

# 6

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## Hansen's RBC Model

While Kydland and Prescott [52] published what was the first Real Business Cycle (RBC) model, this model contained quite a number of complications designed to increase the persistence and amplification of the stochastic shocks to technology. The model included time to build for converting investment into capital, so that the new capital that comes available in time  $t$  was built using investment from a number of periods anterior to period  $t$ . In addition, leisure was included in the utility function as a function of current and past time spent not working. Inventories were necessary so that previous investment commitments could be met in a period of a very large negative technology shock. Finally, technology shocks were designed to contain both a temporary and a permanent component. The model produced the required persistence and amplification, but is difficult to interpret in terms of determining which elements of the model are responsible for its success.

Gary Hansen [48] produced a pair of models that have become the reference for RBC studies. The first is a stochastic, variable labor model much like that of section 4.6, and the second is an almost identical model where a type of indivisibility in labor was introduced. Hansen's objective was to show how the addition of indivisible labor could improve the ability of the model to match the second moments of the time series data from the United States.

In this chapter, we present the basic model and find the first-order conditions and the stationary state for capital and labor. The second section introduces the techniques of log linearization, which we then apply to the basic model to get a linear version that permits us to study the dynamics of the system. The solution method we use for the linear version of the models requires finding the roots to a matrix quadratic equation. Since the model

cannot be solved analytically, we need to specify a version of the model with the parameters given. Calibration is the name used to describe the method of choosing the values of the parameters. One takes values from other studies and then uses some results of simulations (or calculations) of the model to determine the rest. We calibrate the basic model, find the laws of motion (a linear contingent plan), calculate the variances that the model generates for the endogenous set of variables, and compare these to the variances that Hansen found for the United States. We then do the same for Hansen's model with indivisible labor and compare the results to those of the first model. At this stage, the success of an RBC model is determined by how well it can match the variance and covariance structure that comes from the time series of macroeconomic variables of the particular economy being studied.

### 6.1 HANSEN'S BASIC MODEL

In a Robinson Crusoe type economy, the one agent maximizes the discounted utility function

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

where  $c_t$  is time  $t$  consumption and  $l_t$  is time  $t$  leisure, where  $l_t = 1 - h_t$ , and  $h_t$  is time  $t$  labor. The specific utility function that we use is

$$u(c_t, 1 - h_t) = \ln c_t + A \ln(1 - h_t),$$

with  $A > 0$ . The production function is Cobb-Douglas with a stochastic technology,

$$f(\lambda_t, k_t, h_t) = \lambda_t k_t^\theta h_t^{1-\theta},$$

where  $\lambda_t$  is a random technology variable that follows the process

$$\lambda_{t+1} = \gamma \lambda_t + \varepsilon_{t+1},$$

for  $0 < \gamma < 1$ . The  $\varepsilon_t$  shocks are identically and independently distributed, are positive, bounded above, and have a mean of  $1 - \gamma$ . These assumptions imply that the mean of  $\lambda_t$  is 1 and that output cannot go negative. Capital accumulation follows the process

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

and the feasibility constraint is

$$f(\lambda_t, k_t, h_t) \geq c_t + i_t$$

for every period  $t$ .

The model can be written as a Bellman equation of the form

$$V(k_t, \lambda_t) = \max_{c_t, h_t} [\ln c_t + A \ln(1 - h_t) + \beta E_t [V(k_{t+1}, \lambda_{t+1}) | \lambda_t]],$$

subject to

$$\begin{aligned} \lambda_t k_t^\theta h_t^{1-\theta} &\geq c_t + i_t, \\ \lambda_{t+1} &= \gamma \lambda_t + \varepsilon_{t+1}, \text{ and} \\ k_{t+1} &= (1 - \delta)k_t + i_t. \end{aligned}$$

The expectations operator in the value function,  $E_t [V(k_{t+1}, \lambda_{t+1}) | \lambda_t]$ , indicates that expectations are conditional on the realization of  $\lambda_t$ , a fact that comes from the stochastic process for technology. With substitution, the Bellman equation becomes

$$\begin{aligned} V(k_t, \lambda_t) = \max_{k_{t+1}, h_t} & [\ln (\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}) + A \ln(1 - h_t) \\ & + \beta E_t [V(k_{t+1}, \lambda_{t+1}) | \lambda_t]], \end{aligned}$$

with  $k_{t+1}$  and  $h_t$  as the control variables.

The first-order conditions are

$$\frac{\partial V(k_t, \lambda_t)}{\partial k_{t+1}} = 0 = -\frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}} + \beta E_t [V_k(k_{t+1}, \lambda_{t+1}) | \lambda_t],$$

and

$$\frac{\partial V(k_t, \lambda_t)}{\partial h_t} = 0 = (1 - \theta) \frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}} (\lambda_t k_t^\theta h_t^{-\theta}) - A \frac{1}{1 - h_t}.$$

The Benveniste-Scheinkman envelope theorem condition is

$$\frac{\partial V(k_t, \lambda_t)}{\partial k_t} = \frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}} (\theta \lambda_t k_t^{\theta-1} h_t^{1-\theta} + (1 - \delta)).$$

These first-order conditions can be written as

$$\begin{aligned} & \frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}} \\ &= \beta E_t \left[ \frac{\theta \lambda_{t+1} k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + (1 - \delta)}{\lambda_{t+1} k_{t+1}^\theta h_{t+1}^{1-\theta} + (1 - \delta)k_{t+1} - k_{t+2}} | \lambda_t \right], \end{aligned} \tag{6.1}$$

and

$$(1 - \theta)(1 - h_t) (\lambda_t k_t^\theta h_t^{-\theta}) = A (\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}). \quad (6.2)$$

Since, at equality,

$$c_t = \lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1},$$

and with perfect factor markets so that factor rentals equal their marginal products,

$$r_t = \theta \lambda_t k_t^{\theta-1} h_t^{1-\theta}$$

and

$$w_t = (1 - \theta) \lambda_t k_t^\theta h_t^{-\theta},$$

these conditions simplify to

$$\frac{1}{c_t} = \beta E_t \left[ \frac{r_{t+1} + (1 - \delta)}{c_{t+1}} \mid \lambda_t \right]$$

and

$$(1 - h_t) w_t = A c_t.$$

One can solve these two equations for the stationary state case by imposing the conditions that  $\bar{k} = k_t = k_{t+1} = k_{t+2}$  and  $\bar{h} = h_t = h_{t+1}$ . With these restrictions, the expectations operator disappears since expectations equal the realized values in a stationary state. The stationary state problem here is the same as for the deterministic economy. The rest is just algebra and the stationary state value of  $\bar{h}$  is found to be

$$\bar{h} = \frac{1}{1 + \frac{A}{(1-\theta)} \left[ 1 - \frac{\beta \delta \theta}{1 - \beta(1-\delta)} \right]},$$

and the stationary state value of  $\bar{k}$  is determined by

$$\bar{k} = \bar{h} \left[ \frac{\theta \bar{\lambda}}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\theta}}.$$

To determine the behavior of the economy outside of the stationary state, a number of approaches are possible, all of which give approximate solutions.

If the set of possible states of nature for the technology variable were small, one could use the techniques of the previous chapter and calculate an approximate version of the value function and of the plans. Then one could run simulations, using an appropriate random number generator, and find the second moments, the variances and covariances, of the variables of interest in the economy. This method would have two advantages. It would preserve the curvature of the model, and in cases where the curvature is important, it would generate good approximate second moments. It would also allow study of the model in equilibria that are far from the stationary state. Once the value function is calculated, one can begin the economy at any level of capital stock within the domain used to calculate the value functions.

Unfortunately, given the definition of the stochastic process for technology,  $\lambda_{t+1} = \gamma \lambda_t + \varepsilon_{t+1}$ , even a very small dimension for  $\varepsilon_{t+1}$  results in a very large dimension for  $\lambda_t$ . The smallest dimension that  $\varepsilon_{t+1}$  can have and still be a random variable is two. Suppose that the values that are possible for  $\varepsilon_{t+1}$  are  $1 - \gamma + \Delta$  and  $1 - \gamma - \Delta$ , where  $\Delta$  is a very small positive number, and suppose that it can have each of these values with probability .5. Suppose that at time  $t$ ,  $\lambda_t = 1$ . At time  $t + 1$ ,  $\lambda_{t+1}$  can have values  $1 + \Delta$ , and  $1 - \Delta$ . At time  $t + 2$ ,  $\lambda_{t+2}$  can have values  $1 + (1 + \gamma)\Delta$ ,  $1 - (1 - \gamma)\Delta$ ,  $1 + (1 - \gamma)\Delta$  and  $1 - (1 + \gamma)\Delta$ . At time  $t + n$ ,  $\lambda_{t+n}$  can take on  $2^n$  values. As  $n$  goes to infinity so does the number of possible values for  $\lambda_{t+n}$ .

It is still possible to find approximations for the value function as a function of the two variables,  $k_t$  and  $\lambda_t$ . One needs to choose domains for both of these variables so that in simulations it is very unlikely that  $\lambda_t$  will go outside of this domain. Take sufficiently dense discrete samplings of these domains, choose an initial guess for  $V_0(k_t, \lambda_t)$ , and iterate on the value function and the contingent plans. The calculations will need to do two-dimensional interpolations of the previous iteration's value function during the calculations but these calculations, are quite feasible. Modern computing power makes this solution method possible if one is patient with the calculations.

However, even modern computing power begins to lose its usefulness for direct approximations if the model becomes more complicated and the dimension of the state variables increases. In those cases, two forms of linear approximations<sup>1</sup> are available: 1) log-linear approximations of the first-order conditions and the budget constraints and 2) quadratic approximation of the objective function and linearization of the budget constraints to allow the use of linear quadratic dynamic programming. In this chapter we will continue

1. Quadratic approximation techniques are being developed but are outside the scope of this book. See, for example, Kim, Kim, Schaumburg, and Sims [50] and Schmitt-Grohé and Uribe [75]. Quadratic approximation techniques may do a better job of approximating the underlying system when one is a bit further from the stationary state.

with log-linear approximations of the first-order conditions. Linear quadratic dynamic programming will come in Chapter 7.

Before we go on with the log linearization, it helps to simplify the Hansen model by making a couple of substitutions of variables. We will add definitions and additional budget constraints to carry along, but these additions will make the linearization of the model easier. In addition, we consider the problem to be that of a social planner and will write the model in aggregate terms, where variables in capital letters are the aggregate values.<sup>2</sup> Write the first-order conditions<sup>3</sup> (equations 6.1 and 6.2) as

$$1 = \beta E_t \left[ \frac{C_t}{C_{t+1}} (r_{t+1} + (1 - \delta)) \right]$$

and

$$(1 - H_t) (1 - \theta) \frac{Y_t}{H_t} = AC_t,$$

where the budget constraints give

$$C_t = Y_t + (1 - \delta) K_t - K_{t+1}$$

and

$$Y_t = \lambda_t K_t^\theta H_t^{1-\theta},$$

and the rental on capital,  $r_t$ , is

$$r_t = \theta \frac{Y_t}{K_t}.$$

Here,  $r_t$  is equal to the marginal product of a unit of capital. If one owns a unit of capital at the beginning of period  $t$  and rents that capital out in a competitive market,  $r_t$  is the rental that one receives for the use of the capital during that period. These five equations contain the same information as the two first-order conditions and budget constraints in a more simplified format.

## 6.2 LOG LINEARIZATION TECHNIQUES

Handling and solving models with substantial nonlinearity is often difficult. As we showed earlier, when the model is relatively simple, one can find an

2. Applying the second welfare theorem.

3. We are no longer going to write out exactly what information is included in forming the expectation  $E_t$ ; that is why the term “ $|\lambda_t$ ” is no longer included.

approximation to the policy function by recursively solving for the value function. This method works if a model is relatively small and can be written as the solution to a single optimization problem.<sup>4</sup>

Linear models are often much easier to solve, and there exist well-developed methods for solving linear models. The problem is to convert a nonlinear model into a sufficiently good linear approximation so that the solutions to the linear approximation are helpful in understanding the behavior of the underlying nonlinear system. A now standard method for a linear approximation is to log-linearize a model around its stationary state. The assumption is that, if the model is not too far from the stationary state, the linear version that results closely approximates the original model.

### 6.2.1 The Basics of Log Linearization

Consider a nonlinear model that can be represented by a set of equations of the general form

$$F(x_t) = \frac{G(x_t)}{H(x_t)},$$

where  $x_t$  is a vector of the variables of the model that can include expectational variables and lagged variables in addition to contemporaneous variables. The process of log linearization is to first take the logarithms of the functions  $F()$ ,  $G()$ , and  $H()$ , and then take a first-order Taylor series approximation. Taking the logarithms gives

$$\ln(F(x_t)) = \ln(G(x_t)) - \ln(H(x_t)),$$

and taking the first-order Taylor series expansion around the stationary state values,  $\bar{x}$ , gives

$$\begin{aligned} \ln(F(\bar{x})) + \frac{F'(\bar{x})}{F(\bar{x})(x_t - \bar{x})} &\approx \ln(G(\bar{x})) + \frac{G'(\bar{x})}{G(\bar{x})(x_t - \bar{x})} \\ &\quad - \ln(H(\bar{x})) - \frac{H'(\bar{x})}{H(\bar{x})(x_t - \bar{x})}, \end{aligned}$$

where the notation  $X'(\bar{x})$  is used to indicate the gradient at the stationary state. Notice that the model is now linear in  $x_t$ , since  $F'(\bar{x})/F(\bar{x})$ ,  $G'(\bar{x})/G(\bar{x})$ ,  $H'(\bar{x})/H(\bar{x})$ ,  $\ln(F(\bar{x}))$ ,  $\ln(G(\bar{x}))$ , and  $\ln(H(\bar{x}))$  are constants. Given that the

4. However, somewhat more complicated models can be included if simple solutions to secondary optimization problems are included as constraints.

log version of the model holds at the stationary state,

$$\ln(F(\bar{x})) = \ln(G(\bar{x})) - \ln(H(\bar{x})),$$

we can eliminate the three  $\ln(\cdot)$  components, and the equation simplifies to

$$\frac{F'(\bar{x})}{F(\bar{x})}(x_t - \bar{x}) \approx \frac{G'(\bar{x})}{G(\bar{x})}(x_t - \bar{x}) - \frac{H'(\bar{x})}{H(\bar{x})}(x_t - \bar{x}).$$

The implicit assumption is that one is staying close enough to the stationary state,  $\bar{x}$ , so that the second-order or higher terms of the Taylor expansion are small enough to be irrelevant and can safely be left out.

What follows shows how to do this log linearization in two cases that are commonly encountered in economic problems.

