Quantum Hall Effect

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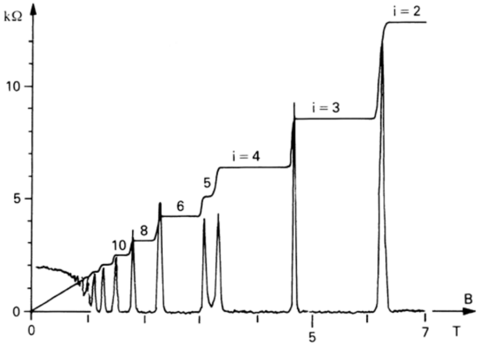
Summary

- The discovery of the Quantum Hall Effect
- Integer Quantum Hall Effect
- Fractional Quantum Hall Effect
- The Laughlin ansatz
- Exitations of the Quantum Hall Fluid

The discovery of the Quantum Hall Effect

Von Kitzling discovers the Integer Quantum Hall Effect in 1980



Trasversal conductance is quantized: $\sigma_{xy} = \frac{e^2}{h} n$ with astonishing precision! Theoretical explaination is needed.

2D electron fluid

Hamiltonian of N electrons (Jelium model) in a uniform magnetic field in 2D:

$$\mathcal{H} = \sum_i \frac{\left[\mathbf{p_i} + \frac{e}{c}\mathbf{A}\left(\mathbf{x_i}\right)\right]^2}{2m} - \sum_i eV(\mathbf{x}_i) + \sum_{i<j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

- The single particle potential may represent disorder superimposed to a lattice potential.
- We will neglet electrons interactions for the moment and treat them perturbatively if needed.

We assume the magnetic field is strong: $\hbar\omega_B \gg E_{int} \gg V$.

Landau levels for one particle Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2$$

We introduce the momentum operator $\pi = \mathbf{p} + \frac{e}{c}\mathbf{A}$. The operators π_x and π_y are *conjugate*. They satisfy the commutation rule:

$$[\pi_x, \pi_y] = ie\hbar B$$

We have $\mathcal{H} = \frac{1}{2m}\pi \cdot \pi$. We define the bosonic operators:

$$a = \frac{1}{\sqrt{2e\hbar B}}\left(\pi_x - i\pi_y\right) \quad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}\left(\pi_x + i\pi_y\right)$$

$\mathcal{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$ with $\omega_B = \frac{eB}{m}$.

Let's introduce another momentum operator: $\tilde{\pi} = \mathbf{p} - \frac{e}{c}\mathbf{A}$.
Together with π they give:

$$[\pi_x, \pi_y] = ie\hbar B \quad [\tilde{\pi}_x, \tilde{\pi}_y] = ie\hbar B$$

$$[\pi_x, \tilde{\pi_x}] = 2ie\hbar \frac{\partial A_x}{\partial x} \quad [\pi_y, \tilde{\pi_y}] = 2ie\hbar \frac{\partial A_y}{\partial y}$$

$$[\pi_x, \tilde{\pi}_y] = [\pi_y, \tilde{\pi_y}] = ie\hbar \left(\frac{\partial A_x}{\partial y} \right)$$

In the symmetric gauge we can diagonalize simultaneusly $\mathcal{H}$ and $\tilde{\pi}$.

$$b = \frac{1}{\sqrt{2e\hbar B}} \left(\tilde{\pi}_x - i\tilde{\pi}_y \right) \quad b^\dagger = \frac{1}{\sqrt{2e\hbar B}} \left(\tilde{\pi}_x + i\tilde{\pi}_y \right)$$

The Hilbert space is built with operators $a, a^\dagger$ and $b, b^\dagger$. The first two are *inter*-Landau levels, the last are *intra*-Landau level.

$$|n, m\rangle = \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n! m!}} |0, 0\rangle$$

There is a unique ground state in the single particle Hilbert space (spanned by momentum eigenstate). Thus there are no other quantum numbers other than n and m.

$$a = \frac{1}{\sqrt{2\pi\hbar B}}\left[-i\hbar\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{eB}{2}(-y - ix)\right]$$

$$z = x - iy \quad \bar{z} = x + iy$$

$$a = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right) \quad a\left|\psi\right\rangle = 0$$

where $\bar{\partial}$ is the derivative with respect to $\bar{z}$. $l_B = \sqrt{\frac{\hbar}{eB}}$ is the magnetic lenght.

