9

Money in the Utility Function

Putting money into a general microfoundations model is not easy. In the cash-in-advance model, it was simply assumed that money had to be used to make certain types of purchases, in our case, consumption goods. There was no real theoretical rationale for that assumption other than the empirical observation that we seem to find money being used on one side of most transactions. If one takes this empirical observation as a given, then the cash-in-advance models are fine.

A second way of introducing money into a model, introduced by Sidrauski [76], is to assume that money provides some service to the economy and that the benefits of that service can be expressed in the utility function. If one assumes that having more real balances means that one will be able to reduce the time and energy spent making transactions, for example, one might include real balances in the utility function as a way of representing these utility gains.[1] There are other ways to rationalize putting money in the utility function. Additional holdings of real balances might provide insurance against certain individual or economy-wide shocks and might permit transactions with unknown individuals (with whom credit transactions might not be feasible nor wise). Whatever the reason for the benefits derived from the holding of real balances, we assume we can approximate them in the utility function.

In this chapter we will use a utility function in which increased holdings of real balances directly increases welfare. Utility that individuals wish to maxi-

1. Of course, another way to do this would be to include transactions costs in the budget constraint and have these costs be a declining function of real balances held at the beginning of the period.

mize is still the present value of an infinite sequence of additively separable subutilities. The subutility function in period t for individual i is of the form

$$u(c_t^i, \frac{m_t^i}{P_t}, l_t^i) = u(c_t^i, \frac{m_t^i}{P_t}, 1 - h_t^i),$$

where the only important change is that we now add real balances of the individual, m_t^i/P_t, as a variable. The rationale for adding real balances to the utility function is the presumption that additional real balances reduce the cost of making transactions or reduce search (since they solve the noncoincidence-of-wants problem that arises in barter trade). One of the benefits of putting money in the utility function is that if there are other assets that individuals can hold, capital, for instance, the model will produce a real rate of return for money that is less than that of the other assets.

This benefit does not come without costs. The model doesn't explain which money should be in the utility function (dollars or pesos, for example). In addition, there is no clear use for money in the model: it doesn't do anything. Just keeping the money in your possession creates the utility. In economies with one good in which all agents are identical, no trades ever take place and money really is not ever used for anything. Nevertheless, as a rough approximation of the gains from using money, and in particular, for giving money value when there are interest-earning assets available, money-in-the-utility-function models are useful.

9.1 THE MODEL

A unit mass of identical households each choose sequences of $\{c_t^i, m_t^i, k_{t+1}^i, h_t^i\}_{t=0}^{\infty}$ to maximize the infinite horizon discounted utility function,

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, \frac{m_t^i}{P_t}, 1 - h_t^i),$$

subject to the sequence of period t budget constraints,

$$c_t^i + k_{t+1}^i + \frac{m_t^i}{P_t} = w_t h_t^i + r_t k_t^i + (1-\delta)k_t^i + \frac{m_{t-1}^i}{P_t} + (g_t - 1)\frac{M_{t-1}}{P_t}, \quad (9.1)$$

where $(g_t - 1)M_{t-1}$ is a lump sum transfer of money from the monetary authority to the household in period t. All other variables are defined as in previous chapters. The subutility function $u(\cdot)$ is

$$u(c_t^i, \frac{m_t^i}{P_t}, 1 - h_t^i) = \ln c_t^i + D \ln\left(\frac{m_t^i}{P_t}\right) + B h_t^i,$$

where D is the positive coefficient on the log of real balances. Assuming indivisible labor of h_0 and an insurance system that provides labor income in period t, whether or not the household is one of those chosen to work in period t, the coefficient B on labor in the subutility function is equal to $B = A\ln(1-h_0)/h_0$. Here there is only one budget constraint for the households since a cash-in-advance constraint no longer applies. With these assumptions, the first-order conditions for the household are

$$\frac{1}{c_t^i} = \beta E_t \frac{P_t}{c_{t+1}^i P_{t+1}} + \frac{DP_t}{m_t^i}, \tag{9.2}$$

$$\frac{1}{c_t^i} = \beta E_t \frac{1}{c_{t+1}^i}\left[r_{t+1} + (1-\delta)\right], \tag{9.3}$$

$$\frac{1}{c_t^i} = -\frac{B}{w_t}. \tag{9.4}$$

Note that in a representative agent economy or an economy where all of a mass $= 1$ of agents are identical, the first-order condition, equation 9.2, can be written in aggregate terms as

$$\frac{1}{P_t C_t} = \beta E_t \frac{1}{C_{t+1}P_{t+1}} + D\frac{1}{M_t}.$$

One can write this equation for the expected values one period into the future as

$$E_t \frac{1}{P_{t+1}C_{t+1}} = \beta E_t \frac{1}{C_{t+2}P_{t+2}} + D\frac{1}{M_{t+1}},$$

and substituting this into the first equation gives

$$\begin{aligned}\frac{1}{P_t C_t} &= \beta E_t \left[\beta E_t \frac{1}{C_{t+2}P_{t+2}} + D\frac{1}{M_{t+1}}\right] + D\frac{1}{M_t} \\ &= \beta^2 E_t \frac{1}{C_{t+2}P_{t+2}} + \beta E_t D\frac{1}{M_{t+1}} + D\frac{1}{M_t}.\end{aligned}$$

