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The Basic Solow Model

Solow [80, 82] introduced a model of economic growth that has served as the basis for most growth theory, for Real Business Cycle models and for New Keynesian modeling. This model is quite simple, but elegant, and generates a number of very specific and testable results about growth. Not all of these results are confirmed by the data, however, and, after the initial enthusiasm for the Solow model, this fact first had the result of causing the model to fall into disuse. Later revisions of the model by Paul Romer [69] and by Robert Lucas [56] have made the model more consistent with the growth data, and it has once again become popular. Kydland and Prescott [52] used a stochastic version of this model to study business cycles and thus started the branch of macroeconomics now known as Real Business Cycle theory (including New Keynesian).

Solow's model is quite simple: there is a constant returns-to-scale production function, a law for the evolution of capital, and a savings rate. Equilibrium conditions are simply that investment equals savings. From this basis, a first-order difference equation for the evolution of capital per worker is found, and the time path of the economy springs from this equation.

1.1 THE BASIC MODEL

The production function is

$$Y_t = A_t F(K_t, H_t),$$

where Y_t is output of the single good in the economy at date t, A_t is the level of technology, K_t is the capital stock, and H_t is the quantity of labor used in production. The production function is homogeneous of degree one with the usual properties. Technology is equal to $A_t = (1+\alpha)^t A_0$, where A_0 is the time 0 level of technology and α is the net rate of growth of technology. Writing this in terms of output per worker, $y_t = Y_t/H_t$, and using the constant returns-to-scale properties of the function $F(K_t, H_t)$,[1] we get

$$y_t = \frac{Y_t}{H_t} = A_t F(\frac{K_t}{H_t}, \frac{H_t}{H_t}) = A_t F(k_t, 1) \equiv A_t f(k_t),$$

where $k_t \equiv K_t/H_t$ is the capital per worker.

Assume that the labor force grows at a constant net rate n, so that $H_{t+1} = (1+n)H_t$, and that capital grows by

$$K_{t+1} = (1-\delta)K_t + I_t,$$

where δ is the rate of depreciation and I_t is investment at time t. With these assumptions, capital per worker grows according to the rule

$$k_{t+1} = \frac{(1-\delta)k_t + i_t}{1+n},$$

where $i_t = I_t/H_t$ is the inversion per worker in time t. Savings is defined as a fixed fraction of output,

$$s_t = \sigma y_t,$$

where σ is the fraction of output per worker that is saved. This assumption about savings is probably the most important simplification of the Solow model. In equilibrium in a closed economy, $i_t = s_t$. First substituting savings for investment in the capital growth rule, replacing savings by the constant times output, and finally replacing output by the production function per worker, one gets the difference equation,

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma(1+\alpha)^t A_0 f(k_t).$$

In the simplest case, where technology growth is zero, $\alpha = 0$, this results in the equation

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma A_0 f(k_t).$$

1. The usual assumptions about the shape of the production function are that $F_i(K_t, H_t) > 0$, $F_{ii}(K_t, H_t) < 0$, $F_{ij}(K_t, H_t) > 0$, for $i = K, H$ and $i \neq j$. $F_i(K_t, H_t) \to \infty$ as the quantity of factor i goes to 0, and $F_i(K_t, H_t) \to 0$ as the quantity of factor i goes to ∞. Also, $F(K_t, 0) = F(0, H_t) = 0$.

A stationary state can be found from this equation for the case where $k_{t+1} = k_t = \bar{k}$. This stationary state occurs when

$$(1+n)\bar{k} = (1-\delta)\bar{k} + \sigma A_0 f(\bar{k}),$$

or when

$$(\delta + n)\bar{k} = \sigma A_0 f(\bar{k}).$$

Given the conditions of the function $f(\cdot)$, there is a stationary state at $\bar{k} = 0$, and one for a positive $\bar{k}$. All economies with $k_0 \neq 0$ converge to the positive stationary state.

The stability conditions of the positive stationary state can be seen from the equation

$$k_{t+1} = g(k_t) = \frac{(1-\delta)k_t + \sigma A_0 f(k_t)}{(1+n)} \tag{1.1}$$

shown in Figure 1.1. Notice that between 0 and the positive $\bar{k}$, the function, $g(k_t)$, is above the 45 degree line, so that k_{t+1} is greater that k_t. In this range, capital per worker is growing and converges to the positive $\bar{k}$. Above the positive $\bar{k}$, the value of the function, $g(k_t)$, is less than k_t, so the capital stock declines, converging to the positive $\bar{k}$.

