

# 14. Transformata di Laplace e risposte in frequenze

Titolo nota

4/11/2008

1. Fourier  $F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$

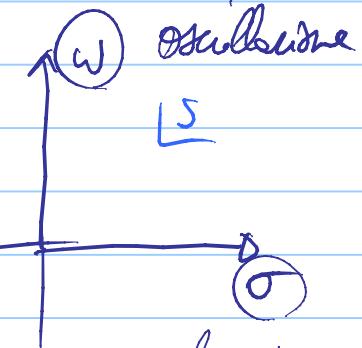
$\omega$  reale  $f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$

$$\int_{-\infty}^{+\infty} e^{j\omega t} d\omega = 2\pi \delta(t)$$

2. Laplace  $s = \sigma + j\omega$  complessa

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

solo  $t > 0$



Cose termine reale divergente e  $e^{-\sigma t}$  è necessario un  $t > t_{min}$ , scelta  $\sigma = 0$

$$\begin{aligned} (\theta(t))f(t) &= \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} \frac{ds}{2\pi j} = \\ &= \int_{-\infty}^{+\infty} F(\sigma_0 + j\omega) e^{\sigma_0 t} e^{j\omega t} \frac{d\omega}{2\pi} \end{aligned}$$

$$\begin{aligned} \text{Dim: } &= \int_{-\infty}^{+\infty} \left( \int_0^{\infty} f(t') e^{-(\sigma_0 + j\omega)t'} dt' \right) e^{\sigma_0 t} e^{j\omega t} \frac{d\omega}{2\pi} = \\ &= \int_0^{\infty} f(t') e^{\sigma_0(t-t')} \int_{-\infty}^{+\infty} e^{j\omega(t-t')} \frac{d\omega}{2\pi} = \\ &= \int_0^{\infty} f(t') e^{\sigma_0(t-t')} \delta(t-t') = \theta(t) \cdot f(t) \cdot 1 \end{aligned}$$

$f(t < 0) = 0$

$$\int fg' = fg - \int f'g$$

### 3. Trasformate notevoli

$$\theta(t) \rightarrow \frac{1}{s} \quad [\text{con } \sigma > 0] \quad \left[ \int_0^\infty e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^\infty \right]$$

$$e^{-at} \rightarrow \frac{1}{s+a} \quad [\text{con } \sigma > -\text{Re } a] \quad \left[ \int_0^\infty e^{-(s+a)t} dt = \frac{e^{-(s+a)t}}{(s+a)} \Big|_0^\infty \right]$$

$$\sin wt = \frac{1}{2j} (e^{jwt} - e^{-jwt}) \rightarrow \frac{1}{2j} \left[ \frac{1}{s-jw} - \frac{1}{s+jw} \right] = \frac{\omega}{s^2 + w^2}$$

$$\cos wt = \frac{1}{2} (e^{jwt} + e^{-jwt}) \rightarrow \frac{1}{2} \left[ \frac{1}{s-jw} + \frac{1}{s+jw} \right] = \frac{s}{s^2 + w^2}$$

$$t \rightarrow \frac{1}{s^2} \quad \left[ \int_0^\infty t e^{-st} dt = -\frac{te^{-st}}{s} \Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} dt = -\frac{e^{-st}}{s^2} \Big|_0^\infty = \frac{1}{s^2} \right]$$

$$\delta(t) \rightarrow 1$$

$$f(t-\tau) \rightarrow F(s) e^{-s\tau} \quad \left[ \int_0^\infty f(t-\tau) e^{-st} dt = \int_{-\tau}^\infty f(t') e^{-st} e^{-s(t'-\tau)} dt' \right]$$

$$f(at) \rightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$f(t) e^{-\alpha t} \rightarrow F(s+\alpha)$$

$$f'(t) \rightarrow sF(s) - f(0) \quad \left[ \int_0^\infty f'(t) e^{-st} dt = f(t) e^{-st} \Big|_0^\infty + \int_0^\infty f(t) e^{-st} dt \right]$$

$$\int_0^t f(t') dt' \rightarrow \frac{1}{s} F(s) \quad \left[ \int_0^\infty dt \int_0^t f(t') dt' e^{-st} dt = -\frac{e^{-st}}{s} \int_0^t f(t') dt' \Big|_0^\infty + \frac{1}{s} \int_0^\infty f(t) e^{-st} dt \right]$$

## 4. Perche' Laplace?

→ analisi dei poli e zeri nel piano complesso delle funzioni di trasferimento definisce il funzionamento del circuito

(t) t < t considerata

$$x(t) \xrightarrow{G} y(t) \quad \text{In generale } Y(t) = \int_0^t G(t, t') X(t') dt'$$