**EXAMPLE 6.1** Consider the Cobb-Douglas production function

$$Y_t = \lambda_t K_t^\theta H_t^{1-\theta}.$$

First one takes the logarithms of both sides of the production function to get

$$\ln Y_t = \ln \lambda_t + \theta \ln K_t + (1 - \theta) \ln H_t,$$

and then the first-order Taylor expansion gives

$$\begin{aligned} \ln \bar{Y} + \frac{1}{\bar{Y}} (Y_t - \bar{Y}) &\approx \ln \bar{\lambda} + \frac{1}{\bar{\lambda}} (\lambda_t - \bar{\lambda}) + \theta \ln \bar{K} + \frac{\theta}{\bar{K}} (K_t - \bar{K}) \\ &\quad + (1 - \theta) \ln \bar{H} + \frac{(1 - \theta)}{\bar{H}} (H_t - \bar{H}). \end{aligned}$$

Since in a stationary state

$$\ln \bar{Y} = \ln \bar{\lambda} + \theta \ln \bar{K} + (1 - \theta) \ln \bar{H},$$

the zero-order terms can be removed to get

$$\frac{1}{\bar{Y}} (Y_t - \bar{Y}) \approx \frac{1}{\bar{\lambda}} (\lambda_t - \bar{\lambda}) + \frac{\theta}{\bar{K}} (K_t - \bar{K}) + \frac{(1 - \theta)}{\bar{H}} (H_t - \bar{H}).$$

Further simplification gives

$$\frac{Y_t}{\bar{Y}} + 1 \approx \frac{\lambda_t}{\bar{\lambda}} + \frac{\theta K_t}{\bar{K}} + \frac{(1 - \theta) H_t}{\bar{H}}.$$

The production function is now expressed as a linear equation.

**EXAMPLE 6.2** Consider the infinite horizon CES utility function of the form

$$\sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\eta}}{1-\eta} - \frac{H_t^{1-\phi}}{1-\phi} \right],$$

subject to the budget constraint

$$C_t = \lambda_t K_t^\theta H_t^{1-\theta} + (1-\delta)K_t - K_{t+1}.$$

The first-order conditions (found by substitution and variational methods) are

$$\beta \left[ \theta \lambda_{t+1} K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta) \right] C_{t+1}^{-\eta} = C_t^{-\eta}$$

and

$$C_t^{-\eta} \left[ (1-\theta) \lambda_t K_t^\theta H_t^{-\theta} \right] = H_t^{-\phi}.$$

The log versions of the budget constraint and the first-order conditions are

$$\begin{aligned} \ln C_t &= \ln \left[ \lambda_t K_t^\theta H_t^{1-\theta} + (1-\delta)K_t - K_{t+1} \right], \\ -\eta \ln C_t &= \ln \beta + \ln \left[ \theta \lambda_{t+1} K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta) \right] - \eta \ln C_{t+1}, \\ -\phi \ln H_t &= -\eta \ln C_t + \ln (1-\theta) + \ln \lambda_t + \theta \ln K_t - \theta \ln H_t. \end{aligned}$$

Taking the first-order Taylor expansion and removing the equalities from the stationary state gives

$$\begin{aligned} \frac{C_t}{\bar{C}} + \frac{\bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta}}{[\bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} - \delta \bar{K}]} &= \frac{(\bar{K}^\theta \bar{H}^{1-\theta})}{[\bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} - \delta \bar{K}]} \lambda_t + \frac{\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta} + (1-\delta)}{[\bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} - \delta \bar{K}]} K_t \\ &\quad + \frac{(1-\theta) \bar{\lambda} \bar{K}^\theta \bar{H}^{-\theta}}{[\bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} - \delta \bar{K}]} H_t - \frac{1}{[\bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} - \delta \bar{K}]} K_{t+1}, \\ -\eta \frac{C_t}{\bar{C}} &= -\frac{2\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta}}{\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta} + (1-\delta)} + \frac{\theta \bar{K}^{\theta-1} \bar{H}^{1-\theta}}{\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta} + (1-\delta)} \lambda_{t+1} \\ &\quad + \frac{(\theta-1) \theta \bar{\lambda} \bar{K}^{\theta-2} \bar{H}^{1-\theta}}{\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta} + (1-\delta)} K_{t+1} + \frac{(1-\theta) \theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{-\theta}}{\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta} + (1-\delta)} H_{t+1} - \eta \frac{C_{t+1}}{\bar{C}}, \end{aligned}$$

and

$$1 - \phi \frac{H_t}{\bar{H}} = -\eta \frac{C_t}{\bar{C}} + \frac{\lambda_t}{\bar{\lambda}} + \theta \frac{K_t}{\bar{K}} - \theta \frac{H_t}{\bar{H}} + (\eta - \phi).$$

The constant terms of some of these equations are messy, but they are constants, and the three equations are all linear.

### 6.2.2 Uhlig's Method of Log Linearization

Harald Uhlig [86] recommends using a simpler method for finding log-linear approximations of functions. His method does not require taking derivatives and gives the same result as the above method, except that the linear model is expressed in terms of log differences of the variables.

Consider an equation of a set of variables  $X_t$ . Define  $\tilde{X}_t = \ln X_t - \ln \bar{X}$ . The tilde variables are the log difference of the original variables from the value  $\bar{X}$ . One can write the original variable as

$$X_t = \bar{X} e^{\tilde{X}_t},$$

since

$$\bar{X} e^{\tilde{X}_t} = \bar{X} e^{\ln X_t - \ln \bar{X}} = \bar{X} e^{\ln X_t / \bar{X}} = \bar{X} \cdot X_t / \bar{X} = X_t.$$

Uhlig's method is to first multiply out all the variables, getting rid of as many variables in the denominator as possible. Then each variable  $X_t$  is replaced by the equivalent  $\bar{X} e^{\tilde{X}_t}$ . Up to this point, one has only done a slight change of variable and all equalities still hold exactly. Now, bring together all the exponential terms that you can. For example,

$$\frac{A_t B_t^\alpha}{C_t^\delta} = \frac{\bar{A} e^{\tilde{A}_t} \bar{B}^\alpha e^{\alpha \tilde{B}_t}}{\bar{C}^\delta e^{\delta \tilde{C}_t}}$$

becomes

$$\frac{\bar{A} \bar{B}^\alpha}{\bar{C}^\delta} e^{\tilde{A}_t + \alpha \tilde{B}_t - \delta \tilde{C}_t}. \quad (6.3)$$

At this point, one applies the linear approximation of the tilde variables. Taking the Taylor expansion of the exponential term around its stationary value gives

$$\begin{aligned} e^{\tilde{A}_t + \alpha \tilde{B}_t - \delta \tilde{C}_t} &\approx e^{\tilde{A} + \alpha \tilde{B} - \delta \tilde{C}} + e^{\tilde{A} + \alpha \tilde{B} - \delta \tilde{C}} (\tilde{A}_t - \tilde{A}) \\ &\quad + \alpha e^{\tilde{A} + \alpha \tilde{B} - \delta \tilde{C}} (\tilde{B}_t - \tilde{B}) - \delta e^{\tilde{A} + \alpha \tilde{B} - \delta \tilde{C}} (\tilde{C}_t - \tilde{C}) \\ &= 1 + \tilde{A}_t + \alpha \tilde{B}_t - \delta \tilde{C}_t, \end{aligned}$$

where terms without date subscripts are the stationary state values of the tilde variables (the log differences). Naturally, we assume that the stationary state

value of the differences is zero. The approximation of expression 6.3 above is

$$\frac{\bar{A}\bar{B}^\alpha}{\bar{C}^\delta} \left( 1 + \tilde{A}_t + \alpha \tilde{B}_t - \delta \tilde{C}_t \right).$$

The example below uses Uhlig's method to find a log-linear approximation for the same Cobb-Douglas production function we did above.

**EXAMPLE 6.3** Consider the Cobb-Douglas production function

$$Y_t = \lambda_t K_t^\theta H_t^{1-\theta}.$$

Substitute  $X_t = \bar{X} e^{\tilde{X}_t}$  for each variable and get

$$\bar{Y} e^{\tilde{Y}_t} = \bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} e^{\tilde{\lambda}_t + \theta \tilde{K}_t + (1-\theta) \tilde{H}_t}.$$

This is approximated by the Taylor expansion (or Uhlig's rules, if one prefers) to get

$$\bar{Y} (1 + \tilde{Y}_t) = \bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta} (1 + \tilde{\lambda}_t + \theta \tilde{K}_t + (1-\theta) \tilde{H}_t).$$

Given that in the stationary state  $\bar{Y} = \bar{\lambda} \bar{K}^\theta \bar{H}^{1-\theta}$ , this simplifies to

$$\tilde{Y}_t = \tilde{\lambda}_t + \theta \tilde{K}_t + (1-\theta) \tilde{H}_t.$$

This result may not seem the same as the one in the example using direct log linearization. However, the result of the first method is

$$\frac{Y_t}{\bar{Y}} + 1 \approx \frac{\lambda_t}{\bar{\lambda}} + \frac{\theta K_t}{\bar{K}} + \frac{(1-\theta) H_t}{\bar{H}},$$

which is given in levels and not changes. Replacing each variable  $X_t$  by the term written as differences,  $X_t \approx \bar{X}(1 + \tilde{X}_t)$ , gives

$$\frac{\bar{Y} (1 + \tilde{Y}_t)}{\bar{Y}} + 1 \approx \frac{\bar{\lambda} (1 + \tilde{\lambda}_t)}{\bar{\lambda}} + \frac{\theta \bar{K} (1 + \tilde{K}_t)}{\bar{K}} + \frac{(1-\theta) \bar{H} (1 + \tilde{H}_t)}{\bar{H}},$$

and this simplifies to

$$\tilde{Y}_t \approx \tilde{\lambda}_t + \theta \tilde{K}_t + (1-\theta) \tilde{H}_t.$$

The advantage of Uhlig's method is that one does not need to take explicit derivatives. The substitutions are direct and mostly mechanical. There are a number of rules that are useful for using this technique. These rules help keep the process simple. However, when comparing the results of log linearization using Uhlig's rules, one needs to remember that the results are given in log differences from the stationary state.

Uhlig's definitions are

$$\tilde{X}_t = \ln X_t - \ln \bar{X}$$

and

$$X_t = \bar{X} e^{\tilde{X}_t}.$$

His rules are

$$e^{\tilde{X}_t + a\tilde{Y}_t} \approx 1 + \tilde{X}_t + a\tilde{Y}_t,$$

$$\tilde{X}_t \tilde{Y}_t \approx 0,$$

$$E_t [ae^{\tilde{X}_{t+1}}] \approx a + aE_t [\tilde{X}_{t+1}].$$

The first rule is the most direct; it is best to try to write as many of the equations in this form as possible. If this is done adequately, the second rule, which says that second-order terms are approximately zero, is not usually necessary. A useful version of the expectations rule is that

$$E_t [X_{t+1}] = \bar{X} \left( 1 + E_t [\tilde{X}_{t+1}] \right).$$

**EXERCISE 6.1** Use Uhlig's method to find the log-linear approximation of the budget constraint and first-order conditions of example 6.2 above.

### 6.3 LOG-LINEAR VERSION OF HANSEN'S MODEL

The five equations of the Hansen model are

$$1 = \beta E_t \left[ \frac{C_t}{C_{t+1}} (r_{t+1} + (1 - \delta)) \right],$$

$$AC_t = (1 - \theta) (1 - H_t) \frac{Y_t}{H_t},$$

$$C_t = Y_t + (1 - \delta)K_t - K_{t+1},$$

$$Y_t = \lambda_t K_t^\theta H_t^{1-\theta},$$

$$r_t = \theta \frac{Y_t}{K_t}.$$

Beginning with the first, we substitute each variable for its log difference around a set of stationary state values, designated by a bar above the variable, and, after taking the linear approximation, get

$$\begin{aligned} 1 &= \beta E_t \left[ \frac{\bar{C}e^{\tilde{C}_t}}{\bar{C}e^{\tilde{C}_{t+1}}} \bar{r} e^{\tilde{r}_{t+1}} + (1 - \delta) \frac{\bar{C}e^{\tilde{C}_t}}{\bar{C}e^{\tilde{C}_{t+1}}} \right] \\ &= \beta E_t \left[ \bar{r} e^{\tilde{C}_t - \tilde{C}_{t+1} + \tilde{r}_{t+1}} + (1 - \delta) e^{\tilde{C}_t - \tilde{C}_{t+1}} \right] \\ &\approx \beta \left( \bar{r} E_t \left[ 1 + \tilde{C}_t - \tilde{C}_{t+1} + \tilde{r}_{t+1} \right] + (1 - \delta) E_t \left[ 1 + \tilde{C}_t - \tilde{C}_{t+1} \right] \right) \\ &= E_t \left[ 1 + \tilde{C}_t - \tilde{C}_{t+1} + \beta \bar{r} \tilde{r}_{t+1} \right], \end{aligned}$$

or

$$0 \approx \tilde{C}_t - E_t \tilde{C}_{t+1} + \beta \bar{r} E_t \tilde{r}_{t+1},$$

where  $\tilde{C}_t = \ln C_t - \ln \bar{C}$ . In this simplification, we used the fact that  $1/\beta = \bar{r} + (1 - \delta)$  in a stationary state equilibrium.

Applying the log linearization techniques to the second equation gives

$$0 \approx \tilde{Y}_t - \frac{\tilde{H}_t}{1 - \bar{H}} - \tilde{C}_t.$$

The next three equations are

$$\begin{aligned} 0 &\approx \bar{Y} \tilde{Y}_t - \bar{C} \tilde{C}_t + \bar{K} \left[ (1 - \delta) \tilde{K}_t - \tilde{K}_{t+1} \right], \\ 0 &\approx \tilde{\lambda}_t + \theta \tilde{K}_t + (1 - \theta) \tilde{H}_t - \tilde{Y}_t, \end{aligned}$$

and

$$0 \approx \tilde{Y}_t - \tilde{K}_t - \tilde{r}_t,$$

where  $\bar{r} = \theta \bar{Y}/\bar{K}$ .

The stochastic process,

$$\lambda_{t+1} = \gamma \lambda_t + \varepsilon_{t+1},$$

can be written as

$$\bar{\lambda} e^{\tilde{\lambda}_{t+1}} = \gamma \bar{\lambda} e^{\tilde{\lambda}_t} + \varepsilon_{t+1},$$

which can be approximated by

$$\bar{\lambda} (1 + \tilde{\lambda}_{t+1}) = \gamma \bar{\lambda} (1 + \tilde{\lambda}_t) + \varepsilon_{t+1},$$

or

$$\tilde{\lambda}_{t+1} = \gamma \tilde{\lambda}_t + \mu_{t+1},$$

where  $\bar{\lambda} = 1$  and  $\mu_{t+1} = \varepsilon_{t+1} - (1 - \gamma)$ . Notice that the change of variable gives  $E_t(\mu_{t+1}) = 0$ .

