14. Trasformata di Laplace e risposta in frequenza

Titolo nota 4/11/2008

1. Fourier $\quad F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$

ω reale $\quad f(t) = \int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}$

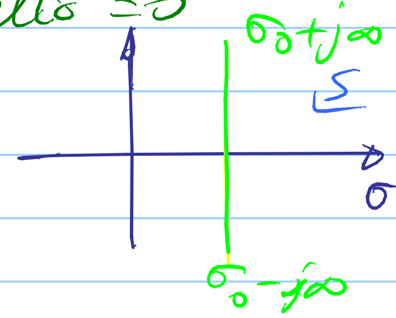
$$\int_{-\infty}^{+\infty} e^{j\omega t}\, d\omega = 2\pi\delta(t)$$

2. Laplace $\quad s = \sigma + j\omega \quad$ complessa

$$F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

$\to$ solo $t>0$

Causa termine reale divergente $e^{-\sigma t}$
è necessario un $t > t_{min}$, scelto $=0$



$$\theta(t) f(t) = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)\, e^{st}\, \frac{ds}{2\pi j} =$$

$$= \int_{-\infty}^{+\infty} F(\sigma_0 + j\omega)\, e^{\sigma_0 t}\, e^{j\omega t}\, \frac{d\omega}{2\pi}$$

Dim: $= \int_{-\infty}^{+\infty} \left(\int_0^\infty f(t')\, e^{-(\sigma_0 + j\omega)t'}\, dt' \right) e^{\sigma_0 t}\, e^{j\omega t}\, \frac{d\omega}{2\pi} =$

$$= \int_0^\infty dt'\, f(t')\, e^{\sigma_0 (t-t')} \int_{-\infty}^{+\infty} e^{j\omega(t-t')}\, \frac{d\omega}{2\pi} =$$

$$= \int_0^\infty dt'\, f(t')\, e^{\sigma_0(t-t')}\, \delta(t-t') = \theta(t) \cdot f(t) \cdot 1$$

$$\boxed{f(t<0) = 0}$$

$$\int f g' = fg - \int f' g$$

3. Trasformate notevoli

$\theta(t) \to 1/s \quad [\text{con } \sigma > 0] \quad \left[\int_0^\infty e^{-st} dt = -\frac{e^{-st}}{s}\Big|_0^\infty\right]$

$e^{-\alpha t} \to \frac{1}{s+\alpha} \quad [\text{con } \sigma > -\text{Re}\,\alpha] \quad \left[\int_0^\infty e^{-(s+\alpha)t} dt = \frac{e^{-(s+\alpha)t}}{-(s+\alpha)}\right]$

$\sin \omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \to \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$

$\cos \omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \to \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$

$t \to 1/s^2 \quad \left[\int_0^\infty t e^{-st} dt = \cancel{-\frac{t e^{-st}}{s}\Big|_0^\infty} + \int_0^\infty \frac{e^{-st}}{s} dt = \right.$

$\left. = -\frac{e^{-st}}{s^2}\Big|_0^\infty = \frac{1}{s^2}\right]$

$\delta(t) \to 1$

$t' = t - \tau$

$f(t-\tau) \to F(s) e^{-s\tau} \quad \left[\int_0^\infty f(t-\tau) e^{-st} dt = \int_{-\tau}^\infty f(t') e^{-st'} e^{-s\tau} dt'\right.$

$f(\alpha t) \to \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$

$f(t) e^{-\alpha t} \to F(s+\alpha)$

$f'(t) \to s F(s) - f(0) \quad \left[\int_0^\infty f'(t) e^{-st} dt = f(t) e^{-st}\Big|_0^\infty + s\int_0^\infty f(t) e^{-st} dt\right]$

$\int_0^t f(t') dt' \to \frac{1}{s} F(s) \quad \left[\int_0^\infty dt \int_0^t f(t') dt' e^{-st} dt = \right.$

$-\frac{e^{-st}}{s}\int_0^t f(t') dt'\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t) e^{-st} dt$

4.- Perché Laplace?

