$$\begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} = F \begin{bmatrix} 1 \\ k_t \end{bmatrix},$$

or

$$\begin{bmatrix} 12.6695 \\ .3335 \end{bmatrix} = \begin{bmatrix} 0.5869 & 0.9537 \\ 0.4146 & -0.0064 \end{bmatrix} \begin{bmatrix} 1 \\ 12.6695 \end{bmatrix}.$$

The values in the policy matrix, F, indicate that with amounts of time t capital above the stationary state, time $t+1$ capital will be above the stationary state value and labor supply will be below the stationary state value.

In this section, we found a solution to a quadratic version of the simple Hansen model. To do this, we put the nonlinearity into the objective function and used very simple budget constraints, the same budget constraints that we used in earlier chapters to get simple Euler equations. This choice made the calculation of the derivatives of the objective function and the M matrix fairly burdensome. Other choices for the linear form of budget constraints and of the objective function for the same model are possible, and we will be using a different separation of objective function and budget constraint in the next section.

7.3 ADDING STOCHASTIC SHOCKS

The easiest way to solve models with stochastic shocks is to put all of the stochastic parts in the linear budget constraints. Since we are limiting ourselves to linear budget constraints, the shocks appear as

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1},$$

where ε_t is an independent and identically distributed random variable with $E_t(\varepsilon_{t+1}) = \vec{0}$,[4] and with a finite, diagonal variance matrix, Σ. C is a matrix that is $m \times n$, where m is the number of state variables and n is the length

4. Here, $\vec{0}$ is a vector of zeros the same dimension as ε_{t+1}.

of the vector of shocks, ε_{t+1}. As before, x_t is the time t state variable and y_t are the control variables. All of the nonlinearity of the model is put into the objective function, $F(x_t, y_t)$. Agents maximize the discounted expected value of the objective function,

$$\max_{y_t} E_0 \sum_{t=0}^{\infty} \beta^t F(x_t, y_t).$$

The expectations operator is included because future values of the objective function will depend on the realizations of the random shocks. Although the shocks will matter for future realizations of the objective function, the problem is written so that the stochastic variables do not appear directly in the objective function.

One first finds a quadratic approximation of the objective function so that in a neighborhood of the stationary state, $(\bar{x}, \bar{y})$,

$$F(x_t, y_t) = F(z_t) \approx z_t' M z_t = [\,1 \quad x_t'\,]\, R \begin{bmatrix} 1 \\ x_t \end{bmatrix} + y_t' Q y_t + 2 y_t' W \begin{bmatrix} 1 \\ x_t \end{bmatrix},$$

where $z_t = [\,1 \quad x_t \quad y_t\,]'$ and M is found from the second-order Taylor approximation of the function F. The model to be solved is

$$E_0 \sum_{t=0}^{\infty} \beta^t z_t' M z_t,$$

subject to the linear budget constraints

$$x_{t+1} = A x_t + B y_t + C \varepsilon_{t+1}.$$

The problem can be written in a recursive form by first defining a value matrix P so that $x_t' P x_t + c$ is the value function, where

$$x_t' P x_t + c = \max_{\{y_s\}_{s=t}^{\infty}} E_0 \sum_{s=t}^{\infty} \beta^{s-t} z_s' M z_s,$$

subject to the budget constraint. This value function is different from earlier versions in that it includes a potential constant term, c. The recursive form (the Bellman equation) is

$$x_t' P x_t + c = \max_{y_t} \left\{ z_t' M z_t + \beta E_0 \left[x_{t+1}' P x_{t+1} + c \right] \right\},$$

subject to

$$x_{t+1} = A x_t + B y_t + C \varepsilon_{t+1}.$$

Substituting the right-hand side of the budget constraint into the expectations part of the Bellman equation gives

$$x_t'Px_t + c = \max_{y_t} \left[z_t'Mz_t + \beta x_t'A'PAx_t + 2x_t'A'PBy_t + \beta y_t'B'PBy_t \right.$$
$$\left. + \beta E_0 \left[\varepsilon_{t+1}'C'PC\varepsilon_{t+1}\right] + \beta c\right]. \tag{7.6}$$

Define the square matrix $G = \left[g_{jk}\right] = C'PC$. Then

$$E_t \left[\varepsilon_{t+1}'C'PC\varepsilon_{t+1}\right] = \sum_j \sum_k E_t \left[\varepsilon_{t+1}^j g_{jk} \varepsilon_{t+1}^k\right] = \sum_j g_{jj} E_t \left[\varepsilon_{t+1}^j \varepsilon_{t+1}^j\right].$$

The last equality holds because $E_t \left[\varepsilon_{t+1}^k \varepsilon_{t+1}^j\right] = 0$, when $k \neq j$. Therefore,

$$E_t \left[\varepsilon_{t+1}'C'PC\varepsilon_{t+1}\right] = \text{trace}\left[C'PC\Sigma\right],$$

which is a constant.[5] Putting this into equation 7.6 gives

$$x_t'Px_t + c = \max_{y_t} \left\{z_t'Mz_t + \beta x_t'A'PAx_t \right.$$
$$\left. + 2x_t'A'PBy_t + \beta y_t'B'PBy_t + \beta\text{trace}\left[C'PC\Sigma\right]\right\} + \beta c,$$

so

$$c = \frac{\beta\text{trace}\left[C'PC\Sigma\right]}{1-\beta}$$

and

$$x_t'Px_t = \max_{y_t} \left[z_t'Mz_t + \beta x_t'A'PAx_t + \beta y_t'B'PBy_t\right]$$
$$= \max_{y_t} \left[x_t'Rx_t + y_t'Qy_t + 2y_t'Wx_t + \beta x_t'A'PAx_t \right.$$
$$\left. + 2x_t'A'PBy_t + \beta y_t'B'PBy_t\right]. \tag{7.7}$$

The first-order conditions of this problem are

$$\left[Q + \beta B'PB\right] y_t = -\left[W + \beta B'PA\right] x_t,$$

which are exactly the same as in the nonstochastic problem. The policy function, F, where

$$y_t = Fx_t = -\left[Q + \beta B'PB\right]^{-1} \left[W + \beta B'PA\right] x_t,$$

5. The trace(A) of a square matrix A is the sum of its diagonal elements.

is also exactly the same as in the nonstochastic problem.

