Macroeconomics II
OLG models: the savings decision

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Why study these models

- The savings rate s is no longer a constant
- It is determined by choices of individuals
- These models are sometimes included inside RBC models
    - for example, in models where colateral is needed to borrow to buy capital
    - where you want people to have a finite life or probability of death
- Useful models: a large literature uses these models
    - education
    - growth
    - assets holding and money
    - retirement

References:

Peter Diamond (1965) "National debt in a neoclassical growth model,"AER p.1126-1150.

McCandless and Wallace (1991) Introduction to Dynamic Macroeconomic Theory,

The Environment

- Time is discrete: $t = 0, 1, 2, 3, 4, ...\infty$
- Individuals live 2 periods (smallest number to have a "future")
  - Generation t is born in period t, new generation in $t + 1$
  - Lives in periods t and $t + 1$
  - They are dead in periods $t + 2$ and onwards
  - There are $N(t)$ members of generation t
- Individuals have preferences: person h of generation t

$$u_t^h(c_t^h(t), c_t^h(t+1))$$

- Individuals have a labor endownment:
  - today, no labor supply decision, as a simplification, they supply all

$$h_t^h = \left[h_t^h(t), h_t^h(t+1)\right]$$

  - our future models will have both consumption and labor decisions
  - Capital is saved (or bought) by young at end of period t and used in $t + 1$

The Environment

- Production technology (constant returns to scale)

$$Y(t) = F(K(t), H(t))$$

- Total labor supplied in period t

$$H(t) = \sum_{h=1}^{N(t)} h_t^h(t) + \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

Useful to defind labor supplied by each generation in period t

$$H_t(t) = \sum_{h=1}^{N(t)} h_t^h(t) \qquad H_{t-1}(t) = \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

Feasibility constraint

- Feasibility constraint for period t (assume that all $K(t)$ depreciates in period t)

$$Y(t) = F(K(t), H(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1)$$

- how would this change if not all capital depreciates?
- if only $\delta K(t)$ depreciates, the old could sell the $(1-\delta) K(t)$ that remains to young
- what would the price of the old capital be?
- How do budget constraints of young and old change?

The economy (a market economy)

- Budget constraint of a member h of generation t when young

$$w_t h_t^h(t) = c_t^h(t) + l^h(t) + k^h(t+1)$$

  - w_t are the wages paid at time t
  - $k^h(t+1)$ is their savings in the form of capital for use in period $t+1$
  - $l^h(t)$ is lending or borrowing (if negative) of young in period t
  - Can only lend to own generation (why?)
  - so an equilibrium condition is

$$0 = \sum_{h=1}^{N(t)} l^h(t)$$

- Budget constaint of member h generation t when old

$$c_t^h(t+1) = w_{t+1} h_t^h(t+1) + r_t l^h(t) + rental_{t+1} k^h(t+1)$$

  - r_t is the interest rate on private loans
  - $rental_{t+1}$ is the rental rate on capital when it is being used
  - here we assume that capital depreciates 100% in the period of use

Factor market conditions

- Perfectly compeditive factor markets
- So, if firms maximize profits, we get the conditions that

$$w_t = F_H(K(t), H(t))$$

and

$$rental_t = F_K(K(t), H(t))$$

- Factor rentals equal their marginal products

Lifetime budget constraint and Arbitrage conditions

- Combine the two budget constrains through $l_h(t)$
- assume perfect foresight (not reasonable but easy to do)
- The lifetime budget constraint is

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{r_t} &= w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t} \\ &\quad - k^h(t+1)\left[1 - \frac{rental_{t+1}}{r_t}\right] \end{aligned}$$

- In perfect foresight economy, all assets have same risk (none, that is what perfect foresight implies) so must have same return (for both to be held)

Lifetime budget constraint and Arbitrage conditions

- Arbitrage condition (actually NO arbitrage condition)

$$rental_{t+1} = r_t$$

  - What if not true: two options $rental_{t+1} < r_t$ or $rental_{t+1} > r_t$
  - If $rental_{t+1} < r_t$
    - Everyone wants to lend and no one holds capital
    - But marginal product of capital $\longrightarrow \infty$ when $K(t) \longrightarrow 0$
    - Not an equilibrium (no demand)
  - If $rental_{t+1} > r_t$
    - Everyone wants to borrow an infinite amount to buy capital
    - This can't be an equilibrium (no supply)
  - Only $rental_{t+1} = r_t$ remains as possibility

Lifetime budget constraint II

- Given that

$$rental_{t+1} = r_t$$

- The lifetime budget constraint simplifies to

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t}$$

- The present value of lifetime consumption equals the present value of lifetime wage income

Definition of an equilibrium

A *competitive equilibrium* is a sequence of prices, $\{w_t, rental_t, r_t\}_{t=0}^{\infty}$, and quantities, $\left\{\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty}$, such that each member h of each generation $t > 0$ maximizes the utility function,

$$u_t^h(c_t^h(t), c_t^h(t+1))$$

subject to the lifetime budget constraint,

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t}$$

and the equilibrium conditions,

$$\begin{aligned} r_t &= rental_{t+1} \\ w_t &= F_H(K(t), H(t))) \\ rental_t &= F_K(K(t), H(t)) \end{aligned} \qquad \begin{aligned} H(t) &= \sum_{h=1}^{N(t)} h_t^h(t) + \sum_{h=1}^{N(t-1)} h_{t-1}^h(t \\ K(t) &= \sum_{h=1}^{N(t-1)} k_{t-1}^h(t) \end{aligned}$$

hold in every period.

