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On the Algebraic Structure of Higher-Spin Field Equations and New Exact Solutions

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Ph.D. Thesis

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Abstract

This Thesis reviews Vasiliev's approach to Higher-Spin Gauge Theory and contains some original results concerning new exact solutions of the Vasiliev equations and the representation theory of the higher-spin algebra.

The review part covers the various formulations of the free theory as well as Vasiliev's full nonlinear equations, in particular focusing on their algebraic structure and on their properties in various space-time signatures.

Then, the original results are presented. First, the 4D Vasiliev equations are formulated in space-times with signatures $(4-p,p)$ and non-vanishing cosmological constant, and some new exact solutions are found, depending on continuous and discrete parameters: (a) an $SO(4-p,p)$-invariant family of solutions; (b) non-maximally symmetric solutions with vanishing Weyl tensors and higher-spin gauge fields, that differ from the maximally symmetric background solutions in the auxiliary field sector; and (c) solutions of the chiral models with an infinite tower of Weyl tensors proportional to totally symmetric products of two principal spinors. These are apparently the first exact 4D solutions with non-vanishing massless higher-spin fields.
Finally, a generalized harmonic expansion of the Vasiliev's master zero-form is performed as a map from the associative algebra $\mathcal{A}$ of operators on the singleton phase space to representations of the background isometry algebra that include one-particle states along with linearized runaway solutions. Such Harish-Chandra modules are unitarizable in a $\mathrm{Tr}_{\mathcal{A}}$-norm rather than in the standard Killing norm. We also take the first steps towards a regularization scheme for handling strongly coupled higher-derivative interactions within this operator formalism.

Ai miei genitori, e a Chloé

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Contents

Chapter 1

Introduction

Higher-Spin (HS) Fields have attracted the attention of theoretical physicists since the very early days of Relativistic Field Theory. As soon as the importance of space-time isometries was exploited, and free propagating particles were associated to the solutions of invariant equations under the corresponding symmetry group, it became natural to investigate the properties of relativistic fields of more general types. Indeed, the relativity principle reduced the classification of linear relativistic wave equations to the classification of the unitary irreducible representations (UIRs) of the Poincaré group, and the latter where found to be fully specified by the two quantum numbers of mass and spin [5]. Therefore, the very existence of massive and massless UIRs of the Poincaré group with arbitrary spin (or, more properly, helicity, in the massless case) was, and keeps on being, the first motivation for a study of the corresponding field theories.

The massless, discrete-helicity case has received the greatest attention due to the local symmetry principles associated to it. The gauge invariance that the field equations acquire in the massless limit is a signal that they actually involve more variables than the physical degrees of freedom (dof). In group-theoretical terms, this means that the space of solutions no longer corresponds to an irreducible representation of the Poincaré group, but rather to an indecomposable one, in which the unphysical polarizations form an invariant submodule. The role of the gauge symmetry is precisely that of factoring out such submodule, thereby defining the gauge field via an equivalence class.

The huge interest in gauge theories of course arises from the fact that massless lower-spin fields (*i.e.*, spin $s \leq 2$) are known to describe the fundamental interactions, encoded in the Standard Model and in Einstein's General Relativity (GR): electroweak and strong interactions are based on abelian and non-abelian spin-1 gauge fields, that implement $SU(3) \times SU(2) \times U(1)$ local symmetry, and gravity on a spin-2 gauge field, that essentially implements diffeomorphism invariance. In Classical and Quantum Field Theory (QFT), local symmetries put strong restrictions on, or completely determine, the dynamics of fundamental constituents and the possible interactions between them - and have

indeed led theoretical physicists to spectacular predictions.
By now, the classical lower-spin gauge theories are very well known, and explain an amazing variety of very different phenomena. But the history of Physics is, to a good extent, a history of unification: formerly separated areas of investigation were shown to admit a description within a single conceptual and formal framework, which proved to be successful in increasing their predictive power. The last century achieved a partial success in unifying the fundamental interactions mediated by spin-1 massless particles within the framework of QFT. However, joining gravity in this technical and conceptual scheme still faces enormous difficulties.
Indeed, a number of very important differences arise already at the classical level, when one compares gauge fields of spin 1 and 2: first, the former is associated to a gauge parameter $\epsilon(x)$ which is an arbitrary scalar function of space-time coordinates, and the corresponding local symmetries are therefore *internal*, while spin-2 gauge fields are associated to a vector gauge parameter $\epsilon^{\mu}(x)$, which can be identified with an infinitesimal change of coordinates, *i.e.*, with *space-time* symmetries. This fact is at the root of many differences - that, for example, account for the known subtleties in the *tout-court* interpretation of gravity as a gauge theory in the Yang-Mills sense - that we shall examine in detail in a broader and more general context. Let us only stress here the well-known fact that, while consistent self-interactions of non Abelian spin-1 gauge fields demand finitely many nonlinear terms (up to the quartic order in the Lagrangian), the same request for a spin-2 field actually implies infinitely many nonlinear corrections of higher and higher order! Furthermore, the propagators of massless fields in QFT show that the exchange of *even*-spin massless boson gives rise to universally attractive static potentials, while *odd*-spin gauge fields mediate attractive and repulsive static interactions among particles of unlike and like charges, respectively. Finally, a QFT of spin-1 gauge fields, both abelian and non-abelian, has been constructed and has been shown to be renormalizable, while gravity is plagued by non-renormalizable infinities.
The fundamental difficulty of the many attempts at combining internal and space-time symmetries into a bigger, more fundamental symmetry group, within a purely field-theoretical scheme, were encoded into the famous Coleman-Mandula theorem [7] in 1967: roughly speaking, it states that, under certain assumptions which at that time seemed reasonable for any physically relevant field theory, any S-matrix - governing interactions among quantum fields - symmetric under an algebra $\mathfrak{g}$ which is bigger than the direct sum of the Poincaré (or at most the conformal) algebra $\mathcal{P}$ and some internal symmetry algebra (spanned by elements that commute with the generators of $\mathcal{P}$) has to be trivial. However, one very important exception was indeed found a few years later: supersymmetry was the first instance of a global symmetry that is not forbidden by the Coleman-Mandula theorem and that transforms fields of different spin into one another. Such a way out was found by explicitly evading one of the hypothesis of the theorem: in particular, by introducing Grassmann-odd generators, *i.e.*, spin-1/2 fermionic parameters, and constructing a *graded* symmetry algebra, that acts via commutators and anti-commutators. The realm of the

physically interesting theories was therefore expanded, and the new landmarks were put down with the formulation of the Haag-Lopuszanski-Sohnius theorem [8], stating that, under similar assumptions, the maximal symmetry of a nontrivial S-matrix is the direct sum of the superconformal algebra and some internal symmetry algebra.
In 1976, the discovery of supergravity [23], a theory with *local* supersymmetry, renewed the interest for Higher-Spin Gauge Theories (HSGT). Indeed, simple supergravity can be obtained from the requirement of a consistent gravitational coupling (a crucial requirement for any matter or gauge field, due to the universality of the gravitational interaction) for a massless spin-3/2 field, the gravitino, whose free equation had been written down already in 1941 [4]. In some sense this was, historically, the first example of a HSGT, of a theory which is invariant under a local transformations that rotates fields of different spins among themselves, albeit a particularly simple one, for a reason that we shall explain later. Since then, higher spins are fields with spin $s \geq 5/2$. The contributions of the fermionic superpartners were actually improving the quantum behavior of the theory, compared to pure gravity, but the theory still seemed to be divergent[1].
Therefore, Higher-Spin Gauge Theories correspond to field theories that include massless fields of various spins s and that are invariant under more general local symmetry principles associated with parameters that are spin-$(s-1)$ fields. Although there is, at present, no experimental evidence for fundamental particles of higher spin, from a purely field theoretical point of view the study of a more general case may teach some important lesson on the known gauge theories, and can lead to a better understanding of the gauge principle. Moreover, our experience with supergravity suggests that the quantum theory of a bigger symmetry multiplet, that contains the spin-2 gauge field together with other fields of different spins, might behave better than the known gravity theories, and hopefully be free of infinities.

Nowadays, the dreams of unifications of the fundamental interactions and of a finite quantum theory of gravity both seem within reach in String Theory - for the very reason that the fundamental constituent is not a point particle but a one-dimensional extended object of length $l_s \sim 10^{-33} cm$. The string length, indeed, not only acts as a natural UV cutoff, but also provides some necessary conditions for unification in a very natural way, compared to the case of point-like particles. Indeed, in the stringy framework, fields of different spins arise as vibration modes of the string, with the spin naturally carried by Fourier coefficients that carry indices related to target space-time coordinates. However, such a framework leads to a much richer set of fields than the familiar low-spin ones. As a generic vibration of a classical string, with certain boundary condition, admits an expansion over an infinite set of harmonics with higher and higher frequency, in the same way the spectrum of a quantum mechanical relativistic string naturally involves *infinitely many* excitations that can be identified with different particles of higher and higher mass

[1] More recent investigations [24], however, suggest that the divergences of supergravity are not as severe as were originally thought, and that $\mathcal{N} = 8$ supergravity might even be finite!

and spin. The length and tension T of the string are related as (in units $\hbar = c = 1$)

$$l_s = \sqrt{2\alpha'} = \frac{1}{\sqrt{\pi T}} \ , \tag{1.0.1}$$

where α' is a constant, the so-called *Regge slope*, while the mass and spin of string excitations are related among themselves and to the tension as

$$m^2 \sim \frac{1}{\alpha'}(s - a) \ , \qquad a \ = \ \begin{cases} 1 \ , \ \text{open string} \\ 2 \ , \ \text{closed string} \end{cases} \ , \tag{1.0.2}$$

where m^2 is the squared mass of string states, s is their spin, and a a constant. The string tension is typically assumed to be of the order of the Planck mass, $M_{Pl} \sim 10^{19} GeV$, and is therefore so high that, at low energies, only the modes with low frequency can be excited, that exactly correspond to the massless fields - spin-1 and spin-2 gauge fields, in the open and closed string sector, respectively - that mediate the known long-range interactions[2]. Moreover, in the superstring spectrum, whose critical dimension is $D = 10$, a massless spinor with $s = 3/2$ is also present. The low-energy superstring dynamics is therefore well-approximated by ten-dimensional supergravity, while all higher-spin excitations have very high masses and essentially decouple. However, an infinite tower of higher-spin fields is necessarily present in String Theory, and is crucial for its consistency. Indeed, from a field theoretical point of view, the finiteness of the Theory, due to the natural cutoff given by the finite string length, translates into the inevitable appearance of an infinite set of higher-spin fields, with certain fine-tuned masses, whose contribution to the quantum perturbative expansion cancels exactly the divergencies of the lower-spin sector. A study of the properties of HS fields may therefore lead to a better understanding of String Theory. This becomes particularly true when referred to its high-energy regime, where HS fields can indeed contribute significantly to the dynamics. Despite many efforts, however, much less is known about strings in this limit than in the supergravity approximation, but nevertheless interesting observations have been made about the appearance of some intriguing symmetry enhancements. Indeed, hints of a higher-spin symmetry have been found in the study of string scattering amplitudes in the high-energy limit. This fact, together with our knowledge of spin-1 gauge theories - according to which the quantum theory of a massive boson is renormalizable only if the mass is generated via spontaneous breaking of the gauge symmetries -, led many to think that String Theory might in fact correspond to a Higher-Spin Gauge Theory in some spontaneously broken phase. All the extra symmetries should therefore be recovered in the tensionless limit, in which the

[2] In the bosonic string spectrum there is also a tachyon, which is a signal of the instability of the vacuum around which the theory is expanded. Here we do not comment further on this subtlety, but just remind the reader that the tachyon is anyway not present in the spectrum of superstrings. Moreover, let us also mention here that theories of open strings only are inconsistent, and necessarily a closed string sector needs to be introduced. In other words, in some sense String Theory predicts the existence of gravity.

whole tower of string excitations become massless and each corresponding field mediates new long-range interactions. If this is the case, the study of HSGT in relation with the tensionless limit of strings might unravel the true symmetries on which String Theory is based, and give some guiding principle for exploring the physics of the so-far elusive eleven-dimensional M-theory that has been recognized to underlie the different ten-dimensional Superstring Theories. Furthermore, the latter are still lacking a background independent formulation, where space-time is a concept that only emerges from the dynamics of the various string excitations and is not fixed *a priori*. There are reasons to believe that this is essentially a technical problem, rather than an intrinsic weakness of String Theory: in particular, it might depend on the fact that, as mentioned before, it is only the low-energy dynamics of strings - that we are accustomed to describe with tools and concepts derived from the known low-spin gauge theories - that has been widely explored, while a more "fundamental" description may need to take into account the contribution of HS excitations to the space-time texture. The search for such a description may also therefore benefit from the developments of HSGT, since, as we shall see, classical interacting HS field equations exist that are fully background independent, in a way that is at the same time compatible with HS gauge symmetries.

As mentioned above, the idea that String Theory is a broken phase of a more symmetric HSGT has been somehow in the back of many theoretical physicists' mind, but has never found a truly quantitative formulation. Nowadays, however, the level reached by our knowledge of HSGT and of the so-called AdS/CFT correspondence has enabled to recognize, at least at a kinematical level, signatures of the huge Higgs mechanism (that some authors called *La Grande Bouffe*) expected to take place in switching on the tension, giving mass to all higher-spin states and leaving massless only the low-spin excitations that mediate the known long-range interactions. A very important window for the study of both String Theory and quantum gauge field theories, the holographic AdS/CFT dualities relate, in their most general version, type IIB Superstring Theory in space-time geometries that are asymptotically Anti-de Sitter (AdS) times a compact space to conformal field theories. More precisely, the AdS/CFT correspondence [29] is a conjectured equivalence, at the level of partition functions, between the type IIB Superstring Theory on an $AdS_5 \times S^5$ background and the $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory (SYM), with $SU(N)$ gauge group, living on the four-dimensional boundary of AdS_5. The equivalence is, in fact, a *duality*: that is, the string and field theoretic pictures are two descriptions of the same physics, such that when one is weakly coupled, the dual one is strongly coupled. Although the4 correspondence is supposed to hold for every regime of both descriptions, *i.e.*, for every value of the relevant parameters, it has been mainly tested in the limit of large N, large 't Hooft coupling $\lambda = g^2_{YM} N$ and high tension $T \sim \sqrt{\lambda}$, in which the bulk (string) side is well approximated by classical IIB supergravity and the boundary theory is a strongly coupled $\mathcal{N} = 4$ SYM in the planar limit (in which there is now some evidence that it is an integrable theory).

However, the correspondence also offers a powerful framework to explore the physics of strings in the tensionless limit, that is dual to a weakly coupled or free boundary theory. In such a limit, the bulk theory cannot be approximated by supergravity anymore, since the mass of HS excitations becomes small, and, on the boundary side, the free SYM admits an infinite number of conserved currents of all spins. In the spirit of the correspondence, the dual bulk theory should therefore be a theory of interacting gauge fields of higher spin, based on some non-abelian infinite-dimensional extension of the AdS superalgebra. Recent investigations have indeed found tracks of a HS Higgs mechanism in the fact that, switching on a small λ for the SYM, all the currents that correspond to bulk HS fields acquire anomalous dimensions, which is equivalent to a spontaneous breaking of all local HS bulk symmetries to the finite-dimensional ones of the supergravity theory. There are also very interesting issues of integrability, on the two sides of the correspondence, that show remarkable similarities: particularly intriguing are, for example, those between the construction of Yangian algebras [41] and HS algebras.

The study of String Theory in the tensionless limit and of the HS dynamics has also been carried on, to some extent, in the framework of String Field Theory (SFT), a second quantized approach to strings. Interacting HSGT and SFT show very interesting similarities, since they both deal with similar variables (the *string field* and the *master fields*, that we will present later on, that both include space-time fields of all spins in an expansion over higher and higher powers of oscillators), that have similar associative but noncommutative composition laws (a $\star$-product that essentially implements an (infinite) matrix multiplication), and since their field equations share the simple and elegant form of zero-curvature conditions. Therefore, also certain methods for finding exact solutions are very common in spirit in the two settings (and are also common to noncommutative field theories, in general), as well as the language and main tools.
In fact, one expects that SFT in the tensionless limit reduces somehow to a HSGT. However, there are a number of difficulties in relating explicitly the two theories. To begin with, important corners are still missing, on both sides: Closed SFT (CSFT) is, under many respects, much more complicated than Open SFT (OSFT), and its present formulation is still incomplete; on the other hand, the full classical theory of HS fields at all orders in interactions has only been worked out for a particular class of fields, those represented by totally symmetric tensors, that is exhaustive, up to dualities, only in four dimensions. For this reason, interacting HSGT deals today essentially with the fields that belong to a single Regge trajectory of the open string spectrum, *i.e.*, obtained acting on the vacuum with powers of a single oscillator, like

$$\varphi^{\mu_1\ldots\mu_s} \;=\; \alpha_{-1}^{\mu_1}\ldots\alpha_{-1}^{\mu_s}\,|0\rangle\ , \tag{1.0.3}$$

while ten-dimensional Superstring theories also contain mixed symmetry tensor excitations, obtained acting on the vacuum with several oscillators. In $D>4$, such vibrating modes describe degrees of freedom that are independent from those encoded in totally

symmetric fields, and have therefore necessarily to be taken into account in the dynamics. It is possible that the tensionless limit of SFT will be clearer once a more complete formulation of CSFT and HSGT will be available.
However, the study of the tensionless limit of the *free* string field equations, that can be consistently truncated to totally symmetric fields, has already given some positive results: in particular, it has been shown that in such a limit they reduce to equations for triplets of fields that describe the propagation of massless fields of all spins and that exactly reduce to the *unconstrained* free HS field equations of Francia and Sagnotti that we shall describe in Chapter 2 (see [22], also for generalizations to mixed symmetry free equations and (A)dS background, and for further references).

History and properties of Higher-spin Gauge Theories

From what has been said so far, one should appreciate the importance of a better understanding of HS dynamics, and how HSGT are connected to many important aspects and open questions of the current research in String Theory. In particular, interacting HSGT restricted to the totally symmetric sector, offers a somewhat simplified framework for the study of certain string excitations when collapsed to zero mass. Indeed, the research in HS fields has been carried out since many years independently of String Theory, and produced results that are very interesting in their own right, for the reasons mentioned in the beginning of this Introduction.
Already in 1939, Fierz and Pauli wrote free equations for massive HS fields in flat spacetimes. For bosons, the correct number of polarizations was found to be propagated by a real, totally symmetric traceless and divergenceless field $\Phi_{\mu_1\ldots\mu_s}$ satisfying a massive Klein-Gordon equation. Analogously, massive spin-(s+1/2) fields were described by a totally symmetric, γ-traceless and divergenceless spinor-tensor $\Psi_{\mu_1\ldots\mu_s}$ satisfying a massive Dirac equation. Already at this level, some difficulty related to the peculiarities of HS fields was recognized: indeed, limiting here the discussion to bosonic fields for brevity, while the equations of motion were evidently a generalization of the Proca construction for the massive photon, a corresponding variational principle seemed not easy to find. Only in 1974 Singh and Hagen [12, 13] were able to cast the free theory of a massive spin-s field in Lagrangian form, by making use of a set of auxiliary fields of all spins $s-2, s-1, ..., 0$. The massless limit of this formulation was studied for the first time in 1978 by Fronsdal, for bosons, and by Fang and Fronsdal for fermions [14, 15, 16, 17]. It was found that, as the mass tends to zero, all auxiliary fields of the massive bosonic Singh-Hagen lagrangian decouple from the physical field $\Phi_{\mu_1\ldots\mu_s}$, except for the one with spin $s-2$. The latter can then be combined with the spin-s field in a totally symmetric *doubly traceless* field $\varphi_{\mu_1\ldots\mu_s}$ satisfying an equation that is a straightforward generalization of Maxwell's and linearized Einstein's equations: in particular, it possesses an abelian gauge symmetry under

$$\delta\varphi_{\mu_1\ldots\mu_s}(x) \;=\; \partial_{\mu_1}\epsilon_{\mu_2\ldots\mu_s}(x) + \text{symmetrizations}\;, \tag{1.0.4}$$

where the gauge parameter is a totally symmetric *traceless* tensor field of rank $s-1$. It was also found that such construction admitted an extension to (A)dS background, that essentially amounted to replace standard derivatives with (A)dS-covariant ones and to add a mass-like term that encoded the coupling of the spin-s field with the constant curvature scalar R.
The Fronsdal equations marked the beginning of the studies on HSGT. However, the constraints on the gauge parameter and on the field, that were crucial for the gauge invariance of the field equations and the lagrangian, were somehow unsatisfactory signs of an incomplete formulation; indeed, there are no such algebraic constraints in the free string field equations. These were the main motivations for a deeper investigations of the free HS dynamics, that was carried on mainly by Francia and Sagnotti [19, 20], and Bekaert and Boulanger [47]: the result was a more general formulation, that achieves *unconstrained* gauge invariance and, in doing so, elucidates the geometry of HS gauge fields. For instance, the resulting Francia-Sagnotti unconstrained equations for free spin-s massless fields are *non-local* generalizations of the Maxwell and linearized Einstein ones, but this time written in terms of proper HS curvatures, that were first constructed by de Wit and Freedman [18]. The conventional Fronsdal formulation can be recovered via a gauge fixing. Moreover, the non-locality is pure gauge, and can therefore be removed by introducing, for every spin-s field, a compensator field of rank $s-3$. An important feature of such equations, among others, is that they allowed a direct, successful comparison with those of the triplets coming from free SFT [22]. More recently, the Francia-Sagnotti equations, in both their local and non-local forms, were also cast in a lagrangian form, by making use of the compensator and of an additional auxiliary field, a spin-$(s-4)$ Lagrange multiplier[3].

Notwithstanding the many motivations for a careful study of a full HSGT, for decades there has not been much progress in the construction of interactions of massless HS fields, among themselves and with low-spin gauge fields. The "HS interaction problem" was already recognized from the early analysis of Fierz and Pauli, that studied the coupling of HS fields to the electromagnetic field, and was later formalized in general no-go theorems - such as those due to Coleman-Mandula and Haag-Lopuszanski-Sohnius already cited and another due to Weinberg and Witten [9] - that seemed to rule out a consistent nontrivial embedding of the lower-spin symmetries into some bigger symmetry algebra mixing fields of different spins, therefore forbidding the possibility of couplings with HS fields that would not break the gauge symmetries. A special attention was devoted to the coupling to gravity, due to its universality: the analysis of Aragone-Deser and Aragone-Laroche [10] showed that indeed the coupling of the spin-2 field with a gauge field with spin $s \geq 5/2$ is inconsistent with the gauge symmetries, and this essentially because the presence of

[3] It should be also mentioned that the free HS off-shell formulation for symmetric tensors had also been worked out previously by Pashnev and Tsulaia using BRST techniques: the spin-s lagrangian, however, involved a number of auxiliary fields that grew proportionally to s.

"too many" indices attached to a massless field inevitably leads to the appearance of the Weyl tensor in the gauge variation of the covariantized action for the spin-s field - and the Weyl tensor cannot be cancelled by any variation of the gravitational part of the action. More precisely, the argument proceeds as follows. To introduce the coupling with gravity, it is necessary to introduce a coupling with the spin-2 field by covariantizing derivatives, $\partial \longrightarrow D = \partial + \Gamma$, both in the lagrangian and in the gauge transformations. The lagrangian for the spin-s field φ contains the terms $\frac{1}{2}(D\varphi)^2 - \frac{s}{2}(D \cdot \varphi)^2$, and the gauge transformation becomes $\delta\varphi_{\mu_1\mu_2...\mu_s} = D_{(\mu_1}\epsilon_{\mu_2...\mu_s)}$. The variation of the lagrangian amounts to a commutator of covariant derivatives, that is proportional to the Riemann tensor acting on the gauge parameter,

$$\delta\mathcal{L} = \Re...(\epsilon...D\varphi...) \neq 0 \; . \tag{1.0.5}$$

For $s > 2$ the *full* Riemann tensor - both its trace (Ricci tensor) and its traceless part (Weyl tensor) - contributes to this variation, and this is what makes a consistent coupling impossible, since it is only the trace part that can be compensated by varying the gravitational action. It is interesting to note that the case of spin 3/2 is the last one (together of course with spin 2 self-couplings) in which a consistent coupling is possible, and indeed gives rise to simple supergravity. Indeed, the variation of the covariantized Rarita-Schwinger lagrangian

$$I_{RS} \;=\; \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \tag{1.0.6}$$

under $\delta\psi_\mu^\alpha = D_\mu \epsilon^\alpha$ is proportional to

$$\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu D_\rho \epsilon \;\sim \bar{\psi}_\mu \gamma^{\mu\nu\rho} [D_\nu D_\rho] \epsilon \;\;\sim\;\; \bar{\psi}_\mu \gamma^{\mu\nu\rho} \Re_{\nu\rho,\sigma\tau} \gamma^{\sigma\tau} \epsilon \;\;\sim\;\; \bar{\psi}_\mu (Ric)^\mu{}_\nu \gamma^\nu \epsilon \; , \tag{1.0.7}$$

where we have used γ-matrices identities in the last step. Thanks to the low rank of the gravitino, the only nontrivial part of the variation that survives is proportional to the Ricci tensor, and can be compensated by a corresponding supersymmetry transformation of the metric,

$$\delta g_{\mu\nu} \;=\; \frac{i}{2}(\bar{\psi}_\mu \gamma_\nu \epsilon + \bar{\psi}_\nu \gamma_\mu \epsilon) \; , \tag{1.0.8}$$

in the spin-2 lagrangian, that is proportional to the Einstein field equation, involving the Ricci tensor. Already for $s = 5/2$ the eq. (1.0.7) does not hold anymore, and a Weyl contribution remains on the right hand side.

At the beginning of the Eighties, however, the work of a few different research groups showed that interactions of HS fields among themselves are indeed possible, provided one relaxes certain hypotheses - hidden or explicit - on which the no-go theorems and arguments were crucially based. More in detail, in [36, 37, 38, 39] certain vertices among HS fields only in flat space were constructed by explicitly evading one key assumption of

the Coleman-Mandula theorem and its generalizations, *i.e.*, by dealing with an *infinite-dimensional* symmetry algebra. These were the first works in which it was recognized that non-abelian HS gauge transformations do not close on a finite set of generators, or, equivalently, that as soon as one introduces a gauge field of spin higher than 2 the closure of gauge transformations forces one to introduce other fields of higher and higher spins, with no upper bound. It was also noticed that, in general, vertices involving HS fields should involve higher derivatives, as one would guess by considering that a Lorentz -invariant cubic coupling involving fields of spins 0-0-s involves at least s derivatives of the scalar fields in order to saturate the indices of the spin-s gauge field.
A consistent coupling with gravity was obtained, a few years later, by Fradkin and Vasiliev [61] reconsidering the Aragone-Laroche problem in the framework of a perturbative expansion around a *nonflat* background - in particular, expanding the metric around a constant curvature (A)dS space-time, that, being Weyl-flat, has the virtue of allowing the free propagation of massless HS fields without breaking the local HS symmetries, as again can be understood from equation (1.0.5). Such a setting evades, in the first place, the no-go theorems, since they assumed $\mathfrak{iso}(3,1)$ as space-time isometry algebra and, more generally, since they are all S-matrix arguments, while there is no S-matrix in (A)dS. Moreover, a nonvanishing cosmological constant Λ enables to construct naturally higher-derivative cubic s-s-2 vertices that restore the gauge invariance of the spin-s lagrangian coupled to gravity, at least to the first nontrivial order in interactions. Indeed, the important fact is that, if $\Lambda \neq 0$, it is possible to expand in powers of the fluctuation R of the Riemann tensor around a nonvanishing background curvature $R^{(0)}_{\mu\nu,\rho\sigma} = (\Lambda/3)\left(g^{(0)}_{\mu\rho}g^{(0)}_{\nu\sigma} - g^{(0)}_{\nu\rho}g^{(0)}_{\mu\sigma}\right)$, and to write nonminimal coupling terms that are, schematically, of the form

$$\mathcal{L}^{int} = \sum_{A,B} \alpha(A,B)\Lambda^{-\left[\frac{(A+B)}{2}\right]} D^A\varphi D^B\varphi R \ , \tag{1.0.9}$$

i.e., an appropriate combination of (A)dS-covariant derivatives of the spin-s field with certain coefficients $\alpha(A,B)$, with A and B limited by the condition $A+B \leq s$. The gauge variation of such terms again produces commutators of two covariant derivative. However, the latter is now, to lowest order, schematically,

$$[D,D] \ \sim \ \Lambda g^{(0)} g^{(0)} + \text{higher order terms} \ , \tag{1.0.10}$$

and this in turn implies

$$\delta\mathcal{L}^{int} \sim R\, D\varphi\, \epsilon \ , \tag{1.0.11}$$

where the dependence on Λ has disappeared. A proper choice of the coefficients $\alpha(A,B)$ is therefore enough to cancel (1.0.5) and to restore gauge invariance. Notice that, as observed before, since in a fully interacting theory all spins $s > 2$ must be included, the number of derivatives is not bounded from above! In other words, there is strong evidence

that interacting HSGT are *nonlocal*[4]. Moreover, HS vertices in general do not admit a flat limit $\Lambda \rightarrow 0$, *i.e.*, a nonvanishing cosmological constant is necessary for consistent HS interactions. The two facts are connected, at least in a field theoretic context: indeed, higher-derivative couplings need to be rescaled with negative powers of a dimensionful parameter, and the only such parameter available in HSGT is indeed the cosmological constant. However, the situation is somewhat different in String Theory, where a natural dimensionful parameter exists also in flat space, the string length α', or its inverse, the string tension T. We will not comment further here on such issue, but just recall that recent results [46], obtained with BRST techniques, show that indeed, in field theory, the gravitational couplings obtained by Fradkin and Vasiliev are the only "minimal" ones (where minimal here means obtained by covariantizing derivatives with Γ).
Thus, interestingly, a phase with unbroken HS gauge symmetries seems to be related with a constant curvature gravitational background. Amusingly, this is not in contradiction with the fact that we observe, today, a very small - almost zero - positive cosmological constant. Indeed, is HS symmetries play a role in Nature, they are presently broken, and the spontaneous symmetry breaking mechanism might be also responsible for a redefinition of Λ through the vacuum values of the Higgs-like fields.

Still, the fact that interaction terms do not make sense in flat space-time can be taken as an indication of the importance of not forcing *a priori* the cosmological constant to vanish. Similar considerations motivated Fronsdal to undertake a series of works on field theory in four-dimensional AdS space-time, starting from a detailed study of the unitary irreducible representations (UIRs) of its isometry group $SO(3,2)$. The greatest, striking difference with respect to flat space-time was found by Flato and Fronsdal to be that massless (and massive) particles in AdS are not fundamental representations, but arise from the tensor product of two (or more) ultra-short fundamental UIRs called *singletons* [31]. The latter had been first discovered in 1963 by Dirac [30], and describe conformal particles (a scalar and a spinor) living on the boundary of AdS. As such, they do not admit a flat limit, although their tensor product - that decomposes, under the adjoint action of the algebra $\mathfrak{so}(3,2)$, into the direct sum of massless representations of all spins - does. More in detail, the singleton representation becomes a trivial representation under translations in the contraction $\Lambda \rightarrow 0$ of the AdS isometry algebra to the Poincaré algebra $\mathcal{P}$. The study of QFT in space-times with a nonvanishing cosmological constant may therefore reveal some interesting subtlety, as indeed has already been pointed out in several contexts.

Summarizing, the early analysis on interactions among massless HS fields had shown that:

[4]Nonlocal theories do not automatically suffer from the higher-derivative problem. For instance, in some cases like String Field Theory, the problem is somehow cured [42, 43, 44] if the free theory is well-behaved and if non-locality is treated perturbatively (see [45] for a comprehensive review on this point).

1. A consistent interacting HSGT requires the simultaneous introduction of infinitely many gauge fields of all spins. We shall be more precise on what "all" means, here. However, an infinite-dimensional non-abelian HS algebra governing such interactions can be postulated to underlie such a theory, and this necessarily has to be spanned by generators of higher and higher ranks;

2. The interaction with gravity is consistent with the HS gauge symmetries only on a nonflat gravitational background, *i.e.*, in the presence of a nonvanishing cosmological constant Λ, since interaction terms are nonanalytical in Λ;

3. HS interaction vertices require higher derivatives of the physical fields involved. This property is strictly connected with the previous one, since, in order for the physical dimension of the lagrangian to be preserved with more than two derivatives, a dimensionful parameter must enter the vertices, and Λ is the only natural candidate in a field theoretical context.

Properties 1 and 3 together imply that a consistent interacting HSGT is nonlocal. This fact, together with the consideration that fields of all spins must be introduced, seems very reminiscent of how HS fields appear in String Theory, together with low-spin fields as vibration modes of extended fundamental objects, that are in general p-dimensional (p-branes) [32]. Notice moreover that the three properties listed above suggest that, in a theory with unbroken HS symmetries that mix together fields of different spin, higher and lower-derivative term must come on an equal footing. In other words, there is no small parameter to set up a low-energy effective action scheme, *i.e.* an expansion in derivatives.

For all these reasons, it should by now be clear that interacting HSGT are a very challenging problem already at the classical level. Indeed, by the mid Eighties, it was clear that the construction of a full theory required a more systematic approach. This was developed essentially by Vasiliev [79, 80, 81, 82, 83, 84] (for reviews, see [48, 49, 50, 52]), and it is at present available at least for the class of totally symmetric fields.

This line of research began with a series of works by Vasiliev and collaborators, where the free theory was cast in a frame-like formalism generalizing the Einstein-Cartan approach to gravity. At the same time, Fradkin and Vasiliev used this approach to construct cubic interactions for HS fields, within the frame formalism, using an action that was a HS generalization of the MacDowell-Mansouri action for (super)gravity, *i.e.*, a bilinear of type $\int R \wedge R$ in suitable HS curvatures R [61, 60, 64]. This frame-like approach had the advantage of giving hints on the possible structure of a non-abelian HS algebra. Indeed, the reformulation of the free dynamics of a spin-s gauge field in terms of a set of one-form connections $(e^{a_1\dots a_{s-1}}, \omega^{a_1\dots a_{s-1},b_1}, \dots, \omega^{a_1\dots a_{s-1},b_1\dots b_{s-1}})$[5] instead of the the metric-like fields $\varphi_{\mu_1\dots\mu_s}$, gave a hint on the structure of the generators of the algebra, much in the same

[5]As it will be explained in detail in Chapter 2, the indices a_i and b_i are tangent-space $\mathfrak{so}(D-1,1)$-vector indices, if the background manifold is $(A)dS_D$, and in particular the one-form connections are

way as the formulation of gravity in terms of the frame field and of the Lorentz connection $(e^a, \omega^{a,b})$ bears a direct relation with the generators of the Poincaré algebra (P_a, M_{ab}), via the correspondent non-abelian Yang-Mills-like connection $\Omega = -i(e^a P_a + \frac{1}{2}\omega^{a,b} M_{ab})$. In other words, this approach exploits the geometry of the group manifold, of which the generators are a basis of tangent vectors and gauge fields - physical and auxiliary ones - enter as the dual, cotangent basis of one-forms. This framework was suitable for building a HSGT, since it provided a systematic algorithm to gauge a Lie algebra *à la* Yang-Mills, and at the same time was powerful enough to treat in a uniform way gravity and internal symmetries[6].
The resulting candidate HS algebras should therefore have the same index structure of the set of one-form connections and should be constructed as appropriate infinite-dimensional extensions of the AdS isometry algebra. Such requirements were essentially realized in terms of a suitable quotient of the enveloping algebra of the latter: the generators were therefore identified as certain projections of all possible products of the generators P_a and M_{ab} of the AdS isometry algebra[7]. Moreover, again in remarkable analogy with String Theory, it was found that such algebras admit internal extensions corresponding to Chan-Paton algebras [35] (i.e., only the classical $\mathfrak{su}(n)$, $\mathfrak{usp}(n)$ and $\mathfrak{so}(n)$ are admitted)[8]. Gauge fields of all spins could therefore enter the equations as coefficients of the expansion of an adjoint one-form, called *master one-form*, over the generators of the HS algebra. However, in order to describe the free dynamics in terms of the correct number of degrees of freedom, the above mentioned action had to be supplemented by certain torsion-like constraints that were not following from its variation. Moreover, there was still no systematic way of building interactions to all orders. Most importantly, it was found [65] that the physical spectrum of fields encoded in the free equations could fit a unitary irreducible representation of the HS algebra only if it contained a scalar.
It was therefore necessary to add a zero-form containing a scalar in the game. It was then noted that, at the level of field equations, the free dynamics admitted a useful reformulation in terms of the master one-form and of a *master zero-form*, valued in a peculiar UIR of the HS algebra called *twisted-adjoint*, that included infinitely many auxiliary fields for every massless physical spin-s field. Constraints that were naturally included in the

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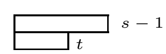
valued in all possible irreducible representations encoded in the Young diagrams 

, where $0 \leq t \leq s-1$.

[6]Indeed, diffeomorphisms are automatically included as field-dependent gauge transformations, as we will see later on.

[7]Similar considerations extend to the case of other signatures, in particular to the dS case. However, AdS is the suitable background for supersymmetric extensions of HS algebras, that also have been constructed. The study of the Vasiliev equations in other signatures will be one of the subjects of this Thesis.

[8]Interestingly, for this reason one would think that Vasiliev equations might encode the dynamics of tensionless *open* strings, rather than that of closed strings. This does not mean, however, that the spin-2 field that appears in the equations cannot correspond to the graviton, since in the tensionless limit there is also the possibility of a mixing between open and closed string states [105].

system of equations relate such auxiliary fields to all on-shell nontrivial combination of derivatives of the physical fields, *i.e.*, the scalar field, all spin-s Weyl tensors and their derivatives to all orders. Although there is no related action principle, such a system of equations is a sort of covariant first-order reformulation of the dynamics, called *unfolded formulation*, that presents several advantages:

- HS field equations are written in a manifestly HS-covariant way, in terms of objects that have "simple" transformation properties under the HS algebra, and without contracting space-time indices with the (inverse) metric tensor. This latter property is of great importance, as it enables to treat the spin-2 field on an equal footing with all the other fields in a system of equations that can encode nontrivial dynamics. As these equations only involve differential forms, they are manifestly diffeomorphism invariant.

- In such a scheme, the field equations are reformulated as certain consistent zero-curvature and covariant constancy conditions, defining a *free differential algebra* (FDA) [117] - *i.e.*, some sort of generalization to forms of arbitrary degree of the dual formulation of an algebra through Maurer-Cartan one-forms. The problem of finding consistent interactions is therefore reduced to finding consistent deformations of the FDA. However, such systems are strongly constrained, which makes it easy to control gauge symmetries, as they are a direct consequence of the consistency of the field equations. Finally, this setting enables the construction of gauge-invariant deformations in an expansion in powers of the zero-form.

- The twisted-adjoint zero-form guarantees a uniform treatment of all higher-derivative interaction terms. Indeed, the latter are expressed as gauge-invariant multi-linear combinations of fields arising from multiple powers of the zero-form only or with one power of the one-form. It is only on-shell, on the constraints that are contained in the unfolded system, that the infinitely many auxiliary fields sitting in every spin-s sector of the zero-form are solved in terms of derivatives of the physical fields.

In some sense, therefore, the unfolded formulation combines the virtues of first order, Hamilton-like formulation of the dynamics with jet-space methods. All relevant derivatives of the fields are hidden in certain extra-variables, sitting in the twisted adjoint: this implies, in particular, that the dynamical problem is well-posed prior to specifying a background metric, and that to set it up in this formulation it is in principle sufficient to specify the values of all the zero-forms at a point x_0 in space-time. Since the set of zero-forms is infinite-dimensional (for every fixed spin), there is indeed room for nontrivial dynamics in the flatness conditions of the unfolded system, because the fluctuating fields can be reconstructed in an arbitrary neighborhood of x_0 via a Taylor expansion, which is exactly what solving such first order equation does. Locally, the dependence on space-time coordinates is therefore purely auxiliary, and in this way this formulation can achieve

perfect background independence. We shall examine carefully all these statements in the remainder of the Thesis.
Moreover, if the coordinate dependence is locally pure gauge, one can always introduce additional dependence from some extra coordinates without changing the physics, as long as one correspondingly enlarges the original FDA with equations that express the dependence on such extra-coordinates in terms of the original physical degrees of freedom. This fact, combined with the fact that consistency implies gauge-invariance, makes it possible to "resum" the perturbation series in powers of the master zero-form. More precisely, one can enlarge the space-time manifold with a set of auxiliary *noncommutative* directions Z, and assign to all the fields a dependence on such variables. The noncommutative nature of such variables essentially ensures that the restriction of the enlarged system to the physical subspace $Z = 0$ is a nontrivial deformation of the original system. It is then possible to write a constraint that essentially solves the Z-dependence in terms of the twisted adjoint zero-form (that contains the physical dof) as a consistent equation - in other words, the new equations are an enlargement of the original FDA. This ensures that the solutions of such equations, *i.e.*, the deformations of the original FDA, enter as solutions of a consistent equation, and therefore, for the properties of FDA, are automatically gauge-invariant. Notice that relating the Z-dependence of adjoint fields with the twisted adjoint zero-forms gives a nontrivial condition, and indeed constrains the form of the commutation relations of the Z variables. Therefore, the whole perturbative series in the zero-forms is encoded in one shot as a solution of a consistent flatness condition in this auxiliary set of noncommutative directions Z. Solving this equation order by order in the twisted adjoint zero-forms and substituting for the Z dependence in the remaining equations projected onto $Z = 0$ gives the long sought for interacting equations for massless HS fields.
Despite their relatively simple and elegant form, however, Vasiliev equations encode the dynamics of a system of formidable complexity: to begin with, they involve infinitely many physical fields, and the proliferation of auxiliary fields makes it difficult to read directly the interaction vertices among them. Furthermore, it is not known, at present, how to derive the unfolded equations from a conventional action principle, nor, consequently, how to quantize such a theory.

However, the property of homotopy invariance of the equations, *i.e.*, the fact that all the local data are encoded in the equations projected onto a point in space-time, can be read, in the full system, as the fact that nontrivial dynamics can either be encoded in the space-time equations (obtained after solving for the Z-dependence) or, equivalently, in the Z-fiber over a fixed point in space-time (after gauging away, locally, the dependence on space-time coordinates). In other words, the unfolded system involves some sort of duality between the space-time evolution and the fiber evolution of the fields, that it is possible to use to one's advantage. For example, as we shall see in Chapter 6, this observation leads to a general, very efficient way of finding *exact* solutions of the full equations! Indeed, the fiber equations do not involve any space-time derivatives, and are purely algebraic equations that are in principle more easily solvable than the full space-time equations - that, moreover, are only given as a perturbative expansion. The first example of a nontrivial (*i.e.*, other than the zeroth-order solution representing the AdS background) solution was found for the three-dimensional theory by Prokushkin and Vasiliev [86]. A few years later, solutions of this type were elevated by Sezgin and Sundell to solutions of the four-dimensional case and were shown to admit some interesting cosmological interpretation [97]. They were found by imposing symmetry requirements on the fiber-projection of the master-fields, and were shown to describe a Lorentz-invariant deformation of the vacuum consisting of a scalar field profile over an asymptotically AdS metric. Recently, a BTZ black hole solution of the Vasiliev equations in three dimensions was also found [98]. Possibly, the algebraic methods developed so far, can elevate it to the dynamically more interesting case of four dimensions.

Original content of this Thesis

The research of new exact solutions of Vasiliev's equations is of crucial importance for a better understanding of the dynamics of higher-spin gauge fields, much in the same way as the study of the Schwarzschild solution was of extreme importance to uncover some peculiar feature of gravity. In the original part of this Thesis, I shall describe new exact solutions that have been found in collaboration with Ergin Sezgin and Per Sundell [99]. In this work, first the Vasiliev equations and the correspondent symmetry algebras were generalized to space-times with signature $(4-p,p)$ and nonvanishing cosmological constant in $D=4$, and then certain families of exact solutions of the equations have been found in the different resulting models. Among them are chiral models in Euclidean $(4,0)$ and Kleinian $(2,2)$ signatures involving half-flat gauge fields. Apart from the maximally symmetric solutions, including de Sitter spacetime, we find:

- $SO(4-p,p)$ invariant deformations, depending on a continuous and infinitely many discrete parameters, including a degenerate metric of rank one;
- Non-maximally symmetric solutions with vanishing Weyl tensors and higher spin gauge fields, that differ from the maximally symmetric solutions in the auxiliary field sector;
- Solutions of the chiral models furnishing higher-spin generalizations of Type D gravitational instantons [100], with an infinite tower of Weyl tensors proportional to totally symmetric products of two principal spinors. These are apparently the first exact 4D solutions with nonvanishing massless HS fields.

We shall present the details of such solutions in Chapter 6. We shall also comment on the construction of certain HS invariants that have been constructed in [97], and that are crucial to distinguish gauge-inequivalent solutions and to characterize them physically. Regrettably, however, at present no "complete" set of observables is known, and in particular the only invariants that are available are built from the master zero-form only. As we shall see, already certain vacuum solutions found in [99] cannot be distinguished from the trivial ones that represent maximally symmetric space-times, although it seems unlikely that gauge transformations can connect these two vacua.

Partly motivated by the study of Vasiliev equations in different signatures, the construction of a precise map in D dimension between the $\mathfrak{so}(D;\mathbb{C})$-covariant operators included in the master fields of the Vasiliev system and the states in the complex lowest (and highest) weight module representations of spin s was developed in [134], in collaboration with Per Sundell. Roughly speaking, to each generator of the twisted adjoint representation there corresponds a "coherent" superposition of infinitely many states, and, viceversa, to every state in the lowest weight modules there corresponds a nonpolynomial combinations of generators. The map can be formulated at the level of complex representations, and can then be restricted to different real forms corresponding to models that admit AdS_D, S^D, dS_D, H_D solutions. In the first of such cases, that of the real form $\mathfrak{so}(D-1,2)$, the lowest weight modules, that correspond to the massless representations appearing in the tensor product of two singletons, are unitary representations. As we shall illustrate, such state-operator correspondence enables to read directly the on-shell content of the adjoint one-form and of the twisted adjoint zero-form master-field in terms of irreps of the background isometry algebra. Roughly speaking, it exhibits the physical excitations that one would discover from the unfolded system after solving the various torsion-like constraints that express the auxiliary fields in terms of the physical ones. Or again, in other words, it connects the standard first-quantized description of localized fluctuations with the master-fields entering the unfolded description.
Such a mapping provides some insight into various features of Vasiliev equations. For example, it shows that, while the on-shell content of the twisted adjoint zero-form can be analyzed in terms of the tensor product of two singletons, that of the adjoint one-form is related to the finite-dimensional $\mathfrak{so}(D+1;\mathbb{C})$-modules that arise from the tensor product of a singleton and his negative-energy counterpart, called *anti-singleton*. Another outcome is that Vasiliev equations in a given signature can describe not only the corresponding UIRs: indeed, for every spin s, a bigger indecomposable module (containing a lowest-spin module, together with the more familiar lowest and highest-energy modules) sits in principle in the master-fields and all the states there contained can, *a priori*, take part in the dynamics. Finally, the problem of potential local divergencies in HSGT, due to the contribution of an arbitrary number of derivatives to some interaction vertices (as, for example, in the scalar-field corrections to the stress-energy tensor calculated in [102]), is mapped into the problem of divergent products of nonpolynomial combinations of gen-

erators. This is however a somewhat more transparent setting, and indeed we make a proposal for an explicit regularization scheme.

Structure of this Thesis

The structure of this Thesis more or less follows the line of this Introduction.
From Chapter 2 to Chapter 5 we review in some detail the main features of interacting HSGT. We begin by recalling the main features of the free HS Field Theory in Chapter 2: in particular, the free dynamics is reviewed both in its metric formulation (concentrating our attention on the Fronsdal and the Francia-Sagnotti local equations) and in the frame formulation. As a preparation to the latter, we also describe the MacDowell-Mansouri-Stelle-West formulation of gravity. In Chapter 3 we turn our attention to the structure of the HS algebras that lie at the heart of the Vasiliev's system: in order to be as general as possible, all its main properties will be examined at the level of the complex abstract algebra, and only at a second stage we introduce the real forms and the oscillator realizations that will be crucial in the formulation of the interacting equations. We also discuss in some detail the construction of the representations that will be of interest in the remainder of the Thesis. Chapter 3 also presents some essential material and notation for Chapter 7. Chapter 4 is then devoted to the unfolded formulation of the free field equations for arbitrary spin: the general scheme of *free differential algebras* is first described, and the features of the unfolded systems are discussed. Then, some lower-spin examples are given, and the unfolding procedure is analyzed in detail. Finally, the free spin-s unfolded system is presented. In Chapter 5, finally, the nonlinear Vasiliev equations are reviewed, in their four-dimensional realization that makes use of a simple oscillator realization of the HS algebra, described earlier in Chapter 3. The issue of uniqueness of interaction terms is also discussed in some detail, as well as a perturbative expansion scheme that makes contact with the free unfolded equations by linearizing the full system around the AdS background solution.
Chapters 6 and 7 contain the original results of this Thesis. In the first, we formulate the four-dimensional Vasiliev equations in arbitrary signature, discuss some new features that emerge in the various cases, and then find the new exact solutions mentioned above. In the second, we elaborate further on the representation theory underlying the Vasiliev system and construct the above-mentioned *reflection map* that will enable us to connect the physical excitations to the basis of monomials of the twisted adjoint zero-form. A number of tools and concepts that are instrumental for such analysis is first introduced, and then some outcome of this mapping is presented. Finally, we draw some Conclusions.
The Thesis also includes nine appendices, that provide some background material or contain the detailed steps of some calculations that are used in the main text. Some of the material contained in Chapters 2 and 4 is based on the review paper [52]. The original results contained in Chapter 6 were found in [99], while those of Chapter 7 in [134] and

[135]. However, we warn the reader that the paper [134] was not yet completed by the time this Thesis was typed, and therefore only a subset of the results there included can actually be found in Chapter 7.

Chapter 2

Free Fields

In this chapter, we shall review the formulation of free equations for massless fields of arbitrary spin s, both in the metric and in the frame formalism. We shall concentrate however only on bosonic fields, and only on the aspects of the free theory that will be directly of relevance to the following, therefore not mentioning interesting features that have an importance of their own and that are essential to the contemporary developments of the subject. We therefore refer the interested reader to the original papers [19, 20, 104, 105, 47] and to some reviews [21, 103].

2.1 Metric Formulation

The Fronsdal formulation of linear HS gauge theories is somehow the most straightforward generalization of the Maxwell and linearized Einstein equations, in the metric formalism, to HS fields represented by totally symmetric rank-s Lorentz tensors. The free dynamics of a integer spin-s gauge field $\varphi_{\mu(s)} \equiv \varphi_{\mu_1\ldots\mu_s}$ in D-dimensional Minkowski space-time is encoded in the equation

$$\mathcal{F}_{\mu(s)} \;\equiv\; \Box\varphi_{\mu_1\ldots\mu_s} - s\,\partial_{(\mu_1}\partial\cdot\varphi_{\mu_2\ldots\mu_s)} + \frac{s(s-1)}{2}\,\partial_{(\mu_1}\partial_{\mu_2}\varphi'_{\mu_3\ldots\mu_s)} \;=\; 0\ , \tag{2.1.1}$$

where the indices within parentheses are intended to be totally symmetrized with unit strength and the prime over fields indicates that a trace is being taken, $\varphi'_{\mu_3\ldots\mu_s)} \equiv \eta^{\mu_1\mu_2}\varphi_{\mu_1\mu_2\ldots\mu_s}$. It can be easily checked that, under the local transformation

$$\delta\varphi_{\mu_1\ldots\mu_s}(x) \;=\; s\,\partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}(x)\ , \tag{2.1.2}$$

that is an natural generalization of linearized diffeomorphisms to a totally symmetric rank-$(s-1)$ gauge parameter $\epsilon_{\mu_1\ldots\mu_{s-1}}$, the variation of (2.1.1) is

$$\delta\mathcal{F}_{\mu_1\ldots\mu_s} \;=\; \frac{s(s-1)(s-2)}{2}\,\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\epsilon'_{\mu_4\ldots\mu_s)}\ . \tag{2.1.3}$$

Thus, in order to achieve invariance of Fronsdal's kinetic operator $\mathcal{F}_{\mu_1\ldots\mu_s}$, it is necessary to restrict the gauge freedom to *traceless* gauge parameters,

$$\epsilon'_{\mu(s-3)}(x) \;=\; 0 \; . \tag{2.1.4}$$

Notice that, as announced, the Maxwell and linearized Einstein equations are particular cases of (2.1.1) for $s = 1$ and $s = 2$, respectively. Of course, there the gauge-invariance is fully unconstrained, *i.e.*, no algebraic constraints are imposed on the gauge parameters, due to the low-rank of the latter. Indeed, it is only from spin-3 onwards that the condition (2.1.4) is nontrivial.
More precisely, the Frondal equations are similar to the free Einstein's equations in vacuum $R^{lin}_{\mu\nu} = 0$, both sharing the feature of being non-lagrangian equations. The "kinetic-operator" that follows from the variation of the Einstein-Hilbert action is, in fact, the divergenceless tensor $\mathcal{G}^{lin}_{\mu\nu} = R^{lin}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{lin}$. For spin s the situation is again more subtle: an action principle for the Fronsdal equations indeed exists and is a generalization of the Fierz-Pauli action (that is, the linearized Einstein-Hilbert action),

$$\begin{aligned} S_2^{(s)}[\varphi] \;=\; -\frac{1}{2}\int d^Dx\,\Big(&\partial_\nu\varphi_{\mu_1\ldots\mu_s}\partial^\nu\varphi^{\mu_1\ldots\mu_s} \\ &-\frac{s(s-1)}{2}\,\partial_\nu{\varphi^\lambda}_{\lambda\mu_3\ldots\mu_s}\partial^\nu{\varphi_\rho}^{\rho\mu_3\ldots\mu_s} + s(s-1)\,\partial_\nu{\varphi^\lambda}_{\lambda\mu_3\ldots\mu_s}\partial_\rho\varphi^{\nu\rho\mu_3\ldots\mu_s} \\ &-s\,\partial_\nu{\varphi^\nu}_{\mu_2\ldots\mu_s}\partial_\rho\varphi^{\rho\mu_2\ldots\mu_s} - \frac{s(s-1)(s-2)}{4}\,\partial_\nu{\varphi^{\nu\rho}}_{\rho\mu_2\ldots\mu_s}\partial_\lambda{\varphi_\sigma}^{\lambda\sigma\mu_2\ldots\mu_s}\Big) \end{aligned} \tag{2.1.5}$$

It can be rewritten as

$$\begin{aligned} S_2^{(s)}[\varphi] \;&\sim\; \int d^Dx\,\varphi^{\mu_1\ldots\mu_s}\mathcal{G}_{\mu_1\ldots\mu_s} \\ &=\; \varphi^{\mu_1\ldots\mu_s}\left(\mathcal{F}_{\mu_1\ldots\mu_s} - \frac{s(s-1)}{4}\eta_{(\mu_1\mu_2}\mathcal{F}'_{\mu_3\ldots\mu_s)}\right)\;, \end{aligned} \tag{2.1.6}$$

where $\mathcal{G}_{\mu_1\ldots\mu_s}$ is a generalized linearized Einstein tensor. Now, using (2.1.2), integrating by parts in the action, and taking the constraint (2.1.4) into account, it is easy to find that its gauge invariance rests crucially on the divergence-free nature of $\mathcal{G}$. However, one gets

$$\partial^\nu\mathcal{G}_{\nu\mu_1\ldots\mu_{s-1}} \;=\; -\frac{(s-1)(s-2)(s-3)}{4}\,\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\varphi''_{\mu_4\ldots\mu_{s-1})}\;, \tag{2.1.7}$$

so that, in order to have a gauge-invariant spin-s free Fronsdal Lagrangian, one has to supplement the theory with an additional constraint on the fields, declaring them to be represented by *doubly traceless* tensors,

$$\varphi''_{\mu(s-4)} \;=\; 0\;, \tag{2.1.8}$$

which is a nontrivial condition for $s \geq 4$.

Indeed, the Fronsdal equations with their restricted gauge-invariance propagate the correct number of degrees of freedom associated to a massless field of spin s, in any dimension. To see this, one can perform an analysis which is again a straightforward generalization of the one that is proper of the low-spin cases. First, we introduce a generalized de Donder gauge condition,

$$\mathcal{D}_{\mu_1 \ldots \mu_{s-1}} \;=\; \partial \cdot \varphi_{\mu_1 \ldots \mu_{s-1}} - \frac{s-1}{2}\, \partial_{(\mu_1} \varphi'_{\mu_2 \ldots \mu_{s-1})} \;=\; 0 \;, \tag{2.1.9}$$

that reduces (2.1.1) to the usual wave equation

$$\Box \varphi_{\mu(s)} \;=\; 0 \;. \tag{2.1.10}$$

The gauge variation of (2.1.9),

$$\delta \mathcal{D}_{\mu(s-1)} \;=\; \Box \epsilon_{\mu(s-1)} \;, \tag{2.1.11}$$

allows for a residual gauge symmetry with parameters that satisfy themselves a wave equation,

$$\Box \epsilon_{\mu(s-1)} \;=\; 0 \;. \tag{2.1.12}$$

Notice also that, by virtue of (2.1.8), the de Donder condition is traceless[1],

$$\mathcal{D}'_{\mu(s-3)} = -\frac{s-3}{2}\, \partial_{(\mu_1} \varphi''_{\mu_2 \ldots \mu_{s-3})} \;=\; 0 \;. \tag{2.1.13}$$

Being doubly traceless, the spin-s field admits the decomposition

$$\varphi_{\mu(s)} \;=\; \phi_{\mu(s)} + \eta_{(\mu_1 \mu_2} \sigma_{\mu_3 \ldots \mu_s)} \;, \tag{2.1.14}$$

where both $\phi_{\mu(s)}$ and $\sigma_{\mu(s-2)}$ are traceless tensors. Correspondingly, the residual gauge transformation (2.1.12) splits into

$$\delta \phi_{\mu(s)} \;=\; s \partial_{\{\mu_1} \epsilon_{\mu_2 \ldots \mu_s\}} \;, \tag{2.1.15}$$

where $\{\ldots\}$ denotes the symmetric traceless projection, and

$$\delta \sigma_{\mu(s-2)} \;=\; \frac{s(s-1)}{2D + 4(s-2)}\, \partial \cdot \epsilon_{\mu(s-2)} \;. \tag{2.1.16}$$

[1]As usual, eqs. (2.1.11) and (2.1.12) are also the conditions that ensure that the de Donder gauge (2.1.9) is a good gauge, since it does not contain more conditions than there are independent components of the gauge parameter, and a parameter that enables to impose it is the solution of a wave equation, that exists under very general conditions.

The divergence of the gauge parameter can be used to set $\sigma_{\mu(s-2)}$ to zero. Therefore, the number of independent degrees of freedom propagated by the Fronsdal equations equals the number of independent component of $\phi_{\mu(s)}$ minus the number of independent constraints (2.1.9) minus the number of independent components of the leftover divergenceless gauge parameter. In $D = 4$, for example, this number is[2] $(s+1)^2 - s^2 - (s^2 - (s-1)^2) = 2$, as expected.

Although Fronsdal's formulation captures the fundamental features of the massless HS free dynamics, it is desirable to overcome the need for the algebraic constraints on the gauge field and parameter, for a number of reasons: first, as already commented in the Introduction, to establish a more direct contact with String Theory and String Field Theory, whose vibration modes are represented by unconstrained tensors; second, because, as our experience with low-spin gauge theories suggests, a restricted gauge invariance is a sign that the field equations are written in terms on non-fully invariant objects. This expectation was indeed found to be true first in [19], where nonlocal unconstrained equations were written in terms of the proper HS curvatures - generalizations of the Maxwell field strength and the Riemann tensor (a local equivalent version of the free unconstrained equations was proposed in the first two references in [47], see the third reference therein for a review). The nonlocality was moreover shown to be pure gauge, and Fronsdal's formulation was recovered via a gauge fixing. An equivalent, *local unconstrained* formulation for the free dynamics of a massless spin-s field was first obtained in [20] and rests on the fact that the variation (2.1.3) can be canceled by introducing in the equations a spin-$(s-3)$ *compensator* field , $\alpha_{\mu(s-3)}$, transforming as the trace of the parameter. The Fronsdal equations are substituted with the system

$$
\begin{aligned}
\mathcal{F}_{\mu(s)} &= \frac{s(s-1)(s-2)}{6}\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\alpha_{\mu_4\ldots\mu_s)} \ , && (2.1.17)\\
\varphi''_{\mu(s-4)} &= 4\partial\cdot\alpha_{\mu(s-4)} + (s-4)\partial_{(\mu_1}\alpha'_{\mu_2\ldots\mu_{s-4})} \ , && (2.1.18)
\end{aligned}
$$

that is invariant under the *unconstrained* gauge transformations

$$
\begin{aligned}
\delta\varphi_{\mu(s)} &= s\partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)} \ , && (2.1.19)\\
\delta\alpha_{\mu(s-3)} &= 3\epsilon'_{\mu(s-3)} \ . && (2.1.20)
\end{aligned}
$$

Notice that (2.1.18) follows from (2.1.17) by using the Bianchi identity (2.1.7), which ensures the compatibility of the system. The latter equation characterizes α as a Stueckelberg-like field, and therefore shows that it is unphysical, since it can be gauged away by fixing

[2]We recall that a totally symmetric rank-s tensor in D dimensions, has $\binom{s+D-1}{D-1}$ independent components (see, for example, [59]). The number of independent components of a totally symmetric traceless rank-s tensor in D dimensions therefore follows immediately subtracting from these the $\binom{s+D-1}{D-1}$ independent trace constraints. This gives, in $D = 4$, $\binom{s+3}{3} - \binom{s+1}{3} = (s+1)^2$.

the trace of the gauge parameter to be $\epsilon'_{\mu(s-3)} = \frac{1}{3}\alpha_{\mu(s-3)}$. This reduces the system to the Fronsdal equations shown above. Interestingly, the local unconstrained formulation can be shown to follow from the variation of a minimal local lagrangian [104] that only makes use of an additional, spin-$(s-4)$ Lagrange multiplier to impose the constraint (2.1.17).

2.1.1 Interlude 1: Maximally Symmetric Space-times

We have already commented in the Introduction on the importance of the (A)dS background for building consistent HS interactions. Indeed, the same reasoning can be extended to any maximally symmetric background - which is a space-time whose metric has the maximum number, $\frac{D(D+1)}{2}$, of isometries in D dimensions - with nonvanishing cosmological constant. Notable examples we shall deal with later on are, together with $(A)dS_D$, their euclidean versions: the hyperbolic space H_D, obtained from AdS_D through a "Wick rotation "of the time direction, and the sphere S^D, obtained from dS_D through a "Wick rotation" of the time direction. In their turn, AdS_D and dS_D are connected by a change in the sign of the curvature (*i.e.*, of the cosmological constant), and the same is true for H_D and S^D. In other words, all such spaces admit a unified description characterized by two relevant parameters: the signature of their tangent-space metric η_{ab} and the sign of the cosmological constant. The simplest one is given in terms of flat coordinates that describe the embedding of any D-dimensional maximally symmetric space-time in a flat, $(D+1)$-dimensional one via the condition

$$k\eta_{ab}x^a x^b + z^2 \;=\; L^2 \qquad a,b = 0,1,...,D-1\ , \tag{2.1.21}$$

where for the moment we do not specify the signature of $\eta_{\mu\nu}$, with the flat embedding space metric

$$ds^2 \;=\; \eta_{ab}dx^a dx^b + \frac{1}{k}dz^2\ . \tag{2.1.22}$$

Only the sign of the curvature constant k will be of relevance, since any rescaling with a positive factor can be absorbed into the definition of the coordinates x^μ. Solving z from (2.1.21), differentiating and substituting dz^2 in (2.1.22) one gets

$$ds^2 \;=\; \eta_{ab}dx^a dx^b + k\frac{\eta_{ac}\eta_{bd}x^c x^d}{L^2 - k\eta_{ab}x^a x^b}dx^a dx^b\ , \tag{2.1.23}$$

from which it follows that the metric for a maximally symmetric space can be written as

$$g_{ab} \;=\; \eta_{ab} + k\frac{\eta_{ac}\eta_{bd}x^c x^d}{L^2 - k\eta_{ab}x^a x^b}\ , \tag{2.1.24}$$

that has the inverse

$$g^{ab} \;=\; \eta^{ab} - k\frac{x^a x^b}{L^2}\ . \tag{2.1.25}$$

It is now simple to calculate the Christoffel connection,

$$\Gamma^c_{ab} \;=\; \frac{1}{2}g^{cd}(\partial_b g_{da} + \partial_a g_{db} - \partial_d g_{ab}) \;=\; \frac{k}{L^2}x^c g_{ab} \;, \tag{2.1.26}$$

and the Riemann tensor

$$R^c_{dab} \;=\; \partial_a \Gamma^c_{db} - \partial_b \Gamma^c_{da} + \Gamma^c_{ea}\Gamma^e_{db} - \Gamma^c_{eb}\Gamma^e_{da} = \frac{k}{L^2}(\delta^c_a g_{db} - \delta^c_b g_{da}) \;, \tag{2.1.27}$$

so that

$$R_{cdab} \;=\; \frac{k}{L^2}(g_{ca}g_{db} - g_{cb}g_{da}) \;. \tag{2.1.28}$$

The Ricci tensor is

$$R_{ab} \equiv R^c_{acb} \;=\; \frac{k}{L^2}(D-1)g_{ab} \;, \tag{2.1.29}$$

and the curvature scalar

$$R \equiv R^a_a \;=\; \frac{k}{L^2}D(D-1) \;. \tag{2.1.30}$$

Therefore, the Riemann tensor for a constant curvature space-time is completely determined by the curvature scalar R, and

$$R_{cdab} \;=\; \frac{1}{D(D-1)}R(g_{ca}g_{db} - g_{cb}g_{da}) \;, \tag{2.1.31}$$

which implies that, for maximally symmetric space-times, the Weyl tensor vanishes identically

$$\begin{aligned} C_{cdab} \;=\;& R_{cdab} + \frac{1}{D-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) \\ &+ \frac{1}{(D-1)(D-2)}R(g_{ca}g_{db} - g_{cb}g_{da}) \;=\; 0 \;. \end{aligned} \tag{2.1.32}$$

Moreover, the curvature scalar is proportional to k, whose sign therefore discriminates between the different types of such space-times: $k = 0$ represents a flat space-time with metric $\eta_{\mu\nu}$ of arbitrary signature ; $k = +1$ (-1) represents a positive (negative) curvature space-time with tangent space metric $\eta_{\mu\nu}$. If the latter is fixed to be euclidean, then $k = 1$ (-1) corresponds to a S^D (H_D) space, while for minkowskian tangent space metric one $k = 1$ (-1) corresponds to the dS (AdS) space-time.

All such space-times are solutions of Einstein equations in absence of matter and in presence of a cosmological constant Λ,

$$R_{ab} - \frac{1}{2}g_{ab}R \;=\; -\Lambda g_{ab} \;, \tag{2.1.33}$$

that are extrema of the Einstein-Hilbert action

$$S \;=\; \frac{1}{16\pi G_D}\int d^Dx\sqrt{-g}(R-2\Lambda)\ . \tag{2.1.34}$$

From (2.1.29) and (2.1.30) it follows that

$$R_{ab}-\frac{1}{2}g_{ab}R \;=\; -k\frac{(D-1)(D-2)}{2L^2}g_{ab}\ , \tag{2.1.35}$$

and by comparison with (2.1.33) one has

$$\Lambda \;=\; k\frac{(D-1)(D-2)}{2L^2}\ , \tag{2.1.36}$$

from which one reads that the sign of Λ is related to that of k, i.e., of the curvature, for any[3] $D>2$. Thus, S^D and dS_D space-times have a positive cosmological constant, and H_D and AdS_D a negative one.

A presentation of (2.1.21) that puts the AdS case in greater evidence is given by

$$X^AX^B\eta_{AB} \;\equiv\; \eta_{ab}x^ax^b-\lambda^2z^2 \;=\; -\lambda^2L^2\ , \qquad A,B=0,1,...,D-1,0'\ , \tag{2.1.37}$$

where λ is a complex parameter and again of $\lambda^2=-1/k$ only the sign matters (positive for AdS), and the ambient-space metric is

$$ds^2 \;= dX^AdX^B\eta_{AB} \;\equiv\; \eta_{ab}dx^adx^b-\lambda^2dz^2 \;\equiv\; -\tau^2dt^2+\delta_{rs}x^rx^s-\lambda^2dz^2 \tag{2.1.38}$$

Therefore, the embedding space-time has metric $\eta_{AB}\;=\;(\eta_{ab},-\lambda^2)\;=\;(-\tau^2,\delta_{rs},-\lambda^2)$. Notice that the embedding direction is a time for the AdS case, and this is the reason for denoting it with $0'$ (we shall use this label for all signatures anyway, for the sake of uniformity); similarly, the tangent-space metric is denoted with an η_{ab} for all cases, including the euclidean ones. Equation (2.1.37) clearly shows that the isometry algebra of the different D-dimensional maximally symmetric space-times is the one preserving the quadratic form at its left hand side, which, for general tangent-space signature is $\mathfrak{so}(p',D+1-p')$ with $p'\le D+1$. In particular, the AdS isometry algebra corresponds to the case $p'=D-1$, $\mathfrak{so}(D-1,2)$, while dS to $p'=D$. Moreover, one can describe the different manifolds encoded in (2.1.37) as the coset spaces

$$\begin{aligned} AdS_D \;&=\; \frac{\mathfrak{so}(D-1,2)}{\mathfrak{so}(D-1,1)} \qquad & dS_D \;&=\; \frac{\mathfrak{so}(D,1)}{\mathfrak{so}(D-1,1)} \\ H_D \;&=\; \frac{\mathfrak{so}(D,1)}{\mathfrak{so}(D)} \qquad & S^D \;&=\; \frac{\mathfrak{so}(D)}{\mathfrak{so}(D-1)}\ , \end{aligned} \tag{2.1.39}$$

[3] $D=1,2$ are trivial cases, since in $D=1$ there is no curvature, and in $D=2$, although a curvature can be defined, the Einstein-Hilbert action, that encodes the dynamics of the gravitational field, is a topological invariant, the Euler characteristic.

where in all cases one factors out, from the isometry algebra of the embedding metric η_{AB}, the one of the tangent space metric η_{ab}.

The commutation relations that define the isometry algebra of (2.1.37) are

$$[M_{AB}, M_{CD}] \;=\; i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} + \eta_{AC}M_{BD} + \eta_{BC}M_{AD}) \; . \tag{2.1.40}$$

Splitting the indices as $A = (a, 0')$ and defining the translation generator to be $P_a = \frac{1}{\lambda L}M_{0'a}$ this can be rephrased as

$$[M_{ab}, M_{cd}] \;=\; 4i\eta_{[c|[b}M_{a]|d]} \; , \qquad [M_{ab}, P_c] \;=\; 2i\eta_{c[b}P_{a]} \; , \tag{2.1.41}$$

$$[P_a, P_b] \;=\; \frac{i}{\lambda^2 L^2}M_{ab} \; , \tag{2.1.42}$$

which exhibits the difference with respect to the Poincaré algebra, that can be obtained from the previous equations via an Inönü-Wigner contraction (*i.e.*, in the limit $\frac{\lambda}{L} \to 0$, or $L \to \infty$),

$$[M_{ab}, M_{cd}] \;=\; 4i\eta_{[c|[b}M_{a]|d]} \; , \qquad [M_{ab}, P_c] \;=\; 2i\eta_{c[b}P_{a]} \; , \tag{2.1.43}$$

$$[P_a, P_b] \;=\; 0 \; . \tag{2.1.44}$$

In most of the review part of this Thesis we will mainly focus on the AdS case, but for the bosonic HSGT everything can be rephrased for dS and for the euclidean signatures as well. One is mostly interested in the AdS case for the reason that it is more suitable for supersymmetric extensions. Furthermore, a change in the signature affects the Representation Theory: as it is well-known, for example, dS and AdS have rather different unitary representations (for dS there are unitary irreducible representations the energy of which is not bounded from below). Nevertheless, in the original part of this Thesis we will explicitly let the signature be arbitrary: examining HSGT in this more general setting will also prove to be interesting in finding certain new solutions to the Vasiliev equations, as announced already in the Introduction.

2.1.2 Free Equations in $(A)dS$ Space-time

The same reasoning (see, in particular, (1.0.5)) that led to recognize the importance of a nonflat background to build consistent HS interactions, also leads to the conclusion that the free propagation of HS fields in a gravitational background that solves the Einstein equations is consistent with the HS gauge symmetries only if the solution is Weyl-flat, *i.e.*, if the Weyl tensor calculated with the background metric is identically zero, which is indeed the case for the (A)dS space-times - on which we focus here for definiteness, keeping in mind that everything can be immediately rephrased in the corresponding euclidean signatures. Therefore, it is useful to look at the form of Fronsdal's equations in the presence of a cosmological constant.