Per incidenze temporali  $G(t, t') = G(t - t')$

$$Y(t) = \int_0^t G(t-t') X(t') dt' = \int_0^t G(\tau) X(t-\tau) d\tau \quad \begin{matrix} \text{note scambio} \\ \text{estremi} \\ \tau = t - t' \end{matrix}$$

↑ funzione di trasferimento del sistema

$$\begin{aligned} \tilde{Y}(s) &= \int_0^\infty dt Y(t) e^{-st} = \int_0^\infty dt e^{-st} \int_0^t G(\tau) X(t-\tau) d\tau = \\ &= \int_0^\infty dt e^{-st} \int_0^\infty \delta(t-\tau) G(\tau) X(t-\tau) d\tau = \\ &= \int_0^\infty \int_0^\infty \delta(t-\tau) X(t-\tau) e^{-s(t-\tau)} e^{-s\tau} G(\tau) dt d\tau = \\ (t=t-\tau) &= \int_0^\infty \left[ \int_{-\infty}^\infty \delta(t) X(t') e^{-st'} dt' \right] e^{-s\tau} G(\tau) d\tau = \end{aligned}$$

$$= \tilde{G}(s) \cdot \tilde{X}(s)$$

↑ funzione di Green del sistema  
o funzione di trasferimento.

Risposta alla delta:  $x(t) = f(t)$ ,  $\tilde{x}(s) = 1$

$$\tilde{Y}(s) = \tilde{G}(s) \quad \rightarrow \text{Mandando un impulso}$$

$$Y(t) = g(s) \quad \text{molto breve attorno in} \\ \text{avanti direttamente } G(t)$$

Risposta al gradino  $\tilde{X}(s) = \frac{Y(s)}{s}$

$$\tilde{Y}(s) = \frac{\tilde{G}(s)}{s}$$

Impedenze di  $R, C, L$

$$R \rightarrow R$$

$$Z_C = \frac{1}{sC}$$

$$Z_L = sL$$

$$V = RI$$

$$V = \frac{1}{C} \int_0^t I d(t)$$

$$V = L \frac{dI}{dt}$$

$$(I(0)=0 \quad Q(0)=0)$$

$s \rightarrow j\omega$  per avere le risposte in freq.

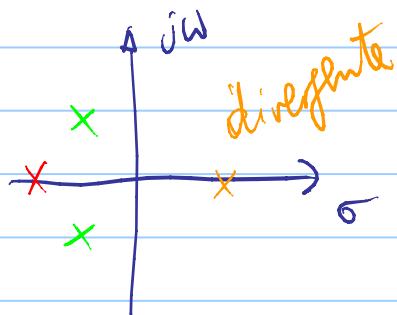
## 5. Poli e zeri

Per una rete lineare la funzione di trasferimento diventa un rapporto di polinomi

$$\tilde{G}(s) = \frac{\sum_{k=0}^m a_k s^k}{\sum_{k=0}^n b_k s^k} = \frac{a_m}{b_n} \frac{(s-z_1)(s-z_2) \cdots (s-z_m)}{(s-p_1)(s-p_2) \cdots (s-p_n)}$$

Poiché i coefficienti  $a, b$  sono reali,  $z_i$  e  $p_i$  sono o reali o complessi coniugati.

$$e^{sxt} \rightarrow \frac{1}{s-\alpha} \Rightarrow \operatorname{Re}(\alpha) < 0 \text{ per avere circuiti stabili}$$

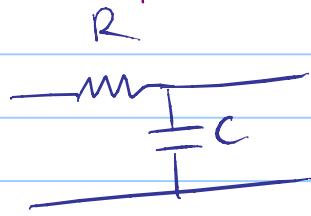


$\operatorname{Re}(\alpha) > 0 \rightarrow \text{oscillatori}$

$$e^{sxt} \sin \omega_0 t \rightarrow \frac{\omega_0}{(s-\alpha)^2 + \omega_0^2}$$

