4

Recursive Deterministic Models

In some infinite horizon economies, the *nature* of the optimization problem that individuals face does not depend on the period in which they are making their decisions. For example, in a Solow-type model where the subutility functions are the same in each period, where the discount factor is constant though time, and where the production functions are the same in every period, the problem that individuals face is the same, independent of the period in which they have to solve this problem. The problems that we solved in the previous chapter using variational methods required these characteristics because we were solving for stationary states. Yet, for those economies, even on a path that is not a stationary state, the nature of the problem does not change with time. What changes from period to period are the initial conditions, the values of the variables that have been determined by the past or by nature.

These kinds of problems can also be solved with *recursive methods*. One of the advantages of recursive methods is that we can solve directly for time paths that are not stationary states. As we will see later, recursive methods also permit the inclusion, in a very natural way, of stochastic shocks. With recursive methods, one looks for a *policy function*, a mapping from the initial conditions, given by the past or the present, to a set of decisions about what to do with the variables we can choose during this period. Because these are normally infinite horizon problems, how we will want to behave in the future matters in determining what we want to do today. Since what one will want to do in the future matters and the whole future time path can be determined, the recursive methods we describe are also known as *dynamic programing*.

4.1 STATES AND CONTROLS

It is helpful to separate the set of variables that we are using into state variables and control variables. In some period t, the *state variables* are those whose values are already determined, either by our actions in the past or by some other process (such as nature). Normally, for a growth model of the type we have been working with, the capital stock that we inherit from the past must be considered a state variable. One might also think that the technology level (recall our earlier model where technology was stochastic) in each period is determined by nature and therefore, in any period, the agents living in that economy must take it as a given.

The past values of other variables might be important as well. In models with habit formation,[1] past consumption matters in determining utility, so a subutility function for period t might be written as

$$u(c_t - \xi c_{t-1}).$$

In that case, an individual's utility depends on consumption today relative to past consumption. This is why it is called habit formation: since the consumer formed the "habit" of consuming a given amount in the previous period, welfare improves or declines relative to this habitual consumption. Since period $t-1$ consumption has already been determined, it must be considered a state variable for decisions that are to be made in period t.

The *control variables* in period t are those variables whose values individuals explicitly choose in that period with the goal of maximizing some objective function. Frequently, a modeler has a choice about which variables will be states and which will be control variables.

Consider the simple version of our Robinson Crusoe model. In that model, the objective function our Robinson Crusoe wants to maximize is the discounted lifetime utility of consumption,

$$\max \sum_{i=0}^{\infty} \beta^i u(c_{t+i}),$$

subject to the budget restrictions

$$k_{t+1} = (1-\delta)k_t + i_t$$

and

$$y_t = f(k_t) = c_t + i_t.$$

1. For an example of models with habit formation, see Amato and Laubach [1].

In this model, the capital stock inherited from period $t-1$, k_t, is clearly a state variable; it is predetermined and known. However, there are a number of ways of choosing a control variable.

Robinson Crusoe could directly choose consumption, c_t. Consumption appears directly in the objective function. Once consumption in period t is determined, the second budget constraint determines investment, i_t, and the first budget constraint then determines the capital stock that will be available in period $t+1$, k_{t+1}. In this case, consumption is the control variable.

An alternative would be for Robinson Crusoe to choose the amount of capital that will be available in period $t+1$, k_{t+1}. In that case, a combined budget constraint could be written as

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1},$$

and after substitution, the objective function is

$$\max \sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) + (1-\delta)k_{t+i} - k_{t+i+1}).$$

In this case, k_{t+1} is the control variable in period t.

Whatever our choice of a control variable, there must be enough budget constraints or market conditions so that the values of the rest of the relevant variables in period t are determined. What may be surprising is that the choice of control variables can matter in how easily we can solve our models. Some choices will simply be more convenient than others. This claim will be demonstrated explicitly further on.

4.2 THE VALUE FUNCTION

Assume that it is possible to calculate the value of the discounted value of utility that an agent receives when that agent is maximizing the infinite horizon objective function subject to the budget constraints. For our case of the Robinson Crusoe economy with fixed labor supply, this value is clearly a function of the initial per worker capital stock, k_t. As shown above, we can write out a version of this problem where the R.C. economy is using the capital stock to be carried over to the next period, k_{t+1}, as the control variable. For that example, the value of utility is equal to

$$V(k_t) = \max_{\{k_s\}_{s=t+1}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+1+i} + (1-\delta)k_{t+i}), \tag{4.1}$$

where we denote the value of the discounted utility by $V(k_t)$, to stress that it is a function of the value of the initial capital stock, k_t. For any value of k_t, limited to the appropriate domain, the value of the *value function*, $V(k_t)$, is the discounted value of utility when the maximization problem has been solved and when k_t was the initial capital stock.