The interaction with the fixed gravitational background is introduced, as usual, by covariantizing derivatives with respect to the (A)dS Christoffel connection calculated in (2.1.26), $\partial \rightarrow \nabla \equiv \partial + \Gamma$. Moreover,

$$\varphi'_{\mu_3\ldots\mu_s} = g^{\mu_1\mu_2}\varphi_{\mu_1\ldots\mu_s} \ , \tag{2.1.45}$$

where g is the (A)dS metric tensor, and we are assuming $\varphi'' = 0$ and $\epsilon' = 0$. Now, in the (A)dS background, these two conditions are no longer sufficient to ensure the invariance under the covariantized spin-s gauge tranformation

$$\delta\varphi_{\mu(s)} \ = \ s\nabla_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)} \ , \tag{2.1.46}$$

since the covariant derivatives do not commute. Indeed,

$$[\nabla_\mu, \nabla_\nu]\,\varphi_{\rho_1\ldots\rho_s} \ = \ \frac{s}{L^2}\left(g_{\nu(\rho_1|}\,\varphi_{\mu|\rho_2\ldots\rho_s)} \ - \ g_{\mu(\rho_1|}\,\varphi_{\nu|\rho_2\ldots\rho_s)}\right) \ , \tag{2.1.47}$$

for AdS ($\lambda^2 = 1$), while the analog for dS can be obtained from (2.1.47) changing sign to the curvature, *i.e.*, by continuing λ to imaginary values ($\lambda^2 = -1$). The direct substitution of (2.1.46) in the covariantized Fronsdal kinetic operator,

$$\mathcal{F}^{\text{cov}}_{\mu_1\ldots\mu_s}(\varphi) \ = \ \Box\varphi_{\mu_1\ldots\mu_s} - s\nabla_{(\mu_1}\nabla\cdot\varphi_{\mu_2\ldots\mu_s)} + \frac{s(s-1)}{2}\left\{\nabla_{(\mu_1},\nabla_{\mu_2}\right\}\,\varphi'_{\mu_3\ldots\mu_s)} \tag{2.1.48}$$

(where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$), produces terms such as

$$s\left[\Box, \nabla_{(\mu_1}\right]\,\epsilon_{\mu_2\ldots\mu_s)} + \frac{1}{L^2}s(s-1)(D+s-3)\,\nabla_{(\mu_1}\,\epsilon_{\mu_2\ldots\mu_s)} \ . \tag{2.1.49}$$

To eliminate these terms it is necessary to modify the kinetic operator with appropriate terms of order $1/L^2$ that cancel the variation of (2.1.48) and vanish in the flat limit $L \rightarrow \infty$. By explicitly calculating the commutator in (2.1.49) one can check that the invariant Fronsdal equation in AdS_D is

$$\mathcal{F}^L_{\mu(s)} \ \equiv \ \mathcal{F}^{\text{cov}}_{\mu(s)} \ - \ \frac{1}{L^2}\left\{\left[(3-D-s)(2-s)-s\right]\,\varphi_{\mu(s)} \ + \ \frac{s(s-1)}{4}\,g_{(\mu_1\mu_2}\,\varphi'_{\mu_3\ldots\mu_s)}\right\} \ = \ 0 \ . \tag{2.1.50}$$

Notice that, although we deal with massless fields, requiring invariance of the Fronsdal equations in a space-time with nonvanishing cosmological constant results in the appearance of a mass-like term, that in fact originates from the coupling with the (constant) space-time curvature. One can repeat now for (2.1.50) the same considerations made above for the flat case. Again, the Fronsdal equations are non-lagrangian, and one can define a generalized Einstein tensor

$$\mathcal{G}^L_{\mu(s)} \;=\; \mathcal{F}^L_{\mu(s)} \;-\; \frac{s(s-1)}{4}\, g_{(\mu_1\mu_2}\, \mathcal{F}'^L_{\mu_3\ldots\mu_s)} \;, \tag{2.1.51}$$

in terms of which one can construct a Lagrangian from which (2.1.50) follows. Finally, a local unconstrained formulation in (A)dS with the aid of a compensator field is also available [20], and again the presence of a cosmological constant results in extra-terms of order $1/L^2$. In particular, if the trace of the gauge parameter is not constrained to vanish,

$$\delta\mathcal{F}^L_{\mu(s)} \;=\; \frac{s(s-1)(s-2)}{2}\left(\nabla_{(\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\epsilon'_{\mu_4\ldots\mu_s)} - \frac{4}{L^2} g_{(\mu_1\mu_2}\,\nabla_{\mu_3}\epsilon'_{\mu_4\ldots\mu_s)}\right) \tag{2.1.52}$$

and therefore the compensator form of the equations should be

$$\mathcal{F}^L_{\mu(s)} \;=\; \frac{s(s-1)(s-2)}{6}\left(\nabla_{(\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\alpha_{\mu_4\ldots\mu_s)} - \frac{4}{L^2} g_{(\mu_1\mu_2}\,\nabla_{\mu_3}\alpha_{\mu_4\ldots\mu_s)}\right) , \tag{2.1.53}$$

$$\varphi''_{\mu(s-4)} \;=\; 4\nabla\cdot\alpha_{\mu(s-4)} + (s-4)\nabla_{(\mu_1}\alpha'_{\mu_2\ldots\mu_{s-4})} \;, \tag{2.1.54}$$

with the gauge symmetries

$$\delta\varphi_{\mu(s)} \;=\; s\nabla_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)} \;, \tag{2.1.55}$$

$$\delta\alpha_{\mu(s-3)} \;=\; 3\epsilon'_{\mu(s-3)} \tag{2.1.56}$$

that are again consistent by virtue of the Bianchi identities.

2.2 Frame Formulation

It is well-known that gravity admits a (first order) formulation in terms of a frame field and a Lorentz connection, in which the gauging of the Poincaré or (A)dS tangent-space isometry algebra is manifest, and similar to the familiar Yang-Mills case. This fact makes it interesting to examine whether the free massless HS theory admits a reformulation in terms of one-form connections bearing a direct relationship to the generators of an underlying symmetry algebra. In other words, such a reformulation can give some hints towards the construction of an appropriate HS symmetry algebra. If this is the case, indeed, then the free equations could be interpreted as the linearization of interacting equations that involve such one-forms valued in a nonabelian algebra that admits generators with the same index structure as that of the internal (tangent-space) indices of the connections.

We shall first review how the frame formulation works for gravity, by especially recalling the MacDowell-Mansouri and Stelle-West formulations, that are of special interest for HS extensions. Then, we shall extend our considerations to HSGT, describing the approach to the free theory first developed in [55, 57, 62].

2.2.1 Interlude: Gravity à la MacDowell - Mansouri - Stelle - West

Einstein's theory of gravity is a non-abelian gauge theory of a spin-two particle, in the same way (at least to a good extent, see Appendix A for more comments) as Yang-Mills theories are non-abelian gauge theories of spin-one particles. Local symmetries of Yang-Mills theories originate from the internal global symmetries. Similarly, the gauge symmetries of Einstein gravity in the vielbein formulation[4] originate from global space-time symmetries of its most symmetric vacua. These symmetries are manifest in the formulation of MacDowell, Mansouri, Stelle and West [33, 34].

This section is devoted to this formulation. First, the Einstein-Cartan formulation of gravity is reviewed and the link with the Einstein-Hilbert action without cosmological constant is explained. Then, the same approach is extended to include a cosmological constant. We review also an elegant action for gravity, written by MacDowell and Mansouri, and its improved version introduced by Stelle and West, where the covariance under all symmetries is made manifest.

- *Gravity as a Poincaré gauge theory*

The basic idea is as follows: instead of considering the metric $g_{\mu\nu}$ as the dynamical field, two new dynamical fields are introduced: the vielbein or frame field e^a_μ and the Lorentz connection $\omega^{L\,ab}_\mu$.

The relevant fields appear via the one-forms $e^a = e^a_\mu\, dx^\mu$ and $\omega^{L\,ab} = -\omega^{L\,ba} = \omega^{L\,ab}_\mu\, dx^\mu$. The number of 1-forms is equal to $D + \frac{D(D-1)}{2} = \frac{(D+1)D}{2}$, which is the dimension of the Poincaré group $ISO(D-1,1)$. So they can be collected into a single 1-form taking values in the Poincaré algebra as $\omega = -i(e^a P_a + \frac{1}{2}\,\omega^{L\,ab} M_{ab})$, where P_a and M_{ab} generate $\mathfrak{iso}(D-1,1)$ (see (2.1.44)). The corresponding curvature is the two-form:

$$R = d\omega + \omega^2 \equiv -i(T^a P_a + \frac{1}{2}\, R^{L\,ab} M_{ab})\,, \tag{2.2.1}$$

where T^a is the torsion, given by

$$T^a = D^L e^a = de^a + \omega^{L\,a}{}_b e^b\ , \tag{2.2.2}$$

and $R^{L\,ab}$ is the Lorentz curvature

$$R^{L\,ab} = D^L \omega^{L\,ab} = d\omega^{L\,ab} + \omega^{L\,a}{}_c \omega^{L\,cb}\,, \tag{2.2.3}$$

[4]See *e.g.* [53] for a pedagogical review on the gauge theory formulation of gravity and some of its extensions, like supergravity.

as follows from the Poincaré algebra (2.1.44). Torsion and Lorentz curvature are the invariant tensors under the symmetries of the theory, i.e., diffeomorphisms and local Lorentz symmetry.

To make contact with the metric formulation of gravity, one must assume that the frame e^a_μ has maximal rank d so that it gives rise to the non-degenerate metric tensor $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. Moreover, the appearance of the extra local lorentz symmetry, with gauge connection $\omega^{L\,ab}$ and antisymmetric parameter $\epsilon^{ab} = -\epsilon^{ba}$, is exactly what enables to gauge away from e^a_μ its antisymmetric part, therefore reducing the field components to the $D(D+1)/2$ of the metric tensor. One can also require the absence of torsion, $T_a = 0$. Then one solves this constraint and expresses the Lorentz connection in terms of the frame field, $\omega^L = \omega^L(e, \partial e)$. It can be checked that the tensor $R_{\mu\nu,\,\rho\sigma} = e^a_\mu e^b_\nu R^L_{ab\,\rho\sigma}$ expressed solely in terms of the metric is the familiar Riemann tensor.

The first order action of the frame formulation of gravity is due to Weyl [54]. In any dimension $D > 1$ it can be written in the form

$$S[\, e^a_\mu, \omega^{L\,ab}_\mu] = \frac{1}{2\kappa^2} \int_{\mathcal{M}^D} R^{L\,bc} e^{a_1} \dots e^{a_{D-2}} \epsilon_{a_1 \dots a_{D-2} bc}\ , \tag{2.2.4}$$

where $\epsilon_{a_1 \dots a_D}$ is the invariant tensor of the special linear group $SL(D)$ and κ^2 is the gravitational constant, so that κ has dimension $(length)^{\frac{D}{2}-1}$. The Euler-Lagrange equations of the Lorentz connection

$$\frac{\delta S}{\delta \omega^{L\,bc}} \propto \epsilon_{a_1 \dots a_{D-2} bc}\, e^{a_1} \dots e^{a_{D-3}} T^{a_{D-2}} = 0 \tag{2.2.5}$$

imply that the torsion vanishes. The Lorentz connection is then an auxiliary field, which can be removed from the action by solving its own (algebraic) equations of motion. The action $S = S[\, e\, ,\, \omega^L(e, \partial e)\,]$ is now expressed only in terms of the vielbein. Actually, only combinations of vielbeins corresponding to the metric appear and the action $S = S[\, g_{\mu\nu}]$ coincides indeed with the second order Einstein-Hilbert action.

The Minkowski space-time solves $R^{L\,ab} = 0$ and $T^a = 0$. It is the most symmetrical solution of the Euler-Lagrange equations, whose global symmetries form the Poincaré group. The gauge symmetries of the action (2.2.4) are the diffeomorphisms and the local Lorentz transformations. Together, these gauge symmetries correspond to the gauging of the Poincaré group (see Appendix A for more comments).

- *Gravity as an* $\mathfrak{so}(D-1,2)$ *gauge theory*

It is rather natural to reinterpret P_a and M_{ab} as the generators of the AdS_D isometry algebra $\mathfrak{so}(D-1,2)$. The curvature $R = d\omega + \omega^2$ then decomposes as $R = -i(T^a P_a + \frac{1}{2} R^{ab} M_{ab})$, where the Lorentz curvature $R^{L\,ab}$ is deformed to

$$R^{ab} \equiv R^{L\,ab} + R^{cosm\,ab} \equiv R^{L\,ab} + \Lambda\, e^a e^b\, , \tag{2.2.6}$$

since (2.1.44) is deformed to (2.1.42).

MacDowell and Mansouri proposed an action [33], that can be built from the product of two curvatures (2.2.6) in $D = 4$

$$S^{MM}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^4} R^{a_1a_2}R^{a_3a_4}\epsilon_{a_1a_2a_3a_4}\,. \tag{2.2.7}$$

Expressing R^{ab} in terms of $R^{L\,ab}$ and $R^{cosm\,ab}$ by (2.2.6), the Lagrangian is the sum of three terms: a term $R^L R^{cosm}$, which is the previous Lagrangian (2.2.4) without cosmological constant, a cosmological term $R^{cosm}\,R^{cosm}$ and a Gauss-Bonnet term $R^L R^L$. The latter contains higher-derivatives but does not contribute to the equations of motion because it is a topological invariant.

The MacDowell-Mansouri action admits a higher dimensional generalization [89]

$$S^{MM}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^F} R^{a_1a_2}R^{a_3a_4}e^{a_5}\dots e^{a_F}\epsilon_{a_1\dots a_F}\,. \tag{2.2.8}$$

The AdS_D space-time is defined as the most symmetrical solution of the Euler-Lagrange equations. As explained in more detail later on, it is a solution of the system $R^{ab} = 0$, $T^a = 0$ such that $\mathrm{rank}(e^a_\mu) = d$. The Gauss-Bonnet term

$$S^{GB}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{L\,a_1a_2}R^{L\,a_3a_4}e^{a_5}\dots e^{a_d}\epsilon_{a_1\dots a_d}\,.$$

is not topological beyond $D = 4$, and therefore the field equations resulting from the action (2.2.8) are different from the Einstein equations in D dimensions. However, the difference involves nonlinear terms that do not contribute to the free spin 2 equations [89], apart from replacing the cosmological constant Λ by $\frac{2(D-2)}{D}\Lambda$ (in such a way that no correction appears in $D = 4$, as expected). One way to see this is by considering the action

$$S^{nonlin}[\,e,\omega] = S^{GB}[\,e,\omega] + \frac{D-4}{4\kappa^2}\int_{\mathcal{M}^D}\Big(\frac{2}{D-2}\,R^{L\,a_1a_2}e^{a_3}\dots e^{a_D} + \frac{\Lambda}{D}\,e^{a_1}\dots e^{a_D}\Big)\epsilon_{a_1\dots a_D}, \tag{2.2.9}$$

which is the sum of the Gauss-Bonnet term plus terms of the same type as the Einstein-Hilbert and cosmological terms (note that the latter are absent when $D = 4$). The variation of (2.2.9) is equal to

$$\delta S^{nonlin}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^D} R^{a_1a_2}R^{a_3a_4}\delta(e^{a_5}\dots e^{a_D})\,\epsilon_{a_1\dots a_D}\,, \tag{2.2.10}$$

when the torsion is required to be zero (i.e., applying the 1.5 order formalism to see that the variation over the Lorentz connection does not contribute). Indeed, the variation of the action (2.2.9) vanishes when $D = 4$, but when $D > 4$ the variation (2.2.10) is bilinear in the AdS_D field strength R^{ab}. Since the AdS_D field strength is zero in the vacuum AdS solution, the action S^{nonlin} only contributes to corrections of the field equations which are nonlinear in the fluctuations near the AdS background, having no effect on the free spin 2 equations. As a consequence, at the linearized level the Gauss-Bonnet term does not affect the form of the free spin 2 equations of motion, it merely redefines an overall factor in front of the action and the cosmological constant via $\kappa^2 \to (\frac{D}{2}-1)\kappa^2$ and $\Lambda \to \frac{2(D-2)}{D}\Lambda$, respectively (as can be seen by substituting S^{GB} in (2.2.8) with its expression in terms of S^{nonlin} from (2.2.9)). Beyond the free field approximation the corrections to Einstein's field equations resulting from the action (2.2.8) are nontrivial for $D>4$ and nonanalytic in Λ (as can be seen from (2.2.10)), with no smooth flat limit. As will be shown later, this is analogous to the structure of HS interactions which also contain terms with higher derivatives and negative powers of Λ. The important difference is that in the case of gravity one can subtract the term (2.2.9) without destroying the symmetries of the model, while this is not possible in HS gauge theories. The flat limit $\Lambda \to 0$ is perfectly smooth at the level of the algebra (*e.g.* $\mathfrak{so}(D-1,2) \to \mathfrak{iso}(D-1,1)$ for gravity, see Section 2.1.1) and at the level of the free equations of motion, but it may be singular at the level of the action and nonlinear field equations.

- *MacDowell-Mansouri-Stelle-West gravity*

The gauge symmetries of the MacDowell-Mansouri action (2.2.7) are diffeomorphisms and local Lorentz transformations. It is however possible to make the $\mathfrak{so}(D-1,2)$ symmetry manifest by combining the vielbein and the Lorentz connection into a single field $\omega = -i\omega_\mu{}^{AB}dx^\mu M_{AB}$. The fiber indices A,B now run from 0 to D. They are raised and lowered by the invariant mostly minus metric η_{AB} of $\mathfrak{so}(D-1,2)$.

In order to promote local $\mathfrak{so}(D-1,2)$ transformations to gauge symmetries, an additional field has to be introduced: the time-like vector V^A called *compensator*[5]. The compensator vector is constrained to have a constant norm ρ,

$$V^A V^B \eta_{AB} = \rho^2\,. \tag{2.2.11}$$

As we shall see, the constant ρ is related to the cosmological constant according to

$$\rho^2 = -\Lambda^{-1}\,. \tag{2.2.12}$$

The MMSW action is ([34] for $D=4$ and [89] for arbitrary D)

$$S^{MMSW}[\,\omega^{AB},V^A] = -\frac{\rho}{4\kappa^2}\int_{\mathcal{M}^D} \epsilon_{A_1\ldots A_{D+1}} R^{A_1A_2}R^{A_3A_4}E^{A_5}\ldots E^{A_D}V^{A_{D+1}}\,, \tag{2.2.13}$$

[5]This compensator field compensates additional symmetries serving for them as a Higgs field. It should *not* be confused with the homonymous - but unrelated - field $\alpha_{\mu(s-3)}$ introduced in the previous Subsection.

where the curvature or field strength R^{AB} is defined by

$$R^{AB} \equiv d\omega^{AB} + \omega^{AC}\omega_C{}^B$$

and the frame field E^A by

$$E^A \equiv DV^A = dV^A + \omega^A_{\ B}V^B\,.$$

Furthermore, in order to make contact with Einstein gravity, two constraints are imposed: (i) the norm of V^A is fixed, and (ii) the frame field E^A_μ is assumed to have maximal rank equal to D. As the norm of V^A is constant, the frame field satisfies

$$E^A V_A = 0\,. \tag{2.2.14}$$

If the condition (2.2.11) is relaxed, the norm of V^A corresponds to an additional dilaton-like field [34].

Let us now analyze the symmetries of the MMSW action. The action is manifestly invariant under

- Local $\mathfrak{so}(D-1,2)$ transformations:

$$\delta\omega^{AB}(x) = D\epsilon^{AB}(x)\,, \qquad \delta V^A(x) = -\epsilon^{AB}(x)V_B(x)\,; \tag{2.2.15}$$

- Diffeomorphisms:

$$\delta\omega^{AB}_\nu = \partial_\nu(\xi^\mu)\omega^{AB}_\mu + \xi^\mu\partial_\mu\omega^{AB}_\nu\,, \qquad \delta V^A = \xi^\nu\partial_\nu V^A\,. \tag{2.2.16}$$

Let us define the covariantized diffeomorphism as the sum of a diffeomorphism with parameter ξ^μ and a $\mathfrak{so}(D-1,2)$ local transformation with parameter $\epsilon^{AB}(\xi^\mu) = -\xi^\mu\omega^{AB}_\mu$. The effect of this transformation is thus

$$\delta^{cov}\omega^{AB}_\mu = \xi^\nu R^{AB}_{\nu\mu}\,, \qquad \delta^{cov}V^A = \xi^\nu E^A_\nu \tag{2.2.17}$$

by (2.2.15)-(2.2.16).

The compensator vector is pure gauge. Indeed, by local $O(D-1,2)$ rotations one can gauge fix $V^A(x)$ to any values with $V^A(x)V_A(x) = \rho^2$. In particular, one can reach the standard gauge

$$V^A = \rho\,\delta^A_{0'}\,. \tag{2.2.18}$$

Taking into account (2.2.14), one observes that the covariantized diffeomorphism also makes it possible to gauge fix fluctuations of the compensator $V^A(x)$ near any fixed value. Since the full list of symmetries can be represented as a combination of covariantized diffeomorphism, local Lorentz symmetry and diffeomorphisms, in the standard gauge

(2.2.18) the algebra of gauge symmetries is broken to the local $\mathfrak{so}(D-1,1)$ algebra and diffeomorphisms. In the standard gauge, one therefore recovers the field content and the gauge symmetries of the MacDowell-Mansouri action. Let us note that covariantized diffeomorphisms (2.2.17) do not affect the connection ω_μ^{AB} if it is flat (*i.e.* has zero curvature $R_{\nu\mu}^{AB}$). In particular covariantized diffeomorphisms do not affect the background AdS geometry.

To show the equivalence of the action (2.2.13) with the action(2.2.8), it is useful to define a Lorentz connection by

$$\omega^{L\ AB} \equiv \omega^{AB} - \rho^{-2}(E^A V^B - E^B V^A)\,. \tag{2.2.19}$$

In the standard gauge, the curvature can be expressed in terms of the vielbein $e^a \equiv E^a = \rho\, \omega^a{}_{\hat{d}}$ and the non-vanishing components of the Lorentz connection $\omega^{L\ ab} = \omega^{ab}$ as

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^{aC}\omega_C{}^b = d\omega^{L\ ab} + \omega^{L\ a}{}_c \omega^{L\ cb} - \rho^{-2} e^a e^b = R^{L\ ab} + R^{cosm\ ab}\,, \\ R^{a0'} &= \rho^{-1} T^a\,. \end{aligned}$$

Inserting these gauge fixed expressions into the MMSW action yields the action (2.2.8), where $\Lambda = -\rho^{-2}$. The MMSW action thus reduces to (2.2.8) after partially fixing the gauge symmetry.

Let us now consider the vacuum equations $R^{AB}(\omega_0) = 0$. They are equivalent to $T^a = 0$ and $R^{ab} = 0$ and, under the condition that $\mathrm{rank}(E_\nu^A) = d$, they uniquely define the local geometry of AdS_D with parameter ρ, in a coordinate independent way. The solution ω_0 obviously satisfies the equations of motion of the MMSW action. To find the symmetries of the vacuum solution ω_0, one first notes that vacuum solutions are sent to vacuum solutions by diffeomorphisms and local AdS transformations, because they transform the curvature homogeneously. Since covariantized diffeomorphisms do not affect ω_0, in order to find symmetries of the chosen solution ω_0 it is enough to check its transformation law under local $\mathfrak{so}(D-1,2)$ transformation. Indeed, adjusting an appropriate covariantized diffemorphism it is always possible to keep the compensator invariant.

The solution ω_0 is invariant under local $\mathfrak{so}(D-1,2)$ transformations if and only if the parameter $\epsilon^{AB}(x)$ satisfies

$$0 = D_0\epsilon^{AB}(x) = d\epsilon^{AB}(x) + \omega_0{}^A{}_C(x)\,\epsilon^{CB}(x) - \omega_0{}^B{}_C(x)\,\epsilon^{CA}(x)\,. \tag{2.2.20}$$

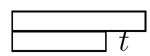
This equation fixes the derivatives $\partial_\mu\epsilon^{AB}(x)$ in terms of $\epsilon^{AB}(x)$ itself. In other words, once $\epsilon^{AB}(x_0)$ is chosen for some x_0, $\epsilon^{AB}(x)$ can be reconstructed for all x in a neighborhood of x_0, since by consistency[6] *all* derivatives of the parameter can be expressed in terms of the

[6]The identity $D_0^2 = R_0 = 0$ ensures consistency of the system (2.2.20), which is overdetermined because it contains $\frac{D^2(D+1)}{2}$ equations for $\frac{D(D+1)}{2}$ unknowns. Consistency in turn implies that higher space-time derivatives $\partial_{\nu_1}\ldots\partial_{\nu_n}\epsilon^{AB}(x)$ obtained by hitting (2.2.20) $n-1$ times with $D_{0\nu_k}$ are guaranteed to be symmetric in the indices $\nu_1\ldots\nu_n$.

parameter itself. The parameters $\epsilon^{AB}(x_0)$ remain arbitrary, and are indeed parameters of the global symmetry $\mathfrak{so}(D-1,2)$. This means that, as expected for AdS space-time, the symmetry of the vacuum solution ω_0 is the global $\mathfrak{so}(D-1,2)$.

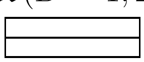
The lesson is that, to describe a gauge model that has a global symmetry h, it is useful to reformulate it in terms of the gauge connections ω and curvatures R of h in such a way that the zero curvature condition $R=0$ solves the field equations and provides a solution with h as its global symmetry. If a symmetry h is not known, this observation can be used the other way around: by reformulating dynamics à la MacDowell-Mansouri one might guess the structure of an appropriate curvature R and thereby the non-abelian algebra h.

2.2.2 Frame-like Formulation of Free HS Dynamics

It is possible to parallel the frame formulation of gravity for HS fields. The doubly-traceless metric-like HS gauge field $\varphi_{\mu_1\ldots\mu_s}$ is replaced by a frame-like field $A_\mu{}^{a_1\ldots a_{s-1}}$, a Lorentz-like connection $A_\mu{}^{a_1\ldots a_{s-1},\,b}$ [55] and an extra set of connections $A_\mu{}^{a_1\ldots a_{s-1},\,b_1\ldots b_t}$, where $t=2,\ldots,s-1$ [56, 57]. All fields are traceless in the fiber indices a,b, which have the symmetry of the Young tableaux  $s-1$, where $t=0$ for the frame-like field and $t=1$ for the Lorentz-like connection. The metric-like field arises as the completely symmetric part of the frame field [55],

$$\varphi_{\mu_1\ldots\mu_s} = A_{\{\mu_1,\,\mu_2\ldots\mu_s\}}\,,$$

where all fiber indices have been lowered using the AdS or flat frame field $e_0{}^a_\mu$ defined in Section 2.2.1. The fiber tracelessness of the frame field implies automatically that the field $\varphi_{\mu_1\ldots\mu_s}$ is doubly traceless.

The frame-like field and other connections are then combined [89] into a connection one-form $A^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ (where $A,B=0,\ldots,d$) taking values in the irreducible $\mathfrak{so}(D-1,2)$-module characterized by the two-row traceless rectangular Young tableau  of length $s-1$, that is

$$\begin{aligned} A_\mu^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} = A_\mu^{\{A_1\ldots A_{s-1}\},B_1\ldots B_{s-1}} &= A_\mu^{A_1\ldots A_{s-1},\{B_1\ldots B_{s-1}\}}\,, \\ A_\mu^{\{A_1\ldots A_{s-1},A_s\}\,B_2\ldots B_{s-1}} = 0\,, \quad A_\mu^{A_1\ldots A_{s-3}\,C}{}_{C,}{}^{B_1\ldots B_{s-1}} &= 0\,. \end{aligned} \tag{2.2.21}$$

One also introduces a time-like vector V^A of constant norm ρ. The component of the connection $A^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ that is most parallel to V^A is the frame-like field

$$A^{A_1\ldots A_{s-1}} = \omega^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}} V_{B_1}\ldots V_{B_{s-1}}\,,$$

while the less V-longitudinal components are the other connections. Note that the contraction of the connection with more than $s-1$ compensators V^A is zero by virtue of (2.2.21). Let us be more explicit in a specific gauge. As in the MMSW gravity reformulation, one can show that V^A is a pure gauge field and that one can reach the standard gauge $V^A = \delta^A_{0'}\rho$ (the argument will not be repeated here). In the standard gauge, the frame field and the connections are given by

$$\begin{aligned} A^{\;a_1\ldots a_{s-1}} &= \rho^{s-1} A^{\;a_1\ldots a_{s-1},\,0'\ldots 0'} \\ A^{\;a_1\ldots a_{s-1},\,b_1\ldots b_t} &= \rho^{s-1-t}\,\Pi A^{\;a_1\ldots a_{s-1},\,b_1\ldots b_t\,0'\ldots 0'} \end{aligned}$$

where the powers of ρ originate from a corresponding number of contractions with the compensator vector V^A and Π is a projector to the Lorentz-traceless part of a Lorentz tensor, which is needed for $t \geq 2$. These normalization factors are consistent with the fact that the auxiliary fields $A_\mu^{\;a_1\ldots a_{s-1},\,b_1\ldots b_t}$ will be found to be expressed via t partial derivatives of the frame field $A_\mu^{\;a_1\ldots a_{s-1}}$ (ρ is a length scale) at the linearized level.

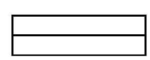
The linearized field strength or curvature is defined as the $\mathfrak{so}(D-1,2)$ covariant derivative of the connection $A^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$, *i.e.* by

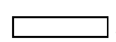
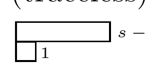
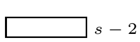
$$\begin{aligned} F_1^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} &= D_0 A^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} \\ &= dA^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} + {\omega_0}^{A_1}{}_C A^{CA_2\ldots A_{s-1},\,B_1\ldots B_{s-1}} + \ldots \\ &\qquad + {\omega_0}^{B_1}{}_C A^{A_1\ldots A_{s-1},\,CB_2\ldots B_{s-1}} + \ldots\ , \end{aligned} \tag{2.2.22}$$

where the dots stand for the terms needed to get an expression symmetric in $A_1\ldots A_{s-1}$ and $B_1\ldots B_{s-1}$, and ${\omega_0}^A{}_B$ is the $\mathfrak{so}(D-1,2)$ connection associated to the AdS space solution, as defined in Section 2.2.1. The connection $A_\mu^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ has dimension $(length)^{-1}$ in such a way that the field strength $F_{\mu\nu}^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ has proper dimension $(length)^{-2}$.

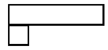
As $(D_0)^2 = R_0 = 0$, the linearized curvature F_1 is invariant under Abelian gauge transformations of the form

$$\delta A^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}} = D_0\epsilon^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}\,. \tag{2.2.23}$$

The gauge parameter $\epsilon^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ has the symmetry  and is traceless.

Before writing the action, let us analyze the frame field and its gauge transformations, in the standard gauge. According to the usual multiplication rule for $\mathfrak{so}(n)$-irreducible Young diagrams [52, 66, 59], the frame field $A_\mu^{\;a_1\ldots a_{s-1}}$ contains three irreducible (traceless) Lorentz components characterized by the symmetry of their indices:  ,  and  , where the last tableau describes the trace component of the frame field $A_\mu^{\;a_1\ldots a_{s-1}}$. Its gauge transformations are given by (2.2.23) and read

$$\delta A^{a_1\ldots a_{s-1}} = D_0^L\epsilon^{a_1\ldots a_{s-1}} - e_{0\;c}\epsilon^{a_1\ldots a_{s-1},\,c}\,.$$

The parameter $\epsilon^{a_1\dots a_{s-1},\,c}$ is a generalized local Lorentz parameter. It allows us to gauge away the traceless component  of the frame field. The other two components of the latter just correspond to a completely symmetric doubly-traceless Fronsdal field $\varphi_{\mu_1\dots\mu_s}$. The remaining invariance is then the Fronsdal gauge invariance (**??**) with a traceless completely symmetric parameter $\epsilon^{a_1\dots a_{s-1}}$.

- *Action for HS gauge fields*

For a given spin s, the most general $\mathfrak{so}(D-1,2)$-invariant action that is quadratic in the linearized curvatures (2.2.22) and, for the rest, built only from the compensator V^C and the background frame field $E_0^B = D_0 V^B$ is

$$S_2^{(s)}[\, A_\mu^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}\,, \omega_0^{AB}\,, V^C\,] \;=\; \frac{1}{2}\sum_{p=0}^{s-2} a(s,p)\, S^{(s,p)}[\, A_\mu^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}\,, \omega_0^{AB}\,,\, V^C\,] \tag{2.2.24}$$

where $a(s,p)$ is the *a priori* arbitrary coefficient of the term

$$\begin{aligned} S^{(s,p)}[\,\omega, V\,] \;&=\; \epsilon_{A_1\dots A_{D+1}}\int_{M^D} E_0^{A_5}\dots E_0^{A_D}V^{A_{D+1}}V_{C_1}\dots V_{C_{2(s-2-p)}}\,\times \\ &\qquad \times F_1^{A_1B_1\dots B_{s-2},\,A_2C_1\dots C_{s-2-p}D_1\dots D_p}\,F_1{}^{A_3}{}_{B_1\dots B_{s-2},}{}^{A_4C_{s-1-p}\dots C_{2(s-2-p)}}{}_{D_1\dots D_p}\,. \end{aligned}$$

This action is manifestly invariant under diffeomorphisms, local $\mathfrak{so}(D-1,2)$ transformations (2.2.15) and Abelian HS gauge transformations (2.2.23) that leave invariant the linearized HS curvatures (2.2.22). Having fixed the AdS_D background gravitational field ω_0^{AB} and the compensator V^A, diffeomorphisms and local $\mathfrak{so}(D-1,2)$ transformations break down to the AdS_D global symmetry $\mathfrak{so}(D-1,2)$.

The connections $A_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$ can be expressed via t derivatives of the frame-like field, according to HS analogues of the torsion constraint. Therefore the coefficients $a(s,p)$ must be chosen in such a way that the Euler-Lagrange derivatives are non-vanishing only for the frame field and the first connection ($t=1$). All other fields, *i.e.* the connections $A_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$ with $t>1$, appear only through total derivatives. They are called extra fields[7]. This requirement guarantees that higher-derivative terms are absent in the free theory and fixes uniquely the spin-s free action up to an overall coefficient $b(s)$. More precisely, the coefficient $a(s,p)$ is essentially a relative coefficient given by [89]

$$a(s,p) = b(s)(-\Lambda)^{-(s-p-1)}\frac{(D-5+2(s-p-2))!!\,(s-p-1)}{(s-p-2)!}$$

where $b(s)$ is the arbitrary spin-dependent factor.

[7]The extra fields show up in the non-linear theory and are responsible for the higher-derivatives as well as for the terms with negative powers of Λ in the interaction vertices.

The equations of motion for $A_\mu{}^{a_1\ldots a_{s-1},\,b}$ are equivalent to the "zero-torsion condition"

$$F_{1\,A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}V^{B_1}\ldots V^{B_{s-1}} = 0\,.$$

They imply that $A_\mu{}^{a_1\ldots a_{s-1},\,b}$ is an auxiliary field that can be expressed in terms of the first derivative of the frame field. Substituting the found expression for $A_\mu{}^{a_1\ldots a_{s-1},\,b}$ into the HS action yields an action only expressed in terms of the frame field and its first derivative, modulo total derivatives. As gauge symmetries told us, the action actually depends only on the completely symmetric part of the frame field, *i.e.* the Fronsdal field. Moreover, the action (2.2.24) has the same gauge invariance as Fronsdal's one, and hence it must be proportional to the Fronsdal action (2.1.5) because the latter is fixed up to an overall factor by the requirements of being gauge invariant and of second order in derivatives of the field [58].

Chapter 3

HS Algebras and Representation Theory

In the previous section, the dynamics of free spin-s gauge fields has been expressed as a theory of one-forms, whose $\mathfrak{so}(D-1,2)$ fiber indices have symmetries characterized by two-row rectangular Young tableaux. This suggests that there exists a non-Abelian HS algebra $h \supset \mathfrak{so}(D-1,2)$ that admits a basis formed by a set of elements $\widehat{T}_{A_1 \ldots A_{s-1}, B_1 \ldots B_{s-1}}$ in irreducible representations of $\mathfrak{so}(D-1,2)$ characterized by such Young tableaux. More precisely, the basis elements $\widehat{T}_{A_1 \ldots A_{s-1}, B_1 \ldots B_{s-1}}$ satisfy the following properties $\widehat{T}_{\{A_1 \ldots A_{s-1}, A_s\} B_2 \ldots B_{s-1}} = 0$, $\widehat{T}_{A_1 \ldots A_{s-3} CC,}{}^{B_1 \ldots B_{s-1}} = 0$, and the basis contains the $\mathfrak{so}(D-1,2)$ basis elements $\widehat{T}_{A,B} = -\widehat{T}_{B,A}$ such that all generators transform as $\mathfrak{so}(D-1,2)$ tensors

$$[\widehat{T}_{C,D}\, ,\, \widehat{T}_{A_1 \ldots A_{s-1}, B_1 \ldots B_{s-1}}] = i\eta_{DA_1} \widehat{T}_{CA_2 \ldots A_{s-1}, B_1 \ldots B_{s-1}} + \ldots \, . \tag{3.0.1}$$

The question is whether a non-Abelian algebra h with these properties really exists. If it does, the Abelian curvatures F_1 can be understood as resulting from the linearization of the non-Abelian field curvatures $F = dA + A^2$ of h with the h gauge connection $A = \omega_0 + A^{(lin)}$, where ω_0 is some fixed flat (*i.e.* vanishing curvature) zero-order connection of the subalgebra $\mathfrak{so}(D-1,2) \subset h$ and $A^{(lin)}$ is the first-order dynamical part which describes massless fields of various spins[1].

Summarizing, in a more general language: assuming that there exists full equations with local HS symmetry h, and that such equations admit some vacuum solution that breaks the local h symmetry to a global one ($\mathfrak{so}(D-1,2)$ in the AdS_D case); a perturbative expansion around such vacuum yields the linearized field equations seen in the previous

[1]Notice that now we are extending the notation previously used for HS one-form connections to the fluctuational part of fields of all spins $s \geq 1$, maintaining the notation ω_0 only for the background gravitational field that is a solution of $R_0 = d\omega_0 + \omega_0^2 = 0$.

Chapter, with massless HS fields that possess local abelian gauge symmetry parameters in h. In such a scheme, a candidate non-abelian HS algebra should satisfy the following requirements:

- In order to be able to interpret the model in terms of relativistic fields carrying some mass and spin, the vacuum solutions has to be invariant under some space-time isometry algebra (like $\mathfrak{so}(D-1,2)$) $g \subset h$.
- h must admit massless unitary representations that contain all the gauge fields in the model and, possibly, some lower spin fields with no associated gauge symmetries *admissibility* criterion).

The HS algebras with the above mentioned properties were originally found for the case of AdS_4 [62, 63, 64, 65] in terms of spinor algebras. Then this construction was extended to HS algebras in AdS_3 [67, 68, 69] and to $4D$ conformal HS algebras [70, 71] equivalent to the AdS_5 algebras of [73]. $D = 7$ HS algebras [74] were also built in spinorial terms. Conformal HS conserved currents in any dimension, generating HS symmetries with the parameters carrying representations of the conformal algebra $\mathfrak{so}(D,2)$ described by various rectangular two-row Young tableaux, were found in [75]. The realization of the conformal HS algebra h in any dimension in terms of a quotient of the universal enveloping algebra was given by Eastwood in [76].

Here we first illustrate the construction of an "abstract" HS algebra, starting from the associative enveloping algebra of $\mathfrak{so}(D-1,2)$, and factoring out an appropriate ideal. Later on, in view of the presentation of the full Vasiliev equations, we shall review the oscillator realization first given in [72] (see also [52]), which is based on vector oscillator algebra (*i.e.*, Weyl algebra). Finally, we shall present the original four-dimensional spinor oscillator realization of the four-dimensional algebra, which is the simplest of all in that the above-mentioned ideal is automatically factored out.

3.1 Complex HS Algebra

As one of the requirements for a HS algebra in D dimensions is that of being an infinite-dimensional extension of the isometry algebra of a maximally symmetric space-time (see Section 2.1.1), our starting point is the latter and its defining commutation relations. To keep the discussion completely general and valid for any signature, we will mostly work, unless explicitly stated, at the level of *complex* Lie algebras, and only later specialize to the different real forms (*i.e.*, to the different signatures, or to the different maximally symmetric backgrounds mentioned in Chapter 2) imposing reality conditions on generators.

The complex Lie algebra $\mathfrak{so}(D+1;\mathbb{C})$ has generators M_{AB} obeying

$$[M_{AB}, M_{CD}] \;=\; 4i\eta_{[C|[B}M_{A]|D]} \;, \tag{3.1.1}$$

where $A = (a, 0')$,$a = (0, r)$, $r = 1, \dots, D-1$ and

$$\eta_{AB} \;=\; \mathrm{diag}(\eta_{ab}, -1) \;, \qquad \eta_{ab} \;=\; \mathrm{diag}(-1, \delta_{rs}) \;. \tag{3.1.2}$$

Although we work at the complex level, as outlined in Section 3.2 the above choice of signature is convenient for describing Harish-Chandra modules, that will be of relevance in the following, and for examining the unitarity properties of the representations for different real forms of the algebra.

The relevant infinite-dimensional extension of $\mathfrak{so}(D+1;\mathbb{C})$ we are looking for is based on its universal enveloping algebra. The universal enveloping algebra $\mathcal{U}$ of $\mathfrak{so}(D+1;\mathbb{C})$ is the associative algebra with product $\star$ generated by the unity $\mathbb{1}$ and monomials in M_{AB} modulo the commutation rule (3.1.1). As a consequence, a basis for $\mathcal{U}$ is given by the unity and symmetrized products of M_{AB}. As we shall deal all the time with this specific ordering prescription, it is useful to adopt a convenient notation: we will henceforth denote with M_{AB} commuting variables that are symbols of the corresponding operators, and implement the operator product on them through a $\star$-product law. The latter is defined in such a way that the $\star$-product of two symbols of operators is the symbol of the product of the two operators. Therefore, the commutation relation (3.1.1) becomes now

$$[M_{AB}, M_{CD}]_\star \;=\; M_{AB}\star M_{CD} - M_{CD}\star M_{AB} \;=\; 4i\eta_{[C|[B}M_{A]|D]} \;, \tag{3.1.3}$$

and the totally symmetrized products of operators can simply be denoted by juxtaposition of the commuting variables M_{AB}. With this convention, the definition of $\mathcal{U}$ is

$$\mathcal{U} \;=\; \bigoplus_{n=0}^{\infty}\mathcal{U}_n \;, \qquad \mathcal{U}_n \;=\; \left\{X_n \;=\; x^{A_1B_1,\dots,A_nB_n}M_{A_1B_1}\cdots M_{A_nB_n}\right\} , \tag{3.1.4}$$

$$M_{A_1B_1}\cdots M_{A_nB_n} \;\equiv\; \begin{cases} \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n} M_{A_{\pi(1)}B_{\pi(2)}}\star\cdots\star M_{A_{\pi(n)}B_{\pi(n)}} & \text{for } n=1,2,\dots\,, \\ \mathbb{1} & \text{for } n=0 \end{cases} \tag{3.1.5}$$

where $x^{A_1B_1,\dots,A_nB_n}$ are complex coefficients, and we note that there is no separate symmetry on A and B indices. The $\star$-product $X_m \star X_n$, which is computed by repeated symmetrization using the commutation rule (3.2.1), yields the "classical" product X_mX_n together with terms of lower order, since each commutation removes one generator. For example,

$$M_{AB}\star M_{CD} \;=\; \frac{1}{2}\{M_{AB}, M_{CD}\}_\star + \frac{1}{2}[M_{AB}, M_{CD}]_\star \;=\; M_{AB}M_{CD} + 2i\eta_{[C|[B}M_{A]|D]} \tag{3.1.6}$$

The map

$$\tau(X_n) \;=\; (-1)^n X_n \;, \tag{3.1.7}$$

that is, $\tau(X(M_{AB})) = X(-M_{AB})$ where $X(M_{AB})$ is a symmetrized function, is an (involutive) anti-automorphism of the $\star$-product, *i.e.*

$$\tau(X \star Y) \;=\; \tau(Y) \star \tau(X) \;. \tag{3.1.8}$$

However, the universal enveloping algebra of $\mathfrak{so}(D+1;\mathbb{C})$ does not satisfy our requirement for being a candidate HS algebra, since the generators (3.1.4) do not really match the conditions we had fixed at the beginning of this chapter. In general, indeed, a symmetrized monomial of degree $n \geq 2$ is reducible under $\mathfrak{so}(D+1;\mathbb{C})$, as it contains both trace parts and irreps labeled by Young diagrams with more than two rows. However, both of them can be absorbed into the ideal

$$\mathcal{I}[V] \;=\; \{X \;=\; V \star X' \text{ for } X' \in \mathcal{U}\} \;, \tag{3.1.9}$$

where $V = \lambda^{AB} V_{AB} + \lambda^{ABCD} V_{ABCD}$ with $\lambda^{AB}, \lambda^{ABCD} \in \mathbb{C}$, and

$$V_{AB} \;\equiv\; \tfrac{1}{2} M_{(A}{}^{C} M_{B)C} - \frac{1}{2(D+1)} \eta_{AB} M^{CD} \star M_{CD} \;, \tag{3.1.10}$$

$$V_{ABCD} \;\equiv\; M_{[AB} M_{CD]} \;. \tag{3.1.11}$$

The generator V_{AB} absorbs the traces, while V_{ABCD} absorbs the Young diagrams with with more than two rows (as it is clear from (3.1.18)) - *i.e.*, the unwanted elements are solved by such constraints in terms of allowed ones. So far, we have a chain of proper ideals, namely $\mathcal{U} \supset \mathcal{U}' \supset \mathcal{I}[V]$, where $\mathcal{U}' = \mathcal{U} \setminus \mathbb{1}$. Factoring out $\mathcal{I}[V]$ induces the infinite-dimensional unital associative quotient algebra

$$\mathcal{A} \;\equiv\; \frac{\mathcal{U}}{\mathcal{I}[V]} \;, \tag{3.1.12}$$

and we shall use the notation

$$X \simeq X' \qquad \Leftrightarrow \qquad X - X' \in \mathcal{I}[V] \;. \tag{3.1.13}$$

The constraints $V_{AB} \simeq 0$ and $V_{ABCD} \simeq 0$ *together* fix the values of the Casimir operators[2]

$$C_{2n}[\mathfrak{so}(D+1;\mathbb{C})] \;\equiv\; \tfrac{1}{2} M_{A_1}{}^{A_2} \star M_{A_2}{}^{A_3} \star \cdots \star M_{A_{2n}}{}^{A_1} \;, \qquad n = 1, 2, \ldots \;. \tag{3.1.14}$$

In what follows we shall denote the restriction of the Casimir operators to a representation $\mathfrak{R}$ of $\mathfrak{so}(D+1;\mathbb{C})$ by $C_{2n}[\mathfrak{so}(D+1;\mathbb{C})|\mathfrak{R}]$, or simply $C_{2n}[\mathfrak{R}]$ in case there is no risk of

[2]For $D = 5$, also the cubic Casimir operator plays a role.

confusion. To begin with, the higher-order operators C_{2n} with $n > 1$ can be rewritten in terms of C_2 using $V_{AB} \simeq 0$, which implies

$$M_A{}^B \star M_{BC} \;=\; \frac{i(D-1)}{2} M_{AC} - \frac{2}{D+1}\eta_{AC} C_2 \;, \tag{3.1.15}$$

so that, for example,

$$C_4[\mathcal{A}] \;=\; \frac{2}{D+1} C_2[\mathcal{A}]^2 + \frac{(D-1)^2}{4} C_2[\mathcal{A}] \;. \tag{3.1.16}$$

The operator $C_2[\mathcal{A}]$ (and hence all $C_{2n}[\mathcal{A}]$) is fixed by compatibility between $V_{AB} \simeq 0$ and $V_{ABCD} \simeq 0$, which requires

$$0 \;\simeq\; M_A{}^B \star V_{BCDE} \;\simeq\; \left(\mu^2 - \epsilon_0\right) \star \eta_{A[C} M_{DE]} \;, \qquad \mu^2 \;\equiv\; -\frac{2C_2[\mathcal{A}]}{D+1} \;, \tag{3.1.17}$$

where we have used (3.1.15) and

$$V_{ABCD} \;=\; M_{[AB} \star M_{CD]} \;=\; M_{[AB} \star M_{C]D} - i\eta_{D[A} M_{BC]} \;, \tag{3.1.18}$$

and we have also introduced the parameter

$$\epsilon_0 \;=\; \frac{D-3}{2} \;. \tag{3.1.19}$$

Thus, one finds, for example,

$$C_2[\mathfrak{so}(D+1;\mathbb{C})|\mathcal{A}] \;=\; -\epsilon_0(\epsilon_0+2) \;, \tag{3.1.20}$$

$$C_4[\mathfrak{so}(D+1;\mathbb{C})|\mathcal{A}] \;=\; -\epsilon_0(\epsilon_0+2)(\epsilon_0^2+\epsilon_0+1) \;, \tag{3.1.21}$$

and one can calculate higher-order Casimir operators as well, by using (3.1.15) recursively.

We can choose a canonical leveled basis for $\mathcal{A}$ as follows:

$$\mathcal{A} \;=\; \bigoplus_{n=0}^{\infty} \mathcal{A}_n \;, \qquad \mathcal{A}_n \;\simeq\; \left\{ X_n \;=\; x^{A(n),B(n)} \widehat{T}_{A(n),B(n)} \right\} \;, \tag{3.1.22}$$

where $x^{A(n),B(n)}$ are traceless type (n,n) tensors[3], and

$$\widehat{T}_{A(n),B(n)} \;=\; M_{\{A_1B_1} \cdots M_{A_nB_n\}} \;=\; M_{\{A_1B_1} \star \cdots \star M_{A_nB_n\}} \;, \tag{3.1.23}$$

[3]Throughout this Thesis, tensors with the symmetry of the Young diagram of height ν with n_i cells in the ith row ($i = 1, \ldots, \nu$) are referred to as type $(n_1, \ldots, n_\nu)$ tensors (where thus $n_1 \geq n_2 \geq \cdots \geq n_\nu$ are positive integers). We work with *normalized and mostly symmetric Young projections*

$$\mathbf{P}_{n_1,n_2,\ldots,n_\nu} = \frac{1}{\prod_{\text{cells}}(\text{hook-lengths})} \prod_{\text{rows } i} \mathbf{S}_i \prod_{\text{columns } j} \mathbf{A}_j \;,$$

where we use the convention that curly brackets, $\{\cdots\}$, enclose $\mathfrak{so}(D+1;\mathbb{C})$ irreducible, *i.e.* traceless and Young-projected, groups of indices. More explicitly,

$$\begin{aligned}\widehat{T}_{A(n),B(n)} &= M_{\langle A_1B_1}\star\cdots\star M_{A_nB_n\rangle}\\ &\quad+\sum_{k=1}^{[n/2]}\kappa_{n;k}\eta_{\langle A_1A_2}\eta_{B_1B_2}\cdots\eta_{A_{2k-1}A_{2k}}\eta_{B_{2k-1}B_{2k}}M_{A_{2k+1}B_{2k+1}}\star\cdots\star M_{A_nB_n\rangle}\ ,\end{aligned}\tag{3.1.24}$$

where we use the convention that hooked brackets, $\langle\cdots\rangle$, enclose $\mathfrak{sl}(D+1;\mathbb{C})$ irreducible, *i.e.* Young projected, groups of indices, and the coefficients $\kappa_{n;k}$ are fixed by

$$\eta^{CD}\widehat{T}_{A(n),B(n-2)CD} = 0\ .\tag{3.1.25}$$

We note that $V_{AB}\simeq 0$ implies that the trace parts in (3.1.24) only involve lower-order enveloping-algebra monomials in rectangular Young projections. From (3.1.23) and onwards, we shall always use the convention that repeated indices that are denoted by a single letter and distinguished by subindices are always symmetrized, so that[4]

$$M_{A_1}{}^{B_1}\cdots M_{A_n}{}^{B_n} \equiv M_{(A_1}{}^{(B_1}\cdots M_{A_n)}{}^{B_n)} = M_{(A_1}{}^{(B_1}\star\cdots\star M_{A_n)}{}^{B_n)}\ ,\tag{3.1.26}$$

that is,

$$M_{A_1B_1}\cdots M_{A_nB_n} = M_{\langle A_1B_1}\cdots M_{A_nB_n\rangle} = M_{\langle A_1B_1}\star\cdots\star M_{A_nB_n\rangle}\ .\tag{3.1.27}$$

For example, the simplest case is given by

$$\widehat{T}_{A(2),B(2)} = M_{A_1B_1}\star M_{A_2B_2}-\frac{2}{D(D+1)}(\eta_{A_1A_2}\eta_{B_1B_2}-\eta_{A_1B_1}\eta_{A_2B_2})C_2[\mathcal{A}]\ .\tag{3.1.28}$$

The $\mathfrak{so}(D+1;\mathbb{C})$-transformations take the form

$$\mathrm{Ad}_{M_{AB}}(T_{C(n),D(n)}) = 2in\eta_{[B|\{C_1}T_{|A]|C(n-1),D(n)\}}+2in\eta_{[B|\{D_1}T_{C(n),|A]|D(n-1)\}}\ ,\tag{3.1.29}$$

where the adjoint action of $\mathcal{A}$ on itself is defined by

$$\mathrm{Ad}_X(Y) = [X,Y]_\star\ .\tag{3.1.30}$$

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where $\mathbf{S}_i$ and $\mathbf{A}_j$ are symmetrizers and anti-symmetrizers, respectively, acting on the indices of the ith row ($i=1,\dots,\nu$) and jth column ($j=1,\dots,n_1$). Thus, a type $(n_1,\dots,n_\nu)$ tensor $T_{A^1(n_1),\dots,A^\nu(n_\nu)}$ has ν groups of symmetrized indices $A^i(n_i)=A^i_1\dots A^1_{n_i}$, subject to the over-symmetrization rule

$$T_{\cdots,(A^i_1\dots A^i_{n_i},A^{i+1}_1)A^{i+1}_2\dots A^{i+1}_{n_{i+1}},\cdots} = 0\ ,\qquad i=1,\dots,\nu-1\ .$$

When $A=1,\dots,N$, such tensors are $\mathfrak{sl}(N;\mathbb{C})$ irreps, and $\mathfrak{so}(N;\mathbb{C})$ irreps when they are traceless, of highest weight $(n_1,\dots,n_\nu)$.

[4]We note that prior to using this convention, the type (n,n) Young projection of $M_{A_1B_1}\cdots M_{A_nB_n}$ (no symmetry on A and B indices!) equals $\frac{2^n}{n+1}M_{A_1B_1}\cdots M_{A_nB_n}$ (symmetry on the A and B indices!).

Therefore, the generators $\widehat{T}_{A(n),B(n)}$ of $\mathcal{A}$ indeed have the correct index structure and transformation properties to be candidate generators of the HS extension of $\mathfrak{so}(D+1;\mathbb{C})$ we were looking for. Let us now look at some other properties that will be important for the following. We can also define the anti-commutator action

$$\mathrm{Ac}_X(Y) \;=\; \{X,Y\}_\star \;, \tag{3.1.31}$$

with the closure

$$[\mathrm{Ac}_X,\mathrm{Ac}_Y] \;=\; \mathrm{Ad}_{[X,Y]_\star} \;. \tag{3.1.32}$$

From $V\simeq 0$ it follows that

$$\mathrm{Ac}_{M_{AB}}(\widehat{T}_{C(n),D(n)}) \;=\; 2\Delta_n\widehat{T}_{[A|\{C(n),|B]D(n)\}}+2\lambda_n\eta_{[A\{C_1}\eta_{|B]\{D_1}\widehat{T}_{C(n-1),D(n-1)\}} \;, \tag{3.1.33}$$

with (suppressing the anti-symmetry on AB)

$$\begin{aligned}
&\eta_{[A\{C_1}\eta_{|B]\{D_1}\widehat{T}_{C(n-1),D(n-1)\}} \;=\; \eta_{AC_1}\eta_{BD_1}\widehat{T}_{C(n-1),D(n-1)}\\
&+\beta_n\Big(\eta_{C_1C_2}\eta_{BD_1}\widehat{T}_{AC(n-2),D(n-1)}+\eta_{AC_1}\eta_{D_1D_2}\widehat{T}_{C(n-1),BD(n-2)}\\
&+\eta_{C_1D_1}\eta_{C_1A}\widehat{T}_{BC(n-2),D(n-1)}+\eta_{C_1D_1}\eta_{D_1B}\widehat{T}_{C(n-1),AD(n-2)}\Big)\\
&+\alpha_n\Big(\eta_{C_1C_2}\eta_{D_1D_2}\widehat{T}_{AC(n-2),BD(n-2)}-\eta_{C_1D_1}\eta_{C_2D_2}\widehat{T}_{AC(n-2),BD(n-2)}\Big) \;,
\end{aligned} \tag{3.1.34}$$

where the coefficients

$$\begin{aligned}
\Delta_n &= 2\frac{n+1}{n+2}\;,\\
\alpha_n &= \frac{1}{4}\frac{(n-1)^2}{(n+\epsilon_0-1)(n+\epsilon_0-\frac{1}{2})}\;, \qquad \beta_n \;=\; -\frac{1}{2}\frac{n-1}{n+\epsilon_0-1}\;,
\end{aligned} \tag{3.1.35}$$

are fixed by traceless type $(n+1,n+1)$ Young projection (so that $\mathbf{P}(M_{AB}\widehat{T}_{C(n),D(n)}) = \Delta_n\mathbf{P}\widehat{T}_{[A|\{C(n),|B]D(n)} = \Delta_n T_{AC(n),BD(n)}$ where $\mathbf{P}\equiv\mathbf{P}_{AC(n),BD(n)}$), while the coefficient

$$\lambda_n \;=\; -\frac{1}{2}\frac{n(n+1)(n+\epsilon_0-1)}{n+\epsilon_0+\frac{1}{2}} \;, \tag{3.1.36}$$

can be computed either by solving the trace conditions on (3.1.33), or by demanding closure under (3.1.32) and (3.1.29), that is

$$[\mathrm{Ac}_{M_{AB}},\mathrm{Ac}_{M_{CD}}](\widehat{T}_{E(n),F(n)}) \;=\; 4i\eta_{BC}\mathrm{Ad}_{M_{AD}}\widehat{T}_{E(n),F(n)} \;, \tag{3.1.37}$$

(where the separate anti-symmetry on AB and CD has been suppressed). In order to apply the first method, one first substitutes the $\widehat{T}$-elements on the left-hand and right-hand sides of (3.1.33) by their trace expansions (3.1.24) up to $\mathcal{O}(\eta^2)$ and $\mathcal{O}(\eta)$, respectively. One then contracts the equation by η^{BD_1} using $V \simeq 0$ and (3.1.25), respectively, to simplify the left-hand and right-hand sides. The second method, on the other hand, relies entirely on the $\mathfrak{so}(D+1;\mathbb{C})$-covariance of the whole procedure of factoring out $\mathcal{I}[V]$, and does not require any further use of $V \simeq 0$. Instead, equation (3.1.37) yields a recursive relation between λ_n and λ_{n-1} that can be solved given the initial datum $\lambda_0 = 0$. There is also a third method of computing λ_n, namely to reduce (3.1.33) under $\mathfrak{so}(D+1;\mathbb{C}) \to \mathfrak{so}(D;\mathbb{C})$, as we shall discuss in the next Section.

As can be seen already by comparing (3.1.20) and (3.1.21) to (C.0.3) and (C.0.4), the values of the Casimir operators in $\mathcal{A}$ are equal those assumed in the scalar-singleton lowest-weight space $\mathfrak{D}_0 \equiv \mathfrak{D}(\epsilon_0;(0))$ described briefly in Appendix C, and that we shall look at more closely later in this Chapter, *i.e.*

$$C_{2n}[\mathfrak{so}(D+1;\mathbb{C})|\mathcal{A}] \;=\; C_{2n}[\mathfrak{so}(D+1;\mathbb{C})|\epsilon_0;(0)] \;. \tag{3.1.38}$$

In fact, the ideal $\mathcal{I}[V]$ is isomorphic to the scalar-singleton annihilator in $\mathcal{U}$, *i.e.*

$$\mathcal{I}[V] \;\simeq\; \mathcal{I}[\mathfrak{D}_0] \;, \tag{3.1.39}$$

where $\mathcal{I}[\mathfrak{D}]$, for given lowest-weight space $\mathfrak{D}$, is the ideal consisting of all elements in $\mathcal{U}$ that annihilate *all* states in $\mathfrak{D}$. To show this, one first derives the lemma[5] that if X belongs to a tensorial $\mathfrak{so}(D+1;\mathbb{C})$ irrep, then $X \in \mathcal{I}[\mathfrak{D}_0]$ iff $X|\epsilon_0,(0)\rangle = 0$. Next one verifies that $V|\epsilon_0,(0)\rangle = 0$.

As found in [106, 107] (see also [112] for a more recent application in the context of affine extensions of $\mathfrak{so}(D+1;\mathbb{C})$), at the level of lowest-weight spaces, the V_{AB} constraint is by itself sufficient to uniquely select the scalar singleton (and also the spinor singleton in $D=4$), and the V_{ABCD} constraint then follows automatically. In the associative algebra $\mathcal{A}$, on the other hand, which does not refer explicitly to lowest-weight spaces, the values of C_{2n} are instead fixed (to be those of the singleton representation) by combining the V_{AB} *and* V_{ABCD} constraints. The enveloping-algebra construction thus rests on a weaker set of assumptions than the lowest-weight construction, and hence $\mathcal{A}$ has potentially an algebraically richer structure than the space of operators on $\mathfrak{D}_0$, as we shall explore in more detail in Chapter 7.

[5]The scalar singletons $\mathfrak{D}^{\pm}(\pm\epsilon_0;(0))$ and the 4D spinor singleton $\mathfrak{D}^{\pm}(\pm(\epsilon_0+1/2);(1/2))$ are annihilated by the ideal $\mathcal{I}[V]$, as we shall see in the next Section.

3.1.1 $\mathfrak{so}(D;\mathbb{C})$-Covariant Form of the Quotient Algebra

Next, we turn to a $\mathfrak{so}(D;\mathbb{C})$-covariant description of the quotient algebra $\mathcal{A}$. We begin by splitting M_{AB} into $\mathfrak{so}(D;\mathbb{C})$ generators M_{ab} and translations

$$P_a \;=\; M_{0'a}\ , \tag{3.1.40}$$

obeying

$$[M_{ab},M_{cd}]_\star \;=\; 4i\eta_{[c|[b}M_{a]|d]}\ ,\qquad [M_{ab},P_c]_\star \;=\; 2i\eta_{c[b}P_{a]}\ , \tag{3.1.41}$$

$$[P_a,P_b]_\star \;=\; iM_{ab} \tag{3.1.42}$$

(comparing with the general notation of (2.1.42) we are now choosing $L=1$ and $\lambda=1$). By definition, the translations are odd under the automorphism π of $\mathcal{A}$ (and $\mathcal{U}$), *viz.*

$$\pi(P_a) \;=\; -P_a\ ,\qquad \pi(M_{ab}) \;=\; M_{ab}\ ,\qquad \pi(X\star Y) \;=\; \pi(X)\star\pi(Y)\ . \tag{3.1.43}$$

The constraints $V_{AB}\simeq 0$ and $V_{ABCD}\simeq 0$ then decompose into

$$V_{0'0'} \;=\; \tfrac{1}{2}(P^a\star P_a-\mu^2)\ \simeq 0\ , \tag{3.1.44}$$

$$V_{0'a} \;=\; \tfrac{1}{4}\{M_a{}^b,P_b\}_\star\ \simeq\ 0\ , \tag{3.1.45}$$

$$V_{ab} \;=\; \tfrac{1}{2}(M_{(a}{}^c\star M_{b)c}-P_{(a}\star P_{b)}+\mu^2\eta_{ab})\ \simeq 0\ , \tag{3.1.46}$$

$$V_{abcd} \;=\; M_{[ab}\star M_{cd]}\simeq 0\ ,\qquad V_{0'abc} \;=\; -P_{[a}\star M_{bc]} \;=\; 0\ , \tag{3.1.47}$$

where μ^2 is defined in (3.1.17). As shown in Appendix B, the constraints (3.1.45) and (3.1.46) follow from (3.1.44), and (3.1.47) is equivalent to $P_{[a}\star P_b\star P_{c]}\simeq 0$. The value of μ^2 is determined from

$$P^a\star P_{[a}\star P_b\star P_{c]}\ \simeq\ \tfrac{i}{6}(\mu^2-\epsilon_0)M_{bc}\ . \tag{3.1.48}$$

Thus, the ideal $\mathcal{I}[V]$ can be given the Lorentz covariant presentation

$$P^a\star P_a\ \simeq\ \epsilon_0\ ,\qquad P_{[a}\star P_b\star P_{c]}\ \simeq\ 0\ . \tag{3.1.49}$$

and we note the auxiliary trace constraints:

$$P^a\star M_{ab}\ \simeq\ M_{ba}\star P^a\simeq i(\epsilon_0+1)P_b\ , \tag{3.1.50}$$

$$M_{(a}{}^c\star M_{b)c}\ \simeq\ -P_{(a}\star P_{b)}+\epsilon_0\eta_{ab}\ . \tag{3.1.51}$$

Correspondingly, the $\mathfrak{so}(D+1;\mathbb{C})$-covariant expansion of the quotient algebra $\mathcal{A}$ given in (3.1.22) reduces to the following $\mathfrak{so}(D;\mathbb{C})$-covariant expansion

$$X \;=\; \sum_{n\geq m\geq 0} X^{a(n),b(m)}T_{a(n),b(m)}\ , \tag{3.1.52}$$

where $X^{a(n),b(m)}$ are traceless type (n,m) tensors, and

$$T_{a(n),b(m)} \equiv \widehat{T}_{\{a(n),b(m)\}0'(n-m)} = M_{\{a_1b_1}\cdots M_{a_mb_m}P_{a_{m+1}}\cdots P_{a_n\}} \ , \qquad (3.1.53)$$

where $\widehat{T}_{A(n),B(n)}$ are defined in (3.1.23) and the curly brackets indicate traceless type (n,m) projection. We note that $T_{a(n),b(m)}$ is a linear recombination of $\widehat{T}_{a(n),b(m-2k)0'(n-m+2k)}$, $k=0,1,\dots,[m/2]$, of the form[6]

$$T_{a(n),b(m)} = \widehat{T}_{\langle a(n),b(m)\rangle 0'(n-m)} + \sum_{k=1}^{[m/2]}\kappa_{n,m;k}\widehat{T}_{\langle a(n),b(m-2k)0'(n-m+2k)}\eta_{b_1b_2}\cdots\eta_{b_{2k-1}b_{2k}\rangle} \ , \qquad (3.1.54)$$

where the $\langle\cdots\rangle$ indicate type (n,m) Young projection, and the coefficients $\kappa_{n,m;k}$ are fixed by the requirement that $T_{a(n),b(m)}$ be traceless. For example, as shown in Appendix B, the simplest case is given by

$$T_{a(n)} = P_{\{a_1}\cdots P_{a_n\}} \simeq P_{(a_1}\star\cdots\star P_{a_n)} + \kappa_{n,0;1}\eta_{(a_1a_2}P_{a_3}\star\cdots\star P_{a_n)} + \mathcal{O}(\eta^2) \ , \qquad (3.1.55)$$

with

$$\kappa_{n,0;1} = -\frac{(n+1)n(n-1)(n+4\epsilon_0-2)}{48(n+\epsilon_0-\frac{1}{2})} \ . \qquad (3.1.56)$$

The $O(D;\mathbb{C})$-transformations are given by

$$\mathrm{Ad}_{M_{ab}}(T_{c(n),d(m)}) = 2in\eta_{[b|\{c_1}T_{|a]c(n-1),d(m)\}} + 2im\eta_{[b|\{d_1}T_{c(n),|a]d(m-1)\}} \ . \qquad (3.1.57)$$

In Chapter 7 we shall need the explicit form of the anti-commutator

$$\mathrm{Ac}_{P_c}(T_{a(s+k),b(s)}) = 2\Delta_{s+k,s}T_{c\{a(s+k),b(s)\}} + 2\lambda_k^{(s)}\eta_{c\{a}T_{a(s+k-1),b(s)\}} \ , \qquad (3.1.58)$$

with

$$\begin{aligned}\eta_{c\langle a}T_{a(s+k-1),b(s)\rangle} &= \eta_{ca}T_{a(s+k-1),b(s)} + \alpha_{s+k,s}\eta_{a(2)}T_{a(s+k-2)c,b(s)} + \\ &\quad +\beta_{s+k,s}\eta_{a(2)}T_{a(s+k-2)b,cb(s-1)} + \gamma_{s+k,s}\eta_{ab}T_{a(s+k-1),cb(s-1)} \ , \qquad (3.1.59)\end{aligned}$$

where the coefficients

$$\Delta_{s+k,k} = \frac{(k+2)(k+s+1)}{(k+1)(k+s+2)} \ , \qquad (3.1.60)$$

$$\alpha_{s+k,s} = -\frac{1}{2}\frac{s+k-1}{s+k+\epsilon_0-\frac{1}{2}} \ , \qquad (3.1.61)$$

$$\beta_{s+k,s} = \frac{1}{2}\frac{(s+k-1)s}{(s+k+\epsilon_0-\frac{1}{2})(2s+k+2\epsilon_0-1)} \ , \qquad (3.1.62)$$

$$\gamma_{s+k,s} = -\frac{s}{2s+k+2\epsilon_0-1} \ , \qquad (3.1.63)$$

[6]In what follows, it is important that (3.1.53), or (3.1.54), is a strong equality, *i.e.* it holds in $\mathcal{U}$ without the need to remove terms in the ideal $\mathcal{I}[V]$.

are fixed by the traceless type $(s+k,s)$ Young projection (so that $\mathbf{P}(P_c T_{a(s+k),b(s)}) = \Delta_{s+k,s}\mathbf{P}T_{c\{a(s+k),b(s)\}} = \Delta_{s+k,s}T_{ca(s+k),b(s)}$ where $\mathbf{P}\equiv\mathbf{P}_{ca(s+k),b(s)}$), while the coefficient

$$\lambda^{(s)}_k \;=\; \frac{1}{8}\frac{k(k+s+1)(k+2s+2\epsilon_0-1)}{k+s+\epsilon_0+\frac{1}{2}}\ , \tag{3.1.64}$$

can be computed in four ways: i) reducing (3.1.33); ii) solving the trace conditions on (3.1.58) (using (3.1.49), (3.1.50) and (3.1.51)); iii) demanding closure under

$$[\mathrm{Ac}_{P_a},\mathrm{Ac}_{P_b}](T_{c(s+k),d(s)}) \;=\; i\mathrm{Ad}_{M_{ab}}(T_{c(s+k),d(s)})\ ; \tag{3.1.65}$$

or iv) solving the twisted-adjoint Casimir relation (*i.e.* the mass-formula for Weyl tensors) that we shall discuss in Section 3.1.2. The methods (ii) and (iii) are examined in detail in the case of $s=0$ in Appendix B.

In Chapter 7 we shall also need the anti-commutator

$$\mathrm{Ac}_{M_{ab}}(T_{c(n)}) \;=\; 2T_{c(n)[a,b]}+2\rho_n\eta_{[a|\{c_1}T_{c(n-1)\},|b]}\ , \tag{3.1.66}$$

with (suppressing the anti-symmetry on ab)

$$\eta_{a\{c_1}T_{c(n-1)\},b} \;=\; \eta_{ac_1}T_{c(n-1),b}-\frac{1}{4}\frac{n}{n+\epsilon_0-\frac{1}{2}}\eta_{c_1c_2}T_{c(n-2)a,b}\ , \tag{3.1.67}$$

where the coefficient ρ_n is

$$\rho_n \;=\; -\frac{(n-1)n(n+1)}{n+\epsilon_0+\frac{1}{2}}\ , \tag{3.1.68}$$

as shown in Appendix B.

3.1.2 HS Algebras. Adjoint and Twisted-Adjoint Master Fields

We are finally ready to define a HS Lie algebra. The associative algebra $\mathcal{A}$ plays a central role in Vasiliev's frame-like formulation of higher-spin gauge theory: indeed, the vielbein $e=dx^\mu e_\mu{}^a P_a$ and the Lorentz connection $\omega=dx^\mu\omega_\mu{}^{ab}M_{ab}$ can be encoded, together with an infinite tower of higher-spin gauge fields, in a *master one-form* A taking values in the *adjoint representation of an infinite-dimensional higher-spin Lie-algebra* extension of $\mathfrak{so}(D+1;\mathbb{C})$. The minimal extension, that is unique in the sense that $\mathfrak{so}(D+1;\mathbb{C})$ is its maximal finite-dimensional Lie subalgebra, is given by

$$\mathfrak{ho}(D+1;\mathbb{C}) \;=\; \{Q\in\mathcal{A}:\tau(X)\;=\;-X\}\ , \tag{3.1.69}$$

where τ is the anti-automorphism defined in (3.1.7), and with Lie bracket induced by the associative $\star$-product, *viz.*

$$\mathrm{Ad}(Q)(Q') \;=\; [Q,Q']_\star \;=\; Q\star Q' - Q'\star Q\ . \tag{3.1.70}$$

Decomposing $\mathfrak{ho}(D+1;\mathbb{C})$ under $\mathfrak{so}(D+1;\mathbb{C})$, leads to an expansion into *finite-dimensional levels*,

$$\mathfrak{ho}(D+1;\mathbb{C}) \;=\; \bigoplus_{\ell=0,1,2,\dots}^{\infty} \mathcal{L}_\ell\ , \tag{3.1.71}$$

where the ℓth level is spanned by monomials of degree $2\ell+1$, *i.e.*

$$Q_\ell \;=\; Q^{A(2\ell+1),B(2\ell+1)}\widehat{T}_{A(2\ell+1),b(2\ell+1)} \;=\; Q^{A(2\ell+1),B(2\ell+1)}M_{A_1B_1}\star\cdots M_{A_nB_n}\ , \tag{3.1.72}$$

with $T_{A(2\ell+1),b(2\ell+1)}$ defined in (3.1.23), and where $Q^{A(2\ell+1),B(2\ell+1)}$ are traceless type $(2\ell+1,2\ell+1)$ tensors. As expected, the Lie bracket mixes the levels as follows (see [76, 66, 77, 52] and also [101] for a more recent discussion)

$$[Q_\ell,Q_{\ell'}]_\star \;=\; \sum_{\ell''=|\ell-\ell'|}^{\ell+\ell'} Q_{\ell''}\ . \tag{3.1.73}$$

Although, to the best of our knowledge, the explicit form of the structure coefficients in the $\widehat{T}_{A(2\ell+1),B(2\ell+1)}$-basis has not yet been worked out explicitly, this formula shows that the HS algebra constructed above recovers the feature that was long [39, 40, 38] known to characterize HS interactions: as soon as one massless field with spin $s\geq 3$ enters the game, one must introduce infinitely many spins (at least all even integer spins) as required by the closure of the gauge algebra.

