# Macroeconomics II
# Variational methods (Soviet)

Prof. George McCandless

June 21, 2022

# Synthisis of Variational methods

- Method of optimal control developed by the Soviets in the 1950's
- Breaks infinite horizon problem into one involving three periods
  - a function in time t-1, t, and t+1
- Assume you know what the values were at time t-1
- Assume you know what you want the value to be at time t+1
- solve for optimal behavior at time t
- if you want a stationary state, let $x_{t-1} = x_t = x_{t+1} = x$ and solve for the $x$'s that work
- Non-stationary states can be found
- Have beginning t=0 values and final t=N values (N is number of periods you are going to have (years, days, minutes, depends on the problem)
- Choose some initial path between start $= x_0$ and finish $= x_N$
- Use $x_0$ and $x_2$ to find a new $x_1^*$, then use $x_1^*$ and $x_3$ to find a new $x_2^*$, continue to $x_N$
- repeat backwards (from $x_N$ to $x_0$) and forwards until path converges

# Robinson Crusoe economy

- Only one agent (who we call Robinson Crusoe) with utility

$$\sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$

- Notice that it is a discounted infinite horizon utility (and here no labor decision, could be)
- Budget constaints

$$k_{t+1} = (1-\delta)k_t + i_t$$

and

$$y_t = f(k_t) \geq c_t + i_t$$

# Variational methods

## Solving for a stationary state

1. Assume that values for endogenous variables for periods $s-1$ and $s+1$ are known
2. Solve for values in period $s$
3. For a stationary state, use solution for $s$ to find
   - Find equilibrium where values for $s-1$, $s$, and $s+1$ are the same
   - This is a stationary state that meets the equilibrium conditions in every period
4. Stationary states are not the only use of Variational methods, we use it to get first order conditions for approximate models that we will use later
5. Also, if the values at starting and end points are known, variational methods can be used to solve for the optimal path between those points by choosing an initial (probably wrong) path and then start from that path to successively find better ones (which converge to the optimal path

# General framework

- Optimization problem: ($F\left(x_{t+i}, x_{t+1+i}\right)$ is the objective function)

$$\max_{\{x_x\}_{s=t}^{\infty}} \sum_{i=0}^{\infty} \beta^i F\left(x_{t+i}, x_{t+1+i}\right).$$

- The Euler condition (a necessary condition)

$$0 = F_2\left(x_{s-1}, x_s\right) + \beta F_1\left(x_s, x_{s+1}\right)$$

- Note that the Euler condition has $x_{s-1}$, $x_s$ and $x_{s+1}$ as variables. Can solve for $x_s$ as a function of $x_{s-1}$ and $x_{s+1}$.
- Transversality condition

$$\lim_{s \to \infty} \beta^s F_1\left(x_s, x_{s+1}\right) x_s = 0.$$

# Example RC economy

1. Substitute budget constraints into utility function. Get:

$$\sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+i+1} + (1-\delta)k_{t+i})$$

2. Note each objective function is a function of $k_{t+i}$ and $k_{t+i+1}$
3. First order conditions are

$$\begin{aligned} 0 &= \beta^{s-t} u'(f(k_s) - k_{s+1} + (1-\delta)k_s) \left(f'(k_s) + (1-\delta)\right) \\ &\quad -\beta^{s-t-1} u'(f(k_{s-1}) - k_s + (1-\delta)k_{s-1}). \end{aligned}$$

4. Notice that the first order conditions are functions of $k$ at times $s-1$, $s$, and $s+1$.

# Example economy (continued)

1. Which can be written as the Euler equation (a necessary condition)

$$f'(k_s) + (1-\delta) = \frac{u'(f(k_{s-1}) - k_s + (1-\delta)k_{s-1})}{\beta u'(f(k_s) - k_{s+1} + (1-\delta)k_s)}.$$

2. In a stationary state, $k_{s+1} = k_s = k_{s-1} = \overline{k}$, so the above becomes

$$f'(\overline{k}) = \frac{1}{\beta} - 1 + \delta$$

   The value of $\overline{k}$ is implicitly defined
3. Savings in a stationary state is $\overline{s} = \delta \overline{k}$