$\left(l_B\bar{\partial} + \frac{z}{4l_B}\right)\psi = 0$ solved by $f(z)e^{-\frac{|z|^2}{4l_B^2}}$.

We repeat for $b = -i\sqrt{2}\left(l_B \partial + \frac{\bar{z}}{4 l_B}\right)$, the unique solution to $b\,|\psi\rangle = 0$ is given by: $\psi_{LLL}, m = 0 \propto e^{-\frac{|z|^2}{4 l_B^2}}$ acting with $b^\dagger$ gives a base of the LLL:

$$\psi_{LLL}, m \propto \left(\frac{z}{l_B}\right)^m e^{-\frac{|z|^2}{4 l_B^2}}$$

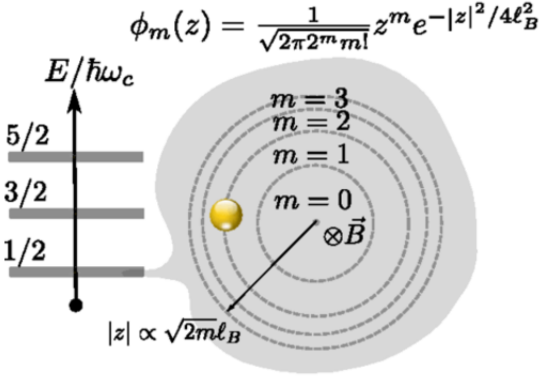
These states are also eigenstates of the angular momentum $J = i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \hbar\left(z\partial - \bar{z}\bar{\partial}\right)$:

$$J\psi_{LLL}, m = \hbar m \psi_{LLL}, m$$

Lowest Landau Level

$$\psi_{LLL}(z) = f(z) e^{-\frac{|z|^2}{4 l_B^2}} \text{ with } f(z) \text{ holomorphic.}$$

ψ_{LLL}, m is a ring of radius $r = \sqrt{2m}l_B$ thus the number of states that we can fill in a circe of area $A = \pi r^2$ is $N = \frac{eBA}{h}$.



Negletting interation

If we have an integer number of filled Landau levels the state is non degenerate and gapped.
If we apply first order perturbation theory:

$$|\psi'\rangle = |\psi\rangle + \sum_n \frac{\langle\psi| V_{int} |\psi_n\rangle}{E_n - E_0} |\psi_n\rangle + O(V_{int}^2)$$

$$\frac{\langle\psi| V_{int} |\psi_n\rangle}{E_n - E_0} \propto \frac{NE_{Coloumb}}{N_{exc}\hbar\omega_B} << 1$$

The **mixing** with a state having a significative number of exitations is **suppressed**.
If the filling fraction is ν the state is $\binom{N}{\nu N}$-degenerate. The way they mix is determined by the interactions.

Naive calculation of σ_{xy}

$$\mathcal{H} = \sum_i \frac{\left[\mathbf{p_i} + \frac{e}{c}\mathbf{A}\left(\mathbf{x_i}\right)\right]^2}{2m} - eE \cdot x$$

We choose the Landau gauge and write $\psi_{n,k}$ with n the Landau level and k the y momentum. The perturbed wavefunction can be compute exactly. The current is:

$$I_y = -\frac{e}{m}\sum_{n=1}^{\nu}\sum_k \langle\psi_{n,k}| - i\hbar\frac{\partial}{\partial y} + exB\,|\psi_{n,k}\rangle = -\nu\frac{E}{B}$$

$$\sigma_{xy} = \nu\frac{e^2}{h}$$

But this argument doesn't explain the existance of the plateau.

Classical electron following a cyclotron orbits is described by:

$$x(t) = X - R\sin(\omega_B t + \phi) \quad y(t) = Y + R\cos(\omega_B t + \phi)$$

where X and Y are the coordinates center of the orbit.

$$X = x - \frac{\dot{y}}{\omega_B} \quad Y = y + \frac{\dot{x}}{\omega_B}$$

If we quantize...

$$X = x - \frac{\pi_y}{\omega_B} = -\frac{\tilde{\pi}_y}{eB} \quad Y = y + \frac{\pi_x}{\omega_B} = \frac{\tilde{\pi}_x}{eB}$$

$$i\hbar\dot{X} = [X, H] = 0 \quad i\hbar\dot{Y} = [Y, H] = 0$$

The center of the orbit is conserved under time evolution.