The ℓth level decomposes further under $\mathfrak{so}(D;\mathbb{C})$ into traceless type $(s-1,t)$ tensors with $s=2\ell+2$ and $0\leq t\leq 2\ell+1$, so that the minimal adjoint one-form is

$$A \;=\; \sum_{s=2,4,6,\dots} A_{(s)}\ , \tag{3.1.74}$$

$$A_{(s)} \;=\; -i\sum_{t=0}^{s-1} dx^\mu A_{\mu,a(s-1),b(t)}(x)M^{a_1b_1}\cdots M^{a_tb_t}P^{a_{t+1}}\cdots P^{a_{s-1}}\ . \tag{3.1.75}$$

where $A_{(2)}$ contains the vielbein and $\mathfrak{so}(D;\mathbb{C})$ connection,

$$A_{(2)} \;=\; e+\omega \;=\; -i(e^aP_a+\tfrac{1}{2}\omega^{ab}M_{ab})\ . \tag{3.1.76}$$

It is also convenient to collect the higher-spin gauge fields as

$$W \;=\; \sum_{s=4,6,\dots} A_{(s)}\ . \tag{3.1.77}$$

Notice that, needless to say, for every spin-s sector $A_{(s)}$ contains precisely the fields discussed in Section 2.2. We will later see that, starting from Vasiliev's equations, and treating $A_{(2)}$ exactly (assuming $e_\mu{}^a$ to be invertible) and W perturbatively, one can derive a weak-field expansion in which all component gauge fields are auxiliary except the metric

$$g_{\mu\nu} \;=\; e_\mu{}^a e_{\nu a} \;, \tag{3.1.78}$$

and the symmetric rank-s tensor gauge fields

$$\phi_{a(s)} \;=\; (e^{-1})_{(a}{}^\mu W_{\mu,a(s-1))} \;, \qquad s = 4, 6, \ldots \;. \tag{3.1.79}$$

However, the master one-form (7.6.13), with its component field (7.6.14), cannot be the only ingredient in a fully interacting HS gauge theory. Indeed, we have not yet made sure that our candidate HS symmetry algebra satisfies the second requirement, *i.e.*, that it admits a unitary representation that contains all the gauge fields that we have examined in 2.2. As we will show in Section 3.2, massless UIRs of $\mathfrak{so}(D-1,2)$ and of its infinite-dimensional HS extensions necessarily include a *scalar field*, which of course cannot sit in the master one-form, but needs to be accommodated in a zero-form. Moreover, as we shall discuss in detail in the next Chapter, the Vasiliev equations have been written in a certain first-order form (called *unfolded* formulation) in which a zero-form, transforming in a peculiar representation of the $\mathfrak{so}(D-1,2)$ algebra and its infinite-dimensional HS extensions, plays a crucial role. Therefore, aside from the adjoint master one-form A, the admissibility criterion and the unfolded formulation[7] require a *master zero-form* Φ taking values in a *twisted-adjoint representation of the higher-spin Lie algebra*. The minimal twisted-adjoint representation is given by

$$\mathcal{T}(D+1;\mathbb{C}) \;=\; \{S \in \mathcal{A} : \tau(S) \;=\; \pi(S)\} \;, \;, \tag{3.1.80}$$

where π is the $\mathcal{A}$-automorphism defined in (3.1.43), and the higher-spin representation is defined by

$$\widetilde{\mathrm{Ad}}_Q(S) \;=\; [Q,S]_\pi \;=\; Q \star S - S \star \pi(Q) \;. \tag{3.1.81}$$

The twisted-adjoint representation decomposes under $\mathfrak{so}(D+1;\mathbb{C})$ into *infinite-dimensional levels*,

$$\mathcal{T}(D+1;\mathbb{C}) \;=\; \bigoplus_{\ell=-1,0,2,\ldots} \mathcal{T}_\ell \;, \tag{3.1.82}$$

spanned by $O(D;\mathbb{C})$-covariant elements, *viz.*

$$S_\ell \;=\; \bigoplus_{k=0}^{\infty} S^{a(s+k),b(s)} T_{a(s+k),b(s)} \;, \qquad s \equiv 2\ell + 2 \;, \tag{3.1.83}$$

[7] Interestingly, in a recent paper [101] it has been proposed that full higher-spin dynamics in even space-time dimensions can be induced by starting from a Chern-Simons-like theory in one higher dimension based on an adjoint one-form only.

where $S^{a(s+k),b(s)} \in \mathbb{C}$ and $T_{a(s+k),b(s)}$ is given by (3.1.53). The twisted-adjoint transformations mixes the levels as follows:

$$\widetilde{\mathrm{Ad}}_{Q_\ell}(S_{\ell'}) = \sum_{\ell''=\max(-1,\ell'-\ell)}^{\ell+\ell'} S_{\ell''} , \tag{3.1.84}$$

where the higher bound on ℓ'' follows immediately, while the the lower bound follows from the contraction rules (3.1.44)-(3.1.47).

The expansion of the minimal twisted-adjoint zero-form reads

$$\Phi = \sum_{s=0,2,4,...} \Phi_{(s)} , \tag{3.1.85}$$

$$\Phi_{(s)} = \sum_{k=0}^{\infty} \frac{i^k}{k!} \Phi^{a(s+k),b(s)}(x) M_{a_1 b_1} \cdots M_{a_s b_s} P_{a_{s+1}} \cdots P_{a_{s+k}} . \tag{3.1.86}$$

As we shall see, in the above-mentioned weak-field expansion of the Vasiliev equations, the component fields $\Phi_{a(s),b(s)}$ become *generalized spin-s Weyl tensors* for $s = 2, 4, \ldots$, and a *physical scalar* for $s = 0$,

$$\phi = \Phi |_{P_a = M_{ab} = 0} , \tag{3.1.87}$$

while $\Phi_{a(s+k),b(s)}$ for $k = 1, 2, \ldots$ become auxiliary fields, given by the kth derivatives of $\Phi_{a(s),b(s)}$ on-shell.

There are many ways to extend the minimal model. In some sense, the simplest extension is to add all odd spins $s = 1, 3, 5, ...$, *i.e.* half-integer levels $\ell = -1/2, 1/2, 3/2, \ldots$, leading to the (non-minimal) adjoint and twisted-adjoint modules

$$\mathfrak{ho}_1(D+1;\mathbb{C}) = \mathcal{A}, \tag{3.1.88}$$

$$\mathcal{T}_1(D+1;\mathbb{C}) = \mathcal{A} , \tag{3.1.89}$$

with representations given by (3.1.70) and (3.1.81), respectively. These modules decompose under $\mathfrak{ho}(D+1;\mathbb{C})$ into

$$\mathfrak{ho}_1(D+1;\mathbb{C}) = \mathcal{A}_- \oplus_s \mathcal{A}_+ , \qquad \mathcal{A}_+ \equiv \mathfrak{ho}(D+1;\mathbb{C}) , \tag{3.1.90}$$

$$\mathcal{T}_1(D+1;\mathbb{C}) = \mathcal{T}_+(D+1;\mathbb{C}) \oplus_{\mathfrak{ho}} \mathcal{T}_-(D+1;\mathbb{C}) , \qquad \mathcal{T}_+(D+1;\mathbb{C}) \equiv \mathcal{T}(D+1;\mathbb{C}), \tag{3.1.91}$$

where $\oplus_s$ denotes the semi-direct sum, $\mathcal{A}_\pm = \{Q \in \mathcal{A} : \tau(Q) = \mp Q\}$ and $\mathcal{T}_\pm(D+1;\mathbb{C}) = \{S \in \mathcal{A} : \tau(S) = \pm\pi(S)\}$. The spaces $\mathcal{A}_\pm$ and $\mathcal{T}_\pm(D+1;\mathbb{C})$ contain the integer (+) and half-integer (−) adjoint and twisted-adjoint levels, respectively, associated to gauge fields and Weyl tensors with even (+) and odd (−) spins, and the representations are given by $[\mathcal{A}_\pm, \mathcal{A}_\sigma]_\star = \mathcal{A}_{\pm\sigma}$ and $[\mathcal{A}_\pm, T_\sigma(D+1;\mathbb{C})]_\pi = T_{\pm\sigma}(D+1;\mathbb{C})$ for $\sigma = \pm$.

3.2 Lowest-Weight and Highest-Weight Representations of $\mathfrak{so}(D+1;\mathbb{C})$

In this Section we recall the construction and the main features of the representations of the $\mathfrak{so}(D+1;\mathbb{C})$ algebra that will be of interest in the following. In particular, we want to focus on the massless irreducible representations of that algebra, *i.e.*, on those irreps that describe massless fields in nonflat maximally symmetric space-times, as they will also be irreps of the HS extension of their isometry algebras. All the representations that will be of interest to us in this Chapter are lowest-weight (or highest-weight) representations, where the energy operator $E = P_0 = M_{0'0}$ is bounded from below (or above) and the energy levels consist of finitely many spins (*i.e.* tensorial or tensor-spinorial representations of the $\mathfrak{so}(D-1;\mathbb{C})$-subalgebra generated by M_{rs}). Among these one finds finite-dimensional tensorial and tensor-spinorial representations, which arise as invariant subspaces containing both lowest-weight and highest-weight states, as well as infinite-dimensional representations arising in harmonic analysis of linearized field equations on maximally symmetric spaces (of various signatures) with nonvanishing cosmological constant. As we shall see in Chapter 7, the linearized fields also contain lowest-spin modules, which contain neither highest-weight nor lowest-weight states, although for fixed spin they consist of finitely many energy levels.

We shall first characterize finite-dimensional and infinite-dimensional highest and lowest representations of the complex algebra $\mathfrak{so}(D+1;\mathbb{C})$. The choices of real forms and related unitarity issues will be discussed in Subsection 3.2.3.

In the standard basis, the $\mathfrak{so}(D+1;\mathbb{C})$ commutation rules read

$$[M_{RS}, M_{TU}]_\star \;=\; 4i\delta_{[T|[S}M_{R]|U]} \;, \tag{3.2.1}$$

where $R = 1, \ldots, D+1$ and

$$\delta_{RS} \;=\; \mathrm{diag}(+\cdots+) \;. \tag{3.2.2}$$

Representations of $\mathfrak{so}(D+1;\mathbb{C})$ can be described starting from left modules consisting of eigenstates of the "diagonal" generators

$$(M_{D+3-2k,D+2-2k} - \lambda_k) \mid (\lambda_1, \ldots, \lambda_\nu)\rangle \;=\; 0 \;, \qquad k = 1, \ldots, \nu = [(D+1)/2] \;, \tag{3.2.3}$$

on which the remaining generators act as suitable raising and lowering operators. A particular class of representations are the *highest weight representations*. These arise assuming the existence of a highest weight state $|(n_1, \ldots, n_\nu)\rangle$ annihilated by all raising operators. From it, the lowering operators generate a module $\mathfrak{V}(n_1, \ldots, n_\nu)$, known as the *Verma module*. For generic values of $(n_1, \ldots, n_\nu)$, it is irreducible (and hence infinite-dimensional). However, for special values, it contains at least one excited state, referred to as a singular vector, that is annihilated by all the raising operators. The singular vectors generate an invariant submodule $\mathfrak{N}(n_1,\ldots,n_\nu)$, and as a result the highest weight representation is now defined as the quotient

$$\mathfrak{D}(n_1,\ldots,n_\nu) \;=\; \frac{\mathfrak{V}(n_1,\ldots,n_\nu)}{\mathfrak{N}(n_1,\ldots,n_\nu)}\ , \tag{3.2.4}$$

which is irreducible, and infinite-dimensional or finite-dimensional depending on $(n_1,\ldots,n_\nu)$. The finite-dimensional *tensorial representations* arise for integer highest weights obeying

$$n_1 \geq n_2 \geq \cdots \geq n_\nu \geq 0\ , \tag{3.2.5}$$

corresponding to an $\mathfrak{so}(D+1;\mathbb{C})$ tensor with symmetry properties given by the Young projection corresponding to the diagram with n_i cells in the ith row. In these representations, the repeated action of any lowering operator on any state sooner or later generates states in $\mathfrak{N}(n_1,\ldots,n_\nu)$. Thus, there exists a lowest weight state, actually given by $|\,(-n_1,-n_2,\ldots,-n_\nu)\rangle$, that is annihilated by all lowering operators. Thus, the finite-dimensional representations are *highest and lowest weight* spaces, and they are invariant under the mirror reflection $\lambda_1 \to -\lambda_1$.

Infinite-dimensional representations of interest for Field Theory in a D-dimensional space-time arise in the case that

$$\mathbf{s}_0 \;=\; (n_2,\ldots,n_\nu)\ , \tag{3.2.6}$$

remains an integer (or half-integer) highest weight, while

$$e_0 \;=\; -n_1 \tag{3.2.7}$$

becomes a sufficiently large positive number (for fixed $\mathbf{s}_0$). Then $M_{D+1,D}$ becomes unbounded from below, *i.e.* $\mathfrak{D}(n_1,\ldots,n_\nu)$ no longer contains any lowest weight state, and thereby becomes infinite-dimensional. The mirror reflection $\lambda_1 \to -\lambda_1$ now sends the highest weight space $\mathfrak{D}(n_1,\ldots,n_\nu)$ to an infinite-dimensional lowest weight space, denoted by $\mathfrak{D}^-(-n_1,\ldots,n_\nu)$. Thus, identifying

$$E \;=\; -M_{D+1,D}\ , \tag{3.2.8}$$

as the field theory Hamiltonian, and the generators

$$M_{rs}\ , \qquad r,s = 1,\ldots,D-1\ , \tag{3.2.9}$$

as the orbital plus internal angular momenta, the highest weight $|(-e_0,n_2,\ldots,n_\nu)\rangle$ becomes a ground state, or lowest energy state, with energy e_0 and spin $\mathbf{s}_0$. It is convenient to "Wick-rotate" the two spatial directions $D+1$ and D into two time-like directions, that we shall denote by 0 and $0'$, and describe the weight space starting from the commutation rules of $\mathfrak{so}(D+1;\mathbb{C})$ in the "two-time" basis (3.2.1). The energy operator therefore is

$$E \;=\; M_{0'0} \;=\; P_0 \;, \tag{3.2.10}$$

and the energy raising and lowering operators are identified with

$$L_r^{\pm} \;=\; M_{0r} \mp iM_{0'r} \;=\; M_{or} \mp iP_r \;, \tag{3.2.11}$$

leading to the following energy graded decomposition of the commutation rules (3.2.1):

$$[L_r^-, L_s^+] \;=\; 2iM_{rs} + 2\delta_{rs}E \;, \tag{3.2.12}$$

$$[E, L_r^{\pm}] \;=\; \pm L_r^{\pm} \;, \tag{3.2.13}$$

$$[M_{rs}, M_{tu}] \;=\; 4i\delta_{[t|[s}M_{r]|u]} \;, \tag{3.2.14}$$

$$[M_{rs}, L_t^{\pm}] \;=\; 2i\delta_{t[s}L_{r]}^{\pm} \;. \tag{3.2.15}$$

The highest weight state $|(n_1,\ldots,n_\nu)\rangle$ of the standard basis is a lowest weight state of the two-time basis, and vice versa, and in order to avoid confusion we shall use the notation

$$|e_0;\mathbf{s}_0\rangle \;\equiv\; |(-e_0, n_2,\ldots,n_\nu)\rangle \;. \tag{3.2.16}$$

In order to accommodate also the negative energy states (resulting from the reflections in weight space), one also needs to define highest weight states. Thus, in general, we have *highest weight states* (+) and *lowest weight states* (−), obeying

$$(E-e_0)|e_0;\mathbf{s}_0\rangle^{\pm} \;=\; 0 \;, \qquad L_r^{\mp}|e_0;\mathbf{s}_0\rangle^{\pm} \;=\; 0 \;. \tag{3.2.17}$$

Moreover, since $\mathbf{s}_0$ is, by assumption, a positive integer highest weight of $\mathfrak{so}(D-1;\mathbb{C})$, one may, without loss of generality, replace the original Verma module $\mathfrak{V}^{\pm}(-e_0,\mathbf{s}_0)$ by the *generalized Verma module* defined by

$$\mathfrak{C}(e_0;\mathbf{s}_0)^{\pm} \;=\; \frac{\mathfrak{V}^{\pm}(-e_0,\mathbf{s}_0)}{\mathfrak{N}[\mathfrak{so}(D-1;\mathbb{C})]} \;, \tag{3.2.18}$$

where $\mathfrak{N}[\mathfrak{so}(D-1;\mathbb{C})]$ consists of all $\mathfrak{so}(D-1;\mathbb{C})$ modules generated from states that are singular with respect to $\mathfrak{so}(D-1;\mathbb{C})$. In other words, the generalized Verma module is a particular example of a *Harish-Chandra module*[8], with energies bounded from below by

[8]Given a Lie algebra $\mathfrak{g}$, the definition of a Harish-Chandra module is a more general one, which does not a priori involve any highest or lowest-weight state, but only a certain slicing of an infinite-dimensional irreducible $\mathfrak{g}$-module $\mathfrak{R}$. In particular, the slicing $\mathfrak{R}|_{\mathfrak{h}}$ of $\mathfrak{R}$ under a subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is said to be *admissible* if it contains only finite-dimensional $\mathfrak{h}$ irreps, sometimes referred to as $\mathfrak{h}$-types, with finite multiplicities, *viz.*

$$\mathfrak{R}|_{\mathfrak{h}} \;=\; \bigoplus_{\kappa}\mathrm{mult}(\kappa)\mathfrak{R}_\kappa \;, \qquad \mathrm{mult}(\kappa)\dim\mathfrak{R}_\kappa < \infty \;, \tag{3.2.19}$$

where κ are referred to as the *compact weights* of $\mathfrak{h}$. If all $\mathfrak{h}$-types are generated by the universal enveloping algebra $\mathcal{U}[\mathfrak{g}]$ of $\mathfrak{g}$ starting from a finite number of $\mathfrak{h}$-types then $\mathfrak{R}$ is referred to as a *Harish-Chandra module* (see [134] and the general treatise [136] for more details).

e_0 (+) or above (-) by $-e_0$, consisting of all states that are generated by the action of (only) the $L_r^\pm$ operators on $|e_0;\mathbf{s}_0\rangle^\pm$, *i.e.*

$$\mathfrak{C}^\pm(e_0;\mathbf{s}_0) \;=\; \mathrm{span}_{\mathbb{C}}\left\{L^\pm_{r_1}\cdots L^\pm_{r_n}|e_0;\mathbf{s}_0\rangle^\pm\right\}_{n=0}^\infty \;. \tag{3.2.20}$$

One refers to $\mathfrak{so}(2;\mathbb{C})_E\oplus\mathfrak{so}(D-1;\mathbb{C})_{M_{rs}}$ as the *compact subalgebra*, and to the corresponding basis of $\mathfrak{C}^\pm(e_0;\mathbf{s}_0)$, consisting of states $\{|e;\mathbf{s}\rangle^\pm\}$ labeled by energy eigenvalues e and spins $\mathbf{s}$, as the *compact basis.* The various weights of a Harish-Chandra module, that result from extracting the $\mathfrak{so}(D-1;\mathbb{C})$-irreducible parts from the states in (3.2.20), can therefore be represented as discrete dots filling a wedge (and its negative energy reflection) in the ($\mathfrak{so}(D-1;\mathbb{C})$-)spin/energy plane. For example, concentrating our attention on the lowest weight modules and starting, for definiteness, from a lowest weight state $|e_0;(s_0,0,...,0)\rangle_{t(s_0)}$, the first excited energy level consists of

$$L_r^+|e_0;\mathbf{s}_0\rangle^+_{t(s_0)} \;=\; \left\{\begin{array}{c} |e_0+1;(s_0+1,0,...,0)\rangle^+_{rt(s_0)} = L^+_{\{r}|e_0;(s_0,0,...,0)\rangle^+_{t(s_0)\}}\;, \\ |e_0+1;(s_0,1,0,...,0)\rangle^+_{t(s_0),r} = L^+_{\{r}|e_0;(s_0,0,...,0)\rangle^+_{t(s_0)\}}\;, \\ |e_0+1;(s_0-1,0,...,0)\rangle^+_{t(s_0-1)} = L^+_r|e_0;\mathbf{s}_0\rangle^+_{rt(s_0-1)}\;, \end{array}\right. \tag{3.2.21}$$

where the brackets $\{...\}$ embrace the indices that are Young projected and traceless according to the various projections indicated in the middle terms of the equality above.

In general, the Harish-Chandra module $\mathfrak{C}^\pm(e_0;\mathbf{s}_0)$ is not irreducible, as it may contain singular vectors, *i.e.* excited states $|e_0';\mathbf{s}_0'\rangle^\pm$ with $e_0'>e_0$ that are annihilated by $L_r^\mp$,

$$L_r^\mp|e_0';\mathbf{s}_0'\rangle^\pm \;=\; 0\;. \tag{3.2.22}$$

This can happen, for instance, when certain relations are imposed between e_0 and $\mathbf{s}_0$. In a definite signature such a constraint can arise from the requirement of unitarity, as we shall see. It is clear from the commutation relations and the definition of Harish-Chandra modules, however, that for $e_0\gg s_0$ there cannot be singular vectors $|e_0';\mathbf{s}_0'\rangle^\pm$: indeed, by construction excited states have $e_0'-\mathbf{s}_0'\geq e_0-s_0\gg 1$, and, schematically,

$$L_r^-|e_0';\mathbf{s}_0'\rangle^+ \;=\; L^+...L^+(2iM+2E)L^+...L^+|e_0;\mathbf{s}_0\rangle^+ + ... \tag{3.2.23}$$

where the commutation relation (3.2.12) has been used and the ellipsis stand for other similar terms with all possible powers of L^+ on the left and on the right of $2iM+2E$. But if $e_0\gg s_0$, the action of E and M_{rs} can only extract strictly positive eigenvalues, and there is no chance that (3.2.22) be verified. On the other hand, this argument shows that lowering e_0 to some critical value $e_{0,\mathrm{crit}}=e_{0,\mathrm{crit}}(\mathbf{s}_0)$ a singular vector may appear. Similarly (interchanging $-$ and $+$) for highest-weight modules.

The singular vectors generate Harish-Chandra submodules

$$\mathfrak{C}^\pm(e_0';\mathbf{s}_0') \;=\; \mathrm{span}_{\mathbb{C}}\left\{L^\pm_{r_1}\cdots L^\pm_{r_n}|e_0';\mathbf{s}_0'\rangle^\pm\right\}_{n=0}^\infty \;, \tag{3.2.24}$$

and $\mathfrak{C}^{\pm}(e_0;\mathbf{s}_0)$, that contains them, is therefore an *indecomposable* module: that is to say, the action of the noncompact generators $L_r^{\pm}$ can bring from $\mathfrak{C}^{\pm}(e_0;\mathbf{s}_0)$ to the inside of the singular submodule, but not back out of the latter (because of (3.2.22)). This implies that the direct sum of such submodules is an ideal $\mathfrak{I}^{\pm}(e_0;\mathbf{s}_0) \subset \mathfrak{C}^{\pm}(e_0;\mathbf{s}_0)$ that can be factored out consistently, leaving the *irreducible* lowest and highest weight spaces

$$\mathfrak{D}^{\pm}(e_0;\mathbf{s}_0) \;=\; \frac{\mathfrak{C}^{\pm}(e_0;\mathbf{s}_0)}{\mathfrak{I}^{\pm}(e_0;\mathbf{s}_0)} \;. \tag{3.2.25}$$

The indecomposable structure can be also be presented by making use of the semi-direct sum symbol $\oplus_s$, as $\mathfrak{C}^{\pm}(e_0;\mathbf{s}_0) = \mathfrak{I}^{\pm}(e_0;\mathbf{s}_0) \oplus_s \mathfrak{D}^{\pm}(e_0;\mathbf{s}_0)$. The lowest and highest weight spaces are isomorphic, namely

$$\mathfrak{D}^{\mp}(e_0;\mathbf{s}_0) \;=\; \pi(\mathfrak{D}^{\pm}(-e_0;\mathbf{s}_0)) \;, \tag{3.2.26}$$

where π is defined by acting over $\mathfrak{so}(D+1;\mathbb{C})$ as the automorphism

$$\pi(L_r^{\pm}) \;=\; L_r^{\mp} \;, \qquad \pi(E) \;=\; -E \;, \qquad \pi(M_{rs}) \;=\; M_{rs} \;, \tag{3.2.27}$$

and on lowest and highest weight states as the reflection

$$\pi(|e_0,\mathbf{s}_0\rangle^{\pm}) \;=\; |-e_0,s_0\rangle^{\mp} \;. \tag{3.2.28}$$

Of importance for our constructions is also the existence of the linear anti-automorphism τ, defined by

$$\tau(M_{AB}) \;=\; -M_{AB} \;, \tag{3.2.29}$$

and

$$\tau(|e_0,\mathbf{s}_0\rangle^{\pm}) \;=\; \varphi(e_0,\mathbf{s}_0)\,{}^{\mp}\langle -e_0,\mathbf{s}_0| \;, \tag{3.2.30}$$

where ${}^{\pm}\langle e_0,\mathbf{s}_0|$ are the ground states of the dual weight spaces $(\mathfrak{D}^{\pm}(e_0,\mathbf{s}_0))^{\star}$, and $\varphi(e_0,\mathbf{s}_0)$ is a phase factor. Lowest weight states and spaces will sometimes be written without the $+$ superscript, and in that case the highest weight dittos will be denoted by a tilde instead of the $-$ superscript.

After this general description, we now turn to examine some important examples.

First, let us exemplify the lowest weight description of the $\mathfrak{so}(D+1;\mathbb{C})$ vector $\Delta(1,0) = \mathfrak{D}(-1;(0))$. The singular vector is $L^+_{\{r}L^+_{s\}}|-1;(0)\rangle$, and $\mathfrak{I}(-1;(0)) \simeq \mathfrak{C}(1;(2))$ contains all states with energy $e \geq 2$, since

$$L_r^- L_s^+ L_t^+ L_t^+ |-1;(0)\rangle \;=\; -6L^+_{\{r}L^+_{s\}}|-1;(0)\rangle \;, \tag{3.2.31}$$

leaving $\mathfrak{D}(-1,0)$ consisting of $|-1;(0)\rangle$, $|0;(1)\rangle = L_r^+|-1;(0)\rangle$ and $|1;(0)\rangle = L_r^+L_r^+|-1;(0)\rangle$. We note that $L_r^+|1;(0)\rangle \in \mathfrak{I}(-1;(0))$ so that $L_r^+|1;(0)\rangle \simeq 0$ in $\mathfrak{D}(-1;(0))$, reflecting the fact that the finite-dimensional representations contain lowest as well as highest weights.

Of the infinite-dimensional cases, those of main interest to us are the singleton and massless representations,

$$\text{scalar and spinor singletons} \;:\; e_0 = s_0+\epsilon_0\ , \qquad s_0 = 0,\tfrac{1}{2}\ , \tag{3.2.32}$$

$$\text{massless particles} \;:\; e_0 = s_0+2\epsilon_0\ , \qquad s_0 = 1,\tfrac{3}{2},2,\tfrac{5}{2},\dots\ , \tag{3.2.33}$$

where $\epsilon_0=(D-3)/2$, and the singular vectors are

$$\text{Singletons } s_0=0: \qquad |\epsilon_0+2;(0)\rangle = L_r^+L_r^+|\epsilon_0;(0)\rangle\ , \tag{3.2.34}$$

$$s_0=\tfrac{1}{2}: \qquad |\epsilon_0+\tfrac{3}{2};(\tfrac{1}{2})\rangle_\alpha = (\gamma_r)_\alpha{}^\beta L_r^+|\epsilon_0+\tfrac{1}{2};(\tfrac{1}{2})\rangle_\beta \tag{3.2.35}$$

$$\text{Massless } s_0=1,2,\dots: \qquad |s_0+2e_0+1;(s_0-1)\rangle_{r_1\dots r_{s_0-1}} = L_r^+|s_0+2e_0;(s_0)\rangle_{rr_1\dots r_{s_0-1}}\ , \tag{3.2.36}$$

$$s_0=\tfrac{3}{2},\tfrac{5}{2},\dots: \qquad |s_0+2e_0+1;(s_0-1)\rangle_{\alpha r_1\dots r_{s_0-3/2}} = L_r^+|e_0;(s_0)\rangle_{\alpha rr_1\dots r_{s_0-3/2}}\ , \tag{3.2.37}$$

with γ_r given by $\mathfrak{so}(D-1)$ Dirac matrices. The $\mathfrak{so}(D-1;\mathbb{C})$ representations of the ground states are given by

$$M_{rs}|e_0;(s_0)\rangle^\pm_{t_1\dots t_{s_0}} = 2is_0\delta_{s\{t_1}|e_0;(s_0)\rangle^\pm_{t_2\dots t_{s_0}\}r}\ , \qquad s_0 = 0,1,2,\dots\ , \tag{3.2.38}$$

$$M_{rs}|e_0;(s_0)\rangle^\pm_{\alpha,t_1\dots t_{s_0-1/2}} = 2i(s_0-1/2)\delta_{s\{t_1}|e_0;(s_0)\rangle^\pm_{t_2\dots t_{s_0-1/2}\}r} \tag{3.2.39}$$

$$- \tfrac{i}{2}(\gamma_{rs})_\alpha{}^\beta|e_0;(s_0)\rangle^\pm_{\beta,t_1\dots t_{s_0-1/2}}\ , \qquad s_0 = \tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\dots\ , \tag{3.2.40}$$

where the curly brackets indicate the symmetric and traceless projection on $t_1\dots t_{s_0}$, and $\gamma_{rs}=\gamma_{[r}\gamma_{s]}$. The phase factors in (3.2.30) are

$$\tau(|\pm\epsilon_0;(0)\rangle^\pm) = {}^\mp\langle\mp\epsilon_0;(0)|\ , \tag{3.2.41}$$

$$\tau(|\pm(\epsilon_0+\tfrac{1}{2});(1/2)\rangle^\pm) = i\ {}^\mp\langle\mp(\epsilon_0+\tfrac{1}{2});(1/2)|\ , \tag{3.2.42}$$

$$\tau(|\pm(s_0+2\epsilon_0);(s_0)\rangle^\pm) = (-)^{s_0}\ {}^\mp\langle\mp(s_0+2\epsilon_0);(s_0)|\ , \qquad s_0=1,2,\dots\ , \tag{3.2.43}$$

$$\tau(|\pm(s_0+2\epsilon_0);(s_0)\rangle^\pm) = (-)^{s_0-1/2}i\ {}^\mp\langle\mp(s_0+2\epsilon_0);(s_0)|\ , \qquad s_0=1/2,3/2,\dots\ , \tag{3.2.44}$$

and the action of τ on dual states is fixed by demanding that

$$\tau^2 = \begin{cases} \mathrm{Id} & \text{integer spin} \\ -\mathrm{Id} & \text{half-integer spin} \end{cases}\ . \tag{3.2.45}$$

We now devote a more detailed discussion to the infinite-dimensional representations that will be of importance in the following.

3.2.1 Massless Irreducible Representations

Even a proper definition of "masslessness" in maximally symmetric spaces with nonvanishing cosmological constant is nontrivial. All sensible definitions have in common the feature that the massless representation should correspond, in the flat limit $\Lambda \to 0$, to a massless representation of the Poincaré algebra, but this does not fix uniquely the notion of masslessness, and additional conditions have to be introduced (see, for example, [107] and references therein). Stronger definitions of masslessness correspond to the concepts of *conformal masslessness* and *composite masslessness*: the first is related to the property of unique extension from representations of $\mathfrak{so}(D+1;\mathbb{C})$ to a *singleton* representation of the conformal group [108], while the second characterizes massless particles as composites of two singletons [31], as we shall review in the next Subsection. The two notions coincide only in $D = 3, 4$.

The latter definition is quite natural as it implies that the appearance and factorization of the singular ideal $\mathfrak{I}^{\pm}(e_0; \mathbf{s}_0)$ corresponds, for $s_0 \geq 1$ to the appearance of a gauge symmetry and elimination of gauge modes. Let us show how this happens for the critical value $e_{0,\mathrm{crit}} = s_0 + 2\epsilon_0$ mentioned above, henceforth restricting our attention, for simplicity, to the case of totally symmetric representations $\mathfrak{D}(e_0; (s_0, 0, ..., 0)) \equiv \mathfrak{D}(e_0; s_0)$ and to integer spins only.

Indeed, for $e_{0,\mathrm{crit}} = s_0 + 2\epsilon_0$ and $s_0 > 1$ a singular vector appears at the first excited level, with quantum numbers

$$|e_0'; s_0'\rangle_{r(s_0')} \;=\; |e_0 + 1; s_0 - 1\rangle_{r(s_0-1)} \;=\; L_t^+ |e_0; s_0\rangle_{tr(s_0-1)} \;. \tag{3.2.46}$$

This simply follows from the assumption that $|e_0; s_0\rangle$ is a lowest weight state and from the algebra,

$$\begin{aligned} L_q^- (L_t^+ |e_0; s_0\rangle_{tr(s_0-1)}) \;&=\; (2iM_{qt} + 2\delta_{qt} E)|e_0; s_0\rangle_{tr(s_0-1)} \\ =\; 2(e_0 - s_0 - 2\epsilon_0)|e_0; s_0\rangle_{tr(s_0-1)} \;&=\; 0 \qquad \text{for} \quad e_0 = s_0 + 2\epsilon_0 \;, \end{aligned} \tag{3.2.47}$$

as can be easily checked recalling also that $|e_0; s_0\rangle_{ttr(s_0-2)} = 0$. The whole Harish-Chandra module $\mathfrak{D}(s_0 + 1; s_0 - 1)$ built on top of the singular vector, and that encodes the gauge modes of the massless field of spin s_0 decouples completely from the rest of the representation, and can therefore be consistently factored out. The leftover irreducible weight space (illustrated in Figure 3.1 for the case of the massless spin-1 field in four dimensions) is formed by states that are "divergence-free" ($L_p^+ |e; s\rangle_{pr(s-1),q(t)} = L_p^+ |e; s\rangle_{r(s),pq(t-1)} = 0$) and where each dot has multiplicity one[9] [110].

The elements of the Harish-Chandra modules (3.2.20) can be identified with the modes

[9]In the sense of the Harish-Chandra module, of course, where every weight always carries a finite-dimensional irrep of $\mathfrak{so}(D-1;\mathbb{C})$.

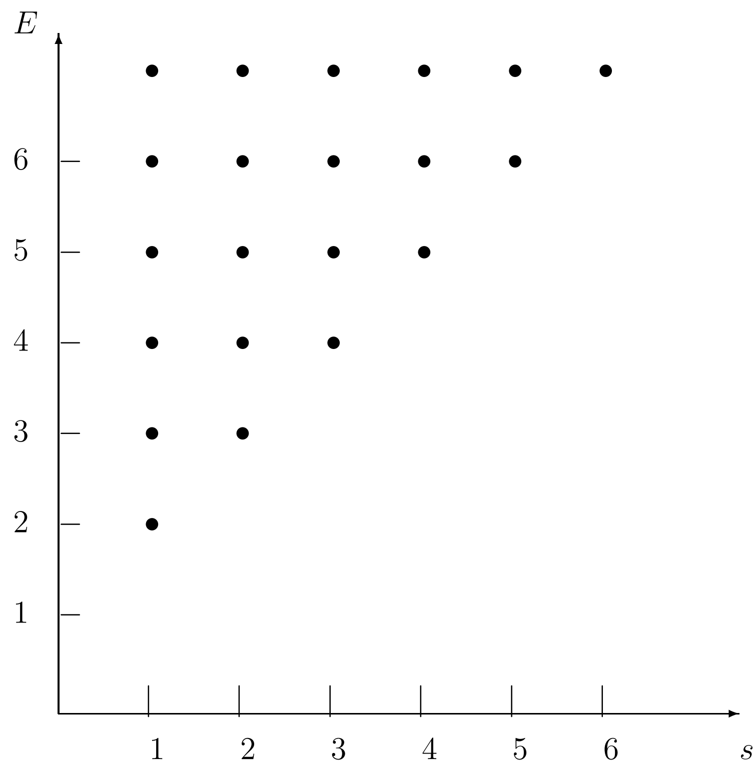


Figure 3.1: Weight diagram of the massless UIR of spin-1 $\mathfrak{D}(2,1)$ in four dimensions.

of a free quantum one-particle state (or anti-particle state)[10] on a maximally symmetric space with nonvanishing cosmological constant. The corresponding field equations for general signature can be obtained describing such space as the coset $\frac{\mathfrak{so}(D+1;\mathbb{C})}{\mathfrak{so}(D;\mathbb{C})}$, that, in the various signatures of interest, gives rise to the manifolds shown in (2.1.39). The D'Alembertian operator is related to the difference of the quadratic Casimir operators $C_2[\mathfrak{so}(D+1;\mathbb{C})]$ and $C_2[\mathfrak{so}(D;\mathbb{C})]$. For a general signature Σ of η_{AB} and Σ' of η_{ab}, and in the notation of Section 2.1.1, this amounts to

$$\lambda^2 L^2 \nabla^a \nabla^b \eta_{ab} \;=\; C_2[\mathfrak{so}(\Sigma)] - C_2[\mathfrak{so}(\Sigma')] \;, \tag{3.2.48}$$

which characterizes the mass-shell condition as

$$\lambda^2 \tau^2 L^2 m^2 \;=\; C_2[\mathfrak{so}(\Sigma)] - C_2[\mathfrak{so}(\Sigma')] \;, \tag{3.2.49}$$

[10]Strictly speaking, such a terminology should be applied to the elements of *unitary* modules, that we shall examine later. With such a proviso, we shall anyway extend it to every state of a Harish-Chandra module.

where m is the mass of a spin-s particle in the nonflat maximally symmetric background at hand (that includes also the mass-like term that originates from the coupling to the background curvature) and we have temporarily reinstated, for clarity, the factors of the radius of curvature L^2 elsewhere taken equal to one. In the compact basis and in lowest or highest weight states $\mathfrak{D}^{\pm}(e_0; \mathbf{s}_0)$, one can evaluate the Casimir operator of $\mathfrak{so}(D+1;\mathbb{C})$ as in (C.0.3), obtaining

$$m^2 \ = \ e_0(e_0 \mp (D-1)) + C_2[\mathfrak{so}(D-1;\mathbb{C})|(\mathbf{s}_0)] - C_2[\mathfrak{so}(D;\mathbb{C})|(\mathbf{s}_0)_D] \ , \qquad (3.2.50)$$

where $(\mathbf{s}_0)_D$ is an $\mathfrak{so}(D;\mathbb{C})$-irrep. One can check from here that the value m^2 of the mass-like term, obtained in Section 2.1.2 as the one preserving the gauge invariance of the Fronsdal equations, corresponds to the one obtained from (3.2.50) at the critical energy $e_0 = s_0 + 2\epsilon_0$ of the massless lowest weight representations $\mathfrak{D}(s_0 + 2\epsilon_0; s_0)$. Notice however that, in general, (3.2.50) is a quadratic equation for e_0, and admits therefore two roots. When there are two real roots, they correspond to two different solutions of the field equations with different boundary conditions at spatial infinity[11]. For fixed spin, such "conjugate" representations $\mathfrak{D}(E_0; s_0)$ are those with the same value of the Casimir operator but different values of energy: from (C.0.1) it is clear that these must have $E_0 = D - 1 - e_0$ (here for lowest weight representations). As we shall see, unitarity introduces some other bound on the value of e_0 at fixed s_0, and in general this rules out one of the two solutions as nonunitary (the one with lower energy). One important exception that we shall encounter is the scalar field in four dimension, that comes in two varieties, $\mathfrak{D}(1,0)$ and $\mathfrak{D}(2,0)$, related to Neumann and Dirichlet boundary conditions, respectively, and both unitary.

Not having any associated gauge symmetries, a massless scalar can only be defined in accordance to the criteria of conformal or composite masslessness. As we shall see, the latter leads, in general, to a scalar representation $\mathfrak{D}(2\epsilon_0; 0)$, and possesses a weight lattice that is half-filled (see Fig. 3.2), compared to that of the gauge fields, due to the fact that, as the lowest weight state $|2\epsilon_0; 0\rangle$ has spin 0, there is no way of forming a spin-0 combination at the first excited level (*i.e.*, with a single energy raising operator L^+).

In four dimensions, the composite massless scalars are of two types, as we shall see, and coincide with the conformal massless ones. Indeed, the conformal coupling gives a (fake) mass term of the form

$$m^2 \ = \ \frac{R}{6} \ = \ -\frac{12}{6} \ = -2 \ , \qquad (3.2.51)$$

where eq. (2.1.30) has been used to determine the curvature scalar R. Substituting in

[11]This is of course strictly true only in the signature $(D-1,2)$, i.e., in the AdS_D case. However, a notion of boundary can be found also in other signatures: for example, in the case of S^D, it is naturally associated with solutions of the field equations that diverge at the poles, and that are therefore well-behaved only if the latter are cut out, which introduces a boundary.

(3.2.50) and solving for e_0 gives $\epsilon_0 = 1, 2$ (and notice that, in four dimensions $2\epsilon_0 = 1$). We shall soon examine the composite interpretation of these two solutions.

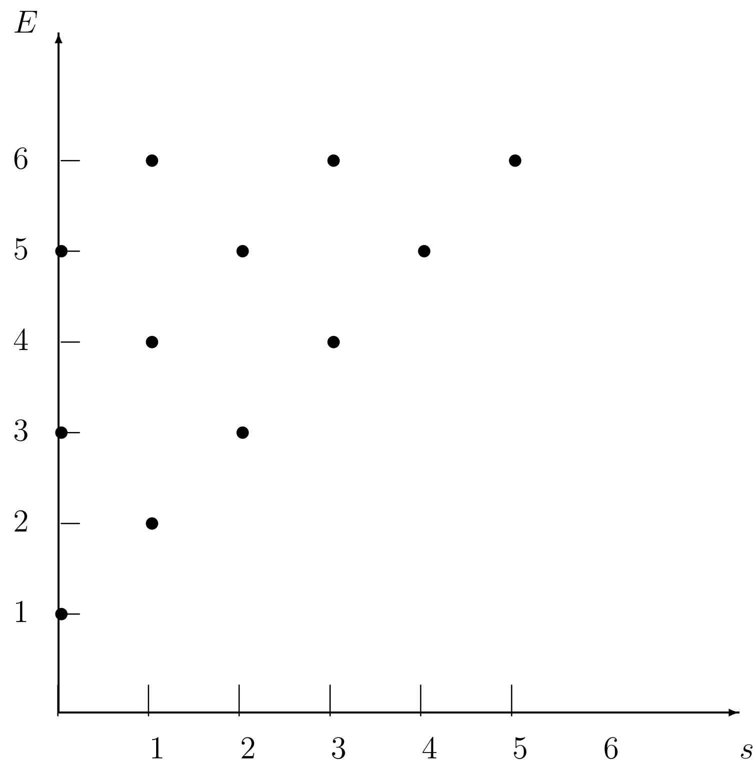


Figure 3.2: Weight diagram of the massless scalar $\mathfrak{D}(1, 0)$ in four dimensions.

3.2.2 Singleton Representations

The most remarkable fact about massless representations in a maximally symmetric background with nonvanishing cosmological constant is that they are not fundamental! This is a dramatic difference with respect to the case of a flat space-time, and relies on the fact that the fundamental representation (unitary in the AdS case) is a a very special one, an ultra-short representation that admits no flat limit, called *singleton*. Such representations were first discovered by Dirac [30], who was quite intrigued by their properties, subsequently studied by Fronsdal and collaborators [16, 31, 108], and then later found a very natural arena in String Theory in the context of the AdS/CFT correspondence and of recent studies on the tensionless limit of strings [111].

Let us examine what happens to the scalar weight space in the case that e_0 is lowered to the value ϵ_0: remarkably, also the Harish-Chandra module of a scalar field becomes indecomposable, and a singular vector appears at the second excited level,

$$|e_0', s_0'\rangle \;=\; |\epsilon_0+2, 0\rangle \;=\; L_r^+ L_r^+ |\epsilon_0, 0\rangle \; . \tag{3.2.52}$$

Indeed,

$$\begin{aligned} L_r^- L_s^+ L_s^+ |e_0, 0\rangle &= (2iM_{rs} + 2\delta_{rs}E) L_s^+ |e_0, 0\rangle + 2L_r^+ E |e_0, 0\rangle \\ &= (4e_0 - 4\epsilon_0) L_r^+ |\epsilon_0, 0\rangle \;=\; 0 \end{aligned} \tag{3.2.53}$$

admits precisely the solution above, $e_0 = \epsilon_0$. The factorization of the scalar singular submodule $\mathfrak{D}(\epsilon_0+2; 0)$ from the Harish-Chandra module of a scalar representation leads to an ultra-short irrep

$$\mathfrak{D}_0 \;\equiv\; \mathfrak{D}(\epsilon_0; 0) \;=\; \mathrm{span}_{\mathbb{C}} \left\{ L^+_{\{r_1} L^+_{r_2} ... L^+_{r_n\}} |e_0, 0\rangle \right\}_{n=0}^{\infty} \tag{3.2.54}$$

that consists of a single line in the weight space (see Fig. 3.3), hence the name singleton. The physical meaning of this fact is that, for every excitation in such a representation, the energy is always proportional to the $\mathfrak{so}(D-1;\mathbb{C})$-spin: this means that there are no radial excitations, *i.e.*, that such a representation only consists of boundary degrees of freedom. In other words, the factorization of the singular submodule does not correspond, as it was the case for $s_0 \geq 1$, to the elimination of gauge modes, but to the absence of bulk degrees of freedom! Indeed, the singular vector (3.2.52) can be shown to be related to $(D-1)$-dimensional equations of motion of a conformal scalar field living at the boundary. This property of being boundary objects gives a kinematical reason for the unobservability of singletons. The defining property that, for each energy level $\mathfrak{D}^{(n)}(\epsilon_0; 0)$,

$$E\mathfrak{D}^{(n)}(\epsilon_0; 0) \;=\; (\epsilon_0 + n)\mathfrak{D}^{(n)}(\epsilon_0; 0) \; , \tag{3.2.55}$$

also gives a reason why such representations do not admit a flat limit. Indeed, recall that for Poincaré irreps one has a continuous tower of modes for every value of spin, which is exactly what one gets from the flat limit of the massless irreps found above[12]. However, having a single energy eigenvalue for any given value of the spin makes the excitations in the singleton representation peculiar, and makes its flat limit result in a representation of the Poincaré algebra that is trivial on the translations.

Nonetheless, as previously announced, the most remarkable properties of the scalar singleton is that the tensor product of two such representations (sometimes refereed to as *doubleton*) decomposes, under the action of $\mathfrak{so}(D+1;\mathbb{C})$, into the direct sum of all bosonic

[12]The discrete spectrum of energy for every value of spin is related to the presence of a boundary, which is in its turn related to a finite radius of curvature L - essentially, dealing with fields in a space-time of constant curvature is analogous to analyzing waves in a box.

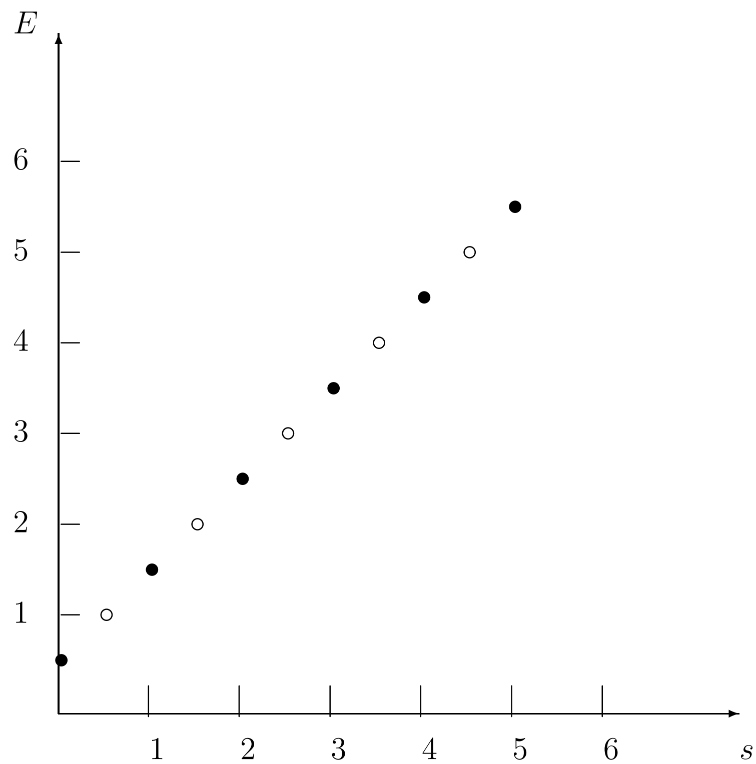


Figure 3.3: Weight diagrams of the scalar singleton $\mathfrak{D}(1/2,0)$ ($\bullet$) and of the spinor singleton $\mathfrak{D}(1,1/2)$ ($\circ$) in four dimensions.

massless representations, as first discovered in the case of AdS_4 by Flato and Fronsdal [31] and later extended to D dimensions [106, 107, 66, 111],

$$\mathfrak{D}_0 \otimes \mathfrak{D}_0 \ = \bigoplus_{s_0=0,1,2,\dots} \mathfrak{D}(s_0 + 2\epsilon_0, (s_0)) \ . \tag{3.2.56}$$

The product above can be decomposed into the symmetric and antisymmetric parts, that contain the even and odd massless spins, respectively,

$$[\mathfrak{D}_0 \otimes \mathfrak{D}_0]_S \ = \bigoplus_{s_0=0,2,4,\dots} \mathfrak{D}(s_0 + 2\epsilon_0, (s_0)) \tag{3.2.57}$$

$$[\mathfrak{D}_0 \otimes \mathfrak{D}_0]_A \ = \bigoplus_{s_0=1,3,5} \mathfrak{D}(s_0 + 2\epsilon_0, (s_0)) \ . \tag{3.2.58}$$

The composite massless lowest weight states can therefore be written as a superposition of states in the doubleton,

$$|s+2\epsilon_0;(s)\rangle_{12;r(s)} \;=\; f_{r(s)}(1,2)|2\epsilon_0;(0)\rangle_{12}|2\epsilon_0;(0)\rangle_{12}\ , \tag{3.2.59}$$

where the composite operator[13] $f_{r(s)}(1,2)$ is given by

$$f_{r(s)}(1,2) \;=\; (-1)^s f_{r(s)}(2,1) \;=\; \sum_{k=0}^{s} f_{s;k}(L^+_{\{r_1}\cdots L^+_{r_k})(1)(L^+_{r_{k+1}}\cdots L^+_{r_s\}})(2)\ , \tag{3.2.60}$$

$$f_{s;k} \;=\; (-1)^s f_{s;s-k} \;=\; \binom{s}{k}\frac{(1-s-\epsilon_0)_k}{(\epsilon_0)_k}\ . \tag{3.2.61}$$

Eq. (3.2.50) determines the mass of a free scalar singleton field Ψ to be $m^2_\Psi = -\frac{(D-3)(D+1)}{4}$, while its field equation is

$$\left(\Box-\frac{(D-3)(D+1)}{4}\right)\Psi \;=\; 0\ , \tag{3.2.62}$$

and belongs to the ideal $\mathfrak{D}(\epsilon_0+2,(0))$.

We note also that $\mathfrak{so}(D+1;\mathbb{C})$ does not act transitively on the singleton weight space. The smallest Lie algebra with this property is the minimal bosonic HS algebra $\mathfrak{ho}(D+1;\mathbb{C})$ defined in (3.1.69).

As announced earlier in this Chapter, the singleton representation is also uniquely defined by the condition that it be annihilated by every combination of generators of the enveloping algebra of $\mathfrak{so}(D+1;\mathbb{C})$ that is in the ideal $\mathcal{I}[V]$ defined in (3.1.9). Indeed, one can characterize $\mathcal{I}[V]$ as the annihilating (left and right) ideal of the singleton,

$$\mathcal{I}[V] \;=\; \mathrm{span}_{\mathbb{C}}\left\{X\in\mathcal{U}\ :\ X|\psi\rangle=0\ ,\forall|\psi\rangle\in\mathfrak{D}_0\right\}\ , \tag{3.2.63}$$

and to prove this it is sufficient to show that the generating elements V_{AB} and V_{ABCD} annihilate every element in the singleton representation. This is easily done splitting the index $A=(+,-,r)$, where $X_\pm\equiv X_0\pm iX_{0'}$, and noting that $c_{--}=L^+_rL^+_r$, that exactly coincides with the combination of energy-raising operators giving rise to the singular vector (3.2.52) and is therefore consistently set to zero in the singleton representation with all the submodule built on top of it. Moreover, using $\mathfrak{so}(D+1;\mathbb{C})$ rotations, this conclusion can be extended to all the independent $\mathfrak{so}(D-1;\mathbb{C})$-components into which c_{AB} is broken as a consequence of the index splitting. Similarly, one can show that also V_{ABCD} gives zero acting on every state in the singleton.

[13]The coefficients $f_{s;k}$ are fixed by the condition $(L^-_r(1)+L^-_r(2))|s+2\epsilon_0;(s)\rangle_{12;r(s)}=0$, which is equivalent to $a_kf_{s;k}+a_{s-k+1}f_{s;k-1}=0$, where $a_k=2k(k+\epsilon_0-1)$, with solution $f_{s;k}=(-1)^sf_{s;s-k}=(-1)^k\frac{a_{s-k+1}\cdots a_s}{a_k\cdots a_1}f_{s;0}$, taking the form (7.3.30) for $f_{s;0}=1$.

Importantly, in $D = 4$ the scalar singleton $\mathfrak{D}(1/2, 0)$ is not the only irrep satisfying (3.2.63): one can show indeed that the latter condition admits another *spinor singleton* irrep $D_{1/2} = \mathfrak{D}(1, 1/2)$, whose lowest weight state is a $\mathfrak{so}(3;\mathbb{C})$-spinor representation $|1, 1/2\rangle_i$. Moreover, one can also prove that $C_2[\mathfrak{so}(5;\mathbb{C}); (1/2, 0)] = C_2[\mathfrak{so}(5;\mathbb{C}); (1, 1/2)]$. This implies, among other things, that also the tensor product of two spinor singletons decomposes into integer-spin massless representations, and in particular

$$\mathfrak{D}_{1/2} \otimes \mathfrak{D}_{1/2} \; = \; \mathfrak{D}(2, 0) \oplus \bigoplus_{s_0=0,1,2,\ldots} \mathfrak{D}(s_0 + 1, s_0) \; , \tag{3.2.64}$$

while the tensor product of a scalar and a spinor singleton gives rise to the half-integer-spin massless representations,

$$\mathfrak{D}_0 \otimes \mathfrak{D}_{1/2} \; = \bigoplus_{2s_0=1,3,\ldots} \mathfrak{D}(s_0 + 1, s_0) \; . \tag{3.2.65}$$

Notice that scalar and spinor doubletons admit the same decomposition except for the scalar sector, where the former contains the parity-invariant scalar $\mathfrak{D}(1, 0)$ and the latter the pseudo-scalar $\mathfrak{D}(2, 0)$. We shall make use of this particular feature of four dimensions in Chapter 7.

3.2.3 Real forms and unitary representation

So far, our analysis has been carried out at the complex level, and with no notion of unitarity. We shall now examine in detail the various different real forms of the $\mathfrak{so}(D{+}1;\mathbb{C})$ algebra with the different signatures $(p', D{+}1{-}p')$ that will be of relevance in the following, and in which of these the representations shown above are unitary. In particular, we shall look at how the splitting

$$\mathfrak{g} \; = \; \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- \; , \tag{3.2.66}$$

into the compact subalgebra $\mathfrak{g}_0 = \mathfrak{so}(D-1;\mathbb{C}) \oplus \mathfrak{so}(2;\mathbb{C})$ and the noncompact parts $\mathfrak{g}_\pm = \mathrm{span}_{\mathbb{C}} \{L_r^\pm\}$ can be performed, and examine the effect of a σ automorphism that acts on the generators in such a way as to always arrive at a positive definite standard inner product.

- $\mathfrak{so}(D-1, 2)$ with tangent space signature $(D-1, 1)$ (AdS_D) : Here we have $\eta_{AB} = (-1, \delta_{rs}, -1)$.

  We begin by assuming $M^\dagger_{AB} = M_{AB}$. One can realize the splitting (3.2.66) taking

$$E \; = \; M_{0'0} \; , \qquad M_{rs} \tag{3.2.67}$$

as the generators of the compact subalgebra, and

$$L_r^{\pm} \ = \ M_{0r} \mp i M_{0'r} \tag{3.2.68}$$

as the energy-raising and energy-lowering operators. Moreover, we define $P_r = M_{0'r}$ as the spatial AdS_D translations. One can check that such definitions indeed satisfy the algebra

$$\begin{aligned} \left[L_r^-, L_s^+\right] &= 2iM_{rs} + 2\delta_{rs}E \ , \\ \left[E, L_r^{\pm}\right] &= \pm L_r^{\pm} \ , \end{aligned} \tag{3.2.69}$$

along with the reality conditions

$$E^\dagger \ = \ E \ , \qquad (L_r^{\pm})^\dagger \ = \ L_r^{\mp} \ . \tag{3.2.70}$$

The latter relation ensures that one can build a lowest weight module $\{L_{r_1}^+ ... L_{r_n}^+ |e_0, s_0\rangle\}$ of states with positive norm. This, together with the hermitian nature of the generators M_{AB}, implies the unitarity of such a representation, at least for suitable values of e_0, s_0.

- $\mathfrak{so}(D+1)$ with tangent space signature $(D, 0)$ (S^D) : $\eta_{AB} = \delta_{AB}$.

  Again we begin by assuming the hermiticity of the generators M_{AB}. Here the splitting is completely arbitrary, due to the compactness of the algebra. One way of realizing the algebra (3.2.69) is to choose

$$L_r^{\pm} \ = \ iM_{0r} \mp M_{0'r} \ , \qquad E \ = \ M_{0'0} \ . \tag{3.2.71}$$

  However, here the reality conditions are such that

$$E^\dagger \ = \ E \ , \qquad (L_r^{\pm})^\dagger \ = \ -L_r^{\mp} \ , \tag{3.2.72}$$

  and the latter relation prevents from constructing Fock space states with positive norm. This is independent of the particular chosen realization of the splitting (3.2.66). The only way to recover this property is to twist the reality conditions of the generators, in such a way as to get rid of the minus sign in the reality conditions of the energy-raising and energy-lowering operators. This amounts to the requirement that

$$M_{AB}^\dagger \ = \ \sigma(M_{AB}) \ , \tag{3.2.73}$$

where in this case

$$\sigma \;=\; \begin{cases} +1 \ , \ \text{on } M_{rs}, M_{0'0} = E \\ -1 \ , \ \text{on } M_{0r}, M_{0'r} = P_r \end{cases} \ . \tag{3.2.74}$$

To reiterate, in the euclidean case one can have a positive definite lowest weight module only by means of a "Wick rotation" of the algebra that leaves hermitian only the generators $M_{rs}, M_{0'0}$, spanning the subalgebra $\mathfrak{so}(D-1) \oplus \mathfrak{so}(2)$. In any case, we cannot have hermitian generators and states with positive norm at the same time, so such lowest weight representations cannot be unitary, for any value of e_0, s_0[14].

- $\mathfrak{so}(D-1,1)$ with tangent space signature $(D,0)$ (H_D) : $\eta_{AB} = (\delta_{ab}, -)$.

  The energy-raising and energy-lowering operators and the energy operator can here be defined as

$$L_r^{\pm} \;=\; iM_{0r} \mp iM_{0'r} \ , \qquad E \;=\; iM_{0'0} \ . \tag{3.2.75}$$

We moreover define the space translations as $P_r = M_{0'r} = \frac{1}{2i}(L_r^- - L_r^+)$. These satisfy the algebra (3.2.69), but their reality conditions are

$$E^\dagger \;=\; -E \ , \qquad (L_r^\pm)^\dagger \;=\; -L_r^\pm \ , \tag{3.2.76}$$

as long as one insists in having hermitian M_{AB} generators. The twist which is needed here to have positive norms is

$$M_{AB}^\dagger \;=\; \sigma(M_{AB}) \ , \tag{3.2.77}$$

where in this case

$$\sigma \;=\; \begin{cases} +1 \ , \ \text{on } M_{rs}, M_{0'r} = P_r \\ -1 \ , \ \text{on } M_{0r}, M_{0'0} \sim E \end{cases} \ . \tag{3.2.78}$$

In this case, the σ-twist acts with an additional minus sign on the energy and space translation generators, with respect to the $\mathfrak{so}(D+1)$ case, so that they exchange their σ-eigenvalue. This means that the leftover hermitian subalgebra changes, in this case, and is indeed $\mathfrak{so}(D-1,1)$. As in the previous case, the representation of the algebra on the lowest weight module defined above cannot be made unitary.

[14]Note that this argument only holds for infinite-dimensional modules. Finite-dimensional unitary representations, which are lowest-and-highest-weight modules, are not ruled out. This is essentially because, by π-invariance, their weight diagram is symmetrical with respect to the $\{E = 0\}$ axis, which in turn implies that the lowest-weight state has a negative energy eigenvalue that compensates for the minus sign appearing in the hermiticity condition (3.2.72), leading to positive norms.

- $\mathfrak{so}(D,1)$ with tangent space signature $(D-1,1)$ (dS_D) : $\eta_{AB} = (-,\delta_{rs},+)$.

  One can define

$$L_r^{\pm} \ = \ iM_{0'r} \pm iM_{0r} \ , \qquad E \ = \ iM_{0'0} \ , \tag{3.2.79}$$

  that satisfy the algebra (3.2.69), and $P_r = M_{0'r}$. The reality conditions are, in this case,

$$E^{\dagger} \ = \ -E \ , \qquad (L_r^{\pm})^{\dagger} \ = \ -L_r^{\pm} \ . \tag{3.2.80}$$

  Again, to have positive definite norms, one needs to twist the reality conditions on generators, in the following way:

$$M_{AB}^{\dagger} \ = \ \sigma(M_{AB}) \ , \tag{3.2.81}$$

  where now

$$\sigma \ = \ \pi \ = \ \begin{cases} +1 \ , \ \text{on } M_{rs}, M_{0r} \\ -1 \ , \ \text{on } M_{0'r} = P_r, M_{0'0} = E \end{cases} . \tag{3.2.82}$$

  Note that, in accordance with the action of the π map previously defined, here the whole space-time translation operator $P_a = (E, P_r)$ becomes non hermitian as a consequence of the twist. This amounts to say that the hermitian subalgebra is, in this case, $\mathfrak{so}(D-1,1)$. The lowest weight realization of the dS_D isometry algebra is then nonunitary.

It is therefore possible, in AdS_D, that the lowest weight representations $\mathfrak{D}(e_0,(s_0))$ presented above are unitary, at least for certain values of e_0 and s_0. One way to check this is to check whether the norms of the various states of the representation at hand are all positive, assuming that the norm of the lowest weight state is, *e.g.*, $\langle e_0,(s_0)|e_0,(s_0)\rangle = 1$. Equivalently, a unitarity bound can be derived imposing that all states within a certain lowest weight representation have the same value of the Casimir operator C_2. Let us look, for example at a scalar representation $\mathfrak{D}(e_0,(0))$, and in particular let us compare the value of C_2 on the lowest weight state and on the scalar excited state $|e_0+2,(0)\rangle$ obtained acting on the lowest weight state with $L_r^+L_r^+$. We obtain

$$e_0(e_0-D+1) \ = \ (e_0+2)(e_0-D+3) + |L_r^-|e_0+2,(0)\rangle|^2 \ , \tag{3.2.83}$$

so that the requirement of unitarity implies the following lower bound

$$e_0 \ \geq \ \epsilon_0. \tag{3.2.84}$$

Note that this inequality is saturated for $e_0 = \epsilon_0$, *i.e.*, for $L_r^-|e_0+2,(0)\rangle = 0$, which tells us that the scalar singleton is the representation that saturates the unitarity bound for scalars.

A similar reasoning can be carried out for more complicated cases [109, 110]. For a spin-(1/2) representation, the bound is

$$e_0 \ \geq \ \epsilon_0 + \frac{1}{2} \ , \tag{3.2.85}$$

which shows that the spinor singleton in $D = 4$ ($\epsilon_0 = 1/2$) is again at the boundary of unitarity. For representations with $s_0 \geq 1$, on the other hand, one gets

$$e_0 \ \geq \ s_0 + 2\epsilon_0 \ , \tag{3.2.86}$$

from which one sees that the massless representations $\mathfrak{D}(s_0 + 2\epsilon_0; (s_0))$ are unitary.

One can also notice that most of the "conjugate" representations $\mathfrak{D}(D-1-e_0;(s_0))$ are nonunitary, with the exception of the conjugate scalar fields $\mathfrak{D}(1,0)$ and $\mathfrak{D}(2,0)$ in four dimensions. Given that the scalar singleton is a unitary representation in any D, the massless lowest weight representations into which the doubleton spectrum decomposes are necessarily unitary in AdS_D but nonunitary in other signatures.

However, the most important conclusion of this section is that the spectrum of free equations we presented in Section 2.2 contains indeed physical massless fields of every spin s, each occurring once, and that, in order for it to fit a unitary module of the infinite-dimensional extension $\mathfrak{ho}(D-1,2)$ of the $\mathfrak{so}(D-1,2)$ background isometry algebra it is necessary that also a scalar enter the HS equations, since a scalar fields always appears in the $\mathfrak{so}(D-1,2)$-decomposition of the tensor product of the fundamental UIR of the HS algebra[15], as it appears in (3.2.56). This gives a rationale for the introduction of a master zero-form (7.6.15), that can contain such a scalar in a natural HS-covariant "master field". We shall examine further reasons for this choice in the next Chapter.

3.3 Oscillator realizations

The aim of this Section is to introduce two useful oscillator realizations of the HS algebra, that are very much related to the singleton representation. As we shall examine in Chapter 5, oscillator realizations are crucial for going to full nonlinear level in the Vasiliev equations, essentially because they allow to take "square roots" of the HS algebra generators. The first oscillator realization we shall recall makes use of "vector" oscillators Y_i^A,

[15] Or in the symmetric part of such tensor product if we compare with the minimal HS algebra (3.1.69).

that have a vector index of $\mathfrak{so}(D-1,2)$ and a doublet index of $\mathfrak{sp}(2)$: they underlie a D-dimensional formulation of Vasiliev equations [72, 52], despite some subtlety due to a redundancy that the $\mathfrak{sp}(2)$ index brings in [66, 52, 77]. The latter is automatically absent in the four-dimensional spinor oscillator realization, based on $\mathfrak{sl}(2;\mathbb{C})$-doublet oscillators, which entered the first formulation of the Vasiliev equations in $D=4$ [81, 84, 82, 49, 103], and that we shall review later on. For the moment, we restrict our considerations to the AdS case only.

3.3.1 Vector oscillator realization

As stressed in [111], one can describe the singleton as a massless particle living on the *Dirac hypercone* $X^A X_A = 0$ in $\mathbb{R}^{D-1,2}$, where $X^A = \sqrt{2}Y_1^A$ and $P^A = \sqrt{2}Y_2^A$ are its phase-space coordinates. Upon quantization, one imposes the commutation relations

$$[Y_i^A, Y_j^A]_\star \;=\; 2i\epsilon_{ij}\eta^{AB} \;, \tag{3.3.1}$$

where $\epsilon_{ij} = -\epsilon ji$ is the invariant tensor of $\mathfrak{sp}(2)$ and we use the conventions $\epsilon^{ij}V_j = V^i$, $V^j\epsilon ji = V_i$, $\epsilon^{ik}\epsilon_{jk} = \delta^i_j$. We are here, again, making use of a $\star$-product defined on the oscillators that implements the operator product on Weyl-ordered (i.e., totally symmetric) combinations of oscillators, that we will simply denote by juxtaposition,

$$Y^A_{i_1}...Y^A_{i_n} \;=\; \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n} Y^{A_{\pi_1}}_{i_{\pi_1}}\star ...\star Y^{A_{\pi_n}}_{i_{\pi_n}} \;. \tag{3.3.2}$$

The totally symmetric ordering of oscillators is preferred in that it preserves $\mathfrak{sp}(2)$-covariance, differently from other choices such as the normal ordering of the creation/annihilation operators $X^A \pm iP^a \sim a, a^\dagger$. More generally, for two Weyl-ordered functions of oscillators,

$$f(Y)\star g(Y) \;=\; f(Y)\exp\left(i\epsilon^{ij}\eta^{AB}\frac{\overleftarrow{\partial}}{\partial Y^{Ai}}\frac{\overrightarrow{\partial}}{\partial Y^{Bj}}\right)g(Y)\;, \tag{3.3.3}$$

which admits the equivalent integral presentation

$$f(Y)\star g(Y) \;=\; \frac{1}{\pi^{2(D+1)}}\int dSdT\, f(Y+S)\, g(Y+T)\exp(-iS_i^A T_A^i)\;. \tag{3.3.4}$$

The constraints ensuring that this particle be massless, $P^2 \approx 0$, and that it live on the hypercone $X^2 \approx 0$ (i.e., in one dimension less), together with their commutator, proportional to $X_A, P_A \approx 0$, which imposes symmetry under dilatations and thus independence on the radial direction, actually characterize such particle as a conformal massless particle in $(D-1)$-dimensions, *i.e.*, a singleton. One can see this also from the fact that the constraints above generate $\mathfrak{sp}(2)$, since they correspond to the three independent components in

$$K_{ij} \;=\; \frac{1}{2}Y_i^A Y_{Aj} \;, \tag{3.3.5}$$

with commutation relations

$$[K_{ij}, K_{kl}]_\star \;=\; 4i\epsilon_{(i|(k}K_{l)|j)} \;, \tag{3.3.6}$$

which means that one can impose the constraints above by declaring that the every state $|\psi\rangle$ in the Hilbert space of the conformal particle is annihilated by K_{ij}. Notice also that, with vector oscillators, the generators of $\mathfrak{so}(D-1,2)$ can be realized as the bilinears[16]

$$M_{AB} \;=\; \frac{1}{2}Y_A^i Y_{iB} \;, \tag{3.3.7}$$

and let us also introduce the combinations

$$L_{ij,AB} \;\equiv\; \frac{1}{2}Y_{Ai}Y_{Aj} - \frac{\eta_{AB}}{D+1}K_{ij} \;, \tag{3.3.8}$$

which are traceless in A, B. With the help of these constructs, one can realize the generators V_{AB} of the annihilating ideal of the singleton $\mathcal{I}[V]$ as

$$V_{AB} \;=\; K^{ij} \star L_{ij,AB} \;=\; L_{ij,AB} \star K^{ij} \;, \tag{3.3.9}$$

which means that imposing the $\mathfrak{sp}(2)$-invariance of the states amounts to describe the singleton weight space! Indeed, the other constraint that generates $\mathcal{I}[V]$ is trivially satisfied in this setting, due to the fact that out of an $\mathfrak{sp}(2)$-doublet such as Y_i^A one cannot form combination with more than two antisymmetric indices A, B. Finally, the quadratic Casimir operator matches that of the singleton, since

$$C_2 \;=\; \frac{1}{2}M^{AB} \star M_{AB} \;=\; \frac{1}{2}K^{ij} \star K_{ij} - \epsilon_0(\epsilon_0 + 2) \;, \tag{3.3.10}$$

which gives the desired result on every state $|\psi\rangle$ such that $K^{ij}|\psi\rangle = 0$.

The realization of a HS algebra follows straightforwardly from the realization of the generators of $\mathfrak{so}(D-1,2)$. It is sufficient to consider arbitrary functions of the oscillators $f(Y)$ subject to the conditions of being $\mathfrak{sp}(2)$-singlets,

$$[K_{ij}, f(Y)]_\star \;=\; 0 \;, \tag{3.3.11}$$

[16] Note also that $[M_{AB}, K_{ij}]_\star = 0$.

This restricts the generators of such an associative algebra, that we shall denote with $\mathcal{S}$, to be products of oscillators with symmetry properties encoded into two-rows rectangular $\mathfrak{so}(D-1,2)$-Young diagrams. However, the algebra formed by such objects is still reducible, as it contains the left and right ideal $\mathcal{I}'$ spanned by all elements $g(Y)$ proportional to the $\mathfrak{sp}(2)$ generator, *i.e.* of the form $g_{ij} \star K^{ij} = K^{ij} \star g_{ij}$. Due to the definition of K_{ij} (6.2.27), all traces of two-row Young diagrams are contained in $\mathcal{I}'$. After factoring it out, the resulting associative algebra $\mathcal{A} = \mathcal{S}/\mathcal{I}'$ contains only all traceless two-row rectangular Young diagrams, and coincides with the associative algebra $\mathcal{A}$ defined in (3.1.12), generated by (3.1.23). One can then realize from it the minimal bosonic HS algebra $\mathfrak{ho}(D-1,2)$ (3.1.69) defining the action of the antiautomorphism of the oscillator algebra (3.3.1) τ on functions of the oscillators,

$$\tau(f(Y_i^A)) \ = \ f(iY_i^A) \ . \tag{3.3.12}$$

Moreover, the reality conditions are

$$(Y_i^A)^\dagger \ = \ Y_i^A \ . \tag{3.3.13}$$

It is possible in particular to realize the AdS-translation generator projecting one index onto the embedding direction, which can be done covariantly making use of the compensator V_A introduced in Section 2.2,

$$P_a \ = \ \frac{1}{2} Y_A^i Y_{Bi} V^B \ , \tag{3.3.14}$$

which in the standard gauge becomes

$$P_a \ = \ \frac{1}{2} Y_a^i Y_{0'i} \ \equiv \ \frac{1}{2} Y_a^i y_i \ . \tag{3.3.15}$$

The action of the π automorphism is

$$\pi(f(Y_i^A)) \ = \ \pi(f(Y_i^a, y_i)) \ = \ f(Y_i^a, -y_i) \ , \tag{3.3.16}$$

i.e., π acts as a parity in the embedding direction. The realization of the adjoint master one-form therefore follows immediately putting (3.3.7) and (3.3.15) into (7.6.13) and (7.6.14), and similarly for the twisted adjoint master zero-form (7.6.15) and (7.6.16). Notice that one can obtain real component fields with the conditions (3.3.13) imposing

$$A^\dagger \ = \ -A \ , \qquad \Phi^\dagger \ = \ \pi(\Phi) \ . \tag{3.3.17}$$

As mentioned in the first Section of this Chapter, referring to the singleton by factoring out its annihilating ideal is much more convenient than doing it by working at the level of Hilbert spaces, especially in view of formulating HS-covariant full field equations that one would like to write in a manifestly background covariant form. However, how to project out the traces, concretely, at the level of the full field equations involves some subtleties [52, 77, 66]. Such factorization is however very important in the Vasiliev equations, as it encodes standard second order dynamical equations in a set of HS-covariant first order curvature constraints.