- See book for Robinson Crusoe with labor decision included

# Transversality condition

- Transversality condition for example economy

$$\lim_{i\to\infty} \beta^i u' \left(f\left(k_{t+i}\right) - k_{t+i+1} + (1-\delta)k_{t+i}\right) \left(f'\left(k_{t+i}\right) + (1-\delta)\right) k_{t+i} = 0$$

- In a stationary state with $\overline{k} > 0$

$$\begin{aligned} & \lim_{i\to\infty} \beta^i u' \left(f\left(\overline{k}\right) - \delta\overline{k}\right) \left(f'\left(\overline{k}\right) + (1-\delta)\right) \overline{k} \\ = & \lim_{i\to\infty} \beta^i u' \left(f\left(\overline{k}\right) - \delta\overline{k}\right) \left(\frac{1}{\beta} - 1 + \delta + (1-\delta)\right) \overline{k} \\ = & \lim_{i\to\infty} \beta^{i-1} u' \left(f\left(\overline{k}\right) - \delta\overline{k}\right) \overline{k} \to 0 \end{aligned}$$

- because $u'\left(f\left(\overline{k}\right) - \delta\overline{k}\right)\overline{k}$ is a finite constant for $\overline{k} > 0$

# Competitive market economy

- Many agents
  - Households
  - Firms
- Markets for goods, labor, and rental of capital
- Individual take aggregate values as given
- Decentralized decision making

# Competitive market economy

- Unit mass of agents
  - indexed from 0 to 1 continuously

$$H_t = \int_0^1 h_t^i di \text{ and } K_t = \int_0^1 k_t^i di$$

- Include individual labor supply decision

$$u\left(c_t^i, l_t^i\right) = u\left(c_t^i, 1 - h_t^i\right)$$

- subject to the budget constaints

$$c_t^i = w_t h_t^i + r_t k_t^i + i_t^i$$

$$k_{t+1}^i = (1-\delta) k_t^i + i_t^i$$

$$w_t = f_h\left(K_t, H_t\right)$$

$$r_t = f_h\left(K_t, H_t\right)$$

# Set up problem as Legrange problem

$$
\begin{aligned}
L &= \sum_{t=0}^{\infty} \beta^t \left[ u\left(c_t^i, 1-h_t^i\right) - \lambda_t^1 \left(k_{t+1}^i - (1-\delta)\, k_t^i - i_t^i\right) \right. \\
&\quad \left. -\lambda_t^2 \left(f_h\left(K_t, H_t\right) h_t^i + f_k\left(K_t, H_t\right) k_t^i - c_t^i - i_t^i\right)\right]
\end{aligned}
$$

- Difference between $h_t^i$ and $H_t$ and $k_t^i$ and $K_t$
- $H_t$ and $K_t$ are the economy wide (aggregate) values of labor and capital
- In optimization, $H_t$ and $K_t$ are considered given
- Equilibrium conditions are applied after optimization

# First order conditions

$$
\begin{aligned}
\frac{\partial L}{\partial c_t^i} &= u_c\left(c_t^i, 1-h_t^i\right) + \lambda_t^2 = 0 \\
\frac{\partial L}{\partial h_t^i} &= -u_h\left(c_t^i, 1-h_t^i\right) - \lambda_t^2 f_h\left(K_t, H_t\right) = 0 \\
\frac{\partial L}{\partial k_{t+1}^i} &= -\lambda_t^1 + \beta\lambda_{t+1}^1\left(1-\delta\right) - \beta\lambda_{t+1}^2 f_k\left(K_{t+1}, H_{t+1}\right) = 0 \\
\frac{\partial L}{\partial i_t^i} &= \lambda_t^1 + \lambda_t^2 = 0
\end{aligned}
$$

# First order conditions (continued)

These simplify to

$$
\begin{aligned}
u_c\left(c_t^i, 1-h_t^i\right) &= \lambda_t^1 = -\lambda_t^2 \\
\frac{u_h\left(c_t^i, 1-h_t^i\right)}{u_c\left(c_t^i, 1-h_t^i\right)} &= f_h\left(K_t, H_t\right) \\
\frac{u_c\left(c_t^i, 1-h_t^i\right)}{u_c\left(c_{t+1}^i, 1-h_{t+1}^i\right)} &= \beta\left[f_k\left(K_{t+1}, H_{t+1}\right) + (1-\delta)\right]
\end{aligned}
$$

