Dynamic macroeconomics
Class 1: Introduction and first models

Prof. George McCandless

June 7, 2022

What we will do today

1. Organization of course
   1. Useful to know Matlab (to numerically solve models)
   2. Structure of classes (mix of economic theory and practical application)
   3. Exercises each week (to be turned in)
   4. Final exam (take home solving a model)
2. Today's class
   1. Definition of our topic
   2. Correction of data (Hordrick-Prescott filter and Hamiltons evaluation)
   3. First model (Solow)
      1. build model
      2. study stationary states
      3. make log-linear version to study stochastic properties

Dynamic Macroeconomics

Why dynamic models

- Much of what we are interested in in economics occurs over time
  - inflation, growth, asset pricing
    - models where expectations about the future matter
  - dynamic effects of government policies
  - models with dynamic stochastic elements
- Various models for handling this
  - Older "Keynesian" reduced form models
  - Models that can be resolved as a single recursive problem
    - via Bellman equations or variational methods
  - Models that need to be approximated (why?)
    - linear approximations
    - quadratic or higher approximations
    - other non-linear techniques
- Need modeling techniques and solution techniques

RBC (DSGE) models

1. Based on growth models (Solow)
2. Kydland and Prescott (won Nobel prize for this)
    - First to explicitly attempt to confront Lucas' critique
    - Their model is quite complicated (to generate persistence)
        - complicated solution technique
    - Our models will be based on model by Hansen
3. We use lots of other economic models
    - Overlapping generations
    - Models of money (we use cash in advance but in utility in book)
    - Credit markets (simple banks)
    - Restrictions in prices and wages (Keynesian models)
    - Open economy models
        - open financial markets
        - international trade (capital goods, etc)

What are RBC (DSGE) models?

1. Micro-foundations based models
2. Full specification of
   1. preferences
   2. production technologies
   3. budget constraints
3. Dynamic
   1. Infinite horizon (no end point nor end point problems)
   2. expectations (rational, learning, "bounded" rationality)
      - if bounded, need to specify how (limited data?, limited reasoning?)
4. Solve linear or quadratic approximation of model
   1. Dynamic programing (Bellmans equation or variational methods)
   2. Log-linearization of model
   3. Linear quadratic techniques
   4. projection techniques (using Chebyshev polynomials, for example)
5. Get time path of economy

A successful RBC model

1. Captures the second moments of the economy
2. Captures correlation with output
3. Mimics the impulse-response functions of the economy
   - Why this may be hard
4. Rules of Prescott
   1. One needs to know the question one wants to ask
   2. Model should be designed for that question
   3. Model should use off-the-shelf technology as much as possible
   4. Model should be explicitly dynamic with expectations included
5. Should forecast well (still working on this: ECB)

What data do we use

1. Model is compared to the data
   1. How well do the model's second moments match those of data
      - how do model's shocks compare to those from data
   2. Compare impulse response of model to those of data (which IR of model)
2. Data is normally filtered
   1. Most models are solved around stationary state
   2. Mimic stationary state using filter
      1. Remove trends
      2. Remove long run growth
3. Hodrick-Prescott Filter is frequently used
   - Hamilton's critique of HP filter

Hodrick-Prescott Filter I

1. Want to find **economic cycles, want to remove trends**
2. Times series data can be seen as of the form

$$Y_{t+1} = \alpha Y_t + \beta t + \varepsilon_t$$

3. α is the auto-regressive coefficient
4. β is the linear trend (one could add quadratic trends, etc.)
5. What is the value of α?
    1. If $|\alpha| \geq 1$ time path is explosive (called unit root or random walk when $|\alpha| = 1$)
    2. if $|a| < 1$ time path stable
        1. $\alpha^n \longrightarrow 0$ as $n \longrightarrow \infty$

Hodrick-Prescott Filter II

1. One wants to remove trends from data, which is y_t
2. Formula for H-P filter

$$\min_{\{u_t\}} \frac{1}{T} \sum_{t=1}^{T} (y_t - u_t)^2 + \frac{\lambda}{T} \sum_{t=2}^{T-1} \left[(u_{t+1} - u_t) - (u_t - u_{t-1})\right]^2$$