One of the important results of the Solow growth model is that, if all economies have access to the same technology, poorer ones (those with less initial

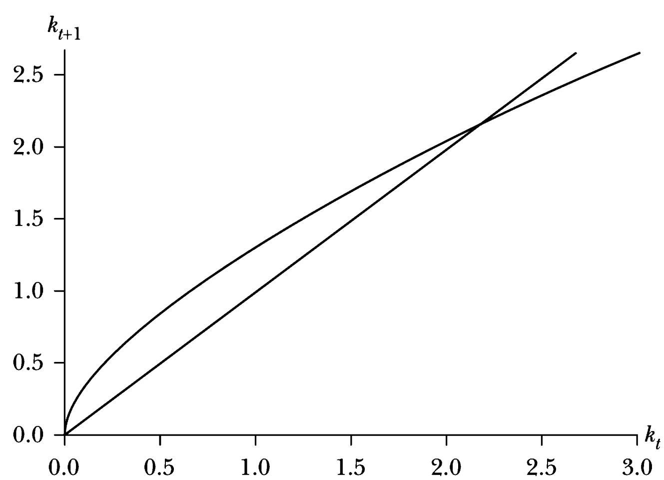

FIGURE 1.1 Stability conditions

capital) will grow faster than richer ones (those with more initial capital). Let $\gamma_t = k_{t+1}/k_t$ be the gross growth of the capital stock from time t to time $t+1$. Using equation 1.1, we can write the growth rate as

$$\gamma_t = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + \sigma A_0 f(k_t)}{(1+n)k_t}.$$

To see how the growth rate depends on the initial capital stock, we take the derivative of γ_t with respect to k_t. This results in

$$\frac{d\gamma_t}{dk_t} = \frac{\sigma A_0}{(1+n)k_t^2}\left[f'(k_t)k_t - f(k_t)\right],$$

which is negative when $k_t > 0$. The growth rate of capital declines as the capital stock increases. Since output per worker is increasing in per worker capital stock, the growth rate of output per worker also declines.

The main results of the simplest version of the Solow model are, when all countries have access to the same technology and all have the same savings rate, that all countries converge to the same levels of capital and output per worker and that poorer countries grow faster than richer ones.

EXERCISE 1.1 Suppose that in every period an economy needs to pay 4 percent of its output to the rest of the world as interest payments on past government debt. The government collects the funds for the transfer using lump sum taxes. How is the growth rate of this country different from one that does not need to make the payments?

EXERCISE 1.2 What happens to the rental rate on capital (the marginal product of capital) as an economy grows from some initial per worker capital stock below the positive stationary state?

1.2 TECHNOLOGICAL GROWTH

With a constant rate of technological growth, all economies converge toward what is called a *balanced growth path*. This is a growth path where the growth rate of capital and output are constant. For an economy with a Cobb-Douglas production function, $f(k_t) = k_t^\theta$, where θ is the fraction of the economy's income that goes to capital, this constant growth path is easy to find. We look for an equilibrium where capital grows at some unknown constant rate, $\gamma = k_{t+1}/k_t$. Using the difference equation with technological growth and normalizing initial capital, $A_0 = 1$, we get

$$\gamma = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + \sigma(1+\alpha)^t k_t^\theta}{(1+n)k_t} = \frac{(1-\delta)}{(1+n)} + \frac{\sigma(1+\alpha)^t}{(1+n)k_t^{1-\theta}}.$$

Rewriting this, we get

$$k_t = \left[\frac{\sigma(1+\alpha)^t}{(1+n)\gamma - (1-\delta)}\right]^{\frac{1}{1-\theta}} = (1+\alpha)^{\frac{t}{1-\theta}} \left[\frac{\sigma}{(1+n)\gamma - (1-\delta)}\right]^{\frac{1}{1-\theta}}.$$

This implies that along a balanced growth path, the constant growth rate of capital per worker, γ, must be equal to

$$\lambda = \frac{k_{t+1}}{k_t} = \frac{(1+\alpha)^{\frac{t+1}{1-\theta}} \left[\frac{\sigma}{(1+n)\gamma-(1-\delta)}\right]^{\frac{1}{1-\theta}}}{(1+\alpha)^{\frac{t}{1-\theta}} \left[\frac{\sigma}{(1+n)\gamma-(1-\delta)}\right]^{\frac{1}{1-\theta}}} = (1+\alpha)^{\frac{1}{1-\theta}}.$$