Since $V(k_t)$ is a function, its value can be found for any permitted value of k_t. In particular, the value of the function can be found for the value of k_{t+1} that was chosen in period t. This is possible because the economy is recursive as mentioned above. In period $t+1$, the value of k_{t+1} is given (it is a state variable) and the problem to be solved is simply the maximization of utility beginning in period $t+1$. The maximization problem can be written as

$$V(k_{t+1}) = \max_{\{k_s\}_{s=t+2}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+1+i}) - k_{t+2+i} + (1-\delta)k_{t+1+i}), \tag{4.2}$$

and its value, $V(k_{t+1})$, is a function of the stock of capital per worker at time $t+1$.

By separating the period t problem from that of future periods, we can rewrite the value function of equation 4.1 as

$$\begin{aligned} V(k_t) = \max_{k_{t+1}} \Bigg[& u\left(f(k_t) - k_{t+1} + (1-\delta)k_t\right) \\ & + \max_{\{k_s\}_{s=t+2}^{\infty}} \sum_{i=1}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+1+i} + (1-\delta)k_{t+i}) \Bigg]. \end{aligned}$$

Adjusting the indices of the second part gives

$$\begin{aligned} V(k_t) = \max_{k_{t+1}} \Bigg[& u\left(f(k_t) - k_{t+1} + (1-\delta)k_t\right) \\ & + \beta \max_{\{k_s\}_{s=t+2}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+1+i}) - k_{t+2+i} + (1-\delta)k_{t+1+i}) \Bigg]. \end{aligned}$$

The summation in the last part of this equation is simply the value function $V(k_{t+1})$ that we wrote out in equation 4.2. Making the substitution, the value function in equation 4.1 can be written recursively as

$$V(k_t) = \max_{k_{t+1}} \left[u(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V(k_{t+1})\right]. \tag{4.3}$$

An equation of the form of equation 4.3 is known as a Bellman equation (Bellman [7]). It presents exactly the same problem as that shown in equation 4.1, but written in a recursive form. Writing out the problem recursively makes it conceptually simpler. The value of the choice variable, k_{t+1}, is being chosen to maximize an objective function of only a single period. The problem is reduced from one of infinite dimensions to one of only one dimension. However, the simplification comes at a cost. The problem is now one where both the time t one-period problem, $u(f(k_t) - k_{t+1} + (1-\delta)k_t)$, and the discounted value function evaluated at k_{t+1}, $\beta V(k_{t+1})$, are included. The complication is that the value of the function $V(k_{t+1})$ evaluated at k_{t+1} is not yet known. If it were known, then the value of the function $V(k_t)$ would also be known—it is the same function—and solving the maximization problem at time t would be trivial.

To proceed, we assume that the value function $V(\cdot)$ exists and has a first derivative. We can then proceed with the one-period maximization problem of equation 4.3 by taking the derivative of that equation with respect to k_{t+1}. The resulting first-order condition is

$$0 = -u'(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V'(k_{t+1}). \tag{4.4}$$

Unfortunately, we seem not to have progressed very far. The first-order condition contains $V'(k_{t+1})$,the derivative of value function $V()$, evaluated at k_{t+1}. This is inconvenient since we need to know the derivative of $V()$ to be able to determine the same function $V()$. We do not know that derivative.

Under certain conditions, and this model has been written so that the conditions hold, one can find the derivative of $V()$ simply by taking the partial derivative of the value function as written in equation 4.3 with respect to k_t. Theorems that provide the sufficient conditions for getting a derivative and that tell us how to find it are called *envelope theorems*.[2] This partial derivative is

$$V'(k_t) = u'(f(k_t) - k_{t+1} + (1-\delta)k_t)\left(f'(k_t) + (1-\delta)\right).$$

We can substitute this definition of the derivative (evaluated at k_{t+1}) into the first-order condition shown in equation 4.4 to get an Euler equation of

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left(f'(k_{t+1}) + (1-\delta)\right).$$

In a stationary state, where $c_t = c_{t+1}$, this Euler equation is

$$\frac{1}{\beta} - (1-\delta) = f'(\bar{k}).$$

Using recursive methods, we find that for a stationary state, the rental on capital is equal to the net interest rate implicit in the discount factor plus the depreciation rate. This is the same condition that we found when we solved for the stationary state using variational methods.

2. The envelope theorems we need for fairly standard economies are given in Benveniste and Scheinkman [8] and for more general economies in Milgrom and Segal [65].