3.3.2 4D spinor oscillator realization

The factorization is automatic in the realization of four-dimensional HS algebras in terms of commuting spinor oscillators $y_\alpha, \bar{y}_{\dot{\alpha}}$ of $\mathfrak{sl}(2;\mathbb{C})$ (first proposed in [63, 64]) satisfying

$$[y_\alpha, y_\beta]_\star \;=\; 2i\epsilon_{\alpha\beta}\ , \qquad [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}]_\star \;=\; 2i\epsilon_{\dot{\alpha}\dot{\beta}}\ , \tag{3.3.18}$$

that is to say

$$y_\alpha \star y_\beta \;=\; y_\alpha y_\beta + i\epsilon_{\alpha\beta}\ , \qquad \bar{y}_{\dot{\alpha}} \star \bar{y}_{\dot{\beta}} \;=\; \bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} + i\epsilon_{\dot{\alpha}\dot{\beta}}\ , \tag{3.3.19}$$

$$y_\alpha \star \bar{y}_{\dot{\beta}} \;=\; y_\alpha \bar{y}_{\dot{\beta}}\ , \tag{3.3.20}$$

where $\epsilon_{\alpha\beta}$ is the invariant tensor of $\mathfrak{sl}(2;\mathbb{C})$ and our spinor conventions are collected in Appendix E. These are particular cases of the most general $\star$-product rule

$$f(y,\bar{y}) \star g(y,\bar{y}) \;=\; f(y,\bar{y}) e^{-i(\overleftarrow{\partial}^\alpha \overrightarrow{\partial}_\alpha + \overleftarrow{\partial}^{\dot{\alpha}} \overrightarrow{\partial}_{\dot{\alpha}})} g(y,\bar{y})\ , \tag{3.3.21}$$

where $\partial_\alpha \equiv \frac{\partial}{\partial y^\alpha}$, or, equivalently,

$$(f \star g)(y,\bar{y}) \;=\; \int \frac{d^4u d^4v}{(2\pi)^4}\, f(y+u, \bar{y}+\bar{u})\, g(y+v, \bar{y}+\bar{v}) \exp i(u^\alpha v_\alpha + \bar{u}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}})\ . \tag{3.3.22}$$

The realization of the Lorentz and AdS translation generators is

$$M_{ab} \;=\; -\frac{1}{8}\Big[\,(\sigma_{ab})^{\alpha\beta} y_\alpha y_\beta + (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \tilde{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}}\,\Big]\ , \qquad P_a \;=\; \frac{1}{4}(\sigma_a)^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}\ . \tag{3.3.23}$$

To realize the HS algebra, here it is sufficient to consider the associative algebra spanned by all possible monomials in oscillators,

$$T_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} \;\equiv\; T_{\alpha(n),\dot{\alpha}(m)}\ , \tag{3.3.24}$$

that have spin $s = \frac{n+m}{2}$, whose elements are therefore all possible functions

$$f(y,\bar{y}) \;=\; \sum_{n,m} f^{\alpha(n),\dot{\alpha}(m)} T_{\alpha(n),\dot{\alpha}(m)}\ . \tag{3.3.25}$$

Compared to the vector oscillator realization, however, this is a simpler setting, due to the lack of the additional $\mathfrak{sp}(2)$ redundancy. In particular, here traces are automatically factored out, since for commuting spinors $y^\alpha y_\alpha = \bar{y}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}} = 0$. In other words, the elements (3.3.25) indeed span the four-dimensional associative algebra $\mathcal{A}$ (3.1.12). Here one can introduce the antiautomorphism τ as

$$\tau(f(y,\bar{y})) \;=\; f(iy, i\bar{y})\ , \tag{3.3.26}$$

and the automorphisms $\pi, \bar{\pi}$, distinguishing Lorentz rotations and AdS translations, as

$$\pi(f(y,\bar{y})) \;=\; f(-y,\bar{y}) \;, \qquad \bar{\pi}(f(y,\bar{y})) \;=\; f(y,-\bar{y}) \;. \tag{3.3.27}$$

Therefore, by imposing the τ-condition (3.1.69) on the elements (3.3.25) one truncates the model to generators with $m+n=2,6,10,...$, that exactly correspond to gauge fields with spin $2,4,6,...$, *i.e.*, one obtains the minimal bosonic HS algebra $\mathfrak{ho}(3,2)$. The adjoint master one-form can be written as

$$A(x;y,\bar{y}) \;=\; \frac{1}{2i}\sum_{\ell=0}^{\infty} A^{(\ell)} \;, \tag{3.3.28}$$

where each level ℓ is expanded as

$$A^{(\ell)}(x;y,\bar{y}) \;=\; \sum_{n+m=4\ell+2} \frac{1}{n!m!} dx^\mu A_\mu^{(\ell)\,\alpha_1...\alpha_n\dot{\alpha}_1...\dot{\alpha}_m}(x) y_{\alpha_1}...y_{\alpha_n}\bar{y}_{\dot{\alpha}_1}...\bar{y}_{\dot{\alpha}_m} \;, \tag{3.3.29}$$

while imposing $\tau(\Phi)=\pi(\Phi)$ one arrives at the realization of the twisted adjoint master zero-form,

$$\Phi(x;y,\bar{y}) \;=\; \sum_{\ell=-1}^{\infty} \Phi^{(\ell)} \;, \tag{3.3.30}$$

where

$$\Phi^{(\ell)}(x|y,\bar{y}) \;=\; \sum_{|n-m|=4\ell+4} \frac{1}{n!m!} \Phi^{(\ell)\,\alpha_1...\alpha_n\dot{\alpha}_1...\dot{\alpha}_m}(x)\, y_{\alpha_1}...y_{\alpha_n}\bar{y}_{\dot{\alpha}_1}...\bar{y}_{\dot{\alpha}_n} \;. \tag{3.3.31}$$

We are also assuming the reality condition

$$y_\alpha^\dagger \;=\; \bar{y}_{\dot{\alpha}} \;. \tag{3.3.32}$$

In Chapter 6 we will also examine more general spinor oscillator realizations for the various signatures $\mathfrak{so}(p',5-p')$.

Another useful oscillator realization that we will make use of is given in terms of linear combinations of the $y,\bar{y}$ oscillators that correspond to creation/annihilation operators building up the Fock space of states of the scalar and spinor four-dimensional singletons. In particular, we introduce the $\mathfrak{su}(2)$-doublet $a_i, a^{\dagger\, i}$, with $a^{\dagger\ i}=(a_i)^\dagger$ and $i=1,2$ as

$$a_1 \;=\; \frac{1}{2}(y_1+i\bar{y}_{\dot{2}}) \;, \qquad a^{\dagger\,1} \;=\; \frac{1}{2}(\bar{y}_{\dot{1}}-iy_2) \;, \tag{3.3.33}$$

$$a_2 \;=\; \frac{1}{2}(-y_2+i\bar{y}^{\dot{1}}) \;, \qquad a^{\dagger\,2} \;=\; \frac{1}{2}(-\bar{y}_{\dot{2}}-iy_1) \;. \tag{3.3.34}$$

They satisfy the following Heisenberg algebra

$$[a_i, a^{\dagger\, j}]_\star \;=\; \delta_i^j \;, \tag{3.3.35}$$

and in terms of them the $\mathfrak{so}(3,2)$ generators can be expressed as

$$E \;=\; \frac{1}{2}(a^{\dagger\, i}a_i + 1)\;, \qquad M_{rs} \;=\; \frac{i}{2}(\sigma_{rs})_i{}^j a^{\dagger\, i}a_j \;, \tag{3.3.36}$$

$$L_r^+ \;=\; \frac{i}{2}(\sigma_r)_{ij}a^{\dagger\, i}a^{\dagger\, j}\;, \qquad L_r^- = \frac{i}{2}(\sigma_r)^{ij}a_i a_j \;. \tag{3.3.37}$$

In terms of such oscillators one can build a Fock space on top of the scalar singleton lowest weight state, which is declared to be annihilated by the a_i, $a_i|1/2,0\rangle = 0$,

$$\mathcal{F} \;=\; \text{span}\left\{a^{\dagger\, i_1}...a^{\dagger\, i_n}|1/2,0\rangle\right\}_{n=0}^{\infty}\;, \tag{3.3.38}$$

that is reducible under the action of $\mathfrak{so}(3,2)$ generators: indeed, being the latter bilinears in oscillators, their action splits the Fock space into the two irreducible even and odd subspaces,

$$\mathcal{F} \;=\; \mathcal{F}_{even}\oplus\mathcal{F}_{odd}\;, \tag{3.3.39}$$

that contain states with even or odd number of oscillators, respectively. Notice that the lowest weight state of the latter is

$$|1,1/2\rangle^i \;=\; a^{\dagger\, i}|1/2,0\rangle\;, \tag{3.3.40}$$

i.e., the lowest weight state of the spinor singleton irrep $\mathfrak{D}(1,1/2)$. The two singleton irreps are therefore connected very naturally in the oscillator realization, in which a number of other relations are manifest: for example, the lowest weights of the two scalar $\mathfrak{D}(1,0)$ and $\mathfrak{D}(2,0)$, that are both composites and read indeed (see (3.2.56) and (3.2.64))

$$|1,0\rangle \;=\; |1/2,0\rangle_1|1/2,0\rangle_2\;, \qquad |2,0\rangle \;=\; |1,1/2\rangle_1^i|1,1/2\rangle_{2\,i}\;, \tag{3.3.41}$$

are also related by the oscillator combination

$$|2,0\rangle \;=\; -y|1,0\rangle\;, \qquad y \;=\; a_i^\dagger(1)a^{\dagger\, i}(2)\;. \tag{3.3.42}$$

Similar relations will be useful in Chapter 7.

Chapter 4

Unfolded formulation

As recalled in the Introduction, although certain gauge-invariant vertices involving massless HS fields can be, and indeed were, obtained in the early and mid-Eighties, addressing the full "HS-interaction problem" is much more demanding, for reasons that we briefly repeat here for the reader's convenience:

1. A consistent interacting HSGT requires the simultaneous introductions of infinitely many gauge fields of all spins;
2. The interaction with gravity is consistent with the HS gauge symmetries only on a nonflat gravitational background, *i.e.*, in presence of a nonvanishing cosmological constant Λ, since interaction terms are nonanalytical in Λ;
3. HS interaction vertices require higher derivatives of the physical fields involved. This property is strictly connected with the previous one, since, in order for the physical dimension of the lagrangian to be preserved with more than two derivatives, a dimensionful parameter must enter the vertices, and Λ is the only candidate in a field theoretical context.

A step forward was described in the previous Chapter, where an infinite-dimensional non-abelian HS algebra was constructed. As in YM theories and in gravity, the invariance under the proper local gauge transformations is the key to determine the form of the interaction terms. However, the above-mentioned peculiar features are important differences that make a "traditional" analysis unyielding: first of all, a full HSGT will involve an infinite tower of fields of different spin and, correspondingly, infinitely many gauge symmetries, which makes impossible an order-by-order analysis in terms of each different gauge fields. In order to overcome such a problem, it is necessary to work with the proper variables, *i.e.*, with some proper "superfields"[1] that have nice transformations properties under the local HS symmetry, and that therefore enable to control at once the whole tower of massless fields. We will see that this also leads to a natural way out of the third problem, namely the complication introduced by the fact that unbroken HS symmetry implies no bound on derivatives in the vertices: indeed, a clever first-order reformulation of the dynamics that involves crucially the "superfields" as main variables will offer a way out of this by enabling to "hide" higher derivatives in the component fields. As the order of derivatives in the interaction vertices grows with the spin, it is intuitively clear that the "superfields" involved not only will have to encode infinitely many components to accommodate all spins, but also, presumably, infinitely many components for each spin-s sector, since each spin can a priori interact with any other one in couplings featuring higher and higher derivatives of the lower spin field.

The end result will be that the required "superfields" will be the adjoint master one-form and the twisted adjoint master zero-form introduced in the previous chapter, and that the proper first-order formulation of the dynamics is the so-called *unfolded formulation* (in which the twisted adjoint plays a crucial role), that enables to write HS field equations in the form of zero-curvature constraints. This is particularly appealing since such form of the equations makes it easy to control gauge-invariance, and this is especially important in view of the search for consistent nonlinear deformations. Indeed, such a formalism is today the only approach to full HS field equations, although an action principle from which to derive them is still not known. Moreover, *unfolding* just means that every field enters the field equations together with all its "descendants", *i.e.*, with all its derivatives: although this looks unconventional, this will enable a "canonical", uniform treatment of HS interactions. Finally, the introduction of infinitely many derivatives will turn out not to be redundant, and the zero-curvature equations will turn out to encode nontrivial dynamics thanks to trace constraints on the component fields.

The unfolded formulation is a particular and extremely interesting case of more general constructions known as *free differential algebras* (FDA) that we shall present first. As we shall see, its peculiarity lies in the introduction of the infinite-dimensional set of twisted-adjoint zero-forms within the general FDA scheme. Indeed, as previously mentioned, this idea, due to M. A. Vasiliev and first suggested in [79], is crucial for encoding nontrivial dynamics in a set of zero-curvature conditions.

[1]We are borrowing this term from the well-known case of supersymmetric theories, although, as it will be made clear in the rest of the Chapter, here we do not mean that supercharges enter among the symmetry generators of the theory (although they could, in principle, since supersymmetric extensions of HS gauge theories have indeed been constructed).

4.1 Free Differential Algebras

FDA were first introduced in physics by D'Auria and Frè [117] (see [118] for a review) as a way to formulate various supergravity theories containing differential forms of higher degree through zero-curvature equations. These generalize the Maurer-Cartan equations that define an algebra through the dual cotangent basis of one-forms of the corresponding Lie group manifold.

Let us consider an arbitrary set of differential $p-$forms $W^\alpha \in \Omega^{p_\alpha}(\mathcal{M}^D)$ with $p_\alpha \geq 0$ (0-forms are included) and α is an index enumerating various forms, which, in principle, may range in the infinite set $1 \leq \alpha < \infty$.

Let $R^\alpha \in \Omega^{p_\alpha+1}(\mathcal{M}^D)$ be generalized curvatures defined by the relations

$$R^\alpha = dW^\alpha + G^\alpha(W^\beta)\,, \tag{4.1.1}$$

where $G^\alpha(W^\beta)$ are some power series in W^β built with the aid of the exterior product of differential forms (that is understood wherever is needed),

$$G^\alpha(W^\beta) \;=\; \sum_{n=1}^{\infty} f^\alpha_{\beta_1\ldots\beta_n} W^{\beta_1}\ldots W^{\beta_n} \;. \tag{4.1.2}$$

The (anti)symmetry properties of the structure constants $f^\alpha_{\beta_1\ldots\beta_n}$ are such that $f^\alpha_{\beta_1\ldots\beta_n} \neq 0$ for $p_a + 1 = \sum_{i=1}^n p_{\beta_i}$, and the permutation of any two indices β_i and β_j brings a factor of $(-1)^{p_{\beta_i}p_{\beta_j}}$ (in the case of bosonic fields, *i.e.* with no extra Grassmann grading in addition to that of the exterior algebra).

A function $G^\alpha(W^\beta)$ satisfying the generalized Jacobi identity

$$G^\beta \frac{\delta^L G^\alpha}{\delta W^\beta} \equiv 0 \tag{4.1.3}$$

(the derivative with respect to W^β is acting from the left) defines a free differential algebra[2]. We emphasize that the property (4.1.3) is a condition on the function $G^\alpha(W)$ to be satisfied identically for all W^β. It is equivalent to the following generalized Jacobi identity on the structure coefficients

$$\sum_{n=0}^{m}(n+1) f^\gamma_{[\beta_1\ldots\beta_{m-n}} f^\alpha_{\gamma\beta_{m-n+1}\ldots\beta_m\}} \;=\; 0\;, \tag{4.1.4}$$

[2]We remind the reader that a differential d is a Grassmann odd nilpotent derivation of degree one, *i.e.* it satisfies the (graded) Leibnitz rule and $d^2 = 0$. A differential algebra is a graded algebra endowed with a differential d. Actually, the "free differential algebras" (in physicist terminology) are more precisely christened "graded commutative free differential algebra" by mathematicians (this means that the algebra does not obey algebraic relations apart from graded commutativity). In the absence of 0-forms, which however play a key ròle in the unfolded dynamics construction, the structure of these algebras is classified by Sullivan [119].

where the brackets [...} denote an appropriate (anti)symmetrization of all indices β_i. Strictly speaking, the generalized Jacobi identities (4.1.3) have to be satisfied only at $p_\alpha < D$ for the case of a D-dimensional manifold $\mathcal{M}^D$ where any $(D+1)$-form is zero. We shall call a free differential algebra universal if the generalized Jacobi identity holds for all values of the indices, *i.e.*, independently of a particular choice of space-time dimension. The HS free differential algebras discussed in this paper belong to the universal class.

The property (4.1.3) guarantees the generalized Bianchi identity

$$dR^\alpha = R^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}\,,$$

which tells us that the differential equations on W^β

$$R^\alpha = 0 \tag{4.1.5}$$

are consistent with $d^2 = 0$ and supercommutativity. Conversely, the property (4.1.3) is necessary for the consistency of eq. (4.1.5).

One defines the gauge transformations as

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}\,, \tag{4.1.6}$$

where $\varepsilon^\alpha(x)$ has form degree equal to $p_\alpha - 1$ (so that 0-forms W^α do not give rise to any gauge parameter). With respect to these gauge transformations the generalized curvatures transform as

$$\delta R^\alpha = -R^\gamma \frac{\delta^L}{\delta W^\gamma}\left(\varepsilon^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}\right)\,,$$

due to the property (4.1.3). This implies the gauge invariance of the equations (4.1.5). Also, since the equations (4.1.5) are formulated entirely in terms of differential forms, they are explicitly general coordinate invariant. In fact, the diffeomorphisms are incorporated in the gauge group, since the Lie derivative $\mathcal{L}_\xi W^\alpha \equiv \{d, i_\xi\} W^\alpha$, where i_ξ is the inner derivative with respect to a vector field $\xi = \xi^\mu \partial_\mu$, is equivalent, up to vanishing curvatures, to a field-dependent gauge transformation with parameters $\epsilon^\alpha = i_\xi W^\alpha$ (see Appendix A for an example).

4.1.1 Unfolding strategy

Unfolding means reformulation of the dynamics of one or another system in the form (4.1.5) which, as we explain below, is always possible by virtue of introducing enough auxiliary fields. Note that, according to (4.1.1), in this approach exterior differential of all fields is expressed in terms of the fields themselves, a feature that we had already encountered in Section 2.2 in the context of the MMSW-reformulation of gravity.

The case of a FDA that only contains one-forms coincides with the usual Maurer-Cartan dual formulation of an algebra. Indeed, let h be a Lie (super)algebra, a basis of which is the set $\{T_\alpha\}$, and $\omega = \omega^\alpha T_\alpha$ be a 1-form taking values in h. Choosing $G(\omega) = \omega^2 \equiv \frac{1}{2}\omega^\alpha\omega^\beta[T_\alpha, T_\beta]$, then eq. (4.1.5) with $W = \omega$ is the zero-curvature equation $d\omega + \omega^2 = 0$, and imposes the Maurer-Cartan equations on ω^α,

$$d\omega^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma}\omega^\beta\omega^\gamma \; = \; 0 \; . \tag{4.1.7}$$

The relation (4.1.3) then amounts to the usual Jacobi identity for the Lie algebra h. In the same way, (4.1.6) is the usual gauge transformation of the connection ω,

$$\delta\omega^\alpha \; = \; D\epsilon^a \; = \; d\epsilon^\alpha + f^\alpha_{\beta\gamma}\omega^\beta\epsilon^\gamma \; . \tag{4.1.8}$$

Note again that the whole philosophy of the MMSW reformulation of gravity examined in Section 2.2 consisted in a reformulation of gravity in AdS as a FDA with only one-forms, in such a way that the background emerges in a coordinate-independent way as a maximally symmetric solution ω_0 of the zero-curvature field equation with its global stability algebra h that solves $\delta\omega_0^\alpha = 0$. Recall that also free HS fields were analyzed in Section 2.2.2 as fluctuations around such a background.

If now the set W^α also contains some p-forms denoted by $\mathcal{C}^i$ (*e.g.* 0-forms) and if the functions G^i are linear in ω and C,

$$G^i = \omega^\alpha (T_\alpha)^i{}_j \mathcal{C}^j \; , \tag{4.1.9}$$

then the relation (4.1.3) implies that the coefficients $(T_\alpha)^i{}_j$ define some matrices T_α forming a representation T of h, acting in a module V where the $\mathcal{C}^i$ take their values. The corresponding equation (4.1.5) is a covariant constancy condition $D_\omega \mathcal{C} = 0$, where $D_\omega \equiv d + \omega$ is the covariant derivative in the h-module V, and admits the gauge symmetry

$$\delta\mathcal{C}^i \; = \; \epsilon^\alpha (T_\alpha)^i{}_j \mathcal{C}^j \; . \tag{4.1.10}$$

Suppose now that $\mathcal{C}^i$ are zero-forms C^i. Notice that, if we pursue the strategy of perturbatively expanding our FDA around a vacuum solution ω_0, and we treat both the remaining one-forms ω and the zero-forms as small fluctuations, the equation for the one-forms becomes

$$d\omega^\alpha + f^\alpha_{\beta\gamma}\omega_0^\beta\omega^\gamma + f^\alpha_{\beta\gamma i}\omega_0^\beta\omega_0^\gamma C^i \; = \; 0 \; . \tag{4.1.11}$$

We note that the vacuum equation $d\omega_0 + \omega_0^2 = 0$ together with (4.1.11) and the zero-form equation $dC^i + G^i = 0$ form a consistent set of equations: indeed, the latter ensures compatibility of the second with $d^2 = 0$, while the first is the consistency condition of the last (and also its own consistency condition). We shall see in the next Sections that free

HS field dynamics can be reformulated in this way - as, in fact, every dynamical system, upon the addition of sufficiently many auxiliary fields.

One may wonder how the set of equations

$$d\omega_0 + \omega_0^2 \;=\; 0\,, \tag{4.1.12}$$

$$D_{\omega_0} C \;=\; 0 \tag{4.1.13}$$

could describe any dynamics, giving that it implies that (locally) the connection ω_0 is pure gauge and C is covariantly constant, so that

$$\omega_0(x) \;=\; g^{-1}(x)\, dg(x)\,, \tag{4.1.14}$$

$$C(x) \;=\; g(x)\, \mathrm{C}\,, \tag{4.1.15}$$

where $g(x)$ is some function of the position x taking values in the Lie group associated with h (by exponentiation), and C is a constant vector of the h-module T. Since the gauge parameter $g(x)$ does not carry any physical degrees of freedom, all physical information is contained in the value $C(x_0) = g(x_0)\mathrm{C}$ of the 0-form $C(x)$ in a fixed point x_0 of space-time. But as one shall see in the next Section, if the 0-form $C(x)$ somehow parametrizes all derivatives of the original dynamical fields, then, supplemented with some algebraic constraints (that, in turn, single out an appropriate h-module), it can actually describe nontrivial dynamics. Indeed, the restrictions imposed on the values of some 0-forms at a fixed point x_0 can lead to a nontrivial dynamics if the set of 0-forms is rich enough to describe all space-time derivatives of the dynamical fields in a fixed point of space-time, provided that the constraints just single out those values of the derivatives which are compatible with the original dynamical equations. Knowing a solution (4.1.15) one knows all derivatives of the dynamical fields compatible with the field equations, and can therefore reconstruct these fields by analyticity in some neighborhood of x_0.

The p-forms with $p > 0$, that also satisfy a zero-curvature condition, are still pure gauge in this setting. As will be clear from the examples below, the meaning of the 0-forms C contained in $\mathcal{C}$ is that they describe all gauge invariant degrees of freedom (*e.g.* the spin-0 scalar field, the spin-1 Maxwell field strength, the spin-2 Weyl tensor, etc., and all their on-mass-shell nontrivial derivatives). When the gauge invariant 0-forms are identified with derivatives of the gauge fields which are $p > 0$ forms, this is expressed by a deformation of the equation of the latter,

$$D_{\omega_0}\mathcal{C} = P(\omega_0)\mathcal{C}\,, \tag{4.1.16}$$

where $P(\omega_0)$ is a linear operator (depending on ω_0 at least quadratically) acting on $\mathcal{C}$, as seen explicitly in (4.1.11). If the deformation is trivial, one can get rid of the terms on the right-hand-side of (4.1.16) by a field redefinition. The interesting case therefore is when the deformation is nontrivial. A useful criterium for telling whether the deformation

(4.1.16) is trivial or not is given in terms of the σ_- cohomology, discussed at length in [52].

Let us now stress some of the advantages of the unfolded formulation (see [92] for more comments) to understand why it is a useful for gauge theories in general and, in particular, HS gauge theories:

- As we have already stressed elsewhere, HS gauge transformations mix fields of different spins: in particular, the metric is not left invariant. This conflicts with the standard implementation of general covariance in General Relativity, where the inverse metric plays a crucial role. Therefore, the unfolded formulation, where manifest gauge invariance and invariance under diffeomorphisms (*i.e.*, coordinate independence) is achieved using the exterior algebra formalism and without any need *a priori* for singling out the metric, is perfectly suited for the study of gauge invariant theories in the framework of gravity and, in particular, HS gauge theories.

- As seen above, in the topologically trivial situation, the degrees of freedom are concentrated in the zero-forms at any point p in space-time. Indeed, the unfolded curvature constraints solve such zero-forms in terms of all on-shell nontrivial derivatives of the physical fields, that can be therefore reconstructed in a neighborhood of p. This implies that, in order to describe a system with an infinite number of degrees of freedom, it is necessary to work with an infinite set of zero-forms that spans an infinite-dimensional module of the space-time symmetry algebra g[3]. On the other hand, if the set of zero-forms is finite, the corresponding unfolded system is topological, describing at most a finite number of degrees of freedom.

- The unfolded formulation is thus an ultra-local approach to the dynamical problem that is particularly appealing if one is aiming at a background independent formulation of gauge theories: indeed, thanks to the the introduction of an infinite set of zero-forms, it is possible to achieve a generally covariant and dynamically nontrivial formulation of gauge theories where the metric is treated on an equal footing with the other fields. This is to be contrasted with what happens in Chern-Simons-like theories that, although diffeomorphism invariant without involving contractions of the indices by the inverse metric, are topological[4]. This in particular means that in the full HS equations of motion the inverse vielbein never appears, and therefore

[3]We will indeed construct, in Chapter 7, a mapping between the operators contained in the twisted adjoint zero-form at a fixed point in space-time and the states of the doubleton spectrum, *i.e.*, various massless irreducible representation $(\mathfrak{D}_0 \otimes \mathfrak{D}_0)_S = \bigoplus_{s_0=0,2,4,\dots} \mathfrak{D}(s_0 + 2\epsilon_0, s_0)$ (see also [90] for related statements in the context of conformal HS symmetries.).

[4]As pointed out in [93] however, it is possible to encode local degrees of freedom in a Chern-Simons (CS) theory by expanding it around a *non*-maximally symmetric solution. This is at the root of an attempt [101] towards the formulation of an action principle for full HSGT starting from a CS-like action in odd dimensions that does not make use of zero-forms.

the theory also naturally incorporates classical solutions with degenerate metrics (as we shall see explicitly in Chapter 6), that have long been conjectured to be of importance in quantum gravity as they can mediate space-time topology changes (see [94] and references therein).

- Equations (4.1.14) and (4.1.15) show, in particular, that the unfolded formulation based on universal FDAs makes the dependence on space-time coordinates purely auxiliary. The dynamics is entirely encoded in the functions $G^\alpha(W)$. This fact proves to be extremely useful in the search for consistent HS interactions: indeed, it means that one can search for them looking for deformations of the $G^\alpha(W)$ that still respect a generalized Jacobi identity, *i.e.*, that preserve the consistency of the system. As we shall see in the next Chapter, it also enables one to express the whole infinite perturbative series of nonlinear corrections to the free unfolded system as solution of some additional equations of an enlarged unfolded system, where differentials and differential forms live in a larger space: this does not spoil consistency, nor alters the local dynamics, that is still determined by the zero-forms at a point in space-time, as long as the additional equations locally reconstruct the dependence on the additional coordinates in terms of the original degrees of freedom.

- The unfolded formulation is in principle available for any dynamical system, provided one introduces additional auxiliary variables, since, as stressed in [91], it is nothing but a generally covariant first-order formalism.

We now have to determine what is the appropriate infinite-dimensional module of the space-time symmetry algebra in which the zero-forms have to take values, in order to encode nontrivial dynamics. This will be explained through the unfolding of free lower-spin fields.

4.2 Unfolding of lower spins

In this Section, we first show how one can indeed unfold an arbitrary system [85] and then apply such technique for the free spin 0 and spin 2 system [52].

4.2.1 General procedure

Let $\omega_0 = e_0^a\, P_a + \frac{1}{2}\omega_0^{ab} M_{ab}$ be a vacuum gravitational gauge field taking values in some space-time symmetry algebra s. Let $C^{(0)}(x)$ be a given space-time field satisfying some dynamical equations to be unfolded. Consider for simplicity the case where $C^{(0)}(x)$ is a 0-form. The general procedure of unfolding free field equations goes schematically as follows:

For a start, one writes the equation

$$D_0^L C^{(0)} \; = \; e_0^a \; C_a^{(1)} \, , \tag{4.2.1}$$

where D_0^L is the covariant Lorentz derivative and the field $C_a^{(1)}$ is auxiliary. Next, one checks whether the original field equations for $C^{(0)}$ impose any restrictions on the first derivatives of $C^{(0)}$. More precisely, some part of $\partial_\mu C^{(0)}$ might vanish on-mass-shell (*e.g.* for Dirac spinors). These restrictions in turn impose some restrictions on the auxiliary fields $C_a^{(1)}$. If these constraints are satisfied by $C_a^{(1)}$, then these fields parametrize all on-mass-shell nontrivial components of first derivatives.

Then, one writes for these first level auxiliary fields an equation similar to (4.2.1)

$$D_0^L C_a^{(1)} = e_0^b \; C_{a,b}^{(2)} \, , \tag{4.2.2}$$

where the new fields $C_{a,b}^{(2)}$ parametrize the second derivatives of $C^{(0)}$. Once again one checks (taking into account the Bianchi identities) which components of the second level fields $C_{a,b}^{(2)}$ are non-vanishing provided that the original equations of motion are satisfied.

This process continues indefinitely, leading to a chain of equations having the form of some covariant constancy condition for the chain of fields $C^{(m)}_{a_1,a_2,\ldots,a_m}$ ($m \in \mathbb{N}$) parametrizing all on-mass-shell nontrivial derivatives of the original dynamical field. By construction, this leads to a particular unfolded equation (4.1.5) with G^i in (4.1.1) given by (4.1.9). As explained in Section 4.1, this means that the set of fields realizes some module T of the space-time symmetry algebra s. In other words, the fields $C^{(m)}_{a_1,a_2,\ldots,a_m}$ are the components of a single field C living in the infinite-dimensional s–module T. Then the infinite chain of equations can be rewritten as a single covariant constancy condition $D_0 C = 0$, where D_0 is the s-covariant derivative in T.

4.2.2 The example of the scalar field

For simplicity, for the remaining of this Section, we will consider a flat space-time background. The Minkowski solution can be written as

$$\omega_0 = dx^\mu \delta^a_\mu P_a \tag{4.2.3}$$

i.e. the flat frame is $(e_0)^a_\mu = \delta^a_\mu$ and the Lorentz connection vanishes. The equation (4.2.3) corresponds to the pure gauge solution (4.1.14) with

$$g(x) = \exp(x^\mu \, \delta^a_\mu \, P_a) \, , \tag{4.2.4}$$

where the space-time Lie algebra s is identified with the Poincaré algebra $iso(d-1,1)$.

As a preliminary to the gravity example considered in the next subsection, the simplest field-theoretical case of unfolding is reviewed, *i.e.* the unfolding of a massless scalar field $\phi(x)$, which was first described in [85]. The "unfolding" of the massless Klein-Gordon equation

$$\Box\Phi(x) = 0 \tag{4.2.5}$$

is relatively easy to work out, so we give directly the final result and we comment about how it is obtained afterwards.

To describe dynamics of the spin zero massless field $\Phi(x)$, let us introduce the infinite collection of 0-forms $\Phi_{a_1\dots a_n}(x)$ $(n = 0, 1, 2, \dots)$ which are completely symmetric traceless tensors

$$\Phi_{a_1\dots a_n} = \Phi_{\{a_1\dots a_n\}}\ ,\quad \eta^{bc}\Phi_{bca_3\dots a_n} = 0\,. \tag{4.2.6}$$

The "unfolded" version of the Klein-Gordon equation (4.2.5) has the form of the following infinite chain of equations

$$d\Phi_{a_1\dots a_n} = e_0^b\Phi_{a_1\dots a_n b}\quad (n = 0, 1, \dots)\,, \tag{4.2.7}$$

where we have used the opportunity to replace the Lorentz covariant derivative D_0^L by the ordinary exterior derivative d. It is easy to see that this system is formally consistent because applying d on both sides of (4.2.7) does not lead to any new condition,

$$d^2\Phi_{a_1\dots a_n} = -e_0^b d\Phi_{a_1\dots a_n b} = e_0^b e_0^c d\Phi_{a_1\dots a_n bc} = 0 \quad (n = 0, 1, \dots)$$

since $e_0^b e_0^c = -e_0^c e_0^b$ because e_0^b is a 1-form. As we know from Section 4.1, this property implies that the space T of 0-forms $\Phi_{a_1\dots a_n}$ spans some representation of the Poincaré algebra $\mathfrak{iso}(D-1,1)$. In other words, T is an infinite-dimensional $\mathfrak{iso}(D-1,1)$-module[5].

To show that this system of equations is indeed equivalent to the free massless field equation (4.2.5), let us identify the scalar field $\Phi(x)$ with the member of the family of 0-forms $\Phi_{a_1\dots a_n}(x)$ at $n = 0$. Then the first two equations of the system (4.2.7) read

$$\partial_\nu\Phi = \Phi_\nu\ ,$$

$$\partial_\nu\Phi_\mu = \Phi_{\mu\nu}\ ,$$

where we have identified the world and tangent indices via $(e_0)_\mu^a = \delta_\mu^a$. The first of these equations just tells us that Φ_ν is the first derivative of Φ. The second one tells us that $\Phi_{\nu\mu}$ is the second derivative of Φ. However, because of the tracelessness condition (4.2.6) it imposes the Klein-Gordon equation (4.2.5). It is easy to see that all other equations in (4.2.7) express highest tensors in terms of the higher-order derivatives

$$\Phi_{\nu_1\dots\nu_n} = \partial_{\nu_1}\dots\partial_{\nu_n}\Phi \tag{4.2.8}$$

[5]Strictly speaking, to apply the general argument of Section 4.1 one has to check that the equation remains consistent for any flat connection in $\mathfrak{iso}(D-1,1)$. It is not hard to see that this is true indeed.

and impose no new conditions on Φ. The tracelessness conditions (4.2.6) are all satisfied once the Klein-Gordon equation is true. From this formula it is clear that the meaning of the 0-forms $\Phi_{\nu_1\ldots\nu_n}$ is that they form a basis in the space of all on-mass-shell nontrivial derivatives of the dynamical field $\Phi(x)$ (including the derivative of order zero which is the field $\Phi(x)$ itself).

Let us note that the system (4.2.7) without the constraints (4.2.6), which was originally considered in [88], remains formally consistent but is dynamically empty just expressing all highest tensors in terms of derivatives of Φ according to (4.2.8). This simple example illustrates how algebraic constraints like tracelessness of a tensor can be equivalent to dynamical equations.

In a parallel fashion, one can also check that indeed the unfolded dynamical problem is well-posed once one gives the values of all the zero-forms at a point in space-time. In fact, one can show that, again due to the trace constraints that the zero-forms satisfy, this is equivalent to the Cauchy problem. To simplify matters, let us apply the previous considerations to a 2-dimensional flat space-time, $a = 0, 1$. We suppose then that all $\Phi_{a(n)}(p)$, where the space-time point p has coordinates $p = (t_0, x_0)$, are given. On the constraints that the unfolded system imposes these are set equal to all the derivatives (4.2.8). However, because of trace constraints, there are fewer independent derivatives, and they "spread", "unfold" the local initial data onto the space-like line (hypersurface, in higher dimensions) $\{t = t_0\}$. In particular: for $n = 0$ one is given $\Phi(p)$; $n = 1$ fixes $\dot{\Phi}(p)$ and $\partial\Phi(p)$ (where we use the shorthand notations $\dot{\Phi}$ and $\partial\Phi$ for derivatives with respect to t and x, respectively); the independent local data for $n = 2$ is $\Phi_{00}(p) = \Phi_{11}(p) = \partial^2\Phi(p)$ and $\Phi_{01}(p) = \partial\dot{\Phi}(p)$; for $n = 3$ is $\Phi_{000}(p) = \Phi_{110}(p) = \partial^2\dot{\Phi}(p)$ and $\Phi_{001}(p) = \Phi_{111}(p) = \partial^3\Phi(p)$; and so on. In other words, the independent local data fixes *all* spatial derivatives of Φ and $\dot{\Phi}$, *i.e.*, it is equivalent to the standard initial data of the Cauchy problem, Φ and $\dot{\Phi}$ on the equal-time surface $\{t = t_0\}$. Notice however that the unfolded formulation is more general, since giving a set of zero-forms at a point in space-time can be done prior to specifying a metric (the zero-form indices are fiber indices only) and in a coordinate-independent way. Again, this is because the first-order unfolded equations involve a trading of space-time indices for fiber (tangent-space) indices, that makes it possible to encode a nontrivial dynamics into an algebraic constraint. This reasoning can be extended to arbitrary space-time dimensions.

The above considerations can be simplified further by means of introducing the auxiliary coordinate u^a and the generating function

$$\Phi(x,u) \;=\; \sum_{n=0}^{\infty} \frac{1}{n\,!}\Phi_{a_1\ldots a_n}(x)u^{a_1}\ldots u^{a_n} \tag{4.2.9}$$

with the convention that

$$\Phi(x,0) = \Phi(x)\,.$$

This generating function accounts for all tensors $\Phi_{a_1\dots a_n}$ once the tracelessness condition is imposed, which in these terms implies that

$$\Box_u \Phi(x,u) \equiv \frac{\partial}{\partial u^a}\frac{\partial}{\partial u_a}\Phi = 0\,. \tag{4.2.10}$$

In other words, the $\mathfrak{iso}(D-1,1)$-module T is realized as the space of harmonic formal power series in u^a. Eqns. (4.2.7) then acquire the simple form

$$\frac{\partial}{\partial x^\mu}\Phi(x,u) = \delta^a_\mu \frac{\partial}{\partial u^a}\Phi(x,u)\,. \tag{4.2.11}$$

From this realization one concludes that the translation generators in the infinite-dimensional module T of the Poincaré algebra are realized as translations in u–space, *i.e.*

$$P_a = -\frac{\partial}{\partial u^a}\,,$$

so that eqn. (4.2.11) reads as a covariant constancy condition (4.1.13)

$$d\Phi(x,u) + e_0^a P_a \Phi(x,u) = 0\,. \tag{4.2.12}$$

One can find a general solution of eq. (4.2.12) in the form

$$\Phi(x,u) = \Phi(x+u,0) = \Phi(0,x+u)$$

from which it follows in particular that

$$\Phi(x) \equiv \Phi(x,0) = \Phi(0,x) = \sum_{n=0}^{\infty}\frac{1}{n!}\Phi_{\nu_1\dots\nu_n}(0)x^{\nu_1}\dots x^{\nu_n}\,. \tag{4.2.13}$$

From (4.2.6) and (4.2.8) one can see that this is indeed the Taylor expansion for any solution of the Klein-Gordon equation which is analytic in $x_0 = 0$. Moreover one can recognize the equation (4.2.13) as a particular realization of the pure gauge solution (4.1.15) with the gauge function $g(x)$ of the form (4.2.4).

The example of a free scalar field is so simple that one might think that the unfolding procedure is always a trivial mapping of the original equation, in this case (4.2.5), to the equivalent one, here (4.2.10), in terms of additional variables. This is not true, however, for the less trivial cases of dynamical systems in nontrivial backgrounds and, especially, for nonlinear systems. The situation here is analogous to that in the Fedosov quantization prescription [95] which reduces the nontrivial problem of quantization in a curved background to the standard problem of quantization of the flat phase space, that, of course, becomes an identity when the ambient space itself is flat. It is worth to mention that this parallelism is not occasional because, as one can easily see, the Fedosov quantization prescription provides a particular case of the general unfolding approach [78] in the dynamically empty situation (*i.e.*, with no dynamical equations imposed).

The unfolded free scalar field can also be used as a prototype example of how the ultra-local unfolded approach to the dynamical problem can be mapped to the standard Cauchy problem, and viceversa. The key point is the possibility of encoding a Taylor expansion into an infinite-dimensional fiber at a point p: the corresponding degrees of freedom can be unfolded in space-time via an expansion in derivatives of the physical field, that is the content of the system (4.2.7). We shall encounter again a similar mechanism in Chapter 7, when we shall see that space-time local fluctuation fields, with different boundary conditions, are in correspondence with certain well-defined nonpolynomial combinations of operators of the HS algebra that are their fiber-(or tangent-space) duals, in the same way as (4.2.9) and (4.2.13) are dual via (4.2.12).

4.2.3 The example of gravity

The set of fields in the Einstein-Cartan's formulation of gravity comprises the frame field e_μ^a and the Lorentz connection ω_μ^{ab}. One assumes that the torsion constraint $T_a = 0$ is satisfied, in order to express the Lorentz connection in terms of the frame field. The Lorentz curvature can be expressed as $R^{ab} = e_c e_d \, R^{[ab]\,;\,[cd]}$, where $R^{ab\,;\,cd}$ is a rank four tensor with indices in the tangent space and which is antisymmetric both in ab and in cd, having the symmetries of the tensor product $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array} \otimes \begin{array}{|c|}\hline c\\\hline d\\\hline\end{array}$. The algebraic Bianchi identity $e_b R^{ab} = 0$, which follows from the zero torsion constraint, forces the tensor $R^{ab\,;\,cd}$ to possess the symmetries of the Riemann tensor, *i.e.* $R^{[ab\,;\,c]d} = 0$. More precisely, it carries an irreducible representation of $GL(D)$ characterized by the Young tableau $\begin{array}{|c|c|}\hline a & c\\\hline b & d\\\hline\end{array}$ in the antisymmetric basis. The vacuum Einstein equations state that this tensor is traceless, so that it is actually irreducible under the pseudo-orthogonal group $O(D-1,1)$ on-mass-shell. In other words, the Riemann tensor is equal on-mass-shell to the Weyl tensor.

For HS generalization, it is more convenient to use the symmetric basis. In this convention, the Einstein equations can be written as

$$T^a = 0\,, \qquad R^{ab} \;=\; e_c\, e_d\, \Phi^{ac,\,bd}\,, \tag{4.2.14}$$

where the 0-form $\Phi^{ac,\,bd}$ is the Weyl tensor in the symmetric basis. More precisely, the tensor $\Phi^{ac,bd}$ is symmetric in the pairs ac and bd and it satisfies the algebraic identities

$$\Phi^{(ac,\,b)d} = 0\,, \qquad \eta_{ac}\Phi^{ac,\,bd} = 0\,.$$

Notice that, while dynamically equivalent to the vanishing of the Ricci tensor, this formulation of the vacuum Einstein equations is more suitable for HS extensions, in that it is written purely in terms of differential forms, and does not involve the inverse metric.

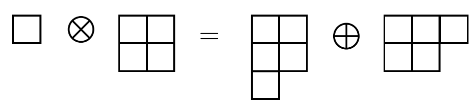
In other words, the metric is not given any special role, and this makes this formulation of the spin-2 field equations manifestly compatible with a symmetry that mixes different spins.

Let us now start the unfolding of linearized gravity around the Minkowski background described by a frame 1-form e_0^a. The linearization of the second equation of (4.2.14) is

$$R_1^{ab} \;=\; e_{0\,c}\, e_{0\,d}\, \Phi^{ac,\,bd}\,, \tag{4.2.15}$$

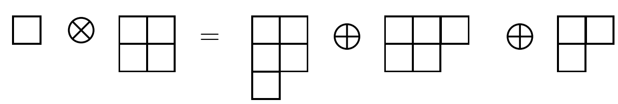
where R_1^{ab} is the linearized Riemann tensor. This equation is a particular case of eq. (4.1.16). What is lacking at this stage is the additional set of equations containing the differential of the Weyl 0-form $\Phi^{ac,\,bd}$. Since we do not want to impose any additional dynamical restrictions on the system, the only restrictions on the derivatives of the Weyl 0-form $\Phi^{ac,\,bd}$ may result from the Bianchi identities for (4.2.15).

A priori, the first Lorentz covariant derivative of the Weyl tensor is a rank-five tensor in the following representation



(4.2.16)

decomposed according to irreducible representations of $\mathfrak{sl}(D)$. Since the Weyl tensor is traceless, the right hand side of (4.2.16) contains only one nontrivial trace, that is for traceless tensors we have the $\mathfrak{so}(D-1,1)$ Young decomposition by adding a three cell hook tableau, *i.e.*



The linearized Bianchi identity $dR_1^{ab} = 0$ leads to

$$e_{0\,c} e_{0\,d}\, d\Phi^{ac,\,bd} = 0\,. \tag{4.2.17}$$

The components of the left-hand-side, written in the basis $dx^\mu dx^\nu dx^\rho$, have the symmetry property corresponding to the tableau

$$\begin{array}{|c|c|}\hline \mu & a \\ \hline \nu & b \\ \hline \rho & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;\sim\; \partial_{[\rho}\Phi^{a}{}_{\mu}{}^{,b}{}_{\nu]}\,,$$

which also contains the single trace part with the symmetry properties of the three-cell hook tableau.

Therefore the consistency condition (4.2.17) states that in the decomposition (4.2.16) of the Lorentz covariant derivative of the Weyl tensor, the first term vanishes and the second term is traceless but otherwise arbitrary. Let $\Phi^{abf,\,cd}$ be the traceless tensor corresponding to the second term in the decomposition (4.2.16) of the Lorentz covariant derivative of the Weyl tensor. This is equivalent to saying that

$$d\Phi^{ac,\,bd} \;=\; e_{0\,f}\,(2\Phi^{acf,\,bd} + \Phi^{acb,\,df} + \Phi^{acd,\,bf})\,,$$

where the right hand side is fixed by the Young symmetry properties of the left hand side modulo an overall normalization coefficient. This equation looks like the first step (4.2.1) of the unfolding procedure, with $\Phi^{acf,\,bd}$ irreducible under $\mathfrak{so}(D-1,1)$.

One should now perform the second step of the general unfolding scheme and write the analogue of (4.2.2). This process goes on indefinitely. To summarize the procedure, one can analyze the decomposition of the k-th Lorentz covariant derivatives (with respect to the Minkowski vacuum background, so they commute) of the Weyl tensor $\Phi^{ac,\,bd}$. Taking into account the Bianchi identity, the decomposition goes as follows

$$\ydiagram{k}\; k \;\otimes\; \ydiagram{2,2} \;\cong\; \ydiagram{k+2,2}\; k+2 \tag{4.2.18}$$

As a result, one obtains

$$d\Phi^{a_1\ldots a_{k+2},\,b_1b_2} \;=\; e_{0\,c}\Big(\,(k+2)\,\Phi^{a_1\ldots a_{k+2}c,\,b_1b_2} \;+\; \Phi^{a_1\ldots a_{k+2}b_1,\,b_2c} \;+\; \Phi^{a_1\ldots a_{k+2}b_2,\,b_1c}\Big)\,,$$
$$(0\leq k\leq\infty)\,, \tag{4.2.19}$$

where the fields $\Phi^{a_1\ldots a_{k+2},\,b_1b_2}$ are in the irreducible representation of $o(d-1,1)$ characterized by the traceless two-row Young tableau on the right hand side of (4.2.18), *i.e.*

$$\Phi^{(a_1\ldots a_{k+2},\,b_1)b_2} = 0\,,\quad \eta_{a_1a_2}\Phi^{a_1a_2\ldots a_{k+2},\,b_1b_2} = 0\,.$$

Note that, as expected, the system (4.2.19) is consistent with $d^2\Phi^{a_1\ldots a_{k+2},\,b_1b_2} = 0$.

As in the spin-zero case, the meaning of the zero-forms $\Phi^{a_1\ldots a_{k+2},\,b_1b_2}$ is that they form a basis in the space of all on-mass-shell nontrivial gauge invariant combinations of the derivatives of the spin-2 gauge field.

4.3 Free massless equations for any spin

In order to follow the strategy exposed in Subsection 4.1.1 and generalize the example of gravity treated along these lines in Subsection 4.2.3, we shall begin by writing unfolded HS field equations in terms of the linearized HS curvatures (2.2.22). This result is called "central on-mass-shell theorem". It was originally obtained in [56, 78] for the case of $D=4$ and then extended to any D in [57, 89]. That these HS equations of motion indeed reproduce the correct physical degrees of freedom can be shown via an elegant cohomological approach explained in [52].

The linearized curvatures $F_1^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ were defined in (2.2.22). They decompose into the linearized curvatures with Lorentz (*i.e.*, V^A transverse) fibre indices which have the symmetry properties associated with the two-row traceless Young tableau $\begin{array}{l}\boxed{}\,{\scriptstyle s-1}\\ \boxed{}\,{\scriptstyle t}\end{array}$. It is convenient to use the standard gauge $V^A = \delta_{0'}^A$ (we also normalize V to unity). In the Lorentz basis, the linearized HS curvatures have the form

$$F_1^{a_1\dots a_{s-1},\,b_1\dots b_t} \;=\; D_0^L A^{a_1\dots a_{s-1},\,b_1\dots b_t} \;+\; e_{0\,c}\,A^{a_1\dots a_{s-1},\,b_1\dots b_t c} \;+\; O(\Lambda)\,. \tag{4.3.1}$$

For simplicity, in this section we discard the complicated Λ–dependent terms which do not affect the general analysis, *i.e.* we present explicitly the flat-space part of the linearized HS curvatures. It is important to note however that the Λ–dependent terms in (4.3.1) contain the field $A^{a_1\dots a_{s-1},\,b_1\dots b_{t-1}}$ which carries one index less than the linearized HS curvatures. The explicit form of the Λ–dependent terms is given in [57].

For $t = 0$, these curvatures generalize the torsion of gravity, while for $t > 0$ the curvature corresponds to the Riemann tensor. In particular, as it is shown in [52], the analogues of the Ricci tensor and scalar curvature are contained in the curvatures with $t = 1$ while the HS analog of the Weyl tensor is contained in the curvatures with $t = s-1$. (For the case of $s = 2$ they combine into the level $t = 1$ traceful Riemann tensor.)

The first on-mass-shell theorem states that the following free field equations in Minkowski or $(A)dS$ space

$$F_1^{a_1\dots a_{s-1},\,b_1\dots b_t} \;=\; \delta_{t,\,s-1}\;\, e_{0\,c}\,e_{0\,d}\;\Phi^{a_1\dots a_{s-1}c,\,b_1\dots b_{s-1}d}\,, \qquad (0 \le t \le s-1) \tag{4.3.2}$$

properly describe completely symmetric gauge fields of generic spin $s \ge 2$. This means that they are equivalent to the proper unfolded version of the Fronsdal equations in any dimension, supplemented with certain algebraic constraints on the auxiliary HS connections which express the latter via derivatives of the dynamical HS fields. The zero-form $\Phi^{a_1\dots a_s,\,b_1\dots b_s}$ is the spin-s Weyl-like tensor. It is irreducible under $\mathfrak{so}(D-1,1)$ and is characterized by a rectangular two-row Young tableau $\begin{array}{l}\boxed{}\,{\scriptstyle s}\\ \boxed{}\,{\scriptstyle s}\end{array}$. The field equations generalize (4.2.15) of linearized gravity. The equations of motion put to zero all curvatures with $t \neq s-1$ and require $\Phi^{a_1\dots a_s,\,b_1\dots b_s}$ to be traceless.

The analysis of the Bianchi identities of (4.3.2) works for any spin $s \ge 2$ in a way analogous to gravity. The final result is the following equation [89] which presents itself like a covariant constancy condition

$$\begin{aligned} 0 = D_0\Phi^{a_1\dots a_{s+k},\,b_1\dots b_s} \;\;\equiv\;\; & D_0^L\Phi^{a_1\dots a_{s+k},\,b_1\dots b_s} \\ & - e_{0\,c}\Big((k+2)\Phi^{a_1\dots a_{s+k}c,\,b_1\dots b_s} \;+\; s\,\Phi^{a_1\dots a_{s+k}\{b_1,\,b_2\dots b_s\}c}\Big) + O(\Lambda)\,, \\ & (0 \le k \le \infty)\,, \end{aligned} \tag{4.3.3}$$

where $\Phi^{a_1\dots a_{s+k},\,b_1\dots b_s}$ are $\mathfrak{so}(D-1,1)$ irreducible (*i.e.*, traceless) tensors characterized by the Young tableaux $\begin{array}{l}\boxed{}\,{\scriptstyle s+k}\\ \boxed{}\,{\scriptstyle s}\end{array}$. They describe on-mass-shell nontrivial k-th derivatives of the spin-s Weyl-like tensor, thus forming a basis in the space of gauge invariant combinations of $(s+k)$-th derivatives of a spin s HS gauge field. The system (4.3.3) is the generalization of the spin-0 system (4.2.7) and the spin-2 system (4.2.19) to arbitrary spin and to AdS background (the explicit form of the Λ–dependent terms is given in [89]). Let us stress that for $s \geq 2$ the infinite system of equations (4.3.3) is a consequence of (4.3.2) by the Bianchi identity. For $s = 0$ and $s = 1$, the system (4.3.3) contains the dynamical Klein-Gordon and Maxwell equations, respectively. Note that (4.3.2) makes no sense for $s = 0$ because there is no spin-0 gauge potential while (4.3.3) with $s = 0$ reproduces the unfolded spin-0 equation (4.2.7) and its AdS generalization. For the spin-1 case, (4.3.2) only defines the spin-1 Maxwell field strength $\Phi^{a,b} = -\Phi^{b,a}$ in terms of the potential A_μ. The dynamical equations for spin 1, *i.e.* the Maxwell equations, are contained in (4.3.3). The fields $\Phi^{a_1\cdots a_{k+1},\, b}$, characterized by the Lorentz irreducible (*i.e.* traceless) two-row Young tableaux with one cell in the second row, form a basis in the space of on-mass-shell nontrivial derivatives of the Maxwell tensor $\Phi^{a,\,b}$.

Since k goes from zero to infinity for any fixed s in (4.3.3), in agreement with the general arguments of Section 4.1, each irreducible spin-s submodule of the twisted adjoint representation is *infinite-dimensional*. This means that, in the unfolded formulation, the dynamics of any fixed spin-s field is described in terms of an infinite set of fields related by the first-order unfolded equations. Of course, to make it possible to describe a field-theoretical dynamical system with an infinite number of degrees of freedom, the set of auxiliary zero-forms associated with all gauge invariant combinations of derivatives of dynamical fields should be infinite-dimensional.

It is clear that the complete set of zero-forms $\Phi^{a_1\cdots a_{s+k},\, b_1\ldots b_s} \sim \begin{array}{l}\boxed{}\; s+k\\ \boxed{}\; s\end{array}$ covers the set of all two-row Young tableaux. This suggests that the Weyl-like zero-forms take values in the linear space of $\mathfrak{ho}$ that obviously forms an $\mathfrak{so}(D-1,1)$- (*i.e.* Lorentz) module. Following the strategy sketched in Subsection 4.1.1, one can expect that they belong to an $\mathfrak{so}(D-1,2)$-module. But the idea to use the adjoint representation of $\mathfrak{ho}(D-1,2)$ defined in (3.1.69) does not work because, according to the commutation relation (3.0.1), the commutator of the background gravity connection $\omega_0 = \omega_0^{AB}\widehat{T}_{AB}$ with a generator of $\mathfrak{ho}$ preserves the rank of the generator, while the covariant derivative D in (4.3.3) acts on the infinite set of Lorentz tensors of infinitely increasing ranks. Fortunately, the appropriate representation only requires a slight modification compared to the adjoint representation. As it is clear by looking at (3.1.80-7.6.16), the representation we are looking for is nothing but the *twisted adjoint* presented in Section 3.1.2, that naturally encodes the set of zero-forms that are needed for the unfolding for every spin-s sector and transforms through the π-twisted adjoint action (3.1.80). Indeed, the whole set of equations (4.3.2) and (4.3.3) can be written in quite a compact form in terms of the adjoint master one-form (7.6.13) and of the twisted adjoint master zero-form (7.6.15) as

$$F_1 \;\equiv\; dA + \{\omega_0, A\}_\star \;=\; e_0^a\, e_0^b\, \frac{\partial \Phi}{\partial M^{ab}}\bigg|_{P^a=0} \;, \tag{4.3.4}$$

$$D_0\Phi \;\equiv\; d\Phi + [\omega_0, \Phi]_\pi \;=\; 0 \;, \tag{4.3.5}$$

where we recall that the anticommutator of the two one-forms in (4.3.4) implements the adjoint action (3.0.1). The two linearized equations above admit the gauge symmetries

$$\delta^{(0)}A \;=\; D_0\epsilon \;=\; d\epsilon + [\omega_0, \epsilon] \;, \qquad \delta^{(0)}\Phi \;=\; 0 \;, \tag{4.3.6}$$

where ϵ is an adjoint gauge parameter of $\mathfrak{ho}(D-1,2)$, and therefore contains all the linearized gauge transformations $\{\epsilon^{a(s-1),b(t)}\}$ for the set of one-form connections $\{A^{a(s-1),b(t)}\}$ that span the spin-s sector (7.6.14) of the master one-form, and that are needed for the frame-like description of a spin-s field. The equations above contain all the curvature constraints of the frame formulation recalled in Section 2.2.2, supplemented with the unfolding of the Weyl "source term" that follows from the Bianchi identities. As already stressed before, the physical degrees of freedom contained in the system can be conveniently analyzed through the so-called σ_--cohomology involving gauge parameters, gauge potentials, curvatures and Bianchi identities. The details can be found in [52] (see also [116, 115, 103]), but the end result is essentially that solving the generalized torsion constraints (contained in the equations (4.3.2) for $t < s-1$) expresses certain ($\mathfrak{so}(D-1,1)$-irreducible) components of the auxiliary fields contained in each spin-s sector in terms of the generalized frame field $A_\mu^{a(s-1)}$; all the remaining ones are pure gauge, and can be eliminated by fixing the Stückelberg-type gauge parameters (all those with $t \geq 1$), except for the totally symmetric component of the generalized frame field, *i.e.*, the Fronsdal doubly traceless field $\varphi_{\mu(s)} = (e_0^{-1})_{(\mu_1}{}^{a_1} ... (e_0^{-1})_{\mu_{s-1}}{}^{a_{s-1}} A_{\mu_s),a(s-1)}$. The latter carries therefore the physical degrees of freedom, and is accompanied by the remaining gauge symmetry $\epsilon^{a(s-1)}$, which coincides with the true (*i.e.*, differential, and not shift symmetry) Fronsdal traceless gauge parameter. Moreover, the remaining components of the curvatures $F_{1\,\mu\nu}^{a(s-1),b(t))}$ are set to zero by the Bianchi identities except for the $F_1^{a(s),b(s)}$ that is left free to fluctuate.

Eqns. (4.3.4), (4.3.5) provide the unfolded form of the free equations of motion for completely symmetric massless fields of all spins in any dimension. This fact is referred to as central on-mass-shell theorem because it plays a distinguished role in various respects. The idea of the proof is explained in [52] (see references therein for the original papers). Let us note that the right-hand-side of eq. (4.3.4) is a particular realization of the deformation terms (4.1.11) in free differential algebras.

4.3.1 A few Remarks

As shown in the examples given in this Chapter, in the unfolded formulation the trace constraints on the tangent-space indices of the fields involved are crucial to ensure that the first-order system of zero-curvature (or covariant constancy) conditions is not dynamically empty. Indeed, they are responsible for encoding the second-order physical field equation in the system. Once such algebraic constraints are taken into account and one works with traceless one-forms and zero-forms, however, the free unfolded system can be shown to be equivalent to the Fronsdal equations, and in this sense one can say that trace constraints put the system on-shell - and, conversely, that the unfolded equations with traceful fields describe an off-shell (or, better to say, nondynamical) system.

However, it is important to stress at this point that, as discovered first in [77], the free unfolded equations can nicely recover also the local, unconstrained Francia-Sagnotti equations (2.1.17), and also their $(A)dS$ version (2.1.53). This can be achieved by declaring only the zero-forms to be traceless in the internal indices, and letting the trace parts in the one-forms free to adjust to their source terms in (4.3.2). Let us examine briefly how this can happen by looking at the example of spin 3 in flat space-time. Let $\varphi_{\mu ab}$ be the totally symmetric $(3,0)$ component of the frame-like field. After solving some generalized torsion constraints and using the shift symmetries, one gets

$$A^{(2,2)}_{\mu,ab,cd} \;=\; \gamma_{2,2}\,\partial_{\langle c}\partial_d\varphi_{ab\rangle\mu} \;, \tag{4.3.7}$$

where $\gamma_{2,2}$ is a coefficient, unimportant for this discussion, and we recall that the indices enclosed in the brackets $\langle ...\rangle$ are $\mathfrak{sl}(D-1,1)$-projected. By tracing two indices this becomes

$$A^{(2,2)}_{\mu,ab,c}{}^{c} \;=\; \frac{\gamma_{2,2}}{3}K_{\mu,ab} \;, \tag{4.3.8}$$

with

$$K_{\mu,ab} \;=\; \Box^2\varphi_{\mu ab} - 2\partial_{(a}\partial\cdot\varphi_{b)\mu} + \partial_a\partial_b\varphi'_\mu \;. \tag{4.3.9}$$

The curvature constraints involving such components of the gauge potentials read

$$\partial_{[\mu}A^{(2,2)}_{\nu],ab,c}{}^{c} \;=\; 0 \;, \tag{4.3.10}$$

since the trace of the zero-form on the right hand side vanishes. This implies that the curl of the tensor $K_{\mu,ab}$ vanishes, i.e., that the latter is a pure gradient,

$$K_{\mu,ab} \;=\; \partial_\mu\beta_{ab} \;. \tag{4.3.11}$$

This admits the two projections $\mathfrak{sl}(D-1,1)$-irreducible projections $(3,0)$ and $(2,1)$. The latter is a consistency condition that admits the solution

$$\beta^{(2,0)}_{ab} \;=\; \partial\cdot\varphi_{ab} - 2\partial_{(a}\varphi'_{b)} + \partial_a\partial_b\alpha \;, \tag{4.3.12}$$

where the last term is a homogeneous solution parametrized by an unconstrained field α. Finally, substituting back into the solution (4.3.11) of the curvature constraint and taking the $(3,0)$ projection one indeed finds the equations (2.1.17),

$$\Box\varphi_{\mu ab} - 3\partial_{(a}\partial\cdot\varphi_{b\mu)} + 3\partial_{(a}\partial_b\varphi'_{\mu)} \;=\; \partial_{(a}\partial_b\partial_{\mu)}\alpha \;. \tag{4.3.13}$$

Moreover, in this case the leftover gauge parameter is, as the adjoint fields, an $\mathfrak{sl}(D-1,1)$-irreducible parameter. In other words, the system has an unconstrained gauge symmetry. The compensator therefore enters the unfolded constraints as a cohomologically exact part [120] in the solutions of a trace part of the curvature constraints.

This consideration points towards the fact that under many respects, including a comparison with String Field Theory, it may be important to have a formulation of the unfolded equations in terms of master fields that contain traceful generators and coefficients. The appropriate HS algebra corresponds to the enveloping algebra of $\mathfrak{so}(D-1,2)$ (or $\mathfrak{iso}(D-1,1)$ as well, as long as the free theory is concerned) modulo the V_{ABCD} element only, and not also V_{AB}, leading to fields and generators represented by traceful two-row Young diagrams. Notice moreover that this is exactly what the vector oscillator realization, treated in Subsection 3.3.1 enables one to do, prior to factoring out the $\mathfrak{sp}(2)$ ideal. Then, one can put the system on-shell, *i.e.*, enforcing the equations of motion on the physical field, by projecting out the ideal in the zero-form sector only, if one wants to get the unconstrained formulation; or, alternatively, both in the one-form and the zero-form sector if one wants to obtain the Fronsdal equations. On the other hand, the spinor oscillator formulation in intrinsically on-shell, since trace parts are automatically factored out, as explained in Subsection 3.3.2.

We are finally ready to see how interaction terms can be added. The next step will be therefore to promote the linearized curvature F_1 and the covariant derivative D_0 to full ones $F = dA + A \star A$ and $D = d + [A,]_\pi$, involving the whole $\mathfrak{ho}(D-1,2)$-connection A, and to interpret (4.3.4) and (4.3.5) as the linearization of a full system

$$\mathcal{F} \;\equiv\; F + \mathcal{J}(A,\Phi) \;=\; 0 \;, \tag{4.3.14}$$

$$\mathcal{D}\Phi \;\equiv\; D\Phi + \mathcal{P}(A,\Phi) \;=\; 0 \;, \tag{4.3.15}$$

where $\mathcal{J}$ and $\mathcal{P}$ are non-linear zero-form deformations obeying the compatibility condition (4.1.3) and having the correct physical weak field limit (4.3.4) and (4.3.5), that excludes the trivial solution $\mathcal{J} = \mathcal{P} = 0$. These functions are at least cubic and written using the wedge product (and index contractions by the flat metric η_{ab} and possibly the anti-symmetric tensor $\epsilon_{a_1 \dots a_D}$). Thus $\mathcal{J}$ is quadratic in A and at least linear in Φ, while $\mathcal{P}$ is linear in A and at least quadratic in Φ. Notice that such nonlinearities are required to maintain consistency: indeed,

$$D^2\Phi \sim F\Phi \sim \Phi^2 + \text{higher order terms} \;, \tag{4.3.16}$$

being F at least of first-order in Φ.

Before looking for the consistent nonlinear deformations, one can make one more observation about the unfolding procedure. As already stresses in the scalar field example, at the free level unfolding may seem a cumbersome reformulation of the dynamics. The point is that at the free level, and also at the interacting level for lower-spin theories, higher-derivative interactions do not enter the physical field equations: indeed, all the one-form connections $A_\mu^{a(s-1),b(t)}$ with $t \geq 2$, that are solved from torsion constraints as

$$A^{\,a_1\dots a_{s-1},\,b_1\dots b_t} = \Pi\left(\Lambda^{-t/2}\frac{\partial}{\partial x^{b_1}}\cdots\frac{\partial}{\partial x^{b_t}}A^{\,a_1\dots a_{s-1}}\right) + \text{lower derivative terms}\,, \quad (4.3.17)$$

and the infinite chain of equations that constrains higher-rank zero-forms $\Phi_{a(s+k),b(s)}$ have no effect on the dynamics. In particular, $\Phi_{a(s+k),b(s)}$ with $k+s>1$ do not enter the contorsion or the stress-energy tensor for matter-coupled gravity or supergravity, nor the HS free theories, as we have seen. However, this is not the case for interacting HS gauge theories, where higher-derivative interaction vertices naturally enter the physical field equations already at second order in the weak-field expansion: this was seen, for example, in [61, 39, 40, 38] among other works, and the quadratic scalar-field contribution to T_{ab} (*i.e.* the scalar-scalar-graviton vertex) in $D=4$ resulting from the full Vasiliev equations was computed in [102] (using a specific physical gauge choice made at the level of the full Vasiliev equations). The result is an infinite series of higher-derivative interactions given by various contractions of $\nabla^m_{a(m)}\phi\nabla^n_{b(n)}\phi$ for arbitrarily high values of $m+n$. Higher-derivative interaction, compensated by the inverse of the cut-off mass scale M, do arise in lower-spin field theories, but they only yield small corrections to classical solutions in which derivatives are small in units of M. On the other hand, in a theory with local unbroken HS symmetries, higher-derivative interactions are part of the minimally coupled microscopic theory. Interacting HS can in principle give rise to many complicated higher-derivative interaction terms in the physical field equation: the unfolded formulation offers a way of handling them systematically, in a first-order form: higher derivatives are hidden in the auxiliary higher-rank zero-forms, that enter the field equations through consistent deformations of the FDA describing the free system. Therefore, the problem is reduced to find such deformations, since gauge invariance is guaranteed by consistency of the system (*i.e.*, the deformed gauge symmetry follows automatically through the relation (4.1.6), and does not need to be guessed).

Chapter 5

Nonlinear equations

5.1 Preliminaries

In the last Chapter, we have recalled the unfolded formulation of the free theory. We still have to make use of the non-abelian HS algebra constructed in Chapter 3 to write nonlinear corrections. We are now in a position similar to having the linearized vacuum Einstein equations for the gravitational field $h_{\mu\nu}$ (where of course $h_{\mu\nu}$ represents fluctuations over a fixed background metric that can be chosen to be Minkowski, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) and wanting to get the full nonlinear Einstein equations without the possibility of resorting to geometrical concepts. This program was indeed completed successfully for gravity several years after the formulation of General Relativity (see [121, 122, 123] and references therein). The idea was to proceed by consistency: as soon as one allows quadratic terms, the free equations of motion become inconsistent, since while the linearized Einstein tensor is divergenceless, this is not the case for its source, the stress-energy tensor, as long as one does not introduce the contribution of gravity to the latter, as calculated from the free action. However, in order for the new equation to be derived from a Lagrangian, one must add cubic terms to the free Lagrangian that respect gauge invariance (up to $\mathcal{O}(h^2)$). But the variation of such cubic Lagrangian introduces new terms, that spoil consistency. This process goes on indefinitely, and the infinite series of nonlinear corrections sums up to the full Einstein equations, where they appear through a crucial contribution of the inverse metric.

A similar program would still be hardly tractable for higher spins, due to the presence of infinitely many fields and symmetries one should control. As explained in the previous Chapter, the unfolded formalism and the identification of the convenient master fields one should work with bring a great simplification. Pursuing this approach (first proposed in [79]), the "HS interaction problem" was solved, for totally symmetric massless fields, by Vasiliev at the beginning of the Nineties [81]. As shown above, indeed, in his formulation the problem is reduced to find nonlinear corrections to (4.3.4) and (4.3.5) that preserve consistency in the sense of (4.1.3).

Let us begin our analysis by observing that, in the unfolded setup, the natural expansion parameter is the zero-form Φ: indeed, containing all the (generalized) Weyl tensors and their derivatives, it is the variable that measures the deviation from the background solution; moreover, as already stressed below (4.3.15), in order to preserve the form-degree of the curvature constraints, arbitrary nonlinearities can only involve zero-forms contained in Φ, and not A. Indeed, consistent deformations of the free FDA at the lowest order in curvatures were found in [79, 80]. However, to find the whole series a more systematic approach was necessary, and that was also developed exploiting the power of the unfolded formalism.

One could obtain all consistent nonlinear corrections in one shot if some deformation of the HS algebra $\mathfrak{ho} \to \widehat{\mathfrak{ho}}$ existed such that the full nonlinear constraints (4.3.15) could be seen as zero-curvature and covariant constancy conditions for the deformed algebra [51]; *i.e.*, if $\mathcal{F}$ and $\mathcal{D}$ could be seen as curvature and covariant derivative with respect to the connection $\widehat{A}$ of $\widehat{\mathfrak{ho}}$[1]. In which case, the latter should contain all the nonlinearities in the zero-forms contained in Φ with indices contracted in various ways: $\widehat{A} = A + A\Phi + A\Phi\Phi + ...$, schematically. At the same time, the gauge symmetries get modified correspondingly. How can one construct such a deformation?

The basic trick is to introduce an additional set of auxiliary, noncommutative coordinates Z and express the entire series of nonlinear corrections as solution of a consistent equation with respect to them. Such deformations will by this mechanism be guaranteed to be consistent and, therefore, will preserve the gauge invariance. Therefore, one enlarges the space-time with additional variables Z, $x \longrightarrow (x, Z)$, and lets all the variables and differentials acquire a dependence on them:

$$d \longrightarrow \widehat{d} = d + d_Z \ , \tag{5.1.1}$$

$$A(x; M_{ab}, P_a) \longrightarrow \widehat{A}(x; Z; M_{ab}, P_a) \tag{5.1.2}$$

$$= i \sum_s \sum_{t=0}^{s-1} dx^\mu A_{\mu, a(s-1), b(t)}(x; Z) M^{a_1 b_1} \cdots M^{a_t b_t} P^{a_{t+1}} \cdots P^{a_{s-1}} \tag{5.1.3}$$

$$\Phi(x; M_{ab}, P_a) \longrightarrow \widehat{\Phi}(x; Z; M_{ab}, P_a) \tag{5.1.4}$$

$$= \sum_s \sum_{k=0}^{\infty} \frac{i^k}{k!} \Phi^{a(s+k), b(s)}(x; Z) M_{a_1 b_1} \cdots M_{a_s b_s} P_{a_{s+1}} \cdots P_{a_{s+k}} \ , \tag{5.1.5}$$

where the coefficients are themselves power series in the Z variables. The ordinary space-

[1]In this chapter and in the next one we denote with a hat the variables valued in the deformed algebra that enter Vasiliev's nonlinear equations, and that should not be confused with the hatted generators that appeared in Chapter 3.

time corresponds to the subspace $\{Z = 0\}$. Notice, however, that due to the noncommutative nature of the new coordinates the pull-back of the extended unfolded system to $\{Z = 0\}$ does not correspond to the original system,

$$d\widehat{A} + \widehat{A} \star \widehat{A}\big|_{Z=0} \neq dA + A \star A \ , \tag{5.1.6}$$

since there will always be infinitely many contributions to the identity coming from total contractions of the Z variables. But how is this connected to the nonlinear corrections in Φ?

The crux of the matter is now to extend the FDA with a constraint (a consistent one, by definition of FDA) that relates every contraction of Z to Φ,

$$\frac{\partial}{\partial Z}\widehat{A} \sim \Phi + ... \ , \tag{5.1.7}$$

(where the ellipsis stands, possibly, for higher order terms) thereby solving all the dependence on the extra variables in terms of the physical degrees of freedom of the original system. Therefore, the infinitely many terms that correct the pure space-time curvature $dA + A \star A$ in (5.1.6) are all expressed in terms of Φ and build up the deformations $\mathcal{J}(A, \Phi)$ and $\mathcal{P}(A, \Phi)$ of equation (4.3.15). In order to write (5.1.7) in a consistent way, it has to be cast in the form of a curl in Z-space, a curvature with indices taking values in the noncommutative subspace Z. The solutions of (5.1.7) will therefore be automatically consistent with the gauge symmetries, that will turn out to be correspondingly deformed but not broken down, which is essential in order to be able to interpret the nonlinear theory in terms of the same degrees of freedom present at the free level.

If such a Z-extension can be found, then the problem is solved, since we have deformed the theory with infinitely many nonlinear corrections while keeping gauge symmetries and diffeomorphism invariance. Further constraints come from the fact that it must be possible to relate, in this extension, the adjoint and twisted adjoint representations, as the condition (5.1.7) is crucial. Note that in the linearized approximation (4.3.4) and (4.3.5) this problem is absent, since the transformation of the zero-forms produces terms of second order: in other words, because (4.3.6) are valid, up to second order terms. However, at the full level, in order to write a consistent equation like (5.1.7), it is necessary that it involve some mapping between adjoint and twisted adjoint representations. As we shall examine in Chapter 7 in greater detail, such a map has to act on the "square root" of the $\mathfrak{so}(D-1, 2)$ generators that build up the expansion of Φ.