$s = \alpha \pm j\omega_0$

## RC pone basso



$$G(s) = \frac{V_{SC}}{R + \frac{1}{sC}} = \frac{1}{s/w_0 + 1}$$

$$w_0 = \frac{1}{RC}$$

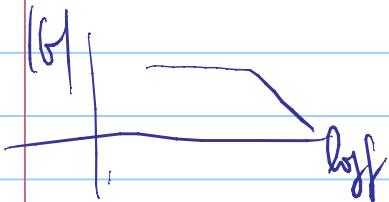
1 polo  $s = -w_0$

$-6\text{ dB/ottava}$   
 $w > w_0$

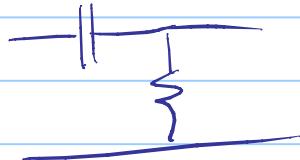
$$|G(\omega)| (\text{dB}) = -20 \log_{10} \sqrt{1 + \left(\frac{\omega}{w_0}\right)^2}$$

$$\left| \frac{s/w_0 + 1}{1} \right| = \left| \frac{\omega/w_0 + 1}{1} \right| = \sqrt{1 + \left(\frac{\omega}{w_0}\right)^2}$$

$$\tan \varphi = \frac{Im}{Re} = \frac{\omega}{w_0} \rightarrow \varphi(\omega) = -\tan^{-1} \frac{\omega}{w_0}$$

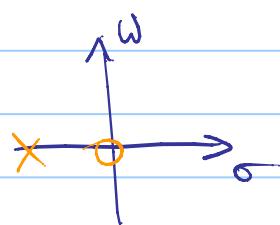


## RC pone alto



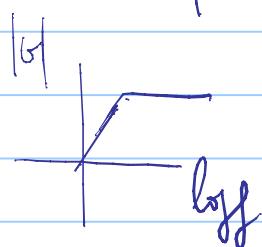
$$G(s) = \frac{R}{R + \frac{1}{sC}} = \frac{s/w_0}{s/w_0 + 1} = \frac{1}{1 + \frac{w_0}{s}}$$

1 zero  $s = 0$   
1 polo  $s = -w_0$



$$G(\omega) (\text{dB}) = -20 \log_{10} \sqrt{1 + \left(\frac{w_0}{\omega}\right)^2}$$

$$\varphi = \frac{w_0}{\omega} \rightarrow \varphi(\omega) = \tan^{-1} \left( \frac{w_0}{\omega} \right) \quad 6\text{ dB/ottava}$$

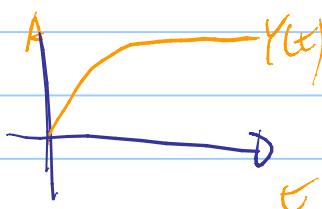


## Risposte al gradino

pone basso

$$\tilde{x}(s) = \frac{1}{s} \quad \tilde{y}(s) = \frac{1}{s} \frac{1}{s/w_0 + 1} = \left( \frac{1}{s} - \frac{1/w_0}{s/w_0 + 1} \right) =$$

$$y(t) = \Theta(t) \left[ 1 - e^{-w_0 t} \right]$$

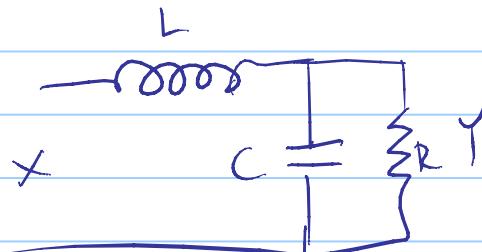


per le soluzioni

$$\tilde{Y}(s) = \frac{1}{s} \cdot \frac{s/\omega_0}{s/\omega_0 + 1} = \frac{1}{s + \omega_0}$$

$$Y(t) = e^{-\omega_0 t} \theta(t)$$

RLC



$$Z_{RC} = \frac{1}{sC + \frac{1}{R}} = \frac{R}{1 + sRC}$$

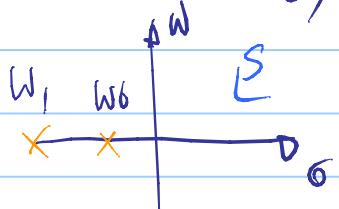
$$\tilde{G}(s) = \frac{Z_{RC}}{sL + Z_{RC}} = \frac{R}{R + sL(1 + sRC)} = \frac{1}{1 + \frac{sL}{R}(1 + sRC)}$$

$$\omega_C = \frac{1}{RC} \quad \omega_L = \frac{R}{L} \quad \omega_L \omega_C = \frac{1}{LC} \quad \omega_L / \omega_C = \frac{R^2 C}{L}$$

$$\tilde{G}(s) = \frac{\frac{1}{\omega_L \omega_C}}{\frac{s^2}{\omega_L \omega_C} + \frac{s}{\omega_L} + 1} = \frac{\omega_L \omega_C}{(s - \omega_0)(s + \omega_1)}$$

$$\omega_{0,1} = \frac{-\omega_C \pm \sqrt{\omega_C^2 - 4\omega_L \omega_C}}{2} = \frac{\omega_C}{2} \left[ -1 \pm \sqrt{1 - \frac{\omega_L}{\omega_C}} \right]$$