The matrix $P = \lim P_k$ as $k \to \infty$ is found by making an initial guess for P_0 and iterating on the matrix Ricotti equation,

$$P_{k+1} = R + \beta A' P_k A - \left(\beta A' P_k B + W'\right) \left[Q + \beta B' P_k B\right]^{-1} \left(\beta B' P_k A + W\right).$$

This Ricotti equation is found by substituting the policy function into equation 7.7 and is the same equation as in the nonstochastic case.

To simulate the time path for a stochastic economy of the kind we have been studying in this section, one begins with an initial value for the state variables x_0, and uses the policy function, F, to find the appropriate values for the control variables, y_0. Next, a set of observations of the random variable, ε_1, is taken from the appropriate distribution. The values of the state variables in period 1 are found from

$$x_1 = A x_0 + B y_0 + C \varepsilon_1.$$

The calculation is repeated to generate the time series. This process is equal to finding the process $\{x_t\}$ given a sequence of $\{\varepsilon_{t+1}\}$ using the laws of motion,

$$x_{t+1} = [A + BF] x_t + C \varepsilon_{t+1}.$$

7.3.1 The Example Economy

For the example economy of Chapter 6, the random process for technology is

$$\lambda_{t+1} = \gamma \lambda_t + \widehat{\varepsilon}_{t+1},$$

where $\widehat{\varepsilon}_{t+1}$ has a mean of $1 - \gamma$, and the production function is

$$y_t = \lambda_t k_t^{\theta} h_t^{1-\theta}.$$

Here we let k_t and λ_t be the state variables and k_{t+1} and h_t be the controls for the basic Hansen model. We would like the error, ε_{t+1}, to have a mean of zero, so we add a constant to the random process and can write it as

$$\lambda_{t+1} = (1 - \gamma) + \gamma \lambda_t + \varepsilon_{t+1},$$

where ε_{t+1} has a mean of zero and finite variance.

Agents maximize

$$u(k_t, \lambda_t, k_{t+1}, h_t) = \sum_{t=0}^{\infty} \beta^t \left[\ln(\lambda_t k_t^{\theta} h_t^{1-\theta} + (1-\delta) k_t - k_{t+1}) + A\left(1 - h_t\right)\right],$$

subject to the budget constraints

$$k_{t+1} = k_{t+1}$$

and

$$\lambda_{t+1} = (1-\gamma) + \gamma\lambda_t + \varepsilon_{t+1}.$$

Define $z_t = [\,x_t' \quad y_t'\,]'$, where the state variables are $x_t = [\,1 \quad k_t \quad \lambda_t\,]'$ and the control variables are $y_t = [\,k_{t+1} \quad h_t\,]'$. The "1" is in the first row of x_t for two reasons. One is to allow a constant term in the matrix version of the budget constraint for λ_{t+1}, and the other is, as described above, to include the constant and linear terms of the Taylor approximation of the objective function. Using these definitions, the budget constraints are written as

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1},$$

or as

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\gamma & 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}.$$

The second-order Taylor series expansion of the objective function is

$$\begin{aligned} u(\cdot) \approx{} & \ln\left(\bar{\lambda}\bar{k}^{\theta}\bar{h}^{1-\theta} - \delta\bar{k}\right) + A\ln(1-\bar{h}) \\ & + \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right]\left(k_t - \bar{k}\right) \\ & + \frac{\bar{y}}{\bar{c}}\left(\lambda_t - \bar{\lambda}\right) - \frac{1}{\bar{c}}\left(k_{t+1} - \bar{k}\right) \\ & + \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right]\left(h_t - \bar{h}\right) \\ & + \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(\lambda_t - \bar{\lambda}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(h_t - \bar{h}\right) \end{bmatrix}' \begin{bmatrix} a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(\lambda_t - \bar{\lambda}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(h_t - \bar{h}\right) \end{bmatrix}, \end{aligned}$$

where the elements of the matrix, a_{ij}, are the same as in the previous example from section 7.2.1 (and their subindices are the same as in that example) and the elements $\widehat{a}_{ij}$, where either i or $j = \lambda$ as part of the subindex, are the new ones related to the new variable, λ_t. The difference between this Taylor expansion and the previous one is the addition of the $(\lambda_t - \bar{\lambda})$ terms. The "$\widehat{a}_{ij}$" terms are

$$\widehat{a}_{1\lambda} = \widehat{a}_{\lambda 1} = -\frac{\bar{y}}{\bar{c}^2}\left[\theta \frac{\bar{y}}{\bar{k}} + (1-\delta)\right] + \theta \frac{1}{\bar{c}}\frac{\bar{y}}{\bar{k}},$$

$$\widehat{a}_{\lambda\lambda} = -\frac{\bar{y}^2}{\bar{c}^2},$$

$$\widehat{a}_{2\lambda} = \widehat{a}_{\lambda 2} = \frac{\bar{y}}{\bar{c}^2},$$

and

$$\widehat{a}_{3\lambda} = \widehat{a}_{\lambda 3} = -(1-\theta)\frac{1}{\bar{c}^2}\frac{\bar{y}^2}{\bar{h}} + (1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}}.$$