Moreover, other important constraints follow from the fact that the appropriate noncommutative extension must be such that the full equations reproduce the free equations (4.3.4) and (4.3.5) once linearized around the AdS vacuum solution and pulled back on the space-time manifold. It turns out that this requirement can be satisfied once one assumes that the Z coordinates have a nontrivial contraction with the oscillators that realize the $\mathfrak{so}(D-1,2)$ generators. This fact in its turn implies that, in order to preserve covariance, the Z-coordinates must have the same index structure of the oscillators, *i.e.*, the sought-after noncommutative extension must correspond to a doubling of the oscillators. Associativity then fixes the commutation relations of the Z coordinates among themselves. The final result is that they will satisfy an algebra that is isomorphic to the Heisenberg algebra of the oscillators (it only differs by a sign). A generalized $\star$-product can then be defined, that controls contractions among oscillators and Z coordinates and therefore defines a proper composition rule for the elements of an extended oscillator algebra $\widehat{\mathfrak{ho}}(D-1,2)$.

Indeed, an appropriate Z-extension has been constructed, originally in the four-dimensional spinor oscillator realization and later also in the D-dimensional vector oscillator formalism (see [111] for a geometric derivation of the Z extension). In the following, we will present in detail only the first realization, as some of the results of this Thesis, discussed in Chapter 6, only rest on them. The D-dimensional realization of Vasiliev equations, first proposed in [72], rests on the doubling of the vector oscillators presented in Section 3.3.1. Although many of the steps performed in the following can be repeated in that setting, an important subtlety arises in factoring out the traces (*i.e.*, in putting the system on-shell). There are in principle two different ways to do this: the first one (called *weak projection*, and proposed in [66, 52]) consists in factoring out elements proportional to the (extended) $\mathfrak{sp}(2)$ generators, at the full level, both in the one-form and in the zero-form sectors by multiplying the whole equations by a projector $\widehat{M}$ such that $\widehat{K}_{ij}\star\widehat{M}=0$; the second one (called *strong projection*, and proposed in [77]), somehow in the spirit of the unfolding, consists in factoring out traces only in the master zero-form by multiplying it with the projector $\widehat{M}$. As mentioned in the previous Chapter, at the free level the first formulation results in the Fronsdal equations, while the second leads to the unconstrained equations of Francia and Sagnotti. However, at the full level the strong projection may suffer from potential divergencies arising in the expansion in powers of the projected master zero-form $M\star\Phi$, since $M\star M$ diverges. We will not comment further on this issue, but send to the paper cited above for reference.

5.2 Z-Extension and Vasiliev Equations in (3 + 1) Dimensions

It is now time to examine more quantitatively how one can obtain the nonlinear Vasiliev equations, working with the $SL(2;\mathbb{C})$-doublet spinor oscillator realization of Subsection 3.3.2. The free equations one would like to make contact with, *i.e.*, (4.3.4) and (4.3.5) read, in such realization,

$$F_1(x;y,\bar{y}) \;\equiv\; dA+\{\Omega,A\}_\star \;=\; \frac{i}{4}\,e_0^{\alpha\dot{\alpha}}\,e_{0\,\dot{\alpha}}^{\beta}\,\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\,\Phi(y,0)-\text{h.c.}\;, \tag{5.2.1}$$

$$D_0\Phi(x;y,\bar{y}) \;\equiv\; d\Phi+[\Omega,\Phi]_\pi \;=\; 0\;, \tag{5.2.2}$$

where the background connection is

$$\Omega \;=\; \frac{1}{4i}dx^{\mu}\left[\omega_{\mu}^{\alpha\beta}\; y_{\alpha}y_{\beta}+\overline{\omega}_{\mu}{}^{\dot\alpha\dot\beta}\;\bar y_{\dot\alpha}\bar y_{\dot\beta}+2e_{\mu}^{\alpha\dot\beta}\; y_{\alpha}\bar y_{\dot\beta}\right] \tag{5.2.3}$$

and our spinor conventions are listed in Appendix E.

To formulate the nonlinear field equations we double the oscillators by adding the $SL(2;\mathbb{C})$-doublet spinors z_α their hermitian conjugates $\bar z_{\dot\alpha}$ to the theory. Together with the y_α they generate an oscillator algebra with non-commutative and associative $\star$-product defined by

$$y_\alpha\star y_\beta \;=\; y_\alpha y_\beta+i\epsilon_{\alpha\beta}\;,\qquad y_\alpha\star z_\beta \;=\; y_\alpha z_\beta-i\,\epsilon_{\alpha\beta}\;, \tag{5.2.4}$$

$$z_\alpha\star y_\beta \;=\; z_\alpha y_\beta+i\,\epsilon_{\alpha\beta}\;,\qquad z_\alpha\star z_\beta \;=\; z_\alpha z_\beta-i\,\epsilon_{\alpha\beta}\;, \tag{5.2.5}$$

and

$$\bar y_{\dot\alpha}\star\bar y_{\dot\beta} \;=\; \bar y_{\dot\alpha}\bar y_{\dot\beta}+i\epsilon_{\dot\alpha\dot\beta}\;,\qquad \bar z_{\dot\alpha}\star\bar y_{\dot\beta} \;=\; \bar z_{\dot\alpha}\bar y_{\dot\beta}-i\epsilon_{\dot\alpha\dot\beta}\;, \tag{5.2.6}$$

$$\bar y_{\dot\alpha}\star\bar z_{\dot\beta} \;=\; \bar y_{\dot\alpha}\bar z_{\dot\beta}+i\epsilon_{\dot\alpha\dot\beta}\;,\qquad \bar z_{\dot\alpha}\star\bar z_{\dot\beta} \;=\; \bar z_{\dot\alpha}\bar z_{\dot\beta}-i\epsilon_{\dot\alpha\dot\beta}\;, \tag{5.2.7}$$

where, as usual, juxtaposition denotes symmetrized, or Weyl-ordered, products. Equivalently, Weyl-ordered functions obey[2]

$$\begin{aligned}&\widehat f(y,\bar y,z,\bar z)\;\star\;\widehat g(y,\bar y,z,\bar z) \qquad (5.2.8)\\ =\;&\int\frac{d^4\xi d^4\eta}{(2\pi)^4}\;e^{i\eta^\alpha\xi_\alpha+i\bar\eta^{\dot\alpha}\bar\xi_{\dot\alpha}}\;\widehat f(y+\xi,\bar y+\bar\xi,z+\xi,\bar z-\bar\xi)\;\widehat g(y+\eta,\bar y+\bar\eta,z-\eta,\bar z+\bar\eta)\;,\end{aligned}$$

which extends the definition (3.3.22) of the $\star$-product to functions of all $(y,\bar y,z,\bar z)$ oscillators, denoted with the hats, while functions of only y_α and $\bar y_{\dot\alpha}$ will be unhatted. Notice the peculiar difference in sign in the contraction of two z oscillators compared with that of two y oscillators, and the opposite sign of the contractions in $y_\alpha\star z_\beta$ and $z_\alpha\star y_\beta$, so that the commutation relations are

$$[z_\alpha,z_\beta]_\star \;=\; -2i\epsilon_{\alpha\beta}\;,\qquad [y_a,z_\beta]_\star \;=\; 0\;, \tag{5.2.9}$$

together with their hermitian conjugates. Although y and z oscillators commute, it is crucial, as said in the beginning of this Chapter, that they have a nontrivial contraction, as we shall stress again later on.

The definitions of the master fields correspondingly extend to the *adjoint* one-form $\widehat A$ and the *twisted-adjoint* zero-form $\widehat\Phi$ defined by

$$\widehat A \;=\; dx^\mu\widehat A_\mu(x;y,\bar y,z,\bar z)+dz^\alpha\widehat A_\alpha(x;y,\bar y,z,\bar z)+d\bar z^{\dot\alpha}\widehat A_{\dot\alpha}(x;y,\bar y,z,\bar z)\;, \tag{5.2.10}$$

$$\widehat\Phi \;=\; \widehat\Phi(x;y,\bar y,z,\bar z)\;, \tag{5.2.11}$$

[2] The integration measure is defined by $d^4\xi=d^2\xi^1d^2\xi^2$, where $d^2z=idz\wedge d\bar z=2dx\wedge dy$ for $z=x+iy$. With this normalization, $\mathbb{I}\star\widehat f=\widehat f$.

where x^μ are coordinates on a commutative base manifold (which can, but need not, be fixed to be four-dimensional space-time). One also defines the total exterior derivative

$$d = dx^\mu \partial_\mu + dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}} , \qquad (5.2.12)$$

with the property $d(\widehat{f} \wedge \star \, \widehat{g}) = (d\widehat{f}) \wedge \star \, \widehat{g} + (-1)^{\deg \widehat{f}} \widehat{f} \wedge \star \, d\widehat{g}$ for general differential forms. In what follows we shall again suppress the $\wedge$ symbol as we did so far. The master fields can be made subject to the following discrete symmetry conditions[3] [48, 129]

$$\text{Minimal model } (s = 0, 2, 4, ...) \ : \ \tau(\widehat{A}) = -\widehat{A} , \qquad \tau(\widehat{\Phi}) = \bar{\pi}(\widehat{\Phi}) , \qquad (5.2.13)$$

$$\text{Non-minimal model } (s = 0, 1, 2, 3, ...) \ : \ \pi\bar{\pi}(\widehat{A}) = \widehat{A} , \qquad \pi\bar{\pi}(\widehat{\Phi}) = \widehat{\Phi} , \qquad (5.2.14)$$

where τ is the $\star$-product algebra anti-automorphism defined by

$$\tau(\widehat{f}(y, \bar{y}; z, \bar{z})) = \widehat{f}(iy, i\bar{y}; -iz, -i\bar{z}) , \qquad (5.2.15)$$

and π and $\bar{\pi}$ are two involutive $\star$-product automorphisms defined by

$$\pi(\widehat{f}(y, \bar{y}; z, \bar{z})) = \widehat{f}(-y, \bar{y}; -z, \bar{z}) , \qquad \bar{\pi}(\widehat{f}(y, \bar{y}; z, \bar{z})) = \widehat{f}(y, -\bar{y}; z, -\bar{z}) . \qquad (5.2.16)$$

We note that

$$\tau(\widehat{f} \star \widehat{g}) = (-1)^{\deg(\widehat{f})\deg(\widehat{g})} \tau(\widehat{g}) \star \tau(\widehat{f}) , \qquad (5.2.17)$$

$$\pi(\widehat{f} \star \widehat{g}) = \pi(\widehat{f}) \star \pi(\widehat{g}) , \qquad (5.2.18)$$

$$\bar{\pi}(\widehat{f} \star \widehat{g}) = \bar{\pi}(\widehat{f}) \star \bar{\pi}(\widehat{g}) , \qquad (5.2.19)$$

and that $\tau^2 = \pi\bar{\pi}$. As mentioned in the previous Section, the crucial feature that selects the realization of the noncommutative extension as a doubling of the oscillators, with the specific $\star$-product rule (5.2.8), is that the $\pi, \bar{\pi}$ automorphisms, that distinguish adjoint and twisted adjoint, are *inner* and can be generated by conjugation with the functions κ and $\bar{\kappa}$ given by

$$\kappa = \exp(iy^\alpha z_\alpha) , \qquad \bar{\kappa} = \exp(-i\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}) , \qquad (5.2.20)$$

such that

$$\kappa \star \widehat{f}(y, z) = \kappa \widehat{f}(z, y) , \qquad \widehat{f}(y, z) \star \kappa = \kappa \widehat{f}(-z, -y) , \qquad \kappa \star \widehat{f} \star \kappa = \pi(\widehat{f}) , \qquad (5.2.21)$$

$$\bar{\kappa} \star \widehat{f}(\bar{y}, \bar{z}) = \bar{\kappa} \widehat{f}(-\bar{z}, -\bar{y}) , \qquad \widehat{f}(\bar{y}, \bar{z}) \star \bar{\kappa} = \bar{\kappa} \widehat{f}(\bar{z}, \bar{y}) , \qquad \bar{\kappa} \star \widehat{f} \star \bar{\kappa} = \bar{\pi}(\widehat{f}) . \qquad (5.2.22)$$

[3]The exterior derivative obeys $\tau d = d\tau$ and $\pi d = d\pi$, and the τ and π maps do not act on the commutative coordinates.

In other words, κ provides the map between adjoint and twisted adjoint representation that is necessary to relate the Z-dependence to the physical degrees of freedom of the theory, and to make contact with the linearized equations (5.2.1) and (5.2.2). Notice also that, as one can check from (5.2.8),

$$\kappa \star \kappa \ = \ 1 \ , \qquad \bar{\kappa} \star \bar{\kappa} \ = \ 1 \ . \tag{5.2.23}$$

Let us note that *a priori* the $\star$-product (5.2.8) is well-defined for the algebra of polynomials (which means that the $\star$-product of two polynomials is still a polynomial). Thus the $\star$-product admits an ordinary interpretation in terms of oscillators, as long as we deal with polynomial functions. But κ is not a polynomial because it contains an infinite number of terms with higher and higher powers of $y^\alpha z_\alpha$. Thus, *a priori*, the $\star$-product with κ may give rise to divergencies arising from the contraction of an infinite number of terms (for example, an infinite contribution may appear in the zeroth order like a sort of vacuum energy). What singles out the particular $\star$-product (5.2.8) is that this does not happen for the class of functions which extends the space of polynomials to include κ and similar functions. More precisely, as was shown originally in [86], the $\star$-product (5.2.8) is well-defined for the class of functions, called "regular", that can be expanded into a finite sum of functions f of the form

$$f(Z,Y) = P(Z,Y) \int_{M^n} d^n t \ \rho(t) \exp\Big(i\phi(t) y^\alpha z_\alpha\Big) \, , \tag{5.2.24}$$

where the integration is over some compact domain $M^n \subset \mathbb{R}^n$ parametrized by the coordinates t_i ($i = 1, \ldots, n$), the functions $P(Z,Y)$ and $\phi(t)$ are arbitrary polynomials of $(Z,Y) \equiv (z, \bar{z}, y, \bar{y})$ and t_i, respectively, while $\rho(t)$ is integrable in M^n. The key point of the proof is that the $\star$-product (5.2.8) is such that the exponential in the Ansatz (5.2.24) never contributes to the quadratic form in the integration variables simply because $\xi^\alpha \xi_\alpha = \eta^\alpha \eta_\alpha = 0$. As a result, a $\star$-product of two elements (5.2.24) never develops an infinity and the class (5.2.24) turns out to be closed under $\star$-multiplication pretty much as ordinary polynomials.

The full field equations are

$$\widehat{F} \ = \ \frac{i}{4}\Big[dz^\alpha \wedge dz_\alpha \widehat{\Phi} \star \kappa + d\bar{z}^{\dot\alpha} \wedge d\bar{z}_{\dot\alpha} \widehat{\Phi} \star \bar{\kappa}\Big] \ , \tag{5.2.25}$$

$$\widehat{D}\widehat{\Phi} \ = \ 0 \ , \tag{5.2.26}$$

where the curvatures and gauge transformations are given by

$$\widehat{F} \ = \ d\widehat{A} + \widehat{A} \star \widehat{A} \ , \qquad \delta_{\widehat{\epsilon}} \widehat{A} \ = \ \widehat{D}\widehat{\epsilon} \tag{5.2.27}$$

$$\widehat{D}\widehat{\Phi} \ = \ d\widehat{\Phi} + [\widehat{A}, \widehat{\Phi}]_\pi \ , \qquad \delta_{\widehat{\epsilon}} \widehat{\Phi} \ = \ -[\widehat{\epsilon}, \widehat{\Phi}]_\pi \ , \tag{5.2.28}$$

with

$$[\widehat{f},\widehat{g}]_\pi \;=\; \widehat{f}\star\widehat{g}-(-1)^{\deg(\widehat{f})\deg(\widehat{g})}\widehat{g}\star\pi(\widehat{f})\;. \tag{5.2.29}$$

Notice that the (5.2.25) and (5.2.26) are indeed consistent: the second guarantees the consistency of the first, since

$$\widehat{D}\widehat{F} \;=\; 0 \;=\; \frac{i}{4}\left[dz^\alpha\wedge dz_\alpha\widehat{D}\widehat{\Phi}\star\kappa+d\bar{z}^{\dot{\alpha}}\wedge d\bar{z}_{\dot{\alpha}}\widehat{D}\widehat{\Phi}\star\bar{\kappa}\right]\;, \tag{5.2.30}$$

where we have taken into account that $\widehat{D}\kappa=dz^\alpha\frac{\partial}{\partial z^\alpha}\kappa$, and that the latter term (and its hermitian conjugate) do not contribute to (5.2.30) because they involve a triple antisymmetrization $dz^\alpha\,dz^\beta\,dz^\gamma$, which vanishes identically; on the other hand, the first ensures the consistency of the second, since

$$\widehat{D}^2\widehat{\Phi} \;=\; 0 \;=\; [\widehat{F},\widehat{\Phi}]_\pi \;=\; \frac{i}{4}dz^\alpha\wedge dz_\alpha\left(\widehat{\Phi}\star\kappa\star\widehat{\Phi}-\widehat{\Phi}\star\pi(\widehat{\Phi}\star\kappa)\right)-\text{h.c.}\;, \tag{5.2.31}$$

where the vanishing of the last two terms can be checked by using the properties of κ (5.2.21). The consistency of the extended FDA (5.2.25) and (5.2.26) with respect to both x and Z variables ensures that one can solve for $\widehat{A}$ in terms of $\widehat{\Phi}$ from the first, and then for $\widehat{\Phi}$ in terms of the "initial condition" $\widehat{\Phi}(Z=0)|_p=\Phi|_p$ (up to an extended gauge transformation) from the second. This implies that the extended unfolded system is equivalent to the nonlinear space-time system (4.3.15), since they both have the same local data.

In components, the constraints read

$$\widehat{F}_{\mu\nu} \;=\; 0\;,\qquad \widehat{D}_\mu\widehat{\Phi}\;\equiv\;\partial_\mu\widehat{\Phi}+[\widehat{A}_\mu,\widehat{\Phi}]_\pi\;=\;0\;, \tag{5.2.32}$$

$$\widehat{F}_{\mu\alpha} \;=\; 0\;,\qquad \widehat{F}_{\mu\dot{\alpha}}\;=\;0\;, \tag{5.2.33}$$

$$\widehat{F}_{\alpha\beta} \;=\; -\tfrac{i}{2}\epsilon_{\alpha\beta}\widehat{\Phi}\star\kappa\;,\qquad \widehat{F}_{\dot{\alpha}\dot{\beta}}\;=\;-\tfrac{i}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\widehat{\Phi}\star\bar{\kappa}\;, \tag{5.2.34}$$

$$\widehat{F}_{\alpha\dot{\alpha}} \;=\; 0\;, \tag{5.2.35}$$

$$\widehat{D}_\alpha\widehat{\Phi}\;\equiv\;\partial_\alpha\widehat{\Phi}+\widehat{A}_\alpha\star\widehat{\Phi}+\widehat{\Phi}\star\pi(\widehat{A}_\alpha)\;=\;0\;, \tag{5.2.36}$$

$$\widehat{D}_{\dot{\alpha}}\widehat{\Phi}\;\equiv\;\partial_{\dot{\alpha}}\widehat{\Phi}+\widehat{A}_{\dot{\alpha}}\star\widehat{\Phi}+\widehat{\Phi}\star\bar{\pi}(\widehat{A}_{\dot{\alpha}})\;=\;0\;, \tag{5.2.37}$$

where (5.2.37) can be derived using $\pi\bar{\pi}(\widehat{A}_{\dot{\alpha}}) = -\widehat{A}_{\dot{\alpha}}$. Introducing [81]

$$\widehat{S}_\alpha \;=\; z_\alpha - 2i\widehat{A}_\alpha \;, \qquad \widehat{S}_{\dot{\alpha}} \;=\; \bar{z}_{\dot{\alpha}} - 2i\widehat{A}_{\dot{\alpha}} \;, \tag{5.2.38}$$

the component form of the equations carrying at least one spinor index now take the form

$$\partial_\mu \widehat{S}_\alpha + [\widehat{A}_\mu, \widehat{S}_\alpha]_\star \;=\; 0 \;, \qquad \partial_\mu \widehat{S}_{\dot{\alpha}} + [\widehat{A}_\mu, \widehat{S}_{\dot{\alpha}}]_\star \;=\; 0 \;, \tag{5.2.39}$$

$$[\widehat{S}_\alpha, \widehat{S}_\beta]_\star \;=\; -2i\epsilon_{\alpha\beta}(1 - \widehat{\Phi} \star \kappa) \;, \qquad [\widehat{S}_{\dot{\alpha}}, \widehat{S}_{\dot{\beta}}]_\star \;=\; -2i\epsilon_{\dot{\alpha}\dot{\beta}}(1 - \widehat{\Phi} \star \bar{\kappa}) \;, \tag{5.2.40}$$

$$[\widehat{S}_\alpha, \widehat{S}_{\dot{\beta}}]_\star \;=\; 0 \;, \tag{5.2.41}$$

$$\widehat{S}_\alpha \star \widehat{\Phi} + \widehat{\Phi} \star \pi(\widehat{S}_\alpha) \;=\; 0 \;, \tag{5.2.42}$$

$$\widehat{S}_{\dot{\alpha}} \star \widehat{\Phi} + \widehat{\Phi} \star \bar{\pi}(\widehat{S}_{\dot{\alpha}}) \;=\; 0 \;. \tag{5.2.43}$$

5.2.1 Lorentz-Covariance and Uniqueness

The introduction of the extra $(z_\alpha, \bar{z}_{\dot{\alpha}})$-oscillators, which are $\mathfrak{sl}(2;\mathbb{C})$-doublets, requires a corresponding modifications of the Lorentz generators, that were realized at the free level as in (3.3.23). Indeed, due to the commutativity of y and z oscillators (5.2.9), it is impossible to rotate the content in $z, \bar{z}$ of the extended variables $\widehat{A}$ and $\widehat{\Phi}$ if we do not extend the realization of the Lorentz generators. This can be done easily, however, by noticing that z-oscillators satisfy an algebra which is identical to that of y-oscillators, up to a sign.This means that the appropriate generalization of Lorentz rotations is given by

$$\widehat{M}_{\alpha\beta} \;=\; y_\alpha y_\beta - z_\alpha z_\beta \;, \qquad \widehat{M}_{\dot{\alpha}\dot{\beta}} \;=\; \bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} - \bar{z}_{\dot{\alpha}}\bar{z}_{\dot{\beta}} \;. \tag{5.2.44}$$

In terms of such generators, indeed, one gets the usual Lorentz transformation

$$\delta_{\hat{\epsilon}_0} y_\alpha \;\equiv\; [\frac{1}{4i}\Lambda^{\beta\gamma}\widehat{M}_{\beta\gamma} - \text{h.c.}, y_\alpha]_\star \;=\; \Lambda_\alpha{}^\beta\, y_\beta \;, \tag{5.2.45}$$

$$\delta_{\hat{\epsilon}_0} z_\alpha \;\equiv\; [\frac{1}{4i}\Lambda^{\beta\gamma}\widehat{M}_{\beta\gamma} - \text{h.c.}, z_\alpha]_\star \;=\; \Lambda_\alpha{}^\beta\, z_\beta \;. \tag{5.2.46}$$

But this only takes care of the internal $\mathfrak{sl}(2;\mathbb{C})$-doublet indices of the master fields, while the extension also brought in *external* $\mathfrak{sl}(2;\mathbb{C})$-doublet indices associated with the Z-space parts of the one-form connection $\widehat{A}$ introduced in (5.2.10) and the corresponding Z-space "covariant derivative" $\widehat{S}_\alpha, \widehat{S}_{\dot{\alpha}}$ (5.2.38). Such indices are not rotated properly by the modified Lorentz generators (5.2.44), since, as one can read from (5.2.27), the resulting transformation is not homogenous,

$$\delta_{\widehat{\epsilon}_0}\widehat{A}_\alpha \;=\; [\widehat{A}_\alpha,\widehat{\epsilon}_0]_\star+\frac{1}{2i}\Lambda_\alpha{}^\beta\, z_\beta\ . \tag{5.2.47}$$

It is important to be able to modify the local-Lorentz parameter $\widehat{\epsilon}_0$ in such a way as to get a standard homogeneous transformation law under local Lorentz, since in order to make contact with the free equations (4.3.4) and (4.3.5) a physical gauge must be maintained (as we will see in the next Section) that enables to solve the Z-space connection entirely in terms of Φ in perturbation theory. To see how this can be done, let us first rewrite the last equation in terms of $\widehat{S}_\alpha$,

$$\delta_{\widehat{\epsilon}_0}\widehat{A}_\alpha \;=\; \Lambda_\alpha{}^\beta\widehat{A}_\beta+[\widehat{A}_\alpha,\widehat{\epsilon}_0]_\star+\frac{1}{2i}\Lambda_\alpha{}^\beta\,\widehat{S}_\beta\ , \tag{5.2.48}$$

where it is clear that the latter term is the one that must be eliminated. Now, the crucial point is that a consistent modification of the Lorentz generators that rotates properly the external spinor indices fixes the form of the Vasiliev equations up to field redefinitions. Indeed, notice that the only nontrivial equations (that imply that the whole system is not pure gauge) (5.2.40) have the form of a deformed Heisenberg algebra (see [49] and references therein)

$$[\hat{y}_\alpha,\hat{y}_\beta] \;=\; 2i\varepsilon_{\alpha\beta}(1+\nu k)\ ,\quad \{\hat{y}_\alpha,k\} \;=\; 0\ ,\quad k^2 \;=\; 1\ , \tag{5.2.49}$$

(where ν is a real number) which, together with (5.2.42) and (5.2.43) that can be rewritten as

$$\{\widehat{S}_\alpha,\widehat{\Phi}\star\kappa\}_\star \;=\; 0 \;=\; \{\widehat{S}_{\dot\alpha},\widehat{\Phi}\star\bar\kappa\}_\star\ , \tag{5.2.50}$$

implies that the generator $\frac{1}{2}\{\widehat{S}_\alpha,\widehat{S}_\beta\}_\star$ and its hermitian conjugate satisfy the $\mathfrak{sl}(2;\mathbb{C})$ algebra and rotate properly the external spinor indices

$$\left[\frac{1}{2}\{\widehat{S}_\alpha,\widehat{S}_\beta,\}_\star,\widehat{S}_\gamma\right]_\star \;=\; -4i\widehat{S}_{(\alpha}\epsilon_{\beta)\gamma}\ . \tag{5.2.51}$$

Therefore, defining [96]

$$\widehat{\epsilon}_{\rm extra} \;=\; \frac{1}{8i}\Lambda^{\alpha\beta}\{\widehat{S}_\alpha,\widehat{S}_\beta,\}_\star\ , \tag{5.2.52}$$

one has

$$[\widehat{A}_\alpha,\widehat{\epsilon}_{\rm extra}]_\star \;=\; -\frac{1}{2i}\Lambda_\alpha{}^\beta\,\widehat{S}_\beta\ , \tag{5.2.53}$$

that removes the unwanted term in (5.2.48). The necessary modification of the Lorentz generator that implements Lorentz-covariance in the full equations is therefore

$$\widehat{\epsilon}_L \equiv \widehat{\epsilon}_0 + \widehat{\epsilon}_{\rm extra} \;=\; \frac{1}{4i}\Lambda^{\alpha\beta}\left(\widehat{M}_{\alpha\beta} + \frac{1}{2}\{\widehat{S}_\alpha, \widehat{S}_\beta,\}_\star\right) - \text{h.c.}\ . \tag{5.2.54}$$

What is important for the issue of uniqueness of the interaction terms is that the need to maintain Lorentz-covariance in the full equations only allows source terms of the type in (5.2.25) and no hermitian modifications[4] such as $dz^\alpha\, d\bar{z}^{\dot\beta} H_{\alpha\dot\beta}(\widehat{\Phi}\star\kappa)$. Finally, the local-Lorentz transformations of the extended master fields are

$$\delta_{\widehat{\epsilon}_L}\widehat{\Phi} \;=\; -[\widehat{\epsilon}_0, \widehat{\Phi}]_\star\ , \tag{5.2.55}$$

$$\delta_{\widehat{\epsilon}_L}\widehat{A}_\alpha \;=\; [\widehat{A}_\alpha, \widehat{\epsilon}_0]_\star + \Lambda_\alpha{}^\beta \widehat{A}_\beta\ , \tag{5.2.56}$$

$$\delta_{\widehat{\epsilon}_L}\widehat{A}_\mu \;=\; [\widehat{A}_\mu, \widehat{\epsilon}_0]_\star + \frac{1}{4i}\partial_\mu\Lambda^{\alpha\beta}\left(\widehat{M}_{\alpha\beta} + \frac{1}{2}\{\widehat{S}_\alpha, \widehat{S}_\beta,\}_\star\right)\ . \tag{5.2.57}$$

The space-time constraints, that we will examine in the next Section, will be left invariant by the pulled-back local-Lorentz transformation

$$\delta_{\widehat{\epsilon}_L}\Phi \;=\; -[\epsilon_0, \Phi]_\star\ , \tag{5.2.58}$$

$$\delta_{\widehat{\epsilon}_L}\widehat{A}_\mu \;=\; [A_\mu, \epsilon_0]_\star + \frac{1}{4i}\partial_\mu\Lambda^{\alpha\beta}\left[y_\alpha y_\beta - 4\left(\widehat{A}_\alpha\star\widehat{A}_\beta - \frac{\partial}{\partial y^\alpha}\widehat{A}_\beta\right)_{Z=0} - \text{h.c.}\right], \tag{5.2.59}$$

where $\epsilon_0 \equiv \frac{1}{2i}\lambda^{\alpha\beta}y_\alpha y_\beta - \text{h.c.}$. So the transformations of the gauge fields have acquired a complicate field dependent part. However, the latter can be reabsorbed into a redefinition of the Lorentz connection $\omega_\mu^{\alpha\beta}$ inside A_μ, since the quantity $\omega_\mu + K_\mu$, with

$$\omega_\mu \;=\; \frac{1}{4i}\omega_\mu^{\alpha\beta}y_\alpha y_\beta - \text{h.c.} \tag{5.2.60}$$

and

$$K_\mu \;=\; \frac{1}{4i}\omega_\mu{}^{\alpha\beta}\ \widehat{S}_\alpha\star\widehat{S}_\beta\Big|_{Z=0} + \frac{1}{4i}\bar\omega_\mu{}^{\dot\alpha\dot\beta}\ \widehat{S}_{\dot\alpha}\star\widehat{S}_{\dot\beta}\Big|_{Z=0} \tag{5.2.61}$$

$$\;=\; i\omega_\mu{}^{\alpha\beta}\ (\widehat{A}_\alpha\star\widehat{A}_\beta - \frac{\partial}{\partial y^\alpha}\widehat{A}_\beta)\Big|_{Z=0} + i\bar\omega_\mu{}^{\dot\alpha\dot\beta}\ (\widehat{A}_{\dot\alpha}\star\widehat{A}_{\dot\beta} - \frac{\partial}{\partial \bar{y}^{\dot\alpha}}\widehat{A}_{\dot\beta})\Big|_{Z=0}\ , \tag{5.2.62}$$

[4]However, it would still possible to substitute $\widehat{\Phi}\star\kappa$ in (5.2.25) by an arbitrary function $\mathcal{V}(\widehat{\Phi}\star\kappa)$, as long as such function is *odd* [84, 49, 96, 103], which ensures that (5.2.42) imply $\widehat{S}_\alpha\star\mathcal{V}(\widehat{\Phi}\star\kappa)+\mathcal{V}(\widehat{\Phi}\star\kappa)\star\widehat{S}_\alpha = 0$, which is crucial for rotating properly the external spinor indices, as we shall see. Notice, however, that introducing higher odd powers of $\widehat{\Phi}\star\kappa$ would only affect higher order interaction terms, while leaving unaltered the free equations (4.3.4) and (4.3.5). The Vasiliev equations correspond therefore to a "minimal" choice in this sense, that reproduces the correct free field dynamics in the linearized approximation.

transforms as

$$\delta_{\widehat{\epsilon}_L}[\omega_\mu + K_\mu] \;=\; [\omega_\mu + K_\mu, \epsilon_0]_\star \frac{1}{4i}\partial_\mu \Lambda^{\alpha\beta}\left[y_\alpha y_\beta - 4\left(\widehat{A}_\alpha \star \widehat{A}_\beta - \frac{\partial}{\partial y^\alpha}\widehat{A}_\beta\right)_{Z=0} - \text{h.c.}\right] \tag{5.2.63}$$

and therefore $\delta_{\widehat{\epsilon}_L}[A_\mu - \omega_\mu - K_\mu] = [A_\mu - \omega_\mu - K_\mu, \epsilon_0]_\star$, which means that every field inside $A_\mu - \omega_\mu - K_\mu$ is a Lorentz tensor. The space-time nonlinear constraints will thus involve

$$A_\mu \;\equiv \widehat{A}_\mu\big|_{Z=0} \;=\; e_\mu + \omega_\mu + W_\mu + K_\mu \ , \tag{5.2.64}$$

where we have singled out the vielbein part e_μ and the higher-spin ($s \geq 4, 6, ...$ for the minimal and $s \geq 3, 4, ...$ for the nonminimal bosonic model) fields W_μ, and we conclude that the fields, at the nonlinear level, transform as Lorentz tensors under an *undeformed* local-Lorentz symmetry ϵ_0, implemented through a field-dependent Lorentz connection $\omega_\mu + K_\mu$.

5.3 Perturbative Expansion

We now want to show that the Vasiliev equations admit a perturbative expansion in the master zero-form Φ that, to the first order, reproduces the free equations (4.3.4) and (4.3.5). In performing such an expansion we shall encounter and examine in greater detail all the issues discussed qualitatively in Section 5.1. The strategy will be to solve the Z-dependence from the components of the equations that have at least one spinor index, in terms of the initial condition $\Phi = \widehat{\Phi}|_{Z=0}$ and then plug the solution into the pure space-time components (5.2.32) [84, 96].

At zeroth order in Φ, a natural vacuum solution of the equation is AdS_4 space-time, around which we shall expand the full equations, in the end. Indeed, for $\Phi = 0$ (5.2.34) (or (5.2.40)) implies

$$\widehat{S}^{(0)} \;=\; dz^\alpha\, z_\alpha + d\bar{z}^{\dot{\alpha}}\, \bar{z}_{\dot{\alpha}} \ , \tag{5.3.1}$$

(where the superscript refers to the order zero in Φ) since $\widehat{A}^{(0)}_\alpha$ is then a flat connection, and one is choosing the gauge condition

$$\widehat{A}^{(0)}_\alpha \;=\; \widehat{A}^{(0)}_{\dot{\alpha}} \;=\; 0 \tag{5.3.2}$$

(which in its turn implies $\frac{\partial}{\partial y^\alpha}\widehat{A}_\beta|_{Z=0} = 0$, that simplifies the expressions (6.2.59)). Plugging into (5.2.39) one gets then $[\widehat{A}^{(0)}_\mu, z_a]_\star = 0$, which implies that the space-time component of the master one-form, to zeroth order in Φ, is Z-independent. Therefore, one Z-independent solution of $\widehat{F}_{\mu\nu} = 0$ is the AdS_4 connection

$$\widehat{A}^{(0)}_\mu \;=\; \Omega_\mu \;=\; \frac{1}{4i}\left[\omega_\mu^{\alpha\beta}\; y_\alpha y_\beta + \overline{\omega}_\mu{}^{\dot{\alpha}\dot{\beta}}\; \bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} + 2e_\mu^{\alpha\dot{\beta}}\; y_\alpha \bar{y}_{\dot{\beta}}\right] \ , \tag{5.3.3}$$

which thus appears as a natural vacuum solution of the full equations. Notice that the symmetry $\epsilon^{gl}(Z,Y;x)$ of this vacuum solution is just $\mathfrak{ho}(3,2)$. Indeed, the vacuum symmetry parameters $\epsilon^{gl}(Z,Y;x)$ must satisfy

$$[S^{(0)}\,,\epsilon^{gl}]_\star = 0\,, \qquad D_0(\epsilon^{gl}) = 0\,. \tag{5.3.4}$$

The first of these conditions implies that $\epsilon^{gl}(Z,Y;x)$ is Z–independent, *i.e.*, $\epsilon^{gl}(Z,Y;x) = \epsilon^{gl}(Y;x)$ while the second reconstructs the dependence of $\epsilon^{gl}(Y;x)$ on space-time coordinates x in terms of the values of the constant element $\epsilon^{gl}(Y;x_0) \in \mathfrak{ho}(3,2)$ at any fixed point x_0 of space-time.

Now we have to investigate whether the free equations on AdS_4 emerge from the full system as first order corrections to such vacuum solution. Before doing that, however, we shall show that one can also expand the Vasiliev equations in a HS-covariant curvature expansion treating the gauge fields exactly. In general, we set up an expansion scheme

$$\widehat{\Phi} = \sum_{n=1}^{\infty} \widehat{\Phi}^{(n)}\ , \quad \widehat{A}_\alpha = \sum_{n=0}^{\infty} \widehat{A}^{(n)}_\alpha\ , \quad \widehat{A}_\mu = \sum_{n=0}^{\infty} \widehat{A}^{(n)}_\mu\ , \tag{5.3.5}$$

where $\widehat{\Phi}^{(n)}$ $(n = 1, 2, 3, ...)$, $\widehat{A}^{(n)}_\alpha$ $(n = 0, 1, 2, ...)$ and $\widehat{A}^{(n)}_\mu$ $(n = 0, 1, 2, ...)$ are functionals which are nth order in Φ and which obey the initial conditions

$$\widehat{\Phi}^{(n)}|_{Z=0} \;=\; \begin{cases} \Phi\ , & n=1 \\ 0\ , & n = 2,3,... \end{cases} \tag{5.3.6}$$

$$\widehat{A}^{(n)}_\mu|_{Z=0} \;=\; \begin{cases} A_\mu\ , & n=0 \\ 0\ , & n = 1,2,3,... \end{cases}\ . \tag{5.3.7}$$

Next, the constraints $\widehat{F}_{\alpha\dot\beta} = 0$, $\widehat{F}_{\alpha\beta} = -\frac{i}{2}\epsilon_{\alpha\beta}\widehat{\Phi}\star\kappa$ and $\widehat{D}_\alpha\widehat{\Phi} = 0$ can be solved in the nth order $(n \geq 1)$ as

$$\widehat{\Phi}^{(1)} \;=\; \Phi(y,\bar y)\ , \tag{5.3.8}$$

$$\widehat{A}^{(1)}_\alpha \;=\; \partial_\alpha \widehat{\xi}^{(1)} - \frac{i}{2} z_\alpha \int_0^1 tdt\ \Phi(-tz,\bar y)\kappa(tz,y)\ , \tag{5.3.9}$$

and ($n \geq 2$):

$$\begin{aligned}
\widehat{\Phi}^{(n)} &= z^\alpha \sum_{j=1}^{n-1} \int_0^1 dt \Bigg(\widehat{\Phi}^{(j)} \star \bar{\pi}(\widehat{A}_\alpha^{(n-j)}) - \widehat{A}_\alpha^{(n-j)} \star \widehat{\Phi}^{(j)} \Bigg)_{z\to tz, \bar{z}\to t\bar{z}} \\
&\quad + \bar{z}^{\dot{\alpha}} \sum_{j=1}^{n-1} \int_0^1 dt \Bigg(\widehat{\Phi}^{(j)} \star \pi(\widehat{A}_{\dot{\alpha}}^{(n-j)}) - \widehat{A}_{\dot{\alpha}}^{(n-j)} \star \widehat{\Phi}^{(j)} \Bigg)_{z\to tz, \bar{z}\to t\bar{z}} , \qquad (5.3.10) \\
\widehat{A}_\alpha^{(n)} &= \partial_\alpha \widehat{\xi}^{(n)} + z_\alpha \int_0^1 t dt \Bigg(-\tfrac{i}{2}(\widehat{\Phi}^{(n)} \star \kappa) + \sum_{j=1}^{n-1} \widehat{A}^{(j)\beta} \star \widehat{A}_\beta^{(n-j)} \Bigg)_{z\to tz, \bar{z}\to t\bar{z}} \\
&\quad + \bar{z}^{\dot{\beta}} \sum_{j=1}^{n-1} \int_0^1 t dt \left[\widehat{A}_\alpha^{(j)}, \widehat{A}_{\dot{\beta}}^{(n-j)} \right]_{*(z\to tz, \bar{z}\to t\bar{z})} . \qquad (5.3.11)
\end{aligned}$$

We emphasize that in (5.3.10) and (5.3.11) the replacements $(z, \bar{z})$ by $(tz, t\bar{z})$ are to be made *after* the $\star$-products are carried out. The integration functions $\widehat{\xi}^{(n)}$ are gauge artifacts, which can be eliminated by means of Φ-dependent gauge transformations. We therefore impose the gauge conditions [84, 96]

$$\widehat{\xi}^{(n)} = 0 , \qquad n = 1, 2, \ldots \qquad (5.3.12)$$

The gauge conditions (6.2.60) and (5.3.12) are left invariant by Z-independent, and therefore $\mathfrak{ho}(3,2)$-valued, gauge transformations (which in general may be Φ-dependent).

From the constraints $\widehat{F}_{\mu\alpha} = 0$ and $\widehat{F}_{\mu\dot{\alpha}} = 0$ one can solve for the Z-dependence of $\widehat{A}_\mu$. It follows that

$$\begin{aligned}
\widehat{A}_\mu^{(0)} &= A_\mu \qquad (5.3.13) \\
\widehat{A}_\mu^{(n)}(x;Y,Z) &= \frac{i}{2} \int_0^1 dt \Bigg\{ z^\alpha \left(\sum_{j=0}^{n-1} \left[\widehat{A}_\mu^{(j)}, \widehat{A}_\alpha^{(n-j)} \right]_\star \right)(x;Y,Z) \\
&\quad + \bar{z}^{\dot{\alpha}} \left(\sum_{j=0}^{n-1} \left[\widehat{A}_\mu^{(j)}, \widehat{A}_{\dot{\alpha}}^{(n-j)} \right]_\star \right)(x;Y,Z) \Bigg\}_{z\to tz, \bar{z}\to t\bar{z}} , \qquad (5.3.14)
\end{aligned}$$

where we note that the terms $z^\alpha d\widehat{A}_\alpha + \bar{z}^{\dot{\alpha}} d\widehat{A}_{\dot{\alpha}}$ are identically zero by virtue of (5.3.10) and (5.3.11) with the gauge choice (5.3.12), since $z^\alpha z_\alpha = \bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}} = 0$.

Finally, having solved the Z-space part of (5.2.25) and (5.2.26), the remaining constraints $\widehat{F}_{\mu\nu} = 0$ and $\widehat{D}_\mu \widehat{\Phi} = 0$ yield the following space-time full nonlinear field equations[5]:

[5]The integrability of (5.2.25) and (5.2.26) implies that if $\widehat{F}_{\mu\nu}|_{Z=0} = 0$ and $\widehat{D}_\mu \widehat{\Phi}|_{Z=0} = 0$ then $\widehat{F}_{\mu\nu} = 0$

$$F_{\mu\nu} = -2\sum_{n=1}^{\infty}\sum_{j=0}^{n}\left(\widehat{A}^{(j)}_{[\mu}\star\widehat{A}^{(n-j)}_{\nu]}\right)_{|Z=0}\ , \tag{5.3.17}$$

$$D_\mu\Phi = \sum_{n=2}^{\infty}\sum_{j=1}^{n}\left(\widehat{\Phi}^{(j)}\star\bar\pi(\widehat{A}^{(n-j)}_{\mu}) - \widehat{A}^{(n-j)}_{\mu}\star\widehat{\Phi}^{(j)}\right)_{|Z=0}\ , \tag{5.3.18}$$

where

$$F = dA + A\star A\ , \qquad D\Phi = d\Phi + A\star\Phi - \Phi\star\bar\pi(A)\ . \tag{5.3.19}$$

It is important that (5.3.17) and (5.3.18) are integrable equations. As such they are invariant under gauge transformations whose form follows uniquely from their functional variation of (5.3.17) and (5.3.18), according to the general scheme of FDAs. Equivalently, these symmetries can be described as the residual $\mathfrak{ho}(3,2)$-valued gauge transformations discussed above.

Now, to make contact with the free unfolded equations in AdS_4, one can further expand the one-form A around the AdS_4 vacuum connection,

$$A = \Omega + A_1\ , \tag{5.3.20}$$

where A_1 contains all fluctuation fields (including spin 2) over the AdS_4 background and is treated as a weak field.

To the first order in curvatures, $n = 1$, *and* in fluctuations A_1, the space-time components of Vasiliev equations read

$$F_{1\,\mu\nu} = -2\left(\Omega_{[\mu}\star\widehat{\Omega}^{(1)}_{\nu]}\right)_{Z=0}\ , \tag{5.3.21}$$

$$D_{0\,\mu}\Phi = 0\ , \tag{5.3.22}$$

with $F^{(1)} = dA_1 + \{\Omega, A_1\}_\star$, and

$$\widehat{\Omega}^{(1)}(x;Y,Z) = -\frac{1}{2}\int_0^1 dt'\int_0^1 dt\,t\Big\{\left(itt'\,{\omega_0}^{\alpha\beta}z_\alpha z_\beta + {e_0}^{\alpha\dot\beta}z_\alpha\bar\partial_{\dot\beta}\right)\Phi(x;-tt'z,\bar y)\kappa(tt'z,y)$$
$$+\left(itt'\,{\omega_0}^{\dot\alpha\dot\beta}\bar z_{\dot\alpha}\bar z_{\dot\beta} - {e_0}^{\alpha\dot\beta}\bar z_{\dot\beta}\partial_\alpha\right)\Phi(x;y,tt'z)\bar\kappa(tt'\bar z,\bar y)\Big\}\ , \tag{5.3.23}$$

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and $\widehat{D}_\mu\widehat{\Phi} = 0$. Indeed, integrability ensures that

$$[\widehat{S}, dx^\mu dx^\nu\widehat{F}_{\mu\nu}]_\star = 0\ , \tag{5.3.15}$$

$$\widehat{S}\star dx^\mu\widehat{D}_\mu\widehat{\Phi} + dx^\mu\widehat{D}_\mu\widehat{\Phi}\star\bar\pi(\widehat{S}) = 0\ , \tag{5.3.16}$$

which imply that (5.2.32) are covariantly constant in Z space.

where $\partial_\alpha = \frac{\partial}{\partial y^\alpha}$. We can see from here that, had we not postulated the nontrivial contraction rule (6.2.2) and (6.2.4), there would be no chance of obtaining a nontrivial right hand side in (5.3.21), since Ω only contains $y, \bar{y}$ oscillators. Equation (5.3.22) is already of the desired form (5.2.2). Performing the $\star$-product in (5.3.21) and projecting onto $\{Z = 0\}$ one finally obtains (5.2.1) (where we note that what we called A there is a fluctuation filed over AdS_4, and therefore has to be identified with A_1 in this context), which is what we wanted to prove.

Let us note that one could also expand A_μ in terms of HS fields only (as well as higher derivatives of all fields), *i.e.*, by treating exactly the whole vierbein and Lorentz connection while assuming W_μ to include weak fields. This procedure yields a manifestly diffeomorphism and locally Lorentz invariant expansion.

Chapter 6

Exact Solutions

6.1 Introduction

In the last Chapter we have presented the four-dimensional Vasiliev equations, that are naturally formulated in terms of $SL(2;\mathbb{C})$ spinor oscillators in Lorentzian signature $(3,1)$, and we have described an explicit perturbative expansion that makes contact with the known free equations for massless fields of arbitrary spin. As shown above, this procedure amounts to first solving for the dependence from the auxiliary Z-variables and then plugging back in the pure space-time components of the field equations, that can be analyzed order by order in the interactions. In other words, one solves first the evolution along the infinite dimensional Z-fiber over each point of the space-time manifold, which generates infinitely many nonlinear terms in Φ, and then examines the resulting very complicate equations in a subspace that can be taken to be an ordinary four-dimensional space-time. Observe, indeed, that according to the general discussion on unfolded systems of Section 4.1.1, the FDA that corresponds to the Vasiliev equations makes the dependence on space-time coordinates completely auxiliary, and the dynamics is all encoded in the functions $G^{\alpha}(W)$ satisfying the generalized Jacobi identity (4.1.3). Moreover, as it should be clear from the component form of the equations (5.2.32-5.2.37), the only term that introduces a nontrivial dynamics in the system sits in the ZZ component of the curvature: this implies that the equations are homotopy invariant, *i.e.*, the dynamics is preserved by the restriction to a single point in space-time. In other words, one may equivalently analyze the content of the equations by first solving for the space-time dependence from the equations that have at least one space-time index, and then for the Z-evolution. This strategy has moreover the advantage that solving the space-time equations is particularly easy, since they are zero-curvature conditions. The x-dependence is thus all encoded into gauge functions, and the local data (the zero-forms at a fixed space-time point) is the only nontrivial information that enters the Z-space constraints. Now, the latter are pure algebraic equations, that one has a better chance of solving exactly! One may always reconstruct the space-time dependence of the solution at a later stage, by $\star$-multiplying with the gauge function.

This feature of the Vasiliev equations, namely the fact that their projection to the fiber at a given space-time point preserves all the dynamical information, is remarkable, and was indeed exploited for finding the first nontrivial exact solution, other than the AdS background, in [97]. Locally, the solution describes a scalar field in a FRW-like metric with a space-like singularity, that can be resolved by the method of patches. The solution is asymptotically AdS and periodic in time, so that one may think of it as an "instanton universe" inside AdS [97]. More recently, the gauge function method has been used to describe the BTZ black hole metric as a solution to full three dimensional HS gauge theory [98].

This raises the question of how to Wick rotate solutions of the Lorentzian theory into solutions of a Euclidean theory. The main difficulty is to impose proper reality conditions given the doubling of the spinor oscillators due to the Euclidean signature. In this Chapter, we shall review the conclusions of the recent paper [99], where the Vasiliev equations have been formulated using spinor oscillators in Euclidean signature $(4,0)$ and Kleinian signature $(2,2)$ as well, and new nontrivial exact solutions with novel properties, such as the excitation of all higher spin fields, have been found. The difficulty with the Euclidean signatures is resolved taking the master fields to be holomorphic functions of the left-handed and right-handed spinor oscillators subject to pseudo-reality conditions, as we shall see.

In addition to the Euclidean signature, we shall consider the Kleinian signature as well. While in all signatures there is the possibility of a chiral asymmetry, in Euclidean and Kleinian signatures, the extreme case of parity violation involving half-flat gauge fields can also arise. We refer to the latter ones as *chiral models*. In HS gauge theory, the HS algebra valued gauge-field curvatures can be made, say, self-dual, but the model nonetheless contains the anti-self-dual gauge fields through the master zero-form which contains the corresponding Weyl tensor obeying the appropriate field equation. Although this is contrary to what happens in ordinary Euclidean gravity, where the field equations can contain only self-dual fields, it is not a surprise in HS theory since the underlying higher spin algebra, which is an extension of $SO(5)$, does not admit a chiral massless multiplet [1].

There are several reasons that make the investigation of HS theory in Euclidean and Kleinian signatures worthwhile. To begin with, just as the Euclidean version of gravity plays a significant role in the path integral formulation of quantum gravity, it is reasonable to expect that this may also be the case in the quantum formulation of HS theory, despite the fact that an action formulation is yet to be spelled out (see, however, [101] for a recent attempt). For reviews of Euclidean quantum gravity, see, for example, [124] and [125].

[1] We shall leave a more detailed group-theoretical analysis in all signatures to Chapter 7.

Another well known aspect of self-dual field theories is their ability to unify a wide class of integrable systems in two and three dimensions. It would be interesting to extend these mathematical structures to self-dual HS gauge theories to find new integrable systems.

The chiral HS theories in Kleinian signatures may also be of considerable interest in closed $N = 2$ string theory in which the self-dual gravity in $(2,2)$ dimensions arises as the effective target space theory [126]. However, there are some subtleties in treating the picture-changing operators in the BRST quantization which have raised the question of whether there are more physical states [127], and in the case of open $N = 2$ theory an interpretation in terms of an infinite tower of massless higher spin states has been proposed [128]. It would be very interesting to establish whether these theories or their possible variants admit self-dual HS theory in the target space. While the $N = 2$ string theories may seem to be highly unrealistic, it should not be ruled out that they may be connected in subtle ways to all the other string theories which are themselves connected by a web of dualities in M theory.

In this Chapter, we shall take the necessary first steps to start the exploration of the Euclidean and Kleinian HS theories. We shall start by determining the real forms of the complex HS algebra based on an infinite dimensional extension of $SO(5;\mathbb{C})$ and formulate the corresponding higher-spin gauge theories in four-dimensional spacetime with signature $(4-p,p)$. Maximally symmetric four- dimensional constant curvature spacetimes, including de Sitter spacetime, defined by the embedding into five-plane with signature $(5-q,q)$ are readily exact solutions. Fluctuations about these spaces arrange themselves into all the irreducible representations of $SO(5-q,q)$ contained in the symmetric two-fold product of the fundamental singleton representation of this group, each occurring once. The details of this phenomenon will be provided in Chapter 7.

We then devote the rest of the Chapter to finding a class of nontrivial exact solutions of these models, including the Euclidean and chiral cases. The key information about these solutions is encoded in the master zero-form which we recall contains a real ordinary scalar field, and the *Weyl tensors* $\Phi_{\alpha_1\cdots\alpha_{2s}}$ and $\Phi_{\dot\alpha_1,\cdots\dot\alpha_{2s}}$ for spin $s = 2,4,6,...$ in the minimal bosonic model and $s = 1,2,3,4,...$ in a non-minimal bosonic model [48, 129]. Our new exact solutions are constructed by using the oscillators to build suitable projectors, with slightly different properties in the minimal and non-minimal models.

Our exact solutions fall into the following four classes:

<u>Type 0</u>:

These are *maximally symmetric solutions* (see Table 1) with

$$\begin{aligned} \phi(x) &= 0\ , \qquad \Phi_{\alpha_1\cdots\alpha_{2s}} = 0\ , \qquad \Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}} = 0\ , \\ e_\mu^a &= \frac{4\delta_\mu^a}{(1-\lambda^2x^2)^2}\ , \qquad W_\mu{}^{a_1\cdots a_{s-1}} = 0\ , \end{aligned} \tag{6.1.1}$$

describing the symmetric spaces $S^4, H_4, AdS_4, dS_4, H_{3,2} = SO(3,2)/SO(2,2)$, where $|\lambda|$ is the inverse radius of the symmetric space, $x^2 = x^a x^b \eta_{ab}$, and η_{ab} is the tangent space metric. In the above the zero-forms have spin $s = 2, 4, 6, \ldots$ in the minimal model and $s = 1, 2, 3, 4, \ldots$ in the non-minimal model, while for $W_\mu{}^{a_1 \cdots a_{s-1}}$, $s = 4, 6, \ldots$ in the minimal model, and $s = 1, 3, 4, 5, 6, \ldots$ in the non-minimal model.

Type 1:

These solutions, which arise in the minimal models (and therefore are evidently solutions also to the non-minimal models with vanishing odd spins), are $SO(p, 4-p)$ *invariant deformations* of the maximally symmetric solutions with

$$\begin{aligned} \phi(x) &= \nu(1 - \lambda^2 x^2) \;, \qquad \Phi_{\alpha_1 \cdots \alpha_{2s}} = 0 \;, \qquad \Phi_{\dot\alpha_1 \cdots \dot\alpha_{2s}} = 0 \;, \quad (s = 2, 4, \ldots) \\ e_\mu^a &= f_1 \delta_\mu^a + \lambda^2 f_2 x_\mu x^a \;, \qquad W_\mu{}^{a_1 \cdots a_{s-1}} = 0 \quad (s = 4, 6, \ldots) \;, \end{aligned} \tag{6.1.2}$$

where ν is a continuous parameter and f_1, f_2 (see (6.3.85)) are highly complicated functions of x^2, ν, and a set of *discrete* parameters corresponding to whether certain projectors are switched on or off. The metric is Weyl-flat conformal to the maximally symmetric solution with a complicated conformal factor, and note that all the higher spin gauge fields vanish. Interestingly, a particular choice of the discrete parameters yield, in the $\nu \to 0$ limit, the *degenerate metric*:

$$g_{\mu\nu} = \frac{1}{(1 - \lambda^2 x^2)} \frac{x_\mu x_\nu}{\lambda^2 x^2} \;. \tag{6.1.3}$$

Degenerate metrics are known to play a role in topology change in space-time (see, for example, [94], and references therein). Interestingly, here they arise in a natural way by simply taking a certain limit in the parameter space of our solution.

Type 2:

These are solutions of the non-minimal model that are *not* solutions to the minimal model. The spacetime component fields are identical to those of the maximally symmetric Type 0 solutions, but, unlike in the Type 0 solution, the spinorial master one-form is non-vanishing (see (6.3.102)). Even though all odd spin fields are vanishing, the solution exists only for the non-minimal model because the spinorial master field violates the kinematic conditions of the minimal model. In particular, this means that this type of solution cannot be a $\nu \to 0$ limit of the Type 1 solutions. Furthermore, the spinorial master field is parametrized by *discrete* parameters, again associated with projectors.

Type 3:

These are solutions of the *non-minimal chiral models* in Euclidean and Kleinian signatures, in which *all gauge fields are non-vanishing.* These solutions also depend on an infinite set of *discrete parameters* and for simple choices of these parameters we obtain two such solutions, in both of which

$$\phi(x) \ = \ -1 \ , \qquad \Phi_{\alpha_1\cdots\alpha_{2s}} \ = \ 0 \ , \qquad W_\mu{}^{a_1\cdots a_{s-1}} \ \neq \ 0 \ . \tag{6.1.4}$$

In one of the solutions the Weyl tensors and the vierbein take the form

$$\Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}} \ = \ -2^{2s+1}(2s-1)!!\left(\frac{h^2-1}{\epsilon h^2}\right)^s U_{(\dot\alpha_1}\cdots U_{\dot\alpha_s}V_{\dot\alpha_{s+1}}\cdots V_{\dot\alpha_{2s})} \ , \tag{6.1.5}$$

$$e_\mu^a \ = \ \frac{-2}{h^2(1+2g)}\left[g_3\delta_\mu^a + g_4\lambda^2 x_\mu x^a + g_5\lambda^2 (Jx)_\mu (Jx)^a\right] \ , \tag{6.1.6}$$

where h, g, g_3, g_4, g_5 are functions of x^2 defined in (F.0.3), (6.3.126), and the almost complex structure J_{ab} and spinors (U,V) are defined in (F.0.8) and (F.0.11), and $\epsilon = \pm 1$ as explained in Section 3.5. For the other solution we have

$$\Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}} \ = \ -2^{2s+1}(2s-1)!!\left(\frac{1}{\epsilon h^2}\right)^s \bar\lambda_{(\dot\alpha_1}\cdots\bar\lambda_{\dot\alpha_s}\bar\mu_{\dot\alpha_{s+1}}\cdots\bar\mu_{\dot\alpha_{2s})} \ , \tag{6.1.7}$$

$$e_\mu^a \ = \ \frac{-2}{h^2(1+2\tilde g)}\left[\delta_\mu^a + \tilde g_4\lambda^2 x_\mu x^a + \tilde g_5\lambda^2 (\tilde J x)_\mu (\tilde J x)^a\right] \ , \tag{6.1.8}$$

where the functions $\tilde g, \tilde g_4, \tilde g_5$ are defined in (6.3.134), and the almost complex structure $\tilde J_{ab}$ is defined in (F.0.10).

These are remarkable solutions in that they are, to our best knowledge, the first exact solution of higher-spin gauge theory in which higher-spin fields are non-vanishing. We also note that the Weyl tensors in these solutions corresponds to higher-spin generalization of the Type D Weyl tensor that takes the form $\phi_{\dot\alpha\dot\beta\dot\gamma\dot\delta} \sim \lambda_{(\dot\alpha}\lambda_{\dot\beta}\mu_{\dot\gamma}\mu_{\dot\delta)}$, up to a scale factor [131]. Type D instanton solutions of Einstein's equation in Euclidean signature with and without cosmological constant have been discussed in [100]. Our solution provides their higher spin generalization.

After we describe the full HS field equations in diverse signatures in the next Section, repeating some of the steps of the previous Chapter to show where the difference in signature plays a role, we shall present the detailed construction of our solutions in Section 6.3. We shall comment further on these solutions and open problems in the Conclusions to this Chapter.

6.2 The Bosonic 4D Models in Various Signatures

We shall first describe the field equations without imposing reality conditions on the master fields. These conditions will then be discussed separately leading to five different models in four-dimensional space-times with various signatures (see Table 6.1).

6.2.1 The Complex Field Equations

To formulate the complex field equations we use *independent* $SL(2;\mathbb{C})_L$ doublet spinors (y_α, z_α) and $SL(2;\mathbb{C})_R$ doublet spinors $(\bar{y}_{\dot\alpha}, \bar{z}_{\dot\alpha})$, generating an oscillator algebra with non-commutative and associative product $\star$ defined by

$$y_\alpha \star y_\beta \;=\; y_\alpha y_\beta + i\epsilon_{\alpha\beta}\ , \qquad y_\alpha \star z_\beta \;=\; y_\alpha z_\beta - i\,\epsilon_{\alpha\beta}\ , \tag{6.2.1}$$

$$z_\alpha \star y_\beta \;=\; z_\alpha y_\beta + i\,\epsilon_{\alpha\beta}\ , \qquad z_\alpha \star z_\beta \;=\; z_\alpha z_\beta - i\,\epsilon_{\alpha\beta}\ , \tag{6.2.2}$$

and

$$\bar{y}_{\dot\alpha} \star \bar{y}_{\dot\beta} \;=\; \bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}\ , \qquad \bar{z}_{\dot\alpha} \star \bar{y}_{\dot\beta} \;=\; \bar{z}_{\dot\alpha}\bar{y}_{\dot\beta} - i\epsilon_{\dot\alpha\dot\beta}\ , \tag{6.2.3}$$

$$\bar{y}_{\dot\alpha} \star \bar{z}_{\dot\beta} \;=\; \bar{y}_{\dot\alpha}\bar{z}_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}\ , \qquad \bar{z}_{\dot\alpha} \star \bar{z}_{\dot\beta} \;=\; \bar{z}_{\dot\alpha}\bar{z}_{\dot\beta} - i\epsilon_{\dot\alpha\dot\beta}\ . \tag{6.2.4}$$

All the definitions given in Section 5.2 carry over, except that the oscillators y and $\bar{y}$, z and $\bar{z}$ are *not* related by hermitian conjugation and that now the full field equation have independent holomorphic and anti-holomorphic sources,

$$\widehat{F} \;=\; \frac{i}{4}\left[c_1 dz^\alpha \wedge dz_\alpha \widehat{\Phi} \star \kappa + c_2 d\bar{z}^{\dot\alpha} \wedge d\bar{z}_{\dot\alpha} \widehat{\Phi} \star \bar\kappa\right]\ , \tag{6.2.5}$$

$$\widehat{D}\widehat{\Phi} \;=\; 0\ , \tag{6.2.6}$$

where c_1 and c_2 are complex constants. The curvatures and gauge transformations are still given by

$$\widehat{F} \;=\; d\widehat{A} + \widehat{A} \star \widehat{A}\ , \qquad \delta_{\widehat\epsilon}\widehat{A} \;=\; \widehat{D}\widehat{\epsilon} \tag{6.2.7}$$

$$\widehat{D}\widehat{\Phi} \;=\; d\widehat{\Phi} + [\widehat{A}, \widehat{\Phi}]_\pi\ , \qquad \delta_{\widehat\epsilon}\widehat{\Phi} \;=\; -[\widehat{\epsilon}, \widehat{\Phi}]_\pi\ , \tag{6.2.8}$$

with

$$[\widehat{f}, \widehat{g}]_\pi \;=\; \widehat{f} \star \widehat{g} - (-1)^{\deg(\widehat{f})\deg(\widehat{g})}\widehat{g} \star \pi(\widehat{f})\ . \tag{6.2.9}$$

Since $\widehat{\Phi}$ is defined up to rescalings by complex numbers, the model only depends on one complex parameter, that we can take to be

$$c \;=\; \frac{c_2}{c_1}\ . \tag{6.2.10}$$

In components, the constraints read

$$\widehat{F}_{\mu\nu} \;=\; 0\ , \qquad \widehat{D}_\mu\widehat{\Phi} \;\equiv\; \partial_\mu\widehat{\Phi} + [\widehat{A}_\mu, \widehat{\Phi}]_\pi \;=\; 0\ , \tag{6.2.11}$$

$$\widehat{F}_{\mu\alpha} \;=\; 0\ , \qquad \widehat{F}_{\mu\dot\alpha} \;=\; 0\ , \tag{6.2.12}$$

$$\widehat{F}_{\alpha\beta} \;=\; -\tfrac{ic_1}{2}\epsilon_{\alpha\beta}\widehat{\Phi}\star\kappa \;, \qquad \widehat{F}_{\dot\alpha\dot\beta} \;=\; -\tfrac{ic_2}{2}\epsilon_{\dot\alpha\dot\beta}\widehat{\Phi}\star\bar\kappa \;, \tag{6.2.13}$$

$$\widehat{F}_{\alpha\dot\alpha} \;=\; 0 \;, \tag{6.2.14}$$

$$\widehat{D}_{\alpha}\widehat{\Phi} \;\equiv\; \partial_\alpha\widehat{\Phi}+\widehat{A}_\alpha\star\widehat{\Phi}+\widehat{\Phi}\star\pi(\widehat{A}_\alpha) \;=\; 0 \;, \tag{6.2.15}$$

$$\widehat{D}_{\dot\alpha}\widehat{\Phi} \;\equiv\; \partial_{\dot\alpha}\widehat{\Phi}+\widehat{A}_{\dot\alpha}\star\widehat{\Phi}+\widehat{\Phi}\star\bar\pi(\widehat{A}_{\dot\alpha}) \;=\; 0 \;, \tag{6.2.16}$$

where (6.2.16) can be derived using $\pi\bar\pi(\widehat{A}_{\dot\alpha})=-\widehat{A}_{\dot\alpha}$. Introducing again

$$\widehat{S}_\alpha \;=\; z_\alpha-2i\widehat{A}_\alpha \;, \qquad \widehat{S}_{\dot\alpha} \;=\; \bar z_{\dot\alpha}-2i\widehat{A}_{\dot\alpha} \;, \tag{6.2.17}$$

the component form of the equations carrying at least one spinor index now take the form

$$\partial_\mu\widehat{S}_\alpha+[\widehat{A}_\mu,\widehat{S}_\alpha]_\star \;=\; 0 \;, \qquad \partial_\mu\widehat{S}_{\dot\alpha}+[\widehat{A}_\mu,\widehat{S}_{\dot\alpha}]_\star \;=\; 0 \;, \tag{6.2.18}$$

$$[\widehat{S}_\alpha,\widehat{S}_\beta]_\star \;=\; -2i\epsilon_{\alpha\beta}(1-c_1\widehat{\Phi}\star\kappa) \;, \qquad [\widehat{S}_{\dot\alpha},\widehat{S}_{\dot\beta}]_\star \;=\; -2i\epsilon_{\dot\alpha\dot\beta}(1-c_2\widehat{\Phi}\star\bar\kappa) \;, \tag{6.2.19}$$

$$[\widehat{S}_\alpha,\widehat{S}_{\dot\beta}]_\star \;=\; 0 \;, \tag{6.2.20}$$

$$\widehat{S}_\alpha\star\widehat{\Phi}+\widehat{\Phi}\star\pi(\widehat{S}_\alpha) \;=\; 0 \;, \tag{6.2.21}$$

$$\widehat{S}_{\dot\alpha}\star\widehat{\Phi}+\widehat{\Phi}\star\bar\pi(\widehat{S}_{\dot\alpha}) \;=\; 0 \;. \tag{6.2.22}$$

This form of the equations makes the following $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry manifest:

$$\widehat{S}_\alpha \;\to\pm\widehat{S}_\alpha \;, \qquad \widehat{S}_{\dot\alpha} \;\to\; \pm\widehat{S}_{\dot\alpha} \;, \tag{6.2.23}$$

(where the two transformations can be performed independently) keeping $\widehat{A}_\mu$ and $\widehat{\Phi}$ fixed. We note that $\widehat{S}_\alpha\to-\widehat{S}_\alpha$ is equivalent to $\widehat{A}_\alpha\to-\widehat{A}_\alpha-iz_\alpha$, *idem* $\widehat{S}_{\dot\alpha}$ and $\widehat{A}_{\dot\alpha}$.

All component fields are of course complex at this level. Next we shall discuss various reality conditions on the (hatted) master fields that will lead to models with real physical fields living in space-times with different signatures.

6.2.2 Real Forms

In order to define the real forms of the field equations one has to impose reality conditions on both adjoint one-form and twisted-adjoint zero-form, corresponding to suitable real forms of the higher-spin algebra and signatures of spacetime. There are three distinct real forms of the complex higher-spin algebra itself. In two of these cases there are two distinct reality conditions that can be imposed on the zero-form, leading to five distinct models in total, as shown in Table 6.1. The reality conditions are

$$\widehat{A}^\dagger \;=\; -\sigma(\widehat{A}) \;, \qquad \widehat{\Phi}^\dagger \;=\; \sigma(\pi(\widehat{\Phi})) \;, \tag{6.2.24}$$

where the possible actions of the dagger [2] on the spinor oscillators and consequential selections of real forms of $SO(4;\mathbb{C}) \simeq SL(2;\mathbb{C}) \times SL(2;\mathbb{C})$ are given by

$$SU(2)_L \times SU(2)_R \;:\quad (y^\alpha)^\dagger \;=\; y^\dagger_\alpha \;, \quad (z^\alpha)^\dagger \;=\; z^\dagger_\alpha \;, \quad (\bar{y}^{\dot\alpha})^\dagger \;=\; \bar{y}^\dagger_{\dot\alpha} \;, \quad (\bar{z}^{\dot\alpha})^\dagger \;=\; \bar{z}^\dagger_{\dot\alpha} \;, \tag{6.2.25}$$

$$SL(2;\mathbb{C})_{\rm diag} \;:\quad (y^\alpha)^\dagger \;=\; \bar{y}^{\dot\alpha} \;, \quad (z^\alpha)^\dagger \;=\; \bar{z}^{\dot\alpha} \;, \tag{6.2.26}$$

$$Sp(2;\mathbb{R})_L \times Sp(2;\mathbb{R})_R \;:\quad (y^\alpha)^\dagger \;=\; y^\alpha \;, \quad (z^\alpha)^\dagger \;=\; -z^\alpha \;, \quad (\bar{y}^{\dot\alpha})^\dagger \;=\; \bar{y}^{\dot\alpha} \;, \quad (\bar{z}^{\dot\alpha})^\dagger \;=\; -\bar{z}^{\dot\alpha} \;, \tag{6.2.27}$$

and the map σ is given in Table 6.1, with the isomorphism ρ given by

$$\rho(\widehat{f}(y^\dagger_\alpha, \bar{y}^\dagger_\alpha, z^\dagger_\alpha, \bar{z}^\dagger_\alpha)) \;=\; \widehat{f}(y_\alpha, \bar{y}_\alpha, -z_\alpha, -\bar{z}_\alpha) \tag{6.2.28}$$

in the case of $(4,0)$ signature. Note that σ is an oscillator-algebra automorphism in signatures $(3,1)$ and $(2,2)$, while it is an isomorphism in signature $(4,0)$. Here, the $SU(2)$ doublets are pseudo real in the sense that from $(y_\alpha)^\dagger = -y^{\dagger\alpha}$ *idem* $(z_\alpha)^\dagger$, $(\bar{y}_{\dot\alpha})^\dagger$ and $(\bar{z}_{\dot\alpha})^\dagger$ it follows that $(y_\alpha, \bar{y}_{\dot\alpha}; z_\alpha, \bar{z}_{\dot\alpha})$ and $(y^\dagger_\alpha, \bar{y}^\dagger_{\dot\alpha}; z^\dagger_\alpha, \bar{z}^\dagger_{\dot\alpha})$ generate equivalent oscillator algebras with isomorphism ρ. The reality property of the exterior derivative d, defined in (5.2.12), takes the following form in different signatures:

$$\text{Signature } (3,1) \text{ and } (2,2) \;:\quad d^\dagger \;=\; d \;, \tag{6.2.29}$$

$$\text{Signature } (4,0) \;:\quad \rho \circ d^\dagger \;=\; d \circ \rho \;. \tag{6.2.30}$$

We note that the Euclidean case is consistent in the sense that

$$\rho(dz^\alpha)^\dagger \;=\; \rho d^\dagger (z^\alpha)^\dagger \;=\; d\rho(z^\dagger_\alpha) \;=\; -dz_\alpha \tag{6.2.31}$$

[2]The dagger acts as usual complex conjugation on component fields; in this Chapter we shall denote the conjugate of a complex number x by x^*, while reserving the bar for denoting quantities associated with the R-handed oscillators.

is compatible with representing $d\widehat{f}$ using $\partial\widehat{f}/\partial z^\alpha = \frac{i}{2}[z_\alpha, \widehat{f}]_\star$, which yields

$$\rho\left(\frac{i}{2}dz^\alpha[z_\alpha,\widehat{f}]_\star\right)^\dagger \;=\; \frac{i}{2}dz_\alpha\rho\left([\widehat{f}^\dagger, -z^{\dagger\alpha}]_\star\right) \;=\; \frac{i}{2}dz_\alpha[\rho\widehat{f}^\dagger, z^\alpha]_\star \;=\; \frac{i}{2}dz^\alpha[z_\alpha, \rho\widehat{f}^\dagger]_\star \tag{6.2.32}$$

Demanding compatibility between the reality conditions (6.2.24) and the master field equations (6.2.5) and (6.2.6), and using

$$\rho\left((\kappa)^\dagger\right) \;=\; \kappa\ , \qquad \rho\left((idz^\alpha\wedge dz_\alpha)^\dagger\right) \;=\; -idz^\alpha\wedge dz_\alpha\ , \tag{6.2.33}$$

one finds the following reality conditions on the parameters

$$\text{Signature } (3,1) \;:\; c_1^* \;=\; c_2\ , \tag{6.2.34}$$

$$\text{Signature } (4,0) \text{ and } (2,2) \;:\; c_1^* \;=\; c_1\ , \quad c_2^* \;=\; c_2\ . \tag{6.2.35}$$

As a result, the parameter c is a phase factor in Lorentzian signature and a real number in Euclidean and Kleinian signatures. The parameters can be restricted further by requiring invariance under the parity transformation

$$P(y_\alpha) \;=\; \bar{y}_{\dot\alpha}\ , \qquad P\circ d \;=\; d\circ P\ , \qquad P^2 \;=\; \mathrm{Id}\ . \tag{6.2.36}$$

Taking $\widehat{A}$ to be invariant and assigning intrinsic parity $\epsilon = \pm 1$ to $\widehat{\Phi}$,

$$P(\widehat{A}) \;=\; \widehat{A}\ , \qquad P(\widehat{\Phi}) \;=\; \epsilon\widehat{\Phi}\ , \tag{6.2.37}$$

one finds that the master equations are parity invariant provided that

$$c \;=\; \epsilon \;=\; \begin{cases} 1 & \text{Type A model (scalar)} \\ -1 & \text{Type B model (pseudoscalar)} \end{cases} \tag{6.2.38}$$

In Lorentzian signature, there is no loss of generality in choosing $c_1 = c_2 = 1$ in the Type A model and $c_1 = -c_2 = i$ in the Type B model, while in Euclidean and Kleinian signatures one may always take $c_1 = c_2 = 1$ in the Type A model and $c_1 = -c_2 = 1$ in the Type B model. More generally, the parity transformation maps different models into each other as follows,

$$P(c_1) \;=\; \epsilon c_2\ , \qquad P(c_2) \;=\; \epsilon c_1\ , \qquad P(c) \;=\; \frac{1}{c}\ , \tag{6.2.39}$$

leaving invariant the Type A and B models. The *maximally parity violating* cases are

$$\text{Signature } (3,1) \;:\; c \;=\; \exp(i\pi/4)\ , \tag{6.2.40}$$

$$\text{Signature } (4,0) \text{ and } (2,2) \;:\; c \;=\; 0\ . \tag{6.2.41}$$

The case with $c = 0$ shall be referred to as the *chiral model*, that we shall discuss in more detail below.