4.3 A GENERAL VERSION

Let x_t be a vector of the period t state variables and let y_t be a vector of the control variables. In the example above, $x_t = [k_t]$ and $y_t = [k_{t+1}]$.[3] Let $F(x_t, y_t)$ be the time t value of the objective function that is to be maximized. In the example economy, the objective function is the utility function. Given initial values of the state variables, x_t, the problem to be solved at time t is the value function

$$V(x_t) = \max_{\{y_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, y_s),$$

subject to the set of budget constraints given by

$$x_{s+1} = G(x_s, y_s)$$

for $s \geq t$. The objective function, $F(\cdot, \cdot)$, and the budget constraints, $G(\cdot, \cdot)$, are the same for all periods $s \geq t$. Notice that, in general, we permit both time t state variables and control variables to be in the objective function and the budget constraints at time t. Using the same recursive argument that we used above, we can write the value function as a Bellman equation,

$$V(x_t) = \max_{y_t} \left[F(x_s, y_s) + \beta V(x_{t+1}) \right],$$

subject to the budget constraints

$$x_{s+1} = G(x_s, y_s),$$

or, by replacing the future value of the state variables by the budget constraints, as the single problem

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V(G(x_t, y_t)) \right]. \tag{4.5}$$

The solution to this problem gives the values of the control variables as a function of the time t state variables,

$$y_t = H(x_t).$$

We call the function $H(x_t)$ a *policy function*, since it describes how the controls behave as a function of the current state variables, x_t. Equation 4.5 is really a *functional equation*, since it must hold for every value of x_t within the permitted

3. Notice that in this particular case, it turns out that $y_t = x_{t+1}$, that the control at time t becomes the state at time $t + 1$. This is one of the aspects of the specific example that makes it simple to solve and should not be considered the normal relationship between controls and states.

domain. Solutions to functional equations are functions, which is why we call the solution to the Bellman equation a policy *function*. Since the policy function optimizes the choice of the controls for every permitted value of x_t, it must fulfill the condition that

$$V(x_t) = F(x_t, H(x_t)) + \beta V(G(x_t, H(x_t))), \tag{4.6}$$

where maximization is no longer required because it is implicit in the policy function, $H(x_t)$.

To find the policy function, $H(x_t)$, we find the first-order conditions for the problem in equation 4.5 with respect to the control variables. The first-order conditions are

$$0 = F_y(x_t, y_t) + \beta V'(G(x_t, y_t))G_y(x_t, y_t), \tag{4.7}$$

where $F_y(x_t, y_t)$ is the vector of derivatives of the objective function with respect to the control variables, $V'(G(x_t, y_t))$ is the vector of derivatives of the value function with respect to the time $t+1$ state variables, and $G_y(x_t, y_t)$ is the vector of derivatives of the budget constraints with respect to the control variables. We encounter the same problem that we had in the specific example, that we need to know the derivatives of the value function to be able to solve for the policy function, and the value function is unknown.

However, it may be possible to use the envelope theorem from Benveniste-Scheinkman [8] to find an expression for the derivative for the value function. Taking the derivative of the value function, equation 4.5, with respect to the time t state variables, x_t, one gets the Benveniste-Scheinkman envelope theorem,[4]

4. The envelope theorem of Benveniste and Scheinkman [8] gives a set of sufficient assumptions under which one can write the derivative of the value function. We are using a somewhat different notation of the form

$$V(x_t) = \max\left[F\left(x_t, y_t\right) + \beta V(x_{t+1})\right]$$

subject to the budget constraint

$$x_{t+1} = G\left(x_t, y_t\right)$$

that gives the Benveniste-Scheinkman derivative as

$$V'(x_t) = F_x\left(x_t, y_t\right) + \beta V'\left(G\left(x_t, y_t\right)\right) G_x\left(x_t, y_t\right).$$

The following four assumptions are sufficient so that the requirements of Benveniste-Scheinkman are met:

1. $x_t \in X$, where X is a convex set with a nonempty interior
2. $F(\cdot, \cdot)$ is concave and differentiable
3. $G(\cdot, \cdot)$ is concave and differentiable and invertible in y_t
4. $y_t \in Y$, where Y is a convex set with a nonempty interior

$$V'(x_t) = F_x(x_t, y_t) + \beta V'(G(x_t, y_t))G_x(x_t, y_t).$$

If, as can frequently be done, the controls have been chosen so that $G_x(x_t, y_t) = 0$, then it is possible to simplify this expression to

$$V'(x_t) = F_x(x_t, y_t).$$

The first-order conditions of equation 4.7 can be written as

$$0 = F_y(x_t, y_t) + \beta F_x(G(x_t, y_t), y_{t+1})G_y(x_t, y_t).$$

If the function $F_x(G(x_t, y_t), y_{t+1})$ is independent of y_{t+1}, then this equation can be solved for the implicit function, $y_t = H(x_t)$, which is the required policy function. One can substitute this policy function into equation 4.6 and solve for the implicit value function $V(\cdot)$. If the function $F_x(G(x_t, y_t), y_{t+1})$ is not independent of y_{t+1}, then one can solve for the stationary state as we did in the example economy above, using the condition that $y_{t+1} = y_t$ in a stationary state.