The matrix quadratic version of the objective function is $z_t' M z_t$. The dynamic programming problem to be solved is

$$\max_{\{y_t\}} \sum_{t=0}^{\infty} z_t' M z_t,$$

subject to the budget constraints

$$x_{t+1} = A x_t + B y_t + C \varepsilon_{t+1}.$$

The 5×5 matrix M in the quadratic version of the objective function is

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ m_{31} & \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ m_{41} & a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ m_{51} & a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix},$$

where the a_{ij} and $\widehat{a}_{ij}$ elements are defined as above and

$$m_{11} = \ln\left(f(\bar{k}, \bar{h}) - \delta \bar{k}\right) + A \ln(1 - \bar{h})$$
$$- \frac{1}{\bar{c}}\left[\theta \frac{\bar{y}}{\bar{k}} + (1-\delta) - 1\right]\bar{k} - \frac{\bar{y}}{\bar{c}}\bar{\lambda}$$

$$
- \left[(1-\theta)\, \frac{1}{\bar{c}} \frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}} \right] \bar{h}
$$

$$
+ \begin{bmatrix} \bar{k} \\ \bar{\lambda} \\ \bar{k} \\ \bar{h} \end{bmatrix}' \begin{bmatrix} a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ a_{21} & \widehat{a}_{2\lambda} & a_{32} & a_{32} \\ a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \bar{k} \\ \bar{\lambda} \\ \bar{k} \\ \bar{h} \end{bmatrix},
$$

$$
m_{12} = m_{21} = \frac{1}{\bar{c}} \left[\theta \frac{\bar{y}}{\bar{k}} + (1-\delta) \right] - [\, \bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h} \,] \begin{bmatrix} a_{11} \\ \widehat{a}_{\lambda 1} \\ a_{21} \\ a_{31} \end{bmatrix},
$$

$$
m_{13} = m_{31} = \frac{\bar{y}}{\bar{c}} - [\, \bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h} \,] \begin{bmatrix} \widehat{a}_{1\lambda} \\ \widehat{a}_{\lambda\lambda} \\ \widehat{a}_{2\lambda} \\ \widehat{a}_{3\lambda} \end{bmatrix},
$$

$$
m_{14} = m_{41} = -\frac{1}{\bar{c}} - [\, \bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h} \,] \begin{bmatrix} a_{12} \\ \widehat{a}_{\lambda 2} \\ a_{22} \\ a_{32} \end{bmatrix},
$$

and

$$
m_{15} = m_{51} = \left[(1-\theta)\, \frac{1}{\bar{c}} \frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}} \right] - [\, \bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h} \,] \begin{bmatrix} a_{13} \\ \widehat{a}_{\lambda 3} \\ a_{23} \\ a_{33} \end{bmatrix}.
$$

7.3.2 Calibrating the Example Economy

We calibrate this economy with the same values for the parameters as was used in section 7.2.3, adding the parameter $\gamma = .95$. The stationary states are the same as in that section, since in a stationary state for this economy the shocks are zero and the stationary state value for the technology parameter is $\bar{\lambda} = 1$.

We iterate using the equation

$$
P_{k+1} = R + \beta A' P_k A - \left(\beta A' P_k B + W'\right) \left[Q + \beta B' P_k B\right]^{-1} \left(\beta B' P_k A + W\right),
$$

to find the matrix P as

$$P = \begin{bmatrix} -124.0532 & 1.0657 & 15.6762 \\ 1.0657 & -0.0259 & -0.1878 \\ 15.6762 & -0.1878 & -1.9963 \end{bmatrix}$$

and then use

$$y_t = Fx_t = -\left[Q + \beta B'PB\right]^{-1}\left[W + \beta B'PA\right]x_t,$$

to find the policy function F,

$$F = \begin{bmatrix} -0.8470 & 0.9537 & 1.4340 \\ 0.1789 & -0.0064 & 0.2357 \end{bmatrix}.$$

Combining this policy function with the budget constraints, one gets

$$\begin{aligned} \begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ .05 & 0 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8470 & 0.9537 & 1.4340 \\ 0.1789 & -0.0064 & 0.2357 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}, \end{aligned}$$

or, a simple form for the laws of motion is

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.8470 & 0.9537 & 1.4340 \\ .05 & 0 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}.$$

One can think of this last equation in terms of

$$x_{t+1} = \Psi x_t + C\varepsilon_{t+1},$$

and recursively replacing lagged versions of this equation in the right-hand side gives a moving average representation of x_{t+1} as

$$x_{t+1} = \sum_{i=0}^{\infty} \Psi^i C\varepsilon_{t-i} + \Psi^{\infty} x_{-\infty},$$

where we define Ψ^{∞} as the limit of Ψ^i as $i \to \infty$. It turns out that

$$\Psi^{\infty} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{k} & 0 & 0 \\ \bar{\lambda} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12.6695 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which, when multiplied by any $x_{-\infty}$, is equal to $\bar{x}$. This occurs because the first element of every x_t is the constant 1. Using this equation, one can calculate the covariance matrix of x as

$$\begin{aligned} \text{var}(x) &= E(x_{t+1} - \bar{x})(x_{t+1} - \bar{x})' \\ &= E\left[x_{t+1}x'_{t+1} - x_{t+1}\bar{x}' - \bar{x}x'_{t+1} + \bar{x}\bar{x}'\right] \\ &= E\left[x_{t+1}x'_{t+1}\right] - \bar{x}\bar{x}'. \end{aligned}$$

Substituting in the moving average representation of x_{t+1} gives

$$\begin{aligned} \text{var}(x) &= E\left[\left(\sum_{i=0}^{\infty} \Psi^i C\varepsilon_{t-i}\right)\left(\sum_{i=0}^{\infty} \varepsilon'_{t-i}C'\left(\Psi'\right)^i\right) + \Psi^{\infty}x_{-\infty}x'_{-\infty}\left(\Psi^{\infty}\right)'\right] - \bar{x}\bar{x}' \\ &= \sum_{i=0}^{\infty} \Psi^i C\text{var}(\varepsilon_t)C'\left(\Psi'\right)^i, \end{aligned}$$

where $E\left[\Psi^{\infty}x_{-\infty}x'_{-\infty}\left(\Psi^{\infty}\right)'\right] = \bar{x}\bar{x}'$. Because ε_t is a 1×1 vector, $\text{var}(\varepsilon_t)$ is a scalar. Then the $\text{var}(x)$ can be written as

$$\text{var}(x) = \text{var}(\varepsilon_t)\sum_{i=0}^{\infty} \Psi^i CC'\left(\Psi'\right)^i.$$