Applying this substitution repeatedly, one gets a solution for this first-order difference equation of

$$\frac{1}{P_t} = DC_t \sum_{j=0}^{\infty} \beta^j E_t \frac{1}{M_{t+j}}.$$

One can apply the money growth rule,

$$M_{t+1} = g_{t+1} M_t,$$

to get the above equation in terms of a sequence of growth rates of money,

$$\frac{1}{P_t} = \frac{DC_t}{M_t} \sum_{j=0}^{\infty} \beta^j E_t \prod_{k=1}^{j} \frac{1}{g_{t+k}}. \tag{9.5}$$

The price level is forward looking and depends on the current consumption, the amount of money with which people begin the period t, and the entire sequence of expected future growth rates of money.

Since all firms have the same Cobb-Douglas production function, one can aggregate production and treat the economy as if it had the aggregate production function

$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta},$$

with

$$\text{costs} = w_t H_t + r_t K_t.$$

This results in the competitive factor market conditions of

$$r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}$$

and

$$w_t = (1-\theta)\,\lambda_t K_t^{\theta} H_t^{-\theta}.$$

As before, we assume that the stochastic shocks for technology, λ_t, and for money growth, g_t, follow the processes

$$\ln \lambda_t = \gamma \ln \lambda_{t-1} + \varepsilon_t^{\lambda}$$

and

$$\ln g_t = (1-\pi) \ln \bar{g} + \pi \ln g_{t-1} + \varepsilon_t^{g},$$

where $\varepsilon_t^i \sim N(0, \sigma^i)$ for $i = \lambda, g$, and $\bar{g}$ is the stationary state growth rate of money.

Since all of the unit mass of households are alike and the insurance system provides the same income whether the household works or not, aggregation conditions for this economy are

$$C_t = c_t^i,$$

$$M_t = m_t^i,$$

$$H_t = h_t^i,$$

and

$$K_t = k_t^i,$$

for all $t \geq 0$.

9.2 STATIONARY STATES

A stationary state for an economy with a constant rate of money growth $\bar{g}$ is given by a set of constant real variables, $\{\bar{Y}, \bar{C}, \bar{H}, \bar{K}, \bar{w}, \bar{r}\}$, and constant real balances, $\overline{M/P}$, where all the first-order conditions of the households and firms hold, the budget constraint of households holds, and the production function defines output. Since $\overline{M/P}$ is constant in a stationary state, if the money supply grows at the rate $\bar{g}$ in a stationary state, so must prices. Therefore, the stationary state gross inflation rate, $\bar{\pi}$, must be equal to the stationary state gross growth rate of money, $\bar{g}$. The stationary state version of the first-order conditions are

$$\frac{1}{\bar{C}} = \beta \frac{P_t}{\bar{C} P_{t+1}} + D \frac{P_t}{M_t},$$

$$\bar{r} = \frac{1}{\beta} - (1 - \delta),$$

and

$$\bar{C} = -\frac{\bar{w}}{B}.$$

The stationary state budget constraint is simply

$$\bar{Y} = \bar{C} + \delta \bar{K}.$$

The factor market conditions in a stationary state are

$$\bar{r} = \theta \bar{K}^{\theta-1} \bar{H}^{1-\theta}$$

and

$$\bar{w} = (1 - \theta)\, \bar{K}^{\theta} \bar{H}^{-\theta}.$$

The rental rate on capital is given by the second first-order condition. From the equilibrium conditions for the factor markets, we get

$$\bar{w} = (1 - \theta) \left[\frac{\theta}{\bar{r}}\right]^{\frac{\theta}{1-\theta}}.$$

Once the stationary state wage is known, consumption is immediate from the third first-order condition. Since in the stationary state the gross inflation rate, $\pi = P_{t+1}/P_t$, equals the gross growth rate of money, $\bar{g}$, the first first-order condition can be written as

$$1 = \frac{\beta}{\bar{g}} + D\frac{\bar{C}}{\overline{M/P}}.$$

This expression can be rearranged to give

$$\overline{M/P} = D\frac{\bar{g}\bar{C}}{\bar{g} - \beta}. \tag{9.6}$$

Combining the household budget constraint with the expression we get from the production function and the factor market condition,

$$\bar{Y} = \bar{K}\left[\frac{\bar{r}(1-\theta)}{\bar{w}\theta}\right]^{1-\theta}, \tag{9.7}$$

gives

$$\bar{K} = \frac{\bar{C}}{\left[\frac{\bar{r}(1-\theta)}{\bar{w}\theta}\right]^{1-\theta} - \delta}.$$

With this value of $\bar{K}$, one finds hours worked as

$$\bar{H} = \left[\frac{\bar{r}(1-\theta)}{\bar{w}\theta}\right]\bar{K},$$

and output from the above expression, equation 9.7.