As it turns out, along this path output per worker grows by

$$\begin{aligned}\frac{y_{t+1}}{y_t} &= \frac{(1+\alpha)^{t+1}k_{t+1}^\theta}{(1+\alpha)^t k_t^\theta} = (1+\alpha)\left(\frac{k_{t+1}}{k_t}\right)^\theta \\ &= (1+\alpha)\left((1+\alpha)^{\frac{1}{1-\theta}}\right)^\theta = (1+\alpha)^{\frac{1}{1-\theta}},\end{aligned}$$

which is the same rate as the growth rate of capital per worker.

1.3 THE GOLDEN RULE

To maximize welfare in a stationary state, given that welfare is a function of only consumption, one wants to maximize the steady state level of consumption. We consider welfare only in an economy without technological growth. What is a variable in this case is the savings rate. The savings rate that maximizes consumption is known as the *golden rule* savings rate. The golden rule was first developed by Phelps [68]. Since production can be either saved (invested) or consumed, the per worker consumption in each period is equal to

$$c_t = (1-\sigma)y_t = (1-\sigma)A_0 f(\bar{k}),$$

where the use of $\bar{k}$ indicates that we are looking at stationary states. For the model without technical growth, the condition for a stationary state is

$$(\delta + n)\bar{k} = \sigma A_0 f(\bar{k}).$$

Substituting the condition for the stationary state into the equation for consumption yields

$$\bar{c} = A_0 f(\bar{k}) - (\delta + n)\bar{k}.$$

First-order conditions for maximizing consumption give

$$A_0 f'(\bar{k}) = \delta + n.$$

Put the value of $\bar{k}^*$ that solves the above equation in

$$(\delta + n)\bar{k}^* = \sigma A_0 f(\bar{k}^*)$$

and solve for the savings rate, σ. The golden rule value of σ is

$$\sigma = \frac{(\delta + n)\bar{k}^*}{A_0 f(\bar{k}^*)}.$$

1.4 A STOCHASTIC SOLOW MODEL

Adding a stochastic shock to the standard Solow model is relatively simple, although there are a number of ways that it could be done. Among the variables that could be made to follow some simple stochastic process are the technology level, the discount factor, the savings rate, or the growth rate of population. In our version of the model, a technology shock and a shock to the savings rate are essentially identical in the way they affect the evolution of output. We develop a version where technology is stochastic. The effects of stochastic technology growth are the basis for Real Business Cycle theory. Changes in technology that come from trying to match the Solow model to the data are often referred to as the Solow residual (see Solow [81]).

There are several common ways of defining the stochastic process for technology. One is a first-order moving average of the form

$$A_t = \psi \bar{A} + (1 - \psi) A_{t-1} + \varepsilon_t,$$

where $0 < \psi < 1$ and $\bar{A}$ is a non-negative constant. In this case, the probability distribution of ε_t needs to be bounded from below by $-\psi \bar{A}$ or, with some positive probability, technology will become negative. A lower bound is not normally a problem if one is simulating an economy, but it does make the analytics more complicated. An alternative that does not have the lower bound problem is to define

$$A_t = \bar{A} e^{\varepsilon_t},$$

where ε_t has a normal distribution with mean 0. In this case, A_t has a log normal distribution of the form

$$\ln A_t = \ln \bar{A} + \varepsilon_t.$$

We will use the second formulation and suppose that technology follows the stochastic process

$$A_t = \bar{A}e^{\varepsilon_t},$$

where $\bar{A}$ is positive and the random term ε_t has a normal distribution with mean zero. The first-order difference equation that describes the time path of the economy in the model is mechanical,[2] in the sense that the savings rate is a constant, and equation 1.1 is written simply as

$$k_{t+1} = \frac{(1-\delta)k_t + \sigma \bar{A} e^{\varepsilon_t} f(k_t)}{(1+n)}. \tag{1.2}$$

Divide both sides of this equation by k_t to get the growth rate, $\gamma_t = k_{t+1}/k_t$, on the left-hand side, rearrange, and then take logarithms. One ends up with

$$\ln \left[\gamma_t - \frac{1-\delta}{1+n}\right] = \ln \frac{\sigma \bar{A}}{(1+n)} + \ln \frac{f(k_t)}{k_t} + \varepsilon_t. \tag{1.3}$$

As before, we assume a Cobb-Douglas production function, the per worker production function is $f(k_t) = k_t^{\theta}$, and equation 1.3 becomes

$$\ln \left[\gamma_t - \frac{1-\delta}{1+n}\right] = \varphi - (1-\theta)\ln k_t + \varepsilon_t,$$

where the constant $\varphi \equiv \ln\left[\sigma \bar{A}/(1+n)\right]$. The gross growth rate of per worker capital is a nonlinear function of the current per worker capital stock and of the shocks. Variance of the growth rate of per worker capital depends on the initial level of per worker capital as well as the variance of the shocks.

