8

Money: Cash in Advance

Cooley and Hansen [36] introduce money into Hansen's model [48] with indivisible labor. The model of money they present is one with cash in advance (Clower [35] or Lucas and Stokey [58], for example) but where the use of cash is restricted to the purchase of consumption goods. The purchase of investment goods does not require the use of money held over from the previous period but is restricted by an overall, flow budget constraint. Consumption is purchased using money that is held over from the previous period, but capital can be purchased using this period's income. This model adds two new features to our models: money and economy-wide variables. Some changes in techniques are required to handle models with these new features.

Cash-in-advance models require agents to have carried over from the previous period the money they will use in this period to make purchases or to make a subset of their purchases. The story that is often used to describe a cash-in-advance model has a family with two members, one called the shopper and the other the worker. In the beginning of each period, the two members of the family separate. The shopper takes the money the family has (along with any transfer received from the government) and goes to market. Sometimes there is a sequence of markets, first a market in financial assets and then a market in goods. The producer goes to the family factory, uses capital and her/his labor to produce goods, and sells them to shoppers who are not members of the same family. Families cannot consume the good they produce. At the end of the period, the members of the family rejoin and consume the goods that were purchased for consumption. The worker brings home the money from that period's sales and gives it to the shopper who will use it in the next period, along with any money that remains from this day's shopping.

The requirement that money be used to purchase goods, or at least some goods, is simply imposed. Nothing in the model explains why money is used or what particular benefit comes from using money. However, for most practical purposes, the same can be said about how most of us use money day to day. There is nothing in daily life that much explains why we use money except that it is what our employer gives us for the labor we provide and what the grocer accepts in exchange for the food we want to consume. This is usually a good enough reason for using money day by day and the reason we use it in this chapter.[1]

Cash-in-advance models normally have only one good. That is the reason for the taboo on consuming goods produced by the same family. Most families in the real world produce only one market good or service, and the particular good or service that they produce is only an extremely small part of the basket of goods they consume. The prohibition of consuming the family's own production is a way of making a one-good model behave like one with many diversified goods without the complications. Money is buying us the basket of goods we normally consume but do not produce for ourselves.

Adding money to the model creates an additional complication in solving the models. The presence of money puts a friction into the economy so that the equilibrium will not necessarily be that of a frictionless competitive equilibrium. The second welfare theorem allowed us to use a Robinson Crusoe (representative agent) type model to solve for a multi-agent competitive equilibrium because, without frictions, the equilibria would be the same. Here, we need to work explicitly with multiple agents, with markets, and with the friction of money. Each family in the economy will be responding to market variables (the price level) or economy-wide expectations variables (the next period's aggregate capital stock). In each period, each family bases its decisions on these variables. In an equilibrium, the values that the families use for these variables need to be the actual ones (in the case of prices) or the expected value of the economy-wide realization (in the case of the next period's capital). This complication adds a new step to the solution of an equilibrium, one that makes the solution process a bit more difficult, but not excessively so.

8.1 COOLEY AND HANSEN'S MODEL

A number of changes are required in the basic Hansen model to include money. One of the most important is that the model can no longer be solved

1. In some countries this issue is less than clear. One money, pesos, for example, is used for small transactions and another money, dollars, for example, is used for large transactions. There is an extensive and very interesting literature on endogenous money that tries to deal with such issues using search models for money. For some recent papers on this topic, see Lagos and Wright [53] and Zhu and Wallace [91].

by using a representative agent of the style of Robinson Crusoe. We now need to explicitly include markets for labor and capital, with wages and rentals determined by the aggregate labor and capital provided. Individual agents (families or households) take these wages and rentals as given and do not believe that their actions can influence either current or future wages and rentals. Although in equilibrium, the amount of labor provided and the stock of capital are determined by these family decisions, this feedback is not taken into account when the optimization decisions are made. All families are identical and all will end up doing the same thing, but each family is so small that if it chooses to do something else it will not have an effect on aggregate labor or capital. It is useful to think of the economy as being comprised of a continuum of agents indexed by i, where $i = [0, 1]$. There is a unit mass of agents (all points from zero to one) and each agent is simply a point. All agents are identical, so when we sum them (take the integral over i from 0 to 1, since there is a continuum of them), the aggregate behaves like any one of them. Points have measure zero, so if one agent behaves differently from the rest, the aggregate is not affected.

Each family, i, wants to maximize the discounted expected utility function,

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, h_t^i).$$

Utility increases with consumption and declines with labor supplied. As before, families have a maximum of 1 unit of labor in each period, so the leisure they consume, which does give positive utility, is equal to $1 - h_t^i$. Following Cooley and Hansen, the subutility function we use has indivisible labor, where each family signs a contract to provide with probability α_t^i an amount of h_0 units of labor.[2] The amount of indivisible labor, h_0, is the same for all families. Defining h_t^i as the expected labor to be provided in period t, $h_t^i = \alpha_t^i h_0$, the subutility function can be written as

$$u(c_t^i, h_t^i) = \ln c_t^i + \left[A \frac{\ln(1 - h_0)}{h_0} \right] h_t^i.$$

Production in the economy occurs with a Cobb-Douglas aggregate production function,

$$y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}.$$

Technology follows a stochastic process,

2. In Chapter 6, we discussed how this assumption makes convex the set over which one is finding the optimum.

$$\ln(\lambda_{t+1}) = \gamma \ln(\lambda_t) + \varepsilon_{t+1},$$

where the error term is independently and identically distributed as $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$. This assumption implies that the unconditional expectation of $\ln(\lambda_t) = 0$. In a stationary state stochastic variables are equal to their unconditional expectation, so $\ln(\bar{\lambda}) = 0$, and the stationary state value of the level of technology is $\bar{\lambda} = 1$.

Under conditions of perfect competition, the wage rate at time t equals

$$w_t = (1-\theta)\,\lambda_t K_t^{\theta} H_t^{-\theta},$$

and the rental rate is

$$r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}.$$

Notice that it is the amounts of aggregate labor and capital that determine wage and rental rates. Given that the production function displays constant returns to scale, all income to the firm gets paid out in wages and rentals: there are no excess profits to be distributed. The aggregate amount of labor available at time t is equal to

$$H_t = \int_0^1 h_t^i di,$$

and the aggregate amount of capital available at time t is

$$K_t = \int_0^1 k_t^i di.$$

Family i carries over an amount of money from the previous period, m_{t-1}^i, and receives a transfer of money from the government equal to $\left(g_t - 1\right) M_{t-1}$, where M_{t-1} is the per capita money stock at time $t-1$. In this model, since there is a unit mass of families, the per capita variables are equal to the aggregate of the same variable.[3] Notice that, in this model, a family's holdings of money do not determine how much they will receive from the government as a transfer. The cash-in-advance constraint on consumption purchases implies that

$$p_t c_t^i \le m_{t-1}^i + \left(g_t - 1\right) M_{t-1},$$

3. This because there is a mass 1 of individuals, so dividing the aggregate amount of some variable by 1 gives the per capita, which is also the aggregate. This is the benefit of defining the population as being a continuum of mass = 1,

where p_t is the price level in period t, and g_t is the time t gross growth rate of money (equal to 1 plus the net growth rate of money). One can consider cases where the growth rate of money is constant and one is in a stationary state or where it follows a stochastic process and the dynamics are of interest. We will consider an example of each case.

Notice that money transfers to families are a function of the aggregate money supply and not of the family's own money holdings. The results of the model would be different if the money transfer were based on the family's own holdings, since they would take into account how their decision to hold money would affect the transfer.

To keep the solution technique simple, we want a condition under which the cash-in-advance constraint always holds. Cooley and Hansen give this condition as follows: the expected gross growth rate of money, g_t, must be greater than the discount factor, β. If the expected gross growth rate of money were less than β, keeping some of the money acquired last period into the next period would increase consumption more than the discount rate and raise discounted welfare. Therefore, we will limit ourselves to cases where $g_t \geq \beta$. This implies that, in every period t, the shopper spends all the family's money holdings on time t consumption goods.

Families enter a period with holdings of capital, k_t^i, and money, m_{t-1}^i. In addition to the cash-in-advance constraint on consumption purchases, family i faces the flow budget constraint,

$$c_t^i + k_{t+1}^i + \frac{m_t^i}{p_t} = w_t h_t^i + r_t k_t^i + (1-\delta)k_t^i + \frac{m_{t-1}^i + (g_t - 1)M_{t-1}}{p_t}.$$

The right-hand side of the budget constraint sums income from labor (from the labor contract where they get paid whether they work or not) and capital holdings, the capital that remains after depreciation, and the real value of money held at the beginning of the period (but after the government transfer has been made). On the left-hand side of the budget constraint are consumption, the new capital holdings, and the real value of money to be held into the next period. Notice that everything in this budget constraint is measured in real terms (in terms of the one good in the economy).

To be able to solve this model, we need to be able to define a stationary state. When $\bar{g} = 1$, for all t, the stock of money is constant and there exists a natural concept for a stationary state. When $\bar{g} \neq 1$, the stock of money is either growing or shrinking continually over time and we will not, in general, be able to find stationary state values for p_t, m_t^i, or M_t. There are two ways of handling this problem.

One way, that of Cooley and Hansen, is to normalize the three nominal variables in each period by dividing them by M_t, and to define $\hat{p}_t = p_t/M_t$,

$\hat{m}_t^i = m_t^i/M_t$, and $M_t/M_t = 1$. Using these definitions, the cash-in-advance constraint is equal to

$$\frac{p_t}{M_t}c_t^i = \frac{m_{t-1}^i + \left(g_t - 1\right) M_{t-1}}{M_t},$$

which can be written as

$$\hat{p}_t c_t^i = \frac{m_{t-1}^i + \left(g_t - 1\right) M_{t-1}}{g_t M_{t-1}}$$

$$\hat{p}_t c_t^i = \frac{\hat{m}_{t-1}^i + \left(g_t - 1\right)}{g_t}.$$

Dividing each nominal variable by M_t, the household budget constraint can be written as

$$c_t^i + k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = w_t h_t^i + r_t k_t^i + (1-\delta)\, k_t^i + \frac{\hat{m}_{t-1}^i + \left(g_t - 1\right)}{g_t \hat{p}_t}.$$

Writing the model this way allows the nominal variables to have stationary values in an economy with nonzero net money growth. It will also allow us to find a stable solution to the dynamic version of the model.

A second way of normalizing the nominal variables in a stationary state is to divide them by prices and write everything in terms of real balances. In that case, we define the family's real balances as

$$\overline{m/p} = \frac{m_t^i}{p_t}$$

and the economy's real balances as

$$\overline{M/p} = \frac{M_t}{p_t}.$$

Some care needs to be taken with the lagged money variables. In a stationary state, money growth and (as we will show later) inflation are constant through time and equal to each other, $\bar{g} = \bar{\pi}$. Defining the gross inflation from period $t-1$ to period t as π_{t-1}, the stationary state value of a family's money holdings at the beginning of period t are

$$\frac{m_{t-1}^i}{p_t} = \frac{m_{t-1}^i}{\pi_{t-1} p_{t-1}} = \frac{\overline{m/p}}{\bar{\pi}} = \frac{\overline{m/p}}{\bar{g}}.$$

A similar condition holds for M_{t-1}.

Since we are developing the model of Cooley and Hansen, we will use their method in this chapter. In future chapters, it will be more convenient (and possibly more natural for an economist) to write the model in terms of real balances.

With these variables, the equilibrium we want can be found by solving the maximization problem,

$$\max E_0 \sum_{t=0}^{\infty} \left(\beta^t \ln c_t^i + \left[A \frac{\ln(1-h_0)}{h_0} \right] h_t^i \right),$$

subject to the budget constraints

$$c_t^i = \frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t \hat{p}_t}$$

and

$$c_t^i + k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = \left((1-\theta)\, \lambda_t K_t^{\theta} H_t^{-\theta} \right) h_t^i + \left(\theta \lambda_t K_t^{\theta-1} H_t^{1-\theta} \right) k_t^i$$
$$+ \ (1-\delta) k_t^i + \frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t \hat{p}_t}.$$

The law of motion for the stochastic shock is

$$\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1}^{\lambda},$$

and the growth rate for money is either a stationary state rule,

$$g_t = \bar{g},$$

or a stochastic rule

$$\ln g_{t+1} = (1-\pi) \ln \bar{g} + \pi \ln g_t + \varepsilon_{t+1}^{g}.$$

This last expression is the law of motion for money growth. The inclusion of the term $(1-\pi) \ln \bar{g}$ causes this g_t process to have a stationary state value of $\bar{g}$. This can be seen by setting the shocks, ε_{t+1}^{g}, to zero and solving for $g_t = g_{t+1}$.

The aggregation conditions for an equilibrium are

$$K_t = k_t^i,$$

$$H_t = h_t^i,$$

$$C_t = c_t^i,$$

and

$$\hat{M}_t = \hat{m}_t^i = 1.$$

Before solving for the dynamic version of the model, we need to find the stationary state.