In stationary states for this model of money in the utility function, the growth rate of money is completely neutral with respect to the real variables in the economy. This kind of neutrality is sometimes called *superneutrality* since, in the long run, the growth rate of money has no effect at all on the real variables.

Equation 9.6 says that stationary state real money holdings are smaller in economies with higher stationary state growth rates of money. In fact, utility is maximized in a stationary state when the growth rate of money is equal to the discount factor β. This result is well known, for example, Friedman [41] was the first to conjecture this optimality condition. In the model as written here, this is somewhat problematic since equation 9.6 tells us that desired real balances are infinite when $\bar{g} = \beta$. That the desired real balances are infinite come from our use of a separable subutility function with log utility for real balances and does not necessarily hold for other formulations of the subutility

Table 9.1 Stationary state values

Variable	Stationary state value
$\bar{r}$	.035101
$\bar{w}$	2.3706
$\bar{C}$	.9187
$\bar{K}$	12.6707
$\bar{H}$	.3335
$\bar{Y}$	1.2354
$\overline{M/P}$	$\frac{.009187\bar{g}}{\bar{g}-.99}$

function. This problem with desiring an infinite amount of money has been observed before by Brock [17], using a model similar to the one here, and by Bewley [11], using a stochastic model.

For the standard economy with $\beta = .99$, $\delta = .025$, $\theta = .36$, and $B = -2.5805$, the stationary state values for the variables are given in Table 9.1. In order to get the same values when $\bar{g} = 1$ for $\overline{M/P}$ as one would have had in the cash-in-advance model (where $\bar{C} = \overline{M/P}$), we use a value for the parameter in the utility function on real balances of $D = .01$.

9.3 LOG-LINEAR VERSION OF THE MODEL

We find a version of the model in the log-linear values of the variables around their stationary state, $\tilde{X}_t = lnX_t - ln\bar{X}$. The set of variables of the system are the eight variables, $\tilde{K}_t$, $\tilde{M}_t$, $\tilde{P}_t$, $\tilde{r}_t$, $\tilde{w}_t$, $\tilde{C}_t$, $\tilde{Y}_t$, $\tilde{H}_t$, and the two stochastic shock variables, $\tilde{\lambda}_t$ and $\tilde{g}_t$. The log-linear version of the model, found using Uhlig's method, is given in the eight equations,

$$0 = \tilde{C}_t + \left[\frac{\beta}{\bar{g}} + \frac{D\bar{C}}{\overline{M/P}}\right]\tilde{P}_t - \frac{\beta}{\bar{g}}E_t\tilde{C}_{t+1} - \frac{\beta}{\bar{g}}E_t\tilde{P}_{t+1} - \frac{D\bar{C}}{\overline{M/P}}\tilde{M}_t,$$

$$0 = \tilde{w}_t + \beta\bar{r}E_t\tilde{r}_{t+1} - \beta\left[\bar{r} + (1-\delta)\right]E_t\tilde{w}_{t+1},$$

$$0 = \tilde{w}_t - \tilde{C}_t,$$

$$0 = \bar{C}\tilde{C}_t + \bar{K}\tilde{K}_{t+1} - \bar{w}\bar{H}\tilde{w}_t - \bar{w}\bar{H}\tilde{H}_t - \bar{r}\bar{K}\tilde{r}_t - \left[\bar{r} + (1-\delta)\right]\bar{K}\tilde{K}_t, \quad (9.8)$$

$$0 = \tilde{Y}_t - \tilde{\lambda}_t - \theta \tilde{K}_t - (1-\theta)\tilde{H}_t,$$

$$0 = \tilde{r}_t - \tilde{\lambda}_t - (\theta - 1)\, \tilde{K}_t - (1-\theta)\tilde{H}_t,$$

$$0 = \tilde{w}_t - \tilde{\lambda}_t - \theta \tilde{K}_t + \theta \tilde{H}_t,$$

$$0 = \tilde{M}_t - \tilde{g}_t - \tilde{M}_{t-1}. \tag{9.9}$$

In addition, we have the two processes for the log-linear form of the stochastic variables,

$$\tilde{\lambda}_t = \gamma \tilde{\lambda}_{t-1} + \varepsilon_t^{\lambda}$$

and

$$\tilde{g}_t = \gamma \tilde{g}_{t-1} + \varepsilon_t^{g}.$$[2]

The system can be written in terms of three "state" variables,[3] $x_t = [\tilde{K}_{t+1}, \tilde{M}_t, \tilde{P}_t]$, the five "jump" variables, $y_t = [\tilde{r}_t, \tilde{w}_t, \tilde{C}_t, \tilde{Y}_t, \tilde{H}_t]$, and the stochastic variables, $z_t = [\tilde{\lambda}_t, \tilde{g}_t]$, as

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t,$$

$$0 = E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right],$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1},$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \bar{K} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -\left[\bar{r} + (1-\delta)\right]\bar{K} & 0 & 0 \\ -\theta & 0 & 0 \\ (1-\theta) & 0 & 0 \\ -\theta & 0 & 0 \end{bmatrix},$$

2. Recall that $\ln g_t = (1-\pi)\ln \bar{g} + \pi \ln g_{t-1} + \varepsilon_t^g$ can be written as $\ln g_t - \ln \bar{g} = \pi\left[\ln g_{t-1} - \ln \bar{g}\right] + \varepsilon_t^g$, which is equal to $\tilde{g}_t = \gamma \tilde{g}_{t-1} + \varepsilon_t^g$ by the definition of $\tilde{g}_t$.