Figure 1.2 shows three runs of simulations of the exact stochastic version of the Solow model given in equation 1.2. The parameters used are $\bar{A} = 1$, $n = .02$, $\delta = .1$, $\theta = .36$, $\sigma = .2$, and the standard error of the shock is .2.

2. The process is mechanical in the sense that there is no feedback from the technology shock to the decision about savings. A model with habit formation would imply that the savings rate would decline in response to a negative shock to output. Here the savings rate is always σ.

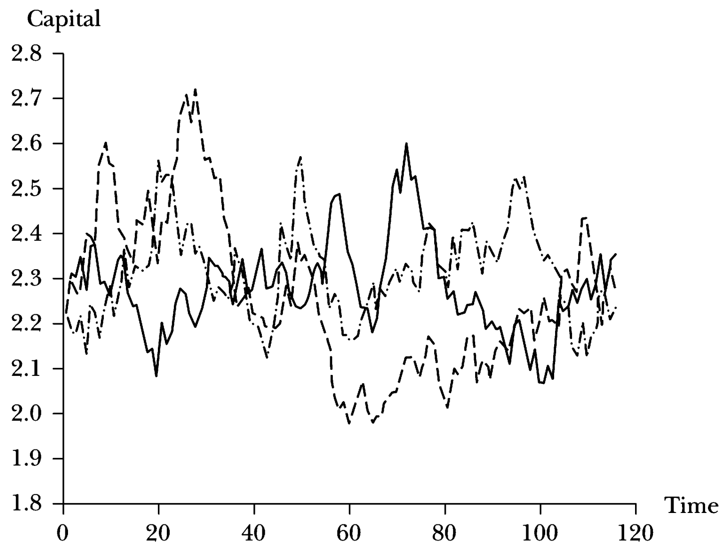

FIGURE 1.2 Three simulations of the exact Solow model

1.5 LOG-LINEAR VERSION OF THE SOLOW MODEL

Given that the model is nonlinear, a simple expression for the variance of per worker capital in terms of the shocks is not easy to find or even to define in general. However, it is possible to find a linear approximation of the model around a stationary state and to use this version to study some of the second-order characteristics of the model. In particular, one is interested in the size of the technology shock that is required so that the variance of output is similar to that of real economies around their long-term trend.[3] We will be using log-linear versions of a number of our future models, so this section is very important for understanding the process of producing a log-linear version of a model.

Define the log differences[4] of a variable, $\tilde{X}_t$, as

$$\tilde{X}_t = \ln X_t - \ln \bar{X},$$

where X_t is the time t value of the variable and $\bar{X}$ is its value in the stationary state. This definition of the log differences lets us write the variable as

3. We speak of variance around the trend since the version we are using here excludes technological growth or any other trend. Various filters (one of the most popular is the Hodrick-Prescott) are used to detrend the series.

4. More extended material on this topic can be found in Uhlig [86] and in Section 6.2.2.

$$X_t = \bar{X} e^{\tilde{X}_t}.$$

To find a log-linear version of a model, it is frequently possible simply to replace the variable by the version in log differences. The following rules for first-order approximations are useful for small values of $\tilde{X}_t$ and $\tilde{Y}_t$:

$$e^{\tilde{X}_t} \approx 1 + \tilde{X}_t,$$

$$e^{\tilde{X}_t + a\tilde{Y}_t} \approx 1 + \tilde{X}_t + a\tilde{Y}_t,$$

$$\tilde{X}_t \tilde{Y}_t \approx 0,$$

$$E_t\left[a e^{\tilde{X}_{t+1}}\right] \approx E_t\left[a\tilde{X}_{t+1}\right] + \text{ a constant.}$$