> **EXERCISE 8.1** Write out the equations for an economy identical to the one above but where the government makes money transfers to each family as a fraction of the money that family is holding when it enters the period.

8.2 FINDING THE STATIONARY STATE

For the two methods we have for approximating the dynamic version of the model, we first need to find the stationary state. As mentioned earlier, variational methods[4] are adequate for finding a stationary state, so we can use these methods here to find the first-order conditions for the families. In addition, since we will later be solving a log-linear approximation of the model (the first-order constraints, the budget constraints, and the equilibrium conditions), the conditions we get here will be used in that part as well. One could get exactly the same first-order conditions using a Bellman equation approach.

The problem that the households solve can be written as the Lagrangian,

$$L = \max_{\left\{c_t^i, k_{t+1}^i, h_t^i, \hat{m}_t^i\right\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t^i + B h_t^i + \chi_t^1 \left(\hat{p}_t c_t^i - \frac{\hat{m}_{t-1}^i + g_t - 1}{g_t} \right) \right.$$
$$\left. + \chi_t^2 \left(k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} - (1-\delta) k_t^i - w_t h_t^i - r_t k_t^i \right) \right].$$

Since the stochastic variables, g_t and λ_t, will be held constant in a stationary state, the expectations operator will drop out. However, we keep it because we will be using these first order-conditions later in the log-linear solution to the model. Solving this Lagrangian yields the first-order conditions

$$\frac{\partial L}{\partial c_t^i} = \frac{1}{c_t^i} + \chi_t^1 \hat{p}_t = 0,$$

$$\frac{\partial L}{\partial h_t^i} = B - \chi_t^2 w_t = 0,$$

4. While not explicitly written here, a transversality condition must hold.

$$\frac{\partial L}{\partial k^i_{t+1}} = \chi^2_t - \beta E_t \chi^2_{t+1} \left[(1-\delta) + r_{t+1} \right] = 0,$$

and

$$\frac{\partial L}{\partial \hat{m}^i_t} = \chi^2_t \frac{1}{\hat{p}_t} - \beta E_t \chi^1_{t+1} \frac{1}{g_{t+1}} = 0.$$

The first two first-order conditions can be solved for the Lagrangian multipliers,

$$\chi^1_t = -\frac{1}{\hat{p}_t c^i_t}$$

and

$$\chi^2_t = \frac{B}{w_t}.$$

We use these to remove the multipliers from the last two first-order conditions. These two equations plus the two budget constraints of the families give the following four equations of the model:

$$\frac{1}{\beta} = E_t \frac{w_t}{w_{t+1}} \left[(1-\delta) + r_{t+1} \right],$$

$$\frac{B}{w_t \hat{p}_t} = -\beta E_t \frac{1}{\hat{p}_{t+1} c^i_{t+1} g_{t+1}},$$

$$\hat{p}_t c^i_t = \frac{\hat{m}^i_{t-1} + g_t - 1}{g_t},$$

and

$$k^i_{t+1} + \frac{\hat{m}^i_t}{\hat{p}_t} = (1-\delta)\, k^i_t + w_t h^i_t + r_t k^i_t$$

In addition, the factor market conditions, from the assumption of perfect competition in the factor markets, are

$$w_t = (1-\theta)\, \lambda_t \left[\frac{K_t}{H_t} \right]^{\theta}$$

and

$$r_t = \theta\lambda_t \left[\frac{K_t}{H_t}\right]^{\theta-1}.$$

In a stationary state, the stochastic technology parameter, λ_t, will be equal to 1 and drop out of the equations.

Since all families are the same, in equilibrium, the individual values will be the same as the aggregate, so

$$C_t = c_t^i,$$

$$H_t = h_t^i,$$

$$K_{t+1} = k_{t+1}^i,$$

and

$$\hat{M}_t = \hat{m}_t^i.$$

In the stationary state, $\bar{K} = K_t = K_{t+1}$, $\bar{H} = H_t = H_{t+1}$, $\bar{C} = C_t = C_{t+1}$, $\bar{w} = w_t = w_{t+1}$, $\bar{r} = r_t = r_{t+1}$, $\hat{p}_t = \hat{p}_{t+1} = \hat{p}$, and $\hat{M}_t = \hat{M}_{t+1} = 1$. The six equations of the model in a stationary state are

$$\frac{1}{\beta} = (1-\delta) + \bar{r},$$

$$\frac{B}{\bar{w}} = -\frac{\beta}{\bar{g}\bar{C}},$$

$$\hat{p}\bar{C} = 1,$$

$$\frac{1}{\hat{p}} = (\bar{r} - \delta)\,\bar{K} + \bar{w}\bar{H},$$

$$\bar{w} = (1-\theta)\left[\frac{\bar{K}}{\bar{H}}\right]^{\theta},$$

and

$$\bar{r} = \theta\left[\frac{\bar{K}}{\bar{H}}\right]^{\theta-1}.$$

The first equation gives us the rental rate on capital as

$$\bar{r} = \frac{1}{\beta} - (1-\delta).$$

We can then find the real wage in terms of the rental rate from the two factor market equations and get

$$\bar{w} = (1-\theta)\left[\frac{\bar{K}}{\bar{H}}\right]^{\theta} = (1-\theta)\left[\frac{\bar{r}}{\theta}\right]^{\frac{\theta}{\theta-1}}.$$

Consumption is

$$\bar{C} = -\frac{\beta\bar{w}}{\bar{g}B}, \tag{8.1}$$

and the price level is

$$\hat{p} = \frac{1}{\bar{C}}.$$

Using the factor market equations, we can write

$$\bar{H} = \left(\frac{\bar{r}}{\theta}\right)^{\frac{1}{1-\theta}}\bar{K}$$

and, using the family's budget constraint, get

$$\bar{K} = \frac{\bar{C}}{\frac{\bar{r}}{\theta} - \delta}.$$

Since stationary state consumption is known, so is equilibrium capital stock and the labor supply. Output can be found from the household budget constraint as

$$\bar{Y} = \bar{C} + \delta\bar{K}.$$

For our normal, quarterly economy, we have been using the parameter values $\beta = .99$, $\delta = .025$, $\theta = .36$, $A = 1.72$, and $h_0 = .583$, so $B = -2.5805$. Using these parameter values, the stationary state values for the variables are those shown in Table 8.1. Notice that real wages and rentals do not depend on the rate of growth of the money supply, but stationary state consumption, price level, capital stock, hours worked, and output do depend on it.

In their paper, Cooley and Hansen have a table that shows how different levels of inflation in the stationary state can result in different values for the real variables. They also calculate the welfare loss that comes from inflation. Since in the stationary state discounted utility is

$$\sum_{i=0}^{\infty}\beta^{i}\left(\ln\bar{C} + B\bar{H}\right) = \frac{\ln\bar{C} + B\bar{H}}{1-\beta},$$

we can easily use our results on the stationary state to construct the table they have (Cooley and Hansen [36], page 743). Table 8.2 shows these results.

The level of utility when the inflation rate is equal to the discount rate is used as the basis from which to measure welfare losses from inflation. The loss is measured as the percentage decline in discounted lifetime utility from the case where the gross growth rate of money is equal to β. Recall that for the cash-in-advance constraint to hold with equality, one needs $\bar{g} \geq \beta$. This is why -4% (where $\bar{g} = .99$) is the lowest inflation rate shown in the table.

One interesting characteristic of this table is that the stationary state values for the case where inflation is -4 percent are almost exactly equal to those of the Hansen model with indivisible labor but without money (the second model in Chapter 6; see Table 6.3). The equilibria are subtly different. Compare the Hansen model with indivisible labor to the Cooley-Hansen model without

Table 8.1 Stationary state as a function of g

Variable	Value in stationary state
$\bar{r}$	.0351
$\bar{w}$	2.3706
$\bar{C}$	$\frac{0.9095}{\bar{g}}$
$\hat{p}$	$1.0995\bar{g}$
$\bar{K}$	$\frac{12.544}{\bar{g}}$
$\bar{H}$	$\frac{0.3302}{\bar{g}}$
$\bar{Y}$	$\frac{1.2231}{\bar{g}}$

Table 8.2 Table as in Cooley and Hansen

Annual inflation	-4%	0%	10%	100%	400%
Corresponding g	β	1	1.024	1.19	1.41
Output	1.2355	1.2231	1.1944	1.0278	0.8674
Consumption	0.9187	0.9095	0.8882	0.7643	0.6450
Investment	0.3168	0.3136	0.3063	0.2635	0.2224
Capital stock	12.6707	12.544	12.2500	10.541	8.8965
Hours worked	0.3335	0.3302	0.3225	0.2775	0.2342
Welfare loss, %	0	.15%	.55%	4.16%	10.29%

inflation. In the Cooley-Hansen model, money is acquired in period t to pay for consumption in period $t+1$, so this time $t+1$ consumption is reduced by the factor β. This shows up in the first-order conditions, where the only difference (with zero inflation) is the addition of a β in the equation relating consumption to inflation (see equations 6.7 and 8.1 for the versions in the stationary state).

> **EXERCISE 8.2** Find the stationary states as a function of the growth rate of money for the economy, where the government makes money transfers to each family as a fraction of the money that family is holding when it enters the period.

8.3 SOLVING THE MODEL USING LINEAR QUADRATIC METHODS

Cooley and Hansen solve their model using linear quadratic techniques, so we will show these methods first. The methods are somewhat more complicated than those given in Chapter 7 because the choices of the families depend on the values of some economy-wide variables, K_t and $\hat{p}_t$. The optimization problem for each family in each period must take these values as given (because each family is too small to have a direct effect on the values of these variables), but the aggregate behavior of the families determines the values of these variables. The optimization problem needs to be separated from the aggregation problem. In this section, we set up this problem and show how to find the approximation to the dynamic model but do not actually calculate the solution.

A second, and normally simpler, method of finding a solution is by using the log linearization techniques shown in Chapter 6. This solution is normally very similar to that found by using linear quadratic methods. In the next section, we use this method to find the dynamic properties of this model.

Given the values for the stationary state of this economy, we look for a solution to an approximate, quadratic version of the model. There are two stages to finding an equilibrium. First, one needs to find the optimal behavior for a family given the aggregate variables for the economy. Then, one uses the decision rules of the families and aggregates them (given that all families in the economy are identical and there is a mass $= 1$ of families) to find the values of the aggregate variables in the economy. Using aggregate versions of the policy functions for families, one needs to find a fixed point for time $t+1$ aggregate capital so that the amount of time $t+1$ aggregate capital that families expect is equal to the sum of capital that all the families in the economy actually hold into time $t+1$. The market variable, the price level, is found simultaneously with the fixed point for capital.

The optimization problem and the aggregation problem need to be separated, with the optimization done before the aggregation. If one were to do the aggregation first, the results of the optimization problem would not be the same as that of an economy of small, atomic families. The second welfare theorem does not hold here because the addition of money creation means that the equilibrium is not that of a pure competitive equilibrium.

8.3.1 Finding a Quadratic Objective Function

The method we have available to us solves a quadratic objective function with linear budget constraints. To make sure that we can use that method, we need to put all the nonlinear restrictions into the objective function. Using the restriction that

$$c_t^i = \frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t \hat{p}_t},$$

one first eliminates the variable c_t from both the objective function and from the second budget constraint. That makes the objective function

$$\max E_0 \sum_{t=0}^{\infty} \left(\beta^t \ln \left[\frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t \hat{p}_t} \right] + \left[A \frac{\ln(1-h_0)}{h_0} \right] h_t^i \right),$$

and the remaining budget constraint

$$k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = \left((1-\theta)\, \lambda_t K_t^{\theta} H_t^{-\theta} \right) h_t^i + \left(\theta \lambda_t K_t^{\theta-1} H_t^{1-\theta} \right) k_t^i + (1-\delta) k_t^i.$$

This budget constraint can be simplified to get

$$k_{t+1}^i - (1-\delta) k_t^i + \frac{\hat{m}_t^i}{\hat{p}_t} = \left(\lambda_t K_t^{\theta} H_t^{1-\theta} \right) \left[(1-\theta) \frac{h_t^i}{H_t} + \theta \frac{k_t^i}{K_t} \right]. \qquad (8.2)$$

Summing across families, the individual variables are replaced by the aggregate, and the budget constraint becomes

$$K_{t+1} + \frac{1}{\hat{p}_t} = \lambda_t K_t^{\theta} H_t^{1-\theta} + (1-\delta) K_t,$$

which can be solved for aggregate labor as

$$H_t = \left[\frac{K_{t+1} - (1-\delta)K_t + \frac{1}{\hat{p}_t}}{\lambda_t K_t^{\theta}} \right]^{\frac{1}{1-\theta}}.$$

Putting this definition for aggregate labor into the individual budget constraint, equation 8.2, gives individual labor supply as

$$h_t^i = \frac{k_{t+1}^i - (1-\delta)k_t^i + \frac{\hat{m}_t^i}{\hat{p}_t} - \theta \left[K_{t+1} - (1-\delta)K_t + \frac{1}{\hat{p}_t} \right] \frac{k_t^i}{K_t}}{(1-\theta)\left[K_{t+1} - (1-\delta)K_t + \frac{1}{\hat{p}_t} \right]^{-\frac{\theta}{1-\theta}} \left[\lambda_t K_t^{\theta} \right]^{\frac{1}{1-\theta}}}.$$