3. As we will see below, $\tilde{P}_t$ is not really a state variable.

$$C=\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ -\bar{r}\bar{K} & -\bar{w}\bar{H} & \bar{C} & 0 & -\bar{w}\bar{H} \\ 0 & 0 & 0 & 1 & -(1-\theta) \\ 1 & 0 & 0 & 0 & -(1-\theta) \\ 0 & 1 & 0 & 0 & \theta \end{bmatrix},$$

$$D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ -1 & 0 \end{bmatrix},$$

$$F=\begin{bmatrix} 0 & 0 & -\frac{\beta}{\bar{g}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G=\begin{bmatrix} 0 & -\frac{D\bar{C}}{\overline{M/P}} & \frac{\beta}{\bar{g}}+\frac{D\bar{C}}{\overline{M/P}} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$H=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

$$J=\begin{bmatrix} 0 & 0 & -\frac{\beta}{\bar{g}} & 0 & 0 \\ \beta\bar{r} & -\beta\left[\bar{r}+(1-\delta)\right] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix}.$$

Solving the above system gives matrix policy equations of the form

$$x_{t+1} = Px_t + Qz_t$$

and

$$y_t = Rx_t + Sz_t.$$

When $\bar{g} = 1$, the P, Q, R, and S matrices are

$$P = \begin{bmatrix} 0.9418 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5316 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.1552 & 0 \\ 0 & 1 \\ -0.4703 & 1.6648 \end{bmatrix},$$

$$R = \begin{bmatrix} -0.9450 & 0 & 0 \\ 0.5316 & 0 & 0 \\ 0.5316 & 0 & 0 \\ 0.0550 & 0 & 0 \\ -0.4766 & 0 & 0 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 1.9417 & 0 \\ 0.4703 & 0 \\ 0.4703 & 0 \\ 1.9417 & 0 \\ 1.4715 & 0 \end{bmatrix}.$$

The policy matrices show a pair of interesting characteristics of the model. First, prices are not really a state variable and, for that reason, the coefficients on prices in the matrices P and R are all zero. Second, the real variables of the economy are not affected by money growth shocks in this model. The only variables that respond to the money growth shocks are M and P. Interestingly, they do not respond in exactly the same way. As was shown earlier, equation 9.5, the current price level is determined by expected future growth in the money

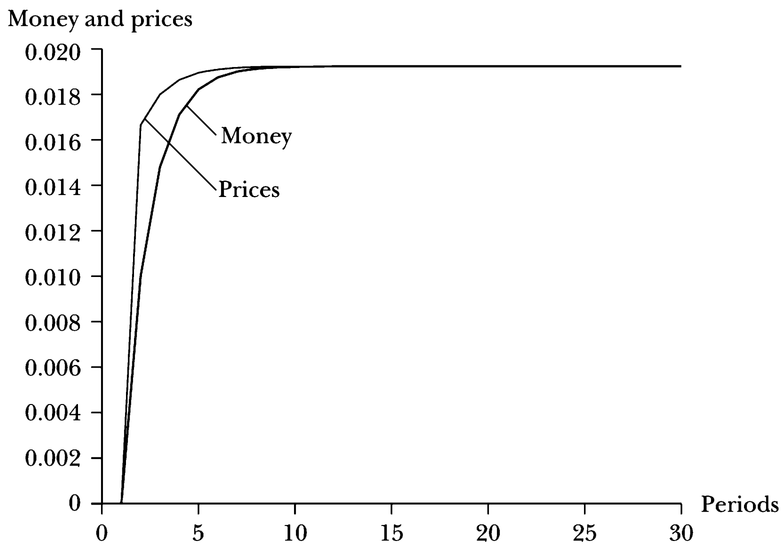

FIGURE 9.1 Response of money and prices to money growth shock

supply. A single shock to the gross growth rate of money continues to cause the money supply to grow for some time. This occurs because of the π parameter on the lagged shock in the money stock growth equation. Prices take this into account and adjust immediately, although not completely because of the β discount factor, to the single shock. Figure 9.1 shows the response functions for the money supply and for forward-looking prices in response to a single shock (impulse) to the growth rate of money. Because prices adjust much more rapidly than does the money stock, real balances decline in response to a money supply growth shock.

The responses of the real variables to a technology shock are the same in this model as they have been in previous models. Figure 9.2 shows these responses. Compare this figure to the Cooley-Hansen model in Figure 8.2. They are identical. The real sides of these two economies are exactly the same.