Define the vector of endogenous variables at time  $t$  as

$$x_t = [\tilde{K}_{t+1} \quad \tilde{Y}_t \quad \tilde{C}_t \quad \tilde{H}_t \quad \tilde{r}_t]',$$

and the exogenous stochastic variable as

$$z_t = \tilde{\lambda}_t.$$

The model given above can be written in matrix form as

$$0 = E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t],$$

where

$$F = \begin{bmatrix} 0 & 0 & -1 & 0 & \beta \bar{r} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{1-H} & 0 \\ -\bar{K} & \bar{Y} & -\bar{C} & 0 & 0 \\ 0 & -1 & 0 & 1-\theta & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \bar{K}(1-\delta) & 0 & 0 & 0 & 0 \\ \theta & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L = [0 \ 0 \ 0 \ 0 \ 0]',$$

$$M = [0 \ 0 \ 0 \ 1 \ 0]'$$

Write the stochastic process as

$$z_{t+1} = Nz_t + \mu_{t+1},$$

with  $E_t(\mu_{t+1}) = 0$ , and, in this case,  $N$  is simply the one-element matrix  $N = [\gamma]$ .

We look for a solution to the problem, matrices  $P$  and  $Q$ , that gives the equilibrium laws of motion,

$$x_t = Px_{t-1} + Qz_t,$$

where the equilibrium described by this process is stable.

Theorem 1 from Uhlig [86] shows that, if a solution exists, the matrix  $P$  can be found by solving the matrix quadratic equation

$$0 = FP^2 + GP + H,$$

and the matrix  $Q$  comes from

$$V\text{vec}(Q) = -\text{vec}(LN + M),$$

where  $\text{vec}(\cdot)$  is columnwise vectorization and

$$V = N' \otimes F + I_k \otimes (FP + G),$$

with  $I_k$  a  $k$ -dimensional identity matrix,  $k$  the number of stochastic variables (the dimension of  $z_t$ ), and  $\otimes$  the Kronecker product.<sup>5</sup> This result can be found by substituting the solution,

$$x_t = Px_{t-1} + Qz_t,$$

and the stochastic process, as expectations,

$$E_t z_{t+1} = Nz_t,$$

5. Given a  $k \times l$  matrix  $A$  and an  $m \times n$  matrix  $B$ , the  $km \times ln$  Kronecker product  $A \otimes B$  is found by multiplying each element of the matrix  $B$  by the entire matrix  $A$ . For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \text{then } A \otimes B = \begin{bmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{bmatrix}.$$

into the equation

$$0 = E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t],$$

to get

$$0 = [FP^2 + GP + H]x_{t-1} + [FPQ + FQN + GQ + LN + M]z_t.$$

For this equation to hold for all permitted values of  $x_{t-1}$  and  $z_t$ , each expression in square brackets must be equal to zero. The vec and Kronecker notations are necessary to solve for  $Q$ . (See Appendix 1 of this chapter for further explanation.) This problem can sometimes be difficult to solve because finding the roots of the matrix quadratic equation in  $P$  can be complex and computationally demanding, especially if the model is large.

### 6.3.1 Solution Using Jump Variables

Solution of the linear version of the economy is usually easier if one defines a vector of endogenous state variables,  $x_t$ , and a vector of other endogenous variables,  $y_t$ , which depend on the values of the state variables. The other endogenous variables are often called the jump variables. The name *jump variables* comes from the saddle point dynamics of continuous time systems. The system is stable (and converges) along the ridge of the saddle path. Normally, the control variables naturally follow the ridge, but the values of other variables need to jump (instantaneously) to get the system on the stable ridge. We are somewhat misusing the name *jump variable* here since, as we saw in the previous section, all variables can be included in what we are calling the “controls” vector.

In the example economy, we could choose

$$x_t = [\tilde{K}_{t+1}]$$

as the one element vector of endogenous state variables and

$$y_t = [\tilde{Y}_t, \tilde{C}_t, \tilde{H}_t, \tilde{r}_t]'$$

as the vector of other endogenous variables. Separating equations that include expectations from those that do not, the linear version of the model can be written as

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t,$$

$$0 = E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t],$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1} \quad E_t(\varepsilon_{t+1}) = 0.$$

For our model,

$$\begin{aligned} A &= [0 \ -\bar{K} \ 0 \ 0]', \\ B &= [0 \ (1-\delta)\bar{K} \ \theta \ -1]', \\ C &= \begin{bmatrix} 1 & -1 & -\frac{1}{1-\bar{H}} & 0 \\ \bar{Y} & -\bar{C} & 0 & 0 \\ -1 & 0 & 1-\theta & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \\ D &= [0 \ 0 \ 1 \ 0]', \\ F &= [0], \quad G = [0], \quad H = [0], \\ J &= [0 \ -1 \ 0 \ \beta\bar{r}], \\ K &= [0 \ 1 \ 0 \ 0], \\ L &= [0], \quad M = [0], \quad N = [\gamma]. \end{aligned}$$

The solution for this economy is a set of matrices,  $P, Q, R$ , and  $S$ , that describe the equilibrium laws of motion,

$$x_t = Px_{t-1} + Qz_t,$$

and

$$y_t = Rx_{t-1} + Sz_t.$$

Note that, for our model, the matrix  $C$  is of full rank<sup>6</sup> and has a well-defined inverse,  $C^{-1}$ .<sup>7</sup> From Corollary 1 of Uhlig [86], if equilibrium laws of motion exist, they must fulfill

$$0 = (F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H,$$

$$R = -C^{-1}(AP + B).$$

$Q$  satisfies the equation

$$\begin{aligned} &\left( N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A) \right) \text{vec}(Q) \\ &= \text{vec} \left( \left( JC^{-1}D - L \right) N + KC^{-1}D - M \right), \end{aligned}$$

6. Almost always. There is a set of measure zero of parameter values that could make it of less than full rank.

7. The problem can usually be solved even if  $C$  is not of full rank, but it is somewhat more complicated. Uhlig [86] considers this case as well.

and

$$S = -C^{-1}(AQ + D).$$

The identity matrix  $I_k$  has dimension  $k \times k$ , where  $k$  is the number of columns in the matrix  $Q$ .

Here, the matrix quadratic equation is quite simple. The matrix,  $P$ , has one element, so we are finding the solution to a standard quadratic equation. The rest is matrix algebra.

### 6.3.2 Calibration of the Log-Linear Model

To find the laws of motion using our numerical technique, we need to fix the values of the parameters of the model and determine the stationary state values of the variables (since many of these show up in the matrices that we are using in the linear version of the model). The source for these values is varied. For some parameters, for example,  $\theta$ , the economic interpretation of the variable is quite clear; with a Cobb-Douglas production function,  $\theta$  is the fraction of national income that goes to capital, and the value can be found easily from aggregate data. Others, such as  $\beta$  or  $\delta$ , have been the objects of a large number of microeconomic-based studies, and there is a set of values that are more or less generally accepted. Still others, such as  $A$ ,  $\gamma$ , and the probability distribution of  $u_{t+1}$ , are open questions, and we need some rules to help us choose them. These parameters we *calibrate*. Since we cannot see the variables directly, we choose values for these variables so that other relationships in the model have values similar to the ones we see in the data. For example, we choose  $A$  so that, given the parameters already chosen, the time spent working is near one-third of total time (approximately 8 hours per day). We choose  $\gamma$  and the distribution of  $u_{t+1}$  so that the variance and covariance of  $y_t$  that comes from multiple simulations of the model are similar to those observed in the data.

Hansen's model is calibrated for quarterly observations of the data. Therefore, we use values for the parameters that are different from those used in earlier chapters. Here  $\beta = .99$ ,  $\delta = .025$ ,  $\theta = .36$ . The equations for the stationary state values of  $\bar{H}$  and  $\bar{K}$  we found above, and are

$$\bar{H} = \frac{1}{1 + \frac{A}{(1-\theta)} \left[ 1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)} \right]} \quad (6.4)$$

and

$$\bar{K} = \bar{H} \left[ \frac{\theta\bar{\lambda}}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\theta}}. \quad (6.5)$$

The stochastic process for technology was chosen so that  $\bar{\lambda} = 1$ . What remains to be determined are the values for  $A$  and  $\gamma$ . Hansen chose a value of  $A = 2$ , which, for his choice of parameters, resulted in  $\bar{H} = 1/3$ , or that, in equilibrium, people spent a third of their time working. For the parameters we are using here, Figure 6.1 shows the relationship between  $A$  and the resulting  $\bar{H}$  that come from equation 6.4. In this equation, all the other parameters are already determined, so there is a simple relationship between the value of  $A$  and that of  $\bar{H}$ , that shown in the figure. For our choice of parameters, a value of  $A = 1.72$  results in about one-third of the available time spent working. We will use this value in our calculations.

The value for  $\gamma$  comes from estimations of a log version of the production function,

$$\ln \lambda_t = \ln Y_t - \theta \ln K_t - (1 - \theta)H_t,$$

using data from the United States. In this series, the first lag autocorrelations coefficient on the series  $\{\lambda_t\}$  is about .95. We use this value so  $\gamma = .95$ .

Choosing a distribution for  $\mu_{t+1}$  is more complicated. Typically, the variance is chosen so that simulations of the model result in a variance output,  $Y_t$ , that is similar to the ones observed in the data from the United States. This, however, is done after the laws of motion have been calculated, so we will determine this value later.

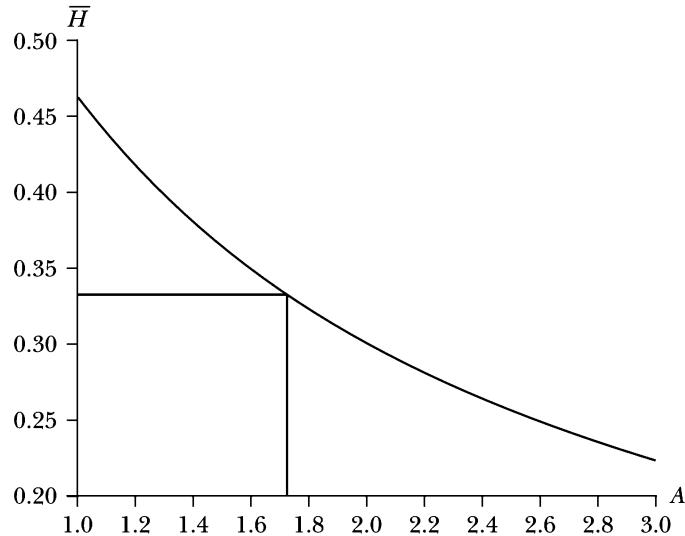


FIGURE 6.1 Finding the value for  $A$

With  $A = 1.72$ ,  $\bar{H} = .3335$ . Using this value in equation 6.5, one finds that  $\bar{K} = 12.6695$ . The stationary state value of output,  $\bar{Y}$ , is found from the production function evaluated at the stationary state values of capital and labor,

$$\bar{Y} = \bar{K}^\theta \bar{H}^{(1-\theta)} = (12.6695)^{.36} (.3335)^{.64} = 1.2353.$$

Stationary state consumption,  $\bar{C}$ , is found from the stationary state version of the budget constraint

$$\bar{Y} + (1 - \delta)\bar{K} = \bar{C} + \bar{K},$$

or

$$\bar{C} = \bar{Y} - \delta\bar{K} = 1.2353 - .025 \times 12.6695 = 0.9186.$$

Using the values for  $\bar{Y}$  and  $\bar{K}$ , the  $\bar{Y}/\bar{K}$  that we need for the matrices is equal to 0.0975. The last stationary state value that we need is that of  $\bar{r}$ , and we have from above that

$$\frac{1}{\beta} = \bar{r} + (1 - \delta) = \theta \frac{\bar{Y}}{\bar{K}} + (1 - \delta),$$

so the two calculations of  $\bar{r}$  can serve as a check to see that we have done our other calculations correctly:

$$\bar{r} = \frac{1}{\beta} - (1 - \delta) = \frac{1}{.99} - 1 + .025 = 0.0351,$$

and

$$\bar{r} = \theta \frac{\bar{Y}}{\bar{K}} = .36 \times \frac{1.2353}{12.6695} = 0.0351.$$

Putting all these values into the matrices  $A$  to  $N$ , we get

$$A = [0 \quad -12.6698 \quad 0 \quad 0]',$$

$$B = [0 \quad 12.3530 \quad .36 \quad -1]',$$

$$C = \begin{bmatrix} 1 & -1 & -1.5004 & 0 \\ 1.2353 & -0.9186 & 0 & 0 \\ -1 & 0 & .64 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

$$\begin{aligned} D &= [0 \ 0 \ 1 \ 0]', \\ F &= [0], \quad G = [0], \quad H = [0], \\ J &= [0 \ -1 \ 0 \ .0348], \\ K &= [0 \ 1 \ 0 \ 0], \\ L &= [0], \quad M = [0], \quad N = [.95]. \end{aligned}$$

Using these matrices to find the quadratic equation, we look for the  $P$  that solves

$$0 = 7.0734 \cdot P^2 - 14.2376 \cdot P + 7.1448.$$

These are  $P = 1.0592$  and  $P = 0.9537$ . The second value of  $P$  results in a stable equilibrium (it has an absolute value less than one), so that is what we use. Continuing with the calculations, we get  $Q = 0.1132$ ,

$$R = [0.2045 \ 0.5691 \ -0.2430 \ -0.7955]',$$

and

$$S = [1.4523 \ 0.3920 \ 0.7067 \ 1.4523]'. \quad$$

The five equations of the laws of motion around the stationary state can be written out as

$$\begin{aligned} \tilde{K}_{t+1} &= 0.9537\tilde{K}_t + 0.1132\tilde{\lambda}_t, \\ \tilde{Y}_t &= 0.2045\tilde{K}_t + 1.4523\tilde{\lambda}_t, \\ \tilde{C}_t &= 0.5691\tilde{K}_t + 0.3920\tilde{\lambda}_t, \\ \tilde{H}_t &= -0.2430\tilde{K}_t + 0.7067\tilde{\lambda}_t, \\ \tilde{r}_t &= -0.7955\tilde{K}_t + 1.4523\tilde{\lambda}_t. \end{aligned}$$

Matlab code for finding these laws of motion is given at the end of this chapter.

### 6.3.3 Variances of the Variables in the Model

One needs to determine a variance for the stock to the technology process that is consistent with the data. Hansen reports that for the data set he was using (quarterly U.S. data from 1955.3 to 1984.1), output had a standard deviation of 1.76 percent. The variables we solve for in our model are log differences around a stationary state,  $\tilde{Y}_t = \ln Y_t - \ln \bar{Y}$ , so a standard error of 1.76 percent in the log of output is simply a standard error of .0176 for  $\tilde{Y}_t$ .