Using this equation to remove labor from the objective function gives

$$\max_{k_{t+1}^i, \hat{m}_t^i} E_0 \sum_{t=0}^{\infty} \left(\beta^t \ln \left[\frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t \hat{p}_t} \right] \right. \tag{8.3}$$

$$\left. + \left[A \frac{\ln(1-h_0)}{h_0} \right] \left[\frac{k_{t+1}^i - (1-\delta)k_t^i + \frac{\hat{m}_t^i}{\hat{p}_t} - \theta \left[K_{t+1} - (1-\delta)K_t + \frac{1}{\hat{p}_t} \right] \frac{k_t^i}{K_t}}{(1-\theta)\left[K_{t+1} - (1-\delta)K_t + \frac{1}{\hat{p}_t} \right]^{-\frac{\theta}{1-\theta}} \left[\lambda_t K_t^{\theta} \right]^{\frac{1}{1-\theta}}} \right] \right).$$

The state variables at time t are $x_t = [\,1 \quad \lambda_t \quad k_t^i \quad \hat{m}_{t-1}^i \quad g_t \quad K_t\,]'$. A family maximizes its objective function over $y_t = [\,k_{t+1}^i \quad \hat{m}_t^i\,]'$, subject to the budget constraints

$$\ln(\lambda_{t+1}) = \gamma \ln(\lambda_t) + \varepsilon_{t+1}^{\lambda},$$

$$k_{t+1}^i = k_{t+1}^i,$$

$$\hat{m}_t^i = \hat{m}_t^i,$$

and

$$\ln g_{t+1} = (1-\pi)\,\bar{g} + \pi \ln g_t + \varepsilon_{t+1}^{g}.$$

In addition, the set of aggregate or market variables is $Z_t = [\,K_{t+1} \quad \hat{p}_t\,]'$, where

$$K_{t+1} = \int_0^1 k_{t+1}^i di,$$

and $\hat{p}_t$ is the time t price level divided by the time t money supply.

The first step is to take the second-order Taylor expansion of equation 8.3 and, using the method of Kydland and Prescott described in Chapter 7, to write the approximate objective function as

$$[\,x_t' \quad y_t' \quad Z_t'\,]\, Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix}.$$

Given this objective function, we want to solve a Bellman equation of the form

$$x_t' P x_t = \max_{y_t} \left[[\,x_t' \quad y_t' \quad Z_t'\,]\, Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix} + \beta E_0 \left[x_{t+1}' P x_{t+1} \right] \right],$$

subject to the budget constraints

$$x_{t+1} = A x_t + B y_t + C Z_t + D \varepsilon_{t+1}.$$

We can rewrite the matrix Q as

$$Q = \begin{bmatrix} R & W' & X' \\ W & T & N' \\ X & N & S \end{bmatrix},$$

and write

$$[\,x_t' \quad y_t' \quad Z_t'\,]\, Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix}$$

as

$$x_t' R x_t + y_t' T y_t + Z_t' S Z_t + 2 y_t' W x_t + 2 Z_t' X x_t + 2 Z_t' N y_t.$$

The second half of the right-hand side of the Bellman equation can be written as

$$\begin{aligned} &\beta E_0 \left[x_{t+1}' P x_{t+1} \right] \\ &= \beta E_0 \left[\left(A x_t + B y_t + C Z_t + D \varepsilon_{t+1} \right)' P \left(A x_t + B y_t + C Z_t + D \varepsilon_{t+1} \right) \right]. \end{aligned}$$

Taking the derivative of the first half (written out) with respect to y_t gives

$$\left(T + T' \right) y_t + 2 W x_t + 2 N' Z_t,$$

and the second half gives

$$\beta\left[2B'PAx_t + 2B'PBy_t + 2B'PCZ_t\right].$$

Combining, and recalling that T is symmetric, we get the first-order condition (the 2's drop out),

$$0 = Ty_t + Wx_t + N'Z_t + \beta\left[B'PAx_t + B'PBy_t + B'PCZ_t\right],$$

or

$$\left(T + \beta B'PB\right)y_t = -\left(W + \beta B'PA\right)x_t - \left(N + \beta B'PC\right)Z_t.$$

When $\left(T + \beta B'PB\right)$ is invertible, this gives

$$y_t = -\left(T + \beta B'PB\right)^{-1}\left(W + \beta B'PA\right)x_t - \left(T + \beta B'PB\right)^{-1}\left(N + \beta B'PC\right)Z_t,$$

as the *linear* policy function, which we can write as

$$y_t = F_1x_t + F_2Z_t,$$

with

$$F_1 = -\left(T + \beta B'PB\right)^{-1}\left(W + \beta B'PA\right)$$

and

$$F_2 = -\left(T + \beta B'PB\right)^{-1}\left(N + \beta B'PC\right).$$

When $\left(T + \beta B'PB\right)$ is not invertible, somewhat different methods need to be used to get a solution. One way to do this is by using the generalized Schur decomposition described in section 6.8.4.

The value function matrix P fulfills

$$\begin{aligned} x_t'Px_t = {} & [\, x_t' \quad \left(F_1x_t + F_2Z_t\right)' \quad Z_t' \,]\, Q \begin{bmatrix} x_t \\ F_1x_t + F_2Z_t \\ Z_t \end{bmatrix} \\ & + \beta\left(\left(A + BF_1\right)x_t + \left(BF_2 + C\right)Z_t\right)' P \left(\left(A + BF_1\right)x_t + \left(BF_2 + C\right)Z_t\right). \end{aligned}$$

8.3.2 Finding the Economy Wide Variables

The above expression of the value function matrix still contains the aggregate variable $Z_t = [\, K_{t+1} \quad \hat{p}_t \,]'$. We want to find an expression that will give Z_t as a function of x_t. We start by noting that integrating the policy variable, y_t, over the unit mass of families gives us the aggregate values for the time t money stock and the time $t+1$ capital. Given that $y_t^i = [\, k_{t+1}^i \quad \hat{m}_t^i \,]'$, the aggregate capital and money stock can be found by integrating over the unit mass of families,

$$\int_0^1 y_t^i di = \begin{bmatrix} \int_0^1 k_{t+1}^i di \\ \int_0^1 \hat{m}_t^i di \end{bmatrix} = \begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix},$$

where $1 = M_t/M_t$, the aggregate value of the $\hat{m}_t^i$ variable. If a policy function,

$$y_t = F_1 x_t + F_2 Z_t,$$

exists, F_1 and F_2 are both linear functions, and the policy function holds in aggregate when we integrate both sides over the unit mass of families. Doing this gives

$$\begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} = F_1 \int_0^1 x_{t+1}^i di + F_2 Z_t. \tag{8.4}$$

Recalling that

$$x_t^i = [\, 1 \quad \lambda_t \quad k_t^i \quad \hat{m}_{t-1}^i \quad g_t \quad K_t \,]',$$

and that the integral of this vector is

$$\hat{x}_t = \int_0^1 x_{t+1}^i di = [\, 1 \quad \lambda_t \quad K_t \quad 1 \quad g_t \quad K_t \,],$$

we can construct a matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

that will give us $\hat{x}_t = G x_t^i$, for all i. The matrix G is a diagonal matrix except in rows 3 and 4, where it takes k_t^i into K_t and m_t^i into 1, respectively.

Using this aggregation matrix, G, equation 8.4 can be written as

$$\begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} = F_1 G x_t^i + F_2 \begin{bmatrix} K_{t+1} \\ \hat{p}_t \end{bmatrix}.$$

If the matrix, F_2, is invertible, we can find the values of the aggregate variables as

$$\begin{bmatrix} K_{t+1} \\ \hat{p}_t \end{bmatrix} = F_2^{-1} \begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} - F_2^{-1} F_1 G x_t^i,$$

or as

$$\begin{bmatrix} K_{t+1} \\ \hat{p}_t \end{bmatrix} = J \begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} + H x_t^i,$$

where $J = [J_{ij}]$ is a 2×2 matrix and $H = [H_{ij}]$ is a 2×6 matrix. Recalling that the first element of x_t^i is always 1, and working through the matrix algebra of the above equation, we can find K_{t+1} and $\hat{p}_t$ as

$$Z_t = F_3 x_t^i,$$

where

$$F_3 = \begin{bmatrix} \frac{H_{11}+J_{12}}{1-J_{11}} & \frac{H_{12}}{1-J_{11}} & \frac{H_{13}}{1-J_{11}} & \frac{H_{14}}{1-J_{11}} & \frac{H_{15}}{1-J_{11}} & \frac{H_{16}}{1-J_{11}} \\ H_{21}+J_{22}+\frac{J_{21}(H_{11}+J_{12})}{1-J_{11}} & H_{22}+\frac{J_{21}H_{12}}{1-J_{11}} & H_{23}+\frac{J_{21}H_{13}}{1-J_{11}} & H_{24}+\frac{J_{21}H_{14}}{1-J_{11}} & H_{25}+\frac{J_{21}H_{15}}{1-J_{11}} & H_{26}+\frac{J_{21}H_{16}}{1-J_{11}} \end{bmatrix}.$$

While the expression for the F_3 matrix is messy, it simply describes a way of solving for the two variables, K_{t+1} and $\hat{p}_t$, in a system of two linear equations when, given x_t^i, Hx_t^i is taken as a constant and where J_{12} and J_{22} are constants in their respective equations.

Given this matrix for finding the aggregate variables, we can write

$$P = [\, I_x \quad F_1' + F_3' F_2' \quad F_3' \,]\, Q \begin{bmatrix} I_x \\ F_1 + F_2 F_3 \\ F_3 \end{bmatrix}$$
$$+ \; \beta \left((A + BF_1) + (BF_2 + C)\, F_3 \right)' P \left((A + BF_1) + (BF_2 + C)\, F_3 \right),$$

where I_x is an identity matrix with dimensions equal to the size of the state variable x_t.

To find P, we use an iterative process. One begins with an initial guess for the value matrix, P_0, calculates initial policy functions, F_1^0 and F_2^0, using that P_0, and then uses these F_1^0 and F_2^0 to calculate the function for finding the economy-wide variables, F_3^0. Once these three matrices are found, they are put into the above equation, along with P_0 on the right-hand side, and the result is the value matrix for the next iteration, P_1. Using this P_1, new calculations are made for F_1^1, F_2^1, and F_3^1, and using P_1, the above equation is used to calculate P_2. The iterations continue until the difference between P_j and P_{j+1} are within the desired tolerances. As one finds the value matrix P, one also finds the linear approximations of the policy functions, F_1 and F_2, and of the function for calculating the economy-wide variables, F_3.

8.4 SOLVING THE MODEL USING LOG LINEARIZATION

The process for finding the equilibrium using second-order Taylor expansions is very intensive in differentiation, since one has to work out by hand the second partial derivatives of an objective function that contains all the nonlinearity of the system. The existence of economy-wide variables adds another level of complexity. A frequently simpler alternative (that we introduced in Chapter 6) is to find a linear version of the model using the method of log linearization on the first-order conditions of the model, the budget constraints, the stochastic processes, and the aggregation rules. One can then solve this linear system for the linear policy functions. That is what we do in this section.