1.5.1 Capital

For the Solow model, we begin with a stochastic, Cobb-Douglas, zero technological growth version of the first-order difference equation,

$$(1+n)\, k_{t+1} = (1-\delta)k_t + \sigma \bar{A} e^{\varepsilon_t} k_t^{\theta},$$

and replace k_j by $\bar{k}e^{\tilde{k}_j}$, where $\tilde{k}_j = \ln k_j - \ln \bar{k}$. This yields

$$(1+n)\, \bar{k} e^{\tilde{k}_{t+1}} = (1-\delta)\bar{k} e^{\tilde{k}_t} + \sigma \bar{A} e^{\varepsilon_t} \bar{k}^{\theta} e^{\theta \tilde{k}_t},$$

which becomes, using the first rule of approximation given above,

$$(1+n)\, \bar{k}\left(1+\tilde{k}_{t+1}\right) = (1-\delta)\bar{k}\left(1+\tilde{k}_t\right) + \sigma \bar{A}\left(1+\varepsilon_t\right)\bar{k}^{\theta}\left(1+\theta\tilde{k}_t\right).$$

In the nonstochastic stationary state

$$(1+n)\, \bar{k} = (1-\delta)\bar{k} + \sigma \bar{A} \bar{k}^{\theta}.$$

Removing the nonstochastic stationary state terms from the approximation above gives

$$(1+n)\, \bar{k}\tilde{k}_{t+1} = (1-\delta)\bar{k}\tilde{k}_t + \sigma \bar{A} \varepsilon_t \bar{k}^{\theta} + \sigma \bar{A} \bar{k}^{\theta} \theta \tilde{k}_t + \sigma \bar{A} \bar{k}^{\theta} \theta \varepsilon_t \tilde{k}_t.$$

Since $\varepsilon_t \tilde{k}_t \approx 0$, this becomes

$$(1+n)\, \bar{k}\tilde{k}_{t+1} = (1-\delta)\bar{k}\tilde{k}_t + \sigma \bar{A} \varepsilon_t \bar{k}^{\theta} + \sigma \bar{A} \bar{k}^{\theta} \theta \tilde{k}_t.$$

Combining terms, we arrive at the first-order difference equation,

$$\tilde{k}_{t+1} = B\tilde{k}_t + C\varepsilon_t, \tag{1.4}$$

where

$$B = \frac{1-\delta}{1+n} + \frac{\theta\sigma\bar{A}\bar{k}^{\theta-1}}{1+n} = \frac{1-\delta}{1+n} + \frac{\theta(\delta+n)}{1+n} = \frac{1+\theta n - \delta(1-\theta)}{1+n} < 1,$$

and

$$C = \frac{\sigma\bar{A}\bar{k}^{\theta-1}}{1+n} = \frac{\delta+n}{1+n}.$$

Recursively substituting $\tilde{k}_{t-j}$, for $j = 0, \ldots, \infty$ in equation 1.4 results in the approximation

$$\tilde{k}_{t+1} = C\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}. \tag{1.5}$$

We can use expression 1.5 to calculate the variance of capital around its stationary state, $\tilde{k}_{t+1}$, as

$$\operatorname{var}\left(\tilde{k}_{t+1}\right) = E\left[\tilde{k}_{t+1}\tilde{k}_{t+1}\right] = E\left[\left(C\sum_{i=0}^{\infty} B^i\varepsilon_{t-i}\right)\left(C\sum_{i=0}^{\infty} B^i\varepsilon_{t-i}\right)\right].$$

If the technology shocks are independent, in the sense that $E\left[\varepsilon_t\varepsilon_s\right] = 0$ if $t \neq s$, the above expression becomes

$$\operatorname{var}\left(\tilde{k}\right) = C^2\sum_{i=0}^{\infty} B^{2i}\operatorname{var}(\varepsilon) = \frac{C^2}{1-B^2}\operatorname{var}(\varepsilon),$$

where

$$\frac{C^2}{1-B^2} = \frac{[\delta+n]^2}{[1+n]^2 - [1+\theta n - \delta(1-\theta)]^2}.$$

Assuming that $\delta = 0.1$, $\sigma = .2$, $n = 0.02$, and $\theta = 0.36$, we find that the variance of capital around its stationary state is 0.0955 times the variance of the shock to technology. Figure 1.3 shows a simulation for the log-linear version of the Solow model along with one of the exact model using the same error process.