The role of disorder

The single particle potential $V(x)$ will surely break the degeneracy of each Landau level (weather it is a lattice or a disorder).

Due to $\hbar\omega_B \gg V$ we neglet mixing of different Landau levels.

Semiclassical approximation

$$|\nabla V| \ll \frac{\hbar\omega_B}{l_B} \quad V(x,y) \sim V(X,Y)$$

$$i\hbar\dot{X} = [X, V(X,Y)] = [X,Y]\frac{\partial V}{\partial Y} = i l_B^2 \frac{\partial V}{\partial Y}$$

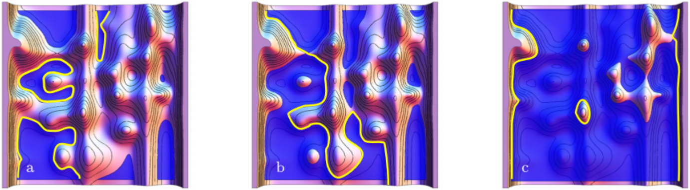
$$i\hbar\dot{Y} = [Y, V(X,Y)] = [Y,X]\frac{\partial V}{\partial X} = -i l_B^2 \frac{\partial V}{\partial Y}$$

Eletrons moves on equipotential lines of $V(x,y)$!

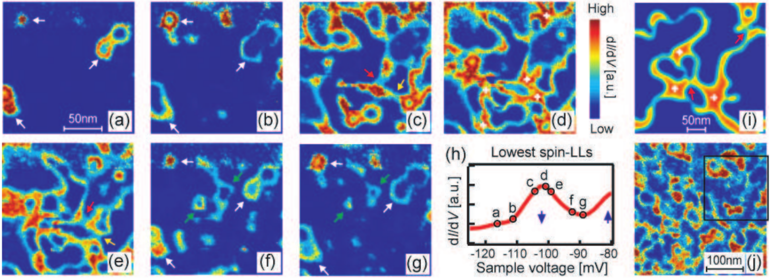
Disorder makes the states localized!

Percolation transition

The kinetic energy of the electron is quenched for each Landau level. Both higher and lower energy states are confined, only middle energy states can move throught the sample.
By slowly changing the Fermi energy the system passes throught a **percolation transition**:



There are only few extended states the majority is localized.

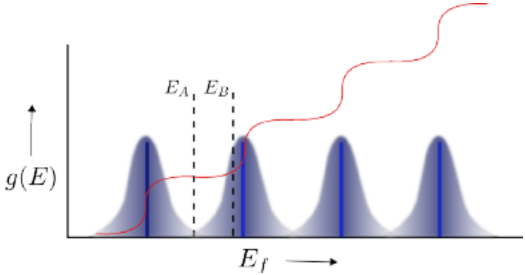


STS measurements by Hashimoto *et al.*, 2008, on a 2D electron system on a n-InSb surface. The figures (a) - (g) show the local DOS at various sample voltages, around the peak obtained from a dI/dV measurement (h). Figure (i) shows a calculated characteristic LDOS, and figure (j) an STS result on a larger scale.

This semiclassical picture should be valid also quantum mechanically!

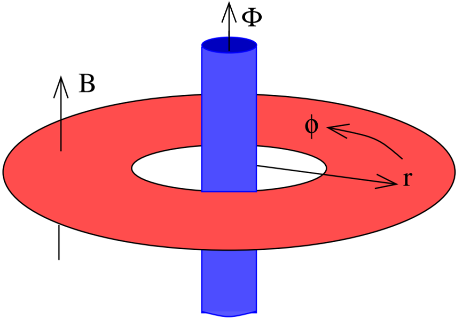
The presence of localized states could explain the plateau while varying **B**. But now we have to explain how the few localized state manage to carry the same current of the full Landau level.

The longitudinal conductivity is zero when the Fermi energy cuts the localized states.