Using the Ψ and C matrices as defined above,

$$\text{var}(x) = \text{var}\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \text{var}(\varepsilon_t)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4728.5 & 148.7 \\ 0 & 148.7 & 10.3 \end{bmatrix}.$$

Since $y_t = Fx_t$, one can find the variance of the control variables (in this case one is really only adding the variance of hours worked), as

$$\text{var}(y) = \text{var}\begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} = F\text{var}(x)F' = \begin{bmatrix} 4728.2 & 6.7 \\ 6.7 & 0.3 \end{bmatrix}.$$

Finding the variance for other variables in this model is more difficult, since the relationships are not linear. Output is equal to $y_t = \lambda_t k_t^{\theta} h_t^{1-\theta}$, and even a

Table 7.1 Statistics from simulations

	Output	Consumption	Invest.	Hours	Capital	Tech.
Standard error	1.42%	0.91%	3.60%	0.54%	1.20%	0.88%
Correl. with y	100%	84%	93%	82%	63%	90%

second-order Taylor approximation of the production function results in a quadratic function.

> **EXERCISE 7.1** Find the second-order Taylor expansion of the Cobb-Douglas production function with stochastic technology, $y_t = \lambda_t k_t^{\theta} h_t^{1-\theta}$. Use the values for the stationary state that we have been using in this chapter.

In order to find the variances for the other variables, we run simulations. Hansen reports data from a sample of 115 observations, so we will run 100 simulations of length 115 using the parameters given above. The one parameter that is yet to be defined is $\text{var}(\varepsilon)$. The value we choose for this parameter is .0000105, so that the standard error of the shock is .0032. This value is chosen so that in the section that follows, on the Hansen model with indivisible labor, the standard error of output is 1.76 percent of its average value. Running the simulations (the shocks come from a normal distribution), the average standard errors of output, consumption, investment, labor hours, capital, and the technology process are given in Table 7.1. Standard errors given are as a percentage of the stationary state value.

7.4 HANSEN WITH INDIVISIBLE LABOR

In his paper, Hansen used the techniques we are describing in this chapter to solve both the basic model and the one with indivisible labor. The model with indivisible labor assumes that an individual who is working provides a predetermined amount of labor, h_0, in a period in which he/she works but only an α_t fraction of the population is working in each period. The process of deciding who works and who does not is perfectly random so that each person has an expected utility function of

$$E \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \alpha_t A \ln(1 - h_0) \right].$$

The total amount of labor that is provided to the firms in period t is $\alpha_t h_0$, and given the amount of capital carried over from the previous period, k_t, production in period t is

$$y_t = \lambda_t f(k_t, \alpha_t h_0) = \lambda_t k_t^\theta \left(\alpha_t h_0\right)^{1-\theta} .$$

The budget constraints in period t are

$$c_t = y_t + (1-\delta)k_t - k_{t+1}$$

and

$$\lambda_{t+1} = (1-\gamma) + \gamma\lambda_t + \varepsilon_{t+1}.$$

The model assumes that there is a mutual insurance program that guarantees every individual the same consumption whether he/she is one of those who must work or not. The only change between this model and the basic model is in the part of the subutility function that is associated with the disutility of labor. Here, the expected disutility of work is what matters, and changes in the amount of work provided shows up in the utility function in a linear form (caused by changes in the variable a_t while $A\ln(1-h_0)$ is a constant). In the basic version of this model, changes in the amount of labor provided by an individual change utility as the logarithm of the change in labor.

Since we want to use the linear quadratic dynamic programming that we developed above to solve this model, we need to put all the nonlinearity into the objective function and have linear budget constraints. We choose $x_t = [\,1 \quad k_t \quad \lambda_t\,]'$ as the state variables (the constant, 1, will serve the same purpose as it did above; it will help with the inclusion of the constant and linear terms of the second-order Taylor expansion) and $y_t = [\,k_{t+1} \quad \alpha_t\,]'$ as the control variables. The objective function we use is

$$E\sum_{t=0}^{\infty}\beta^t\left[\ln(\lambda_t k_t^\theta \left(\alpha_t h_0\right)^{1-\theta} + (1-\delta)k_t - k_{t+1}) + \alpha_t A\ln(1-h_0)\right],$$

subject to the budget constraints

$$k_{t+1} = k_{t+1},$$
$$\lambda_{t+1} = (1-\gamma) + \gamma\lambda_t + \varepsilon_{t+1}.$$

The second-order Taylor expansion of the objective function is

$$
\begin{aligned}
u(\cdot) \approx & \ln\left(\bar{\lambda}\bar{k}^{\theta}\left(\bar{\alpha}h_0\right)^{1-\theta} - \delta\bar{k}\right) + \bar{\alpha}A\ln(1-h_0) \\
& + \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right]\left(k_t - \bar{k}\right) \\
& + \frac{\bar{y}}{\bar{c}}\left(\lambda_t - \bar{\lambda}\right) - \frac{1}{\bar{c}}\left(k_{t+1} - \bar{k}\right) \\
& + \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{\alpha}} + A\ln(1-h_0)\right]\left(\alpha_t - \bar{\alpha}\right) \\
& + \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(\lambda_t - \bar{\lambda}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(\alpha_t - \bar{\alpha}\right) \end{bmatrix}' \begin{bmatrix} a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ a_{21} & \widehat{a}_{2\lambda} & a_{32} & a_{32} \\ a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(\lambda_t - \bar{\lambda}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(\alpha_t - \bar{\alpha}\right) \end{bmatrix},
\end{aligned}
$$

where all parameters are the same as in the basic version of the model, except $\alpha = 0.5721$, $h_0 = 5.83$, and these elements of the a matrix are changed:

$$
a_{13} = a_{31} = -\frac{1}{\bar{c}^2}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right]\left[(1-\theta)\frac{\bar{y}}{\bar{\alpha}}\right] + \frac{1}{\bar{c}}\theta\,(1-\theta)\,\frac{\bar{y}}{\bar{k}\bar{\alpha}},
$$

$$
\widehat{a}_{\lambda 3} = \widehat{a}_{3\lambda} = -\frac{\bar{y}}{\bar{c}^2}\left[(1-\theta)\frac{\bar{y}}{\bar{\alpha}}\right] + (1-\theta)\,\frac{\bar{y}}{\bar{c}\bar{\alpha}},
$$

$$
a_{32} = a_{23} = \frac{1}{\bar{c}^2}\left[(1-\theta)\frac{\bar{y}}{\bar{\alpha}}\right],
$$

and

$$
a_{33} = -\frac{1}{\bar{c}^2}\left[(1-\theta)\frac{\bar{y}}{\bar{\alpha}}\right]^2 - \frac{1}{\bar{c}}\theta\,(1-\theta)\,\frac{\bar{y}}{\bar{\alpha}^2}.
$$

Using the technique of Kydland and Prescott, we define the matrix M for the quadratic approximation of the objective function, $z_t' M z_t$, where $z_t = [\,x_t \quad y_t\,]$, as

$$
M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ m_{31} & \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ m_{41} & a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ m_{51} & a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix},
$$

where

$$
\begin{aligned}
m_{11} = {} & \ln\left(f(\bar{k}, \bar{h}) - \delta\bar{k}\right) + \bar{\alpha} A \ln(1 - h_0) \\
& - \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta) - 1\right]\bar{k} - \frac{\bar{y}}{\bar{c}}\bar{\lambda} \\
& - \left[(1-\theta)\,\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{\alpha}} + A\left(1 - h_0\right)\right]\bar{\alpha} \\
& + \begin{bmatrix} \bar{k} \\ \bar{\lambda} \\ \bar{k} \\ \bar{\alpha} \end{bmatrix}' \begin{bmatrix} a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \bar{k} \\ \bar{\lambda} \\ \bar{k} \\ \bar{\alpha} \end{bmatrix},
\end{aligned}
$$

$$
m_{12} = m_{21} = \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right] - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{\alpha}\,] \begin{bmatrix} a_{11} \\ \widehat{a}_{\lambda 1} \\ a_{21} \\ a_{31} \end{bmatrix},
$$

$$
m_{13} = m_{31} = \frac{\bar{y}}{\bar{c}} - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{\alpha}\,] \begin{bmatrix} \widehat{a}_{1\lambda} \\ \widehat{a}_{\lambda\lambda} \\ \widehat{a}_{2\lambda} \\ \widehat{a}_{3\lambda} \end{bmatrix},
$$

$$
m_{14} = m_{41} = -\frac{1}{\bar{c}} - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{\alpha}\,] \begin{bmatrix} a_{12} \\ \widehat{a}_{\lambda 2} \\ a_{22} \\ a_{32} \end{bmatrix},
$$

and

$$
m_{15} = m_{51} = \left[(1-\theta)\,\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{\alpha}} + A\left(1 - h_0\right)\right] - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h}\,] \begin{bmatrix} a_{13} \\ \widehat{a}_{\lambda 3} \\ a_{23} \\ a_{33} \end{bmatrix}.
$$

Given this definition of M, one wants to choose a sequence of $\{y_t\}$ to maximize

$$
E_0 \sum_{t=0}^{\infty} \beta^t z_t' M z_t = E_0 \sum_{t=0}^{\infty} \beta^t \,[\,x_t' \quad y_t'\,] \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},
$$

subject to the budget constraints

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} + B \begin{bmatrix} k_{t+1} \\ \alpha_t \end{bmatrix} + C\varepsilon_{t+1},$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\gamma & 0 & \gamma \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We look for a policy function, F, that solves the value function, P, where

$$x_t' P x_t = \max_{k_{t+1}, \alpha_t} \left[z_t' M z_t + \beta E_t \left(A x_t + B y_t + C\varepsilon_t \right)' P \left(A x_t + B y_t + C\varepsilon_t \right) \right].$$

As before, P is found by making an initial guess, P_0, and iterating on

$$P_{k+1} = R + \beta A' P_k A - \left(\beta A' P_k B + W' \right) \left[Q + \beta B' P_k B \right]^{-1} \left(\beta B' P_k A + W \right).$$

Once P is known, the policy function is found from

$$y_t = F x_t = -\left[Q + \beta B' P B \right]^{-1} \left[W + \beta B' P A \right] x_t.$$

We want the stationary state to be the same as in the basic example, so we use the same values we found at the end of Chapter 6: $\bar{\alpha} = 572$, and $h_0 = 583$. All other stationary state values are the same as above. Using these values for the stationary state variables, we get

$$P = \begin{bmatrix} -139.5224 & 1.0816 & 14.9763 \\ 1.0816 & -0.0245 & -0.2215 \\ 14.9763 & -0.2215 & -0.8695 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -1.2295 & 0.9418 & 1.9667 \\ 0.0029 & -0.0215 & 0.8418 \end{bmatrix}.$$

We get the laws of motion for this economy by substituting Fx_t for y_t in the budget constraints. This gives

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.2295 & 0.9418 & 1.9667 \\ .05 & 0 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}.$$

We want to compare the results of simulation of this economy with those of the basic model. As with the basic model we used a value for $\text{var}(\varepsilon) = .0000105$, which gives a standard error for the error term of .0032. This standard error results in a standard error for output of 1.76 percent, the same value as Hansen found from his 115 quarterly observations for output from the United States (from third quarter of 1955 to first quarter of 1984). The variance of the error term is calibrated so that the simulations have this standard error for output. The standard errors (and the correlation of the variable with output) for output, consumption, investment, hours worked, capital, and the technology process are given in Table 7.2.