3. The sequence $\{u_t\}$ is the 'trend' to be removed
4. $\frac{1}{T} \sum_{t=1}^{T} (y_t - u_t)^2$ is sum of squared errors
5. $\frac{\lambda}{T} \sum_{t=2}^{T-1} \left[(u_{t+1} - u_t) - (u_t - u_{t-1})\right]^2$ is a discrete version of the second derivative
    1. λ is parameter that controls smoothing
    2. H-P recommend $\lambda = 1600$
6. The detrended data (used for models) is $y_t - u_t$

Example of HP filter on Argentina GDP data

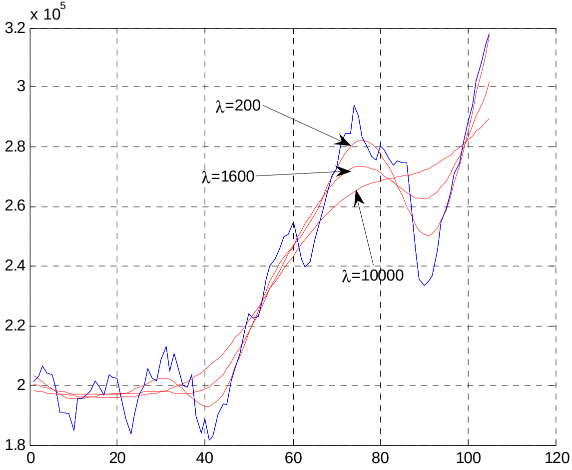


Figure: Three Hodrick-Prescott filters

Filtered data with lambda = 1600

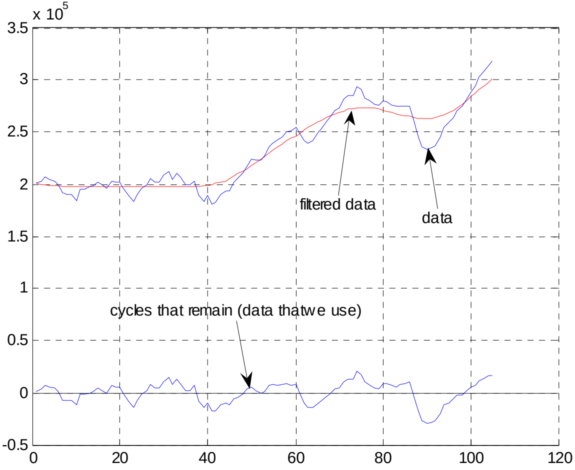


Figure: GDP data, HP trend, and detrended GDP data

Problems with H-P filter - Hamilton

- Not sure what it means
    - Can change characteristics of time series process
        - A martingale (random walk) process can be filtered into a stationary cycle
        - so it can add nonexisting cycles
    - Removes uncertain (low?) frequences from the process
    - What does the data explain after the filtering?
- Endpoint problems
    - Because it is a two sided weighed average of points
        - "predicts" well in sample because future is included in each data point
    - At endpoints, only have one side of average
- Unfortunately, it has become common practice to use H-P filter
- Hamilton recommends using 4 lag (on quarterly data) autoregression then use what is left (the "errors")

Solow's growth model

1. First model we will study
2. Basis for RBC and New Keynesian models
3. Introduce some of the basics of course
    1. Micro foundations
    2. Generates time path
    3. Can find linear approximation
    4. See how much of the cycle the **model** explains

Reference: Robert Solow: A Contribution to the Theory of Economic Growth (1956)

Solow's growth model

- Production function

$$Y_t = A_t F(K_t, N_t)$$

Y_t = output, K_t = stock of capital, N_t = labor, $A_t = (1+\alpha)^t A_0$ = the level of technology

- Properties of $F(\cdot)$

1. twice continuously differentiable
2. homogenous of degree 1

$$\beta F(K_t, N_t) = F(\beta K_t, \beta N_t)$$

3. increasing in both arguments
4. Strictly concave: $F_i > 0$, $F_{ii} < 0$, $F_{ij} > 0$
5. Inada conditions: $F(K_t, 0) = 0$, $F(0, N_t) = 0$, $F_N(K_t, 0) = \infty$, $F_K(0, N_t) = \infty$, $F_N(K_t, \infty) = 0$, $F_K(\infty, N_t) = 0$