The Corbino ring

We deform the Hall bar and its leads:



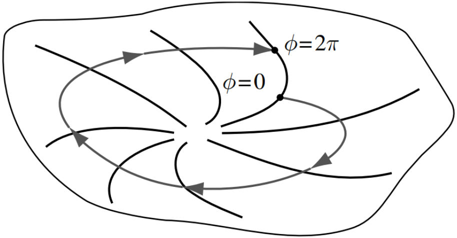
$$\Phi(0) = 0 \quad \Phi(T) = \frac{h}{e} \quad T \gg \frac{1}{\omega_B} \quad \mathcal{E} = -\frac{\Phi_0}{T}$$

$$I = -\frac{ne}{T}, \ \sigma_{xy} = \frac{e^2}{h} n$$

Spectral flow

$$\mathcal{H} = \frac{1}{2m}\left[-\hbar^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \left(-i\frac{\hbar}{r}\frac{\partial}{\partial \phi} + \frac{eBr}{2} + \frac{e\Phi}{2\pi r}\right)^2\right]$$

After inserting one quantum of flux: $\psi(r,\phi) \rightarrow e^{-i\phi}\psi(r,\phi)$



p_ϕ is a **conserved operator** of the adiabatic evolution.

$$\psi_m \rightarrow e^{-i\phi}\psi_{m+1} \qquad r^m \rightarrow r^{m+1}$$

Spectral flow with disorder

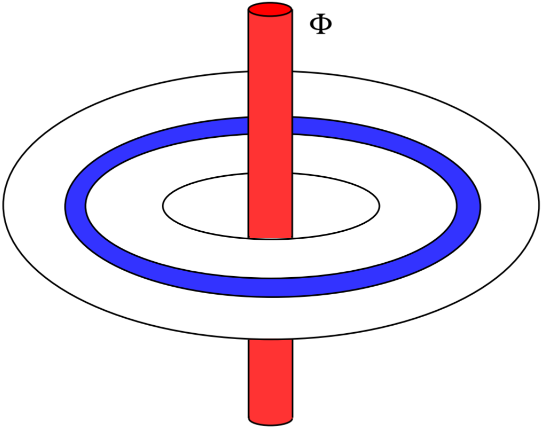
$$\mathcal{H} = \frac{1}{2m}\left[-\hbar^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \left(-i\frac{\hbar}{r}\frac{\partial}{\partial \phi} + \frac{eBr}{2} + \frac{e\Phi}{2\pi r}\right)^2\right] + V(r,\phi)$$

Localized states do not flow:

$$\psi(r,\phi) \to e^{-\frac{ie\Phi\phi}{2\pi\hbar}}\psi(r,\phi)$$

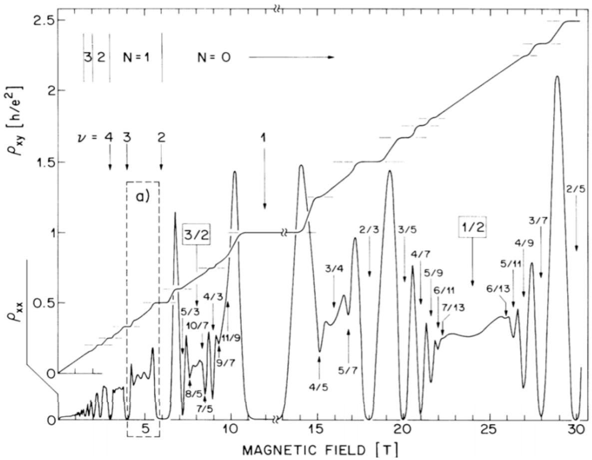
is uniquely defined even for non unit $\frac{e\Phi}{\pi\hbar}$.
Localized states keep their positions, they don't feel the gauge potential.

Spectral flow with disorder



We have a disordered zone pinched between two ordered zone. Disorder doesn't mix the LL. When one level is fully occupied all the m states of the ordered zones are occupied. Exactly one state must flow through the disorded zone after a flux insertion.

In 1982 Stormer and Tsui discover fractional conductance plateau.



- The FQHE should be associated to a fractional filling of the Landau level.
- For an interacting problem it doesn't makes sense to talk about Landau levels occupancy since there will be no more single particle eigenstates.
- We cannot treat the interaction perturbatively, nonetheless we assume the ground state wavefunction to be in the Lowest Landau Level (we again ignore level mixing).

$$\psi = f(z)e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

where z is the set of all the coordinates of the electrons and $f(z)$ is an **holomorphic** function.