9.4 SEIGNIORAGE

Consider the same model as above but where the government finances a stochastic real expenditure g_t by issuing new money in period t. The budget constraint for the government is

$$g_t = \widehat{g}_t \bar{g} = \frac{M_t - M_{t-1}}{P_t}.$$

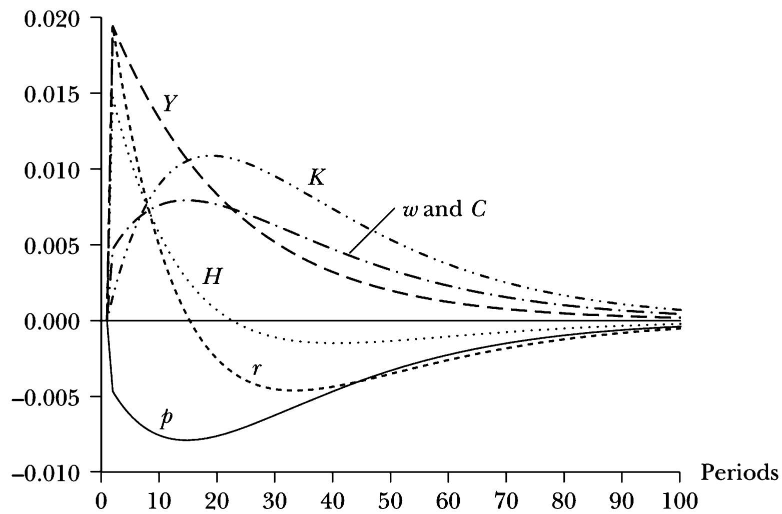

FIGURE 9.2 Responses to a .01 impulse in technology

As in the chapter on cash-in-advance money, we let $\bar{g}$ be the average government deficit financed with money issue and the random variable $\widehat{g}_t$ follows the process

$$\ln \widehat{g}_t = \pi \ln \widehat{g}_{t-1} + \varepsilon_t^g,$$

where $\varepsilon_t^g \sim N(0, \sigma_g)$, with σ_g as the standard error.[4] Define φ_t as the gross growth rate of money that is required to finance the budget deficit in period t. With this definition, $M_t = \varphi_t M_{t-1}$, and

$$g_t = \widehat{g}_t \bar{g} = \frac{(\varphi_t - 1) M_{t-1}}{P_t} = \frac{(\varphi_t - 1)}{\varphi_t} \frac{M_t}{P_t}.$$

This equation will be useful for finding the expression for a stationary state.

The only other change, compared to the previous model, is in the household flow budget constraint, equation 9.1. Households no longer receive direct transfers of money from the government. Instead, the flow of new money into the economy occurs through government purchases of goods. These purchases compete with those of the households. An important assumption here

4. We are committing something of a notational sin here. Before, g_t was the gross growth rate of money; here it stands for real government consumption and is related to the growth rate of money, φ_t, through the government's budget constraint.

is that the government's consumption of goods does not enter directly into household utility. With these considerations, household i's flow budget constraint is

$$c_t^i + k_{t+1}^i + \frac{m_t^i}{P_t} = w_t h_t^i + r_t k_t^i + (1-\delta)k_t^i + \frac{m_{t-1}^i}{P_t}.$$

The first-order conditions for the household are the same as in the earlier version of the money in the utility function model. The only differences in the model are the change in the household budget constraint and in the government's budget constraint. The extra equation, the government's budget constraint, relates money supply growth to government consumption.

9.4.1 The Full Model

The full model for the economy with seigniorage, after the aggregation has been done, is given by

$$\begin{aligned}
\frac{1}{C_t} &= \beta E_t \frac{P_t}{C_{t+1}P_{t+1}} + \frac{DP_t}{M_t}, \\
\frac{1}{w_t} &= \beta E_t \frac{1}{w_{t+1}} \left[r_{t+1} + (1-\delta)\right], \\
w_t &= -BC_t, \\
C_t + K_{t+1} + \frac{M_t}{P_t} &= w_t H_t + r_t K_t + (1-\delta)K_t + \frac{M_{t-1}}{P_t}, \\
Y_t &= \lambda_t K_t^{\theta} H_t^{1-\theta}, \\
r_t &= \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}, \\
w_t &= (1-\theta)\, \lambda_t K_t^{\theta} H_t^{-\theta}, \\
\widehat{g}_t \bar{g} &= \frac{M_t - M_{t-1}}{P_t}.
\end{aligned}$$

This model is the same as the first except for the fourth equation, which does not contain the transfers of money from the government, and the last equation, which can be thought of as either the government's budget constraint or the seigniorage equation.

9.4.2 Stationary States

Stationary states are defined by constant government deficit financed by a constant growth rate of money, constant values for the rest of the real variables, and constant real balances, M_t/P_t. Given an average government deficit, $\bar{g}$, the government budget constraint can be written as

$$\bar{g} = \left(1 - \frac{1}{\bar{\varphi}}\right) \overline{M/P},$$

where, as before, a bar over a variable is its stationary state value. In a stationary state, the household flow budget constraint is

$$\bar{C} + \left[1 - \frac{1}{\bar{\varphi}}\right] \overline{M/P} = \bar{w}\bar{H} + (\bar{r} - \delta)\,\bar{K}.$$

This flow budget constraint implies the feasibility constraint

$$\bar{C} + \bar{g} + \delta\bar{K} = \bar{Y}$$

that output in the stationary states goes to consumption, to government consumption, and to replace the depreciated capital stock.