8.4.1 The Log Linearization

To find the log-linear version of this model, we can use the first-order conditions, the budget constraints, and the equilibrium conditions that we found when we calculated the stationary state. The equations for the household decisions are the two first-order conditions,

$$\frac{1}{\beta} = E_t \frac{w_t}{w_{t+1}} \left[(1-\delta) + r_{t+1} \right]$$

and

$$\frac{B}{w_t \hat{p}_t} = -\beta E_t \frac{1}{\hat{p}_{t+1} c_{t+1}^i g_{t+1}},$$

the cash-in-advance constraint,

$$\hat{p}_t c_t^i = \frac{\hat{m}_{t-1}^i + g_t - 1}{g_t},$$

and the flow budget constraint,

$$k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = (1-\delta)\, k_t^i + w_t h_t^i + r_t k_t^i.$$

The competitive factor market gives us the two equations

$$w_t = (1-\theta)\,\lambda_t \left[\frac{K_t}{H_t}\right]^{\theta}$$

and

$$r_t = \theta\lambda_t \left[\frac{K_t}{H_t}\right]^{\theta-1}.$$

The equilibrium and aggregation conditions are

$$C_t = c_t^i,$$

$$H_t = h_t^i,$$

$$K_{t+1} = k_{t+1}^i,$$

and

$$\hat{M}_t = \hat{m}_t^i = 1.$$

Recall that we have normalized this economy so that $\hat{M}_t = 1$. In addition, we have the two stochastic processes,

$$\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1}^{\lambda}$$

and

$$\ln g_{t+1} = (1-\pi)\ln \bar{g} + \pi \ln g_t + \varepsilon_{t+1}^{g}.$$

Taking log linearizations of the two first-order conditions gives

$$-\tilde{w}_t = \beta E_t \left[\bar{r}\left(\tilde{r}_{t+1} - \tilde{w}_{t+1}\right) - (1-\delta)\,\tilde{w}_{t+1}\right] \tag{8.5}$$

and

$$-\frac{B}{\bar{C}\bar{w}}\left[\tilde{p}_t + \tilde{w}_t\right] = \beta E_t \left[\frac{1}{\bar{g}}\tilde{g}_{t+1}\right], \tag{8.6}$$

having used the cash-in-advance constraint in the form

$$g_t \hat{p}_t c_t^i = \hat{m}_{t-1}^i + g_t - 1.$$

The log-linear version of the cash-in-advance constraint in aggregate gives the conditions that $\tilde{p}_t + \tilde{C}_t = 0$, for all t, and this lets us remove these two variables from the above equation. The real budget constraint is written as

$$\bar{k}\tilde{k}_{t+1} + \frac{\bar{m}}{\bar{p}}\left[\tilde{m}_t - \tilde{p}_t\right] = \bar{w}\bar{h}\left[\tilde{w}_t + \tilde{h}_t\right] + \bar{r}\bar{k}\left[\tilde{r}_t + \tilde{k}_t\right] + (1-\delta)\bar{k}\tilde{k}_t. \tag{8.7}$$

Log-linear versions of the competitive factor market conditions are

$$\bar{r}\tilde{r}_t = \theta \bar{K}^{\theta-1}\bar{H}^{1-\theta}\left[\tilde{\lambda}_t + (\theta-1)\left[\tilde{K}_t - \tilde{H}_t\right]\right] \tag{8.8}$$

and

$$\bar{w}\tilde{w}_t = (1-\theta)\,\bar{K}^{\theta}\bar{H}^{-\theta}\left[\tilde{\lambda}_t + \theta\left[\tilde{K}_t - \tilde{H}_t\right]\right]. \tag{8.9}$$

The stochastic processes of the technology and money supply shocks[5] are

$$\tilde{\lambda}_{t+1} = \gamma\tilde{\lambda}_t + \varepsilon^{\lambda}_{t+1} \tag{8.10}$$

and

$$\tilde{g}_{t+1} = \pi\tilde{g}_t + \varepsilon^{g}_{t+1}. \tag{8.11}$$

The aggregation conditions are

$$\tilde{K}_t = \tilde{k}_t, \tag{8.12}$$

$$\tilde{H}_t = \tilde{h}_t, \tag{8.13}$$

and

$$\tilde{m}_t = 0. \tag{8.14}$$

This last restriction holds because $\hat{M}_t = 1 = \bar{m}\left(1+\tilde{m}_t\right) = 1\left(1+\tilde{m}_t\right)$.

It is useful to eliminate some variables from the model and to reduce the number of expectational variables. One can remove the expectations from the equation

$$-\frac{B}{\bar{C}\bar{w}}\left[\tilde{p}_t + \tilde{w}_t\right] = \beta E_t\left[\frac{1}{\bar{g}}\tilde{g}_{t+1}\right]$$

by using the process for money growth,

$$\tilde{g}_{t+1} = \pi\tilde{g}_t + \varepsilon^{g}_{t+1}.$$

Since the expectation of the error is zero, one can eliminate the expectations operator and get

$$-\frac{B}{\bar{C}\bar{w}}\left[\tilde{p}_t + \tilde{w}_t\right] = \frac{\beta\pi}{\bar{g}}\tilde{g}_t. \tag{8.15}$$

5. The money process, $\ln g_{t+1} = (1-\pi)\ln\bar{g} + \pi\ln g_t + \varepsilon^{g}_{t+1}$, can be written as $\ln g_{t+1} - \ln\bar{g} = \pi\left(\ln g_t - \ln\bar{g}\right) + \varepsilon^{g}_{t+1}$, which, since $\tilde{g}_t = \ln g_t - \ln\bar{g}$, is equal to $\tilde{g}_{t+1} = \pi\tilde{g}_t + \varepsilon^{g}_{t+1}$.

We can remove the individual variables by replacing them with the aggregate variables that they equal in equilibrium, replacing $\tilde{k}_t$ with $\tilde{K}_t$ and $\tilde{h}_t$ with $\tilde{H}_t$. We can also remove the money stock variable since aggregate money must always equal 1 and that implies that $\tilde{m}_t = 0$, always. After these adjustments, we have a system with four equations without expectations,

$$0 = \bar{K}\tilde{K}_{t+1} - \frac{1}{\bar{p}}\tilde{p}_t - \bar{w}\bar{H}\tilde{w}_t - \bar{w}\bar{H}\tilde{H}_t - \bar{r}\bar{K}\tilde{r}_t - \bar{r}\bar{K}\tilde{K}_t - (1-\delta)\bar{K}\tilde{K}_t,$$

$$0 = \tilde{r}_t - \tilde{\lambda}_t - (\theta - 1)\tilde{K}_t + (\theta - 1)\tilde{H}_t,$$

$$0 = \tilde{w}_t - \tilde{\lambda}_t - \theta\tilde{K}_t + \theta\tilde{H}_t,$$

and

$$0 = \tilde{p}_t + \tilde{w}_t - \pi\tilde{g}_t$$

one equation with expectations,

$$0 = \tilde{w}_t + \beta\bar{r}E_t\tilde{r}_{t+1} - E_t\tilde{w}_{t+1},$$

and the two stochastic processes for the shocks to technology and money growth,

$$\tilde{\lambda}_{t+1} = \gamma\tilde{\lambda}_t + \varepsilon^{\lambda}_{t+1}$$

and

$$\tilde{g}_{t+1} = \pi\tilde{g}_t + \varepsilon^{g}_{t+1}.$$

8.4.2 Solving the Log-Linear System

First, write out the system in the form

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t,$$

$$0 = E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right],$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1},$$

where $x_t = [\tilde{K}_{t+1}]'$, $y_t = [\,\tilde{r}_t \quad \tilde{w}_t \quad \tilde{H}_t \quad \tilde{p}_t\,]'$, and $z_t = [\,\tilde{\lambda}_t \quad \tilde{g}_t\,]'$, and where

$$A = \begin{bmatrix} \bar{K} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -(\bar{r}+1-\delta)\,\bar{K} \\ (1-\theta) \\ -\theta \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -\bar{r}\bar{K} & -\bar{w}\bar{H} & -\bar{w}\bar{H} & -\frac{1}{\bar{p}} \\ 1 & 0 & (\theta-1) & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & \pi \end{bmatrix},$$

$$F = [0],$$

$$G = [0],$$

$$H = [0],$$

$$J = [\,\beta\bar{r} \quad -1 \quad 0 \quad 0\,],$$

$$K = [\,0 \quad 1 \quad 0 \quad 0\,],$$

$$L = [\,0 \quad 0\,],$$

$$M = [\,0 \quad 0\,],$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix}.$$

The linear laws of motion that we are looking for are given by the matrices P, Q, R, and S, where

$$x_{t+1} = Px_t + Qz_t$$

and

$$y_t = Rx_t + Sz_t.$$

From Uhlig [86] and Appendix 1 of Chapter 6, we know that P solves the quadratic equation (matrix equation in general but a simple quadratic in this case, since capital is the only state variable)

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0,$$

and that

$$R = -C^{-1}(AP + B),$$

$$\begin{aligned}&\operatorname{vec}(Q)\\ &= \left(N' \otimes (F - JC^{-1}A) + I_k \otimes (FP + G + JR - KC^{-1}A)\right)^{-1}\\ &\quad \times \operatorname{vec}\left(\left(JC^{-1}D - L\right) N + KC^{-1}D - M\right),\end{aligned}$$

and

$$S = -C^{-1}(AQ + D).$$

The results for the model are $P = [0.9418]$, $Q = [\,0.1552 \quad 0.0271\,]$,

$$R = \begin{bmatrix} -0.9450 \\ 0.5316 \\ -0.4766 \\ -0.5316 \end{bmatrix},$$

$$S = \begin{bmatrix} 1.9418 & -0.0555 \\ 0.4703 & 0.0312 \\ 1.4715 & -0.0867 \\ -0.4703 & 0.4488 \end{bmatrix}.$$

The results of the simulations are given in Tables 8.3 and 8.4. In this run, the stationary state inflation rate was set to zero ($\bar{g} = 1$). For the simulations, the standard economy was used except that the standard error of the error terms was set to $\sigma_\lambda = .0036$, and either $\sigma_g = .00$, .01, or .02. In these tables one can see that the increasing money growth shocks have relatively little effect on the production-related variables ($\tilde{Y}$, $\tilde{K}$, $\tilde{H}$, $\tilde{r}$, and $\tilde{w}$) and relatively more on consumption and investment. Not surprisingly, the standard error of prices grows rapidly with the growth in the standard eror of money growth. Since output reacts so little to the money growth shock, it should not be surprising that correlations with output decline across the board as the money growth shock increases.

A particular characteristic of this model is that the dynamics do not depend on the stationary state inflation rate. A brief look at the A to M matrices of the model shows that the level of $\bar{g}$ does not show up explicitly anywhere. It does show up through the stationary state values of $\bar{K}$, $\bar{H}$, and $1/\bar{p}$, since $\bar{K}$ and $\bar{H}$ are determined by a constant divided by $\bar{g}$ and $\bar{p}$ by a constant multiplied by $\bar{g}$, but these stationary state variables all show up in only one equation and every coefficient in that equation gets divided by $\bar{g}$. Therefore, the relationship between variables described by that equation does not change as $\bar{g}$ changes. The log-linear model is the same for all permissible values of $\bar{g}$.

Table 8.3 Standard errors of variables in cash-in-advance model

Variable	$\sigma_\lambda = .0036$ $\sigma_g = 0$	$\sigma_\lambda = .0036$ $\sigma_g = .01$	$\sigma_\lambda = .0036$ $\sigma_g = .02$
$\tilde{Y}$	0.0176	0.0176	0.0178
$\tilde{C}$	0.0098	0.0119	0.0168
$\tilde{I}$	0.0478	0.0496	0.0535
$\tilde{K}$	0.0130	0.0129	0.0130
$\tilde{r}$	0.0147	0.0147	0.0148
$\tilde{w}$	0.0098	0.0098	0.0098
$\tilde{H}$	0.0110	0.0110	0.0112
$\tilde{p}$	0.0098	0.0109	0.0138

Table 8.4 Correlations with output in the CIA economy

Variable	$\sigma_\lambda = .0036$ $\sigma_g = 0$	$\sigma_\lambda = .0036$ $\sigma_g = .01$	$\sigma_\lambda = .0036$ $\sigma_g = .02$
$\tilde{Y}$	1.0000	1.0000	1.0000
$\tilde{C}$	0.8234	0.6666	0.5094
$\tilde{I}$	0.9472	0.9060	0.8030
$\tilde{K}$	0.6166	0.6106	0.5966
$\tilde{r}$	0.7149	0.7173	0.7161
$\tilde{w}$	0.8234	0.8186	0.8045
$\tilde{H}$	0.8753	0.8758	0.8715
$\tilde{p}$	−0.8234	−0.7291	−0.5993

EXERCISE 8.3 Write out a cash-in-advance model in which money is used by the families to purchase consumption goods and by the firms to pay wages. In each case, it is time $t-1$ money, adjusted by any government transfer or tax in period t that is used to make the payments in period t.

8.4.3 Impulse Response Functions

The impulse response functions make very clear some characteristics of this model. The response of the economy to a technology shock is exactly the same as that of the Hansen indivisible labor model of Chapter 6. Figure 8.1 shows the impulse response function (given a .01 shock) from the earlier Hansen model (this is the same as Figure 6.5). Figure 8.2 shows the response of the Cooley-Hansen model to the same shock. The difference in the figures is in the variables that each model is finding. The Hansen indivisible labor model shows the responses of $\tilde{K}$, $\tilde{Y}$, $\tilde{C}$, $\tilde{H}$, and $\tilde{r}$ while the Cooley-Hansen model shows the responses of $\tilde{K}$, $\tilde{r}$, $\tilde{w}$, $\tilde{H}$, and $\tilde{p}$.