Using the model with indivisible labor improves the results with respect to the data in investment and hours worked and reduces the importance of the technology in explaining movements in output. Table 7.3 shows the standard errors (as a fraction of the standard error of output) in the basic model and in the model with indivisible labor. Note the results for investment, hours worked, and technology.

We can get some insight into why this relatively minor change of adding indivisible labor would result in such a large change in the relative standard error of hours by looking at the two Taylor expansions of the objective function and calculating how much utility would change as a result of a small change in the variable related to hours worked. Define the two vectors, χ_h as $\chi_h = \bar{z}_h + [\,0 \quad 0 \quad 0 \quad 0 \quad .01 * \bar{h}\,]$, and χ_α as $\chi_\alpha = \bar{z}_\alpha + [\,0 \quad 0 \quad 0 \quad 0 \quad .01 * \bar{\alpha}\,]$. The

Table 7.2 Standard errors and correlations from U.S. data

	Output	Consumption	Invest.	Hours	Capital	Tech.
Standard error	1.76%	1.06%	4.69%	1.08%	1.45%	0.88%
As % of Y	100%	60%	266%	61%	82%	50%
Correl with Y	100%	82%	93%	84%	61%	91%

Table 7.3 Standard errors from the models

	Output	Consumption	Invest.	Hours	Capital	Tech.
Basic model	100%	63.91%	253.06%	38.06%	84.45%	62.10%
Indivisible labor	100%	60.31%	265.89%	61.11%	82.28%	50.14%

vector χ_i represents a one percent movement in the hours worked variable from the stationary state. The two stationary states are represented by different vectors (with $\bar{h}$ in the basic model and $\bar{\alpha}$ in the indivisible labor model). The one percent increase in labor supplied in the basic model results in a change in utility of -7.3691×10^{-5} units, which is a decline of .0094613 percent in utility from the stationary state. A one percent increase in labor supplied in the indivisible labor model results in a change of utility of -5.2556×10^{-5} units, which is a decline of .0055559 percent from the stationary state. The incorporation of indivisible labor into the model has made utility less sensitive to change in labor and therefore encourages agents to change the labor supplied (or expected labor supplied) more than they would have in the basic model.

7.5 IMPULSE RESPONSE FUNCTIONS

Using the linear policy function, F, and a vector of technology that incorporates a one-time shock to technology in period 2, $\varepsilon_2 = .01$, and the technology law of motion,

$$\lambda_t = (1-\gamma) + \gamma\lambda_{t-1} + \varepsilon_t,$$

one can find the time path of capital and hours worked (the state variables), and, using the production function and the budget constraints, one can then calculate the path of output and consumption. The solution technique used in this chapter finds the values of the variables in levels, and the resulting impulse response functions are those shown in Figure 7.1. To be able to compare the response functions to those in the previous chapter, one needs to find the log differences around the stationary state values of the variables. Figure 7.2 shows the resulting response functions for the basic Hansen model.

These response functions should look familiar. These are very close to the response functions we found using the log-linear method of Chapter 6. Figure 7.3 shows the response functions above compared to those for the same model using Uhlig's log-linear method. The 45 degree line has a little width to it, so the two solution methods don't give exactly the same impulse response functions, but they are very close. One should expect a little

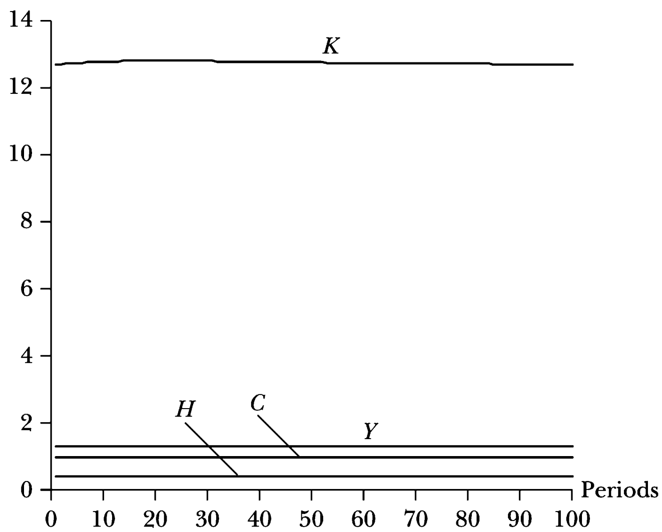

FIGURE 7.1 Impulse response function in levels

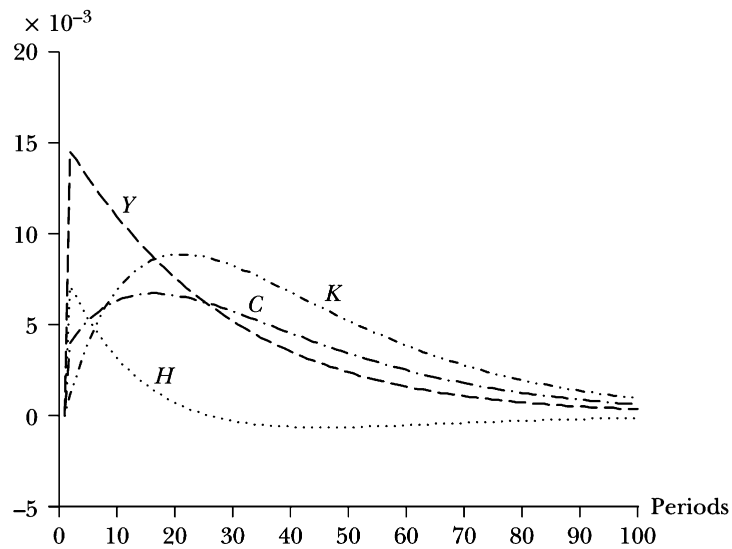

FIGURE 7.2 Responses found using the linear quadratic solution method

difference because of the order in which we took the approximations, but Figure 7.3 shows that, for most purposes, the results are the same.