As before, we find that rentals, wages, and consumption are independent of the rate of growth of the money supply. In particular,

$$\bar{r} = \frac{1}{\beta} - (1 - \delta),$$

$$\bar{w} = (1 - \theta)\left[\frac{\theta}{\bar{r}}\right]^{\frac{\theta}{1-\theta}},$$

and

$$\bar{C} = -\frac{\bar{w}}{B}.$$

From the first-order condition for money holdings, one gets

$$\overline{M/P} = D\frac{\bar{\varphi}\bar{C}}{\bar{\varphi} - \beta},$$

and from the government budget constraint

$$\bar{g} = \left(1 - \frac{1}{\bar{\varphi}}\right) D\frac{\bar{\varphi}\bar{C}}{\bar{\varphi} - \beta} = \frac{(\bar{\varphi} - 1)\,D}{\bar{\varphi} - \beta}\bar{C}.$$

From the factor market conditions,

$$\bar{Y} = \left[\frac{(1 - \theta)\bar{r}}{\theta\bar{w}}\right]^{1-\theta}\bar{K}.$$

Putting these into the feasibility constraint gives

$$\left[1+\frac{(\bar{\varphi}-1)\,D}{\bar{\varphi}-\beta}\right]\bar{C}=\left(\left[\frac{(1-\theta)\bar{r}}{\theta\bar{w}}\right]^{1-\theta}-\delta\right)\bar{K},$$

or

$$\bar{K}=\frac{\left[1+\frac{(\bar{\varphi}-1)D}{\bar{\varphi}-\beta}\right]}{\left[\frac{(1-\theta)\bar{r}}{\theta\bar{w}}\right]^{1-\theta}-\delta}\bar{C}.$$

Once $\bar{K}$ is determined, from the equilibrium condition for the labor market, one gets

$$\bar{H}=\left(\frac{1-\theta}{\bar{w}}\right)^{\frac{1}{\theta}}\bar{K},$$

and from the production function,

$$\bar{Y}=\bar{K}^{\theta}\bar{H}^{1-\theta}.$$

For our standard economy, with $D=.01$, the Bailey curve, the relationship between the stationary state money growth rate (which equals the inflation rate in stationary states) and seigniorage, is shown in Figure 9.3. The stationary state values of variables for this economy at annual inflation rates of 0 percent, 10 percent, 100 percent, and 400 percent are given in Table 9.2.

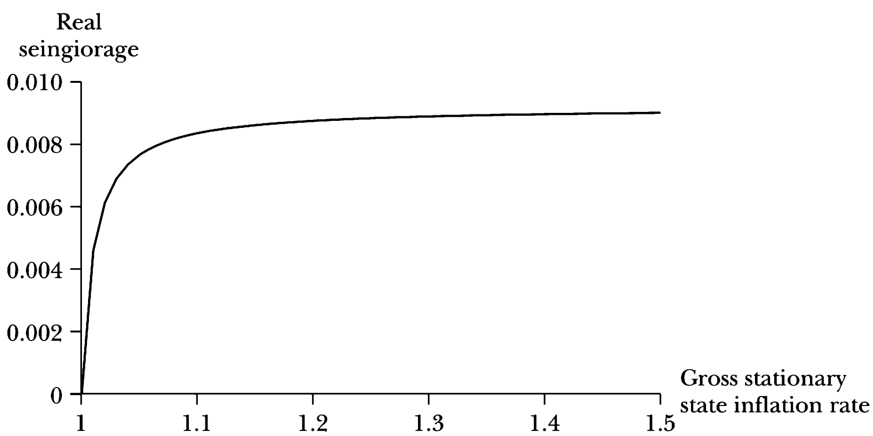

FIGURE 9.3 Real seigniorage

The structure of the model makes consumption constant in stationary states. The rental on capital is determined by the "Fisher" equation. The factor market equations then give us the real wage (since $\bar{r}$ determines the capital-labor ratio). Consumption is determined by the real wage. All three of these variables were determined without any information about the rate of money creation, so they are independent of the rate of money growth. Since consumption is the same in every stationary state, higher seigniorage must come from higher output (which requires more labor and capital at fixed wage-rental ratios). Therefore, seigniorage is generated by higher levels of capital stock and of hours worked that produce additional output to cover the seigniorage and the additional depreciation that must be replaced to be in a stationary state with higher capital stock.