→ analisi dei poli e zeri nel piano complesso delle funzione di trasferimento definisce il funzionamento del circuito

$t' < t$ causalità

$x(t) \longrightarrow \boxed{G} \longrightarrow Y(t)$ In generale $Y(t) = \int_0^t G(t,t')\, x(t')\, dt'$

Per invarianza temporale $G(t,t') = G(t-t')$

Convoluzione

$$Y(t) = \int_0^t \boxed{G(t-t')}\, x(t')\, dt' = \int_0^t G(\tau)\, x(t-\tau)\, d\tau$$

Nota scambio estremi $\tau = t - t'$

funzione di trasferimento del sistema

$$\tilde{Y}(s) = \int_0^\infty dt\, Y(t)\, e^{-st} = \int_0^\infty dt\, e^{-st} \int_0^t G(\tau)\, x(t-\tau)\, d\tau =$$

$$= \int_0^\infty dt\, e^{-st} \int_0^\infty \theta(t-\tau)\, G(\tau)\, x(t-\tau)\, d\tau =$$

$$= \int_0^\infty \int_0^\infty \theta(t-\tau)\, x(t-\tau)\, e^{-s(t-\tau)}\, e^{-s\tau}\, G(\tau)\, dt\, d\tau =$$

$(t' = t - \tau)$

$$= \int_0^\infty \left[\int_{-\tau}^\infty \theta(t')\, x(t')\, e^{-st'}\, dt' \right] e^{-s\tau}\, G(\tau)\, d\tau =$$

limita a 0 l'estremo inferiore

$$= \tilde{G}(s) \cdot \tilde{X}(s)$$

↳ funzione di Green del sistema o funzione di trasferimento.

Risposta alla delta. $x(t) = \delta(t)$ $\tilde{x}(s) = 1$

$\tilde{Y}(s) = \tilde{G}(s)$ → Mandando un impulso molto breve ottengo in uscita direttamente $G(t)$

$Y(t) = G(t)$

Risposta al gradino $\tilde{X}(s) = 1/s$

$$\tilde{Y}(s) = \frac{\tilde{G}(s)}{s}$$

Impedenze di R, C, L

$$R \to R \qquad Z_C = \frac{1}{sC} \qquad Z_L = sL$$

$$V = RI \qquad V = \frac{1}{C}\int_0^t I\, d(t) \qquad V = L\frac{dI}{dt}$$

$(I(0) = 0 \quad Q(0) = 0)$ $\quad s \to j\omega$ per avere le risposte in freq.

5. Poli e zeri

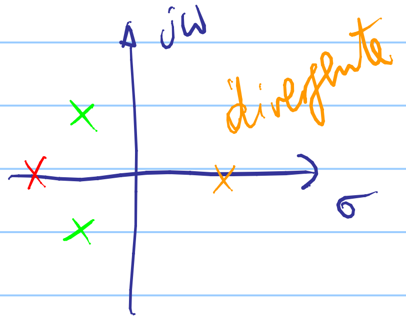
Per una rete lineare la funzione di trasferimento diventa un rapporto di polinomi

$$\tilde{G}(s) = \frac{\sum_{k=0}^{m} a_k s^k}{\sum_{k=0}^{n} b_k s^k} = \frac{a_m}{b_n}\,\frac{\overbrace{(s-z_1)(s-z_2)\cdots(s-z_m)}^{\text{zeri}}}{\underbrace{(s-P_1)(s-P_2)\cdots(s-P_n)}_{\text{poli}}}$$

Poiché i coefficienti a, b sono reali, z_i e P_i sono o reali o complessi coniugati.

$$e^{+\alpha t} \to \frac{1}{s-\alpha} \implies Re(P) < 0 \text{ per avere circuiti stabili}$$

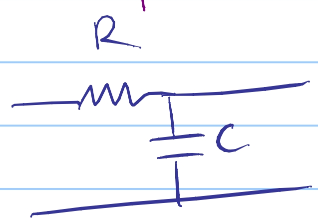
$Re(P) > 0 \to$ oscillatori



$$e^{\alpha t}\sin\omega_0 t \to \frac{\omega_0}{(s-\alpha)^2+\omega_0^2}$$
(oscill. smorzata)

$s = \alpha \pm j\omega_0$

RC passa basso



$$G(s) = \frac{1/sC}{R + 1/sC} = \frac{1}{s/\omega_0 + 1} \qquad \omega_0 = 1/RC$$

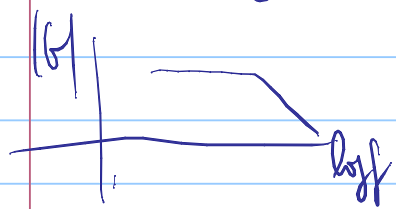
$$\boxed{1 \text{ polo} \quad s = -\omega_0}$$

$$|G(\omega)|\,(dB) = -20 \log_{10} \sqrt{1 + (\omega/\omega_0)^2}$$

-6 dB/ottava

$\omega >> \omega_0$

$$\left|\frac{s}{\omega_0} + 1\right| = \left|\frac{j\omega}{\omega_0} + 1\right| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

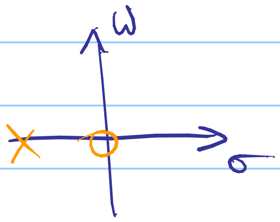


$$tg\,\varphi = \frac{Im}{Re} = -\frac{\omega}{\omega_0} \to \varphi(\omega) = -tg^{-1}\,\omega/\omega_0$$