The Vandermonde determinant

The Slater determinant for a completely filled Landau level is:

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_1 & z_2 & \cdots & z_{n-1} & z_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{n-2} & z_2^{n-2} & \cdots & z_{n-1}^{n-2} & z_n^{n-2} \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_{n-1}^{n-1} & z_{n-1}^{n} \end{pmatrix} = \prod_{i<j}(z_i - z_j)$$

with the addictional exponential factor the state becomes:

$$\psi(z) = \prod_{i<j}(z_i - z_j)\, e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

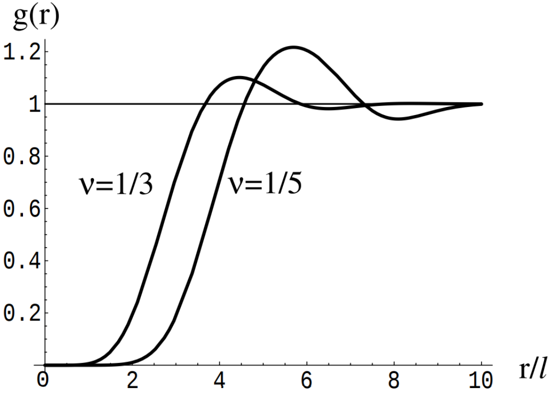
Laughlin ansatz

Laughlin proposed the following wave function to account for the plateau at filling fraction $\nu = \frac{1}{m}$:

$$\psi(z) = \prod_{i<j} (z_i - z_j)^m \, e^{-\sum_i \frac{|z_i|^2}{4 l_B^2}}$$

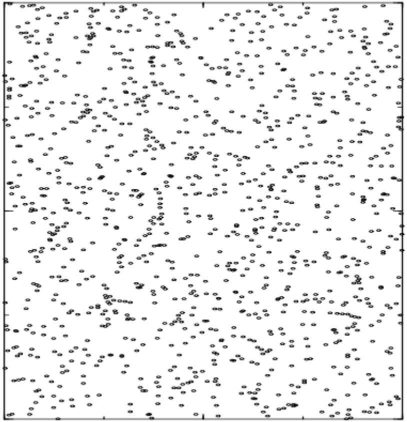
$m = 2k + 1$ because of the Pauli exclusion principle.
The maximum power of z_1 is $m(N-1) \sim mN$ thus the area of the sample is $A = 2\pi m N l_B^2$ thus the total number of states is mN and $\nu = \frac{N}{mN} = \frac{1}{m}$.
What does it mean to have a fractional filling **when single particle states don't make** sense anymore?

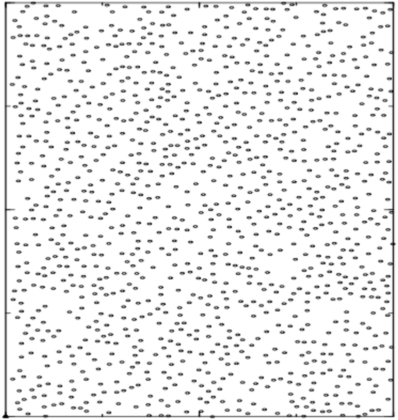
Pair correlation functions for $m = 3$ and $m = 5$:



The Laughlin ansatz effectively keeps the electron apart lowering the Coloumb interaction energy. It's a **liquid phase** of the electron fluid, there is also a competing solid phase: the **Wigner crystal**.

Comparison of a random distribution of electrons (left) and one following the Laughlin ansatz (right). Density fluctuations are **suppressed**.





We can shed light on the Laughlin ansatz by exploiting the analogy with a classical plasma:

$$\hat{n}(z) = \sum_i \delta(z - z_i)$$

$$n(z) = \langle\psi|\, n(z)\, |\psi\rangle = \frac{\int \prod_{i=1}^{N} d^2 z_i n(z) P[z]}{\int \prod_{i=1}^{N} d^2 z_i P[z]}$$

$$P[z] = \prod_{i<j} \frac{|z_i - z_j|^{2m}}{l_B^{2m}} e^{-\sum_i \frac{|z_i|^2}{2 l_B^2}}$$

The plasma analogy

Particles of charge $-m$ inteacting with a uniform background

$$P\left[z\right] = e^{-\beta U(z)}$$

$$U(z) = -m^2 \sum_{i<j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{m}{4 l_B^2} \sum_{i=1}^{N} |z_i|^2 \quad \beta = \frac{2}{m}$$

Background charge density: $\rho_0 = -\frac{1}{2\pi l_B^2}$.