9.4.3 Log Linearization

The log-linear version of this model is the same as that of the previous except for three changes. The household flow budget constraint, equation 9.8, is replaced by

$$0 = \bar{C}\tilde{C}_t + \bar{K}\tilde{K}_{t+1} + \overline{M/P}\tilde{M}_t - \overline{M/P}\left(1 - \frac{1}{\bar{\varphi}}\right)\tilde{P}_t$$
$$- \bar{w}\bar{H}\tilde{w}_t - \bar{w}\bar{H}\tilde{H}_t - \bar{r}\bar{K}\tilde{r}_t - [\bar{r} + (1-\delta)]\,\bar{K}\tilde{K}_t - \frac{\overline{M/P}}{\bar{\varphi}}\tilde{M}_{t-1}.$$

One replaces the money growth rule, equation 9.9, with the seigniorage equation

Table 9.2 Stationary states for different inflation rates

Annual inflation	0%	10%	100%	400%
Corresponding $\bar{\varphi}$	1	1.024	1.19	1.41
Rental	0.0351	0.0351	0.0351	0.0351
Wages	2.3706	2.3706	2.3706	2.3706
Consumption	0.9187	0.9187	0.9187	0.9187
Real balances $=\overline{M/P}$	0.9187	0.2767	0.0547	0.0308
Output	1.2354	1.2441	1.2472	1.2475
Capital	12.6707	12.7601	12.7910	12.7943
Hours worked	0.3335	0.3359	0.3367	0.3368
Seigniorage $= \bar{g}$	0	0.0065	0.0087	0.0090

$$0 = \bar{g}\tilde{g}_t + \overline{M/P}\left(1 - \frac{1}{\bar{\varphi}}\right)\tilde{P}_t - \overline{M/P}\tilde{M}_t + \frac{\overline{M/P}}{\bar{\varphi}}\tilde{M}_{t-1},$$

where $\tilde{g}_t \equiv \ln \widehat{g}_t - \ln 1 = \ln \widehat{g}_t$ is the log of the shock to the government deficit that is financed by seigniorage. The stationary state value of $\widehat{g}_t$ is 1.

Defining x_t , y_t,and z_t as before, so $x_t = [\tilde{K}_{t+1}, \tilde{M}_t, \tilde{P}_t]$, $y_t = [\tilde{r}_t, \tilde{w}_t, \tilde{C}_t, \tilde{Y}_t, \tilde{H}_t]$, and $z_t = [\tilde{\lambda}_t, \tilde{g}_t]$, one can write the model in matrix form,

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t,\\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right],\\
z_{t+1} &= Nz_t + \varepsilon_{t+1},
\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \bar{K} & \overline{M/P} & -\overline{M/P}\left(1 - \frac{1}{\bar{\varphi}}\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -\left[\bar{r} + (1-\delta)\right]\bar{K} & -\frac{\overline{M/P}}{\bar{\varphi}} & 0 \\ -\theta & 0 & 0 \\ (1-\theta) & 0 & 0 \\ -\theta & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ -\bar{r}\bar{K} & -\bar{w}\bar{H} & \bar{C} & 0 & -\bar{w}\bar{H} \\ 0 & 0 & 0 & 1 & -(1-\theta) \\ 1 & 0 & 0 & 0 & -(1-\theta) \\ 0 & 1 & 0 & 0 & \theta \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ -1 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & -\frac{\beta}{\bar{\varphi}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & -\frac{D\bar{C}}{\overline{M/P}} & \frac{\beta}{\bar{\varphi}} + \frac{D\bar{C}}{\overline{M/P}} \\ 0 & 0 & 0 \\ 0 & -\overline{M/P} & \left[1 - \frac{1}{\bar{\varphi}}\right] \overline{M/P} \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\overline{M/P}}{\bar{\varphi}} & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 & -\frac{\beta}{\bar{\varphi}} & 0 & 0 \\ \beta\bar{r} & -\beta\left[\bar{r} + (1-\delta)\right] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \bar{g} \end{bmatrix},$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix}.$$

As before, the solution is a set of policy matrices of the form

$$x_{t+1} = Px_t + Qz_t$$

and

$$y_t = Rx_t + Sz_t.$$

For the policy matrices, it matters what the stationary state is. In the case where $\bar{\varphi} = 1$, there is no seigniorage in the stationary state and the P, Q, R, and S matrices are

$$P = \begin{bmatrix} 0.9418 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5316 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.1552 & 0 \\ 0 & 0 \\ -0.4703 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} -0.9450 & 0 & 0 \\ 0.5316 & 0 & 0 \\ 0.5316 & 0 & 0 \\ 0.0550 & 0 & 0 \\ -0.4766 & 0 & 0 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 1.9417 & 0 \\ 0.4703 & 0 \\ 0.4703 & 0 \\ 1.9417 & 0 \\ 1.4715 & 0 \end{bmatrix}.$$

Around a stationary state with no seigniorage, the economy is completely neutral to seigniorage shocks.