Solow model: population growth

- Constant net growth rate of population (and labor) $= \lambda$
- The labor supply follows the rule:

$$N_{t+1} = (1 + \lambda) N_t$$

Solow model: Capital accumulation process

- Capital follows the accumulation rule

$$K_{t+1} = (1-\delta)\,K_t + I_t$$

- I_t is the investment at time t

Savings rule

1. Savings is a fixed fraction of output (major simplification of Solow)

$$S_t = sY_t$$

   1. No household optimization for savings decision
   2. the parameter s is given exogenously
   3. can fine "golden rule" but not necessarily optimal if world has shocks

2. Equilibrium conditions for closed economy investment-savings

$$S_t = I_t$$

The full model

$$Y_t = A_t F(K_t, N_t)$$
$$K_{t+1} = (1-\delta) K_t + I_t$$
$$S_t = sY_t$$
$$S_t = I_t$$
$$N_{t+1} = (1+\lambda) N_t$$

The first four can be combined to give the equation

$$K_{t+1} = (1-\delta) K_t + sA_t F(K_t, N_t)$$

Put everything into 'per unit of labor' terms

- Divide both sides of

$$K_{t+1} = (1-\delta) K_t + sA_t F(K_t, N_t)$$

by $N_{t+1} = (1+\lambda) N_t$ to get

$$\frac{K_{t+1}}{N_{t+1}} = \frac{(1-\delta) K_t}{(1+\lambda) N_t} + \frac{sA_t F(K_t, N_t)}{(1+\lambda) N_t}$$

- Defining per-worker terms: $k_t = K_t / N_t$, $y_t = Y_t / N_t$, this can be written as

$$k_{t+1} = \frac{(1-\delta)}{(1+\lambda)} k_t + \frac{sA_t}{(1+\lambda)} \frac{F(K_t, N_t)}{N_t}$$

- Because the production function is homogenous of degree 1,

$$\frac{F\left(K_t, N_t\right)}{N_t} = F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = F\left(k_t, 1\right) \equiv f\left(k_t\right),$$

- The Solow difference equation can be written as

$$k_{t+1} = G\left(k_t\right) = \frac{(1-\delta)}{(1+\lambda)} k_t + \frac{sA_t}{(1+\lambda)} f\left(k_t\right)$$

- We will (in class) let $A_t = A_0$, a constant technology
- Model with constant technological growth

$$A_t = (1+\alpha)^t A_0$$

where $\alpha = 0$

- what happens if $\alpha \neq 0$?

- Using the Cobb-Douglas production function and zero technological growth $f(k_t) = A_0 k_t^{\theta}$
- The Solow first difference equation is

$$k_{t+1} = \widehat{G}(k_t) = \frac{(1-\delta)}{(1+\lambda)} k_t + \frac{sA_0}{(1+\lambda)} k_t^{\theta}$$

- Given some initial k_0, the time path for the economy can be found
- Where the solution k to the equation

$$k = \widehat{G}(k) = \frac{(1-\delta)}{(1+\lambda)} k + \frac{sA_0}{(1+\lambda)} k^{\theta}$$

is a stationary state capital stock for the economy

Solow Graph

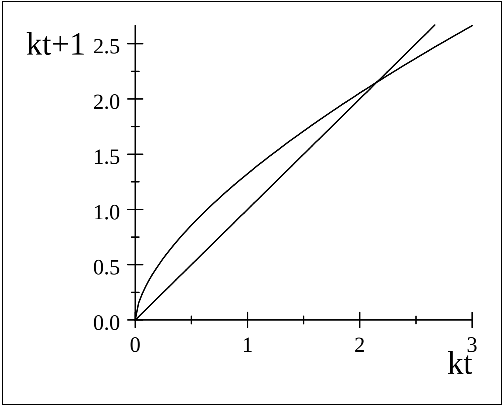


Figure: Stability conditions

- Stability conditions for Solow model
- There are two stationary states
    - zero
    - at

$$k = \left(\frac{sA_0}{\delta + \lambda}\right)^{\frac{1}{1-\theta}}$$

    - this last is an attractor

A result of Solow Model

- Poor countries (those with less capital) grow faster than rich
- Gross growth rate of capital can be written as

$$\gamma_t = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + sA_0 f(k_t)}{(1+\lambda)k_t}.$$

- And taking the derivative with respect to capital:

$$\frac{d\gamma_t}{dk_t} = \frac{sA_0}{(1+\lambda)k_t^2}\left[f'(k_t)k_t - f(k_t)\right] < 0$$

when $k_t > 0$.