We use two of the laws of motion,

$$\begin{aligned}\tilde{K}_{t+1} &= a\tilde{K}_t + b\tilde{\lambda}_t, \\ \tilde{Y}_t &= c\tilde{K}_t + d\tilde{\lambda}_t,\end{aligned}$$

and the process for technology,

$$\tilde{\lambda}_t = \gamma\tilde{\lambda}_{t-1} + \varepsilon_t,$$

to find an expression for determining the standard error of  $\tilde{Y}_t$  as a function of the standard error of  $\varepsilon_t$ . We want to find the standard error for  $\varepsilon_t$ ,  $\sigma_\varepsilon$ , that will result in a standard error of  $\tilde{Y}_t$  equal to .0176 when we use the values for  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $\gamma$  that we found for the laws of motion.

First take the law of motion for capital and substitute the process for technology to get the expression

$$\tilde{K}_{t+1} = a\tilde{K}_t + b\gamma\tilde{\lambda}_{t-1} + b\varepsilon_t.$$

One then recursively substitutes lagged versions of this equation and the process for technology to get an expression for capital in terms of the history of shocks to technology,

$$\tilde{K}_{t+1} = b \sum_{i=0}^{\infty} \sum_{j=0}^i a^j \gamma^{i-j} \varepsilon_{t-i}.$$

Note that the current technology level at time  $t$ ,  $\tilde{\lambda}_t$ , can be written recursively as

$$\tilde{\lambda}_t = \sum_{i=0}^{\infty} \gamma^i \varepsilon_{t-i}.$$

These two expressions (capital lagged one period) can be substituted into the law of motion for output to give

$$\tilde{Y}_t = d\varepsilon_t + \sum_{i=0}^{\infty} \left[ cb \sum_{j=0}^i a^j \gamma^{i-j} + d\gamma^{i+1} \right] \varepsilon_{t-1-i}.$$

Since the shocks to technology are independent, the variance for output is equal to

$$\text{var}\tilde{Y}_t = \left( d^2 + \sum_{i=0}^{\infty} \left[ cb \sum_{j=0}^i a^j \gamma^{i-j} + d\gamma^{i+1} \right]^2 \right) \text{var}\varepsilon_t. \quad (6.6)$$

Table 6.1 Standard errors from model

	$\tilde{Y}_t$	$\tilde{C}_t$	$\tilde{H}_t$	$\tilde{r}_t$	$\tilde{I}_t$
Standard error	$5.484\sigma_\varepsilon$	$4.065\sigma_\varepsilon$	$1.640\sigma_\varepsilon$	$3.492\sigma_\varepsilon$	$11.742\sigma_\varepsilon$
As % of output	100%	74.12%	29.90%	63.67%	214.1%

The term in the parenthesis is difficult to solve but, since the sequence converges, can be approximated to any desired degree of precision. A Matlab program that does this can be found at the end of this chapter. Using the values from the laws of motion above and  $\gamma = .95$ , the expression in parenthesis equals 30.0757.

The standard error of .0176 for output gives  $\text{var}\tilde{Y}_t = .00030976$ , and that implies  $\text{var}\varepsilon_t = .000010299$ , or a standard error for the shock to technology of .0032.

Equation 6.6 can be used, with different values for  $c$  and  $d$ , to find the variances of consumption, hours worked, and the rental on capital. For example, to find the variance of consumption, simply use  $c = .5691$  and  $d = .3920$  along with the original values for  $a$ ,  $b$ , and  $\gamma$  and the value for the variance of the shock that we have just found. The resulting standard errors are reported in Table 6.1. The standard error for investment is found using the budget constraint,

$$\bar{I}\tilde{I}_t = \bar{Y}\tilde{Y}_t - \bar{C}\tilde{C}_t,$$

and gives a law of motion equal to

$$\tilde{I}_t = \hat{c}\tilde{K}_t + \hat{d}\tilde{\lambda}_t,$$

where  $\hat{c} = .2045\bar{Y}/\bar{I} - .5691\bar{C}/\bar{I} = -.8530$  and  $\hat{d} = 1.4523\bar{Y}/\bar{I} - 0.3920\bar{C}/\bar{I} = 4.5277$ . We use these values in equation 6.6 to find the variance for investment.

Hansen calculated the variance of logged, detrended series from the United States from 1955.3 to 1984.1, using quarterly data. As mentioned earlier, he found that the variance of output was 1.76 percent. He claimed that the standard error for the shock to technology should belong to the interval [.007, .01]. Clearly, the number we found, .0032, is much smaller than that. Hansen's data gave standard errors for consumption, hours worked, and investment as a fraction of the standard errors of output that are shown in Table 6.2.<sup>8</sup> The basic Hansen model does reasonably well with consumption but fails rather badly with investment and hours worked.

8. Data for this table comes from Hansen [48], page 321.

Table 6.2 Standard errors from Hansen's data

	$\tilde{Y}_t$	$\tilde{C}_t$	$\tilde{H}_t$	$\tilde{I}_t$
As % of output	100%	73.30%	94.32%	488.64%

#### 6.4 HANSEN'S MODEL WITH INDIVISIBLE LABOR

The structure of manufacturing production in the real world implies that a group of workers need to be present simultaneously to be able to run production lines. This need has led to the organization of labor through contracts for workweeks with a fixed number of hours. Workweeks tend to be 40 hours long, and there are relatively few workers who have the ability of smoothly adjusting the number of hours that they work. Many (if not most) are employed for a full workweek or they are unemployed. Hansen [48] adds this characteristic to a simple dynamic model.

Households sign contracts with firms to provide  $h_0$  units of labor in period  $t$  with probability  $\alpha_t$ . A random process determines if the household is one of those that works in period  $t$ . In each period,  $\alpha_t$  of the households work  $h_0$  hours and  $1 - \alpha_t$  of the households do not work at all. All households get paid the same wages whether they work or not, so the contract is like perfect unemployment insurance. Since  $\alpha_t h_0$  hours of work are being provided to the market, the labor demanded by the firms is  $h_t = \alpha_t h_0$ , and the firms pay a wage  $w_t$ , determined by the marginal product of  $h_t$  units of labor.

There is a good technical reason for introducing these unemployment insurance type contracts into the model. Without them, a family would have to decide to work or not. At a wage low enough, it would choose not to work and live off of the income from its capital holdings. Each family would have some wage,  $w^*$ , below which it would not provide work to the labor force and would have no labor income. The family's goods consumption would take a sharp jump at wage  $w^*$ , and its consumption of leisure would fall sharply at the same time. This jump upward in goods consumption and fall in consumption of leisure makes the consumption set nonconvex. A set is convex if a line joining any two points inside the set is also inside the set. The shaded part in Figure 6.2 is the set of goods consumption. The jump in consumption occurs at point  $w^*$ . Notice that the line from point  $A$  to point  $B$  has a section that is outside the set. Therefore, the goods consumption set is not convex. Utility could be continuous at point  $w^*$ , since the increase in utility from additional consumption is compensated for by the decline in utility from the loss of leisure.

Introducing the labor contracts in which wages are paid to all families but only a fraction of the families end up working (a fraction  $\alpha_t$  chosen randomly

from the set of all families) smooths out the goods consumption set and makes it convex over goods consumption and expected hours worked. Optimization problems have well-defined solutions when the objective functions are concave and the optimization takes place over convex sets. Adding unemployment insurance contracts makes our optimization problem well defined.

The budget constraint for each family is

$$c_t + i_t = w_t h_t + r_t k_t.$$

The expected utility in period  $t$  is equal to

$$u(c_t, \alpha_t) = \ln c_t + h_t \frac{A \ln(1 - h_0)}{h_0} + A(1 - \frac{h_t}{h_0}) \ln(1),$$

where  $\frac{h_t}{h_0} = \alpha_t$ , the probability that a particular household will be chosen to provide labor. Since  $\ln(1) = 0$ , the expected utility in period  $t$  reduces to

$$u(c_t, \alpha_t) = \ln c_t + h_t \frac{A \ln(1 - h_0)}{h_0}.$$

To simplify notation, define the constant

$$B = \frac{A \ln(1 - h_0)}{h_0}.$$

Given initial values for  $k_0$  and  $\lambda_0$ , an infinitely lived household maximizes

$$\max \sum_{t=0}^{\infty} \beta^t [\ln c_t + Bh_t],$$

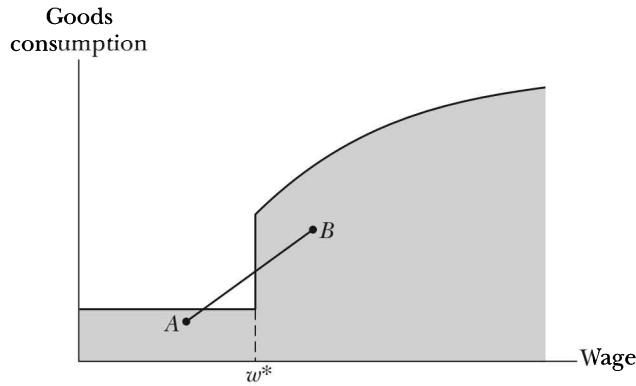


FIGURE 6.2 Nonconvex set

subject to the budget constraints

$$c_t + i_t = w_t h_t + r_t k_t$$

and

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

In this economy, the production function is the same as before, except that  $h_t = \alpha_t h_0$ . Technology follows the same stochastic process,  $\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1}$ .

The social planner's problem (which gives the same solution as the competitive equilibrium) is

$$\max \sum_{t=0}^{\infty} \beta^t [\ln c_t + Bh_t],$$

subject to

$$\lambda_t k_t^\theta h_t^{1-\theta} = c_t + k_{t+1} - (1 - \delta)k_t,$$

and

$$\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1}.$$

Recall that because production is perfectly competitive, the firms make no profits, and all income to the firms is paid out as wages and rentals; that is,

$$\lambda_t k_t^\theta h_t^{1-\theta} = w_t h_t + r_t k_t.$$

This problem can be written as a Bellman equation as

$$V(k_t, \lambda_t) = \max_{c_t, h_t} [\ln [c_t] + Bh_t + \beta E_t V(k_{t+1}, \lambda_{t+1})],$$

subject to

$$\lambda_t k_t^\theta h_t^{1-\theta} = c_t + k_{t+1} - (1 - \delta)k_t,$$

and

$$\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1}.$$

A version of the Bellman equation that is easier to work with is

$$V(k_t, \lambda_t) = \max_{k_{t+1}, h_t} \left\{ \ln [\lambda_t k_t^\theta h_t^{1-\theta} - k_{t+1} + (1 - \delta)k_t] + Bh_t + \beta E_t V(k_{t+1}, \lambda_{t+1}) \right\}.$$

The first-order conditions from this problem, after replacing  $\lambda_t k_t^\theta h_t^{1-\theta} - k_{t+1} + (1 - \delta)k_t$  with  $c_t$ , are

$$0 = \frac{1}{c_t} \left( (1 - \theta) \lambda_t k_t^\theta h_t^{1-\theta} \right) + B,$$

$$0 = -\frac{1}{c_t} + E_t \left[ \frac{1}{c_{t+1}} \theta \lambda_{t+1} k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + (1 - \delta) \right].$$

We use the factor market conditions that wages equal the marginal product of labor and rentals equal the marginal product of capital. In addition, since families are identical and there is a unit mass of them, we replace individual family variables, in lowercase, with the aggregate values, in uppercase. The first-order conditions simplify to

$$1 = \beta E_t \left[ \frac{C_t}{C_{t+1}} (r_{t+1} + (1 - \delta)) \right],$$

$$C_t = -\frac{(1 - \theta) Y_t}{B H_t}.$$

To complete the model, we add the flow budget constraint, in aggregate terms, the production function, and the two factor market conditions,

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t,$$

$$r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta},$$

$$Y_t = \lambda_t K_t^\theta H_t^{1-\theta}.$$

In the first budget constraint, the zero profit condition allowed  $w_t H_t + r_t K_t$  to be replaced by  $Y_t$ . This system is in the same form as the one for the model with divisible labor. The only difference is in the second first-order condition.

#### 6.4.1 Stationary State

The stationary state occurs when  $\bar{X} = X_t = X_{t+1}$ , for all  $t$ , for every variable  $\bar{X}$ , given that all variables in this model are measured in real terms (this will change in future chapters once we introduce money to the models). The stationary state value of technology is  $\bar{\lambda} = 1$ . The stationary state versions of the first-order condition are

$$\frac{1}{\beta} = \bar{r} + (1 - \delta)$$

and

$$\bar{C} = -\frac{(1-\theta)\bar{Y}}{B\bar{H}}. \quad (6.7)$$

The capital market condition and the production function are

$$\bar{r} = \theta\bar{K}^{\theta-1}\bar{H}^{1-\theta},$$

$$\bar{Y} = \bar{K}^\theta\bar{H}^{1-\theta}.$$

The stationary state version of the flow budget constraint is

$$\bar{C} = \bar{Y} - \delta\bar{K}.$$

These can be solved to give

$$\bar{H} = -\frac{(1-\theta)}{B\left(1 - \frac{\delta\theta\beta}{1-\beta(1-\delta)}\right)}. \quad (6.8)$$

Since the rental on capital is known from the first equation, the stationary state capital stock is found from

$$\bar{K} = \left[ \frac{\theta\beta}{1-\beta(1-\delta)} \right]^{\frac{1}{1-\theta}} \bar{H}.$$

The rest of the stationary state values follow directly.

Before we proceed with the laws of motion, it is helpful to recall that  $B$  is a function of the number of hours that a household works when it is its turn to work. The stationary state quantity of labor  $\bar{H}$  is equal to  $\bar{\alpha}h_0$ . We want to choose values of  $\bar{\alpha}$  and  $h_0$ , so that  $\bar{\alpha}h_0 = \bar{H}$ , where  $\bar{H}$  has the value it had in the first Hansen model. With this value of labor, the second equation says that the stationary state capital stock,  $\bar{K}$ , will be the same as well. With capital and labor the same as in the earlier stationary state, the production function and the budget constraint imply that output,  $\bar{Y}$ , and consumption,  $\bar{C}$ , will be the same as in the first model. We want to find the values of  $\bar{\alpha}$  and  $h_0$ . Equation 6.4 gives

$$\bar{H} = \frac{1}{1 + \frac{A}{(1-\theta)} \left[ 1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)} \right]},$$

and we want our equation 6.8 to have the same value for  $\bar{H}$ ; the definition of  $B$  implies that

$$\bar{H} = -\frac{(1-\theta)}{B \left(1 - \frac{\delta\theta\beta}{1-\beta(1-\delta)}\right)} = -\frac{(1-\theta)}{\frac{A \ln(1-h_0)}{h_0} \left(1 - \frac{\delta\theta\beta}{1-\beta(1-\delta)}\right)}.$$

Combining these two equations for  $\bar{H}$  gives

$$\frac{1}{1 + \frac{A}{(1-\theta)} \left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]} = -\frac{(1-\theta)}{\frac{A \ln(1-h_0)}{h_0} \left(1 - \frac{\delta\theta\beta}{1-\beta(1-\delta)}\right)},$$

which simplifies to

$$\frac{h_0}{\ln(1-h_0)} = -\frac{\frac{A}{(1-\theta)} \left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]}{1 + \frac{A}{(1-\theta)} \left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]} = G, \quad (6.9)$$

where the constant  $G$  is comprised of parameters of the model and, with the parameter values we have been using for the first Hansen model,  $G = -.6665$ . In Figure 6.3, the two lines are the right- and left-hand sides of equation 6.9. The lines cross where  $h_0 = .583$ , and that is the value we will use in our exercises. With  $\bar{H} = .3335$  in the first Hansen model,  $\bar{\alpha} = \bar{H}/h_0 = .3335/.583 = .572$ . For those who work in period  $t$ ,  $h_t = .583$ , and there is a 57.2 percent chance that a given household will be supplying labor in period  $t$ . For the model with these values for  $h_0$  and  $\bar{\alpha}$ , Table 6.3 gives the stationary state values for this economy.