If it is not the case that $G_x(x_t, y_t) = 0$, then an alternative solution method is to find an approximation to the value function numerically.[5] Consider some initial guess for the value function, $V_0(x_t)$. It doesn't matter very much what this initial guess, is and a convenient one is to assume that it has a constant value of zero. One can then calculate an updated value function, $V_1(x_t)$, using the formula

$$V_1(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V_0(G(x_t, y_t)) \right],$$

and doing the maximization numerically (using a computer maximization algorithm) over a sufficiently dense set of values from the domain of x_t. This maximization defines, approximately, the function $V_1(x_t)$. Using this new function (and interpolating when necessary), one can update again and get a new approximate value function $V_2(x_t)$, using

$$V_2(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V_1(G(x_t, y_t)) \right]$$

over a dense set of values from the domain of x_t. Repeated application of this process results in a sequence of approximate value functions $\left\{V_i(x_t)\right\}_{i=0}^{\infty}$. Bellman showed that, under a set of conditions that are often met in economic problems, this sequence converges to the value function, $V(x_t)$.

5. The approximation can be done in any case, but if $G_x(x_t, y_t) = 0$, then one can find exact representations of the value function, and approximation is not necessary.

To get some intuition as to why this convergence might occur, consider what happens to the initial guess of the function $V_0(\cdot)$. In the calculation of $V_1(\cdot)$, one is maximizing the real objective function, $F(x_t, y_t)$, and discounting the initial guess, $V_0(\cdot)$, by β, where $0 < \beta < 1$. In the calculation of $V_2(\cdot)$, the real objective function is once again used in the maximization, and the initial guess, now hidden inside $V_1(\cdot)$, has been discounted by $\beta^2 < \beta$. As the process is repeated, the importance of the initial guess, $V_0(\cdot)$, goes to zero and what remains are the effects of the repeated maximizations of the objective function.

It turns out that in the repeated calculations of the value function, one is also calculating repeated approximations to the policy function. The limit of the sequence of y_t's that are found in the maximization process for each x_t are precisely the values that solve $y_t = H(x_t)$ for that value of x_t. The numerical method of successive approximations described above gives both the desired value function and the required policy functions.

4.4 RETURNING TO OUR EXAMPLE ECONOMY

It is useful to write out the example economy showing how each component matches with the general version.

In the example economy that we used in the first section of this chapter, we chose the capital stock at time t to be the state variable at time t, so $x_t = k_t$, and the capital stock at time $t+1$ to be the control variable, so $y_t = k_{t+1}$. In this particular example, the control variable at time t was chosen so that it becomes the state variable at time $t+1$. One should not expect this to be the normal case.

The objective function for the example economy is

$$F(x_t, y_t) = u(f(k_t) - k_{t+1} + (1-\delta)k_t),$$

and the budget constraint is written so that the time $t+1$ state variable is

$$k_{t+1} = x_{t+1} = G(x_t, y_t) = y_t = k_{t+1}.$$

This statement says simply that the time $t+1$ state variable (the part: $k_{t+1} = x_{t+1}$) is equal to the time t budget constraint, which in this case is equal to the time t control variable (that $y_t = G(x_t, y_t) = k_{t+1}$). The first-order condition for this economy is

$$\begin{aligned} 0 &= F_y(x_t, y_t) + \beta V'(G(x_t, y_t))G_y(x_t, y_t) \\ &= -u'(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V'(G(x_t, y_t)) \cdot 1. \end{aligned} \tag{4.8}$$

This choice for the budget constraint is very useful for solving the model. Recall that the Benveniste-Scheinkman envelope theorem gives

$$V'(x_t) = F_x(x_t, y_t) + \beta V'(G(x_t, y_t))G_x(x_t, y_t).$$

For our example, the derivative of the budget constraint with respect to the time t state variable is simply

$$G_x(x_t, y_t) = \frac{\partial x_{t+1}}{\partial x_t} = 0,$$

so that the envelope theorem condition can be simplified to

$$V'(x_t) = F_x(x_t, y_t) = u'(f(k_t) - k_{t+1} + (1-\delta)k_t)\left(f'(k_t) + (1-\delta)\right),$$

and the derivative of the value function is defined in terms of functions that we know. We substitute this into equation 4.8 and get the result that

$$\begin{aligned} 0 = &-u'(f(k_t) - k_{t+1} + (1-\delta)k_t) \\ &+ \beta\left[u'(f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1})\left(f'(k_{t+1}) + (1-\delta)\right)\right]. \end{aligned}$$