- Countries with higher k_t have smaller growth rates of capital (not consistent with data)

A stochastic Solow Model

Suppose that technology is stochastic, one very simple way to do that is to assume that

$$A_t = \overline{A} e^{\varepsilon_t}$$

- $\overline{A}$ is positive
- random term ε_t has a normal distribution with mean zero
- So in stationary state (with $\varepsilon_t = 0$), $e^{\varepsilon_t} = 1$
- e^{ε_t} is never negative (which is good)

The stochastic version of the Solow first order difference equation is

$$k_{t+1} = \frac{(1-\delta)k_t + s\overline{A}e^{\varepsilon_t} f(k_t)}{(1+\lambda)}$$

.

Preview of more interesting shocks

- The stochastic equation on the previous slide is very simple
  - The stochastic process has no persistence
  - the shocks are independent, so each A_t is independent of others
- We can write a stochastic process with more presistence, for example a first order stochastic equation of the form

$$A_t = (1-\gamma)\overline{A} + \gamma A_{t-1} + \varepsilon_t$$

  with ε_t bounded both above and below and with $|\gamma| < 1$.
- Start economy at $\overline{A}$, stationary state. Give positive shock $\varepsilon_t > 0$ at some time t and only at time t, $A_t = \overline{A} + \varepsilon_t$
- at time t+1, $A_{t+1} = (1-\gamma)\overline{A} + \gamma A_t + \varepsilon_{t+1} = (1-\gamma)\overline{A} + \gamma\left(\overline{A} + \varepsilon_t\right) + 0 = \overline{A} + \gamma\varepsilon_t$
- at time t+2, $A_{t+2} = \overline{A} + \gamma^2\varepsilon_t$ and so on.

Version in Logs (model is simpler this way)

1. Divide both sides by k_t
2. replace k_{t+1}/k_t by growth rate γ_t
3. There is a problem with $\ln\left((1-\delta)k_t + s\bar{A}e^{\varepsilon_t}f(k_t)\right)$
4. Take logs of both sides

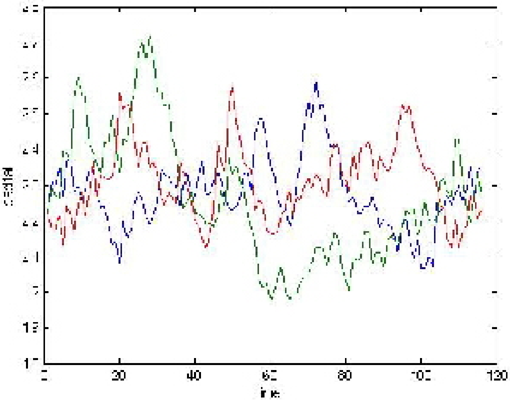
$$\ln\left[\gamma_t - \frac{1-\delta}{1+\lambda}\right] = \ln\frac{s\bar{A}}{(1+\lambda)} + \ln\frac{f(k_t)}{k_t} + \varepsilon_t.$$

5. Use Cobb-Douglas production function, $f(k_t) = k_t^{\theta}$, to get

$$\ln\left[\gamma_t - \frac{1-\delta}{1+\lambda}\right] = \varphi - (1-\theta)\ln k_t + \varepsilon_t,$$

where $\varphi \equiv \ln\left[s\bar{A}/(1+\lambda)\right]$

Three simulations, with computer choosing different random ε_t



Parameters used

$\overline{A} = 1$

$s = .2$

$\lambda = .02$

$\delta = .1$

$\theta = .36$

$\sigma_\varepsilon^2 = .2$

Figure: Three simulations of the exact Solow model

Log-linear version of (no growth trend) Solow model

1. We have a second way to handle dynamics
2. Make an (log) linear approximation of the model
3. Study the dynamics of this linearized model
4. Exists technology for solving dynamic linear models