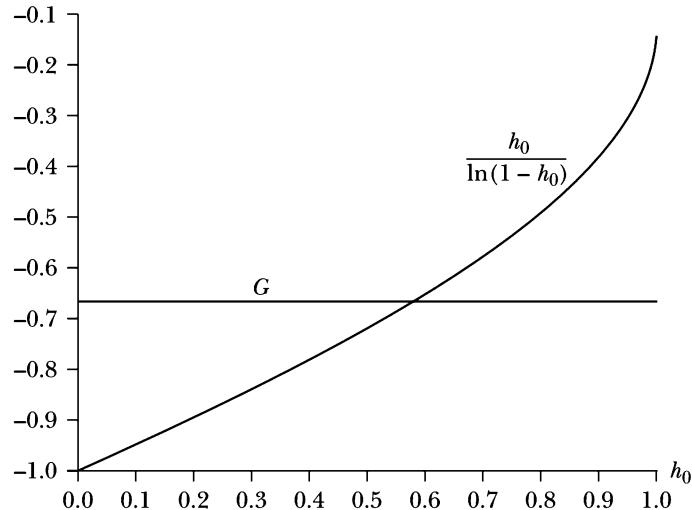


FIGURE 6.3 Finding  $h_0$

Table 6.3 Value of variables in stationary state

Variable	$\bar{H}$	$\bar{K}$	$\bar{Y}$	$\bar{C}$	$\bar{w}$	$\bar{r}$
Value	.3335	12.6698	1.2353	.9186	2.3706	.0351

#### 6.4.2 Log-Linear Version of the Indivisible Labor Model

The log-linear version of Hansen's model with indivisible labor is

$$\begin{aligned} 0 &\approx \tilde{C}_t - E_t \tilde{C}_{t+1} + \beta \bar{r} E_t \tilde{r}_{t+1}, \\ 0 &\approx \tilde{C}_t + \tilde{H}_t - \tilde{Y}_t, \\ 0 &\approx \tilde{Y}_t - \tilde{C}_t + (1 - \delta) \tilde{K}_t - \tilde{K}_{t+1}, \\ 0 &\approx \tilde{Y}_t - \tilde{\lambda}_t - \theta \tilde{K}_t - (1 - \theta) \tilde{H}_t, \end{aligned}$$

and

$$0 \approx \tilde{Y}_t - \tilde{K}_t - \tilde{r}_t.$$

Since there is only one condition (the first) in expectations, we can write the log-linear version of the model as

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\ 0 &= E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \\ z_{t+1} &= Nz_t + \varepsilon_{t+1} \quad E_t(\varepsilon_{t+1}) = 0, \end{aligned}$$

where  $x_t = [\tilde{K}_{t+1}]$ ,  $y_t = [\tilde{Y}_t, \tilde{C}_t, \tilde{H}_t, \tilde{r}_t]$ , and  $z_t = [\tilde{\lambda}_t]$ . For our model,

$$\begin{aligned} A &= [0 \quad -\bar{K} \quad 0 \quad 0]', \\ B &= [0 \quad \bar{K} \quad (1 - \delta) \quad \theta \quad -1]', \\ C &= \begin{bmatrix} 1 & -1 & -1 & 0 \\ \bar{Y} & -\bar{C} & 0 & 0 \\ -1 & 0 & (1 - \theta) & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \\ D &= [0 \quad 0 \quad 1 \quad 0]', \end{aligned}$$

$$F = [0], \quad G = [0], \quad H = [0],$$

$$J = [0 \ -1 \ 0 \ \beta\bar{r}],$$

$$K = [0 \ 1 \ 0 \ 0],$$

$$L = [0], \quad M = [0], \quad N = [\gamma].$$

As before, the solution for this economy is a set of matrices,  $P, Q, R$ , and  $S$ , that describe the equilibrium laws of motion,

$$x_t = Px_{t-1} + Qz_t$$

and

$$y_t = Rx_{t-1} + Sz_t.$$

Given these values of  $h_0$  and  $\bar{\alpha}$ , the only change in the matrices is in the third element of the first row of matrix  $C$ . The rest of the model is unchanged. Solving the model in exactly the same way we did before gives

$$\tilde{K}_{t+1} = .9418\tilde{K}_t + .1552\lambda_t$$

and

$$y_t = R\tilde{K}_t + S\lambda_t,$$

where

$$R = [0.0550, 0.5316, -0.4766, -0.9450]'$$

and

$$S = [1.9418, 0.4703, 1.4715, 1.9417]'.$$

Using these values for the laws of motion, one can calculate, as we did above, the standard errors of the variables in the vector  $y_t$ . Table 6.4 shows these standard errors.

The table shows that the relative standard (relative to output) of labor supply has increased when compared to the basic version of the model. Compared

Table 6.4 Standard errors of indivisible labor model

	$\tilde{Y}_t$	$\tilde{C}_t$	$\tilde{H}_t$	$\tilde{r}_t$	$\tilde{l}_t$
Standard errors	$6.431\sigma_\varepsilon$	$4.081\sigma_\varepsilon$	$3.444\sigma_\varepsilon$	$4.514\sigma_\varepsilon$	$15.722\sigma_\varepsilon$
As % of output	100%	63.46%	53.55%	70.19%	244.5%

to the data, the relative standard error of consumption, investment, and even labor supply are still too low, although investment is better in this model than in the basic one. Consumption standard error is worse than in the basic version. The standard error of the shock to technology,  $\sigma_\epsilon$ , is smaller than in the basic version. Since the standard error of output is  $6.431\sigma_\epsilon = .0176$ ,  $\sigma_\epsilon$  is now only .0027. This version of the model contains substantially more amplification<sup>9</sup> than the basic version, so the shock to technology that is required to account for the standard error in output is only about 85 percent of what it was in the version of this model with divisible labor.

## 6.5 IMPULSE RESPONSE FUNCTIONS

Comparing standard errors and correlations of the model with those from the data is one way of evaluating the performance of the model. At one level, we can say the model is doing a good job of representing the economy if these statistical characteristics, found either analytically as in this chapter or from simulations (as will be done in Chapter 7), are close to those of the economy. In fact, a model is frequently deemed better than others if it does a better job of mimicking the statistics of the economy.

Another way of looking at the dynamic properties of an economy, either a real one or a model, is by studying its impulse response functions. The name tells all. We are interested in observing how a model or an economy responds to an impulse applied to one of its error terms. The economy begins in a stationary state, with all shocks to stochastic processes set to zero (or their mean value if it is not zero) and, since this is a model in log differences from the stationary state, with all variables set to zero. One then applies a small, positive, one-period change to the shock of interest and calculates how the economy responds to this shock. In the log-linear version of Hansen's model, the only stochastic shock is to technology,

$$\tilde{\lambda}_t = \gamma \tilde{\lambda}_{t-1} + \varepsilon_t, \quad (6.10)$$

where the shock is  $\varepsilon_t$ . The time path the economy follows is defined by the laws of motion (the linear policy functions)

$$\tilde{K}_{t+1} = P \tilde{K}_t + Q \tilde{\lambda}_t \quad (6.11)$$

and

$$y_t = R \tilde{K}_t + S \tilde{\lambda}_t, \quad (6.12)$$

9. Amplification can be thought of as the ratio of the variance of important variables of the model, output or prices, for example, to the variance of the shocks.

where  $y_t = [\tilde{Y}_t, \tilde{C}_t, \tilde{H}_t, \tilde{r}_t]'$ . With  $\gamma = .95$ , a one-time shock to technology of  $\varepsilon_t = .01$  applied to equation 6.10 will generate the time path for technology that is shown in Figure 6.4. This time path for technology will generate the time paths for  $\tilde{K}_t$ , and for the variables in  $y_t$  that are shown in Figure 6.5. One first finds the time path for capital by recursively applying equation 6.11. Using these time paths for technology and capital, one repeatedly applies equation 6.12 to find the time paths for the variables in  $y_t$ . These time paths are the responses of the model to the one-time impulse in the technology shock,  $\varepsilon_t$ .

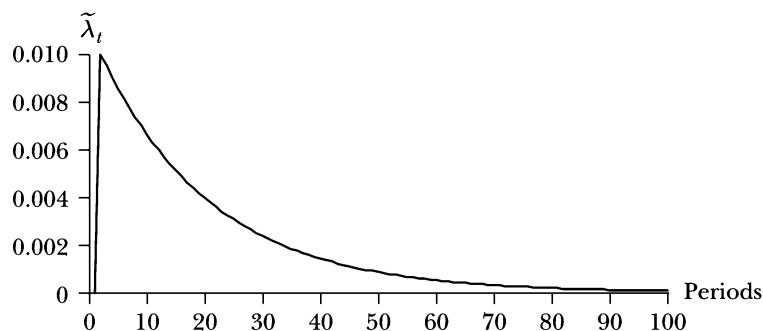


FIGURE 6.4 Response of technology to a .01 impulse

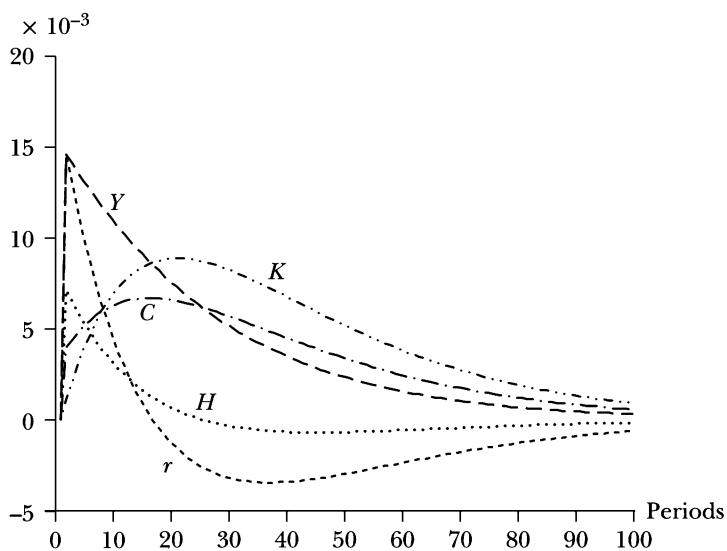


FIGURE 6.5 Responses of Hansen's basic model

Figure 6.6 shows the responses for the same variables to the same shock in Hansen's model with indivisible labor. It should be fairly clear that the model with indivisible labor produces a bigger response than does the basic model.

A useful, but as yet not very common, way to compare the responses of two models to the same impulse is to plot the responses of each variable in one model against the same variable in the second model. Imagine taking Figure 6.5 and Figure 6.6 and plotting them in the same three-dimensional graph where they share the time axis. Figure 6.7 shows this three-dimensional graph. Time goes to the back and to the left. The horizontal axis to the right measures the responses of the basic model. The vertical axis measures the responses of the model with indivisible labor. We now rotate the three-dimensional graph so that the time axis goes directly away from us and we only see a two-dimensional graph with the responses of each variable of the basic model measured on the horizontal axis and those of the model with indivisible labor on the vertical axis. This is shown in Figure 6.8. The relative response can be compared by looking at the line for each variable relative to the 45 degree line (that is drawn in). A response line above the 45 degree line means that the response of the model with indivisible labor for this variable is greater than the model with divisible labor. If a line were in the upper left or the lower right quadrants, then the responses of the corresponding variable would be in the opposite direction in the two models. In this case, all variables respond in the same direction, but

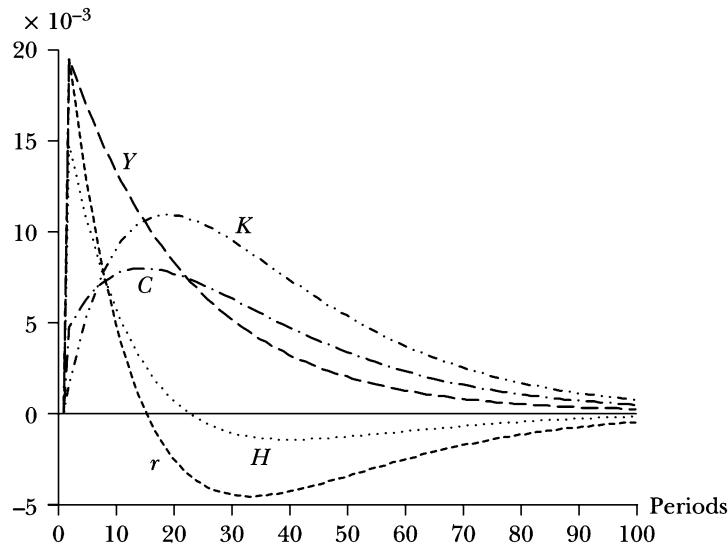


FIGURE 6.6 Responses for Hansen's model with indivisible labor

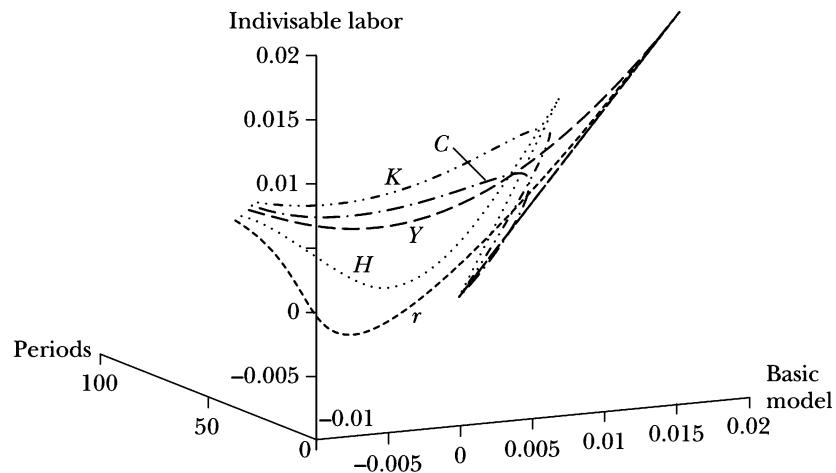


FIGURE 6.7 Responses for both Hansen models

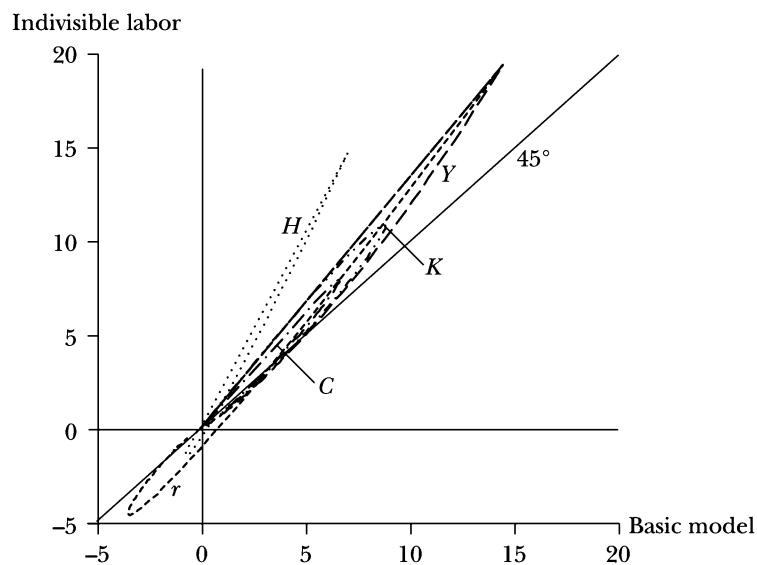


FIGURE 6.8 Comparing the response of the two models

the responses of the variables in the model with indivisible labor are larger. Figure 6.8 illustrates our observation that the economy with indivisible labor has bigger responses to a technology shock and that the biggest differences in the response are in labor,  $H$  in the figure.