This second-order difference equation can be solved for the stationary state, where $\bar{k} = k_t = k_{t+1} = k_{t+2}$, to give

$$f'(\bar{k}) = \frac{1}{\beta} - (1-\delta). \tag{4.9}$$

4.4.1 Another Version of the Same Economy

The example economy can be written with different choices for the control variables. The state variable in this version is still time t capital, $x_t = k_t$, but one can choose time t consumption to be the time t control variable, $y_t = c_t$. In that case, we need to redefine the objective function and the budget constraints. With this definition of controls, the objective function is

$$F(x_t, y_t) = u(c_t),$$

and the budget constraint is

$$k_{t+1} = x_{t+1} = G(x_t, y_t) = f(k_t) + (1-\delta)k_t - c_t.$$

Writing out the model, we have the Bellman equation,

$$V(k_t) = \max_{c_t}\left[u(c_t) + \beta V\left(f(k_t) + (1-\delta)k_t - c_t\right)\right],$$

where we have replaced the time $t+1$ state variable, $x_{t+1} = k_{t+1}$, by the budget constraint in the $V(x_{t+1})$ part of the Bellman equation. It should be clear that the problem given above is the exact same economic problem that we solved in the first version of the example economy.

This version is somewhat less convenient than the earlier version when we try to write out the condition from the Benveniste-Scheinkman envelope theorem. When we take the derivative of the budget constraint with respect to the time t state variable, we get

$$\frac{\partial G(x_t, y_t)}{\partial x_t} = f'(k_t) + (1-\delta),$$

and this is generally not equal to zero. If we then write out the envelope theorem condition, we get

$$\begin{aligned} V'(x_t) &= F_x(x_t, y_t) + \beta V'(G(x_t, y_t))G_x(x_t, y_t) \\ &= \beta V'(f(k_t) + (1-\delta)k_t - c_t)\left(f'(k_t) + (1-\delta)\right), \end{aligned}$$

and we have the derivative of the value function in terms of the derivative of the value function and some other terms, which is no improvement.

One of the important tricks (simplifications) of working with the Bellman equation is to write out the objective function and the budget constraints so that one gets a convenient version of the envelope theorem, that is, so that $G_x(x_t, y_t) = 0$. Doing this usually means putting as much of the model as possible into the objective function and requires keeping the time t state variable out of the budget constraint.

4.5 AN APPROXIMATION OF THE VALUE FUNCTION

As claimed above, we can use numerical methods to find an approximation of the value function (and the policy function) for specific economies. Consider the specific functions of the example economy that we used in Chapter 3 but with fixed labor supply at $h_t = 1$, where the production function is

$$f(k_t) = k_t^\theta,$$

for $0 < \theta < 1$, and the utility function is

$$u(c_t) = \ln(c_t).$$

We can write the Bellman equation as

$$V(k_t) = \max_{k_{t+1}} \left[\ln(k_t^\theta - k_{t+1} + (1-\delta)k_t) + \beta V(k_{t+1})\right].$$

To use the recursive method of calculating the approximate $V(\cdot)$, we need to choose values for δ, θ, β, and a functional form for $V_0(\cdot)$. Let $\delta = 0.1$ and $\theta = 0.36$ as in the example economy of Chapter 1. Let $\beta = .98$, which is consistent with estimations of the discount factor for models with annual data. The simplest form to choose for the initial guess of the value function is the constant function, $V_0(k_{t+1}) \equiv 0$, for all values of k_{t+1}. Using the equation for the stationary state that we found above (equation 4.9), we find that stationary state values for this model are at $k = 0$, and where

$$.36 \cdot \bar{k}^{-.64} = \frac{1}{.98} - (1 - .1),$$

or where $\bar{k} = 5.537$.

After three iterations of the recursive approximation procedure, the calculated value function has moved from a constant at zero (the line labeled $V_0(k_t)$) to the line labeled "third iteration" in Figure 4.1. The discount parameter, β, is close to one so the iterations of the value function converge relatively slowly to the true value function. Figure 4.2 shows how the value function is converging after 240 iterations. Each of the lines shown in the figure represents the results of 48 iterations, so there are a total of 6 lines shown (the initial line at zero and the 5 lines that come from the calculations). The highest line is the last calculated and is marked 240. The number associated with each line is the number of iterations. It should be clear from the graph that the steps are gradually getting smaller as the number of iterations increases and the line moves upward.