RC passa alto

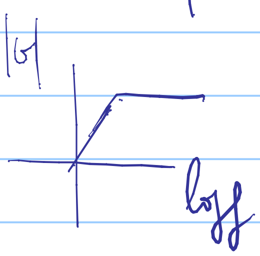
$$G(s) = \frac{R}{R + 1/sC} = \frac{s/\omega_0}{s/\omega_0 + 1} = \frac{1}{1 + \omega_0/s}$$

$$\boxed{\begin{array}{ll} 1 \text{ zero} & s = 0 \\ 1 \text{ polo} & s = -\omega_0 \end{array}}$$



$$G(\omega)\,(dB) = -20 \log_{10} \sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}$$

6dB/ottava

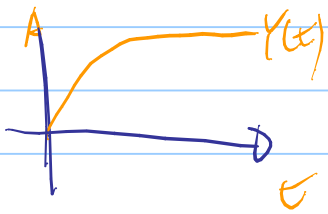


$$\varphi = \frac{\omega_0}{\omega} \to \varphi(\omega) = tg^{-1}\left(\frac{\omega_0}{\omega}\right)$$

Risposta al gradino $\quad \tilde{X}(s) = 1/s$

passa basso

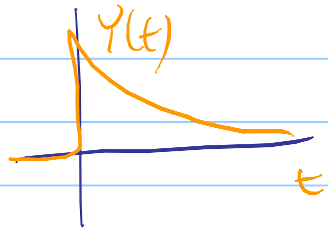
$$\tilde{Y}(s) = \frac{1}{s}\,\frac{1}{s/\omega_0 + 1} = \left(\frac{1}{s} - \frac{1/\omega_0}{s/\omega_0 + 1}\right) =$$



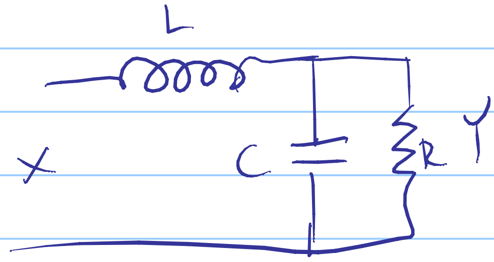
$$Y(t) = \theta(t)\left[1 - e^{-\omega_0 t}\right]$$

passa alto $\tilde{Y}(s) = \frac{1}{s}\,\frac{s/\omega_0}{s/\omega_0+1} = \frac{1}{s+\omega_0}$

$Y(t) = e^{-\omega_0 t}\,\theta(t)$



RLC



$$Z_{RC} = \frac{1}{sC+\frac{1}{R}} = \frac{R}{1+sCR}$$

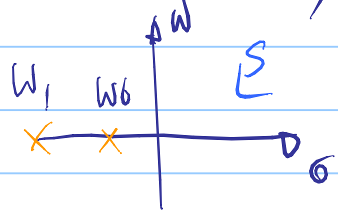
$$\tilde{G}(s) = \frac{Z_{RC}}{sL+Z_{RC}} = \frac{R}{R+sL(1+sRC)} = \frac{1}{1+s\frac{L}{R}(1+sRC)}$$

$\omega_C = \frac{1}{RC}$ $\omega_L = \frac{R}{L}$ $\omega_L\omega_C = \frac{1}{LC}$ $\omega_L/\omega_C = \frac{R^2C}{L}$

$$\tilde{G}(s) = \frac{1}{\frac{s^2}{\omega_L\omega_C}+\frac{s}{\omega_L}+1} = \frac{\omega_L\omega_C}{(s-\omega_0)(s-\omega_1)}$$

$$\omega_{0,1} = \frac{-\omega_C \pm\sqrt{\omega_C^2-4\omega_L\omega_C}}{2} = \frac{\omega_C}{2}\left[-1\pm\sqrt{1-4\frac{\omega_L}{\omega_C}}\right]$$

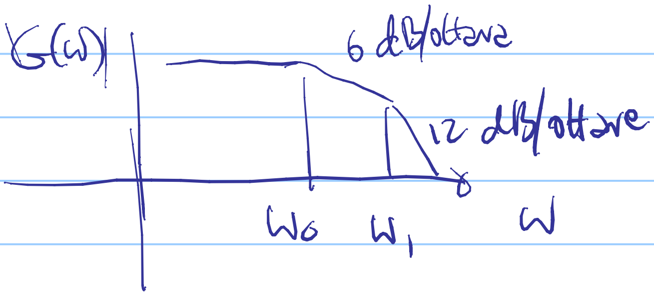
1) se $\Delta>0$ ω_0, ω_1 reali negative $4\frac{R^2C}{L}<1$ $R<\frac{1}{2}\sqrt{\frac{L}{C}}$