The HS equations in Lorentzian signature have the $\mathbb{Z}_2$ symmetry acting as $(\widehat{S}_\alpha, \widehat{S}_{\dot\alpha}) \to (\epsilon\widehat{S}_\alpha, \epsilon\widehat{S}_{\dot\alpha})$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in $(4,0)$ and $(2,2)$ signatures acting as $(\widehat{S}_\alpha, \widehat{S}_{\dot\alpha}) \to (\epsilon\widehat{S}_\alpha, \epsilon'\widehat{S}_{\dot\alpha})$, where $\epsilon = \pm 1$ and $\epsilon' = \pm 1$.

Finally, let us give the reality conditions at the level of the $SO(5;\mathbb{C})$ algebra and its minimal bosonic higher-spin extension. The adjoint representation of the complex minimal bosonic higher-spin Lie algebra is defined by [3]

$$\mathfrak{ho}(5;\mathbb{C}) \;=\; \{Q(y,\bar y) \;:\; \tau(Q) \;=\; -Q\} \;, \tag{6.2.42}$$

and the corresponding minimal twisted-adjoint representation by

$$T[\mathfrak{ho}(5;\mathbb{C})] \;=\; \{S(y,\bar y) \;:\; \tau(S) \;=\; \pi(S)\} \;. \tag{6.2.43}$$

The real forms are defined by

$$\mathfrak{ho}(5-q,q) \;=\; \big\{Q(y,\bar y) \in \mathfrak{ho}(5;\mathbb{C}) \;:\; Q^\dagger \;=\; -\sigma(Q)\big\} \;, \tag{6.2.44}$$

$$T[\mathfrak{ho}(5-q,q)] \;=\; \big\{S(y,\bar y) \in T[\mathfrak{ho}(5-q,q) \;:\; S^\dagger \;=\; \sigma(\pi(S))\big\} \;. \tag{6.2.45}$$

The finite-dimensional $SO(5;\mathbb{C})$ subalgebra is generated by M_{AB}, that we split into Lorentz rotations and translations (M_{ab}, P_a) defined by

$$\pi(M_{ab}) \;=\; M_{ab} \;, \qquad \pi(P_a) \;=\; -P_a \;. \tag{6.2.46}$$

For these generators, which by convention arise in the expansion of the master fields together with a factor of i, the reality condition (6.2.24) implies

$$(M_{AB})^\dagger \;=\; \sigma(M_{AB}) \;. \tag{6.2.47}$$

This condition is solved by

$$M_{ab} \;=\; -\frac{1}{8}\Big((\sigma_{ab})^{\alpha\beta} y_\alpha y_\beta + (\bar\sigma_{ab})^{\dot\alpha\dot\beta} \bar y_{\dot\alpha} \bar y_{\dot\beta}\Big)\,, \quad P_a \;=\; \frac{\lambda}{4} (\sigma_a)^{\alpha\dot\alpha} y_\alpha \bar y_{\dot\alpha} \;, \tag{6.2.48}$$

where the van der Waerden symbols are defined in Appendix E and λ^2 is proportional to the cosmological constant, as shown in Table 6.1. The van der Waerden symbols encode the space-time signature η_{ab}, and the commutation relations among the M_{AB} then fix the signature of the ambient space to be

$$\eta_{AB} \;=\; (\eta_{ab}; -\lambda^2) \;. \tag{6.2.49}$$

[3] A more detailed description of the complex higher-spin algebra and its representations is given in Chapter 3.

6.2.3 The Chiral Model

In the chiral model with $c = 0$, the master field $\widehat{\Phi}$ can be eliminated using (6.2.13), and expressed as

$$\widehat{\Phi} \;=\; (1 + \frac{i}{2}\widehat{S}^{\alpha} \star \widehat{S}_{\alpha}) \star \kappa \;, \tag{6.2.50}$$

where we have chosen $c_1 = 1$ and $\widehat{S}_{\alpha}$ is given by (5.2.38). The remaining independent master-field equations now read

$$\widehat{F}_{\mu\nu} \;=\; 0 \;, \qquad \widehat{D}_{\mu}\widehat{S}_{\alpha} \;=\; 0 \;, \qquad \widehat{D}_{\mu}\widehat{S}_{\dot{\alpha}} \;=\; 0 \;, \tag{6.2.51}$$

$$[\widehat{S}_{\alpha}, \widehat{S}_{\dot{\alpha}}]_{\star} \;=\; 0 \;, \qquad [\widehat{S}_{\dot{\alpha}}, \widehat{S}_{\dot{\beta}}]_{\star} \;=\; -2i\epsilon_{\dot{\alpha}\dot{\beta}} \;, \tag{6.2.52}$$

$$\widehat{S}_{\alpha} \star \widehat{S}^{\beta} \star \widehat{S}_{\beta} + \widehat{S}^{\beta} \star \widehat{S}_{\beta} \star \widehat{S}_{\alpha} \;=\; 4i\widehat{S}_{\alpha} \;. \tag{6.2.53}$$

We note that (6.2.22) holds identically in virtue of $\widehat{S}_{\dot{\alpha}} \star \widehat{\Phi} + \widehat{\Phi} \star \bar{\pi}(\widehat{S}_{\dot{\alpha}}) = [\widehat{S}_{\dot{\alpha}}, 1 + \frac{i}{2}\widehat{S}^{\alpha} \star \widehat{S}_{\alpha}]_{\star} \star \kappa = 0$, where we used $\kappa\bar{\kappa} \star \widehat{S}_{\dot{\alpha}} \star \kappa\bar{\kappa} = -\widehat{S}_{\dot{\alpha}}$ and $[\widehat{S}_{\alpha}, \widehat{S}_{\dot{\alpha}}]_{\star} = 0$. The chiral model can be truncated further by imposing

$$\widehat{A}_{\dot{\alpha}} \;=\; 0 \;, \qquad \frac{\partial}{\partial z^{\dot{\alpha}}}\widehat{A}_{\mu} \;=\; 0 \;, \qquad \frac{\partial}{\partial z^{\dot{\alpha}}}\widehat{A}_{\alpha} \;=\; 0 \;. \tag{6.2.54}$$

In general, the chiral model also has interesting solutions with non-vanishing $\widehat{A}_{\dot{\alpha}}$, since flat connections in non-commutative geometry can be non-trivial.

6.2.4 Comments on Weak-Field Expansion and Spectrum

The procedure, described in great detail in Chapter 5, for obtaining the manifestly diffeomorphism and locally Lorentz invariant weak-field expansion of the physical field equations can be extended straightforwardly to arbitrary signature. The expansion is in terms of spin-s physical fields with $s \neq 2$ as well as higher derivatives of all fields, while the vierbein and Lorentz connection are treated exactly.

In this approach one first solves (6.2.12)–(6.2.16) subject to the initial condition

$$\Phi \;=\; \widehat{\Phi}|_{Z=0} \;, \tag{6.2.55}$$

$$A_{\mu} \;=\; \widehat{A}_{\mu}\Big|_{Z=0} \;=\; e_{\mu} + \omega_{\mu} + W_{\mu} + K_{\mu} \;, \tag{6.2.56}$$

where

$$e_\mu = \frac{1}{2i} e_\mu{}^a P_a \ , \qquad \omega_\mu = \tfrac{1}{4i} \omega_\mu{}^{ab} M_{ab} \ ; \tag{6.2.57}$$

W_μ contains the higher-spin gauge fields (and also the spin $s = 1$ gauge field in the non-minimal model); and we recall the field redefinition

$$K_\mu = \frac{1}{4i} \omega_\mu{}^{\alpha\beta} \, \widehat{S}_\alpha \star \widehat{S}_\beta \Big|_{Z=0} + \frac{1}{4i} \bar{\omega}_\mu{}^{\dot\alpha\dot\beta} \, \widehat{S}_{\dot\alpha} \star \widehat{S}_{\dot\beta} \Big|_{Z=0} \tag{6.2.58}$$

$$= i\omega_\mu{}^{\alpha\beta} \, (\widehat{A}_\alpha \star \widehat{A}_\beta - \frac{\partial}{\partial y^\alpha} \widehat{A}_\beta) \Big|_{Z=0} + i\bar{\omega}_\mu{}^{\dot\alpha\dot\beta} \, (\widehat{A}_{\dot\alpha} \star \widehat{A}_{\dot\beta} - \frac{\partial}{\partial \bar{y}^{\dot\alpha}} \widehat{A}_{\dot\beta}) \Big|_{Z=0} \ . \tag{6.2.59}$$

One also imposes the gauge condition

$$\widehat{A}^{(0)}_\alpha = 0 \ , \qquad \widehat{A}^{(0)}_{\dot\alpha} = 0 \ , \tag{6.2.60}$$

where we have defined the internal flat connection

$$\widehat{A}^{(0)}_\alpha = \widehat{A}_\alpha|_{\Phi=0} \ , \qquad \widehat{A}^{(0)}_{\dot\alpha} = \widehat{A}_{\dot\alpha}|_{\Phi=0} \ . \tag{6.2.61}$$

One then substitutes the resulting $\widehat{\Phi}$ and $\widehat{A}_\mu$, which can be obtained explicitly in a perturbative expansion in Φ, into (5.2.32) and sets $Z = 0$, which yields a manifestly spin-2 covariant complex HS gauge theory on the base manifold. Up to this point the local structure of the base-manifold, nor the detailed structure of the gauge fields, have played any role. To proceed, one may refer to an ordinary spacetime, take $e_\mu{}^a$ to be an (invertible) vierbein, and treat W_μ as a weak field. This allows one to eliminate a large number of auxiliary fields in Φ and W_μ, leaving a model consisting of a physical scalar $\phi = \Phi|_{y=\bar{y}=0}$, the vierbein $e_\mu{}^a$, and an infinite tower of (doubly traceless) HS gauge fields $\phi_{a(s)}$ residing in W_μ.

The gauge choice (6.2.60) is convenient since it implies $\frac{\partial}{\partial y^\alpha} \widehat{A}_\beta|_{Z=0} = 0$ that simplifies the expansion [96]. However, there are also other gauges where $\widehat{A}_\alpha|_{\Phi=0}$ is a flat but nontrivial internal connection, and indeed this will be the case for the Type 1 and Type 2 solutions that we shall present in Section 6.3.

In the leading order in the weak fields, the two-form and one-form constraints for the minimal model read

$$s=2\quad:\quad\left\{\begin{array}{lcllcl}\mathcal{R}_{\alpha\beta,\gamma\delta} & = & c_2\Phi_{\alpha\beta\gamma\delta}\ , & \mathcal{R}_{\dot\alpha\dot\beta,\gamma\delta} & = & 0\ ,\\ \mathcal{R}_{\alpha\beta,\gamma\dot\delta} & = & 0\ , & \mathcal{R}_{\dot\alpha\dot\beta,\gamma\dot\delta} & = & 0\ ,\\ \mathcal{R}_{\alpha\beta,\dot\gamma\dot\delta} & = & 0\ , & \mathcal{R}_{\dot\alpha\dot\beta,\dot\gamma\dot\delta} & = & c_1\Phi_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\ ,\end{array}\right. \tag{6.2.62}$$

$$s=4,6,...\quad:\quad\left\{\begin{array}{lcllcl}F^{(1)}_{\alpha\beta,\gamma_1\dots\gamma_{2s-2}} & = & c_2\Phi_{\alpha\beta\gamma_1\dots\gamma_{2s-2}}\ , & F^{(1)}_{\dot\alpha\dot\beta,\dot\gamma_1\dots\dot\gamma_k\gamma_{k+1}\dots\gamma_{2s-2}} & = & 0\ ,\\ F^{(1)}_{\alpha\beta,\gamma_1\dots\gamma_k\dot\gamma_{k+1}\dots\dot\gamma_{2s-2}} & = & 0\ , & F^{(1)}_{\dot\alpha\dot\beta,\dot\gamma_1\dots\dot\gamma_{2s-2}} & = & c_1\Phi_{\dot\alpha\dot\beta\dot\gamma_1\dots\dot\gamma_{2s-2}}\ ,\end{array}\right. \tag{6.2.63}$$

$$\text{0-forms}\quad:\quad\nabla_\alpha{}^{\dot\alpha}\Phi_{\beta_1\dots\beta_m}{}^{\dot\beta_1\dots\dot\beta_n}\ =\ i\lambda\left(\Phi_{\alpha\beta_1\dots\beta_m}{}^{\dot\alpha\dot\beta_1\dots\dot\beta_n}-mn\epsilon_{\alpha(\beta_1}\epsilon^{\dot\alpha(\dot\beta_1}\Phi_{\beta_2\dots\beta_m)}{}^{\dot\beta_2\dots\dot\beta_n)}\right), \tag{6.2.64}$$

where for higher spins $s=4,6,\dots$ and $k=0,\dots,2s-3$, and for 0-forms $|m-n|=0$ mod 4. In all cases, the zero-form system contains a physical scalar with field equation

$$(\nabla^2+2\lambda^2)\phi\ =\ 0\ . \tag{6.2.65}$$

In the Lorentzian case, where both c_1 and $c_2=c_1^*$ are non-zero, the spin-2 sector consists of gravity with cosmological constant $-3\lambda^2$, and the spin-s sectors with $s=4,6,\dots$ consist of higher-spin tensor gauge fields with critical masses proportional to λ^2. The criticality in the masses, that implies composite masslessness in the case of AdS, holds in the dS case as well, where thus the physical spectrum is given by the symmetric tensor product of two (non-unitary) $SO(4,1)$ singletons (see Chapter 3 for details).

In the Euclidean and Kleinian cases, the parameters c_1 and c_2 are real and independent. In case $c_1c_2\neq 0$, the Lorentzian analysis carries over, leading to a composite massless spectrum given by symmetric tensor products of suitable singletons. However, unlike the Lorentzian case, the spin-s sector of the twisted adjoint representation can be decomposed into left-handed and right-handed sub-sectors of real states, corresponding to $\{\Phi_{\alpha_1\dots\alpha_m,\dot\alpha_1\dots\dot\alpha_n}\}$ with $m-n=\pm 2s$. These sub-sectors mix under HS transformations.

In case either c_1 or c_2, but not both, vanishes, that we shall refer to as the chiral models, the metric and the higher-spin gauge fields become half-flat. For definiteness, let us consider the case $c_2=0$. The components of the zero-form that drop out in the two-form constraint, *i.e.* $\Phi_{\alpha_1\dots\alpha_{2s}}$, now become *independent* physical fields, obeying field equations following from (6.2.64).

6.3 Exact Solutions

In this section we shall give four types of exact solutions to the 4D HS models given in the previous section. The salient features of these are summarized in the Introduction

HSA	Signature η_{ab}	Spinors	λ^2	Reality σ	Symmetric space	Hermitian isometries
$\mathfrak{ho}(5)$	$(4,0)$	$SU(2)_L \times SU(2)_R$	-1	ρ	S^4	$\mathfrak{so}(2) \otimes \mathfrak{so}(3)$
$\mathfrak{ho}(4,1)$	$(4,0)$	$SU(2)_L \times SU(2)_R$	$+1$	$\rho\pi$	H_4	$\mathfrak{so}(3,1)$
$\mathfrak{ho}(4,1)$	$(3,1)$	$SL(2,\mathbb{C})_{\text{diag}}$	-1	π	dS_4	$\mathfrak{so}(3,1)'$
$\mathfrak{ho}(3,2)$	$(3,1)$	$SL(2,\mathbb{C})_{\text{diag}}$	$+1$	id	AdS_4	$\mathfrak{so}(3,2)$
$\mathfrak{ho}(3,2)$	$(2,2)$	$SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$	-1	id	$H_{3,2}$	$\mathfrak{so}(3,2)$

Table 6.1: The minimal bosonic higher-spin algebras $\mathfrak{ho}(p',5-p') \supset \mathfrak{so}(5-p',p')$ in signature $(p,4-p)$ can be realized with spinor oscillators transforming as doublets under the groups listed in the third column. These realizations obey reality conditions $(M_{AB})^\dagger = \sigma(M_{AB})$, with hermitian subalgebras listed above. The symmetric spaces with unit radius have cosmological constant $\Lambda = -3\lambda^2$.

to this Chapter. Here we stress that (a) the Type 0 solutions are maximally symmetric spaces; (b) the Type 1 solutions are $SO(4-p,p)$ invariant deformations of Type 0; (c) the Type 2 solutions, which exist necessarily in the non-minimal model, have vanishing spacetime component fields but non-vanishing spinorial master one-form; (d) the Type 3 solutions, which exist in the non-minimal chiral model only, have the remarkable feature that all higher spin gauge fields are non-vanishing in such a way that the Weyl zero-forms are covariantly constant, in a certain sense that will be explained below. Before we give these four types of solutions we shall describe briefly the method for solving the master field equations using gauge functions.

6.3.1 The Gauge Function Ansatz

In order to construct an interesting class of solutions we shall use the Z-space approach [84, 87, 97] in which the constraints carrying at least one curved space-time index, *viz.*

$$\widehat{F}_{\mu\nu} \;=\; 0\;, \qquad \widehat{D}_\mu \widehat{\Phi} \;=\; 0\;, \tag{6.3.1}$$

$$\widehat{F}_{\mu\alpha} \;=\; 0\;, \qquad \widehat{F}_{\mu\dot{\alpha}} \;=\; 0\;, \tag{6.3.2}$$

are integrated in simply connected space-time regions given the space-time zero-forms at a point p,

$$\widehat{\Phi}' \;=\; \widehat{\Phi}|_p\;, \qquad \widehat{S}'_\alpha \;=\; \widehat{S}_\alpha|_p\;, \qquad \widehat{S}'_{\dot{\alpha}} \;=\; \widehat{S}_{\dot{\alpha}}|_p\;, \tag{6.3.3}$$

and expressed explicitly as

$$\widehat{A}_\mu = \widehat{L}^{-1} \star \partial_\mu \widehat{L} \ , \qquad \widehat{\Phi} = \widehat{L}^{-1} \star \widehat{\Phi}' \star \pi(\widehat{L}) \ , \tag{6.3.4}$$

$$\widehat{S}_\alpha = \widehat{L}^{-1} \star \widehat{S}'_\alpha \star \widehat{L} \ , \qquad \widehat{S}_{\dot\alpha} = \widehat{L}^{-1} \star \widehat{S}'_{\dot\alpha} \star \widehat{L} \ , \tag{6.3.5}$$

where $\widehat{L} = \widehat{L}(x, z, \bar{z}; y, \bar{y})$ is a gauge function, and

$$\widehat{L}|_p = 1 \ , \qquad \partial_\mu \widehat{\Phi}' = 0 \ , \qquad \partial_\mu \widehat{S}'_\alpha = 0 \ , \qquad \partial_\mu \widehat{S}'_{\dot\alpha} = 0 \ . \tag{6.3.6}$$

The internal connections $\widehat{A}_\alpha$ and $\widehat{A}_{\dot\alpha}$ can be reconstructed from $\widehat{S}_\alpha$ and $\widehat{S}_{\dot\alpha}$ using (5.2.38). In particular note the relation

$$\widehat{A}_\alpha = \widehat{L} \star \partial_\alpha \widehat{L} + \widehat{L}^{-1} \star \widehat{A}'_\alpha \star \widehat{L} \ , \tag{6.3.7}$$

from which it follows that

$$\widehat{S}'_\alpha = z_\alpha - 2i \widehat{A}'_\alpha \ . \tag{6.3.8}$$

The remaining constraints in Z-space, *viz.*

$$[\widehat{S}'_\alpha, \widehat{S}'_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 - c_1 \widehat{\Phi}' \star \kappa) \ , \qquad [\widehat{S}'_{\dot\alpha}, \widehat{S}'_{\dot\beta}]_\star = -2i\epsilon_{\dot\alpha\dot\beta}(1 - c_2 \widehat{\Phi}' \star \bar\kappa) \ , \tag{6.3.9}$$

$$[\widehat{S}'_\alpha, \widehat{S}'_{\dot\beta}]_\star = 0 \ , \tag{6.3.10}$$

$$\widehat{S}'_\alpha \star \widehat{\Phi}' + \widehat{\Phi}' \star \pi(\widehat{S}'_\alpha) = 0 \ , \tag{6.3.11}$$

$$\widehat{S}'_{\dot\alpha} \star \widehat{\Phi}' + \widehat{\Phi}' \star \bar\pi(\widehat{S}'_{\dot\alpha}) = 0 \ , \tag{6.3.12}$$

are then to be solved with an initial condition

$$C'(y, \bar{y}) = \widehat{\Phi}'|_{Z=0} \ , \tag{6.3.13}$$

and some assumption about the topology of the internal flat connections

$$\widehat{S}'^{(0)}_\alpha = \widehat{S}'_\alpha|_{C'=0} \ , \qquad \widehat{S}'^{(0)}_{\dot\alpha} = \widehat{S}'_{\dot\alpha}|_{C'=0} \ . \tag{6.3.14}$$

In what follows, we shall restrict the class of solutions further by assuming that

$$\widehat{L} = L(x; y, \bar{y}) \ . \tag{6.3.15}$$

The gauge fields can then be obtained from (6.2.56), (6.2.59) and (6.3.5), *viz.*

$$e_\mu + \omega_\mu + W_\mu = L^{-1} \partial_\mu L - K_\mu \ , \tag{6.3.16}$$

where

$$K_\mu = \frac{1}{4i} L^{-1} \star \left(\omega_\mu{}^{\alpha\beta} \widehat{S}'_\alpha \star \widehat{S}'_\beta + \bar\omega_\mu{}^{\dot\alpha\dot\beta} \widehat{S}'_{\dot\alpha} \star \widehat{S}'_{\dot\beta} \right) \star L \Big|_{Z=0} \ . \tag{6.3.17}$$

Hence, the gauge fields, including the metric, can be obtained algebraically without having to solve any differential equations in space-time.

6.3.2 Ordinary Maximally Symmetric Spaces (Type 0)

The complex master-field equations are solved by

$$\widehat{\Phi} = 0 , \qquad \widehat{S}_\alpha = z_\alpha , \qquad \widehat{S}_{\dot\alpha} = \bar z_{\dot\alpha} , \qquad \widehat{A}_\mu = L^{-1} \star \partial_\mu L , \tag{6.3.18}$$

where the gauge function [87]

$$L(x;y,\bar y) = \frac{2h}{1+h} \exp\left[\frac{i\lambda x^{\alpha\dot\alpha} y_\alpha \bar y_{\dot\alpha}}{1+h}\right] , \tag{6.3.19}$$

gives

$$ds^2_{(0)} = \frac{4dx^2}{(1-\lambda^2 x^2)^2} , \tag{6.3.20}$$

which we identify as the metric of the symmetric spaces listed in Table 6.1 for the different real forms of the model, in stereographic coordinates with inverse radius $|\lambda|$. This metric is invariant under the inversion

$$x^a \to -x^a/(\lambda^2 x^2) , \tag{6.3.21}$$

and H_4 is covered by a single coordinate chart, while the remaining symmetric spaces require two charts, related by the inversion. If we let $\tilde x^a = -x^a/(\lambda^2 x^2)$, the atlases are given by

$$S^4 \quad (\lambda^2 = -1) \;:\; \{x^\mu : 0 \le -\lambda^2 x^2 \le 1\} \cup \{\tilde x^\mu : 0 \le -\lambda^2 \tilde x^2 \le 1\} , \tag{6.3.22}$$

$$H^4 \quad (\lambda^2 = 1) \;:\; \{x^\mu : 0 \le \lambda^2 x^2 < 1\} , \tag{6.3.23}$$

$$dS_4 \quad (\lambda^2 = -1) \;:\; \{x^\mu : -1 < -\lambda^2 x^2 \le 1\} \cup \{\tilde x^\mu : -1 < -\lambda^2 \tilde x^2 \le 1\} , \tag{6.3.24}$$

$$AdS_4 \quad (\lambda^2 = 1) \;:\; \{x^\mu : -1 \le \lambda^2 x^2 < 1\} \cup \{\tilde x^\mu : -1 \le \lambda^2 \tilde x^2 < 1\} , \tag{6.3.25}$$

$$H_{3,2} \quad (\lambda^2 = -1) \;:\; \{x^\mu : -1 < -\lambda^2 x^2 \le 1\} \cup \{\tilde x^\mu : -1 < -\lambda^2 \tilde x^2 \le 1\} , \tag{6.3.26}$$

where the overlap between the charts is given by $\{x^\mu : \lambda^2 x^2 = -1\}$ in the cases of S^4, dS_4, AdS_4 and $H_{3,2}$, and the boundary is $\{x^\mu : \lambda^2 x^2 = 1\}$ in the case of H_4 and $\{x^\mu : \lambda^2 x^2 = 1\} \cup \{\tilde x^\mu : \lambda^2 \tilde x^2 = 1\}$ in the cases of dS_4, AdS_4 and $H_{3,2}$. The $H_{3,2}$ space can be described as the coset $SO(3,2)/SO(2,2)$.

6.3.3 $SO(4-p,p)$ Invariant Solutions to the Minimal Model (Type 1)

Internal Master Fields

A particular class of $SO(4;\mathbb{C})$-invariant solutions is given by the ansatz

$$\widehat{\Phi}' = \nu , \qquad \widehat{S}'_\alpha = z_\alpha \, S(u) , \qquad \widehat{S}'_{\dot\alpha} = \bar z_{\dot\alpha} \, \bar S(\bar u) \tag{6.3.27}$$

where

$$u = y^\alpha z_\alpha \ , \qquad \bar{u} = \bar{y}^{\dot\alpha}\bar{z}_{\dot\alpha} \ . \tag{6.3.28}$$

The above ansatz solves (6.3.10)-(6.3.12). There remains to solve (6.3.9), which now takes the form

$$[\widehat{S}'^\alpha, \widehat{S}'_\alpha]_\star = 4i(1 - c_1\nu e^{iu}) \ , \qquad [\widehat{S}'^{\dot\alpha}, \widehat{S}'_{\dot\alpha}]_\star = 4i(1 - c_2\nu e^{-i\bar{u}}) \tag{6.3.29}$$

Following [86], we use the integral representation

$$S(u) = \int_{-1}^{1} ds\ n(s)\ e^{\frac{i}{2}(1+s)u} \ , \tag{6.3.30}$$

$$\bar{S}(\bar{u}) = \int_{-1}^{1} ds\ \bar{n}(s)\ e^{-\frac{i}{2}(1+s)\bar{u}} \ . \tag{6.3.31}$$

which reduces (6.3.29) to

$$(n \circ n)(t) = \delta(t-1) - \frac{c_1\nu}{2}(1-t) \ , \tag{6.3.32}$$

$$(\bar{n} \circ \bar{n})(t) = \delta(t-1) - \frac{c_2\nu}{2}(1-t) \ . \tag{6.3.33}$$

with $\circ$ defined by [86]

$$(f \circ g)(t) = \int_{-1}^{1} ds \int_{-1}^{1} ds' \delta(t - ss')\ f(s)\ g(s') \ . \tag{6.3.34}$$

Even and odd functions, denoted by $f^\pm(t)$, are orthogonal with respect to the $\circ$ product. Thus, one finds

$$(n^+ \circ n^+)(t) = \iota_0^+(t) - \frac{c_1\nu}{2} \ , \qquad (n^- \circ n^-)(t) = \iota_0^-(t) + \frac{c_1\nu}{2}t \ , \tag{6.3.35}$$

$$(\bar{n}^+ \circ \bar{n}^+)(t) = \iota_0^+(t) - \frac{c_2\nu}{2} \ , \qquad (\bar{n}^- \circ \bar{n}^-)(t) = \iota_0^-(t) + \frac{c_2\nu}{2}t \ , \tag{6.3.36}$$

where

$$\iota_0^\pm(t) = \frac{1}{2}\left[\delta(1-t) \pm \delta(1+t)\right] \ . \tag{6.3.37}$$

One proceeds [86], by writing

$$n^\pm(t) = m^\pm(t) + \sum_{k=0}^{\infty} \lambda_k p_k^\pm \ , \tag{6.3.38}$$

where $m^\pm$ are expanded in terms of $\iota_0^{(\pm)}(t)$ and the functions ($k \geq 1$)

$$\begin{aligned} \iota_k^\sigma(t) &= [\mathrm{sign}(t)]^{\frac{1}{2}(1-\sigma)} \int_{-1}^{1} ds_1 \cdots \int_{-1}^{1} ds_k\ \delta(t - s_1 \cdots s_k) \\ &= [\mathrm{sign}(t)]^{\frac{1}{2}(1-\sigma)} \frac{\left(\log \frac{1}{t^2}\right)^{k-1}}{(k-1)!} \ , \end{aligned} \tag{6.3.39}$$

obeying the algebra ($k, l \geq 0$)

$$\iota^\sigma_k \circ \iota^\sigma_l \;=\; \iota^\sigma_{k+l} \ , \tag{6.3.40}$$

and $p^\sigma_k(t)$ ($k \geq 0$) are the $\circ$-product projectors

$$p^\sigma_k(t) \;=\; \frac{(-1)^k}{k!}\delta^{(k)}(t) \ , \qquad \sigma \;=\; (-1)^k \ , \tag{6.3.41}$$

obeying

$$p^\sigma_k \circ f \;=\; L_k[f] p^\sigma_k \ , \qquad L_k[f] \;=\; \int_{-1}^{1} dt\ t^k f(t) \ . \tag{6.3.42}$$

In particular,

$$p^\sigma_k \circ p^\sigma_l \;=\; \delta_{kl} p^\sigma_l \ . \tag{6.3.43}$$

Substituting the expansion (6.3.38) into (6.3.35) and (6.3.36), one finds, in view of (6.3.40), (6.3.42) and (6.3.43), manageable algebraic equations. Transforming back one finds, after some algebra [97],

$$m(t) \;=\; \delta(1+t) + q(t) \ , \tag{6.3.44}$$

$$q(t) \;=\; -\frac{c_1\nu}{4}\left({}_1F_1\left[\frac{1}{2}; 2; \frac{c_1\nu}{2}\log\frac{1}{t^2}\right] + t\,{}_1F_1\left[\frac{1}{2}; 2; -\frac{c_1\nu}{2}\log\frac{1}{t^2}\right]\right) \ , \tag{6.3.45}$$

and

$$\lambda_k \;=\; -2\theta_k L_k[m] \ , \qquad \theta_k \in \{0, 1\} \ , \tag{6.3.46}$$

where

$$L_k[m] \;=\; (-1)^k + L_k[q] \ , \tag{6.3.47}$$

$$L_k[q] \;=\; -\frac{1+(-1)^k}{2}\left(1 - \sqrt{1 - \frac{c_1\nu}{1+k}}\right) - \frac{1-(-1)^k}{2}\left(1 - \sqrt{1 + \frac{c_1\nu}{2+k}}\right) \tag{6.3.48}$$

The overall signs in $m^\pm$ have been fixed in (6.3.45) by requiring that

$$S(u) = 1 \ \text{ for } \ \nu = 0 \ \text{ and } \ \theta_k = 0 \ . \tag{6.3.49}$$

Treating $\bar{n}$ the same way, one finds

$$\bar{m}(t) \;=\; \delta(1+t) + \bar{q}(t) \ , \tag{6.3.50}$$

$$\bar{q}(t) \;=\; -\frac{c_2\nu}{4}\left({}_1F_1\left[\frac{1}{2}; 2; \frac{c_2\nu}{2}\log\frac{1}{t^2}\right] + t\,{}_1F_1\left[\frac{1}{2}; 2; -\frac{c_2\nu}{2}\log\frac{1}{t^2}\right]\right) \ , \tag{6.3.51}$$

$$\bar{\lambda}_k \;=\; -2\bar{\theta}_k L_k[\bar{m}] \ , \qquad \bar{\theta}_k \in \{0, 1\} \ , \tag{6.3.52}$$

$$L_k[\bar{m}] \;=\; (-1)^k + L_k[\bar{q}] \ , \tag{6.3.53}$$

$$L_k[\bar{q}] \;=\; -\frac{1+(-1)^k}{2}\left(1 - \sqrt{1 - \frac{c_2\nu}{1+k}}\right) - \frac{1-(-1)^k}{2}\left(1 - \sqrt{1 + \frac{c_2\nu}{2+k}}\right) \tag{6.3.54}$$

Thus, the internal solution is given by

$$\widehat{\Phi}' = \nu \ , \tag{6.3.55}$$

together with $\widehat{S}'_\alpha$ and $\widehat{S}'_{\dot\alpha}$ as given in (6.3.8) with

$$\widehat{A}'_\alpha = \widehat{A}'^{(reg)}_\alpha + \widehat{A}'^{(proj)}_\alpha \ , \qquad \widehat{A}'_{\dot\alpha} = \widehat{A}'^{(reg)}_{\dot\alpha} + \widehat{A}'^{(proj)}_{\dot\alpha} \ , \tag{6.3.56}$$

$$\widehat{A}'^{(reg)}_\alpha = \frac{i}{2} z_\alpha \int_{-1}^{1} dt \ q(t)\, e^{\frac{i}{2}(1+t)u} \ , \qquad \widehat{A}'^{(reg)}_{\dot\alpha} = \frac{i}{2} \bar z_{\dot\alpha} \int_{-1}^{1} dt \ \bar q(t)\, e^{-\frac{i}{2}(1+t)\bar u} \ , \tag{6.3.57}$$

$$\widehat{A}'^{(proj)}_\alpha = -i z_\alpha \sum_{k=0}^{\infty} \theta_k (-1)^k L_k[m] P_k(u) \ , \qquad \widehat{A}'^{(proj)}_{\dot\alpha} = -i \bar z_{\dot\alpha} \sum_{k=0}^{\infty} \bar\theta_k (-1)^k L_k[\bar m] \bar P_k(\bar u) \ , \tag{6.3.58}$$

where

$$P_k(u) = \int_{-1}^{1} ds \ e^{\frac{i}{2}(1-s)u} p_k(s) = \frac{1}{k!} \left(\frac{-iu}{2}\right)^k e^{\frac{iu}{2}} \ , \tag{6.3.59}$$

$$\bar P_k(\bar u) = \int_{-1}^{1} ds \ e^{-\frac{i}{2}(1-s)\bar u} p_k(s) = \frac{1}{k!} \left(\frac{i\bar u}{2}\right)^k e^{-\frac{i\bar u}{2}} \tag{6.3.60}$$

are projectors in the $\star$-product algebra given by functions of u and $\bar u$, *viz.*

$$P_k \star F = L_k[f] P_k \ , \qquad P_k \star P_l = \delta_{kl} P_k \ , \tag{6.3.61}$$

$$\bar P_k \star \bar F = L_k[\bar f] \bar P_k \ , \qquad \bar P_k \star \bar P_l = \delta_{kl} \bar P_k \ , \tag{6.3.62}$$

for $F(u) = \int_{-1}^{1} ds e^{\frac{i}{2}(1-s)u} f(s)$ and $\bar F(\bar u) = \int_{-1}^{1} ds e^{-\frac{i}{2}(1-s)\bar u} \bar f(s)$ with $L_k[f]$ and $L_k[\bar f]$ given in (6.3.42). The projectors also obey $(u - 2ik) \star P_k = 0$ and $y^\alpha \star P_k \star z_\alpha = i(k+1)(P_{k-1} + P_{k+1})$ with $P_{-1} \equiv 0$. We note the opposite signs in front of s in the exponents of (6.3.30), (6.3.31) and (6.3.59), (6.3.60), resulting in the $(-1)^k$ in the projector part (6.3.58) of the internal connection, which we can thus write as

$$\widehat{A}'^{(proj)}_\alpha = -i z_\alpha \sum_{k=0}^{\infty} \left[\theta_k P_k - \left(1 - \sqrt{1 - \frac{c_1 \nu}{1+2k}}\right) \theta_{2k} P_{2k} + \left(1 - \sqrt{1 + \frac{c_1 \nu}{3+2k}}\right) \theta_{2k+1} P_{2k+1} \right] \ , \tag{6.3.63}$$

$$\widehat{A}'^{(proj)}_{\dot\alpha} = -i \bar z_{\dot\alpha} \sum_{k=0}^{\infty} \left[\bar\theta_k \bar P_k - \left(1 - \sqrt{1 - \frac{c_2 \nu}{1+2k}}\right) \bar\theta_{2k} \bar P_{2k} + \left(1 - \sqrt{1 + \frac{c_2 \nu}{3+2k}}\right) \bar\theta_{2k+1} \bar P_{2k+1} \right] \ , \tag{6.3.64}$$

which are analytic functions of ν in a finite region around the origin. For example, for $c_1 = c_2 = 1$, they are real analytic for $-3 < \mathrm{Re}\nu < 1$, where also the particular solution can be shown to be real analytic [97]. The reality conditions on the θ_k and $\bar{\theta}_k$ parameters are as follows:

$$(4,0) \text{ and } (2,2) \text{ signature} \;:\; \theta_k \;,\quad \bar{\theta}_k \qquad \text{independent}\;, \tag{6.3.65}$$

$$(3,1) \text{ signature} \;:\; \theta_k \;=\; \bar{\theta}_k \;. \tag{6.3.66}$$

Taking $\nu = 0$ there remains only the projector part, leading to the following "vacuum" solutions

$$\widehat{\Phi}' \;=\; 0\;, \tag{6.3.67}$$

$$\widehat{A}'_\alpha \;=\; -iz_\alpha \sum_{k=0}^{\infty} \theta_k \frac{1}{k!} \left(\frac{-iu}{2}\right)^k e^{\frac{iu}{2}}\;, \qquad \widehat{A}'_{\dot\alpha} \;=\; -i\bar{z}_{\dot\alpha} \sum_{k=0}^{\infty} \bar{\theta}_k \frac{1}{k!} \left(\frac{i\bar{u}}{2}\right)^k e^{-\frac{i\bar{u}}{2}}. \tag{6.3.68}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (**??**) acts by

$$\theta_k \;\to\; 1 - \theta_k\;, \qquad \bar{\theta}_k \;\to\; 1 - \bar{\theta}_k\;. \tag{6.3.69}$$

The maximally symmetric spaces discussed in Section 3.2 are recovered by setting $\theta_k = \theta$ and $\bar{\theta}_k = \bar{\theta}$ for all k. In Euclidean and Kleinian signatures, θ and $\bar{\theta}$ are independent, leading to four solutions related by $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformations. In Lorentzian signature, $\theta = \bar{\theta}$ leading to two solutions related by $\mathbb{Z}_2$ symmetry.

Space-time Component Fields

The calculation of the component fields follows the same steps as in [97]. The spin $s \geq 1$ Weyl tensors vanish, while the scalar field is given by

$$\phi(x) \;=\; \nu h^2(x^2) \;=\; \nu(1 - \lambda^2 x^2)\;. \tag{6.3.70}$$

In order to compute the gauge fields, we first need to compute the quantity K_μ given in (6.3.17). This calculation is formally the same as the one spelled out in the case of $\theta_k = \bar{\theta}_k = 0$ in [97], and the result is

$$K_\mu \;=\; \frac{Q}{4i}\omega_\mu^{\alpha\beta} v_\alpha v_\beta + \frac{\bar{Q}}{4i}\bar{\omega}_\mu^{\dot\alpha\dot\beta} \bar{v}_{\dot\alpha}\bar{v}_{\dot\beta}\;, \tag{6.3.71}$$

where

$$Q \;=\; -\frac{(1-a^2)^2}{4}\int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1+s)(1+s')n(s)n(s')}{(1-ss'a^2)^4}\;, \tag{6.3.72}$$

$$\bar{Q} \;=\; -\frac{(1-a^2)^2}{4}\int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1+s)(1+s')\bar{n}(s)\bar{n}(s')}{(1-ss'a^2)^4}\;. \tag{6.3.73}$$

and

$$v_\alpha = (1+a^2)y_\alpha + 2(a\bar{y})_\alpha\ , \qquad \bar{v}_{\dot{\alpha}} = (1+a^2)\bar{y}_{\dot{\alpha}} + 2(\bar{a}y)_{\dot{\alpha}}\ , \tag{6.3.74}$$

with $\bar{a}_{\dot{\alpha}\alpha} = a_{\alpha\dot{\alpha}}$ defined in (F.0.3). We can simplify Q using $n(t) = \delta(1+t) + q(t) + \sum_k \lambda_k p_k(t)$, with $p_k(t)$ given by (6.3.41) and λ_k by (6.3.46) and (6.3.47). After some algebra we find

$$Q(\nu;\{\theta_k\}) = Q^{(reg)}(\nu) + Q^{(proj)}(\nu;\{\theta_k\})\ , \tag{6.3.75}$$

$$Q^{(reg)} = -\frac{(1-a^2)^2}{4}\int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1+s)(1+s')q(s)q(s')}{(1-ss'a^2)^4}\ , \tag{6.3.76}$$

$$\begin{aligned} Q^{(proj)} &= (1-a^2)^2 \sum_{k=0}^{\infty} \frac{4_k a^{2k}}{k!}(\theta_k - \theta_{k+1})^2 \times \\ &\quad \times \big((-1)^k + L_k(q)\big)\big((-1)^{k+1} + L_{k+1}(q)\big)\ , \end{aligned} \tag{6.3.77}$$

where we note that Q depends on θ_k only via $\theta_k - \theta_{k+1}$. The same expression with $q \to \bar{q}$ and $\theta_k \to \bar{\theta}_k$ holds for $\bar{Q}$. The regular part, which was computed in [97], is given by

$$Q^{(reg)} = Q_+^{(reg)} + Q_-^{(reg)}\ , \tag{6.3.78}$$

$$Q_+^{(reg)} = -\frac{(1-a^2)^2}{4}\sum_{p=0}^{\infty}\binom{-4}{2p} a^{4p}\left(\sqrt{1-\frac{c_1\nu}{2p+1}} - \sqrt{1+\frac{c_1\nu}{2p+3}}\right)^2 \tag{6.3.79}$$

$$Q_-^{(reg)} = \frac{(1-a^2)^2}{4}\sum_{p=0}^{\infty}\binom{-4}{2p+1} a^{4p+2}\left(\sqrt{1-\frac{c_1\nu}{2p+3}} - \sqrt{1+\frac{c_1\nu}{2p+3}}\right)^2 \tag{6.3.80}$$

while a similar expression, obtained by replacing $c_1 \to c_2$, holds for $\bar{Q}$.

Since K_μ is bilinear in the y_α and $\bar{y}_{\dot{\alpha}}$ oscillators, it immediately follows that all higher spin fields vanish. Moreover, after some algebra, we find that the vierbein and $\mathfrak{so}(4;\mathbb{C})$ connection are given by

$$e^a = f_1(x^2)dx^a + f_2(x^2)x^a dx^b x_b\ , \tag{6.3.81}$$

$$\omega_{\alpha\beta} = f(x^2)\omega^{(0)}_{\alpha\beta}\ , \qquad \bar{\omega}_{\dot{\alpha}\dot{\beta}} = \bar{f}(x^2)\bar{\omega}^{(0)}_{\dot{\alpha}\dot{\beta}}\ , \tag{6.3.82}$$

where

$$f = \frac{1+(1-a^2)^2\bar{Q}}{[1+(1+a^2)^2 Q]\left[1+(1+a^2)^2\bar{Q}\right] - 16a^4 Q\bar{Q}}\ , \tag{6.3.83}$$

$$\bar{f} = \frac{1+(1-a^2)^2 Q}{[1+(1+a^2)^2 Q]\left[1+(1+a^2)^2\bar{Q}\right] - 16a^4 Q\bar{Q}}\ , \tag{6.3.84}$$

and

$$f_1 + \lambda^2 x^2 f_2 \;=\; \frac{2}{h^2}\;, \qquad f_2 \;=\; \frac{2(1+a^2)^4}{(1-a^2)^2}(fQ+\bar{f}\bar{Q})\;. \tag{6.3.85}$$

By a change of coordinates, the metric can be written locally, in a given coordinate chart, as a foliation

$$ds^2 \;=\; \epsilon d\tau^2 + R^2 d\Omega_3^2\;, \qquad R^2(\tau) \;=\; \eta^2|\sinh^2(\sqrt{\epsilon}\tau)|\;, \tag{6.3.86}$$

where $x^2 = \epsilon\tan^2\frac{\tau}{2}$ with $\epsilon = \pm 1$, and $d\Omega_3^2$ is a three-dimensional metric of constant curvature with suitable signature, and [97]

$$\eta \;=\; \frac{f_1 h^2}{2}\;. \tag{6.3.87}$$

One has the following simplifications in specific models:

$$\text{Type A model:} \qquad Q \;=\; \bar{Q}\;, \qquad \eta \;=\; \frac{1+(1-a^2)^2 Q}{1+(1+6a^2+a^4)Q}\;, \tag{6.3.88}$$

$$\text{Chiral model:} \qquad \bar{Q} \;=\; 0\;, \qquad \eta \;=\; \frac{1+(1-a^2)^2 Q}{1+(1+a^2)^2 Q}\;. \tag{6.3.89}$$

The metric may have conical singularities, namely zeroes $R(\tau_0)=0$ for which $\partial_\tau R|_{\tau_0} \neq 1$ (we note that $\eta|_{\tau=0} = 1$, so that $\tau = 0$ is not a conical singularity). The scale factor depends heavily on ν as well as on the choice of the infinitely many discrete parameters θ_k and $\bar{\theta}_k$. This makes the analysis unyielding, and we shall therefore limit ourselves to the case of vanishing discrete parameters and $|\nu| \ll 1$. The resulting analysis was performed in [**?**] in Lorentzian signature, and it generalizes straightforwardly to Euclidean and Kleinian signatures. To this end, one examines the integrals (6.3.72) and (6.3.73) in the limits $a^2 \to \pm 1$, where there are potentially divergent contributions from the region of the integration domain where s and s' approach ± 1. These contributions are actually finite for $a^2 = 1$, while they diverge as

$$Q^{(reg)} \;=\; \frac{c_1\nu}{6}\log(1+a^2) + \mathcal{O}(\nu^2)\;, \tag{6.3.90}$$

when $a^2 \to -1$ (and the $\mathcal{O}(\nu^2)$ contributions are finite). Focusing on a single chart, as listed in (6.3.22)-(6.3.26), a^2 is bounded from below by $(1-\sqrt{2})(1+\sqrt{2})^{-1}$, and hence, if $|\nu| \ll 1$, then $|Q| \ll 1$, and consequently the factor η defined in (6.3.87) remains finite. Thus, for small enough ν, there are no conical singularities within the coordinate charts. However, they may appear for some finite critical ν.

While the Q functions are complicated for $\nu \neq 0$, they simplify drastically at $\nu = 0$, where we find

$$Q \;=\; -(1-a^2)^2\sum_{k=0}^{\infty}\frac{4_k a^{2k}}{k!}(\theta_k - \theta_{k+1})^2\;. \tag{6.3.91}$$

An analogous expression can be found for $\bar{Q}$. Setting $(\theta_k - \theta_{k+1})^2 = 1$, yields

$$Q = -\frac{1}{(1-a^2)^2} . \tag{6.3.92}$$

If $Q = \bar{Q} = -(1-a^2)^{-2}$, which is necessarily the case in the Lorentzian models, then the equation system for $\omega_{\alpha\beta}$ and $\bar{\omega}_{\dot{\alpha}\dot{\beta}}$ becomes degenerate, and one finds

$$\omega_{\alpha\beta} = -\frac{(1-a^2)^2}{8a^2}\omega^{(0)}_{\alpha\beta} = \frac{(\sigma^{ab})_{\alpha\beta}dx_a x_b}{2x^2} , \tag{6.3.93}$$

$$\bar{\omega}_{\dot{\alpha}\dot{\beta}} = -\frac{(1-a^2)^2}{8a^2}\bar{\omega}^{(0)}_{\dot{\alpha}\dot{\beta}} = \frac{(\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}dx_a x_b}{2x^2} , \tag{6.3.94}$$

leading to the degenerate vierbein

$$e_{\alpha\dot{\alpha}} = -\frac{\lambda x_{\alpha\dot{\alpha}} x^a dx_a}{x^2 h^2} , \tag{6.3.95}$$

and metric

$$ds^2 = \frac{4(x^a dx_a)^2}{\lambda^2 x^2 h^2} . \tag{6.3.96}$$

6.3.4 Solutions of the Non-minimal Model (Type 2)

Internal Master Fields

The non-minimal model admits the following solutions

$$\widehat{\Phi}' = 0 , \qquad \widehat{S}'_{\alpha} = z_{\alpha} \star \Gamma(y,\bar{y}) , \qquad \widehat{S}'_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}} \star \bar{\Gamma}(y,\bar{y}) , \tag{6.3.97}$$

provided that

$$\Gamma \star \Gamma = \bar{\Gamma} \star \bar{\Gamma} = 1 , \qquad [\Gamma, \bar{\Gamma}]_\star = 0 , \qquad \pi\bar{\pi}(\Gamma) = \Gamma , \qquad \pi\bar{\pi}(\bar{\Gamma}) = \bar{\Gamma} \tag{6.3.98}$$

The elements Γ and $\bar{\Gamma}$ can be written as

$$\Gamma = 1 - 2P , \qquad \bar{\Gamma} = 1 - 2\bar{P} , \tag{6.3.99}$$

where $P(y,\bar{y})$ and $\bar{P}(y,\bar{y})$ are projectors obeying

$$P \star P = P , \qquad \bar{P} \star \bar{P} = \bar{P} , \quad [P, \bar{P}]_\star = 0 , \qquad \pi\bar{\pi}(P) = P , \qquad \pi\bar{\pi}(\bar{P}) = \bar{P} \tag{6.3.100}$$

A set of such projectors is described in Appendix G, where we also explain why the projectors can be subject to the τ-conditions of the non-minimal model, given in (5.2.14), but not to those of the minimal model, given in (5.2.13), unless one develops some further formalism for handling certain divergent $\star$-products.

Space-time Component Fields

Turning to the computation of the space components of the master fields, since z_α star-commutes with L, it immediately follows from (6.3.16), (6.2.59) and (6.3.97) that

$$K_\mu \;=\; 0\ . \tag{6.3.101}$$

From (6.3.16) this in turn implies that all HS gauge fields and the spin-1 gauge field vanish, while the metric is that of maximally symmetric spacetime. To that extent, the Type 1 solution looks like the Type 0 solution, but it does differ in an important way, namely, here the internal connection, *i.e.* the spinor component $\widehat{A}_\alpha$ of the master 1-form, is non-vanishing. Indeed, (6.3.97), (6.3.98) and (6.3.8) give the result

$$\widehat{A}_\alpha \;=\; -iz_\alpha \star V(x;y,\bar{y})\ , \qquad \widehat{A}_{\dot\alpha} \;=\; -i\bar{z}_{\dot\alpha} \star \bar{V}(x;y,\bar{y})\ , \tag{6.3.102}$$

where the quantities V and $\bar{V}$, which shall be frequently encountered in what follows, are defined by

$$V \;=\; L^{-1} \star P \star L\ , \qquad \bar{V} \;=\; L^{-1} \star \bar{P} \star L\ . \tag{6.3.103}$$

Their explicit evaluation is given in Appendix H, with the result (H.0.21).

Whilst the internal connection does not turn on any spacetime component fields, it does, however, affect the interactions as it does not obey the physical gauge condition normally used in the weak-field expansion [96], namely that the internal connection should vanish when the zero-form vanishes. In this sense, the internal connection may be viewed as a non-trivial flat connection in the non-commutative space.

6.3.5 Solutions of the Non-minimal Chiral Model (Type 3)

Internal Master Fields

In the case of the non-minimal chiral model, defined in Section 2.3, it is possible to use projectors $P(y,\bar{y})$ to build solutions with non-vanishing Weyl zero-form and higher spin fields. They are

$$\widehat{\Phi}' \;=\; (1-P)\star\kappa\ , \qquad \widehat{S}'_\alpha \;=\; z_\alpha \star P\ , \qquad \widehat{S}'_{\dot\alpha} \;=\; \bar{z}_{\dot\alpha} \star \bar{\Gamma}\ , \tag{6.3.104}$$

where

$$P\star P \;=\; P\ , \qquad \bar{\Gamma}\star\bar{\Gamma} \;=\; 1\ , \qquad [P,\bar{\Gamma}]_\star \;=\; 0\ , \qquad \pi\bar{\pi}(P) \;=\; P\ , \qquad \pi\bar{\pi}(\bar{\Gamma}) \;=\; \bar{\Gamma}\ . \tag{6.3.105}$$

These elements of the $\star$-product algebra can be constructed as in Section 6.3.4 and Appendix G.

For the purpose of exhibiting explicitly the spacetime component fields, we choose to work with the simplest possible projectors, namely

$$P_+(y) \;=\; 2e^{-2\epsilon uv} \;=\; 2e^{\epsilon yby}\ , \tag{6.3.106}$$

$$P_-(\bar y) \;=\; 2e^{-2\epsilon\bar u\bar v} \;=\; 2e^{\epsilon\bar y\bar b\bar y}\ , \tag{6.3.107}$$

where $\epsilon=\pm1$, and u, v, $\bar u$, $\bar v$, $b_{\alpha\beta}$ and $\bar b_{\dot\alpha\dot\beta}$ are defined in Appendices F and G.

Space-time Component Fields

The master gauge field and zero-form are given by

$$e_\mu+\omega_\mu+W_\mu \;=\; e^{(0)}_\mu+\omega^{(0)}_\mu+\frac{\omega_\mu^{\alpha\beta}}{4i}\frac{\partial^2 V}{\partial y^\alpha\partial y^\beta}\ , \tag{6.3.108}$$

and

$$\Phi \;=\; \left[L^{-1}\star(1-P)\star\kappa\star\pi(L)\right]|_{Z=0} \;=\; 1-V|_{y_\alpha=0}\ , \tag{6.3.109}$$

where V is given by (H.0.21) and we have used (5.2.21). Remarkably, since there is no y-dependence in the Weyl zero-form Φ, it is covariantly constant in the sense that $\Phi_{\alpha(m)\dot\alpha(n)}$ vanishes unless $m=0$. Moreover, using (H.0.21), it is straightforward to compute the constant value of the physical scalar field, with the result

$$\phi(x) \;=\; 1-4\sum_{n_1,n_2\in\mathbb{Z}+\frac12}(-1)^{n_1+n_2-\frac{\epsilon_1+\epsilon_2}{2}}\theta_{n_1,n_2}\ , \tag{6.3.110}$$

where θ_{n_1,n_2} are constrained as in (G.0.15). Summing over all n_2, as explained in Appendix G, and using (G.0.17) with $x=0$, *i.e.* $\sum_{k=0}^\infty(-1)^k=\frac12$, one finds that for the reduced projector (G.0.19), the scalar field is given by

$$\phi(x) \;=\; 1-2\sum_{n\in\mathbb{Z}+\frac12}(-1)^{n-\frac{\epsilon}{2}}\theta_n\ , \tag{6.3.111}$$

where θ_n obey the condition given below (G.0.19). Finally setting $\theta_n=\theta(\pm n)$, one ends up with $P=1$, *i.e.* in the Type 0 case, where indeed $\phi(x)=0$.

In the special cases of (6.3.106) and (6.3.107), one finds

$$V_+ \;=\; L^{-1}\star P_+\star L \;=\; 2\exp\left(-\epsilon\frac{[2\bar y\bar a-(1+a^2)y]\,b\,[2a\bar y+(1+a^2)y]}{(1-a^2)^2}\right) \tag{6.3.112}$$

$$V_- \;=\; L^{-1}\star P_-\star L \;=\; 2\exp\left(-\epsilon\frac{[2ya-(1+a^2)\bar y]\,\bar b\,[2\bar a y+(1+a^2)\bar y]}{(1-a^2)^2}\right) \tag{6.3.113}$$

where $a_{\alpha\dot{\alpha}}$ and $b_{\alpha\beta}$ are defined in Appendix F and ϵ is defined in (6.3.106) and (6.3.107). The physical scalar is now given in both cases by

$$\phi(x) \;=\; -1 \;, \tag{6.3.114}$$

and the self-dual Weyl tensors in both cases by ($s = 1, 2, 3, ...$)

$$\Phi_{\alpha(2s)} \;=\; 0 \;, \tag{6.3.115}$$

while the anti-self-dual Weyl tensors take the form

$$\Phi^{+}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \;=\; -2^{2s+1}(2s-1)!!\left(\frac{h^2-1}{\epsilon h^2}\right)^s U_{(\dot{\alpha}_1}\cdots U_{\dot{\alpha}_s}V_{\dot{\alpha}_{s+1}}\cdots V_{\dot{\alpha}_{2s})} \;, \tag{6.3.116}$$

$$\Phi^{-}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \;=\; -2^{2s+1}(2s-1)!!\left(\frac{1}{\epsilon h^2}\right)^s \bar{\lambda}_{(\dot{\alpha}_1}\cdots\bar{\lambda}_{\dot{\alpha}_s}\bar{\mu}_{\dot{\alpha}_{s+1}}\cdots\bar{\mu}_{\dot{\alpha}_{2s})} \;, \tag{6.3.117}$$

with spinors (U, V) defined in (F.0.11).

In the case of $\lambda^2 = 1$ in Euclidean signature, we only need to use one coordinate chart, in which $0 \leq h^2 \leq 1$. The Weyl tensors blow up in the limit $h^2 \to 0$, preventing the solution from approaching H_4 in this limit. In this sense the above solution is a non-perturbative solution without weak-field limit in any region of spacetime. Indeed, in the perturbative weak-field expansion around the H_4 solution, the scalar field has non-vanishing mass, preventing the linearized scalar field from being a non-vanishing constant.

In the case of $\lambda^2 = -1$ in Euclidean signature, the base manifold consists of two charts, covered by the coordinates in (6.3.22). Thus, in each chart we have $1 \leq h^2 < 2$, and so the local representatives (6.3.116) and (6.3.117) of the Weyl tensors are well-defined throughout the base manifold.

Finally, in the case of $\lambda^2 = -1$ in Kleinian signature, one also needs two charts, with $0 \leq h^2 \leq 2$, and hence the Weyl tensors blow up in the limit $h^2 \to 0$, preventing the solution from approaching $H_{3,2}$ in this limit.

From the Weyl tensors, which are not in themselves HS gauge invariant quantities, one can construct an infinite set of invariant (and thus closed) zero-forms [97], namely

$$\mathcal{C}^{-}_{2p} \;=\; \int \frac{d^4y d^4z}{(2\pi)^4}[(\widehat{\Phi}\star\pi(\widehat{\Phi})]^{\star p}\star\kappa\bar{\kappa} \;. \tag{6.3.118}$$

Remarkably, on our solution they all assume the same value, given by the constant value of the scalar field, *viz.*

$$\mathcal{C}^{-}_{2p} \;=\; (1-V)^{\star 2p}|_{y=\bar{y}=0} \;=\; 1-4\sum_{n_1,n_2\in\mathbb{Z}+\frac{1}{2}}(-1)^{n_1+n_2-\frac{\epsilon_1+\epsilon_2}{2}}\theta_{n_1,n_2} \;. \tag{6.3.119}$$

The calculation of the metric in the two models proceeds in a parallel fashion as follows:

The P_+ Solution:

From (6.3.108) and (6.3.112) a straightforward computation yields the result

$$e_{\mu\dot\alpha\alpha} = e^{(0)}_{\mu\dot\alpha\alpha} + 12(1+h)h^{-4}\, b_{(\alpha\beta}(ba)_{\gamma)\dot\alpha}\, \omega_\mu^{\beta\gamma} \ , \tag{6.3.120}$$

$$\omega_{\mu\alpha\beta} = \omega^{(0)}_{\mu\alpha\beta} + 12h^{-4} b_{(\alpha\beta}b_{\gamma\delta)}\, \omega_\mu^{\gamma\delta} \ , \tag{6.3.121}$$

$$\bar\omega_{\mu\dot\alpha\dot\beta} = \bar\omega^{(0)}_{\mu\dot\alpha\dot\beta} + 4(1+h)^2h^{-4}\left[-(\bar a b a)_{\dot\alpha\dot\beta}b_{\gamma\delta} + 2(\bar a b)_{\dot\alpha\gamma}(\bar a b)_{\dot\beta\delta}\right]\omega_\mu^{\gamma\delta} \ . \tag{6.3.122}$$

First we solve for the spin connection from (6.3.121) by inverting the hyper-matrix that multiplies $\omega^{(0)}$, obtaining the result

$$\omega_{\mu\alpha\beta} = g_1\left[\omega^{(0)}_{\mu\alpha\beta} - 8g(b\omega^{(0)}_\mu b)_{\alpha\beta}\right] + g_2 b_{\alpha\beta}b^{\gamma\delta}\omega^{(0)}_{\mu\gamma\delta} \ , \tag{6.3.123}$$

where

$$g_1 = \frac{1}{1-4g^2} \ , \qquad g_2 = \frac{4g}{(1-2g)(1-4g)} \ , \qquad g = h^{-4} \ . \tag{6.3.124}$$

Substituting this result in (6.3.120) then gives the vierbein

$$e_\mu^a = \frac{-2}{h^2(1+2g)}\left[g_3\delta_\mu^a + g_4\lambda^2 x_\mu x^a + g_5\lambda^2(Jx)_\mu(Jx)^a\right] \ , \tag{6.3.125}$$

where

$$g_3 = 1 + 2h^{-2} \ , \qquad g_4 = 2g \qquad g_5 = \frac{6g}{1-4g} \ , \tag{6.3.126}$$

and the spin connections are given in (6.3.123) and (6.3.122). Thus, the metric $g_{\mu\nu} = e_\mu^a e_\nu^b\, \eta_{ab}$ takes the form

$$g_{\mu\nu} = \frac{4}{h^4(1+2g)^2}\left[g_3^2\eta_{\mu\nu} + g_4(\lambda^2x^2g_4 + 2g_3)x_\mu x_\nu + g_5(\lambda^2x^2g_5 + 2g_3)(Jx)_\mu(Jx)_\nu\right] \ . \tag{6.3.127}$$

The vierbein has thus potential singularities at $h^2 = 0$ and $h^2 = 2$. The limit $h^2 \to 0$ is a boundary in the case of $\lambda^2 = 1$ in Euclidean signature and $\lambda^2 = -1$ in Kleinian signature. At these boundaries $e_\mu{}^a \sim h^{-2}x_\mu x^a$, *i.e.* a scale factor times a degenerate vierbein. In the limit $h^2 \to 2$ one approaches the boundary of a coordinate chart in the case of $\lambda^2 = 1$ in Euclidean signature and $\lambda^2 = -1$ in Kleinian signature. Also in this limit, the vierbein becomes degenerate, *viz.* $e_\mu{}^a \sim h^{-2}(Jx)_\mu(Jx)^a$.

The P_- Solution:

A parallel computation that uses (6.3.108) and (6.3.113) yields the result

$$e_{\mu\dot\alpha\alpha} = e^{(0)}_{\mu\dot\alpha\alpha} + 12\lambda^2 x^2 (1+h) h^{-4}\, \tilde b_{(\alpha\beta}(\tilde b a)_{\gamma)\dot\alpha}\, \omega_\mu^{\beta\gamma} \ , \tag{6.3.128}$$

$$\omega_{\mu\alpha\beta} = \omega^{(0)}_{\mu\alpha\beta} + 12(\lambda^2 x^2)^2 h^{-4} \tilde b_{(\alpha\beta}\tilde b_{\gamma\delta)}\, \omega_\mu^{\gamma\delta} \ , \tag{6.3.129}$$

$$\bar\omega_{\mu\dot\alpha\dot\beta} = \bar\omega^{(0)}_{\mu\dot\alpha\dot\beta} + 4(1+h)^2 h^{-4}\left[-(\bar a\tilde b a)_{\dot\alpha\dot\beta}\tilde b_{\gamma\delta} + 2(\tilde b a)_{\gamma\dot\alpha}(\tilde b a)_{\delta\dot\beta}\right]\omega_\mu^{\gamma\delta} \ , \tag{6.3.130}$$

where $\tilde b_{\alpha\beta}$ is defined in (F.0.9). As before, solving for the spin connection from (6.3.129) by inverting the hyper-matrix that multiplies $\omega^{(0)}$, we obtain

$$\omega_{\mu\alpha\beta} = \tilde g_1 \left[\omega^{(0)}_{\mu\alpha\beta} - 8\tilde g (\tilde b \omega^{(0)}_\mu \tilde b)_{\alpha\beta}\right] + \tilde g_2 \tilde b_{\alpha\beta}\tilde b^{\gamma\delta}\omega^{(0)}_{\mu\gamma\delta} \ , \tag{6.3.131}$$

where

$$\tilde g_1 = \frac{1}{1-4\tilde g} \ , \qquad \tilde g_2 = \frac{4\tilde g}{(1-2\tilde g)(1-4\tilde g)} \ , \qquad \tilde g = (\lambda^2 x^2)^2 h^{-4} \ . \tag{6.3.132}$$

Substituting this result in (6.3.128) then gives the vierbein

$$e_\mu^a = \frac{-2}{h^2\,(1+2\tilde g)}\left[\delta_\mu^a + \tilde g_4 \lambda^2 x_\mu x^a + \tilde g_5 \lambda^2 (\tilde J x)_\mu (\tilde J x)^a\right] \ , \tag{6.3.133}$$

where $\tilde J_{ab}$ is defined in (F.0.10)

$$\tilde g_4 = 2\lambda^2 x^2 h^{-4} \qquad \tilde g_5 = \frac{6\lambda^2 x^2 h^{-4}}{1-4\tilde g} \ , \tag{6.3.134}$$

and the spin connections are given in (6.3.131) and (6.3.130). Thus, the metric $g_{\mu\nu} = e_\mu^a e_\nu^b\, \eta_{ab}$ takes the form

$$g_{\mu\nu} = \frac{4}{h^4\,[1+2\tilde g]^2}\left[\eta_{\mu\nu} + \tilde g_4(\lambda^2 x^2 \tilde g_4 + 2) x_\mu x_\nu + \tilde g_5(\lambda^2 x^2 \tilde g_5 + 2)(\tilde J x)_\mu (\tilde J x)_\nu\right] \ . \tag{6.3.135}$$

The vierbein has potential singularities at $h^2 = 0$, $h^2 = 2$ and $h^2 = \frac{2}{3}$. The singularities at $h^2 = 0$ and $h^2 = 2$ are related to degenerate vierbeins exactly as for the P^+ solution. The singularity at $h^2 = \frac{2}{3}$, which arises in the case of $\lambda^2 = 1$ in Euclidean and Kleinian signature, also gives a degenerate vierbein. This is an intriguing situation since the latter degeneration occurs inside the coordinate charts.

Chapter 7

Singletons, Anti-Singletons and HS Master Fields

The aim of this Chapter is to elaborate further on the relevant representations entering the Vasiliev equations. We have seen in Chapter 3 that the spectrum of physical fields that emerges from the analysis of the linearized curvature constraints actually fits the doubleton spectrum (3.2.56) [65, 66]. In AdS_D such states fill up a unitary multiplet of the HS algebra, and are in correspondence with the modes of normalizable fluctuation fields of all spins, with finite Killing energy [133]. In Chapter 4, we have also remarked on the fact that, in an unfolded system, all the local physical degrees of freedom are contained in the zero-form at a point in space-time, that is to say, in the basis monomials of the twisted adjoint zero-form Φ (see eqs. (3.1.83) and (7.6.16)). Our purpose is now to have a closer look to this basis of operators in order to establish a direct correspondence between the master-field description and the massless irreducible representations earlier examined. In other words, we want to construct a map that exhibits the physical content of the master fields, arranged into massless irreps of the background isometry algebras, without solving the various torsion-like constraints of the unfolded system. This aim will be accomplished with the construction of a map that essentially involves two steps:

1. Performing a nontrivial change of basis, that connects the $\mathfrak{so}(D;\mathbb{C})$-covariant tensors of the master fields with the $\mathfrak{so}(2;\mathbb{C})\oplus\mathfrak{so}(D-1;\mathbb{C})$-covariant elements of the massless modules;

2. Defining a *reflection* map that essentially sends states to operators and viceversa, preserving the representation properties.

This mapping will be defined at the level of complex representations, and later restricted to the various real forms of the HS algebra. As we shall see, the outcome will be that to each basis monomial of the twisted adjoint representation there corresponds a "coherent" superposition of infinitely many states, and, viceversa, to every state in the lowest weight modules there corresponds a nonpolynomial combinations of basis monomials.

This analysis provides some insight into various features of Vasiliev equations. For example, it shows that, while the on-shell content of the twisted adjoint zero-form is related to the tensor product of two singletons, that of the adjoint one-form is related to the finite-dimensional $\mathfrak{so}(D+1;\mathbb{C})$-modules that arise from the tensor product of a singleton and its negative-energy counterpart, called *anti-singleton*. Thus, an intertwiner, that transforms singletons in anti-singletons, connects the two modules, in a way that is reminiscent of the role of the operator κ in the full Vasiliev equations.

Another outcome is that the twisted adjoint representation contains not only the massless lowest-weight or highest-weight modules: indeed, for every spin s, a bigger indecomposable module (that contains also a lowest-spin module, rather than lowest energy module) sits in principle in the zero-form master-field, and all its elements can take part in the dynamics. The states in the lowest-spin module do not admit an interpretation as composites of singleton, and correspond to non-normalizable solutions of the free-field equations, with divergent Killing energy. However, it is important to know the full structure of the twisted adjoint module, not only for analyzing the perturbative spectrum of physical excitations in any signature, but also for including in the analysis nonperturbative degrees of freedom, like the static linearized solution $\Phi|_p = \frac{\sinh 4E}{4E}$ found in [97].

Finally, the problem of potential local divergencies in HSGT, due to the contribution of an arbitrary number of derivatives to some interaction vertices (as, for example, in the scalar-field corrections to the stress-energy tensor calculated in [102]), is mapped to the problem of divergent $\star$-products of nonpolynomial combinations of generators. This is however a somewhat more transparent setting, and indeed we make a proposal for an explicit regularization scheme.

This Chapter is organized as follows. We begin by defining some of the key tools that will be useful for establishing the correspondence: a nontrivial trace operation on the algebra $\mathcal{A}$ (defined in (3.1.12) and (3.1.22)), that will endow the latter with a nondegenerate inner product, and certain *reflector* states, that possess a series of useful properties. Indeed, the trace operation on $\mathcal{A}$ can be defined via the expectation value of its elements between such reflector states and their duals, and, more generally, they lie at the heart of the state/operator correspondence mentioned above. For example, they enable a presentation of the master-fields as left bimodule constructs. Then, we proceed to a general definition of the twisted adjoint indecomposable module in compact basis, *i.e.*, in the basis of operators with definite $\mathfrak{so}(2;\mathbb{C})\oplus\mathfrak{so}(D-1;\mathbb{C})$ quantum numbers. As stated above, the latter are given by nonpolynomial combinations of the twisted adjoint basis monomials. We recover, in this fashion, the Flato-Fronsdal theorem and the composite nature of massless lowest-weight modules (and their negative energy counterparts, that are highest-weight modules). Moreover, we also investigate the structure of the lowest-spin module, containing the above mentioned nonperturbative degrees of freedom: in particular, the static linearized solution $\frac{\sinh 4E}{4E}$ will enter here as one of the two static ground states from which the entire compact twisted adjoint module can be generated. We also stress the appearance, in odd $D \geq 7$ dimensions, of scalar submodules that generalize the scalar singleton representations, as their weight diagram consists of p lines ($p = (D-5)/2$), which we shall refer to as *p-linetons*. In $D = 3 + 2p$, $p = 1, 2, ...$, we find similar spin-1 p-linetons that generalize the 5D spin-1 singleton. Next, we examine the inverse procedure of embedding $\mathfrak{so}(D;\mathbb{C})$-tensors into the compact basis as superpositions of infinitely many states. In particular, we give a realization of the (composite) reflector states as $\mathfrak{so}(D;\mathbb{C})$-invariant combinations of states in the massless scalar representations. We also find an analog of the Flato-Fronsdal theorem for the tensor product of the singleton with an anti-singleton: the decomposition is given in terms of *finite-dimensional* modules (*i.e.*, lowest *and* highest-weight modules) of $\mathfrak{so}(D+1;\mathbb{C})$, in terms of which the content of the adjoint master field can be analyzed. We also examine a relation between adjoint and twisted-adjoint representation that provides a direct explanation for the agreement of the eigenvalues of the Casimir operators in the two representations. Finally, we comment on real forms, and on the inner products on $\mathcal{A}$ defined through the trace operation. In $D = 4$, for a composite reflector expanded over states belonging to the scalar and spinor singleton representations, the latter corresponds (modulo overall factors) to the supertrace operation[1] defined in [63] for the oscillator realization. This setting furnishes a particularly simple example of a composite reflector state with finite norm induced by the standard inner product of the singleton representations. Finally, we comment on the issue of divergent $\star$-products of nonpolynomial elements and on a possible regularization scheme for the weak-field expansion of the Vasiliev equations.

The results collected in this Chapter have been obtained in [134] and [135].