## 6.6 REPRISE

Adding indivisible labor to Hansen's simple growth model produces simulations where the variance in labor is much closer to that observed for the United States. Adding the idea of indivisible labor to the economy has a number of advantages, one of the most interesting of which is that it allows one to think about unemployment and how changes in policies can end up changing unemployment. The way that indivisible labor is added results in only one technical change in the model (labor becomes linear in the utility function) and only one change in the log-linear version of the model (the coefficient on labor in the second first-order condition goes from  $1/(1 - \bar{H})$  to 1). This change in the coefficient increases the rate at which labor responds to shocks to the economy and increases the relative variance of labor compared to that of output.

Using first-order (linear) approximations of the model for dynamic simulations as we have done in this chapter is fine if one believes that the economy is quite near the stationary state and the shocks are small enough so that the economy never goes very far from that stationary state. The techniques of log linearization that are shown in this chapter are quite simple, although some care needs to be taken with the algebra. Once the log-linear version of the model was found, the fact that there was only one equation with expectational variables meant that the solution to the quadratic equation (for finding the  $P$  matrix) could be found without matrix methods. In a later chapter we will show how to use matrix methods when there is more than one equation with expectational variables.

The basic theory for solving linear difference equations in a world with rational expectations was developed by Blanchard and Kahn [14]; see Appendix 2 for an explanation of their solution method. Further extensions of what was developed in this chapter and additional detail can be found in Uhlig [86] and Christiano [28].

## 6.7 APPENDIX 1: SOLVING THE LOG-LINEAR MODEL

In this appendix we show how we find the conditions for solving the log-linear version of the model once it has been divided into a set of equations without expectations and a set of equations in expectations.

The model is divided into the sets of matrix equations

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\ 0 &= E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \end{aligned}$$

and a stochastic process

$$z_{t+1} = Nz_t + \varepsilon_{t+1} \quad E_t(\varepsilon_{t+1}) = 0.$$

The division of the model is important in the sense that we try to keep the dimension of the second (the expectational) equation small and, if possible, to have the matrix  $C$  of full rank and, therefore, invertible.

We look for a solution for linear laws of motion of the model of the form

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t, \\ y_t &= Rx_{t-1} + Sz_t. \end{aligned}$$

The problem we have is to find the values for the matrices  $P$ ,  $Q$ ,  $R$ , and  $S$ . We begin by substituting the laws of motion into the two equations above (in some cases twice) and reduce each equation to one in which there are only two variables:  $x_{t-1}$  and  $z_t$ . One needs to use the stochastic process in the expectational equation to replace  $z_{t+1}$  by  $Nz_t + \varepsilon_{t+1}$ . When one takes expectations, the  $\varepsilon_{t+1} = 0$  and disappear. After the substitutions, the two equations are

$$0 = [AP + B + CR]x_{t-1} + [AQ + CS + D]z_t$$

and

$$\begin{aligned} 0 &= [FPP + GP + H + JRP + KR]x_{t-1} \\ &\quad + [FPQ + FQN + GQ + JRQ + JSN + KS + LN + M]z_t. \end{aligned}$$

Both equations must equal zero for any initial condition  $x_{t-1}$  and for any stochastic shock  $z_t$ . Since the period  $t$  shocks are independent of the period  $t - 1$  values of the endogenous variables, the only way that the equations can equal zero is if the parts in square brackets are equal to zero. Given that the parts in square brackets are all equal to zero, we have four equations to find the four matrices we are looking for:

$$0 = AP + B + CR$$

and

$$0 = AQ + CS + D$$

from the first equation, and

$$0 = FP^2 + GP + H + JRP + KR$$

and

$$0 = FPQ + FQN + GQ + JRQ + JSN + KS + LN + M$$

from the second.

We use the first equation to solve for  $R$ ,

$$R = -C^{-1} [AP + B],$$

and the second to find  $S$ ,

$$S = -C^{-1} [AQ + D].$$

Substituting for  $R$  in the third equation results in

$$0 = FP^2 + GP + H - J \left[ C^{-1} [AP + B] \right] P - K \left[ C^{-1} [AP + B] \right],$$

and, after a bit of matrix algebra, this becomes the matrix quadratic equation

$$0 = \left[ F - JC^{-1}A \right] P^2 - \left[ JC^{-1}B - G + KC^{-1}A \right] P - KC^{-1}B + H.$$

Solving the matrix quadratic equation gives the matrix  $P$ , and once we have  $P$ , we use it in the above equation to find  $R$ .

The final expression is found by taking the fourth equation and substituting in for  $S$ . This gives

$$\begin{aligned} 0 &= FPQ + FQN + GQ + JRQ - \\ &\quad J \left[ C^{-1} [AQ + D] \right] N - K \left[ C^{-1} [AQ + D] \right] + LN + M, \end{aligned}$$

which simplifies to

$$\begin{aligned} &\left[ (F - JC^{-1}A) Q \right] N + \left[ FP + G + JR - KC^{-1}A \right] Q \\ &= \left[ JC^{-1}D - L \right] N + KC^{-1}D - M. \end{aligned}$$

This equation presents a problem. We are interested in finding the matrix  $Q$  as a linear function of the known matrices of the system, but given where  $Q$  appears in the equation, there is no simple, direct matrix manipulation that will permit us to solve for it. In the first component of the first line of this

equation,  $Q$  is the second from the last matrix and, in the second component of this line, it is the last element. To solve for  $Q$ , we make use of the following theorem from tensor algebra.<sup>10</sup>

**THEOREM 6.1** *Let  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  be matrices whose dimensions are such that the product  $\mathbf{ABC}$  exists. Then*

$$\text{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A}) \cdot \text{vec}(\mathbf{B})$$

where the symbol  $\otimes$  denotes the Kronecker product.

The operator  $\text{vec}(\mathbf{X})$  of a matrix  $\mathbf{X}$  results in column-wise vectorization of the form

$$\text{vec} \left( \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \end{bmatrix}.$$

The Kronecker product is defined (for the case where  $\mathbf{A}$  is  $2 \times 2$  and  $\mathbf{B}$  is  $3 \times 2$ ) as

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{11}b_{31} & a_{11}b_{32} & a_{12}b_{31} & a_{12}b_{32} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \\ a_{21}b_{31} & a_{21}b_{32} & a_{22}b_{31} & a_{22}b_{32} \end{bmatrix}. \end{aligned}$$

A useful corollary to the above theorem is the special case where  $\mathbf{B}$  has dimension  $n \times r$ , and  $\mathbf{C}$  is an identity matrix of dimension  $n$ ,  $\mathbf{I}_n$ .

**COROLLARY 6.1**

$$\text{vec}(\mathbf{AB}) = (\mathbf{I}_n \otimes \mathbf{A}) \text{vec}(\mathbf{B}).$$

10. The statement of this theorem comes from Hamilton [47], page 265.

The proof of the corollary comes from simply replacing  $\mathbf{C}$  in the theorem by the identity matrix  $\mathbf{I}_n$ .

Using this theorem and corollary, one can write  $[(F - JC^{-1}A) Q] N$  as

$$\text{vec} \left( \left[ (F - JC^{-1}A) Q \right] N \right) = \left( N' \otimes (F - JC^{-1}A) \right) \text{vec}(Q)$$

and  $[FP + G + JR - KC^{-1}A] Q$  as

$$\begin{aligned} & \text{vec} \left( \left[ FP + G + JR - KC^{-1}A \right] Q \right) \\ &= \left( I_k \otimes (FP + G + JR - KC^{-1}A) \right) \text{vec}(Q), \end{aligned}$$

where  $k$  is the number of columns in  $Q$ .

Using these results,

$$\begin{aligned} & \left[ (F - JC^{-1}A) Q \right] N + \left[ FP + G + JR - KC^{-1}A \right] Q \\ &= \left[ JC^{-1}D - L \right] N + KC^{-1}D - M \end{aligned}$$

can be solved for  $\text{vec}(Q)$  by applying the vec operator to both sides of the equation and getting

$$\begin{aligned} & \left( N' \otimes (F - JC^{-1}A) + I_k \otimes (FP + G + JR - KC^{-1}A) \right) \text{vec}(Q) \\ &= \text{vec} \left( \left( JC^{-1}D - L \right) N + KC^{-1}D - M \right). \end{aligned}$$

The matrix  $Q$  is constructed by separating the columns in  $\text{vec}(Q)$ , where

$$\begin{aligned} & \text{vec}(Q) \\ &= \left( N' \otimes (F - JC^{-1}A) + I_k \otimes (FP + G + JR - KC^{-1}A) \right)^{-1} \\ & \quad \times \text{vec} \left( \left( JC^{-1}D - L \right) N + KC^{-1}D - M \right). \end{aligned}$$

Once  $Q$  is defined, so is the matrix  $S$  from

$$S = -C^{-1} [AQ + D].$$

## 6.8 APPENDIX 2: BLANCHARD AND KAHN'S SOLUTION METHOD

The method that we are using in this book for solving linear dynamic stochastic general equilibrium models is known as the undetermined coefficients

method. It was originally presented by McCallum [60] and developed by Christiano [28]. What is probably the clearest exposition of the method is by Uhlig [86], although he used a solution technique different from that of Christiano. Basically, a linear form for the solution is assumed and the method finds the coefficients for the solution of this form. The assumption of a linear form for the solution is not a very great jump, since linear models generally provide linear solutions.

The undetermined coefficients solution method was not the first method in the literature for solving rational expectations models. The first method, in economics, for solving these linear rational expectations models comes from Blanchard and Kahn [14], who used techniques that were in the engineering literature (from Vaughan [88]) and applied them to macroeconomic models.<sup>11</sup> Christiano uses solution techniques similar to those of Blanchard and Kahn for solving his undetermined coefficients problems. A good expanded explanation of these methods can be found in Blake and Fernandez-Corugedo [12].

### 6.8.1 General Version

A linear model can be written (in what is known as a state space representation) as

$$B \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G \varepsilon_t, \quad (6.13)$$

where  $x_t$  is an  $(n \times 1)$  vector of predetermined variables at date  $t$ ,  $y_t$  is an  $(m \times 1)$  vector of non-predetermined variables at time  $t$ ,  $E_t y_{t+1}$  is the  $(m \times 1)$  vector of expectations for the non-predetermined variables at date  $t + 1$ ,  $\varepsilon_t$  is a  $(k \times 1)$  vector of stochastic shocks,  $A$  and  $B$  are  $((n+m) \times (n+m))$  matrices, and  $G$  is an  $((n+m) \times k)$  matrix. The difference between predetermined and non-predetermined variables is that the values of the predetermined variables at time  $t + 1$  do not depend on the values of the time  $t + 1$  shocks, while the values of the non-predetermined variables do depend on them. That is why, at time  $t$ , one can only think of the expectation of the value of a time  $t + 1$  non-predetermined variable. Its realized value will depend on the (unknown) time  $t + 1$  shock. The equations of the system that are used to make the matrices  $A$ ,  $B$ , and  $G$  are ordered so that those with expectations in them are last (given by the bottom rows of the matrices).

If the matrix  $B$  is invertible, then the first-order difference system of equation 6.13 can be written as

11. I thank Andrew P. Blake and Emilio Fernandez-Corugedo for showing me the results given here.

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B^{-1} A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B^{-1} G \varepsilon_t. \quad (6.14)$$

The matrix,  $Z = B^{-1}A$ , can be decomposed into  $Z = M\Lambda M^{-1}$ , where  $\Lambda$  is a matrix with the eigenvalues of the matrix  $Z$  on its diagonal, and where  $M$  is a matrix of the right eigenvectors. Reorder the eigenvalues from smallest to largest (along the diagonal) as  $\bar{\Lambda}$  and the corresponding matrix of eigenvectors as  $\bar{M}$  (with the same reordering, so that each eigenvector is still associated with the same eigenvalue as before). The condition of Blanchard and Kahn for finding a solution to this problem is that the number of eigenvalues that are outside the unit circle (which have an absolute value greater than one) is equal to the number of expectational variables,  $m$ .

When the number of expectational equations is equal to the number of explosive roots (the number of eigenvalues outside the unit circle), one can impose conditions on the equilibrium to guarantee that there exists a stable solution to the economy.