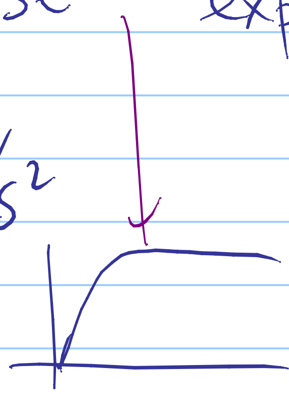
Risposta al gradino

$\sim Ae^{-\omega_0 t}+Be^{-\omega_1 t}$ exp.

Resistenza bassa → sovrasmorzato

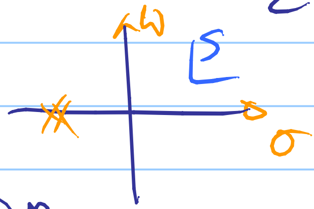


$\sim 1/s^2$

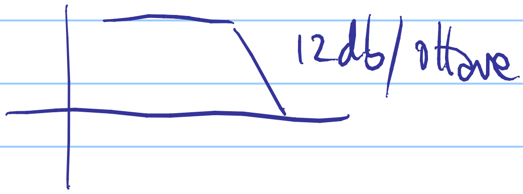


② $\Delta = 0$ $\quad R = \frac{1}{2}\sqrt{\frac{L}{C}}$ $\quad \omega_0, \omega_1 = -\frac{\omega_c}{2} = -2\omega_L$

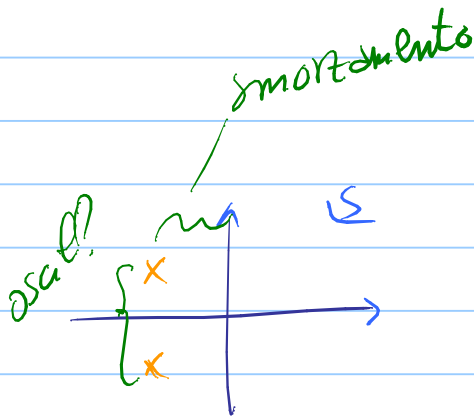
due poli sovrapposti



G(ω)



③ $\Delta < 0$ $\quad \omega_0, \omega_1 = \omega_R \pm j\omega_I$



$\omega_R = -\frac{\omega_c}{2}$ $\quad \omega_I = \frac{\omega_c}{2}\sqrt{4\frac{\omega_L}{\omega_c} - 1}$

Risposta al gradino

$$\frac{1}{(s-\omega_0)(s-\omega_1)} = \frac{1}{(s-\omega_R)^2 + \omega_I^2} \rightarrow \frac{e^{\pm\omega_R t}\,\mathrm{sen}\,\omega_I t}{\omega_I}$$

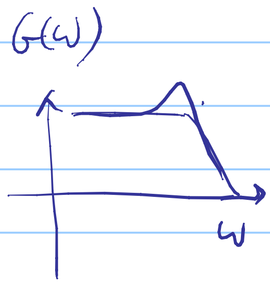
$$\frac{1}{(\omega_0-\omega_1)}\left(\frac{1}{s-\omega_0} - \frac{1}{s-\omega_1}\right) \rightarrow \underset{2j\omega_I}{\frac{1}{\omega_0-\omega_1}}\left[e^{\omega_0 t} \pm e^{\omega_1 t}\right] =$$

$$= \frac{e^{\omega_R t}}{\omega_I}\,\mathrm{sen}\,\omega_I t$$ → oscillazione smorzata

Risposta in frequenza:

$$|j\omega - \omega_R + j\omega_I| = \sqrt{\omega_R^2 + (\omega+\omega_I)^2}$$

$$|j\omega - \omega_R - j\omega_I| = \sqrt{\omega_R^2 + (\omega-\omega_I)^2}$$



$$|G(\omega)| = \frac{1}{\sqrt{(\omega_R^2 + (\omega-\omega_I)^2)(\omega_R^2 + (\omega+\omega_I)^2)}}$$

Poli e zeri reali

la risposta in frequenza in dB è la somma di termini del tipo

$$\pm 20\log_{10}\sqrt{\omega^2+a^2}$$

$$G(dB) = +\sum_{i,\,zeri} 20\log_{10}\sqrt{\omega^2+z_i^2} - \sum_{i,\,poli} 20\log_{10}\sqrt{\omega^2+p_i^2}$$

Se poli e zeri sono distanti tra loro, la pendenza cambia di

+6 dB/ottava passando uno zero

−6 dB/ottava passando un polo.