The plasma is globally neutral. At filling fraction the background charge is screened with a uniform density.

$$n = \frac{1}{2\pi l_B^2 m}$$

Quasiholes

We excite the Laughlin vacuum by adding a "vortex" at position η:

$$\psi\left(z;\eta\right)=\prod_{i=1}^{N}\left(z_i-\eta\right)\prod_{i<j}\left(z_i-z_j\right)^m e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

Charge depletion aroud η (quasihole). Carries a fractional charge $e^*=\frac{e}{m}$.
In the plasma analogy it behaves like an impurity of charge $q=-1$:

$$U(z)=-m^2\sum_{i<j}\log\left(\frac{|z_i-z_j|}{l_B}\right)-m\sum_{i=1}^{N}\log\left(\frac{|z_i-\eta|}{l_B}\right)+\frac{m}{4l_B^2}\sum_{i=1}^{N}|z_i|^2$$

There is something strange with this potential...

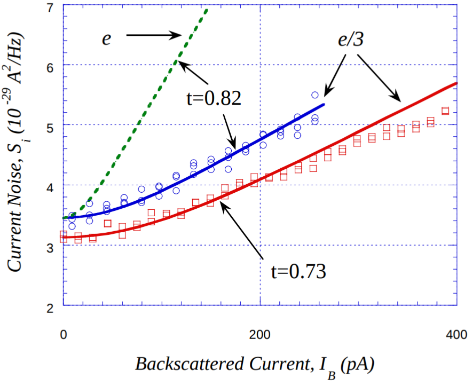
Quasiparticles

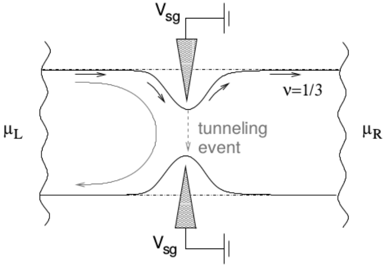
Excitations that carry a charge $e^* = -\frac{e}{m}$. We need to increas the local density of the electrons, thus diminuish the relative angular momentum of a pair of particles. We cannot divide by z (analiticy!).

$$\psi = \left[\prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i} - \bar{\eta}\right)\prod_{k<l}(z_k - z_l)^m\right] e^{-\sum_{i=1}^{n}\frac{|z_i|^2}{4l_B^2}}$$

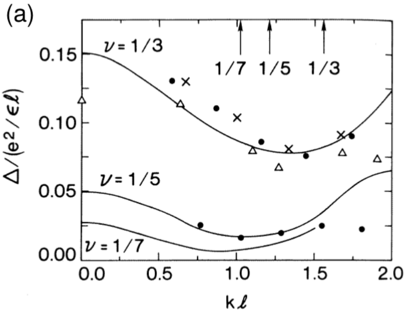
Experimental proof of fractionalization

R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin, and D. Mahalu, "Direct observation of a fractional charge", Nature 389, 162 (1997).





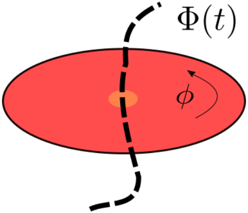
A systematic study of the collective excitations involves taking the immaginary part of the density-density response function $\chi_{nn}(\mathbf{q}, \omega)$.



S. M. Girvin, 1998. Excitations are **gapped**. The liquid is incompressible by definition. Ansatz wavefunction for those excitations (similar to the one by Feynman for superfluids).

Laughlin flux insertion

What is the state after the adiabatic flux insertion if it starts as ψ? If we perform a singular gauge transformation on ψ to insert a flux in the origin we have:



$\mathbf{A} = Arg\,(z)$ is precisely a vortex with $\mathbf{B} = 0$ field and a singularity in the origin.

$$\psi^{'}(z) = \prod \frac{z_i}{|z_i|}\psi(z)$$

This is obviously not in the LLL so Laughlin proposed that the adiabatic evolution would have led to:

$$\psi'(z) = \prod z_i \prod_{i<j} (z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4 l_B^2}}$$

This is consistent with the fact that starting from an eigenstate if there is no level crossing I get in another eigenstate.
A quasihole depletes a charge $\frac{e}{m}$ from its centers, this must have flown through the boundaries and we get $\sigma_{xy} = \frac{e^2}{h}\frac{1}{m}$.