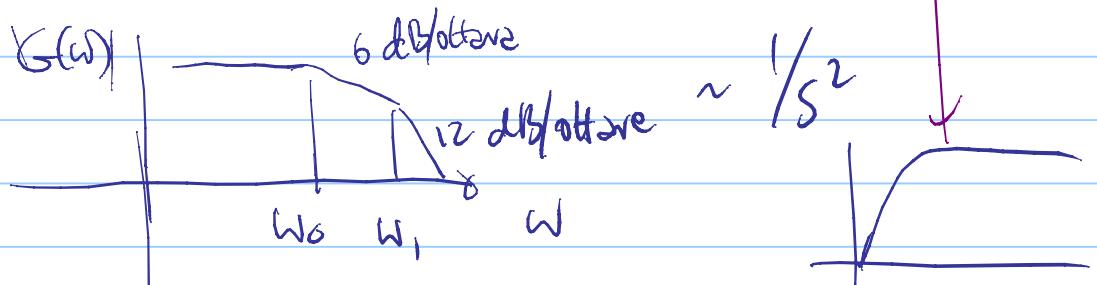
① se  $\Delta > 0$   $\omega_0, \omega_1$  reale negative  $\frac{R^2 C}{L} < 1 \quad R < \sqrt{\frac{L}{C}}$



Risposta al quaderno

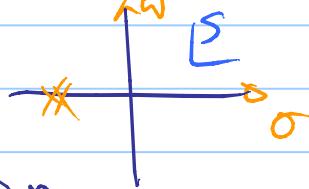
$$\sim A e^{-\omega_0 t} + B e^{-\omega_1 t}$$

Resistenze basse  $\rightarrow$  sorpasso morto



$$\textcircled{2} \quad \Delta = 0 \quad R = \frac{1}{2} \sqrt{\frac{L}{C}} \quad \omega_0, \omega_1 = -\frac{\omega_c}{2} = -2\omega_L$$

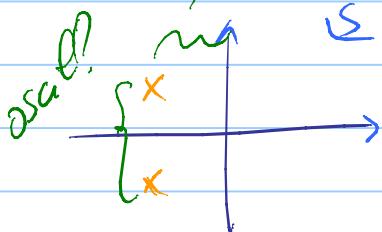
due poli sovrapposti



$$\textcircled{3} \quad \Delta < 0 \quad \omega_0, \omega_1 = \omega_R \pm j\omega_I \quad / \text{smorzamento}$$

$$\omega_R = \frac{\omega_c}{2} \quad \omega_I = \frac{\omega_c}{2} \sqrt{1 + \frac{\omega_0^2}{\omega_c^2} - 1}$$

Risposta al quadro



$$\frac{1}{(s-\omega_0)(s-\omega_1)} = \frac{1}{(s-\omega_R)^2 + \omega_I^2} \rightarrow e^{j\omega t} \frac{\sin \omega_I t}{\omega_I}$$

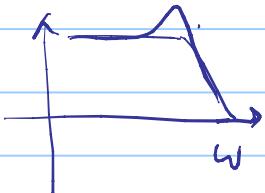
$$\begin{aligned} \frac{1}{(\omega_0 - \omega_1)} \left( \frac{1}{s - \omega_0} - \frac{1}{s - \omega_1} \right) &\rightarrow \frac{1}{\omega_0 - \omega_1} \left[ e^{j\omega_0 t} - e^{j\omega_1 t} \right] = \\ &= \frac{e^{j\omega_0 t}}{\omega_I} \sin \omega_I t \end{aligned} \quad \rightarrow \text{oscillazione smorzata}$$

Risposta in frequenza:

$$|j\omega - \omega_R + j\omega_I| = \sqrt{\omega_R^2 + (\omega + \omega_I)^2}$$

$G(\omega)$

$$|j\omega - \omega_R - j\omega_I| = \sqrt{\omega_R^2 + (\omega - \omega_I)^2}$$



$$|G(\omega)| = \frac{1}{\sqrt{(\omega_R^2 + (\omega - \omega_I)^2)(\omega_R^2 + (\omega + \omega_I)^2)}}$$

Poli e zeri reali

le imposte in frequenza in dB è le somme di termini del tipo

$$\pm 20 \log_{10} \sqrt{w^2 + \alpha^2}$$

$$G(\text{dB}) = + \sum_{i, \text{zeri}} 20 \log_{10} \sqrt{w^2 + z_i^2} - \sum_{i, \text{poli}} 20 \log_{10} \sqrt{w^2 + p_i^2}$$

Se poli e zeri sono distanti tra loro, le pendenze cambia di

+6 dB / otto passando uno zero

-6 dB / otto passando un polo.