1.5.2 Output

To determine the variance of the technology shock relative to that of output, we first find a log-linear version of the production function. This is found from

$$y_t = \bar{A}e^{\varepsilon_t}k_t^{\theta}$$

by replacing the variables with shocks around stationary states to get

$$\bar{y}e^{\tilde{y}_t} = \bar{A}e^{\varepsilon_t}\bar{k}^{\theta}e^{\theta\tilde{k}_t},$$

which is approximated by

$$\bar{y}(1+\tilde{y}_t) = \bar{A}(1+\varepsilon_t)\bar{k}^{\theta}(1+\theta\tilde{k}_t).$$

This can be further simplified by removing the stationary state value of output to get

$$\bar{y}\tilde{y}_t = \bar{A}\bar{k}^{\theta}\varepsilon_t + \bar{A}\bar{k}^{\theta}\theta\tilde{k}_t + \bar{A}\bar{k}^{\theta}\varepsilon_t\theta\tilde{k}_t.$$

Since $\varepsilon_t\tilde{k}_t \approx 0$, and $\bar{y} = \bar{A}\bar{k}^{\theta}$, this expression becomes

$$\tilde{y}_t = \varepsilon_t + \theta\tilde{k}_t.$$

To find the approximate variance of $\widehat{y}_t$, we use the fact that $\tilde{k}_t$ is independent of ε_t (see equation 1.5) and calculate

$$\text{var}\left(\tilde{y}_t\right) = E\left[\tilde{y}_t\tilde{y}_t\right] = E\left[\left(\varepsilon_t + \theta\tilde{k}_t\right)\left(\varepsilon_t + \theta\tilde{k}_t\right)\right] = \text{var}(\varepsilon_t) + \theta^2\text{var}(\tilde{k}_t).$$

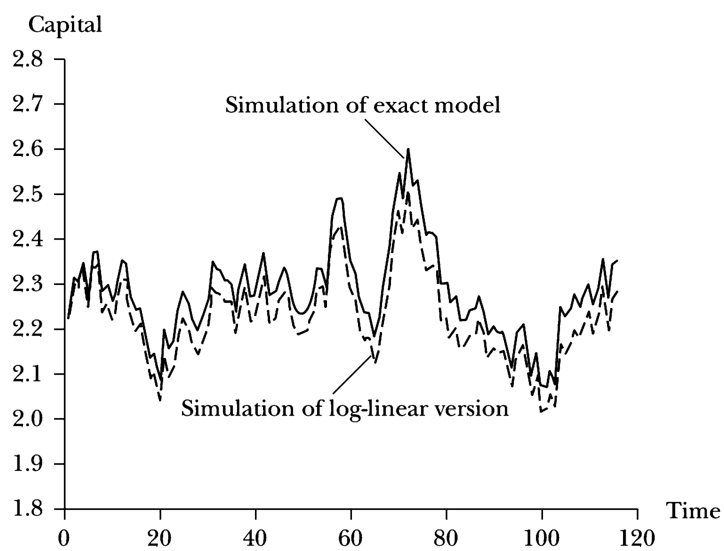

FIGURE 1.3 Simulations of the exact and log-linear Solow model

Using the same values as we did to find the variance of capital relative to that of the technology shock, we find that var $(\tilde{y}_t) = 1.0123$ var(ε_t). This means that for the Solow model, the variance in output is almost exactly equal to the variance in the technology shock. In the model, the shocks show very little persistence. With our parameter values, a technology shock in period t accounts for only 1.23 percent of the variance in output in period $t+1$. The Solow model does not do a good job of explaining either the variance in output or the persistence of output shocks.

EXERCISE 1.3 Find the covariance between the deviation of output from the stationary state at time t, $\tilde{y}_t$, and the deviation of output from the stationary state at time $t-1$, $\tilde{y}_{t-1}$, in terms of the variance of the technology shock.

1.6 REPRISE

The growth model presented in this chapter is not an optimizing model; the quantity of savings is fixed and is not determined by a decision of the consumers or the firms. Nonetheless, a number of rich characteristics can come from the model: issues of stability, importance of capital, and the possibility of producing time series that have some of the properties of those observed for real economies. Solow models fail in some important ways. One of their predictions is that, in general, poor countries should grow faster than rich ones. This prediction is not very well supported by the data (see Parente and Prescott [67], for example).

The log linearization of the model that we did in this chapter is not really necessary for generating stochastic time series, but it gives us an early example of the techniques that we will be using later in the book.

There is an enormous literature on the Solow growth model. For those interested in pursuing this theme, a good standard reference is Barro and Sala-i-Martin [6].