A graph that is essentially identical to Figure 7.3 is found if one graphs the impulse response functions found using the two different techniques for Hansen's model with indivisible labor. For models as simple as those we have developed here, with logarithmic utility functions and Cobb-Douglas production functions, either of the methods produces results that are, given the uncertainties we have about the parameters of the model, the same.

7.5.1 Vector Autoregressions

Comparing the impulse reponse functions generated by the model to those observed in the data is one way of evaluating how well a model is capturing important characteristics of the economy. While comparing covariances of simulations of the model to those of the data is a form of evaluating models that is frequently used, comparing impulse response functions demands more of a model and is a finer measure of success.

The impulse response functions from the data are usually found by estimating vector autoregressions (VARs). The method is well known. One estimates linear equations of the form

$$y_t = A(L)y_{t-1} + \varepsilon_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_n y_{t-n} + \varepsilon_t,$$

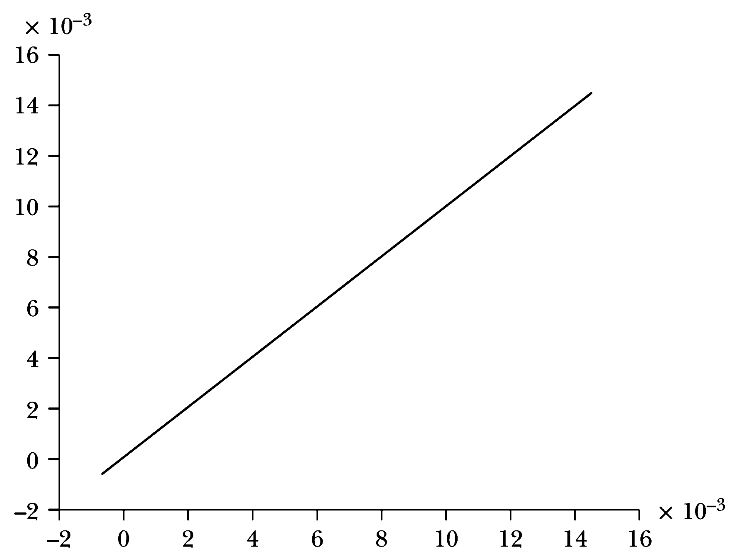

FIGURE 7.3 Comparing the two solution techniques using Hansen's model

where y_t is a vector of variables, $A(L)$ is an nth order matrix polynomial in the lag operator, and ε_t is a vector of error terms that we usually want to have zero mean and a diagonal variance matrix. A lag operator, L, applied to a variable X_t results in

$$LX_t = X_{t-1}.$$

Applied twice gives

$$L^2 X_t = LX_{t-1} = X_{t-2}.$$

The polynomial in the lag operator, $A(L)$, can be written

$$A(L) = A_1 L + A_2 L^2 + \cdots + A_n L^n.$$

In a VAR, current values of each of a set of variables are a function of past values of all the variables in the set. VARs are good for forecasting, since one can find predictions for tomorrow's values for the set of variables using current and past values and then recursively solve for predictions further into the future. In doing these forecasts, the shocks are usually set to zero.

Define $B(L) = 1 - A(L)$. One can rewrite the above equation as

$$B(L) y_t = \varepsilon_t.$$

Let $C(L) = (B(L))^{-1}$ be the inverse of the matrix polynomial $B(L)$. Applying this inverse to both sides of the above equation gives the moving average representation of the vector autoregression,

$$y_t = (B(L))^{-1} B(L) y_t = C(L) B(L) y_t = C(L) \varepsilon_t,$$

or $y_t = C(L)\varepsilon_t$. One can use the matrix polynomial $C(L)$ to find the impulse response functions for the vector autoregressive system. Set all of the error terms to zero except for the one that you wish to study; for example, if we wish to study the results of a shock to the second variable in y_t, we apply

$$\widehat{y}_1 = C(L) \begin{bmatrix} 0 \\ \varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

to get the response of the variables in y_t in the first period. Since all shocks return to zero in periods 2 and beyond, we find the rest of the response function by recursively applying the matrix $A(L)$ to get

$$\widehat{y}_2 = A(L)\widehat{y}_1,$$
$$\widehat{y}_3 = A(L)\widehat{y}_2,$$
$$\vdots$$

Finding the inverse of the $B(L)$ matrix polynomial is not always easy, and the interested reader is encouraged to consult texts in time series such as Hamilton [47] for details.

Unfortunately, for a particular economy, it is sometimes (as is the case here) difficult to identify the shocks from the real world that one is using in a model. What constitute shocks to technology in the real world, how big these shocks are, and what the process of their persistence looks like are all difficult questions. If one cannot find a shock in the economy that approximates well the shock used in the model, then comparing impulse response functions is not possible. If possible, one should try to choose stochastic shocks for the models that have a natural counterpart in the data. For example, some models that include money consider unpredictable changes in short-term interest rates as representing monetary shocks. The residuals of the vector autoregression equation for short-term interest rates provide the required data on this shock.

Sims [77] suggests an alternative method for using VARs to compare impulse response functions. One can find the impulse response functions for the data as described above. Using the model, generate a large number of time series for the simulated economy (using appropriately calibrated variances for the model's shocks). Estimate VARs using this simulated time series and compare them to those generated from the data. Since the times series from the data and those from the simulated economies should have the same variables, the impulse response functions are comparable.

7.6 AN ALTERNATIVE PROCESS FOR TECHNOLOGY

It is sometimes desirable to use a stochastic process that will not go negative. For example, a stochastic process for technology of the form that we have been using,

$$\lambda_{t+1} = (1-\gamma) + \gamma\lambda_t + \varepsilon_{t+1},$$

with $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$, can end up with some values for λ_t that are negative. Since the normal distribution is unbounded below (and above), there is a very small but finite probability that some ε_{t+1} will have a value sufficiently negative so that the associated λ_{t+1} is also negative. There is no economic (nor engineering) sense to a negative level of technology.