In the Cooley-Hansen model, when there is only a technology shock, log-linear version of the cash-in-advance constraint gives us prices and consumption that move exactly opposite from one another. With only a technology shock, the log-linear version of the second first-order condition (equation

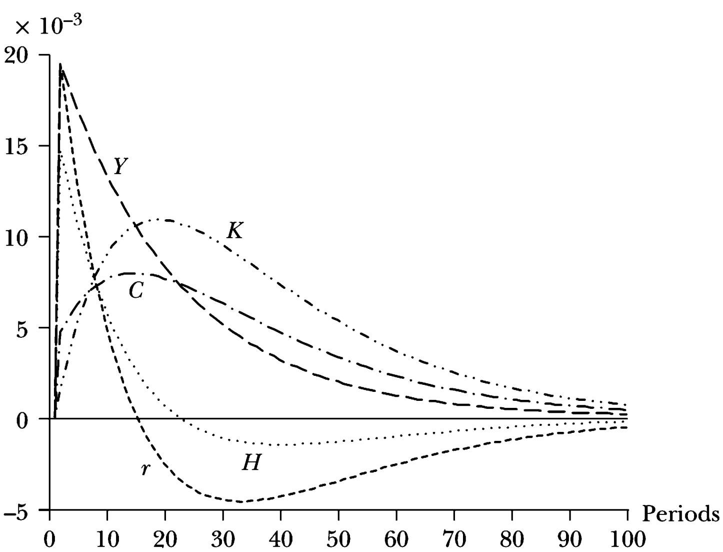

FIGURE 8.1 Response of Hansen's model to technology shock

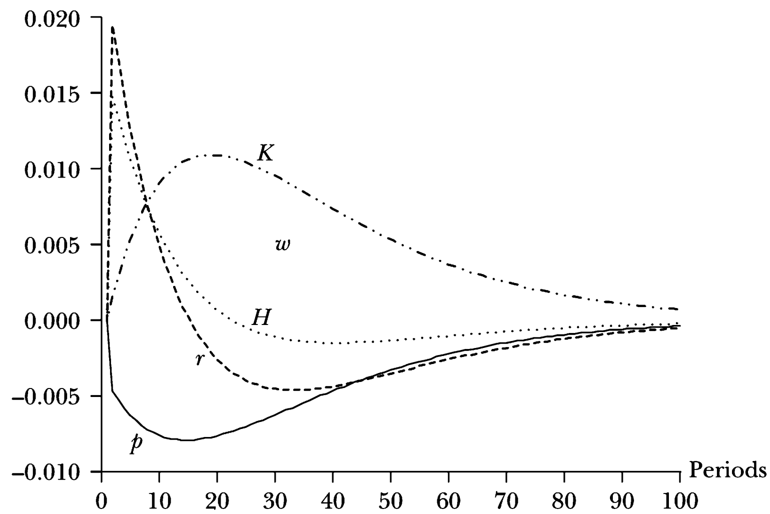

FIGURE 8.2 Response of Cooley-Hansen model to technology shock

8.15) gives us prices and wages that move exactly opposite from one another. This implies that wages and consumption respond in exactly the same way to a technology shock.

The response of the Cooley-Hansen model to a .01 money growth shock is shown in Figure 8.3. The very clear reaction is that of prices, which responds very quickly and almost in the same amount as the money growth shock. Since we have assumed that money growth is less serially correlated than technology (the coefficient on the first lag term is .48 for the money process and .95 for the technology process), the shock dies off much faster than in the case of the technology shock. This is caused almost entirely by the serial correlation in the stochastic process and says little about the propagation characteristics of the model itself. Monetary shocks do have real effects, as can be seen from the responses of capital and labor, but these are relatively small. This is consistent with the relatively small changes that one observes in the standard errors of the real variables as the money shock increases. In particular, output changes are quite small.

8.5 SEIGNIORAGE

A monetary policy that operates in the manner described above is not very common. Governments do make transfers directly to citizens, in the form of unemployment insurance payments, social security payments, and, in less well organized countries, by direct payments by politicians to their friends

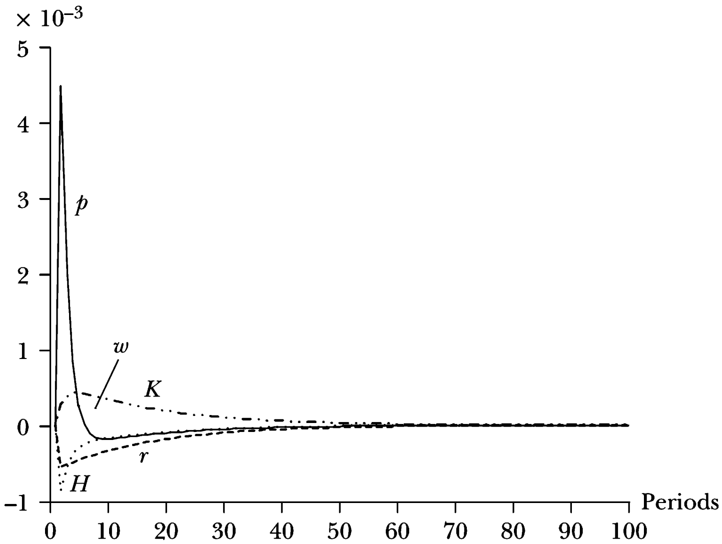

FIGURE 8.3 Response of Cooley-Hansen model to money growth shock

and family. While these payments can make up a relatively large part of the government budget, it is not common that they are financed directly with money issue. In addition, these payments are different from the lump sum monetary transfer that we used in the above model.

In this section we will assume that, instead of simply giving money to the families in the economy, the government runs a fiscal deficit that it finances with the issue of new money. The real value of the goods that the government purchases with the new money is called *seigniorage*. Introducing new money via seigniorage involves two changes in the model.

First, there needs to be a government budget constraint. The government deficit in each period needs to be defined so that the amount of new money that is issued can be determined. The government is consuming goods and these goods reduce those available for private consumption, so in this way seigniorage functions like a tax. To keep the government budget constraint simple, we assume that seigniorage is the only way the government can pay for its goods. Normally, a government uses a wide range of taxes to provide revenue for its expenditures and only resorts to seigniorage when the revenue from other taxes is inadequate. Here we eliminate other taxes and consider government expenditures to be just the deficit that remains, and this deficit is covered by seigniorage.

The second change to the model is that there is no longer a need for the sometimes-difficult-to-justify fiction of direct transfers of money to families. Here the government spends, and the amount of money that enters the economy each period depends on how much the government spends. As mentioned above, it is difficult to find a real-world policy that mimics the direct transfers of money to all the families in the economy, and it is not difficult to find governments that use seigniorage to cover a part of their expenditures. By the way, seigniorage is a taxing method that is particularly effective when there is a large informal sector in the economy. By its very nature, the informal sector can escape normal taxes but tends to use cash extensively. Seigniorage, which operates as a tax on the use of cash, is a way that governments can tax the informal sector. Nicolini [66] shows that countries with larger informal sectors tend to use seigniorage more than those with small informal sectors.

8.5.1 The Model

The budget constraint for the government is

$$g_t = \hat{g}_t \bar{g} = \frac{M_t - M_{t-1}}{p_t},$$

where $g_t = \hat{g}_t \bar{g}$ is the amount of real goods that the government consumes in period t. The amount $\bar{g}$ is the average government deficit, and the shock to the government deficit, $\hat{g}_t$, follows a stochastic process of the form

$$\ln \hat{g}_t = \pi \ln \hat{g}_{t-1} + \varepsilon_t^g,$$

where ε_t^g has the distribution $\varepsilon_t^g \sim N(0, \sigma_g^2)$. This stochastic process implies a stationary state value of 1 for $\hat{g}_t$. $M_t - M_{t-1}$ is the amount of new money that the government issues in period t to finance that period's deficit.

The techniques that we use for solving the model require that there exists a stationary state. As written so far, money and prices can grow without bound if $\bar{g}$ is positive. This is because, on average, the government will have to issue additional new money every period. It is helpful to normalize money and prices by dividing time t money and prices by M_t and by defining φ_t as the time t growth rate of money[6] so that $M_t = \varphi_t M_{t-1}$. Let $\hat{m}_t^i = m_t^i / M_t$, and $\hat{p}_t = p_t / M_t$. Given these definitions, the government budget constraint is written as

$$g_t = \hat{g}_t \bar{g} = \frac{\frac{M_t}{M_t} - \frac{M_{t-1}}{M_t}}{\frac{p_t}{M_t}} = \frac{1 - \frac{1}{\varphi_t}}{\hat{p}_t}.$$

6. The variable φ_t now fills the role that the variable g_t had in the previous sections.

The rest of the model is similar to the previous cash-in-advance model. Families chose a sequence of $\left\{c_t^i, h_t^i, k_{t+1}^i, \hat{m}_t^i\right\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, 1-h_t^i),$$

subject to the sequence of cash-in-advance constraints,

$$\hat{p}_t c_t^i \leq \frac{\hat{m}_{t-1}^i}{\varphi_t},$$

and the sequence of family real budget constraints,

$$k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = w_t h_t^i + r_t k_t^i + (1-\delta) k_t^i.$$

Since money enters the economy through government expenditures, we no longer have the monetary transfers in the family's cash-in-advance constraint. We will assume that the necessary conditions are met so that the cash-in-advance constraint holds with equality.

Since money is used only to purchase goods for consumption by the families or for the deficit of the government, the economy-wide cash-in-advance constraints (at equality) are

$$p_t C_t + p_t g_t = p_t C_t + p_t \hat{g}_t \bar{g} = M_t,$$

or

$$\hat{p}_t C_t + \hat{p}_t \hat{g}_t \bar{g} = 1$$

and

$$p_t C_t = M_{t-1},$$

or, dividing both sides of this equation by M_t,

$$\hat{p}_t C_t = \frac{1}{\varphi_t},$$

where aggregate consumption is defined as $C_t = \int_0^1 c_t^i di$. The real budget constraint for the economy is

$$C_t + K_{t+1} + \hat{g}_t \bar{g} = w_t H_t + r_t K_t + (1-\delta) K_t.$$

The first economy-wide cash-in-advance constraint says that aggregate consumption and government expenditures are paid for with money. The second says that aggregate consumption is paid for with money that families carried over from the previous period. At the end of a period, all the money is in the hands of families (since the workers bring home all the money received from the sale of goods). The third constraint says that real aggregate output,

$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta} = w_t H_t + r_t K_t,$$

is equal to the sum of aggregate consumption, aggregate investment (which equals $K_{t+1} - (1-\delta)K_t$), and real value of government expenditure. This third constraint comes from the assumption of competitive factor markets and a Cobb-Douglas production function. The conditions of competitive factor markets imply that

$$r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}$$

and

$$w_t = (1-\theta)\, \lambda_t K_t^{\theta} H_t^{-\theta},$$

and that firms make zero profits.

Production, wages, and rentals in each period depend on the aggregate capital stock and aggregate amount of hours worked and are taken as given by individual families. In our economy, where all the unit mass of families are identical, the aggregation conditions for capital and hours worked are

$$K_{t+1} = k_{t+1}^i$$

and

$$H_t = h_t^i.$$

Families take wages, w_t, rentals, r_t, prices, $\hat{p}_t$, and their capital stock from the previous period, k_t^i, as given. As in other models with indivisible labor, the subutility function for period t is

$$u(c_t^i, 1-h_t^i) = \ln c_t^i + Bh_t^i,$$

where $B = A \ln(1-h_0)/h_0$. Substituting the family budget constraints into the subutility function, one can write family i's optimization problem as

$$\max_{\left\{\hat{m}_t^i, k_{t+1}^i\right\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln\left(\frac{\hat{m}_{t-1}^i}{\varphi_t \hat{p}_t}\right) + B\left(\frac{k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} - r_t k_t^i - (1-\delta)k_t^i}{w_t}\right)\right].$$

This problem can be written in recursive form as

$$V\left(k_t^i, \hat{m}_{t-1}^i, \lambda_t, \hat{g}_t, K_t\right) = \max_{\hat{m}_t^i, k_{t+1}^i}\left[\ln\left(\frac{\hat{m}_{t-1}^i}{\varphi_t \hat{p}_t}\right)\right.$$

$$+ B\left(\frac{k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} - r_t k_t^i - (1-\delta)k_t^i}{w_t}\right)$$

$$\left. + \beta E_t V\left(k_{t+1}^i, \hat{m}_t^i, \lambda_{t+1}, \hat{g}_{t+1}, K_{t+1}\right)\right],$$

subject to the (degenerate) budget constraints[7]

$$k_{t+1}^i = k_{t+1}^i$$

and

$$\hat{m}_t^i = \hat{m}_t^i.$$

The first-order conditions that come from this problem are

$$\frac{1}{w_t} = \beta E_t\left[\frac{r_{t+1} + 1 - \delta}{w_{t+1}}\right]$$

and

$$-\frac{B}{\hat{p}_t w_t} = \frac{\beta}{\hat{m}_t}.$$

8.5.2 The Stationary State

The stationary state is found by setting $\hat{g}_t = 1$, $\lambda_t = 1$, and φ_t and taking all other real and normalized nominal variables, $\hat{p}$ and $\hat{m}$, as constants. The stationary state versions of the first-order conditions are

$$\frac{1}{\beta} = \bar{r} + (1-\delta)$$

and

$$-\frac{\beta \bar{w}}{B} = \frac{\hat{m}}{\hat{p}}.$$

7. We write the problem this way so that we can get the first-order conditions using the Benveniste-Scheinkman envelope theorem condition.

This last condition may seem strange intuitively. As we shall see shortly, the real wage does not depend on the inflation rate, so the family's desired real money balances at the end of each period are a constant; in particular, desired real money balances do not depend on the inflation rate. This feature of the model comes from the use of a logarithmic utility function.[8] In Appendix 1 of this chapter we find the stationary state for an economy with constant elasticity of substitution subutility for consumption and in that model inflation does have an effect on desired real money balances.