7.1 Non-composite Trace and Reflector

The quotient algebra $\mathcal{A}$ can be equipped with a non-composite trace operation $\mathrm{Tr} : \mathcal{A} \mapsto \mathbb{C}$ defined by the projection[2] on the $\mathfrak{so}(D+1;\mathbb{C})$-singlet X_0 in the basis (3.1.22), or, equivalently (since (3.1.52) and (3.1.53) are strong identities, *i.e.* they hold in $\mathcal{U}$, and X_n contains $X^{(0,0)}$ if and only if $n = 0$), on the $\mathfrak{so}(D;\mathbb{C})$-singlet $X^{(0,0)}$ in the basis (3.1.52),

[1] For further details on the relation between inner products, trace and supertrace operations in the simpler context of one-dimensional Fock spaces we refer to Appendix I.

[2] We note that if Tr is a cyclic trace operation and X is not a singlet then $\mathrm{Tr}[X] = 0$, since X can be written as a commutator. Thus, up to normalization, any trace operation on $\mathcal{A}$ is the projection onto the unity in some basis, where the choice of basis is a crucial part of the definition in the infinite-dimensional case.

viz.[3]

$$\mathrm{Tr}[X] \;=\; X_0 \;=\; X^{(0,0)}\ . \tag{7.1.1}$$

This trace operation is well-defined on $\mathcal{A}$, since $X \simeq X'$ iff X and X' have the same expansions in the bases (3.1.22), or, equivalently (3.1.52). The cyclicity follows from the invariance of Tr under the anti-automorphism τ defined in (3.1.7), that is

$$\mathrm{Tr}[\tau(X)] \;=\; \mathrm{Tr}[X]\ . \tag{7.1.2}$$

Indeed, given two elements $X, Y \in \mathcal{A}$, this implies

$$\mathrm{Tr}[X\star Y] \;=\; \mathrm{Tr}[\tau(X\star Y)] \;=\; \mathrm{Tr}[\tau(Y)\star\tau(X))] \;=\; \mathrm{Tr}[Y\star X]\ , \tag{7.1.3}$$

as can be seen by splitting $X = X_+ + X_-$ with $\tau(X_\pm) = \pm X_\pm$, *idem* Y, and noting that $\mathrm{Tr}(X_\pm \star Y_\mp) = 0$. The trace equips $\mathcal{A}$ with an non-degenerate bi-linear inner product $(X,Y)\mapsto \mathrm{Tr}[X\star Y]$ that is invariant under the adjoint action of $\mathcal{A}$ on itself, which includes not only $\mathfrak{so}(D+1;\mathbb{C})$ but also its higher-spin extension.

The basis elements $T_{A(n),B(n)}$ and $T_{a(n),b(m)}$ defined in (3.1.23) (we suppress the hat on the basis elements (3.1.23) in this Chapter, as that notation will be reserved here to elements of the enlarged algebra (7.1.21)) and (3.1.53), respectively, have inner products

$$\mathrm{Tr}[T_{A(n),B(n)}\star T^{C(m),D(m)}] \;=\; \delta_{mn}\delta^{\{C(n),D(n)\}}_{\{A(n),B(n)\}}\mathcal{N}_{(n,n)_{D+1}}\ , \tag{7.1.4}$$

$$\mathrm{Tr}[T_{a(n),b(m)}\star T^{c(n'),d(m')}] \;=\; \delta_{n,n'}\delta_{m,m'}\delta^{\{c(n),d(m)\}}_{\{a(n),b(m)\}}\mathcal{N}_{(n,m)_D}\ , \tag{7.1.5}$$

where the normalizations are given by

$$\mathcal{N}_{(n,n)_{D+1}} \;=\; \lambda_n\lambda_{n-1}\cdots\lambda_1 \;=\; (-2)^{-n}\frac{n!(n+1)!(\epsilon_0)_n}{(\epsilon_0+\frac{3}{2})_n}\ , \tag{7.1.6}$$

$$\mathcal{N}_{(n)_D} \;=\; \lambda^{(0)}_n\lambda^{(0)}_{n-1}\cdots\lambda^{(0)}_1 \;=\; 8^{-n}\frac{n!(n+1)!(2\epsilon_0)_n}{(\epsilon_0+\frac{3}{2})_n}\ , \tag{7.1.7}$$

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[3]We remark that at the level of the full enveloping algebra $\mathcal{U}[\mathfrak{g}]$ of a general Lie algebra $\mathfrak{g}$, *i.e.* prior to factoring out any other ideal than the commutation rules of $\mathfrak{g}$, the trace operation $\mathrm{Tr}[X] = X_1$, where X_1 is the coefficient of $\mathbb{1}$, is trivial, that is $\mathrm{Tr}[X\star Y] = (X\star Y)_1 = X_1Y_1$. At the level of quotient algebras $\mathcal{U}[\mathfrak{g}]/\mathcal{I}[R]$, where $\mathcal{I}[R]$ is the annihilator of a representation R of $\mathfrak{g}$, the trace Tr is equivalent to the composite trace over R, that is $\mathrm{Tr}_R[X] = \sum_n \langle n^*|X|n\rangle$ where $|n\rangle$ and $\langle n^*|$ are basis elements for R and R^* and $\langle m^*|n\rangle = \delta^m_n$, if R is finite-dimensional. In case R is infinite-dimensional, more care is required. Thus, in the case at hand, we shall distinguish between the non-composite trace Tr on $\mathcal{A}$ (defined in the basis (3.1.22)) and the composite trace $\mathrm{Tr}_{\mathfrak{D}_0}$ over the scalar-singleton lowest-weight space $\mathfrak{D}_0$. We note that the composite trace can be rewritten as $\mathrm{Tr}_R[X] = {}_{12}\langle \mathbb{1}^*|X(1)|\mathbb{1}\rangle_{12}$ where the composite reflectors are defined by $|\mathbb{1}\rangle_{12} = \sum_n |n\rangle_1\otimes|n\rangle_2$ and ${}_{12}\langle\mathbb{1}^*| = \sum_n {}_1\langle n^*|\otimes{}_2\langle n^*|$. Moreover, in case the ideal $\mathcal{I}[R]$ is non-trivial, both types of traces are non-trivial, *i.e.* $\mathrm{Tr}[X\star Y] \neq (\mathrm{Tr}[X])(\mathrm{Tr}[Y])$ *idem* Tr_R.

as can be seen by repeated use of (3.1.33) and (3.1.58), $(x)_n$ being the Pochhammer symbol, $(x)_n = \Gamma(x+n)/\Gamma(x)$. For example,

$$\begin{aligned} \mathrm{Tr}[T_{a(n)} \star T_{b(n)}] &= \mathrm{Tr}[P_{\{a_1} \star \cdots \star P_{a_n\}} \star T_{b(n)}] \\ &= \mathrm{Tr}[P_{\{a_1} \star \cdots \star P_{a_{n-1}} \star \left(T_{n+1} + T_n + \lambda^{(0)}_n \eta_{a_n\}\{b_1} T_{b(n-1)\}}\right)] \\ &= \mathrm{Tr}[P_{\{a_1} \star \cdots \star P_{a_{n-1}} \star \lambda^{(0)}_n \eta_{a_n\}\{b_1} T_{b(n-1)\}}] \ , \end{aligned} \tag{7.1.8}$$

where T_n denote traceless symmetric $\star$-products of n translations, and we have used the fact that $\mathrm{Tr}[T_{n+1} \star T_{n-1}] = \mathrm{Tr}[T_n \star T_{n-1}] = 0$. In particular, using

$$\delta^{\{a(n)\}}_{\{b(n)\}} = \sum_{k=0}^{[n/2]} t_k (\eta^{a(2)}\eta_{b(2)})^k \delta^{a(n-2k)}_{b(n-2k)} \ , \qquad t_k = \frac{(-n)_{2k}}{4^k k! (-n-\epsilon_0+\frac{1}{2})_k} \ , \tag{7.1.9}$$

from which it follows that

$$\delta^{\{0(n)\}}_{\{0(n)\}} = \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{4^k k!(-n-\epsilon_0+\frac{1}{2})_k} = 2^{-n}\frac{(2\epsilon_0+1)_n}{(\epsilon_0+\frac{1}{2})_n} \ , \tag{7.1.10}$$

we obtain

$$\mathrm{Tr}[T_{0(n)} \star T_{0(n)}] = (-1)^n 4^{-2n} \frac{n!(n+1)!(2\epsilon_0)_n (2\epsilon_0+1)_n}{(\epsilon_0+\frac{1}{2})_n(\epsilon_0+\frac{3}{2})_n} \ . \tag{7.1.11}$$

The non-composite trace Tr can equivalently be realized as the expectation value

$$\mathrm{Tr}[X] = {}_{12}\langle \mathbb{1}^* | X(1) | \mathbb{1}\rangle_{12} \ , \tag{7.1.12}$$

where $|\mathbb{1}\rangle_{12} \in \mathcal{B}$ and ${}_{12}\langle\mathbb{1}^*| \in \mathcal{B}^*$ are a *non-composite reflector and non-composite dual reflector* belonging to the $\mathcal{A}$ left and right bimodules[4]

$$\mathcal{B} = \left\{ |\widehat{X}\rangle_{12} : \ V(\xi)|\widehat{X}\rangle_{12} = 0 \ , \xi = 1,2 \right\} \ , \tag{7.1.13}$$

$$\mathcal{B}^* = \left\{ {}_{12}\langle \widehat{X}^*| : \ {}_{12}\langle \widehat{X}^*|V(\xi) = 0 \ , \xi = 1,2 \right\} \ , \tag{7.1.14}$$

(where V is the singleton annihilating ideal defined in (3.1.9), (3.1.10) and (3.1.11)) obeying the overlap conditions

$$(X(1) - (\tau\circ\pi)(X)(2))|\mathbb{1}\rangle_{12} = 0 \ , \quad {}_{12}\langle\mathbb{1}^*|(X(1) - (\tau\circ\pi)(X)(2)) = 0 \tag{7.1.15}$$

[4]In general, given a vector space V that is a left module of an algebra $\mathcal{A}$, the dual V^* carries a natural right action of $\mathcal{A}$, such that $v^*(Xw) = (v^*X)(w)$ for $v^* \in V^*$, $w \in V$ and $X \in \mathcal{A}$. This generalizes straightforwardly to bimodules. A an element $\widehat{X}_{12}$ in a bimodule is referred to as composite in case it can be represented as a sum of factorized elements of the form $u_1 \otimes v_2$. In finite-dimensional bimodules all elements are composite, while this is not necessarily the case for infinite-dimensional bimodules.

for all $X \in \mathcal{A}$ and where $\tau\circ\pi$ is the anti-automorphism composed by the anti-automorphism τ defined in (3.1.7) and the automorphism π defined in (3.1.43), and the pairing between the reflector and its dual is normalized to

$$ {}_{12}\langle \mathbb{1}^*|\mathbb{1}\rangle_{12} \;=\; 1 \;. \qquad (7.1.16)$$

Eq. (7.1.12) is a consequence of the cyclicity property

$$ {}_{12}\langle \mathbb{1}^*|(X\star Y)(1)|\mathbb{1}\rangle_{12} \;=\; {}_{12}\langle \mathbb{1}^*|(Y\star X)(1)|\mathbb{1}\rangle_{12} \;, \qquad (7.1.17)$$

which follows from (7.1.15) and $(\pi\tau)^2 = \mathrm{Id}$ (the cyclicity implies the $\mathfrak{so}(D+1;\mathbb{C})$-invariance ${}_{12}\langle \mathbb{1}^*|[M_{AB}(1),X(1)]_\star|\mathbb{1}\rangle_{12} = 0$ so that if one expands X as in the basis (3.1.22), dropping the ideal parts in view of the definitions (7.1.13) and (7.1.14), one finds ${}_{12}\langle \mathbb{1}^*|X(1)|\mathbb{1}\rangle_{12} = X_0 = \mathrm{Tr}[X]$). We also notice that the overlap conditions (7.1.15) are equivalent to

$$ (M_{ab}(1)+M_{ab}(2))|\mathbb{1}\rangle_{12} \;=\; 0 \;, \qquad (P_a(1)-P_a(2))|\mathbb{1}\rangle_{12} \;=\; 0 \;, \qquad (7.1.18)$$

$$ {}_{12}\langle \mathbb{1}^*|(M_{ab}(1)+M_{ab}(2)) \;=\; 0 \;, \qquad {}_{12}\langle \mathbb{1}^*|(P_a(1)-P_a(2)) \;=\; 0 \;, \qquad (7.1.19)$$

which means that the reflectors are $\mathfrak{so}(D;\mathbb{C})_{\rm diag}$-invariant.

Next we observe that the map π is an outer automorphism of $\mathcal{A}$. Upon defining the operator k by

$$ k\star X \;=\; \pi(X)\star k \;, \qquad k\star k \;=\; \mathbb{1} \;, \qquad \tau(k) \;=\; \pi(k) \;=\; k \;, \qquad (7.1.20)$$

the map π becomes an inner automorphism of the enlarged algebra

$$ \widehat{\mathcal{A}} \;=\; \mathcal{A}\oplus(\mathcal{A}\star k) \;. \qquad (7.1.21)$$

Since $\mathcal{A}\star k$ does not contain the unity, we have

$$ \mathrm{Tr}[\widehat{X}] \;=\; \mathrm{Tr}[X] \;, \qquad (7.1.22)$$

from which it in particular follows that $\mathrm{Tr}[k]=0$. Moreover, since $\pi(V)=V$ it follows that if $|\widehat{X}\rangle_{12}\in\mathcal{B}$ and ${}_{12}\langle\widehat{X}^*|\in\mathcal{B}^*$, then $k(\xi)|\widehat{X}\rangle_{12}\in\mathcal{B}$ and ${}_{12}\langle\widehat{X}^*|k(\xi)\in\mathcal{B}^*$ for $\xi=1,2$. We can now establish a sequence of reflection maps

$$ \mathcal{B}_{12} \;\xrightarrow{R_2}\; \widehat{\mathcal{A}}_{12} \;\xrightarrow{R_1}\; \mathcal{B}^*_{12} \;, \qquad (7.1.23)$$

by making the definitions

$$ {}_{23}\langle \mathbb{1}^*|\star|\mathbb{1}\rangle_{13} \;=\; \mathbb{1}_{12} \;, \qquad (7.1.24)$$

$$ (k(1)-k(2))|\mathbb{1}\rangle_{12} \;=\; 0 \;, \qquad {}_{12}\langle\mathbb{1}|(k(1)-k(2)) \;=\; 0 \;. \qquad (7.1.25)$$

Explicitly, the reflection maps are given by

$$|\widehat{X}\rangle_{12} \;=\; \widehat{X}(1)|\mathbb{1}\rangle_{12}\ , \qquad {}_{12}\langle\widehat{X}^*| \;=\; {}_{12}\langle\mathbb{1}^*|\widehat{X}(1)\ , \tag{7.1.26}$$

$$\widehat{X}_{12} \;=\; {}_{23}\langle\mathbb{1}^*|\star|\widehat{X}\rangle_{13} \;=\; {}_{23}\langle\widehat{X}^*|\star|\mathbb{1}\rangle_{13}\ , \tag{7.1.27}$$

with $\widehat{X} = X + Y\star k \in \widehat{\mathcal{A}}$. We note that from (7.1.18) and (7.1.19) it follows that the quantity $\mathcal{O}_{12} ={}_{23}\langle\mathbb{1}|\star|\mathbb{1}\rangle_{13}$ obeys $\widehat{X}\star\mathcal{O} = \mathcal{O}\star\widehat{X}$ for all $\widehat{X}\in\widehat{\mathcal{A}}$, so that $\mathcal{O} = c\mathbb{1}$, and hence the definition (7.1.24) amounts to setting $c = 1$. We also define the *twisted non-composite reflectors*

$$|\widetilde{\mathbb{1}}\rangle_{12} \;=\; k(1)|\mathbb{1}\rangle_{12} \;=\; k(2)|\mathbb{1}\rangle_{12}\ , \tag{7.1.28}$$

$${}_{12}\langle\widetilde{\mathbb{1}}^*| \;=\; {}_{12}\langle\mathbb{1}^*|k(1) \;=\; {}_{12}\langle\mathbb{1}^*|k(2)\ , \tag{7.1.29}$$

obeying the overlap conditions

$$(X(1)-\tau(X)(2))|\widetilde{\mathbb{1}}\rangle_{12} \;=\; 0\ , \qquad {}_{12}\langle\widetilde{\mathbb{1}}^*|(X(1)-\tau(X)(2)) \;=\; 0\ , \tag{7.1.30}$$

which in particular imply that the twisted reflectors are $\mathfrak{so}(D+1;\mathbb{C})_{\rm diag}$-invariant, and having the normalizations

$${}_{12}\langle\widetilde{\mathbb{1}}^*|\widetilde{\mathbb{1}}\rangle_{12} \;=\; 1\ , \qquad {}_{12}\langle\widetilde{\mathbb{1}}^*|\mathbb{1}\rangle_{12} \;=\; {}_{12}\langle\mathbb{1}^*|\widetilde{\mathbb{1}}\rangle_{12} \;=\; 0\ , \tag{7.1.31}$$

$${}_{23}\langle\widetilde{\mathbb{1}}^*|\star|\widetilde{\mathbb{1}}\rangle_{12} \;=\; \mathbb{1}_{13}\ , \qquad {}_{23}\langle\widetilde{\mathbb{1}}^*|\star|\mathbb{1}\rangle_{12} \;=\; {}_{23}\langle\mathbb{1}^*|\star|\widetilde{\mathbb{1}}\rangle_{12} \;=\; k_{13}\ . \tag{7.1.32}$$

The trace Tr can now be written as

$${\rm Tr}[X] \;=\; {}_{12}\langle\widetilde{\mathbb{1}}^*|X(1)|\widetilde{\mathbb{1}}\rangle_{12}\ , \tag{7.1.33}$$

since ${}_{12}\langle\widetilde{\mathbb{1}}^*|X(1)|\widetilde{\mathbb{1}}\rangle_{12} = {}_{12}\langle\mathbb{1}^*|k(2)X(1)k(2)|\mathbb{1}\rangle_{12} = {}_{12}\langle\mathbb{1}^*|X(1)|\mathbb{1}\rangle_{12} = {\rm Tr}[X]$.

The trace operation Tr induces higher-spin invariant bilinear forms on the adjoint and twisted-adjoint representations:

$$(Q,Q')_{\mathfrak{ho}} \;=\; {\rm Tr}[Q\star Q'] \;=\; {}_{12}\langle Q|Q'\rangle_{12}\ , \tag{7.1.34}$$

$$(S,S')_{\mathcal{T}} \;=\; {\rm Tr}[\pi(S)\star S'] \;=\; {}_{12}\langle\pi(S)|S'\rangle_{12}\ , \tag{7.1.35}$$

where (7.1.26) has been used to define

$$|Q_\pm\rangle_{12} \;=\; Q(1)|\mathbb{1}\rangle_{12} \;=\; \tfrac{1}{2}(Q_\pm(1)\mp\pi(Q_\pm)(2))|\mathbb{1}\rangle_{12}\ , \tag{7.1.36}$$

$$|S_\pm\rangle_{12} \;=\; S_\pm(1)|\mathbb{1}\rangle_{12} \;=\; \tfrac{1}{2}(S_\pm(1)\pm S_\pm(2))|\mathbb{1}\rangle_{12}\ , \tag{7.1.37}$$

and

$${}_{12}\langle Q_\pm| \;=\; {}_{12}\langle\mathbb{1}^*|Q_\pm(1) \;=\; \tfrac{1}{2}{}_{12}\langle\mathbb{1}^*|(Q_\pm(1)\mp\pi(Q_\pm)(2))\ , \tag{7.1.38}$$

$${}_{12}\langle\pi(S_\pm)| \;=\; {}_{12}\langle\mathbb{1}^*|\pi(S_\pm)(1) \;=\; \tfrac{1}{2}{}_{12}\langle\mathbb{1}^*|(\pi(S_\pm)(1)\pm\pi(S_\pm)(2))\ , \tag{7.1.39}$$

carrying the higher-spin representations

$$|\mathrm{Ad}_Q Q'\rangle_{12} \;=\; (Q(1)+\pi(Q)(2))|Q'\rangle_{12} \;\equiv\; \widetilde{Q}|Q'\rangle_{12}\;, \tag{7.1.40}$$

$$|\widetilde{\mathrm{Ad}}_Q S\rangle_{12} \;=\; (Q(1)+Q(2))|S\rangle_{12} \;\equiv\; Q|S\rangle_{12}\;, \tag{7.1.41}$$

and

$${}_{12}\langle \mathrm{Ad}_Q Q'| \;=\; -{}_{12}\langle Q'|\widetilde{Q}\;, \qquad {}_{12}\langle \pi(\widetilde{\mathrm{Ad}}_Q S)| \;=\; -{}_{12}\langle \pi(S)|Q\;. \tag{7.1.42}$$

Using instead the twisted reflector, the mappings read

$$|\widetilde{Q}\rangle_{12} \;=\; Q(1)|\widetilde{\mathbb{1}}\rangle_{12} \;=\; \tfrac{1}{2}(Q(1)-Q(2))|\widetilde{\mathbb{1}}\rangle_{12}\;, \tag{7.1.43}$$

$$|\widetilde{S}_\pm\rangle_{12} \;=\; S_\pm(1)|\widetilde{\mathbb{1}}\rangle_{12} \;=\; \tfrac{1}{2}(S_\pm(1)\pm\pi(S_\pm)(2))|\widetilde{\mathbb{1}}\rangle_{12}\;, \tag{7.1.44}$$

carrying the higher-spin representations

$$|\mathrm{Ad}_Q \widetilde{Q}\rangle_{12} \;=\; Q|\widetilde{Q}'\rangle_{12}\;, \qquad |\widetilde{\mathrm{Ad}}_Q \widetilde{S}\rangle_{12} \;=\; \widetilde{Q}|\widetilde{S}\rangle_{12}\;, \tag{7.1.45}$$

$${}_{12}\langle \mathrm{Ad}_Q \widetilde{Q}'| \;=\; -{}_{12}\langle \widetilde{Q}'|_{12}Q\;, \qquad {}_{12}\langle \widetilde{\mathrm{Ad}}_Q \widetilde{S}| \;=\; -{}_{12}\langle \widetilde{S}|\widetilde{Q}\;. \tag{7.1.46}$$

The ℓth adjoint and twisted-adjoint levels $\mathcal{L}_\ell$ and $\mathcal{T}_\ell$ are finite-dimensional and infinite-dimensional $\mathfrak{so}(D+1;\mathbb{C})$ irreps, respectively. Nonetheless, as shown in Appendix C, the quadratic and quartic Casimir operators, defined in (C.0.1) and (C.0.2), assume the same values in $\mathcal{L}_\ell$ and $\mathcal{T}_\ell$, namely ($s=2\ell+2$)

$$C_2[O(D+1;\mathbb{C})|\ell] \;=\; 2(s-1)(s+2\epsilon_0)\;, \tag{7.1.47}$$

$$C_4[O(D+1;\mathbb{C}|\ell] \;=\; 2(s-1)(s+2\epsilon_0)(s^2+(2\epsilon_0-1)s+2\epsilon_0^2-\epsilon_0+1)\;. \tag{7.1.48}$$

Moreover, as can be seen using (C.0.3) and (C.0.4), these values are also equal to those of the massless lowest-weight spaces $\mathfrak{D}(s+2\epsilon_0;(s))$, *i.e.*

$$C_2[O(D+1;\mathbb{C})|\ell] \;=\; C_2[O(D+1;\mathbb{C})|s+2\epsilon_0;(s)]\;, \tag{7.1.49}$$

$$C_4[O(D+1;\mathbb{C}|\ell] \;=\; C_4[O(D+1;\mathbb{C}|s+2\epsilon_0;(s)]\;. \tag{7.1.50}$$

These agreements follow direct relationships between $\mathcal{L}_\ell$, $\mathcal{T}_\ell$ and $\mathfrak{D}(\pm(s+2\epsilon_0);(s))$ that arise upon going from the $\mathfrak{so}(D;\mathbb{C})$-covariant bases of $\mathcal{L}_\ell$ and $\mathcal{T}_\ell$ to *compact bases* where elements are labeled by quantum numbers of the $\mathfrak{so}(2)\oplus\mathfrak{so}(D-1;\mathbb{C})$ subalgebra (to be identified as the maximal compact subalgebra in the case of two-time signature). In the case of $\mathcal{T}_\ell$, the compact basis elements are series expansions in the covariant basis elements $T_{a(s+k),b(s)}$ (related to Bessel functions). This is a non-trivial change of basis, which corresponds to the harmonic expansion of linearized Weyl tensors. To distinguish between the $\mathfrak{so}(D;\mathbb{C})$-covariant and the compact "slicings" of $\mathcal{T}_\ell$, we thus define the *covariant twisted-adjoint module*

$$\mathcal{T} \;=\; \bigoplus_{\ell} \mathcal{T}_\ell \;=\; \bigoplus_{s} \mathcal{T}_{(s)}\ , \tag{7.1.51}$$

consisting of polynomial elements of the form (3.1.83), and the *compact twisted-adjoint module*

$$\mathcal{M} \;=\; \bigoplus_{\ell} \mathcal{M}_\ell \;=\; \bigoplus_{s} \mathcal{M}_{(s)}\ , \tag{7.1.52}$$

whose elements are polynomial in the compact basis.

We next turn to a more careful analysis of the compact basis elements, the harmonic expansions, and the relations between the compact twisted-adjoint module, the massless weight spaces and the adjoint modules.

7.2 Compact Weight-Space Description of the Master Fields

In this section we shall examine the properties of the adjoint and twisted-adjoint representation spaces in compact bases. As a result, we shall give the explicit embedding of the massless weight spaces into the twisted-adjoint representation, and use this to describe the harmonic expansion of the Weyl tensors. We shall also show how to "glue" the adjoint and twisted-adjoint representations in compact weight space.

7.2.1 Compact Twisted-Adjoint Modules

The *compact twisted-adjoint* $\mathfrak{ho}_1(D+1;\mathbb{C})$ *module* is defined as

$$\mathcal{M} \;=\; \bigoplus_{s=0}^{\infty} \mathcal{M}_{(s)}\ , \tag{7.2.1}$$

$$\mathcal{M}_{(s)} \;=\; \bigoplus_{\substack{e\in\mathbb{Z}\\ s_1\geq s\geq s_2\geq 0}} \mathbb{C}\otimes T^{(s)}_{e;(s_1,s_2)} \tag{7.2.2}$$

where $\mathcal{M}_{(s)}$ are $\mathfrak{so}(D+1;\mathbb{C})$ submodules consisting of components $T^{(s)}_{e;(s_1,s_2)}$ with spin (s_1,s_2) and energy e, that is

$$T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} \;=\; \sum_{n=0}^{\infty} f^{(s)}_{e;(s_1,s_2);n} T_{s;n;(s_1,s_2);r(s_1),t(s_2)}\ , \tag{7.2.3}$$

where

$$T_{s;n;(s_1,s_2);r(s_1),t(s_2)} \ = \ T_{0(n)\{r(s_1),t(s_2)\}0(s-s_2)} \ , \tag{7.2.4}$$

with $T_{a(s_1+n),b(s)}$ defined in (3.1.53) and $\{\cdots\}$ indicating $\mathfrak{so}(D-1)$-traceless type (s_1,s_2)-projection, and $f^{(s)}_{e;(s_1,s_2);n} \in \mathbb{R}$ are determined from

$$\widetilde{\mathrm{Ad}}_E(T^{(s)}_{e;(s_1,s_2)}) \ = \ e\, T^{(s)}_{e;(s_1,s_2)} \ . \tag{7.2.5}$$

Using (3.1.58), which implies

$$\widetilde{\mathrm{Ad}}_E(T_{s;(s_1,s_2);n}) \ = \ \lambda_{s;(s_1,s_2);n}T_{s;(s_1,s_2);n+1} + \lambda'_{s;(s_1,s_2);n}T_{s;(s_1,s_2);n-1} \ , \tag{7.2.6}$$

where the coefficients $\lambda_{s;(s_1,s_2);n}$ and $\lambda'_{s;(s_1,s_2);n}$ are non-vanishing for all n except $\lambda'_{s;(s_1,s_2);0} = 0$, it follows that the generating functions

$$f^{(s)}_{e;(s_1,s_2)}(z) \ = \ \sum_{n=0}^{\infty} f^{(s)}_{e;(s_1,s_2);n} z^n \ , \tag{7.2.7}$$

are *analytical* at $z=0$, and determined *uniquely* up to an overall constant that can be fixed by the normalization condition

$$f^{(s)}_{e;(s_1,s_2)}(0) \ = \ 1 \ . \tag{7.2.8}$$

Under the π-map

$$\pi(T^{(s)}_{e;(s_1,s_2)}) \ = \ (-1)^{s_1-s}T^{(s)}_{-e;(s_1,s_2)} \ , \qquad f^{(s)}_{-e;(s_1,s_2)}(z) \ = \ f^{(s)}_{e;(s_1,s_2)}(-z) \ . \tag{7.2.9}$$

In what follows, we shall use the notation

$$\widetilde{Q}S \ = \ \widetilde{\mathrm{Ad}}_Q(S) \ = \ Q\star S - S\star \pi(Q) \ , \tag{7.2.10}$$

for non-minimal higher-spin generators $Q \in \mathcal{A}$. We note that in $\mathcal{M}_{(s)}$, the condition $s_1 \geq s \geq s_2$ and $s_3 = 0$ implies that

$$\widetilde{L}^{\pm}_r T^{(s)}_{e;(s,s_2);rt(s-1),u(s_2)} \ = \ 0 \qquad \text{for } s_1 = s \geq 1 \text{ and } s_2 < s \ , \tag{7.2.11}$$

$$\widetilde{L}^{\pm}_{[r_1} T^{(s)}_{e;(s_1,s_2);r_2|t(s_1-1),|r_3]u(s_2-1)} \ = \ 0 \qquad \text{for } s_2 \geq 1 \ . \tag{7.2.12}$$

Assuming that the twisted-adjoint action of the Casimir operator $C_{2n}[\mathfrak{so}(D+1;\mathbb{C})]$, defined in (3.1.14), commutes with the summation in (7.2.3), that is

$$C_{2n}[\mathfrak{so}(D+1;\mathbb{C})]\left(T^{(s)}_{e;(s_1,s_2)}\right) \ = \ \sum_{n=0}^{\infty} f^{(s)}_{e;(s_1,s_2)}\tfrac{1}{2}\widetilde{M}_{A_1}{}^{A_2}\cdots\widetilde{M}_{A_{2n}}{}^{A_1}T_{s;n;(s_1,s_2)} \ , \tag{7.2.13}$$

it follows that ($s = 2\ell + 2$)

$$C_{2n}[\mathcal{M}_{(s)}] = C_{2n}[\mathcal{T}_{(s)}] = C_{2n}[\ell] , \tag{7.2.14}$$

where $C_2[\ell]$ and $C_4[\ell]$ are given in (7.1.47) and (7.1.48).

The space $\mathcal{M}$ is a module also under *separate* left and right $\star$-multiplication by (polynomial) generators $Q \in \mathcal{A}$, and as such it splits into *even and odd submodules*,

$$\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^- , \tag{7.2.15}$$

where

$$\mathcal{M}^\pm = \bigoplus_{\substack{e;(s_1,s_2)\\ e+s_1+s_2=\frac{1}{2}(1\mp 1) \bmod 2}} \mathbb{C} \otimes T^{(s)}_{e;(s_1,s_2)} . \tag{7.2.16}$$

As a consequence, also the $\mathfrak{so}(D+1;\mathbb{C})$ submodules split into even and odd parts,

$$\mathcal{M}_{(s)} = \mathcal{M}^+_{(s)} \oplus \mathcal{M}^-_{(s)} . \tag{7.2.17}$$

We propose that $\mathcal{M}^\pm_{(s)}$ are generated by $\mathfrak{so}(D+1;\mathbb{C})$ from the elements with $e = 0$ and minimal $s_1 + s_2$, which we shall refer to as the *static ground states*, namely

$$s = 0 \quad : \qquad T^{(0)}_\pm = T^{(0)}_{0;(\sigma_\pm);t(\sigma_\pm)} , \tag{7.2.18}$$

$$s > 0 \quad : \qquad T^{(s)}_\pm = T^{(s)}_{0;(s,\sigma_\pm);u(s),v(\sigma_\pm)} , \tag{7.2.19}$$

where $\sigma_\pm = (1 \mp 1)/2$. The scalar static ground states are represented by

$$\begin{aligned} f^{(0)}_{0;(0)}(z) &= \sum_{n=0}^\infty \frac{4^{2n}(\epsilon_0+\frac{3}{2})_{2n}}{(2)_{2n}(2\epsilon_0+1)_{2n}} z^{2n} \\ &= {}_2F_3(\frac{2\epsilon_0+3}{4}, \frac{2\epsilon_0+5}{4}; \tfrac{3}{2}, \epsilon_0+\tfrac{1}{2}, \epsilon_0+1; 4z^2) , \end{aligned} \tag{7.2.20}$$

$$\begin{aligned} f^{(0)}_{0;(1)}(z) &= \sum_{n=0}^\infty \frac{(\epsilon_0+\frac{5}{2})_{2n}}{n!(2)_n(\epsilon_0+1)_n(\epsilon_0+2)_n} z^{2n} \\ &= {}_2F_3(\frac{2\epsilon_0+5}{4}, \frac{2\epsilon_0+7}{4}; 2, \epsilon_0+1, \epsilon_0+2; 4z^2) . \end{aligned} \tag{7.2.21}$$

In $D = 4$ these functions take the following simple form:

$$f^{(0)}_{0;(0)}(z) = \frac{\sinh 4z}{4z} , \qquad f^{(0)}_{0;(1)}(z) = \frac{3}{16z^2}(\cosh 4z - \frac{\sinh 4z}{4z}) , \tag{7.2.22}$$

where we note that $f^{(0)}_{0;(0)}(E)$ was found in [97].

We propose that the static ground states $T^{(s)}_{\pm}$ with $s > 0$ can be generated by $\mathfrak{ho}_1(D+1;\mathbb{C})$ starting from the scalar static ground states $T^{(0)}_{\pm}$. Thus, in effect, we propose that $\mathcal{M}^{\pm}$ are generated by $\mathfrak{ho}_1(D+1;\mathbb{C})$ starting from $T^{(0)}_{\pm}$, which therefore serves as ground states of $\mathcal{M}^{\pm}$.

To generate an explicit basis one has to take into account degeneracies of the form

$$\widetilde{L}^{\pm}_t \widetilde{L}^{\mp}_t T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} = \mu^{(s)}_{e;(s_1,s_2)} T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} , \tag{7.2.23}$$

$$\widetilde{x}^{+}\widetilde{x}^{-} T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} = \mu'^{(s)}_{e;(s_1,s_2)} T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} , \tag{7.2.24}$$

$$\widetilde{x}^{\pm}\widetilde{L}^{\mp}_{r_1}\widetilde{L}^{\mp}_{r_2} T^{(s)}_{e;(s_1,s_2);s(s_1),t(s_2)} = \mu''^{(s)}_{e;(s_1,s_2)} \widetilde{L}^{\pm}_{r_1}\widetilde{L}^{\mp}_{r_2} T^{(s)}_{e;(s_1,s_2);s(s_1),t(s_2)} , \tag{7.2.25}$$

for coefficients $\mu^{(s)}_{e;(s_1,s_2)}$, $\mu'^{(s)}_{e;(s_1,s_2)}$ and $\mu''^{(s)}_{e;(s_1,s_2)}$ (that may vanish), and where we have defined

$$\widetilde{x}^{\pm} = \widetilde{L}^{\pm}_r \widetilde{L}^{\pm}_r . \tag{7.2.26}$$

Thus, defining

$$\mathcal{M}^{+}_{(s)} = \mathcal{M}^{+,\geq}_{(s)} \oplus \mathcal{M}^{+,0}_{(s)} \oplus \mathcal{M}^{+,\leq}_{(s)} , \tag{7.2.27}$$

where

$$\mathcal{M}^{+,\geq}_{(s)} = \{T^{(s)}_{e;(s_1,s_2)} \ : \ e \geq s_1+s_2-s\} , \tag{7.2.28}$$

$$\mathcal{M}^{+,0}_{(s)} = \{T^{(s)}_{e;(s_1,s_2)} \ : \ |e| \leq s_1+s_2-s\} , \tag{7.2.29}$$

$$\mathcal{M}^{+,\leq}_{(s)} = \{T^{(s)}_{e;(s_1,s_2)} \ : \ e \leq s_1+s_2-s\} , \tag{7.2.30}$$

and removing degeneracies of the types listed in (7.2.23)–(7.2.25), there should exist *finite* coefficients $\mathcal{C}^{(s)}_{e;(s_1,s_2)}$ such that

$$\mathcal{M}^{+,\geq}_{(s)} \ : \ T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} = \mathcal{C}^{(s)}_{e;(s_1,s_2)}(\widetilde{x}^{+})^p \widetilde{L}^{+}_{\{r_1} \cdots \widetilde{L}^{+}_{r_{s_1-s}} \widetilde{L}^{+}_{t_1} \cdots \widetilde{L}^{+}_{t_{s_2}} T^{(s)}_{0;(s,0);r(s)\}} , \tag{7.2.31}$$

for $p = \frac{1}{2}(e+s-s_1-s_2)$;

$$\mathcal{M}^{+,\leq}_{(s)} \ : \ T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} = \mathcal{C}^{(s)}_{e;(s_1,s_2)}(\widetilde{x}^{-})^p \widetilde{L}^{-}_{\{r_1} \cdots \widetilde{L}^{-}_{r_{s_1-s}} \widetilde{L}^{-}_{t_1} \cdots \widetilde{L}^{-}_{t_{s_2}} T^{(s)}_{0;(s,0);r(s)\}} , \tag{7.2.32}$$

for $p = \frac{1}{2}(-e+s-s_1-s_2)$; and

$$\mathcal{M}^{+,0}_{(s)} \ : \ T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)} = \mathcal{C}^{(s)}_{e;(s_1,s_2)}((\widetilde{L}^{+})^m(\widetilde{L}^{-})^n)_{\{r_1\cdots r_{s_1-s}t_1\cdots t_{s_2}} T^{(s)}_{0;(s,0);r(s)\}} , \tag{7.2.33}$$

for $m=\frac{1}{2}(s_1+s_2-s+e)$ and $n=\frac{1}{2}(s_1+s_2-s-e)$, and where $((\widetilde{L}^+)^m(\widetilde{L}^-)^n)_{r_1\cdots r_{m+n}} = \widetilde{L}^+_{\{r_1}\cdots\widetilde{L}^-_{r_m}\widetilde{L}^-_{r_{m+1}}\cdots\widetilde{L}^-_{r_{m+n}\}}$. We propose a similar generation of the elements in $\mathcal{M}^-_{(s)}$.

Next, to generate $T^{(s)}_{\pm}$ with even spin $s=2p\geq 2$, one may fix coefficients $\xi_{2p;n}$ and $\xi'_{2p;n}$ such that

$$T^{(2p)}_{0;(2p);r(2p)} = \sum_{n=0}^{p-1}\xi_{2p;n}X^n_{r(2p)} , \tag{7.2.34}$$

$$T^{(2p)}_{0;(2p,1);r(2p),s} = \sum_{n=0}^{p-1}\xi'_{2p;n}X^n_{r(2p),s} , \tag{7.2.35}$$

with

$$X^n_{r(2p)} = \widetilde{L}^+_{\{r_1}\widetilde{L}^-_{r_2}\cdots\widetilde{L}^+_{r_{2n-1}}\widetilde{L}^-_{r_{2n}}\widetilde{Q}_{r(2p-2n)\}}T^{(0)}_{0;(0)} , \tag{7.2.36}$$

$$X^n_{r(2p),s} = \widetilde{L}^+_{\{r_1}\widetilde{L}^-_{r_2}\cdots\widetilde{L}^+_{r_{2n-1}}\widetilde{L}^-_{r_{2n}}\widetilde{Q}_{r(2p-2n)}T^{(0)}_{0;(1);s\}} , \tag{7.2.37}$$

where the non-minimal higher-spin generator

$$Q_{r(2n)} = L^+_{\{r_1}\star L^-_{r_2}\star\cdots\star L^+_{r_{2n-1}}\star L^-_{r_{2n}\}} . \tag{7.2.38}$$

The elements $T^{(2p)}_{0;(2p)}$ can also be generated by combining minimal higher-spin transformations with $\mathfrak{so}(D+1;\mathbb{C})$ transformations. Similarly, for odd spin $s=2p+1\geq 3$, there are coefficients $\xi_{2p+1;n}$ and $\xi'_{2p+1;n}$ such that

$$T^{(2p+1)}_{0;(2p+1);r(2p+1)} = \sum_{n=0}^{p-1}\xi_{2p+1;n}X^n_{r(2p+1),s} , \tag{7.2.39}$$

$$T^{(2p+1)}_{0;(2p+1,1);r(2p+1),s} = \sum_{n=0}^{p-1}\xi'_{2p+1;n}X^n_{r(2p+1),s} , \tag{7.2.40}$$

with

$$X^n_{r(2p+1)} = \widetilde{L}^+_{\{r_1}\widetilde{L}^-_{r_2}\cdots\widetilde{L}^+_{r_{2n-1}}\widetilde{L}^-_{r_{2n}}\widetilde{Q}_{r(2p-2n)}T^{(1)}_{0;(1);r_{2p+1}\}} , \tag{7.2.41}$$

$$X^n_{r(2p+1),s} = \widetilde{L}^+_{\{r_1}\widetilde{L}^-_{r_2}\cdots\widetilde{L}^+_{r_{2n-1}}\widetilde{L}^-_{r_{2n}}\widetilde{Q}_{r(2p-2n)}T^{(1)}_{0;(1,1);r_{2p+1},s\}} \tag{7.2.42}$$

where the spin-1 static ground states, in their turn, can be generated from the scalar static ground states. For example, to generate $T^{(1)}_{0;(1)}$ from $T^{(0)}_{0;(1)}$, one may use

$$\widetilde{\mathrm{Ad}}_{EM_{rs}}T^{(0)}_{0;(1);t} = \{EM_{rs},T^{(0)}_{0;(1);t}\} = \delta_{t[s}T^{(1)}_{0;(1);r]} , \tag{7.2.43}$$

as can be seen using $EM_{rs} = E \star M_{rs} = M_{rs} \star E$ and

$$E \star T^{(0)}_{0;(1);r} \;=\; -T^{(0)}_{0;(1);r} \star E \;=\; \tfrac{1}{2}\mathrm{Ad}_E T^{(0)}_{0;(1);r} \;=\; -\tfrac{i}{2} T^{(1)}_{0;(1);r} \ . \tag{7.2.44}$$

The generation of $T^{(1)}_{0;(1,1)}$ from $T^{(0)}_{0;(0)}$ is more involved, since

$$E \star T^{(0)}_{0;(0)} \;=\; \tfrac{1}{2}\mathrm{Ad}_E T^{(0)}_{0;(0)} \;=\; 0 \ . \tag{7.2.45}$$

One may, for example, first use $\mathfrak{so}(D+1;\mathbb{C})$ to transform $T^{(0)}_{0;(0)}$ into $T^{(0)}_{0;(2)}$; then EM_{rs} to go to $T^{(1)}_{0;(2)}$; and finally $\mathfrak{so}(D+1;\mathbb{C})$ to go down to $T^{(1)}_{0;(1,1)}$. We note, however, that $T^{(1)}_{0;(1,1)}$ can be generated immediately by separate $\star$-multiplication from the left. For example,

$$T^{(1)}_{0;(1,1)} \;=\; \frac{(2\epsilon_0+1)(2\epsilon_0+2)}{4\epsilon_0} M_{rs} \star T^{(0)}_{0;(0)} \ , \tag{7.2.46}$$

as can be seen using (3.1.66), which implies

$$M_{rs} \star T_{0(n)} \;=\; \tfrac{1}{2}\mathrm{Ac}_{M_{rs}} T_{0(n)} \;=\; T_{0(n)[r,s]} - \frac{(n-1)n^2(n+1)}{16(n+\epsilon_0-\frac{1}{2})(n+\epsilon_0+\frac{1}{2})} T_{0(n-2)[r,s]} \ , \tag{7.2.47}$$

although one can check that (7.2.46) is not a twisted-adjoint $\mathfrak{ho}_1(D+1;\mathbb{C})$-transformation. We also notice that repeated $\star$ commutation by E, *i.e.* the maps $(\mathrm{Ad}_E)^n : \mathcal{M}_{(s)} \rightarrow \bigoplus_{s'=|s-n|}^{s+n} \mathcal{M}_{(s')}$, assume a relatively simple form, although also Ad_E is not a twisted-adjoint $\mathfrak{ho}_1(D+1;\mathbb{C})$-transformation.

The compact twisted-adjoint modules described contain invariant submodules generated by lowest-weight or highest-weight elements, to which we now turn our attention.

7.3 Composite Lowest-Weight Spaces and Non-Composite Lowest-Spin Spaces

The compact twisted-adjoint modules contain invariant lowest-weight and highest-weight submodules. Suppose $T^{(s)}_{e;(s_1,s_2)} \in \mathcal{M}^{(s)}$ is a lowest-weight state, *i.e.*

$$\widetilde{L}^-_r T^{(s)}_{e;(s_1,s_2);t(s_1),u(s_2)} \;=\; L^-_r \star T^{(s)}_{e;(s_1,s_2);t(s_1),u(s_2)} - T^{(s)}_{e;(s_1,s_2);t(s_1),u(s_2)} \star L^+_r \;=\; 0 \ . \tag{7.3.1}$$

Then the second and quartic Casimir operators are given by, on the one hand, (C.0.3) and (C.0.4), and, on the other hand, (7.2.14), which leads to the necessary conditions

$$x+y+z \;=\; C_2[\ell] \ , \qquad x(x+\Delta)+y(y+\Delta')+z(z+\Delta'') \;=\; C_4[\ell] \ , \tag{7.3.2}$$

where we have defined

$$x = e(e-D+1)\ , \qquad y = s_1(s_1+D-3)\ , \qquad z = s_2(s_2+D-5)\ , \tag{7.3.3}$$

$$\Delta = \tfrac{1}{2}(D-1)(D-2)\ , \qquad \Delta' = \tfrac{1}{2}(D-3)(D-4)-1\ ,$$
$$\Delta'' = \tfrac{1}{2}(D-5)(D-6)-2\ , \tag{7.3.4}$$

and

$$C_2[\ell] = x_0+y_0\ , \qquad C_4[\ell] = x_0(x_0+\Delta)+y_0(y_0+\Delta')\ , \tag{7.3.5}$$

where

$$x_0 = e_0(e_0-D+1)\ , \qquad y_0 = s(s+D-3)\ , \qquad e_0 = s+D-3\ . \tag{7.3.6}$$

Moreover, combining (7.3.1) with the identities (7.2.11) and (7.2.12), respectively, yields

$$e = s+D-3-\frac{s_2}{s} \qquad \text{for } s_1=s\geq 1 \text{ and } s_2<s\ . \tag{7.3.7}$$

$$e = \frac{s_1+s_2+2(D-4)}{D-3} \qquad \text{for } s_2\geq 1\ . \tag{7.3.8}$$

To begin with, let us take $s_2=0$. Then $z=0$ and (7.3.2) have two roots

$$x = x_0\ , \quad y = y_0\ , \qquad \text{and} \quad x = y_0+2-D\ , \quad y = x_0+D-2\ . \tag{7.3.9}$$

The second root corresponds to $s_1=s-1$, which is ruled out for all s, or $s_1=4-D-s$, which is ruled for all s except $s=0$ in $D=4$, where it coincides with the first solution (which is thus a double root). The first root corresponds to $s_1=s$ and $e=s+D-3$ or $e=2-s$. The latter energy level is ruled out for $s\geq 1$ due to the condition (7.3.7). Thus, the admissible lowest weight states with $s_2=0$ are

$$s_1 = s = 0\ , \qquad e = \begin{cases} 2\epsilon_0 \\ 2 \end{cases}\ , \tag{7.3.10}$$

$$s_1 = s \geq 1\ , \qquad e = s+2\epsilon_0\ , \tag{7.3.11}$$

where we note the degeneracy in case $s=0$ and $D=5$. For $s_2\geq 1$ we find the admissible root

$$s_1 = s_2 = s = 1\ , \qquad e = 2\ . \tag{7.3.12}$$

In the scalar sector, the two admissible roots are indeed lowest-weight states, given by

$$f^{(0)}_{2\epsilon_0;(0)}(z) = {}_1F_1(\epsilon_0+\tfrac{3}{2};2;-4z)\ , \tag{7.3.13}$$

$$f^{(0)}_{2;(0)}(z) = {}_1F_1(\epsilon_0+\tfrac{3}{2};2\epsilon_0;-4z)\ , \tag{7.3.14}$$

taking the following particularly simple form in $D = 4$:

$$f^{(0)}_{1;(0)}(z) = e^{-4z} , \qquad f^{(0)}_{2;(1)}(z) = (1-4z)e^{-4z} . \tag{7.3.15}$$

Here we note that the functions $f^{(0)}_{e;(0)}(z)$ are fixed (uniquely) by the twisted-energy condition $(\widetilde{E}-e)T^{(0)}_{e;(0)} = 0$, so that $\widetilde{L}^-_r T^{(0)}_{e;(0)} = 0$ (which can be worked out explicitly using (3.1.58)) becomes an algebraic equation for e with roots $e = 2$ and $e = 2\epsilon_0$.

For $D = 2p + 5$ with $p = 1, 2, 3, ...,$ the Harish-Chandra module $\mathfrak{C}(2; (0))$ contains a singular vector at level $2p$, namely $L^-_r (x^+)^p|2; (0)\rangle = 0$ (where $x = L^+_r L^+_r$), that in its turn generates $\mathfrak{D}(2\epsilon_0; (0))$. The module $\mathfrak{C}(2; (0))$ is isomorphic to the lowest-weight module inside $\mathcal{M}^+_{(0)}$ generated by the twisted-adjoint $\mathfrak{so}(D + 1; \mathbb{C})$ action on $T^{(0)}_{2;(0)}$, where the singular vector obeys

$$\widetilde{L}^-_r (\widetilde{x}^+)^p T^{(0)}_{2;(0)} = 0 , \qquad D = 2p + 5 , \quad p = 1, 2, \ldots . \tag{7.3.16}$$

The lowest-weight space $\mathfrak{D}(2; (0))$ hence occupies p diagonal lines in compact weight space, and we shall refer to it as a scalar p-lineton[5] (see fig. 7.1). Thus, in summary, the scalar compact twisted-adjoint modules contain the following invariant subspaces (where we recall that the $\pm$ on $\mathcal{I}_{(0)}$ denote $\sigma_\pm$-parity while on $\mathfrak{D}$ distinguish a module and its negative-energy counterpart)

$$D = 4, 6, \ldots \ : \ \mathcal{I}^+_{(0)} = \mathfrak{D}^+(2; (0)) \oplus \mathfrak{D}^-(-2; (0)) \tag{7.3.17}$$

$$\mathcal{I}^-_{(0)} = \mathfrak{D}^+(1; (0)) \oplus \mathfrak{D}^-(-1; (0)) , \tag{7.3.18}$$

$$D = 5 \ : \ \mathcal{I}^+_{(0)} = \mathfrak{D}^+(2; (0)) \oplus \mathfrak{D}^-(-2; (0)) \tag{7.3.19}$$

$$\mathcal{I}^-_{(0)} = 0 , \tag{7.3.20}$$

$$D = 7, 9, \ldots \ : \ \mathcal{I}^+_{(0)} = \left[\mathfrak{D}^+(2; (0)) \oplus \mathfrak{D}^-(-2; (0))\right] \tag{7.3.21}$$

$$\oplus_s \left[\mathfrak{D}^+(2\epsilon_0; (0)) \oplus \mathfrak{D}^-(-2\epsilon_0; (0))\right] , \tag{7.3.22}$$

$$\mathcal{I}^-_{(0)} = 0 , \tag{7.3.23}$$

where we recall that $\oplus_s$ denotes a semi-direct sum.

[5] In D dimensions, the p-lineton $\mathfrak{C}(e_0; (0))$ with $e_0 = \epsilon_0 - (p-1)$ has the singular vector $x^p \mid e_0; (0)\rangle \simeq 0$, and therefore consists of p lines in weight space,

$$\mathfrak{D}(e_0; (0)) = \bigoplus_{k=0}^{p-1} \bigoplus_{n=0}^{\infty} |e_0 + k + n; (n)\rangle ,$$

where $|e_0 + k + n; (n)\rangle_{r(n)} = L^+_{r_1} \cdots L^+_{r_n} x^k |e_0; (0)\rangle$. In particular, the 1-lineton coincides with the ordinary singleton. The scalar p-linetons can be thought of as boundary particles satisfying a higher-derivative equation like $\Box^p \phi = 0$.

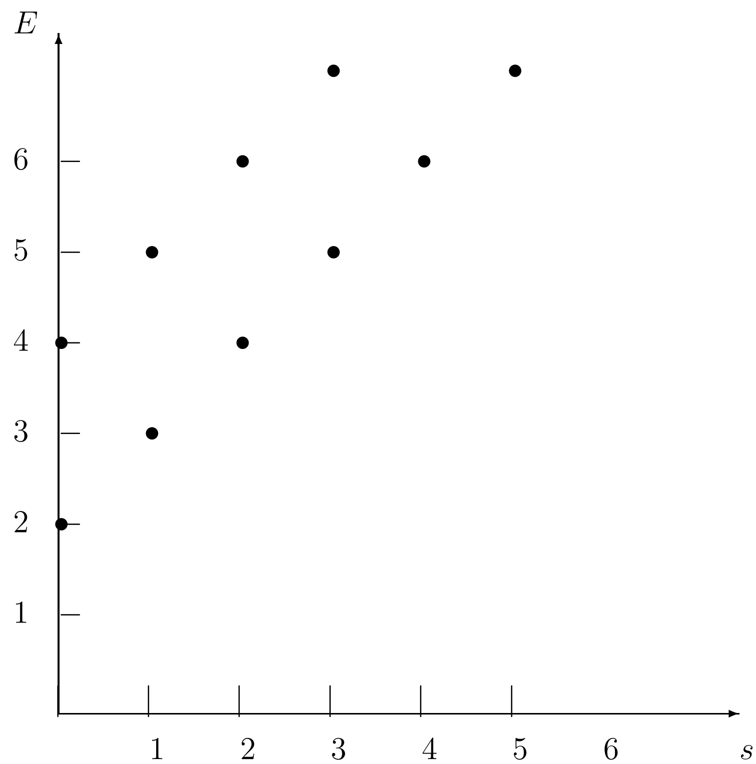

Figure 7.1: Weight diagrams of the scalar 2-lineton in $D = 9$.

Turning to the case of $s > 0$, we first use (3.1.66) to show that the scalar lowest-weight states actually obey slightly stronger conditions that the lowest-weight condition, namely

$$L^-_r \star T^{(0)}_{2\epsilon_0;(0)} = L^-_r \star T^{(0)}_{2;(0)} = 0 \,, \tag{7.3.24}$$

$$T^{(0)}_{2\epsilon_0;(0)} \star L^+_r = T^{(0)}_{2;(0)} \star L^+_r = 0 \,, \tag{7.3.25}$$

and

$$M_{rs} \star T^{(0)}_{2\epsilon_0;(0)} = 0 \,, \tag{7.3.26}$$

while $M_{rs} \star T^{(0)}_{2;(0)}$ is non-vanishing unless $D = 5$. Thus $L^+_{\{r_1} \star \cdots \star L^+_{r_s\}} \star T^{(0)}_{2\epsilon_0;(0)} = \mathcal{C}^{(s)} T^{(s)}_{s+2\epsilon_0;(s);r(s)}$ obeys the lowest-weight condition, and we expect $\mathcal{C}^{(s)}$ to be non-vanishing, so that $T^{(s)}_{s+2\epsilon_0;(s)}$ is indeed a lowest-weight element.

Alternatively, we notice that (7.3.24)–(7.3.26) imply that $|2\epsilon_0;(0)\rangle_{12} = T^{(0)}_{2\epsilon_0;(0)}|\mathbb{1}\rangle_{12}$

obeys

$$L_r^-(\xi)|2\epsilon_0;(0)\rangle_{12} \;=\; M_{rs}(\xi)|2\epsilon_0;(0)\rangle_{12} \;=\; 0\ , \qquad \xi=1,2\ , \tag{7.3.27}$$

so that one may formally reflect the *Flato-Fronsdal formula* [31], which we recall here,

$$|s+2\epsilon_0;(s)\rangle_{12;r(s)} \;=\; f_{(s)}f_{r(s)}(1,2)|2\epsilon_0;(0)\rangle_{12}\ , \tag{7.3.28}$$

where the composite operator[6]

$$f_{r(s)}(1,2) \;=\; (-1)^s f_{r(s)}(2,1) \;=\; \sum_{k=0}^{s} f_{s;k}(L^+_{\{r_1}\cdots L^+_{r_k})(1)(L^+_{r_{k+1}}\cdots L^+_{r_s\}})(2)\ , \tag{7.3.29}$$

$$f_{s;k} \;=\; (-1)^s f_{s;s-k} \;=\; \binom{s}{k}\frac{(1-s-\epsilon_0)_k}{(\epsilon_0)_k}\ , \tag{7.3.30}$$

to obtain the following expression for the lowest-weight elements

$$T^{(s)}_{s+2\epsilon_0;(s);r(s)} \;=\; f_{(s)}\sum_{k=0}^{s}(-1)^{s-k}f_{s;k}L^+_{\{r_1}\star\cdots\star L^+_{r_k}\star T^{(0)}_{2\epsilon_0;(0)}\star L^-_{r_{k+1}}\star\cdots\star L^-_{r_s\}}\ , \tag{7.3.31}$$

where the (finite) renormalization $f_{(s)}$ is fixed by (7.2.8).

Turning to $M_{\{r_1t_1}\star\cdots M_{r_st_s\}}\star T^{(0)}_{2;(0)}$, which are also lowest-weight elements, these must vanish for $s\geq 2$, since $e=2$ and $s_1=s_2=s\geq 2$ are inadmissible compact quantum numbers. On the other hand, for $s=1$ and $D\neq 5$, the element $M_{rs}\star T^{(0)}_{2;(0)}$ is non-vanishing and proportional to $T^{(1)}_{2;(1,1);r,s}$, which is thus a lowest-weight element for $D\neq 5$, and hence it is a lowest-weight element also for $D=5$ (since $\widetilde{L}^-_r T^{(1)}_{2;(1,1)}=0$ for $D\neq 5$ implies that $\widetilde{L}^-_r T^{(1)}_{2;(1,1)}=0$ for $D=5$ by analytical continuation in D). Thus, $\mathcal{M}^+_{(1)}$ contains the generalized Verma module

$$\mathfrak{C}'(2;(1,1)) \;=\; \frac{\mathfrak{C}(2;(1,1))}{\mathcal{I}[V]}\ , \tag{7.3.32}$$

generated by the twisted-adjoint $\mathfrak{so}(D+1;\mathbb{C})$ action on $T^{(1)}_{2;(1,1);r,s}$ (modulo (7.2.11) and (7.2.12) which hold modulo elements in $\mathcal{I}[V]$). For $D=3+2p$, $p=1,2,\ldots$, the lowest-

[6] The coefficients $f_{s;k}$ are fixed by the condition $(L_r^-(1)+L_r^-(2))|s+2\epsilon_0;(s)\rangle_{12;r(s)}=0$, which is equivalent to $a_kf_{s;k}+a_{s-k+1}f_{s;k-1}=0$, where $a_k=2k(k+\epsilon_0-1)$, with solution $f_{s;k}=(-1)^sf_{s;s-k}=(-1)^k\frac{a_{s-k+1}\cdots a_s}{a_k\cdots a_1}f_{s;0}$, taking the form (7.3.30) for $f_{s;0}=1$.

weight state of $\mathfrak{D}(1+2\epsilon_0;(1)) = \mathfrak{D}(1+2p;(1))$ is a singular vector in $\mathfrak{C}'(2;(1,1))$, *viz.*[7]

$$\widetilde{L}_t^-(\widetilde{x}^+)^p \widetilde{L}_s^+ T^{(1)}_{2;(1,1);u,s} \quad = \quad 0 \ , \qquad D = 3+2p \ , \quad p=1,2,\dots \ . \tag{7.3.33}$$

Factoring out the submodule $\mathfrak{N}'(2;(1,1))$ generated from the singular vector, yields the irreducible lowest-weight space

$$\mathfrak{D}'(2;(1,1)) \quad = \quad \frac{\mathfrak{C}'(2;(1,1))}{\mathfrak{N}'(2;(1,1))} \ , \tag{7.3.34}$$

occupying p lines in compact weight space, and which we shall therefore refer to as a spin-1 p-lineton. We note that the 1-lineton in $D=5$ is the spin-1 singleton, that is $\mathfrak{D}'(2;(1,1)) = \mathfrak{D}(2;(1,1))$.

In the case of $D=4$, we note that $\mathfrak{D}(2;(0))$ and $\mathfrak{D}(2;(1,1)) \simeq \mathfrak{D}(2;(1))$ initiate the sequence of composite massless lowest-weight spaces $\mathfrak{D}(s+1;(s,1)) \simeq \mathfrak{D}(s+1;(s))$ contained in the tensor product $\mathfrak{D}_{\frac{1}{2}} \otimes \mathfrak{D}_{\frac{1}{2}}$ of two spinor singletons. We expect that all these lowest-weight spaces are realized in $\mathcal{M}_+$, *i.e.* we expect the following lowest-weight elements

$$D=4: \ T^{(s)}_{s+1;(s);r(s)} = f'_{(s)} \epsilon_{tu\{r_1} \sum_{k=1}^{s} f'_{s;k} L^+_{r_2} \star \cdots \star L^+_{r_k} \star M_{tu} \star T^{(0)}_{2;(0)} \star L^-_{r_{k+1}} \star \cdots \star L^-_{r_s\}} , \tag{7.3.35}$$

where the coefficients $f'_{s;k}$ and (finite) renormalizations $f'_{(s)}$, respectively, are fixed by (7.3.1) and (7.2.8).

[7] At the level of the ordinary Harish-Chandra module $\mathfrak{C}(2;(1,1))$,

$$L_t^- x^p \widetilde{L}_s^+ |2\rangle_{s,u} = -4px^{p-1} L_u^+ L_s^+ |2\rangle_{s,t} + 2p(2p+7-D)x^{p-1} L_t^+ L_s^+ |2\rangle_{s,u} + (10-D)x^p |2\rangle_{t,u} \ ,$$

where $|2\rangle_{s,u} = |2;(1,1)\rangle_{s,u}$. For $D=2p+5$, the (tu)-projection vanishes, while the $[tu]$-projection is proportional to $(D-5)L^+_{[u}|2\rangle_{s,t]}$ that vanishes for $D\neq 5$ only at the level of $\mathfrak{C}'(2;(1,1))$. The 1-lineton in $D=5$ coincides with the spin-1 singleton (which exists already in $\mathfrak{C}(2;(1,1))$).

So far, for $s \geq 1$, we have found the invariant subspaces

$$D = 4 \ : \ \mathcal{I}^+_{(s)} = \mathcal{I}^-_{(s)} = \mathfrak{D}^+(s+1;(s)) \oplus \mathfrak{D}^-(-(s+1);(s)) \ , \tag{7.3.36}$$

$$D = 5 \ : \ \mathcal{I}^+_{(1)} = \left[\mathfrak{D}^+(2;(1,1)) \oplus \mathfrak{D}^-(-2;(1,1))\right] \oplus_s \left[\mathfrak{D}^+(3;(1)) \oplus \mathfrak{D}^-(-3;(1))\right] \ , \tag{7.3.37}$$

$$\mathcal{I}^+_{(s)} \supset \mathfrak{D}^+(s+2;(s)) \oplus \mathfrak{D}^-(-(s+2);(s)) \qquad \text{for } s \geq 2, \tag{7.3.38}$$

$$D = 7,9,\dots \ : \ \mathcal{I}^+_{(1)} = \left[\mathfrak{D}'^+(2;(1,1)) \oplus \mathfrak{D}'^-(-2;(1,1))\right] \oplus_s \left[\mathfrak{D}^+(1+2\epsilon_0;(1)) \oplus \mathfrak{D}^-(-(1+2\epsilon_0);(1))\right] \ , \tag{7.3.39}$$

$$\mathcal{I}^+_{(s)} \supset \mathfrak{D}^+(s+2\epsilon_0;(s)) \oplus \mathfrak{D}^-(-(s+2\epsilon_0);(s) \quad \text{for } s \geq 2 \tag{7.3.40}$$

$$D = 6,8,\dots \ : \ \mathcal{I}^+_{(1)} = \mathfrak{D}^+(2;(1,1)) \oplus \mathfrak{D}^-(-2;(1,1)) \ , \tag{7.3.41}$$

$$\mathcal{I}^-_{(s)} = \mathfrak{D}^+(s+2\epsilon_0;(s)) \oplus \mathfrak{D}^-(-(s+2\epsilon_0);(s)) \quad \text{for } s \geq 1, \tag{7.3.42}$$

where we notice the ordinary spin-1 singleton $\mathfrak{D}(2;(1,1))$ in $D=5$ and the spin-1 p-linetons $\mathfrak{D}'(2;(1,1))$ in $D=7,9,\dots$, whose higher-spin "completion" we leave for future work (for example, in $D=5$, the higher-spin singletons $\mathfrak{D}(s+1;(s,s))$ obey (7.3.7) and (7.3.8) but violate the conditions on C_2 and C_4, and are hence not realized in $\mathcal{I}_{(s)}$).

In summary, taking into account the expected results, the compact twisted-adjoint modules have the following indecomposable structures

$$D = 4,6,\dots \ : \ \mathcal{M}^+ = \mathcal{W}^+ \oplus_s \mathfrak{D}' \ , \tag{7.3.43}$$

$$\mathcal{M}^- = \mathcal{W}^- \oplus_s \mathfrak{D} \ , \tag{7.3.44}$$

$$D = 5,7,\dots \ : \ \mathcal{M}^- = \mathcal{W}^- \ , \tag{7.3.45}$$

$$\mathcal{M}^+ = \mathcal{W}^+ \oplus_s \mathfrak{D}' \oplus_s \mathfrak{D} \ , \tag{7.3.46}$$

where $\mathfrak{D}$ consists of the massless scalar-singleton composites, *i.e.*

$$\mathfrak{D} = \bigoplus_s [\mathfrak{D}^+(s+2\epsilon_0;(s)) \oplus \mathfrak{D}^-(-(s+2\epsilon_0);(s))] \ ; \tag{7.3.47}$$

$\mathfrak{D}'$ contains the higher-spin completion of $\mathfrak{D}^+(2;(0))\oplus\mathfrak{D}^-(-2;(0))$ *i.e.*

$$D=4\ :\ \mathfrak{D}' \ = \ \bigoplus_{s=0}^{\infty}[\mathfrak{D}^+(s+1;(s,1))\oplus\mathfrak{D}^-(-(s+1);(s,1))]\ , \tag{7.3.48}$$

$$D=5\ :\ \mathfrak{D}' \ \supset\ \mathfrak{D}^+(2;(1,1))\oplus\mathfrak{D}^-(-2;(1,1))\ , \tag{7.3.49}$$

$$D=6,7,\dots\ :\ \mathfrak{D}' \ \supset\ [\mathfrak{D}^+(2;(0))\oplus\mathfrak{D}^-(-2;(0))] \oplus[\mathfrak{D}'^+(2;(1,1))\oplus\mathfrak{D}'^-(-2;(1,1))]\ ; \tag{7.3.50}$$

and the remaining quotient spaces $\mathcal{W}^\pm$ are *non-composite lowest-spin modules*. These spaces contain all the states in $\mathcal{M}^{\pm,0}$, *i.e.* with energy $|e|<s_1+s_2-s$, such as static states, and they may also contain states in $\mathcal{M}^{\pm,\geq}$ and $\mathcal{M}^{\pm,\leq}$, but no lowest-weight nor highest-weight states (we notice that if $\mathcal{I}^\pm_{(0)}$ is non-trivial, then $\mathcal{W}^\pm_{(0)}$ contains a finite number of energy levels for fixed s_1).

We stress that, in accordance with our proposal, as given in (7.2.32) and (7.2.33), the spaces $\mathcal{M}^\pm$ are generated by $\mathfrak{ho}_1(D+1;\mathbb{C})$ from the scalar static ground states $T^{(0)}_\pm$ defined in (7.2.18). On the other hand, by the Flato-Fronsdal construction, the space $\mathfrak{D}$ is also generated by $\mathfrak{ho}_1(D+1;\mathbb{C})$ from the scalar lowest-weight element $T^{(0)}_{2\epsilon_0;(0)}$ and highest-weight element $T^{(0)}_{-2\epsilon_0;(0)}$. Similarly, in $D=4$, $\mathfrak{D}'$ is generated by $\mathfrak{ho}_1(D{+}1;\mathbb{C})$ from $T^{(0)}_{2;(0)}$ and $T^{(0)}_{-2;(0)}$ (we expect an analogous generation of $\mathfrak{D}'$ in $D\geq 5$). Correspondingly, in an abbreviated notation where $\mathfrak{so}(D-1;\mathbb{C})$ vector indices are suppressed, elements $S_{\mathcal{W}^\pm}\in\mathcal{W}^\pm$ and $S_{\mathfrak{D}}\in\mathfrak{D}$ can be expanded as

$$S_{\mathcal{W}^\pm}\ =\ \sum_{s=0}^{\infty}\sum_{m,n=0}^{\infty}S^{\pm(s)}_{m,n}(\widetilde{L}^+)^m(\widetilde{L}^-)^n\widetilde{Q}_sT^{(0)}_\pm\ , \tag{7.3.51}$$

$$S_{\mathfrak{D}}\ =\ \sum_{s=0}^{\infty}\sum_{m,p=0}^{\infty}\left[S^{(s)}_{m,p}(\widetilde{x}^+)^p(\widetilde{L}^+)^m\widetilde{R}_sT^{(0)}_{2\epsilon_0;(0)}+\overline{S}^{(s)}_{m,p}(\widetilde{x}^-)^p(\widetilde{L}^-)^m\pi(\widetilde{R}_sT^{(0)}_{2\epsilon_0;(0)})\right] \tag{7.3.52}$$

for $Q_s,R_s\in\mathfrak{ho}_1(D+1;\mathbb{C})$ such that $\widetilde{Q}_sT^{(0)}_\pm=T^{(s)}_\pm$ and $\widetilde{R}_sT^{(0)}_{2\epsilon_0;(0)}=T^{(s)}_{s+2\epsilon_0;(s)}$, and where $S^{(s)}_{m,n}$ and $(S^{(s)}_m,\overline{S}^{(s)}_m)$ are complex coefficients.

7.4 Harmonic Expansion

Physically speaking, the map from $\mathcal{T}_\ell$ to $\mathcal{M}_\ell$, is the *harmonic expansion* of a spin-$(2\ell+2)$ Weyl tensor obeying linearized field equations in the maximally symmetric D-dimensional geometry with cosmological constant (which is also solutions of the higher-spin gauge theory). The corresponding vielbein and spin-connection form the flat $\mathfrak{so}(D{+}1;\mathbb{C})$ connection

$$\Omega\ =\ -i(e^aP_a+\tfrac{1}{2}\omega^{ab}M_{ab})\ =\ L^{-1}\star dL\ , \tag{7.4.1}$$

where the coset element $L \in SO(D+1;\mathbb{C})$, or gauge function, is given in stereographic coordinates by

$$L = \frac{2}{h}\exp\frac{ix^\mu\delta_\mu^a P_a}{h} \;, \qquad h = \sqrt{1-\lambda^2x^2} \;, \qquad x^2 = x^\mu x^\nu \delta_\mu^a\delta_\nu^b\eta_{ab} \;, \quad (7.4.2)$$

for which

$$e_\mu{}^a = \frac{2\delta_\mu^a}{h^2} \;, \qquad \omega_\mu{}^{ab} = \frac{\delta_\mu^{[a}\delta_\nu^{b]}x^\nu}{h^2} \;. \quad (7.4.3)$$

The linearized adjoint one-form A and twisted-adjoint zero-form Φ obey the constraints

$$\begin{aligned} DA &\equiv dA + \{\Omega, A\}_\star \equiv \nabla A - ie^a\{P_a, A\}_\star \\ &= -\tfrac{i}{2}\sum_{s=2,4,6,\dots} e^a\wedge e^b\Phi_{ac(s-1),bd(s-1)}M^{c_1d_1}\cdots M^{c_{s-1}d_{s-1}} \;, \quad (7.4.4) \\ D\Phi &\equiv d\Phi + [\Omega,\Phi]_\pi \equiv \nabla\Phi - ie^a\{P_a,\Phi\}_\star = 0 \;, \quad (7.4.5) \end{aligned}$$

where $D^2=0$, and the consistency of the one-form constraint (7.4.4) follows from $e^a\wedge e^b\wedge e^c\nabla_c\Phi_{ad(s-1),be(s-1)}=0$ which is a consequence of the zero-form constraint (7.4.5) (which is in itself consistent). As shown in Appendix D, inserting the component expansion of Φ into the zero-form constraint and using (3.1.58) yields

$$\nabla_c\Phi_{a(s+k),b(s)} - 2k\Delta_{s+k-1,s}\eta_{c\{a}\Phi_{a(s+k),b(s)\}} + \frac{2\lambda_{k+1}^{(s)}}{k+1}\Phi_{c\{a(s+k),b(s)\}} = 0 \;. \quad (7.4.6)$$

These constraints decompose into the auxiliary field identifications

$$\Phi_{a(s+k),b(s)} = -\frac{k+1}{2\lambda_{k+1}^{(s)}}\nabla_{\{a}\Phi_{a(s+k-1),b(s)\}} \;, \qquad k=1,2,\dots \;, \quad (7.4.7)$$

where we have used $\Phi_{\langle c\langle a(s+k),b(s)\rangle\rangle} = \Phi_{ca(s+k),b(s)}$; and the Bianchi identities and mass-shell conditions

$$\begin{aligned} \nabla_{[\mu}\Phi_{\nu|a(s+k-1),|\rho]b(s-1)} &= 0 \;, \qquad s\geq 1 \;, \quad (7.4.8) \\ (\nabla^2 - m_{s,k}^2)\Phi_{a(s+k),b(s)} &= 0 \;, \quad (7.4.9) \end{aligned}$$

with mass-terms

$$m_{s,k}^2 = -4\epsilon_0 - 2s - (k+2s+2\epsilon_0+1)k \;. \quad (7.4.10)$$

These values are consistent with the Casimir relation

$$\nabla^2\Phi_{a(s+k),b(s)} = \left(C_2[\mathfrak{so}(D+1;\mathbb{C})|\ell] - C_2[\mathfrak{so}(D;\mathbb{C})|(s+k,s)]\right)\Phi_{a(s+k),b(s)} \;, \quad (7.4.11)$$

with $C_2[\mathfrak{so}(D+1;\mathbb{C})|\ell]$ given by (7.1.47) and $C_2[\mathfrak{so}(D;\mathbb{C})|(s+k,s)] = (s+k)(s+k+D-2)+s(s+D-4)$. The Casimir relation follows by re-writing $C_2[\mathfrak{so}(D+1;\mathbb{C})] = C_2[\mathfrak{so}(D;\mathbb{C})] - P^a \star P_a$ in the twisted-adjoint representation as

$$-\mathrm{Ac}_{P^a}\mathrm{Ac}_{P_a}\Phi_{(s)} \;=\; \tfrac{1}{2}(\widetilde{\mathrm{Ad}}_{M_{AB}}\widetilde{\mathrm{Ad}}_{M^{AB}} - \mathrm{Ad}_{M_{ab}}\mathrm{Ad}_{M^{ab}})\Phi_{(s)}\ , \tag{7.4.12}$$

and using the fact that (7.4.5) implies that $\widetilde{\mathrm{Ad}}_{P_a}\Phi_{(s)} = -i\nabla_a\Phi_{(s)}$.