A simple way to ensure that technology will not go negative is to use a different stochastic process, one where the λ_{t+1} will never be negative. One stochastic process that has this characteristic, along with the characteristic that the unconditional expectation of λ_{t+1}, $E\lambda_{t+1}$ is equal to one, is

$$\ln\left(\lambda_{t+1}\right) = \gamma \ln\left(\lambda_t\right) + \varepsilon_{t+1},$$

with $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$. With this process, a very large negative realization for ε_{t+1} implies that λ_{t+1} will be close to zero but positive. The entire sequence of $\{\lambda_t\}$ is guaranteed to be positive. One can check that the mean of λ_{t+1} is 1. Take unconditional expectations of both sides of the process, so that

$$E \ln\left(\lambda_{t+1}\right) = \gamma E \ln\left(\lambda_t\right) + E\varepsilon_{t+1} = \gamma E \ln\left(\lambda_t\right) + 0.$$

Simple manipulation gives

$$E \ln\left(\lambda_{t+1}\right) - \gamma E \ln\left(\lambda_t\right) = (1-\gamma)\ln(\lambda) = 0,$$

so, in a stationary state, $E\ln(\lambda) = 0$. This occurs when $\lambda = 1$.

If we would like to use this process in our basic Hansen model, there is a problem: the budget constraint that we have is not linear in technology, so we cannot immediately use the linear quadratic dynamic programming techniques that we have developed in this section. We need to modify the model slightly to be able to continue using these techniques.

Define the variable $\widehat{\lambda}_t = \ln(\lambda_t)$ and the vector of state variables as $x_t = [\,1 \quad k_t \quad \widehat{\lambda}_t\,]'$. Then one can write the budget constraints, the laws of motion, as

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \widehat{\lambda}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \widehat{\lambda}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}.$$

This way, the budget constraints are linear, although the variable in which they are linear is a variable in logs.

One needs to make a change in the objective function to incorporate this stochastic process. In the objective function,

$$u\left(c_t, h_t\right) = \ln(e^{\widehat{\lambda}_t} k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}) + A\ln(1-h_t),$$

we replace λ_t with $e^{\widehat{\lambda}_t}$, since $e^{\ln(\lambda_t)} = \lambda_t$. This has the effect of complicating a bit the second-order Taylor expansion, but it is normally quite manageable.

7.7 REPRISE

Linear quadratic dynamic programming methods provide an alternative to the linear approximation of the model that we presented in Chapter 6. They have their advantages and limitations. Solving linear quadratic problems is relatively simple once they are formulated. The difficulties arise in the need for being careful in taking the many derivatives that are required for the Taylor approximation and in including economy-wide variables in the problem. Most of the earlier papers in Real Business Cycle theory used this method for finding solutions.

A more extensive development of linear quadratic dynamic programming methods, along with numerous applications, can be found in Ljungqvist and Sargent [54]. Another general reference on dynamic programming with applications to linear quadratic problems is Bertsekas [10].

7.8 MATLAB CODE

The following Matlab program finds the solution to the linear quadratic problem for a standard version of the basic Hansen model. Most of the program involves setting up the matrices. The loop at the end of the program finds the value matrix, and this is used to find the policy function.

```
%this program finds the value function (P matrix) for the linear
%quadratic optimal regulator problem and also calculates the
%resulting policy function
theta=.36;
beta=.99;
delta=.025;
A=1.72;
kbar=12.6695;
hbar=.3335;
ybar=kbar^theta*hbar^(1-theta);
cbar=ybar-delta*kbar;
aa=(theta*ybar/kbar+1-delta);
a(1,1)=-1/(2*cbar*cbar)*aa*aa-1/(2*cbar)*theta*(1-theta)*ybar/
    (kbar*kbar);
a(1,2)=1/(2*cbar*cbar)*aa;
a(2,1)=1/(2*cbar*cbar)*aa;
a(1,3)=-1/(2*cbar*cbar)*aa*(1-theta)*ybar/hbar;
a(1,3)=a(1,3)+1/(2*cbar)*theta*(1-theta)*ybar/(kbar*hbar);
a(3,1)=a(1,3);
a(2,2)=-1/(2*cbar*cbar);
a(2,3)=1/(2*cbar*cbar)*(1-theta)*ybar/hbar;
a(3,2)=a(2,3);
a(3,3)=-1/(2*cbar*cbar)*(1-theta)*ybar/hbar*(1-theta)*ybar/hbar;
```

```
a(3,3)=a(3,3)-1/(2*cbar)*theta*(1-theta)*ybar/(hbar*hbar);
a(3,3)=a(3,3)-A/(2*(1-hbar)*(1-hbar));
x=[kbar kbar hbar]';
m(1,1)=log(kbar^theta*hbar^(1-theta)-delta*kbar)+A*log(1-hbar);
mm1=1/cbar*(theta*ybar/kbar+1-delta);
mm2=(1-theta)*ybar/(cbar*hbar)-A/(1-hbar);
m(1,1)=m(1,1)-mm1*kbar+kbar/cbar-mm2*hbar;
m(1,1)=m(1,1)+(x')*a*x;
m(1,2)=mm1/2-1*a(1:3,1)'*x;
m(2,1)=m(1,2);
m(1,3)=-1/(2*cbar)-1*a(1:3,2)'*x;
m(3,1)=m(1,3);
m(1,4)=mm2/2-1*a(1:3,3)'*x;
m(4,1)=m(1,4);
m(2:4,2:4)=a;
AA=[1 0
    0 0];
B=[0 0
   1 0];
R=m(1:2,1:2);
Q=m(3:4,3:4);
W=m(1:2,3:4)';
P=[1 0
   0 1];
% iterating the Ricotti equation
for i=1:1000
   zinv=inv(Q+beta*B'*P*B);
   z2=beta*AA'*P*B+W';
   P=R+beta*AA'*P*AA-z2*zinv*z2'
end
% finding the policy function
F=-zinv*(W+beta*B'*P*AA)
```