In the case where $\bar{\varphi} = 1.19$, these matrices are

$$P = \begin{bmatrix} 0.9418 & 0 & 0 \\ -0.3243 & 1 & 0 \\ -2.0213 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.1547 & -0.0005 \\ -0.9330 & 0.2178 \\ -5.8433 & 0.3644 \end{bmatrix},$$

$$R = \begin{bmatrix} -0.9477 & 0 & 0 \\ 0.5331 & 0 & 0 \\ 0.5331 & 0 & 0 \\ 0.0523 & 0 & 0 \\ -0.4807 & 0 & 0 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 1.9359 & 0.0011 \\ 0.4735 & -0.0006 \\ 0.4735 & -0.0006 \\ 1.9359 & 0.0011 \\ 1.4624 & 0.0017 \end{bmatrix}.$$

The response function for a seigniorage impulse when $\bar{\varphi} = 1$ is flat except for money and prices. For the case where $\bar{\varphi} = 1.19$, the response of the real variables to a seigniorage shock is shown in Figure 9.4. While the responses are small (the shock is .01), real variables do respond to seigniorage shocks through their effects on government expenditures. The persistence observed in the figure comes from the fact that the coefficient on one period lagged

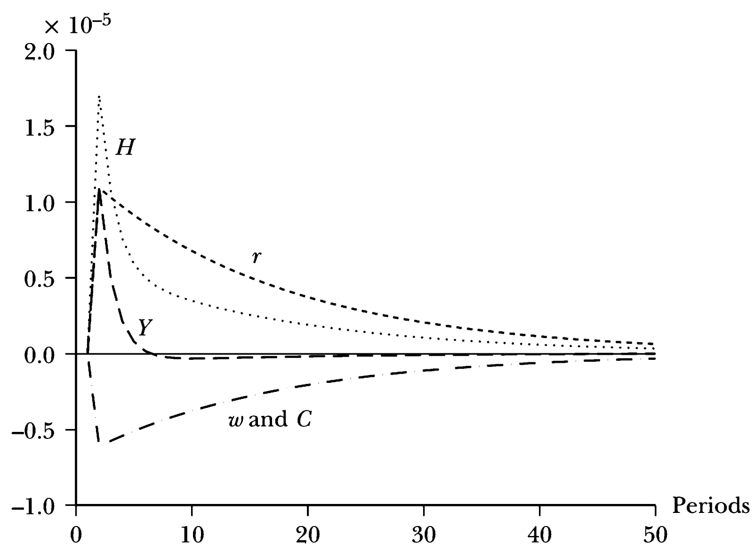

FIGURE 9.4 Response to seigniorage shock, $\bar{\varphi} = 1.19$

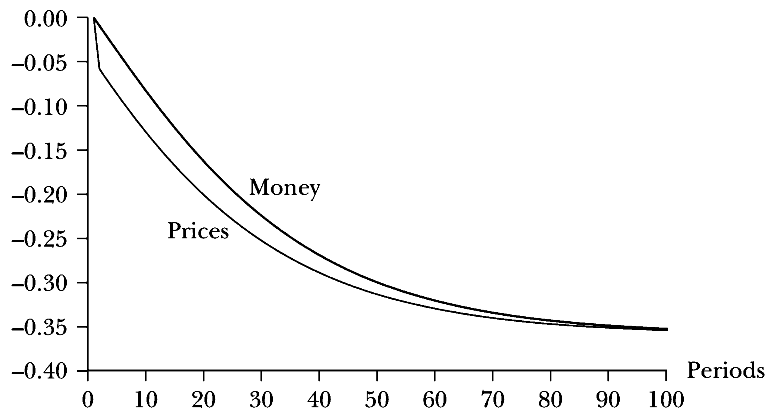

FIGURE 9.5 Responses of money and prices to a technology shock when seigniorage is collected in the stationary state, $\bar{\varphi} = 1.19$

government expenditure is $\pi = .48$ in the stochastic process for government expenditures financed by seigniorage.

Another interesting characteristic of this model is how prices and money respond to a technology shock in the case where the stationary state is one with seigniorage. The responses of money and prices to a .01 technology shock when $\bar{\varphi} = 1.19$ are shown in Figure 9.5. Notice how both money and prices fall to a new level (although the ratio returns to the stationary state value). As before, prices fall faster than does money. What happens here is that the technology shock causes prices to fall (relative to the stationary state inflation) and this means that the government needs to issue less money to cover the constant expenditures that it is purchasing with seigniorage. Both money and prices eventually have the same relative change, and their ratio returns to the stationary state value.

9.5 REPRISE

Putting money in the utility function is a second way of getting money into a macro model. It is simple, and the idea that holding real balances gives utility (possibly by reducing transactions costs) is appealing to many. It creates a demand for money in the most basic way possible; money gives utility. The basic model produces both a stationary state and a dynamic superneutrality of money with respect to the real variables of the system. The dynamic superneutrality persists even when there is a positive inflation in the stationary state.

Putting seigniorage into the model is quite straightforward. With seigniorage, dynamic superneutrality depends on the stationary state. In cases where there is no stationary state financing of government expenditures by seigniorage, the log-linear model displayes dynamic neutrality. When the stationary state is one where stationary state seigniorage financing is positive, then monetary shocks (which are equivalent to changes in the government deficit financed by seigniorage) are not neutral.

More on models with money in the utility function can be found in Sargent [72] and Blanchard and Fischer [13]. A careful critique of money in the utility function models can be found in Karaken and Wallace [49].