The policy function for this economy, which finds the optimizing value of k_{t+1} for each value of k_t, is generated at the same time as the value functions. The one for our example economy, after 240 iterations, is shown in Figure 4.3

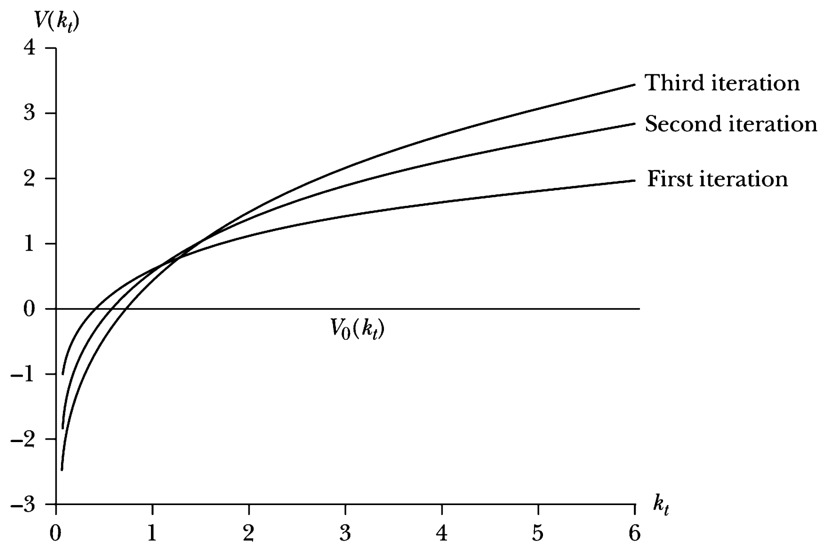

FIGURE 4.1 Value function, first three approximations

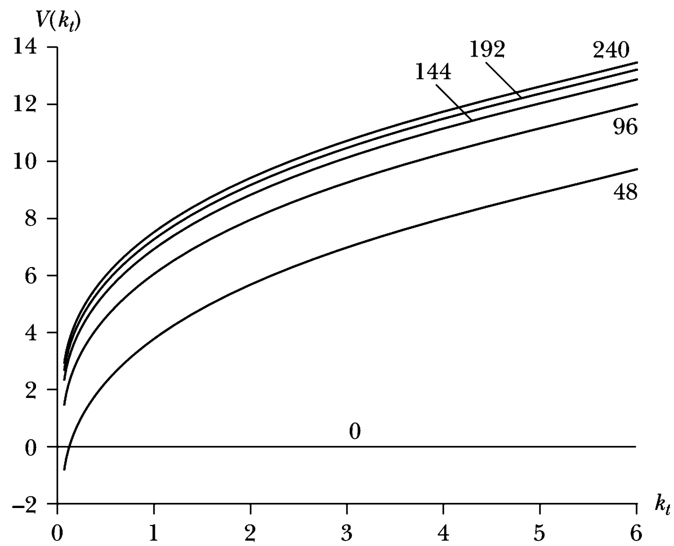

FIGURE 4.2 Approximating the value function

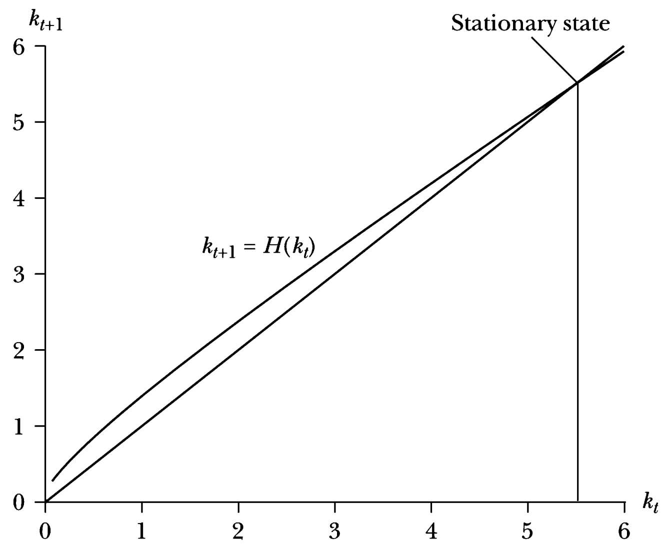

FIGURE 4.3 The policy function after 240 iterations

as the function $k_{t+1} = H(k_t)$. Notice that this function crosses the 45 degree line at the stationary state value of 5.537.

4.6 AN EXAMPLE WITH VARIABLE LABOR

It might be useful to show that recursive methods can be applied to the Robinson Crusoe economy from Chapter 3 where labor was a variable input. Recall that the utility function used there was

$$\sum_{i=0}^{\infty} \beta^i u(c_{t+i}, h_{t+i}),$$

which was maximized subject to the constraints

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + i_t, \\ y_t &= f(k_t, h_t) \geq c_t + i_t, \\ h_t &\leq 1. \end{aligned}$$

This problem can be written quite naturally as the Bellman equation,

$$V(k_t) = \max_{h_t, k_{t+1}} \left[u(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t) + \beta V(k_{t+1}) \right].$$

Here the budget constraint is

$$k_{t+1} = G(k_{t+1}) = k_{t+1},$$

which implies that the condition $G_x(x_t, y_t) = 0$ is met, so Benveniste-Scheinkman's envelope theorem condition has a simple representation.