Since in aggregate $\int_0^1 \hat{m} di = 1$, the second condition gives

$$-\frac{B}{\beta\bar{w}} = \hat{p}.$$

From the conditions for competitive factor markets and the capital first-order condition, we can find the real wage rate as the constant

$$\bar{w} = (1-\theta)\left[\frac{\theta}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{\theta}{1-\theta}},$$

so the ratio of prices to money in this model is constant (and independent of the inflation rate). The government budget constraint is written as

$$\bar{g}\hat{p} = 1 - \frac{1}{\varphi}.$$

Replace $\hat{p}$ in this version of the government budget constraint with the constant $-B/\beta\bar{w}$ (= 1.0995 given the values of the parameters that we have been using), and we get the expression for the growth rate of money (and the inflation rate since $\hat{p}$, the ratio of prices to the money stock, is a constant),

$$\varphi = \frac{\beta\bar{w}}{B\bar{g} + \beta\bar{w}}.$$

For our standard calibrated economy, $\beta = .99$, $B = -2.5805$, $\delta = .025$, and $\theta = .36$, so $\bar{w} = 2.3706$, and

$$\varphi = \frac{2.3469}{2.3469 - 2.5805\bar{g}}.$$

8. When one takes the derivative of $\ln\left(\frac{\hat{m}^i_{t-1}}{\varphi_t\hat{p}_t}\right)$ with respect to money, the result is $\frac{\varphi_t\hat{p}_t}{\hat{m}^i_{t-1}}\frac{1}{\varphi_t\hat{p}_t} = \frac{1}{\hat{m}^i_{t-1}}$, and both the price level and the growth rate of money drop out. Logarithmic utility functions have the property that they allocate fixed fractions of income to each good, independent of prices.

The maximum stationary state seigniorage that the government can obtain in this economy is 0.90947. When the government is collecting this much seigniorage, it does so by creating infinite money growth every period and taxing away all of the consumption of individuals. When $g = 0.90947$, the denominator of the above equation equals zero and φ equals infinity. A Bailey curve, named for Martin Bailey, who first applied this kind of curve to seigniorage in Bailey [5], shows the real seigniorage as a function of the different stationary state inflation rates. The Bailey curve for the economy in this section is shown in Figure 8.4.

We will need the values of $\bar{C}$, $\bar{K}$, and $\bar{H}$ as coefficients in the log-linear version of the model. From above, we know that

$$\hat{p}_t C_t + \hat{p}_t \hat{g}_t \bar{g} = 1,$$

or in a stationary state that

$$\bar{C} + \bar{g} = \frac{1}{\hat{p}}$$

or

$$\bar{C} = -\frac{\beta \bar{w}}{B} - \bar{g} = .90947 - \bar{g}.$$

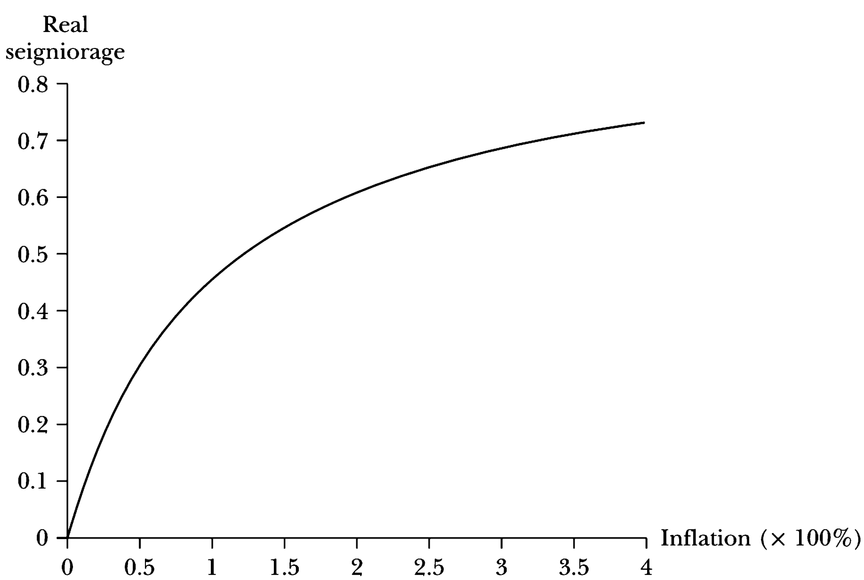

FIGURE 8.4 A Bailey curve

This corresponds with the calculation of maximum seigniorage, since the largest that $\bar{g}$ can be is .90947. It also shows why it is reasonable to speak of seigniorage as a tax on consumption, since in stationary states with higher government deficits, consumption is correspondingly lower. From the competitive factor market condition in a stationary state, we know that

$$\bar{r} = \frac{1}{\beta} - (1-\delta) = \theta \left[\frac{\bar{K}}{\bar{H}}\right]^{\theta-1}.$$

For our standard economy, $\bar{r} = .035101$, and $\bar{K}/\bar{H} = (\theta/\bar{r})^{1/(1-\theta)} = 37.99$. From the aggregate version of the family's real budget constraint in a stationary state, we have

$$\bar{K} + \frac{1}{\hat{p}} = \bar{w}\bar{H} + \bar{r}\bar{K} + (1-\delta)\bar{K},$$

or

$$\frac{1}{\hat{p}} = \left[\bar{w} + (\bar{r}-\delta)\frac{\bar{K}}{\bar{H}}\right]\bar{H}.$$

Therefore,

$$\bar{H} = \frac{\beta\bar{w}}{-B\left[\bar{w} + (\bar{r}-\delta)\frac{\bar{K}}{\bar{H}}\right]} = 0.33020,$$

and $\bar{K} = \left(\bar{K}/\bar{H}\right)\bar{H} = 12.544$.

8.5.3 Log-Linear Version of the Model

The 9 equations for the model in the 9 variables, w_t, r_t, p_t, C_t, K_t, H_t, φ_t, $\hat{g}_t$, and λ_t (after having replaced $\hat{m}_t^i$, k_t^i, and h_t^i with their aggregate values), are the first-order conditions

$$\frac{1}{w_t} = \beta E_t\left[\frac{r_{t+1}+1-\delta}{w_{t+1}}\right]$$

and

$$-\frac{B}{\beta w_t} = \hat{p}_t,$$

the cash-in-advance constraint for families, which we eliminate from the log-linear version of the model because it only determines consumption as a function of the other variables,

$$\hat{p}_t C_t = \frac{1}{\varphi_t},$$

and the cash-in-advance constraint for the government,

$$\hat{p}_t \hat{g}_t \bar{g} = 1 - \frac{1}{\varphi_t},$$

the real budget constraint for the families,

$$K_{t+1} + \frac{1}{\hat{p}_t} = w_t H_t + r_t K_t + (1 - \delta) K_t,$$

the equilibrium condition for competitive factor markets,

$$r_t = \theta \lambda_t K_t^{\theta - 1} H_t^{1-\theta}$$

and

$$w_t = (1 - \theta)\, \lambda_t K_t^{\theta} H_t^{-\theta},$$

and the laws of motion for the stochastic variables,

$$\ln \hat{g}_t = \pi \ln \hat{g}_{t-1} + \varepsilon_t^g$$

and

$$\ln \lambda_t = \gamma \ln \lambda_{t-1} + \varepsilon_t^{\lambda}.$$

Defining a log difference variable $\tilde{X}_t = \ln X_t - \ln \bar{X}$, this system can be written in log-linear form around a stationary state (indicated by a bar over the variable)[9] as

9. Stationary state values for $\hat{g}_t$ and λ_t are 1, so these disappear from the equations.

$$0 = \tilde{w}_t + \bar{r}\beta E_t\tilde{r}_{t+1} - E_t\tilde{w}_{t+1},$$

$$0 = \tilde{w}_t + \tilde{p}_t,$$

$$0 = \bar{p}\bar{g}\left[\tilde{p}_t + \tilde{g}_t\right] - \frac{1}{\bar{\varphi}}\tilde{\varphi}_t,$$

$$0 = \bar{K}\tilde{K}_{t+1} - \frac{1}{\bar{p}}\tilde{p}_t - \bar{w}\bar{H}\left[\tilde{w}_t + \tilde{H}_t\right] - \bar{r}\bar{K}\left[\tilde{r}_t + \tilde{K}_t\right] - (1-\delta)\bar{K}\tilde{K}_t,$$

$$0 = \tilde{r}_t - \tilde{\lambda}_t - (\theta - 1)\tilde{K}_t - (1-\theta)\tilde{H}_t,$$

and

$$0 = \tilde{w}_t - \tilde{\lambda}_t - \theta\tilde{K}_t + \theta\tilde{H}_t.$$

Letting $x_t = \left[\tilde{K}_{t+1}\right]$, $y_t = \left[\tilde{r}_t, \tilde{w}_t, \tilde{p}_t, \tilde{\varphi}_t, \tilde{H}_t\right]'$, and $z_t = \left[\tilde{\lambda}_t, \tilde{g}_t\right]'$, one can write the model as we did earlier in the form

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t,$$

$$0 = E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right],$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1},$$

where

$$A = \begin{bmatrix} 0 \\ 0 \\ \bar{K} \\ 0 \\ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ -\bar{K}\,(\bar{r} + 1 - \delta) \\ (1-\theta) \\ -\theta \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \bar{p}\bar{g} & -\frac{1}{\bar{\varphi}} & 0 \\ -\bar{r}\bar{K} & -\bar{w}\bar{H} & -\frac{1}{\bar{p}} & 0 & -\bar{w}\bar{H} \\ 1 & 0 & 0 & 0 & -(1-\theta) \\ 0 & 1 & 0 & 0 & \theta \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & \bar{p}\bar{g} \\ 0 & 0 \\ -1 & 0 \\ -1 & 0 \end{bmatrix},$$

$$F = [0], \; G = [0], \; H = [0],$$

$$J = [\, \beta\bar{r} \quad -1 \quad 0 \quad 0 \quad 0\,],$$

$$K = [\, 0 \quad 1 \quad 0 \quad 0 \quad 0\,],$$

$$L = [\, 0 \quad 0\,], M = [\, 0 \quad 0\,],$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix}.$$

The calculation of the laws of motion are the same as before: the matrix C is invertible and the vector of state variables (minus the stochastic variables) has only one element. The linear laws of motion that we are looking for are given by the matrices P, Q, R, and S, where

$$x_{t+1} = Px_t + Qz_t$$

and

$$y_t = Rx_t + Sz_t.$$

Using the parameter values for our standard economy and solving the linear system described above, we get the matrices shown in Table 8.5. The results shown are for the cases where stationary state seigniorage is equal to $\bar{g} = \{0, .01, .1\}$.

The shocks to seigniorage only affect the growth rate of money. The endogenous variables modeled here, including the holdings of real balances, do not respond to the shocks on seigniorage. The economy-wide cash-in-advance constraint implies that consumption will change in response to shocks on seigniorage. Since $\hat{p}_t C_t + \hat{p}_t \hat{g}_t \bar{g} = 1$, increasing $\hat{p}_t$ means that C_t must decline for this equation to hold. Seigniorage functions as a tax on consumption. The only other variable that responds is the growth rate of money, φ_t. Changes in money growth pass through to changes in the price level since the ratio of prices to money does not respond to seigniorage.

Table 8.5 Values for matrices in standard economy

$\bar{g}$	0	.01	.1
P	$\begin{bmatrix} .9697 \end{bmatrix}$	$\begin{bmatrix} .9697 \end{bmatrix}$	$\begin{bmatrix} .9697 \end{bmatrix}$
Q	$[\,.07580 \quad 0\,]$	$[\,.07580 \quad 0\,]$	$[\,.07580 \quad 0\,]$
R	$\begin{bmatrix} -0.4300 \\ 0.4781 \\ -0.4782 \\ 0 \\ -0.3282 \end{bmatrix}$	$\begin{bmatrix} -0.4300 \\ 0.4781 \\ -0.4782 \\ -0.0053 \\ -0.3282 \end{bmatrix}$	$\begin{bmatrix} -0.4300 \\ 0.4781 \\ -0.4782 \\ -0.0591 \\ -0.3282 \end{bmatrix}$
S	$\begin{bmatrix} 0.2536 & 0 \\ 0.5802 & 0 \\ -0.5802 & 0 \\ 0 & 0 \\ 1.1662 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2536 & 0 \\ 0.5802 & 0 \\ -0.5802 & 0 \\ -0.0065 & 0.0111 \\ 1.1662 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2536 & 0 \\ 0.5802 & 0 \\ -0.5802 & 0 \\ -0.0717 & 0.1235 \\ 1.1662 & 0 \end{bmatrix}$

8.6 REPRISE

Adding cash-in-advance money to Hansen's indivisible labor model has its complications. Because of the importance of aggregate variables and the price level, one cannot, in general, iterate to find the value function and a nonlinear policy function. Linear quadratic methods give a linear policy function, but finding these are relatively complicated. The most direct method for approxiating the model is to take log-linear approximations of the first-order conditions, the budget constraints, and the equilibrium conditions. There are a variety of methods for solving the linear model; here we use Uhlig's version of the undetermined coefficients method.