How to solve: Individual problem

- Substitute the lifetime budget constraint into the utility function to get

$$\max_{c_t^h(t)} u(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t))$$

- First order condtions are

$$\begin{aligned} & u_1(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)) \\ = \; & r_t u_2(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)), \end{aligned}$$

- One can solve for a savings functions as

$$s_t^h(w_t, w_{t+1}, r_t) = l^h(t) + k^h(t+1).$$

- Note: from budget constraint when young, $s_t^h = w_t h_t^h(t) - c_t^h(t)$ and $h_t^h(t)$ and $h_t^h(t+1)$ are given

How to solve economy: Aggregating savings functions

- Sum savings across members of generation t:

$$S_t(\cdot) \equiv \sum_{h=1}^{N(t)} s_t^h(\cdot) = \sum_{h=1}^{N(t)} l^h(t) + \sum_{h=1}^{N(t)} k^h(t+1).$$

- In equilibrium, total borrowing and lending among generation t is

$$\sum_{h=1}^{N(t)} l^h(t) = 0,$$

- Definition of aggregate capital is

$$K(t+1) = \sum_{h=1}^{N(t)} k^h(t+1),$$

- So the aggregate savings equation is

$$S_t(w_t, w_{t+1}, r_t) = K(t+1).$$

How to solve: Getting a first order difference equation

- From the factor markets we have wages and rentals in terms of capital and labor,

$$\begin{aligned} w_t &= F_H(K(t), H(t))) \\ rental_t &= F_K(K(t), H(t)) \end{aligned}$$

- Substitute these into the aggregate saving equation to get (this ugly equation)

$$S_t(F_H(K(t), H(t)), F_H(K(t+1), H(t+1)), F_K(K(t+1), H(t+1))) \\ = K(t+1)$$

- This can be simplified (since $H(t)$ and $H(t+1)$ are constants) to get a function of the form

$$K(t+1) = G(K(t))$$

- Given some initial $K(0)$, the time path of the economy can be found by repeatedly applying this equation

An example economy

- Let the utility function be

$$u_t^h = u(c_t^h(t), c_t^h(t+1)) = c_t^h(t)c_t^h(t+1)^\beta$$

and the production function be

$$Y_t = F(K(t), H(t)) = K(t)^\theta H(t)^{1-\theta}$$

- After a bunch of algebra,

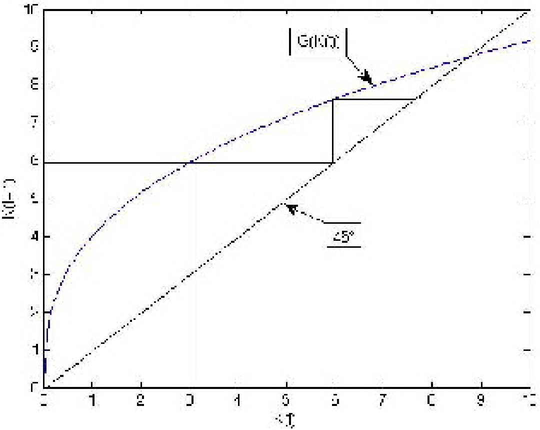
$$K(t+1) = G(K(t)) = \frac{\theta\beta\frac{H_t(t)}{H(t)^\theta}}{\left[\frac{H_t(t+1)}{H(t+1)}\right] + \frac{\theta(1+\beta)}{(1-\theta)}} K(t)^\theta = \kappa K(t)^\theta$$

where κ is a constant equal to

$$\kappa = \frac{\theta\beta\frac{H_t(t)}{H(t)^\theta}}{\left[\frac{H_t(t+1)}{H(t+1)}\right] + \frac{\theta(1+\beta)}{(1-\theta)}}$$

function is

dynamic



$_1$.jpg

Extensions

- The book contains an example of how to make the model stochastic. Where we have $r(t+1)$, $w(t+1)$, and $H(t+1)$ we need to replace them by their expectations, $E_t r(t+1)$, $E_t w(t+1)$, and $E_t H(t+1)$, which will depend on the stochastic process of the technology shock.
- Exercise 1 Find the equilibrium in a constant population economy where, in each period, the government imposes a small lump sum tax, t, on each young person and gives that amount to each old person. Both the young and the old see these taxes as lump sum. Under what conditions can this tax improve welfare and under what conditions will it not?
- Exercise 2: Solve the model of the class but where depreciation is equal to $\delta < 1$, for example, $\delta = .5$. Will output in the stationary state be higher or lower? Will the utility of households in the stationary state be higher or lower? Starting from some K_0 below the stationary state, will growth be faster or slower?