Log-linear version of (no growth trend) Solow model

1. Begin with the Solow difference equation

$$(1+\lambda)\, k_{t+1} = (1-\delta) k_t + s\overline{A} e^{\varepsilon_t} k_t^{\theta},$$

2. Define $\widetilde{k}_t = \ln(k_t) - \ln\left(\overline{k}\right)$, where $\overline{k}$ is the stationary state value of capital, so $\widetilde{k}_t = \ln\left(k_t / \overline{k}\right)$
3. Then $k_t = \overline{k} e^{\widetilde{k}_t}$ given that $e^{\ln((k_t)-\ln\left(\overline{k}\right)} = k_t / \overline{k}$
4. The Solow difference equation is now

$$(1+\lambda)\, \overline{k} e^{\widetilde{k}_{t+1}} = (1-\delta) \overline{k} e^{\widetilde{k}_t} + s\overline{A} e^{\varepsilon_t} \overline{k}^{\theta} e^{\theta \widetilde{k}_t},$$

5. Use the approximation that for small $\widetilde{k}_t$,

$$e^{a\widetilde{k}_t} = 1 + a\widetilde{k}_t$$

6. This approximation comes from a first order Taylor expansion of $e^{a\widetilde{k}_t}$

Log-linear Solow model

1. The approximation of

$$(1+\lambda)\,\overline{k}e^{\widetilde{k}_{t+1}} = (1-\delta)\overline{k}e^{\widetilde{k}_t} + s\overline{Ak}^{\theta}e^{\theta\widetilde{k}_t+e^{\varepsilon_t}},$$

(note that $e^{\varepsilon_t}e^{\theta\widetilde{k}_t} = e^{\theta\widetilde{k}_t+e^{\varepsilon_t}}$) is

$$(1+\lambda)\,\overline{k}\left(1+\widetilde{k}_{t+1}\right) = (1-\delta)\overline{k}\left(1+\widetilde{k}_t\right) + s\overline{Ak}^{\theta}\left(1+\theta\widetilde{k}_t+\varepsilon_t\right).$$

2. The 1's in parenthisis drop out because

$$(1+\lambda)\,\overline{k} = (1-\delta)\overline{k} + s\overline{Ak}^{\theta}.$$

3. The log-linear version of the model is

$$(1+\lambda)\,\overline{k}\widetilde{k}_{t+1} = \left[(1-\delta)\overline{k} + s\overline{Ak}^{\theta}\theta\right]\widetilde{k}_t + s\overline{Ak}^{\theta}\varepsilon_t.$$

Log-linear Solow model (continued)

1. This further simplifies (since $\delta + \lambda = s\overline{Ak}^{\theta-1}$) to

$$\widetilde{k}_{t+1} = \frac{1 - \delta + (\delta + \lambda)\,\theta}{1 + \lambda}\widetilde{k}_t + \frac{\delta + \lambda}{1 + \lambda}\varepsilon_t.$$

2. Notice that

$$\frac{(1 - \delta) + (\delta + \lambda)\,\theta}{1 + \lambda} = \frac{1 + \lambda\theta - \delta\,(1 - \theta)}{1 + \lambda} < 1$$

3. Let $B = \frac{1+\lambda\theta-\delta(1-\theta)}{1+\lambda}$ and $C = \frac{\delta+\lambda}{1+\lambda} < 1$ and time $t + 1$ variation of capital is equal to

$$\widetilde{k}_{t+1} = C\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}$$