There are now two first-order conditions since there are two controls, h_t and k_{t+1}. These conditions are

$$\begin{aligned} \frac{\partial V(k_t)}{\partial h_t} = 0 = {} & u_c(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t) f_h(k_t, h_t) \\ & + u_h(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t) \end{aligned}$$

and

$$\frac{\partial V(k_t)}{\partial k_{t+1}} = 0 = -u_c(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t) + \beta V'(k_{t+1}).$$

The envelope theorem condition is

$$V'(k_t) = u_c(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t) \left(f_k(k_t, h_t) + (1-\delta) \right).$$

These conditions result in the equations

$$\frac{u_h(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t)}{u_c(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t)} = -f_h(k_t, h_t)$$

and

$$\frac{u_c(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t)}{u_c(f(k_{t+1}, h_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}, h_{t+1})} = \beta\left[f_k(k_{t+1}, h_{t+1}) + (1-\delta)\right],$$

which are the same conditions we found for this model in Chapter 3 (equations 3.2 and 3.3) and lead to the same results for a stationary state as we found there.

This model can also be calculated numerically to find approximations for the value function, $V(\cdot)$, and for the *two* policy functions, $k_{t+1} = H^k(k_t)$ and $h_t = H^h(k_t)$. One chooses an initial guess for the value function ($V_0(\cdot) = 0$ is frequently convenient) and repeatedly calculates

$$V_{j+1}(k_t) = \max_{h_t, k_{t+1}} \left[u(f(k_t, h_t) + (1-\delta)k_t - k_{t+1}, h_t) + \beta V_j(k_{t+1})\right],$$

for $j = 0, \ldots$ over a sufficiently dense set of k_t. The sequence of functions, $V_{j+1}(k_t)$, converge to the value function, $V(k_t)$, as $j \to \infty$. As in the case above where labor was fixed, each iteration of this procedure will find the optimizing values for k_{t+1} and h_t for each member of the k_t set that was used. This sequence of functions gives approximations for the policy functions that converge to the policy functions $H^k(\cdot)$ and $H^h(\cdot)$ as $j \to \infty$.

We find the value function for an example economy similar to the one that we used earlier in this chapter: the parameters are $\delta = .1, \theta = .36, \beta = .98$, and $A = .5$, the utility function used is

$$u(c_t, h_t) = \ln(c_t) + A\ln(1 - h_t),$$

and the production function used is

$$f(k_t, h_t) = k_t^{\theta} h_t^{1-\theta}.$$

Figure 4.4 shows the approximate value functions converging upward. The lines shown are $V_m(k_t)$, for iterations numbered $m = 30, 60, 90, \ldots, 240$. Figure 4.5 shows the final policy functions (after 240 iterations) for time $t+1$ capital, $k_{t+1} = H^k(k_t)$, and for time t labor input, $h_t = H^h(k_t)$, along with the 45 degree line so that the value of k_t in the stationary state can be seen. As one might suspect, the amount of labor supplied along an equilibrium path declines as the capital stock increases.