The results show that higher stationary state levels of inflation function as a tax, implying lower output, consumption, and utility. The dynamics of the model are independent of the stationary state inflation rate. There is only one equation in the log-linear version of the model where stationary state values of the variables enter, and all the parameters of this equation are divided by the stationary state inflation rate, leaving the relative effect of each parameter unchanged. The responses to real shocks are exactly the same as those in the Hansen model without money. The responses to money growth shocks are smaller, with prices showing the biggest response.

The cash-in-advance model is one of the workhorses of monetary theory. While it has its problems (money is exchanged only one time per period, for example), it is quite easy to use. Excellent expositions of the theory behind this model can be found in standard graduate macroeconomic texts such as Sargent [72], Blanchard and Fischer [13], or Walsh [89].

8.7 APPENDIX 1: CES UTILITY FUNCTIONS

Instead of the subutility function

$$u(c_t^i, 1 - h_t^i) = \ln c_t^i + Bh_t^i,$$

in this appendix we use the constant elasticity of substitution (in consumption), or CES, function,

$$u(c_t^i, 1 - h_t^i) = \frac{\left(c_t^i\right)^{1-\eta} - 1}{1-\eta} + Bh_t^i,$$

to consider the stationary states in an economy with seigniorage. This function is equal to the logarithmic case when $\eta = 1$. Here we assume that $\eta \neq 1$. The elasticity of substitution is equal to $1/\eta$, so when $\eta > 1$, the indifference curves for consumption between two adjacent periods are flatter than those in the logarithmic case (where the indifference curves are rectangular hyperbolas) and when $\eta < 1$, they are more curved than in the logarithmic case.

The model is the same as before except that the function that families optimize is now

$$\max_{\left\{\hat{m}_t^i, k_{t+1}^i\right\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(\frac{\hat{m}_{t-1}^i}{\varphi_t \hat{p}_t}\right)^{1-\eta} - 1}{1-\eta} + B\left(\frac{k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} - r_t k_t^i - (1-\delta)k_t^i}{w_t}\right)\right],$$

where we used the cash-in-advance constraint,

$$\hat{p}_t c_t^i = \frac{\hat{m}_{t-1}^i}{\varphi_t},$$

to remove consumption and the real budget constraint,

$$k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = w_t h_t^i + r_t k_t^i + (1-\delta)k_t^i,$$

to eliminate hours worked. The first-order conditions are

$$\frac{1}{w_t} = \beta E_t \left[\frac{r_{t+1}}{w_{t+1}} + \frac{1-\delta}{w_{t+1}}\right],$$

as before, and

$$\beta E_0 \frac{\left(\varphi_{t+1}\hat{p}_{t+1}\right)^{\eta-1}}{\left(\hat{m}_t^i\right)^{\eta}} = -\frac{B}{w_t \hat{p}_t}.$$

In a stationary state, the first condition is

$$\frac{1}{\beta} = \bar{r} + (1-\delta),$$

and the second is

$$\left(-\frac{\bar{w}\beta}{B}\right)^{\frac{1}{\eta}} \bar{\varphi}^{1-\frac{1}{\eta}} = \frac{\hat{m}^i}{\hat{p}}.$$

In the aggregate, the second condition becomes

$$\left(-\frac{\bar{w}\beta}{B}\right)^{\frac{1}{\eta}} \bar{\varphi}^{1-\frac{1}{\eta}} = \frac{1}{\hat{p}},$$

or

$$\hat{p} = \left(-\frac{B}{\bar{w}\beta}\right)^{\frac{1}{\eta}} \bar{\varphi}^{\frac{1}{\eta}-1}.$$

One can see from this last equation how when $\eta = 1$ and only when $\eta = 1$, the rate of growth of the money supply drops out of the equation and the price level is independent of the rate of growth of money. The stationary state real wage is as before,

$$\bar{w} = (1-\theta)\left[\frac{\theta}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{\theta}{1-\theta}},$$

and depends only on a set of parameters of the economy that excludes the growth rate of money.

The relationship between the growth rate of money and seigniorage in a stationary state comes from the government's budget constraint,

$$\hat{p}\bar{g} = 1 - \frac{1}{\varphi},$$

and, substituting in $\hat{p}$ from above, the relationship is

$$\left(-\frac{B}{\bar{w}\beta}\right)^{\frac{1}{\eta}} \bar{\varphi}^{\frac{1}{\eta}} \bar{g} = \varphi - 1,$$

or

$$\bar{g} = \left(-\frac{\bar{w}\beta}{B}\right)^{\frac{1}{\eta}} \bar{\varphi}^{-\frac{1}{\eta}} (\varphi - 1) .$$

The relationship between seigniorage and the growth rate of money (the Bailey curve) is shown in Figure 8.5 for stationary state economies where $\eta = .5$, $\eta = 1$, and $\eta = 2$.

With the rental on capital fixed at $\bar{r} = \frac{1}{\beta} - (1 - \delta)$, the stationary state capital-labor ratio is fixed at

$$\frac{\bar{K}}{\bar{H}} = \left[\frac{\theta}{\frac{1}{\beta} - (1 - \delta)}\right]^{\frac{1}{1-\theta}} .$$

Aggregating the real budget constraints

$$k^i_{t+1} + \frac{\hat{m}^i_t}{\hat{p}_t} = w_t h^i_t + r_t k^i_t + (1 - \delta) k^i_t$$

in a stationary state gives

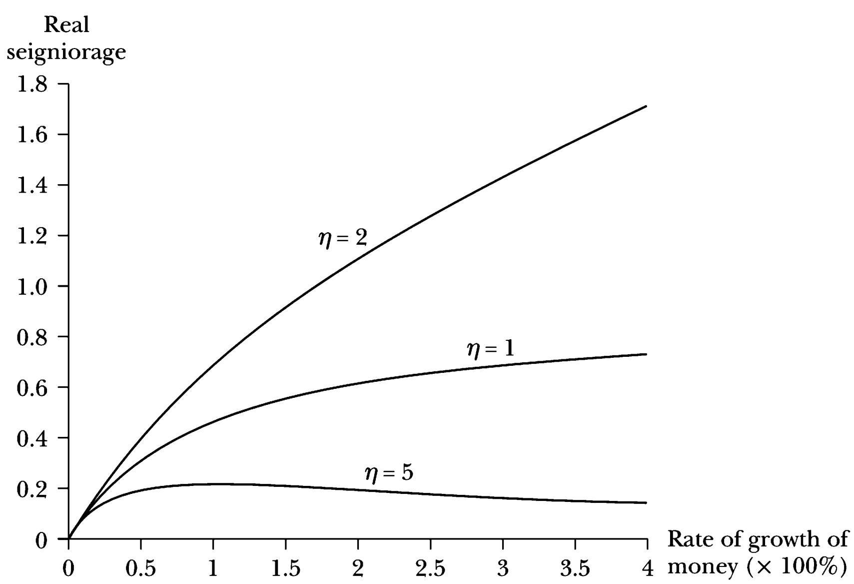

FIGURE 8.5 Bailey curves for a CES utility economy

$$\frac{1}{\hat{p}} = \left[\bar{w} + (\bar{r} - \delta) \frac{\bar{K}}{\bar{H}} \right] \bar{H}.$$

The items in the square brackets are all constant in a stationary state and equal

$$\begin{aligned}\Upsilon &= \left[\bar{w} + (\bar{r} - \delta) \frac{\bar{K}}{\bar{H}} \right] \\ &= \left[(1-\theta) \left[\frac{\theta}{\frac{1}{\beta} - (1-\delta)} \right]^{\frac{\theta}{1-\theta}} + \left(\frac{1}{\beta} - 1 \right) \left[\frac{\theta}{\frac{1}{\beta} - (1-\delta)} \right]^{\frac{1}{1-\theta}} \right] \\ &= 2.7543.\end{aligned}$$

Then

$$\begin{aligned}\bar{H} &= \frac{1}{\Upsilon} \frac{1}{\hat{p}} = \frac{1}{\Upsilon} \left(-\frac{\bar{w}\beta}{B} \right)^{\frac{1}{\eta}} \bar{\varphi}^{1-\frac{1}{\eta}} = \bar{H}_0(\eta) \bar{\varphi}^{1-\frac{1}{\eta}} \\ &= \frac{(0.9095)^{\frac{1}{\eta}}}{2.7543} \bar{\varphi}^{1-\frac{1}{\eta}},\end{aligned}$$

where $\bar{H}_0(\eta)$ is the hours worked in a stationary state with no money growth (and no seigniorage) and is a function of the coefficient on consumption in the utility function, η. Since the capital-labor ratio is constant and output is of constant returns to scale, we know that output follows the rule

$$\bar{Y} = \bar{Y}_0(\eta) \bar{\varphi}^{1-\frac{1}{\eta}}.$$

Using the aggregated cash-in-advance constraint for families in the stationary state,

$$\bar{C} = \frac{1}{\hat{p}\bar{\varphi}},$$

and eliminating prices with

$$\hat{p} = \left(-\frac{B}{\bar{w}\beta} \right)^{\frac{1}{\eta}} \bar{\varphi}^{\frac{1}{\eta}-1},$$

one gets the stationary state consumption as

$$\bar{C} = \left(-\frac{\bar{w}\beta}{B} \right)^{\frac{1}{\eta}} \bar{\varphi}^{-\frac{1}{\eta}} = \bar{C}_0 \bar{\varphi}^{-\frac{1}{\eta}}.$$

Utility in the stationary state is

$$\frac{\left[\frac{\bar{C}_0^{1-\eta}}{1-\eta} + B\bar{H}_0(\eta)\right]\bar{\varphi}^{1-\frac{1}{\eta}} - \frac{1}{1-\eta}}{1-\beta}.$$

Figure 8.6 shows how utility changes in stationary states with different rates of growth of money and with $\eta = .5$ and $\eta = 2$.

The full model is comprised of the first-order conditions

$$\frac{1}{w_t} = \beta E_t \left[\frac{r_{t+1}}{w_{t+1}} + \frac{1-\delta}{w_{t+1}}\right]$$

and

$$\beta E_0 \frac{\left(\varphi_{t+1}\hat{p}_{t+1}\right)^{\eta-1}}{(1)^{\eta}} = -\frac{B}{w_t \hat{p}_t},$$

the cash-in-advance constraint for families and the government, which we will not use in the log-linear version of the model since it only determines consumption as a function of the other variables,

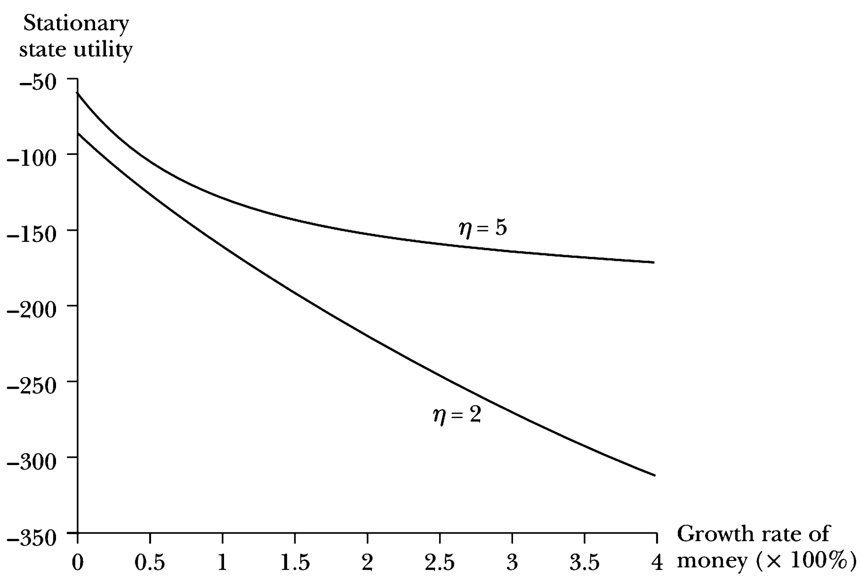

FIGURE 8.6 Utility with seigniorage

$$\hat{p}_t C_t = \frac{1}{\varphi_t}$$

and

$$\hat{p}_t \hat{g}_t \bar{g} = 1 - \frac{1}{\varphi_t},$$

the real budget constraint for the families,

$$K_{t+1} + \frac{1}{\hat{p}_t} = w_t H_t + r_t K_t + (1 - \delta) K_t,$$

the equilibrium condition for competitive factor markets,

$$r_t = \theta \lambda_t K_t^{\theta - 1} H_t^{1-\theta}$$

and

$$w_t = (1 - \theta)\, \lambda_t K_t^{\theta} H_t^{-\theta},$$

and the laws of motion for the stochastic variables,

$$\ln \hat{g}_t = \pi \ln \hat{g}_{t-1} + \varepsilon_t^g$$

and

$$\ln \lambda_t = \gamma \ln \lambda_{t-1} + \varepsilon_t^{\lambda}.$$

All of these equations are the same as in the version of this model with logarithmic utility of consumption except for one of the first-order conditions. Likewise, the log-linear version is the same except that the log-linear version of the aggregate version of the second equation is the expectational equation

$$(\eta - 1)\, (\bar{\varphi}\bar{p})^{\eta - 1} \beta E_t \left(\tilde{\varphi}_{t+1} + \tilde{p}_{t+1}\right) = \frac{B}{\bar{w}\bar{p}} \left(\tilde{w}_t + \tilde{p}_t\right).$$