For the individual decisions we need to include the budget constraint

$$
k_{t+1}^i = (1-\delta)\, k_t^i + f_h\left(K_t, H_t\right) h_t^i + f_k\left(K_t, H_t\right) k_t^i - c_t^i
$$

# Solving the model

- **After and only after** the individual decision rules are found, can one aggregate across individuals
  - Because the $K_t$ and $H_t$ are not controled by individuals
- These are found from the equilibrium conditions

$$K_t = \int_0^1 k_t^i di$$

$$H_t = \int_0^1 h_t^i di$$

- Aggregation condition for consumption

$$C_t = \int_0^1 c_t^i di$$

# Solving the model

- Given that production is homogenous of degree 1

$$f_h(K_t, H_t) H_t + f_k(K_t, H_t) K_t = f(K_t, H_t)$$

- The conditions for equilibrium are

$$\begin{aligned}
\frac{u_h(C_t, 1-H_t)}{u_c(C_t, 1-H_t)} &= f_h(K_t, H_t) \\
\frac{u_c(C_t, 1-H_t)}{u_c(C_{t+1}, 1-H_{t+1})} &= \beta\left[f_k(K_{t+1}, H_{t+1}) + (1-\delta)\right]
\end{aligned}$$

and the budget constraint is

$$K_{t+1} = (1-\delta) K_t + f(K_t, H_t) - C_t$$

# The stationary states

- Solving for the stationary states, one gets

$$
\begin{aligned}
\frac{u_h\left(\overline{C}, 1-\overline{H}\right)}{u_c\left(\overline{C}, 1-\overline{H}\right)} &= f_h\left(\overline{K}, \overline{H}\right) \\
\frac{1}{\beta} - (1-\delta) &= f_k\left(\overline{K}, \overline{H}\right)
\end{aligned}
$$

and the budget constraint is

$$
f\left(\overline{K}, \overline{H}\right) - \delta \overline{K} = \overline{C}
$$

# The stationary states

- Or, the two equations in $H$ and $K$ of

$$\begin{aligned}
\frac{u_h\left(f\left(\overline{K},\overline{H}\right)-\delta\overline{K},1-\overline{H}\right)}{u_c\left(f\left(\overline{K},\overline{H}\right)-\delta\overline{K},1-\overline{H}\right)} &= f_h\left(\overline{K},\overline{H}\right) \\
\frac{1}{\beta}-(1-\delta) &= f_k\left(\overline{K},\overline{H}\right)
\end{aligned}$$

- These are the same two conditions one gets for the Robinson Crusoe economy (when done with labor decision)

# The second welfare theorem

*"The Second Fundamental Welfare Theorem. If household preferences and firm productions sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged."*

- What does the Second Fundamental Welfare Theorem mean?
  - A perfectly competitive economy can be solved as a social planner (one person) problem
  - This need not work if the economy is not perfectly competitive
- The above statement is for a finite dimension economy. Infinite dimensioned economies are a bit more complicated. More assumptions are needed to get competitive equilibrium. In particular, a transversality condition. We simply assume a competitive equilibrium exists

# Solving for time paths (non-stationary)

- Knowing (or guessing) starting and ending points
- One can often approximate the optimal path between these points
- Using the Euler equation and an initial guess for the time path (even a pretty bad one)
- the $x$ axis of the problem is time (N time periods)
  - $x_0$ is the starting point chosen, $x_N$ the ending point
- the $y$ axis is the value of the state variables of interest
- start by using the initial $y$'s at points $x_0$ and $x_2$ and the Euler equation to find $y_1^1$ at point $x_1$
- use $y_1^1$ and $y_3^0$ and the Euler equations to find $y_2^1$
- repeat until using $y_{N-2}^1$ and $y_N^0$ to find $y_{N-1}^1$
- then go backwards from $x_N$ to $x_0$
- repeat until the results don't change much between interations

# Homework

- Exercise: Find the stationary state equilibrium for an economy with

$$u\left(c_t^i, 1-h_t^i\right) = \log c_t^i + A \log\left(1-h_t^i\right),$$

$A = 1.72$, and

$$f\left(K_t, H_t\right) = \gamma K_t^{\theta} H_t^{1-\theta}$$

with $\gamma = 3$ and $\theta = .56$. Use $\beta = .98$ and $\delta = .06$. Find the numerical values of output, consumption, the capital stock, and labor supply in the stationary state using Matlab.