A similar argument goes through in presence of disorder.

Thanks for your attention

Berry phase

Consider a state of M quasiholes:

$$\langle z, \bar{z} | \eta_1, \eta_2, .., \eta_M \rangle = \frac{1}{Z} \prod_{j=1}^{M} \prod_{i=1}^{N} (z_i - \eta_j) \prod_{k<l} (z_k - z_l)^m \, e^{-\sum_i \frac{|z_i|^2}{4 l_B^2}}$$

$$Z = \int \prod d^2 z_i \exp \left(\sum_{i<j} \log |z_i - \eta_j|^2 + m \sum_{k,l} \log |z_k - z_l|^2 - \frac{1}{2 l_B^2} |z_i|^2 \right)$$

What happens if we slowly exchange two quasiparticles? If they are indistinguishable particles the wavefunction stay the same up to a Berry phase.

Berry phase

Let's compute the Berry phase:

$$\mathcal{A}_\eta = -i \langle \eta | \frac{\partial}{\partial \eta} | \eta \rangle = -\frac{i}{2} \frac{\partial \log Z}{\partial \eta} \quad \mathcal{A}_{\bar{\eta}} = +\frac{i}{2} \frac{\partial \log Z}{\partial \bar{\eta}}$$

We use again the plasma analogy!
We add the interaction of the impurities between themselves and with the background. The correct internal energy should be:

$$U(z_k; \eta_i) = -m \sum_{k<l} \log |z_i - \eta_j|^2 - m^2 \sum_{k,l} \log |z_k - z_l|^2 - \frac{1}{2 l_B^2} |z_i|^2 +$$

$$- \sum_{i<j} \log |\eta_i - \eta_j| + \frac{m}{4 l_B^2} \sum_k |z_k|^2 + \frac{1}{4 l_B^2} \sum_i |\eta_i|^2$$

Berry phase

The correct partition function is thus:

$$\int d^2 z_k e^{-\beta U(z_k;\eta_i)} = \exp\left(-\frac{1}{m}\sum_{i<j}\log|\eta_i - \eta_j| + \frac{1}{2ml_B^2}\sum_i |\eta_i|^2\right) Z$$

As long as the plasma is the liquid phase (which is for $m < 70$) because of the screening the partition function should be independent on the position of the impurities η_i, thus we get:

$$Z = C\exp\left(\frac{1}{m}\sum_{i<j}\log|\eta_i - \eta_j| - \frac{1}{2ml_B^2}\sum_i |\eta_i|^2\right)$$

Berry phase

The Berry phase becomes:

$$\mathcal{A}_\eta = -\frac{i}{2m}\sum_{i\neq j}\frac{1}{\eta_i - \eta_j} + \frac{i\bar{\eta}_i}{4ml_B^2}$$

and a similar expression for the anti-holonomic connection. They holds if the particles don't come too close to each other and break the screening approximation.

$$\gamma = -\oint_C \left(\mathcal{A}_\eta d\eta + \mathcal{A}_{\bar{\eta}} d\bar{\eta}\right)$$

Charge of the quasiholes

We consider a loop of one quasihole in a region that doesn't enclose any other quasiparticle.

$$\mathcal{A}_\eta = \frac{i\bar{\eta}}{4ml_B^2} \quad \mathcal{A}_{\bar{\eta}} = \frac{-i\eta}{4ml_B^2}$$

$$\gamma = \frac{e\Psi}{m\hbar}$$

It's the Arhanov-Bohm phase of a particle of charge $\frac{e}{m}$.

Fractional statistics

We generate two quasiparticles at η_1 and η_2. We take η_1 on a journey that encloses η_2.
We neglet the Arhaniv-Bohm phase and get:

$$\gamma = -\frac{1}{2m}\left(\oint \frac{d\eta_1}{\eta_1 - \eta_2} + \oint \frac{d\bar{\eta}_1}{\bar{\eta}_1 - \bar{\eta}_2}\right) = \frac{2\pi i}{m}$$

Quasiholes (and quasiparticles) are nor fermions or bosons. Upon the exchange of two of them the wavefunction gets a phase of $\frac{2\pi i}{m}$.
These particles are called **anyons**.