This is a convergent sequence

Using the above notation

$$\widetilde{k}_{t+1} = \frac{1-\delta+(\delta+\lambda)\,\theta}{1+\lambda}\widetilde{k}_t + \frac{\delta+\lambda}{1+\lambda}\varepsilon_t.$$

is

$$\begin{aligned}
\widetilde{k}_{t+1} &= B\widetilde{k}_t + C\varepsilon_t \\
\widetilde{k}_{t+1} &= B\left(B\widetilde{k}_{t-1} + C\varepsilon_{t-1}\right) + C\varepsilon_t \\
\widetilde{k}_{t+1} &= B^2\widetilde{k}_{t-1} + C\varepsilon_t + BC\varepsilon_{t-1} \\
\widetilde{k}_{t+1} &= B^2\left(B\widetilde{k}_{t-2} + C\varepsilon_{t-2}\right) + C\varepsilon_t + BC\varepsilon_{t-1} \\
\widetilde{k}_{t+1} &= B^3\widetilde{k}_{t-2} + C\varepsilon_t + BC\varepsilon_{t-1} + B^2C\varepsilon_{t-2} \\
\widetilde{k}_{t+1} &= B^3\widetilde{k}_{t-2} + C\left(\varepsilon_t + B\varepsilon_{t-1} + B^2\varepsilon_{t-2}\right)
\end{aligned}$$

Variance of capital

1. The variance of time $t+1$ capital is equal to
2. Since the shocks are time independent,

$$E\left(\varepsilon_i \varepsilon_j\right) = 0 \ \ if \ \ i \neq j, and = \sigma^2 \ \ if \ \ i = j$$

.

3. All cross terms drop out and the sums become

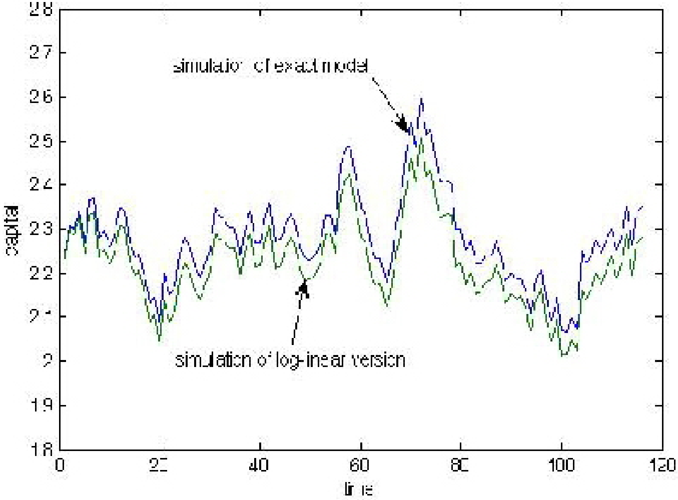
$$var\left(\widetilde{k}_{t+1}\right) = C^2\sigma^2\sum_{i=0}^{\infty} B^{2i} = \frac{C^2}{1-B^2}\sigma^2$$

4. Using the parameters given before, $B = 0.971\,76$ and $C = .044\,11\,8$ so that

$$var\left(\widetilde{k}_{t+1}\right) = \frac{\left(.044\,11\,8\right)^2}{1-\left(0.971\,76\right)^2}\sigma^2 = \frac{.001946\,4}{.05568\,3} = .034955\sigma^2$$

- The variance of capital is very small compared to the technology shock variance

Time path of capital (comparing approximate to exact models)



Variance of output

1. Output comes from the equation

$$y_t = A_t k_t^{\theta} = \overline{A} e^{\varepsilon_t} k_t^{\theta}$$

2. A log-linear approximation of this is

$$\overline{y}\left(1+\widetilde{y}_t\right) \approx \overline{y} e^{\widetilde{y}_t} = \overline{Ak}^{\theta} e^{\theta \widetilde{k}_t + \varepsilon_t} \approx \overline{Ak}^{\theta}\left(1+\theta \widetilde{k}_t + \varepsilon_t\right)$$

which simplifies to

$$\widetilde{y}_t \approx \theta \widetilde{k}_t + \varepsilon_t$$

- The variance of output is almost entirely made up of variance in technology.
- The effect of changes in the capital stock are very small
- Solow model does not explain the persistence of shocks (cycles) in an economy

Homework

Describe the time path for a Solow model where there is an externality for each firm and each firm has the production function

$$y_t^j = A_t k_t^\theta h_t^{1-\theta} K_t^{1-\theta}.$$

Lower case letters are for the firm and upper case letters are for the economy as a whole. The total capital in the economy functions as an externality for each firm. Solve the model in terms of growth rates. How would you get an log linear approximate version of this model?