Letting $x_t = \left[\tilde{K}_{t+1}, \tilde{\varphi}_t\right]$, $y_t = \left[\tilde{r}_t, \tilde{w}_t, \tilde{p}_t, \tilde{H}_t\right]'$, and $z_t = \left[\tilde{\lambda}_t, \tilde{g}_t\right]$, one can write the model as we did above,

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t \left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1},
\end{aligned}$$

but now

$$A = \begin{bmatrix} 0 & -\frac{1}{\bar{\varphi}} \\ \bar{K} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ -\bar{K}\,(\bar{r} + 1 - \delta) & 0 \\ (1-\theta) & 0 \\ -\theta & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & \bar{p}\bar{g} & 0 \\ -\bar{r}\bar{K} & -\bar{w}\bar{H} & -\frac{1}{\bar{p}} & -\bar{w}\bar{H} \\ 1 & 0 & 0 & -(1-\theta) \\ 0 & 1 & 0 & \theta \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & \bar{p}\bar{g} \\ 0 & 0 \\ -1 & 0 \\ -1 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 \\ 0 & (\eta - 1)\,(\bar{\varphi}\bar{p})^{\eta-1}\,\beta \end{bmatrix},\; G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\; H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} \beta\bar{r} & -1 & 0 & 0 \\ 0 & 0 & (\eta - 1)\,(\bar{\varphi}\bar{p})^{\eta-1}\,\beta & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{B}{\bar{w}\bar{p}} & -\frac{B}{\bar{w}\bar{p}} & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\; M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix}.$$

The system was written with the variable $\tilde{\varphi}_t$ as a state variable (part of x_t) so that the matrix C would be square and invertible. Solving the quadratic equation for P is somewhat more complicated because P is now a 2×2 matrix. A method for doing this is from Uhlig [86] and is described in Appendix 2.

Table 8.6 Values for matrices in a CES economy

$\bar{g} = .2$	$\eta = .8, \varphi = 1.3180$	$\eta = 1.8, \varphi = 1.2373$
P	$\begin{bmatrix} 0.9418 & 0 \\ -0.2049 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.9418 & 0 \\ -0.0761 & 0 \end{bmatrix}$
Q	$\begin{bmatrix} 0.1723 & 0.0020 \\ -0.1628 & 0.3282 \end{bmatrix}$	$\begin{bmatrix} 0.1305 & -0.0035 \\ -0.0773 & 0.2232 \end{bmatrix}$
R	$\begin{bmatrix} -0.8612 & 0 \\ 0.4844 & 0 \\ -0.6444 & 0 \\ -0.3456 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.1018 & 0 \\ 0.6198 & 0 \\ -0.3208 & 0 \\ -0.7216 & 0 \end{bmatrix}$
S	$\begin{bmatrix} 2.1478 & -0.0037 \\ 0.3544 & 0.0021 \\ -0.5118 & 0.0322 \\ 1.7934 & -0.0058 \end{bmatrix}$	$\begin{bmatrix} 1.5802 & 0.0083 \\ 0.6736 & -0.0047 \\ -0.3258 & -0.0593 \\ 0.9066 & 0.0130 \end{bmatrix}$

The results for the case where $\bar{g} = .2$ and $\eta = .8$ and $\eta = 1.8$ are given in Table 8.6.

8.8 APPENDIX 2: MATRIX QUADRATIC EQUATIONS

Following Uhlig [86], we look for a solution to a matrix quadratic equation,

$$AP^2 - BP - C = 0,$$

of $P = \Psi\Lambda\Psi^{-1}$, where Λ is a matrix of eigenvalues on the diagonal of the form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \lambda_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix},$$

and Ψ is a matrix with the corresponding eigenvectors. Notice that this form of writing P implies that $P^2 = \Psi\Lambda\Psi^{-1}\Psi\Lambda\Psi^{-1} = \Psi\Lambda^2\Psi^{-1}$, where Λ^2 is a diagonal matrix with the squares of the eigenvalues along the diagonal. Uhlig's method for solving the matrix quadratic equation exploits this feature of the decomposition of P into eigenvalues and eigenvectors.

The matrices A, B, and C are all $n \times n$. Construct the $2n \times 2n$ matrices

$$D = \begin{bmatrix} B & C \\ I & \vec{0} \end{bmatrix}$$

and

$$E = \begin{bmatrix} A & \vec{0} \\ \vec{0} & I \end{bmatrix},$$

where I is an $n \times n$ identity matrix and $\vec{0}$ is an $n \times n$ matrix of zeros. Find the solution to the generalized eigenvalue problem for the matrix pair (D, E). The solution to this problem is a set of $2n$ eigenvalues λ_k and corresponding eigenvectors x_k, such that

$$Dx_k = Ex_k\lambda_k$$

for $k = 1, \ldots, 2n$. Let X be the matrix of all eigenvectors. Assume that there are at least n stable eigenvectors, those whose absolute value is less than one. Order the eigenvalues and their corresponding eigenvectors so that the n stable eigenvalues come first. The eigenvectors are columns, so that the matrix X is

$$X = \begin{bmatrix} x_{1,1} & x_{2,1} & \cdots & \cdots & x_{2n,1} \\ x_{1,2} & x_{2,2} & \vdots & \vdots & x_{2n,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,2n} & x_{2,2n} & \cdots & \cdots & x_{2n,2n} \end{bmatrix}.$$

The matrix X can be broken into four parts, each $n \times n$, of the form

$$X = \begin{bmatrix} X^{11} & X^{21} \\ X^{12} & X^{22} \end{bmatrix}$$

$$= \begin{bmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n,1} & x_{n+1,1} & x_{n+2,1} & \cdots & x_{2n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n,2} & x_{n+1,2} & x_{n+2,2} & \cdots & x_{2n,2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} & x_{n+1,n} & x_{n+2,n} & \cdots & x_{2n,n} \\ x_{1,n+1} & x_{2,n+1} & \cdots & x_{n,n+1} & x_{n+1,n+1} & x_{n+2,1} & \cdots & x_{2n,n+1} \\ x_{1,n+2} & x_{2,n+2} & \cdots & x_{n,n+2} & x_{n+1,n+2} & x_{n+2,2} & \cdots & x_{2n,n+2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{1,2n} & x_{2,2n} & \cdots & x_{n,2n} & x_{n+1,2n} & x_{n+2,2n} & \cdots & x_{2n,2n} \end{bmatrix}.$$

The matrix X^{11} contains the first half of the generalized eigenvectors that correspond to the n stable eigenvalues. The structure of the matrices that were constructed for the generalized eigenvalue problem tell us something about the matrix X^{21}. Solutions to the generalized eigenvalue problem are

$$DX = EX\Delta,$$

where Δ is the diagonal matrix of generalized eigenvalues with the n stable eigenvalues coming first. Using the descriptions of the matrices given above, this corresponds to

$$\begin{bmatrix} B & C \\ I & \vec{0} \end{bmatrix}\begin{bmatrix} X^{11} & X^{21} \\ X^{12} & X^{22} \end{bmatrix} = \begin{bmatrix} A & \vec{0} \\ \vec{0} & I \end{bmatrix}\begin{bmatrix} X^{11} & X^{21} \\ X^{12} & X^{22} \end{bmatrix}\begin{bmatrix} \Delta^1 & \vec{0} \\ \vec{0} & \Delta^2 \end{bmatrix},$$

where Δ^1 contains the stable eigenvalues along the diagonal and zeros everywhere else. Multiplying out the matrices on each side gives

$$\begin{bmatrix} BX^{11} + CX^{12} & BX^{21} + CX^{22} \\ X^{11} & X^{21} \end{bmatrix} = \begin{bmatrix} AX^{11}\Delta^1 & AX^{21}\Delta^2 \\ X^{12}\Delta^1 & X^{22}\Delta^2 \end{bmatrix}.$$

Each of the four $n \times n$ matrices on the left-hand side of the equals sign must be equal to the corresponding matrix on the right-hand side. In particular,

$$X^{11} = X^{12}\Delta^1$$

and

$$BX^{11} + CX^{12} = AX^{11}\Delta^1.$$

Substituting in $X^{12}\Delta^1$ for X^{11} in the second equation gives

$$BX^{12}\Delta^1 + CX^{12} = AX^{12}\Delta^1\Delta^1,$$

and postmultiplying both sides by $\left(X^{12}\right)^{-1}$ gives

$$BX^{12}\Delta^1\left(X^{12}\right)^{-1} + C = AX^{12}\Delta^1\Delta^1\left(X^{12}\right)^{-1}.$$

Define $P = X^{12}\Delta^1\left(X^{12}\right)^{-1}$. Then $P^2 = X^{12}\Delta^1\Delta^1\left(X^{12}\right)^{-1}$ and

$$BP + C = AP^2.$$

Therefore, the solution to the matrix quadratic equation can be found by constructing the matrices D and E and finding the solution to the generalized eigenvalue problem for those matrices as the generalized eigenvector matrix X and the generalized eigenvalue matrix Δ (ordered appropriately, with the stable eigenvalues first). The matrix Δ^1 contains the eigenvalues, and the matrix X^{12} contains the eigenvectors that we use to construct

$$P = X^{12}\Delta^1\left(X^{12}\right)^{-1}.$$

8.9 MATLAB CODE FOR SOLVING THE CES MODEL WITH SEIGNIORAGE

This program finds the solution to a matrix quadratic polynomial in order to solve for the dynamics of the seigniorage model in an economy with CES utility.

```
%solution to the cash in advance model with a CES utility function
%set the parameters and find the stationary state
theta=.36;
beta=.99;
gamma=.95;
delta=.025;
pie=.48;
varphi=1.3180;
eta=.8;
BB=-2.5805;
rbar=1/beta-(1-delta);
wbar=(1-theta)*(theta/rbar)^(theta/(1-theta));
gbar=(-wbar*beta/(BB*varphi))^(1/eta)*(varphi-1);
KoverH=(theta/rbar)^(1/(1-theta));
pbar=(-BB/(wbar*beta))^(1/eta)*(varphi)^(1/eta-1);
hibar=1/(pbar*(wbar+(rbar-delta)*KoverH));
Hbar=hibar;
kibar=KoverH*Hbar;
Kbar=kibar;
cbar=1/(pbar*varphi);
mbar=1;
%build the required matrices A through N
A=[0 -1/varphi
   Kbar 0
   0 0
   0 0];
B=[0 0
   -(rbar+1-delta)*Kbar 0
   1-theta 0
   -theta 0];
C=[0 0 pbar*gbar 0
   -rbar*Kbar -wbar*Hbar -1/pbar -wbar*Hbar
   1 0 0 -(1-theta)
   0 1 0 theta];
```

```
D=[0 pbar*gbar
   0 0
   -1 0
   -1 0];
F=[0 0
   0 (eta-1)];
G=[0 0
   0 0];
H=[0 0
   0 0];
J=[beta*rbar -1 0 0
   0 0 (eta-1) 0];
K=[0 1 0 0
   0 1 1 0];
L=[0 0
   0 0];
M=[0 0
   0 0];
N=[gamma 0
   0 pie];
%Set up and find the solution to the matrix quadratic
polynomial
I1=[1 0
    0 1];
Z1=zeros(2);
invC=inv(C);
psy=F-J*invC*A;
lambda=J*invC*B-G+K*invC*A;
T=K*invC*B-H;
AA1=[lambda T
     I1 Z1];
AA2=[psy Z1
     Z1 I1];
%find the generalized eigenvalues and eigenvectors
[eigvec eigval]=eig(AA1,AA2);
diageigval=diag(eigval)
%select the stable eigenvalues and the corresponding
eigenvalues
iz=find(abs(diageigval)<1);
DD=diag(diageigval(iz));
ei=size(eigvec);
EE=eigvec(ei/2+1:ei,iz);
% find the matrices P, R, Q, and S
P=EE*DD*inv(EE)
R=-invC*(A*P+B)
```

```
I2=[1 0
    0 1];
QQ=kron(N',(F-J*invC*A))+kron(I2,(J*R+F*P+G-K*invC*A));
invQQ=inv(QQ);
QQQ=((J*invC*D-L)*N+K*invC*D-M);
[aa,bb]=size(QQQ);
Qfindvert=[];
for ij=1:bb
   Qfindvert=vertcat(Qfindvert,QQQ(:,ij));
end
Qvert=invQQ*Qfindvert;
Q=[];
for ij=1:bb
   begini=(ij-1)*aa+1;
   endi=ij*aa;
   Q=[Q Qvert(begini:endi,1)];
end
Q
S=-invC*(A*Q+D)
```