To begin with, we consider just the deterministic part of the model and write equation 6.14 as

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \bar{M} \bar{\Lambda} \bar{M}^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

or, after multiplying both sides by  $\bar{M}^{-1}$ , as

$$\bar{M}^{-1} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \bar{\Lambda} \bar{M}^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix}. \quad (6.15)$$

Partitioning the  $\bar{M}^{-1}$  matrix as

$$\bar{M}^{-1} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix}$$

and the matrix  $\bar{\Lambda}$  as

$$\bar{\Lambda} = \begin{bmatrix} \bar{\Lambda}_{11} & 0_{12} \\ 0_{21} & \bar{\Lambda}_{22} \end{bmatrix},$$

where a matrix  $X_{11}$  is  $n \times n$ , a matrix  $X_{12}$  is  $n \times m$ , a matrix  $X_{21}$  is  $m \times n$ , and a matrix  $X_{22}$  is  $m \times m$ . A matrix  $0_{ij}$  is a matrix of zeros of the size that corresponds to  $ij$ . The matrix  $\bar{\Lambda}_{11}$  is a diagonal matrix that contains all the stable eigenvalues of the model, and  $\bar{\Lambda}_{22}$  is a diagonal matrix with all the unstable eigenvalues (those outside the unit circle). Using this partition, equation 6.15 can be written as the two matrix equations

$$[\hat{M}_{11}x_{t+1} + \hat{M}_{12}E_t y_{t+1}] = \bar{\Lambda}_{11} [\hat{M}_{11}x_t + \hat{M}_{12}y_t] \quad (6.16)$$

and

$$[\widehat{M}_{21}x_{t+1} + \widehat{M}_{22}E_t y_{t+1}] = \bar{\Lambda}_{22} [\widehat{M}_{21}x_t + \widehat{M}_{22}y_t].$$

Given that the elements of the diagonal matrix  $\bar{\Lambda}_{22}$  are all greater than one, if  $[\widehat{M}_{21}x_t + \widehat{M}_{22}y_t]$  is ever nonzero, the model will explode. For there to be a stable solution to this problem,  $[\widehat{M}_{21}x_t + \widehat{M}_{22}y_t] = 0$ , always, and that implies that  $[\widehat{M}_{21}x_{t+1} + \widehat{M}_{22}y_{t+1}] = 0$  as well. For all this to happen, in each period, the non-predetermined variables,  $y_t$ , need to be equal to

$$y_t = -(\widehat{M}_{22})^{-1} \widehat{M}_{21}x_t.$$

This equation solves for the non-predetermined variables in each period as a linear function of the predetermined variables. Since this version of the model is deterministic,

$$E_t y_{t+1} = y_{t+1} = -(\widehat{M}_{22})^{-1} \widehat{M}_{21}x_{t+1},$$

and substituting these two results into equation 6.16, after a bit of algebra, one gets

$$x_{t+1} = [\widehat{M}_{11} - \widehat{M}_{12} (\widehat{M}_{22})^{-1} \widehat{M}_{21}]^{-1} \bar{\Lambda}_{11} [\widehat{M}_{11} - \widehat{M}_{12} (\widehat{M}_{22})^{-1} \widehat{M}_{21}] x_t.$$

### 6.8.2 Stochastic Shocks

When the economy has stochastic shocks, the solution is a bit different. We still consider only the case where  $B$  is invertible, so we can write the model in its stochastic version as

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B^{-1}A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B^{-1}G [\varepsilon_t].$$

Using the exact same eigenvalue-eigenvector decomposition as before, one gets

$$\overline{M}^{-1} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \bar{\Lambda} \overline{M}^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \overline{M}^{-1} B^{-1} G [\varepsilon_t],$$

or

$$\begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{21} & \widehat{M}_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \bar{\Lambda}_{11} & 0_{12} \\ 0_{21} & \bar{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{21} & \widehat{M}_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \widehat{G}_1 \\ \widehat{G}_2 \end{bmatrix} [\varepsilon_t],$$

where everything is as before except that we define the partition

$$\begin{bmatrix} \widehat{G}_1 \\ \widehat{G}_2 \end{bmatrix} = \overline{M}^{-1} B^{-1} G,$$

with  $\widehat{G}_1$  a  $k \times n$  matrix and  $\widehat{G}_2$  a  $k \times m$  matrix. The lower partition, that associated with the eigenvalues outside the unit circle (the exploding ones), can be written as

$$\widehat{M}_{21}x_{t+1} + \widehat{M}_{22}E_t y_{t+1} = \bar{\Lambda}_{22} [\widehat{M}_{21}x_t + \widehat{M}_{22}y_t] + \widehat{G}_2 [\varepsilon_t]$$

or, letting  $\lambda_t = \widehat{M}_{21}x_t + \widehat{M}_{22}y_t$ , as

$$E_t \lambda_{t+1} = \bar{\Lambda}_{22} \lambda_t + \widehat{G}_2 [\varepsilon_t].$$

Since the  $\bar{\Lambda}_{22}$  are all greater than one, this can be solved forward (see Sargent [71], for example) to get

$$\lambda_t = - \sum_{i=0}^{\infty} \bar{\Lambda}_{22}^{-i-1} \widehat{G}_2 E_t [\varepsilon_{t+i}].$$

Given the expectations on future shocks, the current value of  $\lambda_t$  is found from a converging sequence. If, as we normally set up a model, the expected future values of the shocks are zero, then this equation can be written as

$$\lambda_t = -\bar{\Lambda}_{22}^{-1} \widehat{G}_2 [\varepsilon_t]$$

or, writing out  $\lambda_t$ , as

$$y_t = -\widehat{M}_{22}^{-1} \widehat{M}_{21}x_t - \widehat{M}_{22}^{-1} \bar{\Lambda}_{22}^{-1} \widehat{G}_2 [\varepsilon_t].$$

The time  $t+1$  values for the predetermined variables are found just as above but using this equation for  $y_t$  and

$$E_t y_{t+1} = -\widehat{M}_{22}^{-1} \widehat{M}_{21}x_{t+1}$$

for the expected value of  $y_{t+1}$ . We use the stable part of the process and, after a bit of algebra, get the equation

$$\begin{aligned} x_{t+1} &= \left[ \widehat{M}_{11} - \widehat{M}_{12} \widehat{M}_{22}^{-1} \widehat{M}_{21} \right]^{-1} \bar{\Lambda}_{11} \left[ \widehat{M}_{11} - \widehat{M}_{12} \widehat{M}_{22}^{-1} \widehat{M}_{21} \right] x_t \\ &\quad - \left[ \widehat{M}_{11} - \widehat{M}_{12} \widehat{M}_{22}^{-1} \widehat{M}_{21} \right]^{-1} \left[ \bar{\Lambda}_{11} \widehat{M}_{12} \widehat{M}_{22}^{-1} \bar{\Lambda}_{22}^{-1} \widehat{G}_2 - \widehat{G}_1 \right] [\varepsilon_t]. \end{aligned}$$

The coefficient on  $x_t$  is exactly the same as in the deterministic case, and we add to that a fairly complicated term for the vector of shocks.

### 6.8.3 Hansen's Model and Blanchard-Kahn

We need to write out the log-linear version of Hansen's model with indivisible labor (ordered with the expectational variables last) in a state space version.

The model is

$$\begin{aligned}\bar{K}\tilde{K}_{t+1} &= \bar{Y}\tilde{Y}_t - \bar{C}\tilde{C}_t + (1-\delta)\bar{K}\tilde{K}_t, \\ \tilde{\lambda}_t &= \gamma\tilde{\lambda}_{t-1} + \varepsilon_t, \\ 0 &= \tilde{\lambda}_t - \theta\tilde{Y}_t + \theta\tilde{K}_t - (1-\theta)\tilde{C}_t, \\ 0 &= \tilde{K}_t + \tilde{r}_t - \tilde{Y}_t, \\ E_t\tilde{C}_{t+1} - \beta\bar{r}E_t\tilde{r}_{t+1} &= \tilde{C}_t.\end{aligned}$$

We have made the system somewhat smaller than the version at the beginning of section 6.4.2 by substituting out  $\tilde{H}_t$ . Write the vector of variables at time  $t+1$  as

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ \tilde{Y}_t \\ E_t \tilde{C}_{t+1} \\ E_t \tilde{r}_{t+1} \end{bmatrix},$$

where  $\tilde{K}_{t+1}$ ,  $\tilde{\lambda}_t$ , and  $\tilde{Y}_t$  are the predetermined variables and  $\tilde{C}_t$  and  $\tilde{r}_t$  are the jump (non-predetermined) variables. The model can be written in the state space form,

$$B \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G [\varepsilon_t], \quad (6.17)$$

where

$$\begin{aligned}B &= \begin{bmatrix} \bar{K} & 0 & -\bar{Y} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & \theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\bar{r}\beta \end{bmatrix}, \\ A &= \begin{bmatrix} (1-\delta)\bar{K} & 0 & 0 & -\bar{C} & 0 \\ 0 & \gamma & 0 & 0 & 0 \\ \theta & 0 & 0 & -(1-\theta) & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\end{aligned}$$

and

$$G = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The basic problem that we have with the model as it is written above is that the  $B$  matrix is not invertible. The method of Blanchard and Kahn using eigenvalues doesn't work for this version of Hansen's model.

#### 6.8.4 The Generalized Schur Method

Luckily, there are ways of decomposing the matrices  $B$  and  $A$  that do let us solve the system. The generalized Schur decomposition gives us what we need. One can follow the same basic logic as in the Blanchard-Kahn problem described above, but using a different way of handling the matrices  $B$  and  $A$ . As an extra benefit, all of the matrices that one gets using the Schur decomposition are real matrices. In the method described above, it is completely possible that some of the eigenvalues and their corresponding eigenvectors are complex. While complex variables are not normally much of a problem for computer programs like Matlab, they do make things a bit more complicated. In this section, all matrices are real ones.

A generalized Schur decomposition takes a pair of square matrices ( $B$  and  $A$ ) and decomposes them (usually using what is called a  $QZ$  algorithm) into the matrices  $T$ ,  $S$ ,  $Q$ , and  $Z$ , where

$$B = QTZ',$$

$$A = QSZ',$$

and  $Q$  and  $Z$  have the special properties that

$$QQ' = Q'Q = I = ZZ' = Z'Z$$

and where both  $T$  and  $S$  are upper triangular matrices.<sup>12</sup> The eigenvalues of the system are given by  $\lambda_{ii} = s_{ii}/t_{ii}$ , where  $s_{ii}$  and  $t_{ii}$  are the associated diagonal

12. An upper triangular matrix has all zero elements below the diagonal. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

is an upper triangular matrix.

elements of matrices  $T$  and  $S$ , respectively. We assume that the program that finds the generalized Schur decomposition produces the output with the matrices  $Q$ ,  $Z$ ,  $S$ , and  $T$  ordered so that the absolute values of the eigenvalues increase for lower rows of the matrices<sup>13</sup>.

The deterministic version<sup>14</sup> of the model of equation 6.17 can be written as

$$QTZ' \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = QSZ' \begin{bmatrix} x_t \\ y_t \end{bmatrix}.$$

Premultiplying both sides by  $Q'$  (which removes  $Q$ , since  $Q'Q = I$ ) and writing out  $Z'$  as a partitioned matrix gives

$$\begin{bmatrix} T_{11} & T_{12} \\ 0_{21} & T_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0_{21} & S_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

or, concentrating just on the equations with explosive eigenvalues, the result is

$$T_{22} \left[ Z'_{21} x_{t+1} + Z'_{22} E_t y_{t+1} \right] = S_{22} \left[ Z'_{21} x_t + Z'_{22} y_t \right].$$

We use the notation that the submatrix  $Z'_{ij}$  is the  $ij$ th part of the partition of the  $Z'$  matrix. To keep this last equation from generating an explosive path, we need the condition

$$Z'_{21} x_t + Z'_{22} y_t = 0.$$

For this condition to hold, the jump variables at time  $t$ ,  $y_t$ , must be equal to

$$y_t = -\left(Z'_{22}\right)^{-1} Z'_{21} x_t = -N x_t,$$

where we define  $N \equiv \left(Z'_{22}\right)^{-1} Z'_{21}$ . Notice that the form of this result is identical to that in the original, eigenvalue-eigenvector version of Blanchard-Kahn that we did above, but here we use the submatrices that come from the Schur decomposition.

Going back to the original model, we write

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ -N x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ -N x_t \end{bmatrix}.$$

13. The `schur` function in Scilab does this automatically, although it puts unit roots at the very bottom.

14. The stochastic version comes shortly.

The top half of this can be written as

$$[B_{11} - B_{12}N]x_{t+1} = [A_{11} - A_{12}N]x_t,$$

so one can solve this as

$$x_{t+1} = [B_{11} - B_{12}N]^{-1} [A_{11} - A_{12}N]x_t.$$

#### SOLVING THE DETERMINISTIC HANSEN MODEL

To find the linear solution of the jump variables as functions of the state variables for Hansen's model with indivisible labor, we use the stationary state values we found in the chapter and get

$$B = \begin{bmatrix} 12.6695 & 0 & -1.2353 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & .36 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -.03475 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 12.353 & 0 & 0 & -.9186 & 0 \\ 0 & .95 & 0 & 0 & 0 \\ .36 & 0 & 0 & -.64 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The generalized Schur decomposition<sup>15</sup> of the matrices  $A$  and  $B$  results in the four matrices  $S$ ,  $T$ ,  $Q$ ,  $Z$ , where

$$S = \begin{bmatrix} 0 & -6.0713 & 2.5534 & -5.6797 & -0.4798 \\ 0 & 5.2880 & -3.3924 & 6.2982 & 0.1563 \\ 0 & 0 & 0.7200 & 0.6793 & -0.0954 \\ 0 & 0 & 0 & 0.9103 & 0.5953 \\ 0 & 0 & 0 & 0 & .8228 \end{bmatrix},$$

<sup>15</sup> For a Matlab program that finds the ordered Schur decomposition, see Anderson et. al. [2]. This Matlab program can be downloaded from Evan Anderson's website at [www.math.niu.edu/~anderson](http://www.math.niu.edu/~anderson).

$$T = \begin{bmatrix} 1.6296 & -6.5107 & 3.6395 & -6.0515 & -0.1748 \\ 0 & 5.6147 & -2.9158 & 5.2866 & -0.2383 \\ 0 & 0 & 0.7579 & 0.6832 & -0.9014 \\ 0 & 0 & 0 & 0.8488 & 0.7907 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.758 & 0.6507 & 0.0124 & 0.0427 & 0 \\ 0 & 0 & -0.6993 & 0.2034 & 0.6853 \\ 0.2209 & 0.2562 & 0.624 & -0.1633 & 0.6853 \\ 0.6137 & 0.7116 & -0.2093 & 0.1115 & -0.2467 \\ 0 & 0.0682 & -0.2786 & -0.958 & 0 \end{bmatrix},$$

and

$$Z = \begin{bmatrix} 0 & 0.6779 & -0.3668 & 0.6371 & 0 \\ 0 & 0 & -0.53 & -0.3051 & 0.7912 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.3604 & -0.4318 & -0.6321 & -0.533 \\ 0 & -0.6407 & -0.631 & 0.3185 & -0.2998 \end{bmatrix}.$$

Notice that  $S$  and  $T$  are upper triangular as required. The eigenvalues of the system are found by dividing the diagonal elements of  $S$  by the corresponding diagonal elements of  $T$ , so that,

$$\text{eigenvalues} = \begin{bmatrix} 0/1.6296 \\ 5.2880/5.6147 \\ 0.7200/0.7579 \\ 0.9103/0.8488 \\ 0.8228/0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9418 \\ 0.9500 \\ 1.0725 \\ \infty \end{bmatrix}.$$

The first three elements in the eigenvalue vector have absolute values less than one, are inside the unit circle, and represent the stable roots of the system. The other two roots are greater than one, are outside the unit circle, and are unstable.