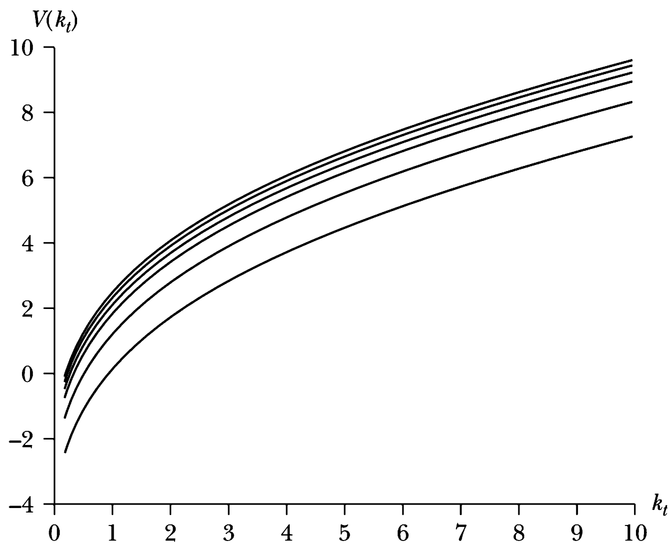

FIGURE 4.4 Approximating the pair of value functions

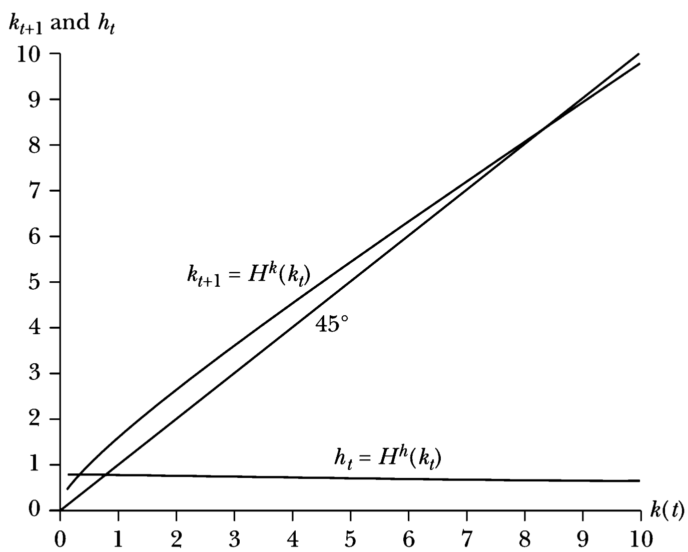

FIGURE 4.5 The two policy functions after 240 iterations

EXERCISE 4.1 A Robinson Crusoe economy has the production function $y_t = k_t^{\theta} h_t^{\psi} n_t^{1-\theta-\psi}$, where n_t is human capital used in production and $0 < \psi < 1$. Human capital grows according to the technology, $n_{t+1} = N(n_t, s_t) = (1-\gamma)\, n_t + \kappa s_t$, where $0 < \gamma < 1$ is the rate of depreciation of human capital, $\kappa > 0$, and $s_t > 0$ is the time spent studying. An individual has an endownment of one unit of labor each period and spends it according to the budget constraint

$$1 = l_t + h_t + s_t.$$

The utility function is a discounted infinite horizon one with the subutility function $u(c_t, l_t) = \ln(c_t) + A\ln(l_t)$. Write out the recursive optimization problem and find the first-order conditions. (As an additional task, modify the Matlab program at the end of this chapter to find the approximate value and policy functions.)

EXERCISE 4.2 Write out the value function for a Robinson Crusoe economy with habit persistence with a utility function of the form $u(c_t, c_{t-1}) = \ln c_t - \xi \ln c_{t-1}$ with $0 < \xi < 1$ and with the rest of the economy the same as the one in section 4.1. Define the state and control variables. Find the approximate value function using Matlab.

4.7 REPRISE

Recursive models have a particular and useful structure. In every period, an agent is solving the same optimization problem: in our case, maximizing the same additive infinite horizon discounted utility function subject to the same budget constraints. What can be different in every period are the state variables, the results of the previous period's decisions and initial conditions that determine the values of some variables today. This structure permits surmising that there exists a function that gives the value of the optimized discounted utility in each period. If such a function exists, the infinite horizon problem can be rewritten as a single-period problem, the Bellman equation.

Unfortunately, the value function is not usually known, but Bellman has shown that, for problems where the functions are given, an iterative process allows us to approximate it with whatever degree of accuracy our computer and patience permits. As we find the value function, we also find the policy functions: the functions that tell us the values we should choose for the choice variables given the state variables.

Since Bellman proved his result, the literature on dynamic programming has grown to be very large. Of particular interest to economists are the books of Ljungqvist and Sargent [54] and of Stokey, Lucas, and Prescott [83]. The first of these books is more accessible and demonstrates a wide range of economic problems that can be treated with these techniques. The second book is technically more difficult and provides the mathematical foundations for the techniques. Only a small part of these books deals with the deterministic problem, so they are also references for future chapters.

4.8 MATLAB CODE FOR FIGURES 4.2 AND 4.3

Main program

```
global vlast beta delta theta k0 kt
hold off
hold all
%set initial conditions
vlast=zeros(1,100);
k0=0.06:0.06:6;
beta=.98;
delta=.1;
theta=.36;
numits=240;
%begin the recursive calculations
for k=1:numits
   for j=1:100
      kt=j*.06;
      %find the maximum of the value function
      ktp1=fminbnd(@valfun,0.01,6.2);
      v(j)=-valfun(ktp1);
      kt1(j)=ktp1;
   end
   if k/48==round(k/48)
      %plot the steps in finding the value function
      plot(k0,v)
      drawnow
   end
   vlast=v;
end
hold off
% plot the final policy function
plot(k0,kt1)
```

Subroutine (valfun.m) to calculate value function

```
function val=valfun(k)
   global vlast beta delta theta k0 kt
```

```
   %smooth out the previous value function
   g=interp1(k0,vlast,k,'linear');
   %Calculate consumption with given parameters
   kk=kt^theta-k+(1-delta)*kt;
   if kk <= 0
      %trick to keep values from going negative
      val=-888-800*abs(kk);
   else
      %calculate the value of the value function at k
      val=log(kk)+beta*g;
   end
%change value to negative since "fminbnd" finds minimum
val=-val;
```