Using these matrices, the process for the values of the non-predetermined variables at time  $t$  is equal to

$$\begin{bmatrix} \tilde{C}_t \\ \tilde{r}_t \\ Y_{t-1} \end{bmatrix} = -N \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} 0.5317 & 0.4468 & 0 \\ -0.9452 & 1.8445 & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}.$$

The deterministic process for the predetermined variables is

$$[B_{11} - B_{12}N] \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ Y_t \end{bmatrix} = [A_{11} - A_{12}N] \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}.$$

Using the matrices from our model, we get

$$\begin{aligned} & \left[ \begin{bmatrix} 12.67 & 0 & -1.24 \\ 0 & 1 & 0 \\ 0 & -1 & .36 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.53 & -0.447 & 0 \\ 0.945 & -1.845 & 0 \end{bmatrix} \right] \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ Y_t \end{bmatrix} \\ &= \left[ \begin{bmatrix} 12.353 & 0 & 0 \\ 0 & .95 & 0 \\ .36 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -.919 & 0 \\ 0 & 0 \\ -.64 & 0 \end{bmatrix} \begin{bmatrix} -0.532 & -0.447 & 0 \\ 0.9452 & -1.845 & 0 \end{bmatrix} \right] \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}, \end{aligned}$$

or

$$\begin{aligned} \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ Y_t \end{bmatrix} &= \left[ \begin{bmatrix} 12.669 & 0 & -1.235 \\ 0 & 1 & 0 \\ 0 & -1 & .36 \end{bmatrix}^{-1} \begin{bmatrix} 11.865 & -0.410 & 0 \\ 0 & 0.95 & 0 \\ 0.0197 & -0.286 & 0 \end{bmatrix} \right] \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} 0.9418 & 0.1475 & 0 \\ 0 & 0.95 & 0 \\ 0.0548 & 1.8446 & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}. \end{aligned}$$

#### THE STOCHASTIC VERSION

The stochastic version of the model is

$$\begin{bmatrix} B & x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G [\varepsilon_t].$$

Using the Schur decomposition of  $B$  and  $A$  that we found above, we have

$$Q TZ' \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = Q SZ' \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G [\varepsilon_t],$$

and multiplying through by  $Q'$  gives

$$\begin{aligned} & \begin{bmatrix} T_{11} & T_{12} \\ 0_{21} & T_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} \\ &= \begin{bmatrix} S_{11} & S_{12} \\ 0_{21} & S_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} Q'_{11} & Q'_{12} \\ Q'_{21} & Q'_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} [\varepsilon_t]. \end{aligned}$$

These submatrices are ordered as above, and those associated with the explosive eigenvalues are in the bottom row. For the model to be nonexplosive, we need either

$$0 = S_{22}Z'_{21}x_t + S_{22}Z'_{22}y_t + [Q'_{21}G_1 + Q'_{22}G_2][\varepsilon_t],$$

or

$$\begin{aligned} y_t &= -[S_{22}Z'_{22}]^{-1}S_{22}Z'_{21}x_t - [S_{22}Z'_{22}]^{-1}[Q'_{21}G_1 + Q'_{22}G_2][\varepsilon_t], \\ &= -(Z'_{22})^{-1}Z'_{21}x_t - (Z'_{22})^{-1}S_{22}^{-1}[Q'_{21}G_1 + Q'_{22}G_2][\varepsilon_t]. \end{aligned}$$

The last line comes from the fact that  $[S_{22}Z'_{22}]^{-1} = (Z'_{22})^{-1}S_{22}^{-1}$ . To keep notation simple, let  $L = (Z'_{22})^{-1}S_{22}^{-1}[Q'_{21}G_1 + Q'_{22}G_2]$ . The solution to the non-predetermined variables can be written as

$$y_t = -Nx_t - L[\varepsilon_t].$$

Because the expected value of the shocks are assumed to be zero, the expectational non-predetermined variables are equal to

$$E_t y_{t+1} = -(Z'_{22})^{-1}Z'_{21}x_{t+1} = -Nx_{t+1},$$

and we write the model as

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I \\ -N \end{bmatrix} x_{t+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} [\varepsilon_t].$$

Replacing  $y_t$  with  $-Nx_t - L[\varepsilon_t]$  gives

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I \\ -N \end{bmatrix} x_{t+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ -N \end{bmatrix} x_t + \begin{bmatrix} G_1 - A_{12}L \\ G_2 - A_{22}L \end{bmatrix} [\varepsilon_t].$$

Using the stable part of the problem (the top half of the partitioned matrices) gives

$$[B_{11} - B_{12}N]x_{t+1} = [A_{11} - A_{12}N]x_t + [G_1 - A_{12}L][\varepsilon_t],$$

or

$$\begin{aligned} x_{t+1} &= [B_{11} - B_{12}N]^{-1}[A_{11} - A_{12}N]x_t + [B_{11} - B_{12}N]^{-1}[G_1 - A_{12}L][\varepsilon_t], \\ &= Cx_t + D[\varepsilon_t], \end{aligned}$$

where  $C = [B_{11} - B_{12}N]^{-1}[A_{11} - A_{12}N]$  and  $D = [B_{11} - B_{12}N]^{-1}[G_1 - A_{12}L]$ . This is the stable solution to the problem of finding the values for the predetermined variables as a function of the shock and the previous values of the predetermined variables.

### SOLVING THE STOCHASTIC HANSEN MODEL

We are looking for two equations. The first gives the values of the non-predetermined variables at time  $t$  as a function of the predetermined variables and the shocks at time  $t$ . This is the equation

$$y_t = -Nx_t - L[\varepsilon_t].$$

The second equation gives the values of the time  $t + 1$  predetermined variables as a function of the time  $t$  predetermined variables and the shocks. This equation is

$$x_{t+1} = Cx_t + D[\varepsilon_t].$$

Using the results of the above section to find a solution for the specific version of Hansen's model gives the same result for  $N = (Z'_{22})^{-1}Z'_{21}$  as was found in the deterministic solution,

$$N = \begin{bmatrix} -0.5317 & -0.4468 & 0 \\ 0.9452 & -1.8445 & 0 \end{bmatrix},$$

and a solution for  $L = (Z'_{22})^{-1}S_{22}^{-1}[Q'_{21}G_1 + Q'_{22}G_2]$  of

$$L = \begin{bmatrix} -0.4704 \\ -1.9416 \end{bmatrix}.$$

The value for  $C = [B_{11} - B_{12}N]^{-1}[A_{11} - A_{12}N]$  is the same as in the deterministic version of the model,

$$C = \begin{bmatrix} 0.9418 & 0.1474 & 0 \\ 0 & .95 & 0 \\ 0.0548 & 1.8445 & 0 \end{bmatrix},$$

and a solution of  $D = [B_{11} - B_{12}N]^{-1}[G_1 - A_{12}L]$  results in

$$D = \begin{bmatrix} 0.1552 \\ 1 \\ 1.9416 \end{bmatrix}.$$

One can write the model as

$$\begin{bmatrix} x_{t+1} \\ y_t \end{bmatrix} = \widehat{R}x_t + \widehat{S}\varepsilon_t,$$

where

$$\widehat{R} = \begin{bmatrix} 0.9418 & 0.1474 \\ 0 & .95 \\ 0.0548 & 1.8445 \\ 0.5317 & 0.4468 \\ -0.9452 & 1.8445 \end{bmatrix}$$

and

$$\widehat{S} = \begin{bmatrix} 0.1552 \\ 1 \\ 1.9416 \\ 0.4704 \\ 1.9416 \end{bmatrix}.$$

Compare these to those found using the Uhlig methods. These are

$$\tilde{K}_{t+1} = .9418\tilde{K}_t + .1552\lambda_t,$$

and for  $y_t = [\tilde{Y}_t, \tilde{C}_t, \tilde{H}_t, \tilde{r}_t]$ ,

$$y_t = R\tilde{K}_t + S\lambda_t,$$

where

$$R = \begin{bmatrix} 0.0550 \\ 0.5316 \\ -0.4766 \\ -0.9450 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1.9418 \\ 0.4703 \\ 1.4715 \\ 1.9417 \end{bmatrix}.$$

The coefficients on  $\tilde{K}_t$  are the same for all shared variables. The coefficients for shared variables in  $\widehat{S}$  are the same as those in  $S$ . The coefficients on  $\lambda_{t-1}$

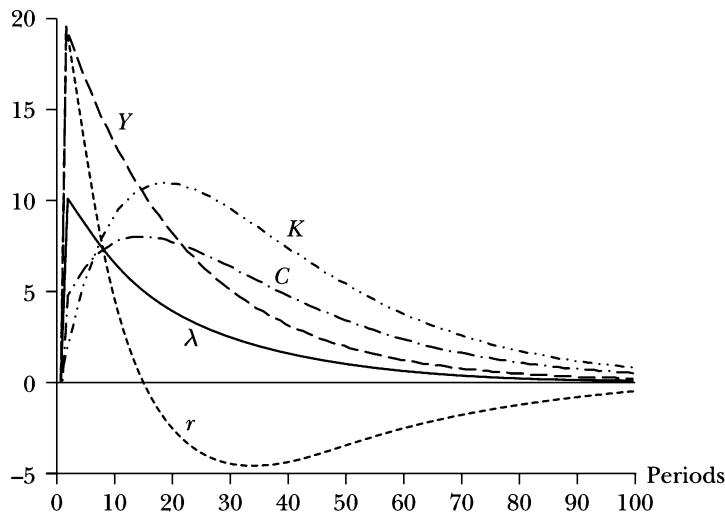


FIGURE 6.9 Responses for Hansen model solved using Schur

in  $\widehat{R}$  are .95 times the value of the corresponding coefficients in  $\widehat{S}$ , taking into account the value of the coefficient  $\gamma$  in the technology shock process.

The impulse-response functions for a shock to technology of .01 are shown in Figure 6.9. Compare these impulse responses to those found using the model solution that comes from Uhlig's solution technique, Figure 6.6. Not surprisingly, they are identical for shared variables.

## 6.9 MATLAB CODE

Three programs are presented here. The first is a basic program for finding the solution to a linear model when there is only one state variable. The example given is for the basic Hansen model but it only requires changing the matrices to get it to solve Hansen's model with indivisible labor. The second program calculates model variances from the laws of motion. The third program is a function that solves a state space version of a linear model using the method of Blanchard and Kahn and is quite general.

### 6.9.1 Solution to Basic Hansen Model

The following code solves the basic Hansen model. The code for finding the solution to the model with indivisible labor is similar. The quadratic equation is scalar, so the solution does not require a matrix technique. The names of the

variables should be clear:  $kbar = \tilde{K}$ . The stable root of the quadratic equation (that with value less than one) is chosen for finding the linear policy functions.

```
% stationary state values are found in another program
A=[0 -kbar 0 0]';
B=[0 (1-delta)*kbar theta -1]';
C=[1 -1 -1/(1-hbar) 0
    ybar -cbar 0 0
    -1 0 1-theta 0
    1 0 0 -1];
D=[0 0 1 0]';
F=[0];
G=F;
H=F;
J=[0 -1 0 beta*rbar];
K=[0 1 0 0];
L=F;
M=F;
N=[.95];
Cinv=inv(C);
a=F-J*Cinv*A;
b=-(J*Cinv*B-G+K*Cinv*A);
c=-K*Cinv*B+H;
P1=(-b+sqrt(b^2-4*a*c))/(2*a);
P2=(-b-sqrt(b^2-4*a*c))/(2*a);
if abs(P1)<1
    P=P1;
else
    P=P2;
end
R=-Cinv*(A*P+B);
Q=(J*Cinv*D-L)*N+K*Cinv*D-M;
QD=kron(N',(F-J*Cinv*A))+(J*R+F*P+G-K*Cinv*A);
Q=Q/QD;
S=-Cinv*(A*Q+D);
```

### 6.9.2 Approximating the Variances

The following code gives the function for calculating the relative variances from the laws of motion.

```
%function to approximate the ratio of the variance of a jump
%variable in the Hansen model to the variance of the shock
%to the technology process. The iteration limit is 1000.
function vr=varratio(a,b,c,d,roe,tol)
vr=0;
```

```

for i=1:1000
    shortsum=0;
    if i==1
        shortsum=1;
    else
        for j=1:(i+1)
            shortsum=shortsum+a^(j-1)*roe^(i+1-j);
        end
    end
    increment=c*b*shortsum+d*roe^i;
    vr=vr+increment*increment;
    if abs(increment)<tol
        'tol achieved'
        increment
        i
        break
    end
end
vr=vr+d*d;

```

### 6.9.3 Code for Appendix 2

The following code finds the  $N$ ,  $L$ ,  $C$ , and  $D$  matrices given the matrices  $A$ ,  $B$ , and  $G$  and the number of expectational variables,  $nx$ . The function, `schurg`, that solves the Schur decomposition is given to us from Anderson.<sup>16</sup> This program is a function that takes the output of Anderson's program and constructs the appropriate solution matrices.

```

function [N,L,C,D,alphabeta]=modelschur(A,B,G,nx)
%This program solves a model of the form
% [ xt+1 ] [xt]
% B[ ]=A[ ]+G[et]
% [Etyt+1] [yt]
%using a Schur decomposition of the matrices B and A
%nx is the number of expecational variables in Etyt+1
% if plotcode=1, impulse response is plotted
%solution is yt = -N xt - L et
%and xt+1 = C xt + D et
[Z,TT,SS,alpha,beta,info,Q] = schurg(B,A);
alphabeta=[alpha beta];
Zp=Z';
Qp=Q';
[a,b]=size(Z);

```

16. The program can be downloaded from [www.math.niu.edu/~anderson](http://www.math.niu.edu/~anderson).

```
N=inv(Zp(a-nx+1:a,a-nx+1:a))*Zp(a-nx+1:a,1:a-nx);  
L=inv(Zp(a-nx+1:a,a-nx+1:a))*inv(SS(a-nx+1:a,a-nx+1:a));  
L=L*(Qp(a-nx+1:a,1:a-nx)*G(1:a-nx,1)+Qp(a-nx+1:a,a-nx+1:a)  
*G(a-nx+1:a,1));  
invBBN=inv(B(1:a-nx,1:a-nx)-B(1:a-nx,a-nx+1:a)*N);  
C=invBBN*(A(1:a-nx,1:a-nx)-A(1:a-nx,a-nx+1:a)*N);  
D=invBBN*(G(1:a-nx,1)-A(1:a-nx,a-nx+1:a)*L);
```