The zero-form constraint (7.4.5) is solved explicitly by

$$\Phi \;=\; L^{-1}\star S\star \pi(L)\ , \tag{7.4.13}$$

where S is a constant twisted-adjoint element. The zero-form $\Phi = \sum_{s=0}^{\infty}\Phi_{(s)}$ can be expanded either covariantly or compactly, *viz.*

$$\begin{aligned}\Phi_{(s)} \;&=\; \sum_{e;s_1\geq s\geq s_2\geq 0}\Phi^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}L^{-1}\star T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}\star\pi(L)\\ &=\; \sum_{e;s_1\geq s\geq s_2\geq 0}\ \sum_{k=s_1-s}^{\infty} T_{a(s+k),b(s)}S^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}D^{(s);a(s+k),b(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}\ ,\end{aligned} \tag{7.4.14}$$

where $S^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}\in\mathbb{C}$ and the generalized harmonic functions ($k\geq s_1-s$)

$$D^{(s);a(s+k),b(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}(x) \;=\; \mathcal{N}^{-1}_{s,k}\ \mathrm{Tr}[T^{a(s+k),b(s)}\star L^{-1}(x)\star T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}\star\pi(L(x))]\ , \tag{7.4.15}$$

as can be seen using (7.1.5) and (7.1.5). The harmonic functions obey the Bianchi identity (7.4.8) and the mass-shell condition (7.4.9) (for fixed $(s);e;(s_1,s_2)$). Representing the trace as the expectation value (7.1.12), and using the overlap condition (7.1.15), which implies $\pi(L)(1)|\mathbb{1}\rangle_{12} = L^{-1}(2)|\mathbb{1}\rangle_{12}$, these harmonic functions can be rewritten as

$$D^{(s);a(s+k),b(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}(x) \;=\; \mathcal{N}^{-1}_{s,k}\,{}_{12}\langle T_{a(s+k),b(s)}|\star L^{-1}(x)\star|(s);e;(s_1,s_2)\rangle_{12;r(s_1),t(s_2)}\ , \tag{7.4.16}$$

where L is given by (F.0.1) with $P_a = P_a(1)+P_a(2)$, and we have used (7.1.26) to define

$$|(s);e;(s_1,s_2)\rangle_{12;r(s_1),t(s_2)} \;=\; T^{(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}(1)\star|\mathbb{1}\rangle_{12}\ , \tag{7.4.17}$$

$$|T_{a(s+k),b(s)}\rangle \;=\; T_{a(s+k),b(s)}(1)\star|\mathbb{1}\rangle_{12}\ . \tag{7.4.18}$$

For $x=0$, the harmonic functions are the (finite) overlaps

$$D^{(s);a(s+k),b(s)}_{e;(s_1,s_2);r(s_1),t(s_2)}(0) \;=\; \delta^{\{a(s+k),b(s)\}_D}_{\{0(n)\{r(s_1),t(s_2)\}_{D-1}0(s-s_2)\}_D}f^{(s)}_{e;(s_1,s_2);n}\ ,\qquad n=s+k-s_1\ , \tag{7.4.19}$$

where $\{\cdots\}_D$ and $\{\cdots\}_{D-1}$, respectively, denote $\mathfrak{so}(D;\mathbb{C})$-traceless and $\mathfrak{so}(D-1;\mathbb{C})$-traceless Young projections.

The harmonic expansions include not only the composite massless lowest-weight spaces $\mathfrak{D}^+(s+2\epsilon_0;(s))$ and highest-weight spaces $\mathfrak{D}^-(-(s+2\epsilon_0);(s))$ but also the non-composite lowest-spin spaces $\mathcal{W}_{(s)}$. In the context of standard second-quantized free higher-spin field theory on AdS_D, the former are unitary spaces containing the one-particle states, corresponding to wave-functions that fall off relatively fast at time-like infinity, while the latter are excluded from the perturbative spectrum since they correspond to non-normalizable wave-functions. However, as we shall see below, the situation is to some extent reversed in the norm induced by the trace operation Tr. Moreover, all elements of $\mathcal{M}$ play a role in the broader context of constructing the space of classical solutions to the full non-linear Vasiliev equations, *i.e.* the covariant phase-space, which is the starting point for the covariant phase-space quantization of higher-spin gauge theory.

Next we shall examine the inverse procedure of expanding the twisted-adjoint $\mathfrak{so}(D;\mathbb{C})$ tensors $|T_{a(s+k),b(s)}\rangle$ and the adjoint $\mathfrak{so}(D+1;\mathbb{C})$ tensors $|Q_\ell\rangle$ in the compact basis.

7.5 Expanding Covariant Tensors in Compact Basis

Twisted-Adjoint $\mathfrak{so}(D;\mathbb{C})$ Tensors as Singleton-Singleton Composites

According to the Flato-Fronsdal formula (7.3.28), the massless lowest-weight states belong to $\mathfrak{D}_0\otimes\mathfrak{D}_0\subset\mathcal{B}$. Letting $|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}$ denote the restriction of the reflector to $\mathfrak{D}_0\otimes\mathfrak{D}_0$, the expansion of the type $(s+k,s)$ twisted-adjoint $\mathfrak{so}(D;\mathbb{C})$ tensor

$$|\Phi_{(s+k,s);\mathfrak{D}_0}\rangle_{12} \;=\; \Phi^{a(s+k),b(s)}|T_{a(s+k),b(s);\mathfrak{D}_0}\rangle_{12} \;=\; \Phi^{a(s+k),b(s)}T_{a(s+k),b(s)}(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} \tag{7.5.1}$$

in the lowest-weight module $\mathfrak{D}(s+2\epsilon_0;(s))$ is given by

$$|\Phi_{(s+k,s);\mathfrak{D}_0}\rangle \;=\; \sum_{s+k\geq j_1\geq s\geq j_2\geq 0}\mathcal{C}^{(s+k,s)}_{(j_1,j_2)}\Phi^{r(j_1),t(j_2)}|(s+k,s);(j_1,j_2)\rangle_{r(j_1),t(j_2)}\ , \tag{7.5.2}$$

where we have made the following definitions: i) the coefficients $\mathcal{C}^{(s+k,s)}_{(j_1,j_2)}$ are overall normalizations; ii) $\Phi^{r(j_1),t(j_2)}=\Phi^{0(s+k-j_1)\{r(j_1),t(j_2)\}_{D-1}0(s-j_2)}$ are the type (j_1,j_2) $\mathfrak{so}(D-1;\mathbb{C})$ *polarization* tensors contained in $\Phi^{a(s+k),b(s)}$; iii) the states

$$|(s+k,s);(j_1,j_2)\rangle_{r(j_1),t(j_2)} \;=\; \psi^{(s+k,s)}_{(j_1,j_2)}(x)|(s);2\epsilon_0+j_1+j_2;(j_1,j_2)\rangle_{r(j_1),t(j_2)}\ , \tag{7.5.3}$$

where[8]

$$|(s);j_1+j_2+2\epsilon_0;(j_1,j_2)\rangle_{r(j_1),t(j_2)} \;=\; L^+_{\{r_1}\cdots L^+_{r_{j_1-s}}L^+_{t_1}\cdots L^+_{t_{j_2}}|2\epsilon_0+s;(s)\rangle_{r(s)\}}\ , \tag{7.5.4}$$

[8] In this section the $\star$ is suppressed in products of L^+ operators. We also notice that $L^+_r=L^+_r(1)+L^+_r(2)$ so that $x=L^+_rL^+_r=2L^+_r(1)L^+_r(2)$ acting on composites.

are normalized type (j_1, j_2) states in $\mathfrak{D}(2\epsilon_0 + s, (s))$ of *minimal* energy; iv) the *dressing* functions

$$\psi^{(s+k,s)}_{(j_1,j_2)}(x) \;=\; \sum_{n=0}^{\infty} \psi^{(s+k,s)}_{(j_1,j_2);n} x^n \;, \qquad x \;=\; y^2 \;=\; L^+_r L^+_r \;, \tag{7.5.5}$$

are determined by the normalization

$$\psi^{(s+k,s)}_{(j_1,j_2)}(0) \;=\; 1 \;, \tag{7.5.6}$$

and by the embedding requirement that $\{|(s+k, s); (j_1, j_2)\rangle\}_{s+k \geq j_1 \geq s \geq j_2}$ furnishes a decomposition of the original type $(s+k, s)$ $\mathfrak{so}(D;\mathbb{C})$ tensor, *i.e.* $(|(j_1, j_2)\rangle \equiv |(s+k, s); (j_1, j_2)\rangle)$

$$M_{0r}|(j_1, j_2)\rangle \;=\; \mathcal{C}^{(s+k,s)}_{(j_1,j_2);(1,0)}|(j_1 + 1, j_2)\rangle + \mathcal{C}^{(s+k,s)}_{(j_1,j_2);(-1,0)}|(j_1 - 1, j_2)\rangle \tag{7.5.7}$$

$$+\mathcal{C}^{(s+k,s)}_{(j_1,j_2);(0,1)}|(j_1, j_2 + 1)\rangle + \mathcal{C}^{(s+k,s)}_{(j_1,j_2);(0,-1)}|(j_1, j_2 - 1)\rangle \;, \tag{7.5.8}$$

with $M_{0r} = \frac{1}{2}(L^+_r + L^-_r)$ and

$$\mathcal{C}^{(s+k,s)}_{(s+k,j_2);(1,0)} \;=\; \mathcal{C}^{(s+k,s)}_{(j_1,s);(0,1)} \;=\; 0 \;, \tag{7.5.9}$$

$$\mathcal{C}^{(s+k,s)}_{(s,j_2);(-1,0)} \;=\; 0 \qquad \text{for } j_2 < s \;, \tag{7.5.10}$$

that enforce the conditions on the ranges of (j_1, j_2) given in (7.5.2). We also notice that the Casimir constraint

$$\begin{aligned}&(C_2[\mathfrak{so}(D;\mathbb{C})] - C_2[\mathfrak{so}(D;\mathbb{C})|(s+k, s)])\\ &\qquad\qquad \psi^{(s+k,s)}_{(j_1,j_2)}(x)|(s); 2\epsilon_0 + j_1 + j_2; (j_1, j_2)\rangle_{r(j_1),t(j_2)} \;=\; 0 \;.\end{aligned} \tag{7.5.11}$$

can be turned into a differential equation in x for the dressing functions. The embedding conditions are equivalent to the condition that the "top" state $|(s + k, s); (s); (s + k, 0)\rangle$ obeys

$$M_{0\{r}|(s + k, s); (s + k, 0)\rangle_{r(s+k)\}} \;=\; 0 \;, \tag{7.5.12}$$

with solution

$$\psi^{(s+k,s)}_{(s+k,0)}(x) \;=\; \Psi^{(0,0)}_{(0,0)}(x) \;=\; \Gamma(\nu + 1) \left(\frac{y}{2}\right)^{-\nu} J_\nu(y) \;, \quad \nu = \epsilon_0 - 1 \;. \tag{7.5.13}$$

In $D = 4$, where the index $\nu = -\frac{1}{2}$, the rescaled Bessel functions become trigonometric, and the reflector takes the simple form

$$D = 4 \;\;:\;\; |\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} \;=\; \cos(y)|1; (0)\rangle_{12} \;. \tag{7.5.14}$$

Let us provide a few more detailed remarks on the basic structure of (7.5.2):

- The ranges of (j_1, j_2) follow from the decomposition formula

$$\widehat{\begin{array}{l}\fbox{}\,{\scriptstyle s+k}\\ \fbox{}\,{\scriptstyle s}\end{array}} = \bigoplus_{\substack{j_1, j_2 \in \mathbb{N} \\ s+k \geq j_1 \geq s \geq j_2 \geq 0}} \begin{array}{l}\fbox{}\,{\scriptstyle j_1}\\ \fbox{}\,{\scriptstyle j_2}\end{array} \ , \tag{7.5.15}$$

where the $\mathfrak{so}(D-1;\mathbb{C})$ Young diagrams on the right-hand side can be obtained by the $\mathfrak{so}(D;\mathbb{C})$ Young diagram (denoted with a hat) on the left-hand side by projecting its indices along the 0-direction in all possible ways compatible with the irreducibility of Young diagrams. We note that for $D = 4$ the type (j_1, j_2) tensors with $j_2 \geq 2$ are trivial, *i.e.* zero-dimensional, as follows from King's rule, which implies that the dimension of a traceless Young-projected $\mathfrak{so}(N;\mathbb{C})$-tensor is zero if the sum of the lengths of the first two columns exceeds N (see, for example, [59]).

- Starting from a state in $\mathfrak{D}(2\epsilon_0 + s; (s))$ at excitation level l, that is

$$|(s); l + s + 2\epsilon_0\rangle_{r(l);t(s)} \ = \ L^+_{r_1} \cdots L^+_{r_l} |2\epsilon_0 + s; (s)\rangle_{t(s)} \ , \tag{7.5.16}$$

and decomposing it under $\mathfrak{so}(D-1;\mathbb{C})$ by extracting traces into powers of $x = L^+_r L^+_r$ (recall that $L^+_r |2\epsilon_0 + s; (s)\rangle_{rt(s-1)} = 0$), one finds

$$|(s); l + s + 2\epsilon_0\rangle \ = \ \sum_{n=0}^{[l/2]} \sum_{p=0}^{\min(s,l-2n)} x^n |(s); 2\epsilon_0 + l - 2n; (s + l - 2n - p, p)\rangle \ . \tag{7.5.17}$$

Thus, a general state in $\mathfrak{D}(s + 2\epsilon_0; (s))$ of type (j_1, j_2) is of the form

$$|(s); (j_1, j_2)\rangle \ = \ \sum_{n=0}^{\infty} \psi_{(j_1,j_2);n} x^n |(s); j_1 + j_2 + 2\epsilon_0; (j_1, j_2)\rangle \ , \tag{7.5.18}$$

where $\psi_{(j_1,j_2);n}$ are arbitrary coefficients and $|(s); j_1 + j_2 + 2\epsilon_0; (j_1, j_2)\rangle$ is the type (j_1, j_2) state of minimal energy given in (7.5.4).

- To show (7.5.13), we use the lemma

$$M_{0\{r} x^n |2\epsilon_0 + p; (p)\rangle_{r(p)\}} \ = \ \tfrac{1}{2}(1 + 4n(n + \epsilon_0 - 1)) L^+_{\{r} x^{n-1} |2\epsilon_0 + p; (p)\rangle_{r(p)\}} \ , \tag{7.5.19}$$

where we notice the independence of $p = s + k$, and we have defined

$$|2\epsilon_0 + p; (p)\rangle_{r(p)} \ = \ L^+_{\{r_1} \cdots L^+_{r_k} |s + 2\epsilon_0; (s)\rangle_{r(s)\}} \ , \tag{7.5.20}$$

with the property $L^-_{\{r_1} |2\epsilon_0 + p; (p)\rangle_{r(p)\}} = 0$. Hence, from the embedding condition (7.5.12), it follows that

$$\left(4x\frac{d^2}{dx^2} + 4\epsilon_0 \frac{d}{dx} + 1\right) \psi^{(s+k,s)}_{(s+k,0)}(x) \ = \ 0 \ , \tag{7.5.21}$$

with solution (7.5.13), as can be seen by the following rescaling and change of variables,

$$\psi^{(s+k,s)}_{(s+k,0)}(x) \;=\; y^{-\nu}J(y)\ , \qquad x=y^2\ , \tag{7.5.22}$$

which brings (7.5.21) to Bessel's differential equation

$$\left(\frac{d^2}{dx^2}+y\frac{d}{dy}+(1-\frac{\nu^2}{y^2})\right)J(y)\ , \qquad \nu=\epsilon_0-1\ . \tag{7.5.23}$$

Thus, the solution of (7.5.21) that is analytical at $x=0$ is given uniquely up to a normalization by (7.5.13).

- For $s=0$, the independence of $\psi^{(k)}_{(k)}(x)$ (where we have dropped the second highest-weight labels) on k, can be shown directly using $P_r=\frac{1}{2i}(L^-_r-L^+_r)=-iM_{0r}+iL^+_r$ and $M_{0r}|(0);(0)\rangle=0$, which implies

$$\begin{aligned} |(k);(k)\rangle &= -i^kP_{\{r_1}\cdots P_{r_k\}}|(0);(0)\rangle && (7.5.24)\\ &= \sum_{p=0}^{k}\binom{k}{p}(-1)^{k-p}L^+_{\{r_1}\cdots L^+_{r_p}M_{0r_{p+1}}\cdots M_{0r_k\}}|(0);(0)\rangle && (7.5.25)\\ &= L^+_{\{r_1}\cdots L^+_{r_k\}}|(0);(0)\rangle\ . && (7.5.26)\end{aligned}$$

Thus, since $|(k);(k)\rangle=\psi^{(k)}_{(k)}(x)L^+_{\{r_1}\cdots L^+_{r_k\}}|2\epsilon_0;(0)\rangle$, it follows that $\psi^{(k)}_{(k)}(x)=\psi^{(0)}_{(0)}(x)$. The energy operator, on the other hand, acts as

$$E\,f(x)|2\epsilon_0+p;(p)\rangle_{r(p)} \;=\; \left(2x\frac{d}{dx}+2\epsilon_0+p\right)f(x)|2\epsilon_0+p;(p)\rangle_{r(p)}\ , \tag{7.5.27}$$

where $|2\epsilon_0+p;(p)\rangle_{r(p)}$ is defined in (7.5.20), which means that the functional form of $\psi^{(k)}_{(j_1)}(x)$ with $j_1<k$ differs from that of $\psi^{(0)}_{(0)}(x)$. For example, from $|(k);(k-1)\rangle\propto E|(k);(k)\rangle$, it follows that

$$\psi^{(k)}_{(k-1)} \;=\; \frac{1}{2\epsilon_0+k}\left(2x\frac{d}{dx}+2\epsilon_0+k\right)\psi^{(0)}_{(0)}(x)\ . \tag{7.5.28}$$

- The embedding formula can easily be adapted to highest-weight spaces by applying the π-map, using $\pi(|s+2\epsilon_0;(s)\rangle)=|-s-2\epsilon_0,(s)\rangle$ and $\pi(L^+_r)=L^-_r$.

Let us demonstrate how we can arrive at the embedding formula and (7.5.13) by using compositeness. To begin with, for $s=k=0$ the embedding formula (7.5.2), where now $\Phi_{(0,0)}\in\mathbb{C}$, amounts to

$$|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} \;=\; (\Phi_{(0,0)})^{-1}|\Phi_{(0,0)}\rangle_{12} \;=\; \mathcal{C}^{(0,0)}_{(0,0)}\psi^{(0,0)}_{(0,0)}(x)|2\epsilon_0;(0)\rangle_{12}\ . \tag{7.5.29}$$

In other words, the $\mathfrak{so}(D;\mathbb{C})$-singlet in $\mathfrak{D}(2\epsilon_0;(0))$ can be identified with the singleton reflector, which can equivalently be described as the map

$$R \ : \ \mathfrak{D}_0 \mapsto \mathfrak{D}_0^* \ , \tag{7.5.30}$$

defined by

$$R(|\epsilon_0;(0)\rangle) \ = \ \langle\epsilon_0;(0)| \ , \tag{7.5.31}$$

and

$$R(X|\epsilon_0;(0)\rangle) \ = \ \langle\epsilon_0;(0)|(\tau\circ\pi)(X) \ , \tag{7.5.32}$$

for $X \in \mathcal{A}$. It follows that

$$R(L_r^\pm) \ = \ -L_r^\mp \ , \qquad R(E) \ = \ E \ , \qquad R(M_{rs}) \ = \ M_{rs} \ , \tag{7.5.33}$$

$$R(|n\rangle) \ = \ (-1)^n\langle n| \ , \tag{7.5.34}$$

where we have defined the following basis elements

$$|n\rangle_{r(n)} \ = \ |n+\epsilon_0;(n)\rangle_{r(n)} \ = \ L^+_{r_1}\cdots L^+_{r_n}|\epsilon_0;(0)\rangle \ , \tag{7.5.35}$$

$$\langle n|_{r(n)} \ = \ \langle n+\epsilon_0;(n)|_{r(n)} \ = \ \langle\epsilon_0;(0)|L^-_{r_1}\cdots L^-_{r_n} \ , \tag{7.5.36}$$

which are traceless as a consequence of the singular vector (3.2.34), and with normalization

$$\langle\epsilon_0;(0)|L^-_{r_1}\dots L^-_{r_n}L^+_{s_1}\dots L^+_{s_n}|\epsilon_0;(0)\rangle \ = \ \mathcal{N}_n\delta^{\{s_1\dots s_n\}}_{\{r_1\dots r_n\}} \ , \qquad \mathcal{N}_n \ = \ 2^n\, n!(\epsilon_0)_n \ , \tag{7.5.37}$$

as can be seen using

$$L^-_r|n\rangle_{s_1\dots s_n} \ = \ 2n(n+\epsilon_0-1)\delta_{r\{s_1}|n-1\rangle_{s_2\dots s_n\}} \ . \tag{7.5.38}$$

For general s and k, we substitute the Flato-Fronsdal formula (7.3.28) into the right-hand side of the embedding formula (7.5.2), which then reads

$$\sum_{s+k\geq j_1\geq s\geq j_2\geq 0} \mathcal{C}^{(s+k,s)}_{(j_1,j_2)}\,\Phi^{r(j_1),t(j_2)}\,\psi^{(s+k,s)}_{j_1,j_2}(x)$$
$$L^+_{\{r_1}\cdots L^+_{r_{j_1-s}}L^+_{t_1}\cdots L^+_{t_{j_2}}f_{r(s)\}}(1,2)|2\epsilon_0;(0)\rangle_{12} \ , \tag{7.5.39}$$

where $L^+_r \ = \ L^+_r(1) + L^+_r(2)$. Turning to the left-hand side, it can be rewritten using (7.4.18) and by decomposing $T_{a(s+k),b(s)}$ under $\mathfrak{so}(D-1;\mathbb{C})$. Schematically, suppressing trace parts in $T_{0(s+k-j_1)\{r(j_1),t(j_2)\}0(s-j_2)}$, the result reads

$$\sum_{s+k\geq j_1\geq s\geq j_2\geq 0} \Phi^{r(j_1),t(j_2)}\left((M_{rs})^{j_2}(M_{r0})^{s-j_2}(P_r)^{j_1-s}E^{k-j_1+s}\right)(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} \ , \tag{7.5.40}$$

where the generators can be (anti)-symmetrized under the exchange of the "flavor' indices 1 and 2 using (7.1.15) together with (3.2.11) and (3.2.12) with the following result:

$$P_r(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} = \tfrac{i}{2}L_r^+|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\,,\quad M_{r0}(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} = -\tfrac{1}{2}(L_r^+(1)-L_r^+(2))|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\,, \tag{7.5.41}$$

$$M_{rs}(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} = (L^+_{[r}(1)L^+_{s]}(2)+L^-_{[r}(1)L^-_{s]}(2))|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\,, \tag{7.5.42}$$

$$E(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12} = \tfrac{1}{2(D-1)}(L_r^+(1)L_r^+(2)-L_r^-(1)L_r^-(2))|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\,. \tag{7.5.43}$$

Proceeding by substituting $|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}=\psi^{(0,0)}_{(0,0)}(x)|2\epsilon_0;(0)\rangle_{12}$ into (7.5.40), and noting that

$$L^-_{[r}(1)L^-_{s]}(2)\psi^{(0,0)}_{(0,0)}(x)|2\epsilon_0;(0)\rangle_{12} = L^+_{[r}(1)L^+_{s]}(2)(D_M\psi^{(0,0)}_{(0,0)})(x)|2\epsilon_0;(0)\rangle_{12})\,, \tag{7.5.44}$$

$$L^-_r(1)L^-_r(2)\psi^{(0,0)}_{(0,0)}(x)|2\epsilon_0;(0)\rangle_{12} = (D_E\psi^{(0,0)}_{(0,0)})(x)|2\epsilon_0;(0)\rangle_{12})\,, \tag{7.5.45}$$

where D_M and D_E are analytical differential operators in x (*i.e.*, with coefficients that are analytical functions of x at $x=0$), one sees that the type-(j_1,j_2) sector of the left-hand side (7.5.40) is of the form

$$\Phi^{r(j_1),t(j_2)}(D^{(s+k,s)}_{(j_1,j_2)}\psi^{(0,0)}_{(0,0)})(x)M_{r(j_1),t(j_2)}(1,2)|2\epsilon_0;(0)\rangle_{12}\,, \tag{7.5.46}$$

where $D^{(s+k,s)}_{(j_1,j_2)}(x)$ is an analytical differential operator in x and $M_{r(j_1),t(j_2)}(1,2)$ is a monomial in $L_r^+(1)$ and $L_r^+(2)$ of degree $2j_2+s-j_2+j_1-s=j_1+j_2$ with flavor-exchange symmetry $M_{r(j_1),t(j_2)}(1,2)=(-1)^sM_{r(j_1),t(j_2)}(2,1)$. Since the right-hand side (7.5.39) is of exactly the same form, we conclude that

$$\left(D^{(s+k,s)}_{(j_1,j_2)}\psi^{(0,0)}_{(0,0)}\right)(x) = \mathcal{C}^{(s+k,s)}_{(j_1,j_2)}\psi^{(s+k,s)}_{(j_1,j_2)}(x)\,. \tag{7.5.47}$$

Specializing to $(j_1,j_2)=(s+k,0)$, there are no contributions to (7.5.40) from M_{rs} and E, so that $D^{(s+k,s)}_{(s+k,0)}$ becomes a constant[9] and

$$\psi^{(s+k,s)}_{(s+k,0)} = \psi^{(0,0)}_{(0,0)}(x)\,,\qquad \mathcal{C}^{(s+k,s)}_{(j_1,j_2)} = D^{(s+k,s)}_{(s+k,s)}. \tag{7.5.48}$$

[9]For $(j_1,j_2)=(s+k,0)$, the schematic expression (7.5.40) is of the form $\Phi^{r(s+k)}(L_r^+(1)+L_r^+(2))^k(L_r^+(1)-L_r^+(2))^s|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}$, with a binomial expansion, to be compared with the "dressed" binomial expansion in the (precise) Flato-Fronsdal formula (7.3.28). There is a precise agreement, however, once the trace corrections to (7.5.40) are included. For example, for $(s+k,s)=(2,0)=(j_1,j_2)$, the precise form of the left-hand side of (7.5.2) reads

$$\Phi^{r(2)}T_{r(2),0(2)}(1) = \tfrac{4}{3}\Phi^{r(2)}(M_{r_10}\star M_{r_20}+\tfrac{1}{2\epsilon_0+1}P_{r_1}\star P_{r_2})\,,$$

as can be seen using (3.1.49) and (3.1.51) to expand the quantity $T_{a(2),b(2)}=\widehat{T}_{\{a(2),b(2)\}_D}$ as follows

$$\tfrac{4}{3}M_{a_1b_1}\star M_{a_2b_2}-\tfrac{4}{3(2\epsilon_0+1)}\left(\eta_{a(2)}P_{b_1}\star P_{b_2}-2\eta_{a_1b_1}P_{(a_2}\star P_{b_2)}+\eta_{b(2)}P_{a_1}\star P_{a_2}\right)+\frac{4\epsilon_0}{3(2\epsilon_0+1)}(\eta_{a(2)}\eta_{b(2)}-\eta_{a_1b_1}\eta_{a_2b_2})\,.$$

Finally, the precise functional form (7.5.13) follows by expanding the unity $1\!\!1_{\mathfrak{D}_0}$ in the basis (7.5.35) and (7.5.36) as

$$1\!\!1_{\mathfrak{D}_0} \;=\; \sum_{n=0}^{\infty} [\mathcal{N}_n]^{-1}\,|n\rangle\langle n| \;=\; \sum_{n=0}^{\infty} \frac{1}{2^n n!(\epsilon_0)_n} L^+_{r_1}\cdots L^+_{r_n}|\epsilon_0;(0)\rangle\langle\epsilon_0;(0)|L^-_{r_1}\cdots L^-_{r_n} \tag{7.5.49}$$

Letting ${}^\times_\times f(L^-_r, L^+_r){}^\times_\times$ denote the standard normal-ordering, and introducing the variable

$$z \;=\; 2L^+_r L^-_r \;, \tag{7.5.50}$$

we can write

$$1\!\!1_{\mathfrak{D}_0} \;=\; \sum_{n=0}^{\infty} \frac{1}{2^n n!(\epsilon_0)_n} {}^\times_\times (L^+_r L^-_r)^n |\epsilon_0;(0)\rangle\langle\epsilon_0;(0)|{}^\times_\times \tag{7.5.51}$$

$$=\; \sum_{n=0}^{\infty} \frac{1}{4^n\, n!(\epsilon_0)_n} {}^\times_\times (\sqrt{z})^{2n} |\epsilon_0,(0)\rangle\langle\epsilon_0,(0)|{}^\times_\times \tag{7.5.52}$$

$$=\; \Gamma(\nu+1)\left(\frac{\sqrt{z}}{2}\right)^{-\nu} {}^\times_\times I_\nu(\sqrt{z})|\epsilon_0,(0)\rangle\langle\epsilon_0,(0)|{}^\times_\times \;, \tag{7.5.53}$$

where $\nu = \epsilon_0 - 1$ and I_ν is the modified Bessel function, related to J_ν according to

$$I_\nu(w) \;=\; \begin{cases} e^{-\frac{i\pi\nu}{2}} J_\nu(e^{\frac{i\pi}{2}} w)\;, & \text{for } -\pi < \arg w \le \frac{\pi}{2} \\ e^{\frac{2i\pi\nu}{3}} J_\nu(e^{-\frac{3i\pi}{2}} w)\;, & \text{for } -\frac{\pi}{2} < \arg w \le \pi \end{cases} \tag{7.5.54}$$

We note that in $D=4$,

$$1\!\!1_{\mathfrak{D}_0} \;=\; {}^\times_\times \cosh\sqrt{z}|\tfrac{1}{2};(0)\rangle\langle\tfrac{1}{2};(0)|{}^\times_\times \;. \tag{7.5.55}$$

Applying the inverse reflection $R^{-1}:\mathfrak{D}_0^* \mapsto \mathfrak{D}_0$ to the dual states in the expansion of $1\!\!1_{\mathfrak{D}_0}$, and denoting the resulting two copies of $\mathfrak{D}_0$ by $\mathfrak{D}_0(\xi)$, $\xi = 1,2$, and using

$$R^{-1}(z) \;=\; -2L^+_r(1)L^+_r(2) \;=\; -x \;, \tag{7.5.56}$$

Thus, substituting $M_{r0} = -\frac{1}{2}(L^+_r + L^-_r)$ and $P_r = \frac{i}{2}(L^+_r - L^-_r)$, and using $(L^\pm_r(1) - L^\mp_r(2))|1\!\!1_{\mathfrak{D}_0}\rangle_{12} = 0$ and $|1\!\!1_{\mathfrak{D}_0}\rangle_{12} = \psi^{(0,0)}_{(0,0)}(x)|2\epsilon_0;(0)\rangle_{12}$, one arrives at

$$\Phi^{r(2)} T_{r(2),0(2)}(1)|1\!\!1_{\mathfrak{D}_0}\rangle_{12}$$

$$=\; \frac{2\epsilon_0}{3(2\epsilon_0+1)}\Phi^{r(2)}\psi^{(0,0)}_{(0,0)}(x)\left(L^+_{r_1}(1)L^+_{r_2}(1) - \frac{2(\epsilon_0+1)}{\epsilon_0} L^+_{r_1}(1)L^+_{r_2}(2) + L^+_{r_1}(2)L^+_{r_2}(2)\right)|2\epsilon_0;(0)\rangle_{12}\;,$$

in agreement with (7.3.28), and we also have $\mathcal{C}^{(2,0)}_{(2,0)} = \frac{2\epsilon_0}{3(2\epsilon_0+1)}$.

which formally implies $R^{-1}(\sqrt{z}) = iy$, we obtain

$$R^{-1}\mathbb{1}_{\mathfrak{D}_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n\, n!(\epsilon_0)_n}(\sqrt{x})^{2n}|\epsilon_0;(0)\rangle_1 \otimes |\epsilon_0;(0)\rangle_2 \tag{7.5.57}$$

$$= \Gamma(\nu+1)\left(\frac{\sqrt{x}}{2}\right)^{-\nu} J_\nu(\sqrt{x})|2\epsilon_0;(0)\rangle_{12} = |\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\ . \tag{7.5.58}$$

7.5.1 Adjoint $\mathfrak{so}(D+1;\mathbb{C})$ Tensors as Singleton-Anti-Singleton Composites

The ℓ-th adjoint level Λ_ℓ decomposes into a finite-dimensional compact lowest and highest-weight space, *viz.* $(s = 2\ell + 2)$

$$\mathcal{L}_\ell = \mathfrak{D}(-(s-1);(s-1)) = \bigoplus_{(e;s)\in\Lambda_\ell} \mathbb{C}\otimes Q_{e;s}\ , \tag{7.5.59}$$

where $Q_{e;s}$ are monomials of degree $2\ell+1$ built from $L^\pm_r$, M_{rs} and E. The lowest-weight element can be written using (7.1.43) as

$$|-(s-1);(s-1)\rangle_{12;r(s-1)} = (L^-_{\{r}\cdots L^-_{r_{s-1}\}})(1)|\widetilde{\mathbb{1}}\rangle_{12}\ , \tag{7.5.60}$$

where the $\mathfrak{so}(D+1;\mathbb{C})$-invariant twisted reflector $|\widetilde{\mathbb{1}}\rangle_{12} = k(1)|\mathbb{1}\rangle_{12}$ was defined in (7.1.28). Thus, the above lowest-weight elements are identified as the singleton-singleton composites

$$|-(s-1);(s-1)\rangle_{12;r(s-1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(\epsilon_0)_n}|n+s-1\rangle^-_{1;r(s-1)t(n)} \otimes |n\rangle^+_{2;t(n)}\ , \tag{7.5.61}$$

where $|n\rangle^+_{r(n)} = |n\rangle_{r(n)}$ is the singleton basis (7.5.35) and the corresponding anti-singleton basis is defined by

$$|n\rangle^-_{r(n)} = k|n\rangle_{r(n)} = L^-_{\{r}\cdots L^-_{r_{s-1}\}}|-\epsilon_0;(0)\rangle^-\ , \tag{7.5.62}$$

such that $L^-_r\cdots L^-_{r_{s-1}}|n\rangle^-_{s(n)} = |n+s-1\rangle^-_{r(s-1)t(n)}$. We notice that (7.5.60) by construction obeys

$$(L^-_r(1) + L^-_r(2))|-(s-1);(s-1)\rangle_{12;r(s-1)} = 0\ , \tag{7.5.63}$$

which one can also check explicitly using (7.5.38). In other words, the analog of the ordinary Flato-Fronsdal formula (7.3.28), which states that $\mathfrak{D}_0\otimes\mathfrak{D}_0$ can be expanded in terms of infinite-dimensional massless lowest-weight spaces, is the following *twisted Flato-Fronsdal formula*:

$$\widetilde{\mathfrak{D}}_0 \otimes \mathfrak{D}_0 \quad = \quad \bigoplus_{s=0}^{\infty} \mathfrak{D}(-(s-1);(s-1)) \ , \tag{7.5.64}$$

which thus states that $\widetilde{\mathfrak{D}}_0 \otimes \mathfrak{D}_0$ can be expanded in terms of finite-dimensional lowest-weight spaces.

7.5.2 On Adjoining the Adjoint and Twisted-Adjoint Representations

The Flato-Fronsdal formulae (7.3.28) and its twisted version (7.5.64) show, respectively, how the massless lowest-weight spaces residing inside the compact twisted-adjoint representation $\mathcal{M}(D+1;\mathbb{C})$ and the adjoint representation $\mathfrak{ho}(D+1;\mathbb{C})$ are mapped to singleton-singleton and anti-singleton-singleton composites. As previously mentioned, if $Q \in \mathfrak{ho}(D+1;\mathbb{C})$, then the element k maps Q to an element $S = Q \star k \in \mathcal{T}(D+1;\mathbb{C})$.

In this section we shall propose another relation between the adjoint and twisted-adjoint representations, that constitutes a "fiber" analog of the unfolding procedure in space-time, and provides a direct explanation for the agreement between the Casimir operators noted in (7.1.47) and (7.1.48). The basic idea is to adjoin the adjoint weight space $\mathcal{L}_\ell = \mathfrak{D}(-(s-1);(s-1))$, given in (7.5.59), to the massless lowest-weight space $\mathfrak{D}(s+2\epsilon_0;(s))$ via the intermediate conjugate massless lowest-weight space $\mathfrak{D}(-s+2;(s))$, whose lowest-weight state has the same quantum numbers of the singular vector

$$|-s+2;(s)\rangle \quad = \quad L^+_{\{r_1}|-s+1;(s-1)\rangle_{r(s-1)\}} \in \mathfrak{I}(-s+1;(s-1)) \ . \tag{7.5.65}$$

We propose that ($x = L^+_r L^+_r$)

$$D \geq 5 \quad : \quad |s+2\epsilon_0;(s)\rangle_{r(s)} \quad = \quad x^{\frac{D-5}{2}} L^+_{t_1} \cdots L^+_{t_s} |2;(s,s)\rangle_{r(s),t(s)} \ , \tag{7.5.66}$$

$$D = 4 \quad : \quad |s+1;(s)\rangle_{r(s)} \quad = \quad \epsilon_{r_1 t_1 u_1} \cdots \epsilon_{r_s t_s u_s} L^+_{u_1} \cdots L^+_{u_{s-1}} |2;(s,1)\rangle_{t(s),u_s} \ , \tag{7.5.67}$$

where $|2;(s,s)\rangle$ and $|2;(s,1)\rangle_{t(s),u_s}$ are the descendants of $|-s+2;(s)\rangle$ given by

$$D \geq 5 \quad : \quad |2;(s,s)\rangle_{r(s),t(s)} \quad = \quad L^+_{\{r_1} \cdots L^+_{r_s}|-s+2;(s,s)\rangle_{t(s)\}} \ , \tag{7.5.68}$$

$$D = 4 \quad : \quad |2;(s,1)\rangle_{r(s),t} \quad = \quad \epsilon_{\{r_1|u_1 v_1} \cdots \epsilon_{|r_{s-1}|u_{s-1} v_{s-1}} L^+_{u_1} \cdots L^+_{u_{s-1}} L^+_{|r_s|}|-s+2;(s,s)\rangle_{v(s-1)|t\}} \tag{7.5.69}$$

We notice that (7.5.66) is a regular enveloping-algebra element in $D = 5, 7, \ldots$ while it is an irregular element (involving a square root) in $D = 6, 8, \ldots$, where we define the action of $L^-_r x^n$ on lowest-weight states by analytical continuation of

$$L^-_r x^n \;=\; x^n L^-_r + 4n x^{n-1}\left(iL^+_s M_{rs} + L^+_r(E + n - \epsilon_0 - 1)\right) \qquad (7.5.70)$$

in n. We have checked that (7.5.66) is a singular vector for $s = 1$ and for $s = 2$ in $D = 5$. We have also checked (7.5.67) for $s = 1$ and $s = 2$. We notice that (7.5.65) is a weight-space analog of an abelian gauge transformation of a $(D-1)$-dimensional gauge field, and that $|2;(s,s)\rangle$ and $|2;(s,1)\rangle$ are the corresponding Weyl and Cotton tensors for $D \geq 5$ and $D = 4$, respectively, inducing the following short exact sequence

$$0 \hookrightarrow |-s+1;(s-1)\rangle \;\to\; |-s+2;(s)\rangle \;\to\; |s+2\epsilon_0;(s)\rangle \;\to\; 0\ . \qquad (7.5.71)$$

7.5.3 On Twisted-Adjoint $\mathfrak{so}(D;\mathbb{C})$ Tensors in Non-Composite Sectors

The embedding formula (7.5.2) describes the expansion of enveloping-algebra operators $T_{a(s+k),b(s)}$ in the massless lowest-weight (or highest-weight) spaces $\mathfrak{D}(s+2\epsilon_0;(s))$, that is, the expansion of the projected state $T_{a(s+k),b(s)}(1)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}$. Likewise, one may consider their expansion around $T^{(0)}_{2;(0)}$ corresponding to the lowest-weight state $|2;(0)\rangle_{12}$, obeying $L^-_r(\xi)|2;(0)\rangle_{12} = 0$ for $\xi = 1, 2$ in view of (7.3.24) and (7.3.25) (although it has not simple composite nature for $D \geq 6$). Denoting the corresponding dressing functions by $\chi^{(s+k,s)}_{(j_1,j_2)}(x)$, the decoupling condition (7.5.12) again implies

$$\chi^{(s+k,s)}_{(s+k,0)}(x) \;=\; \chi^{(0,0)}_{(0,0)}(x)\ , \qquad (7.5.72)$$

where $\chi^{(0,0)}_{(0,0)}(x)$ now obeys (7.5.21) with $\epsilon_0 \to 2 - \epsilon_0$, that is

$$\left(4x\frac{d^2}{dx^2} + 4(2-\epsilon_0)\frac{d}{dx} + 1\right)\chi^{(0,0)}_{(0,0)}(x) \;=\; 0\ , \qquad (7.5.73)$$

related to Bessel functions with index $\nu' = 1 - \epsilon_0$. For $\nu' \neq -1, -2, \ldots$, *i.e.* in all even dimensions and also $D = 5$, the solution analytical at $x = 0$ is the rescaled Bessel function

$$\chi^{(0,0)}_{(0,0)}(x) \;=\; \Gamma(\nu'+1)\left(\frac{\sqrt{x}}{2}\right)^{-\nu'} J_{\nu'}(\sqrt{x})\ , \quad \nu' = -\nu = 1 - \epsilon_0\ . \qquad (7.5.74)$$

In $D = 4$, the dressing functions take the simple form

$$\chi^{(0,0)}_{(0,0)}(x) \;=\; \frac{1}{\sqrt{x}}\sin(\sqrt{x})\ , \qquad (7.5.75)$$

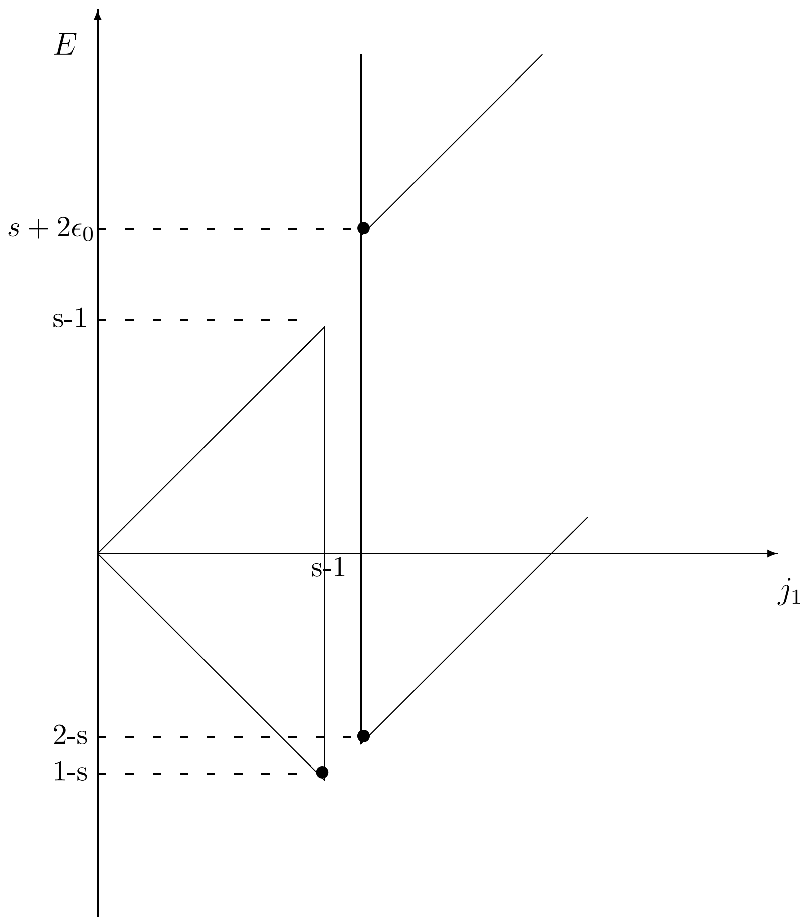

Figure 7.2: The adjoint module $\mathfrak{D}(-(s-1);(s-1))$ is connected via the conjugate massless module $\mathfrak{D}(-(s-2);(s))$ to the massless module $\mathfrak{D}(s+2\epsilon_0;(s))$.

which one identifies with the spinor-singleton reflector

$$D=4 \ : \ |1\!\!1_{\mathfrak{D}_{\frac{1}{2}}}\rangle_{12} \ = \ \chi^{(0,0)}_{(0,0)}(x)|2;(0)\rangle_{12} \ , \qquad |2;(0)\rangle_{12} \ = \ |1;(\tfrac{1}{2})\rangle^i_1 \otimes |1;(\tfrac{1}{2})\rangle_{2,i} \ , \tag{7.5.76}$$

where $i=1,2$ is two-component spinor index.

For $\nu' \neq -1,-2,\dots$, *i.e.* $D=5+2p=7,9,\dots$, $p=1,2,\dots$, the rescaled Bessel function develops a logarithmic branch starting at order $x^p\log x$. Correspondingly, as examined in Subsection 7.3, the module $\mathfrak{D}(2,(0)$ is actually a p-lineton, so that $x^p|2;(0)\rangle$ is a singular vector. Thus, the restricted reflector

$$|1\!\!1'\rangle_{12} \ = \ \chi^{(0,0)}_{(0,0)}(x)|2;(0)\rangle_{12} \ , \tag{7.5.77}$$

is a well-defined $\mathfrak{so}(D;\mathbb{C})$-invariant state in $D=2p$ and $D=5$, while for $D=5+2p$, $p=1,2,\dots$, it appears that some form of logarithmic superpositions of states are required.

The compact twisted-adjoint module $\mathcal{M}$ also contains the lowest-spin modules (7.3.51), whose elements can be reached formally starting from the lowest-weight or highest-weight states by considering dressing functions that are irregular elements of the enveloping algebra. For example, starting from the scalar ground states $|2\epsilon_0;(0)\rangle_{12}$ and $|2;(0)\rangle$, the even static ground state may be represented as either $|0;(0)\rangle_{12} = x^{-\epsilon_0}|2\epsilon_0;(0)\rangle_{12}$ or $|0;(0)\rangle_{12} = x^{-1}|2;(0)\rangle_{12}$, where $x = L_r^+ L_r^+ = 2L_r^+(1)L_r^+(2)$, although the precise meaning of these expressions are not clear to us. Instead, we would like to stress the fact that *all* elements in $\mathcal{M}$ have regular enveloping-algebra presentations.

7.6 Real Forms of the Master Fields

The real forms of the master fields are defined by

$$(A)^\dagger \;=\; -A\;, \qquad (\Phi)^\dagger \;=\; \pi(\Phi)\;, \tag{7.6.1}$$

where $\dagger$ acts as ordinary complex conjugation of the component fields (on real spacetime with $(x^\mu)^* = x^\mu$) and as hermitian conjugation on $\mathcal{A}$ defined by

$$(M_{AB})^\dagger \;=\; \sigma(M_{AB})\;, \tag{7.6.2}$$

where σ is an automorphism of $\mathcal{A}$ with the property that $\sigma\circ\sigma = \mathrm{Id}$, introduced in Chapter 3 and whose action we recall here. Decomposing $M_{AB} \to (M_{ab}; M_{0'a} = P_a) \to (M_{rs}, M_{r0}; P_r, P_0)$, one may consider the following real forms of $\mathfrak{so}(D+1;\mathbb{C}) \supset \mathfrak{so}(D;\mathbb{C})$:

$$\mathfrak{so}(D-1,2)\supset\mathfrak{so}(D-1,1) \;:\; \sigma(M_{AB}) \;=\; M_{AB}\;, \tag{7.6.3}$$

$$\mathfrak{so}(D,1)\supset\mathfrak{so}(D-1,1) \;:\; \sigma(M_{ab};P_a) \;=\; (M_{ab},-P_a)\;, \tag{7.6.4}$$

$$\mathfrak{so}(D,1)\supset\mathfrak{so}(D) \;:\; \sigma(M_{rs},M_{r0};P_r,P_0) \;=\; (M_{rs},-M_{r0};P_r,-P_0)\;, \tag{7.6.5}$$

$$\mathfrak{so}(D+1)\supset\mathfrak{so}(D) \;:\; \sigma(M_{rs},M_{r0};P_r,P_0) \;=\; (M_{rs},-M_{r0};-P_r,P_0)\;, \tag{7.6.6}$$

which we may summarize as

$$\sigma(M_{rs},M_{r0};P_r,P_0) \;=\; (M_{rs},\sigma_0 M_{r0};\sigma_{0'}P_r,\sigma_{0'}\sigma_0 P_0)\;, \qquad \sigma_{0'}\;,\sigma_0 \;=\; \pm 1\;. \tag{7.6.7}$$

Starting from (7.6.3), the three other reality conditions are equivalent to using Wick rotations in either $0'$, or 0, or both $0'$ and 0, respectively, to go to the *real basis* $M^{\mathbb{R}}_{AB}$ obeying

$$(M^{\mathbb{R}}_{AB})^\dagger \;=\; M^{\mathbb{R}}_{AB} \tag{7.6.8}$$

and the commutation rules (3.2.1) with $\eta_{AB} = (-\sigma_{0'};\eta_{ab})$ and $\eta_{ab} = (-\sigma_0, \delta_{rs})$, that is

$$\mathfrak{so}(D-1,2) \supset \mathfrak{so}(D-1,1) \ : \ \eta_{AB} \ = \ (-;\eta_{ab}) \ , \quad \eta_{ab} \ = \ (-,\delta_{rs}) \ , \tag{7.6.9}$$

$$\mathfrak{so}(D,1) \supset \mathfrak{so}(D-1,1) \ : \ \eta_{AB} \ = \ (+;\eta_{ab}) \ , \quad \eta_{ab} \ = \ (-,\delta_{rs}) \ , \tag{7.6.10}$$

$$\mathfrak{so}(D,1) \supset \mathfrak{so}(D) \ : \ \eta_{AB} \ = \ (-;\eta_{ab}) \ , \quad \eta_{ab} \ = \ (+,\delta_{rs}) \ , \tag{7.6.11}$$

$$\mathfrak{so}(D+1) \supset \mathfrak{so}(D) \ : \ \eta_{AB} \ = \ (+;\eta_{ab}) \ , \quad \eta_{ab} \ = \ (+,\delta_{rs}) \ . \tag{7.6.12}$$

The corresponding component fields $A^{\mathbb{R}}_{\mu,a(s-1),b(t)}$ and $\Phi^{\mathbb{R}}_{a(s+k),b(s)}$, defined by the real counterparts of the covariant expansions (7.6.14) and (7.6.16), *i.e.*

$$A \ = \sum_{s=2,4,6,\dots} A_{(s)} \ , \tag{7.6.13}$$

$$A_{(s)} \ = \ -i\sum_{t=0}^{s-1} dx^\mu A^{\mathbb{R}}_{\mu,a(s-1),b(t)}(x) M_{\mathbb{R}}^{a_1b_1}\cdots M_{\mathbb{R}}^{a_tb_t} P_{\mathbb{R}}^{a_{t+1}}\cdots P_{\mathbb{R}}^{a_{s-1}} \ . \tag{7.6.14}$$

$$\Phi \ = \sum_{s=0,2,4,\dots} \Phi_{(s)} \ , \tag{7.6.15}$$

$$\Phi_{(s)} \ = \ \sum_{k=0}^{\infty}\frac{i^k}{k!}\Phi^{\mathbb{R}}_{a(s+k),b(s)}(x) M_{\mathbb{R}}^{a_1b_1}\cdots M_{\mathbb{R}}^{a_sb_s} P_{\mathbb{R}}^{a_{s+1}}\cdots P_{\mathbb{R}}^{a_{s+k}} \ , \tag{7.6.16}$$

are real

$$\left(A^{\mathbb{R}}_{\mu,a(s-1),b(t)}(x^\mu)\right)^* \ = \ A^{\mathbb{R}}_{\mu,a(s-1),b(t)}(x^\mu) \ , \qquad (\Phi^{\mathbb{R}}_{a(s+k),b(s)}(x^\mu))^* \ = \ \Phi^{\mathbb{R}}_{a(s+k),b(s)}(x^\mu) \tag{7.6.17}$$

For the compact basis elements M_{rs}, $E = M_{0'0} = P_0$ and $L^\pm_r = M_{0r} \mp iM_{0'r} = M_{0r} \mp iP_r$, which obey the commutation rules (3.2.12) and (3.2.13) in all signatures, the reality conditions read

$$(M_{rs})^\dagger \ = \ M_{rs} \ , \quad E^\dagger \ = \ \sigma_{0'0}E \ , \quad (L^\pm_r)^\dagger \ = \ \sigma_0 L_r^{\mp\sigma_{0'0}} \ , \qquad \sigma_{0'0} = \sigma_{0'}\sigma_0 \ , \tag{7.6.18}$$

that is

$$\mathfrak{so}(D-1,2) \supset \mathfrak{so}(D-1,1) \ : \ (L^\pm_r)^\dagger \ = \ L^\mp_r \ , \quad (E)^\dagger \ = \ E \ , \tag{7.6.19}$$

$$\mathfrak{so}(D,1) \supset \mathfrak{so}(D-1,1) \ : \ (L^\pm_r)^\dagger \ = \ L^\pm_r \ , \quad (E)^\dagger \ = \ -E \ , \tag{7.6.20}$$

$$\mathfrak{so}(D,1) \supset \mathfrak{so}(D) \ : \ (L^\pm_r)^\dagger \ = \ -L^\pm_r \ , \quad (E)^\dagger \ = \ -E \ , \tag{7.6.21}$$

$$\mathfrak{so}(D+1) \supset \mathfrak{so}(D) \ : \ (L^\pm_r)^\dagger \ = \ -L^\mp_r \ , \quad (E)^\dagger \ = \ E \ , \ . \tag{7.6.22}$$

Correspondingly, the compact basis elements obey

$$(T^{(s)}_{e;(s_1,s_2)})^\dagger \;=\; (\sigma_{0'})^{s_1-s}(\sigma_0)^{s-s_2}T^{(s)}_{\sigma_{0'0}e;(s_1,s_2)}\ , \tag{7.6.23}$$

as can be seen from (7.2.3), where $f^{(s)}_{e;(s_1,s_2);n}\in\mathbb{R}$ and

$$(T_{s;n;(s_1,s_2);r(s_1),t(s_2)})^\dagger \;=\; (\sigma_{0'})^{s_1-s}(\sigma_0)^{s-s_2}(\sigma_{0'0})^nT_{s;n;(s_1,s_2);r(s_1),t(s_2)}\ . \tag{7.6.24}$$

Hence, using also (7.2.9), we see that $\Phi^\dagger=\pi(\Phi)$ implies the following conjugation rules for the harmonic-expansion coefficients defined in (7.4.14):

$$(\Phi^{(s)}_{e;(s_1,s_2)})^* \;=\; (-\sigma_{0'})^{s_1-s}(\sigma_0)^{s-s_2}\Phi^{(s)}_{-\sigma_{0'0}e;(s_1,s_2)}\ . \tag{7.6.25}$$

Alternatively, using the "ladder-operator" bases (7.3.51) and (7.3.52) for $\mathcal{W}^\pm$ and $\mathfrak{D}$, we find the following reality conditions in signatures $\mathfrak{so}(D-1,2)$ and $\mathfrak{so}(D+1)$:

$$\mathcal{W}^\pm,\quad s=0\ :\ (\Phi^{\pm(0)}_{m,n})^* \;=\; (-\sigma_{0'})^{\frac{1\mp1}{2}}(-\sigma_0)^{m+n}\Phi^{\pm(0)}_{n,m}\ , \tag{7.6.26}$$

$$\mathcal{W}^\pm,\quad s>0\ :\ (\Phi^{\pm(s)}_{m,n})^* \;=\; (\sigma_0)^{s+\frac{1\mp1}{2}}(-\sigma_0)^{m+n}\Phi^{\pm(s)}_{n,m}\ , \tag{7.6.27}$$

$$\mathfrak{D},\quad s\geq0\ :\ (\Phi^{(s)}_{m})^* \;=\; (\sigma_0)^{s}(-\sigma_0)^{m}\overline{\Phi}^{(s)}_{m}\ , \tag{7.6.28}$$

and in the signature $\mathfrak{so}(D,1)$:

$$\mathcal{W}^\pm,\quad s=0\ :\ (\Phi^{\pm(0)}_{m,n})^* \;=\; (-\sigma_{0'})^{\frac{1\mp1}{2}}(-\sigma_0)^{m+n}\Phi^{\pm(0)}_{m,n}\ , \tag{7.6.29}$$

$$\mathcal{W}^\pm,\quad s>0\ :\ (\Phi^{\pm(s)}_{m,n})^* \;=\; (\sigma_0)^{s+\frac{1\mp1}{2}}(-\sigma_0)^{m+n}\Phi^{\pm(s)}_{m,n}\ , \tag{7.6.30}$$

$$\mathfrak{D},\quad s\geq0\ :\ (\Phi^{(s)}_{m})^* \;=\; (\sigma_0)^{s}(-\sigma_0)^{m}\Phi^{(s)}_{m}\ ,\qquad (\overline{\Phi}^{(s)}_{m})^* \;=\; (\sigma_0)^{s}(-\sigma_0)^{m}\overline{\Phi}^{(s)}_{m}\ , \tag{7.6.31}$$

where we note the phase factors arising from $\pi((T^{(0)}_\pm)^\dagger)=(-\sigma_{0'})^{\frac{1\mp1}{2}}T^{(0)}_\pm$ and $\pi((T^{(s)}_\pm)^\dagger)=(\sigma_0)^{s+\frac{1\mp1}{2}}T^{(s)}_\pm$, and $\pi((T^{(s)}_{s+2\epsilon_0;(s)})^\dagger)=(\sigma_0)^sT^{(0)}_{-\sigma_{0'0}(s+2\epsilon_0);(s)}$. Hence, positive and negative-energy modes are complex conjugated in signatures $\mathfrak{so}(D-1,2)$ and unconjugated in the signature $\mathfrak{so}(D,1)$.

7.7 Inner Products and Unitarity in Two-Time Signature

The bilinear inner product $(\cdot,\cdot)_{\mathcal{T}}=Tr[\pi(\cdot)\star(\cdot)]$ on the covariant twisted-adjoint module $\mathcal{T}$, consisting of the generalized polynomials defined in (7.1.51), is no longer well-defined on $\mathcal{M}$, due to the non-polynomiality of the compact basis. To define a bilinear form

$(\cdot,\cdot)_{\mathcal{M}}$ on $\mathcal{M}$, we thus declare the twisted-adjoint $\mathfrak{ho}_1(D+1;\mathbb{C})$ action to be self-adjoint, *viz.* :

$$(\widetilde{Q}S,S')_{\mathcal{M}} \;=\; -(S,\widetilde{Q}S')_{\mathcal{M}}\;, \tag{7.7.1}$$

for any $Q\in\mathfrak{ho}_1(D+1;\mathbb{C})$, and define $(S_\pm,S'_\mp)_{\mathcal{M}}=0$ and

$$(S_\pm,S'_\pm)_{\mathcal{M}} \;=\; \tfrac{1}{\mathcal{N}_\pm}\mathrm{Tr}[\pi(S_\pm)\star S'_\pm]\;, \tag{7.7.2}$$

where $\mathcal{N}_\pm$ are the norms of the scalar static ground states in $\mathcal{M}^\pm$, *viz.*

$$\mathcal{N}_+ \;=\; \mathrm{Tr}[T^{(0)}_{0;(0)}\star T^{(0)}_{0;(0)}]\;, \tag{7.7.3}$$

$$\mathcal{N}_- \;=\; \tfrac{1}{D-1}\mathrm{Tr}[T^{(0)}_{0;(1);r}\star T^{(0)}_{0;(1);r}]\;. \tag{7.7.4}$$

Thus, to calculate $(S_\pm,S'_\pm)_{\mathcal{M}}$ one first expands $S_\pm$ and $S'_\pm$ using (7.2.34)–(7.2.35) and (7.2.39)–(7.2.40), and then factors out $\mathcal{N}_\pm$ from $Tr[\pi(S_\pm)\star S'_\pm]$ using (7.7.1). By construction, the resulting bilinear form is finite, symmetric (which follows from $\mathrm{Tr}[\pi(X)]=\mathrm{Tr}[X]$) and $\mathfrak{ho}_1(D+1;\mathbb{C})$-invariant. We note the twisted-adjoint energy conservation law:

$$(S_e,S'_{e'})_{\mathcal{M}} \;=\; \delta_{e+e',0}(S_e,S'_e)_{\mathcal{M}}\;, \tag{7.7.5}$$

provided $\widetilde{E}S_e=\{E,S_e\}_\star=eS_e$ *idem* $S'_{e'}$. We also note that $(\cdot,\cdot)_{\mathcal{M}^+}$ simplifies in view of (7.2.45), which implies

$$\mathrm{Tr}[T^{(0)}_{0;(0)}\star Q_e\star T^{(0)}_{0;(0)}\star Q'_{e'}] \;=\; \delta_{e,0}\delta_{e',0}\mathrm{Tr}[T^{(0)}_{0;(0)}\star Q_0\star T^{(0)}_{0;(0)}\star Q'_0]\;, \tag{7.7.6}$$

provided $\mathrm{Ad}_E Q_e=[E,Q_e]_\star=eQ_e$ *idem* $Q'_{e'}$. Under the indecomposition (7.3.43)–(7.3.46), the bilinear form $(\cdot,\cdot)_{\mathcal{M}}$ splits into a non-degenerate inner product on $\mathcal{W}^\pm$, and a completely degenerate bilinear form on $\mathfrak{D}$ and $\mathfrak{D}'$. On $\mathfrak{D}$, we instead define the non-degenerate inner product

$$(S,S')_{\mathfrak{D}} \;=\; \tfrac{1}{\mathcal{N}_{2\epsilon_0}}\mathrm{Tr}[\pi(S)\star S']\;, \tag{7.7.7}$$

by declaring the twisted-adjoint $\mathfrak{ho}_1(D+1;\mathbb{C})$ action to be self-adjoint, and using the state generation (7.3.52) to factor out

$$\mathcal{N}_{2\epsilon_0} \;=\; \mathrm{Tr}[\pi(T^{(0)}_{-2\epsilon_0;(0)})\star T^{(0)}_{2\epsilon_0;(0)}] \;=\; \mathrm{Tr}[T^{(0)}_{2\epsilon_0;(0)}\star T^{(0)}_{2\epsilon_0;(0)}]\;. \tag{7.7.8}$$

The normalizations $\mathcal{N}_\pm$ and $\mathcal{N}_{2\epsilon_0}$, which thus contain all non-polynomialities, can be computed using (7.1.11) and (3.1.58), which implies

$$\mathrm{Tr}[T_{r0(n)}\star T_{s0(n)}] \;=\; \tfrac{1}{2}\mathrm{Tr}[(\mathrm{Ac}_{P_r}T_{0(n)})\star T_{s0(n)}] \;=\; \mathrm{Tr}[T_{0(n)}\star(\mathrm{Ac}_{P_r}T_{s0(n)})] \tag{7.7.9}$$

$$=\; \lambda^{(0)}_{n+1}\delta_{r\{s}\mathrm{Tr}[T_{0(n)}\star T_{0(n)\}_D}] \;=\; \frac{\lambda^{(0)}_{n+1}}{n+1}\,\delta_{rs}\mathrm{Tr}[T_{0(n)}\star T_{0(n)}]\;, \tag{7.7.10}$$

with $\lambda^{(0)}_{n+1}$ given in (3.1.64), and (7.1.11). For the scalar static ground states, whose generating functions are given in (7.2.20) and (7.2.21), we find the non-oscillating series

$$\mathcal{N}_+ = \frac{\epsilon_0}{\epsilon_0+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{n+\epsilon_0+\frac{1}{2}}{(n+\frac{1}{2})(n+2\epsilon_0)}\ , \tag{7.7.11}$$

which is logarithmically divergent for $\epsilon_0>0$, while

$$\begin{aligned}\mathcal{N}_- &= \frac{4\epsilon_0(\epsilon_0+1)}{(\epsilon_0+\frac{1}{2})(\epsilon_0+\frac{3}{2})}\ \times\\ &\times\ \sum_{n=0}^{\infty}\frac{(n+\frac{\epsilon_0}{2}+\frac{3}{4})(n+\frac{\epsilon_0}{2}+\frac{1}{4})(n+\frac{1}{2})(n+1)}{n+\epsilon_0+1}\frac{[(\epsilon_0+\frac{1}{2})_n]^2[(\frac{1}{2})_n]^2}{(\epsilon_0+1)_n(\epsilon_0+2)_n[(2)_n]^2}\ ,\end{aligned} \tag{7.7.12}$$

which is convergent for $\epsilon_0>0$, as can be seen using $(z)_n=\Gamma(z+n)/\Gamma(z)$ and $\Gamma(z)=\sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}(1+\mathcal{O}(z^{-1}))$, which implies that the summand goes like $n^{-2}(1+\mathcal{O}(n^{-1}))$ for large n. For example, in $D=4$

$$\mathcal{N}_+ = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\ , \tag{7.7.13}$$

$$\mathcal{N}_- = \frac{9}{32}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{3}{2})^2}\ . \tag{7.7.14}$$

For the lowest-weight element, whose generating function is given in (7.3.13), we find the oscillating series

$$\begin{aligned}\mathcal{N}_{2\epsilon_0} &= \sum_{n=0}^{\infty}(-1)^n\frac{(\epsilon_0+\frac{3}{2})_n(2\epsilon_0)_n(2\epsilon_0+1)_n}{n!(2)_n(\epsilon_0+\frac{1}{2})_n}\\ &= {}_3F_2(\epsilon_0+\tfrac{3}{2},2\epsilon_0,2\epsilon_0+1;\epsilon_0+\tfrac{1}{2},2;-1)\ .\end{aligned} \tag{7.7.15}$$

To evaluate the hypergeometric function, we use rewrite it as

$$\begin{aligned}{}_3F_2(\epsilon_0+\tfrac{3}{2},2\epsilon_0,2\epsilon_0+1;\epsilon_0+\tfrac{1}{2},2;x) &= \frac{1}{\epsilon_0+\frac{1}{2}}x^{\frac{1}{2}-\epsilon_0}\frac{d}{dx}\left(x^{\frac{1}{2}+\epsilon_0}{}_2F_1(2\epsilon_0,2\epsilon_0+1;2;x)\right)\\ &= \frac{1+x}{(1-x)^{2\epsilon_0+2}}{}_2F_1\left[1-\epsilon_0,\epsilon_0+\tfrac{1}{2};2;-\frac{4x}{(1-x)^2}\right]\\ &+\frac{8(1-\epsilon_0)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-\epsilon_0)\Gamma(1+\epsilon_0)}\frac{x(1+x)}{(1-x)^{2\epsilon_0+4}}{}_2F_1\left[2-\epsilon_0,\epsilon_0+\tfrac{3}{2};\tfrac{3}{2};\left(\frac{1+x}{1-x}\right)^2\right]\\ &-\frac{4\Gamma(\frac{1}{2})}{\Gamma(\epsilon_0+\frac{3}{2})\Gamma(1-\epsilon_0)}\frac{x}{(1-x)^{2\epsilon_0+3}}{}_2F_1\left[1+\epsilon_0,\tfrac{3}{2}-\epsilon_0;\tfrac{1}{2};\left(\frac{1+x}{1-x}\right)^2\right]\ ,\end{aligned} \tag{7.7.16}$$

with the result ($n = 0, 1, 2, \dots$)

$$\mathcal{N}_{2\epsilon_0} = \frac{\Gamma(\frac{1}{2})}{2^{2\epsilon_0+1}\Gamma(1-\epsilon_0)\Gamma(\epsilon_0+\frac{3}{2})} = \begin{cases} (-1)^n \frac{(2n-1)!!}{2^{3n+2}(n+1)!} & \text{for } \epsilon_0 = \frac{1}{2}+n \\ 0 & \text{for } \epsilon_0 = 1+n \end{cases} \tag{7.7.17}$$

In particular, for $D = 4$ we have

$$\mathcal{N}_1 = \frac{1}{4} . \tag{7.7.18}$$

The inner products between real twisted-adjoint elements, obeying the reality condition (7.6.1), are real,

$$(\Phi^\pm, \Phi^{\pm\prime})_{\mathcal{M}} = \tfrac{1}{\mathcal{N}_\pm}\mathrm{Tr}[(\Phi^\pm)^\dagger \star \Phi^{\pm\prime}] = \tfrac{1}{\mathcal{N}_\pm}\mathrm{Tr}[(\Phi^{\pm\prime})^\dagger \star \Phi^\pm] , \tag{7.7.19}$$

$$(\Phi, \Phi')_{\mathfrak{D}} = \tfrac{1}{\mathcal{N}_{2\epsilon_0}}\mathrm{Tr}[\Phi^\dagger \star \Phi'] = \tfrac{1}{\mathcal{N}_{2\epsilon_0}}\mathrm{Tr}[(\Phi')^\dagger \star \Phi] . \tag{7.7.20}$$

Expanding into components using the bases (7.3.51) and (7.3.52), we find that in the signatures $\mathfrak{so}(D-1,2)$ and $\mathfrak{so}(D+1)$,

$$\begin{aligned}(\Phi^\pm, \Phi^{\pm\prime})_{\mathcal{M}} &= \sum_{m,n;m',n'=0}^{\infty} (\Phi^{\pm(0)}_{m,n})^*(\sigma_0)^{m+n}(-\sigma_{0'})^{\frac{1\mp1}{2}} N^{\pm(0)}_{m,n;m',n'}\Phi^{\pm(s)\prime}_{m',n'} \\ &+ \sum_{s=1}^{\infty}\sum_{m,n;m',n'=0}^{\infty} (\Phi^{\pm(s)}_{m,n})^*(\sigma_0)^{m+n}(\sigma_0)^{s+\frac{1\mp1}{2}} N^{\pm(s)}_{m,n;m',n'}\Phi^{\pm(s)\prime}_{m',n'} ,\end{aligned} \tag{7.7.21}$$

$$\begin{aligned}(\Phi, \Phi')_{\mathfrak{D}} &= \sum_{s=0}^{\infty}\sum_{m,m'=0}^{\infty} \Big((\Phi^{(s)}_m)^*(\sigma_0)^{s+m} M^{(s)}_{m,m'}\Phi^{(s)\prime}_{m'} \\ &+ (\Phi^{(s)\prime}_{m'})^*(\sigma_0)^{s+m'} M^{(s)}_{m',m}\Phi^{(s)}_m \Big) ,\end{aligned} \tag{7.7.22}$$

and that in the signature $\mathfrak{so}(D,1)$,

$$\begin{aligned}(\Phi^\pm, \Phi^{\pm\prime})_{\mathcal{M}} &= \sum_{m,n;m',n'=0}^{\infty} (\Phi^{\pm(0)}_{n,m})^*(\sigma_0)^{m+n}(-\sigma_{0'})^{\frac{1\mp1}{2}} N^{\pm(0)}_{m,n;m',n'}\Phi^{\pm(s)\prime}_{m',n'} \\ &+ \sum_{s=1}^{\infty}\sum_{m,n;m',n'=0}^{\infty} (\Phi^{\pm(s)}_{n,m})^*(\sigma_0)^{m+n}(\sigma_0)^{s+\frac{1\mp1}{2}} N^{\pm(s)}_{m,n;m',n'}\Phi^{\pm(s)\prime}_{m',n'} ,\end{aligned} \tag{7.7.23}$$

$$(\Phi, \Phi')_{\mathfrak{D}} = \sum_{s=0}^{\infty}\sum_{m,m'=0}^{\infty} \Big(\overline{\Phi}^{(s)}_m(-1)^m M^{(s)}_{m,m'}\Phi^{(s)\prime}_{m'} + \overline{\Phi}^{(s)\prime}_{m'}(-1)^{m'} M^{(s)}_{m',m}\Phi^{(s)}_m \Big) , \tag{7.7.24}$$

where the inner product matrices

$$M^{(s)}_{m,m'} \;=\; \Big(T^{(s)}_{-(s+2\epsilon_0);(s)},(\widetilde{L}^-)^m(\widetilde{L}^+)^{m'}T^{(s)}_{s+2\epsilon_0;(s)}\Big)_{\mathfrak{D}}\ , \tag{7.7.25}$$

and

$$N^{\pm(s)}_{m,n;m',n'} \;=\; \Big(T^{(s)}_{\pm},(\widetilde{L}^-)^m(\widetilde{L}^+)^n(\widetilde{L}^+)^{m'}(\widetilde{L}^-)^{n'}T^{(s)}_{\pm}\Big)_{\mathcal{M}}\ . \tag{7.7.26}$$

More explicitly, including also the $\mathfrak{so}(D-1)$ vector indices, and using (7.2.32) and (7.2.33), the inner product matrices $M^{(s)}$ and $N^{+(s)}$ read

$$M^{(s)}_{m,n}((s_1,s_2)|(s_1',s_2')) \;=\; \delta_{m,n}\delta_{s_1,s_1'}\delta_{s_2,s_2'}M^{(s)}(p;(s_1,s_2))\ , \tag{7.7.27}$$

where $p=\frac{m+s-s_1-s_2}{2}$ and

$$M^{(s)}(p;(s_1,s_2))^{r'(s_1),t'(s_2)}_{r(s_1),t(s_2)}$$
$$=\Big(T^{(s)}_{-(s+2\epsilon_0);(s);\{r(s)},(\widetilde{x}^-)^p(\widetilde{L}^-_r)^{s_1-s}(\widetilde{L}^-_{t\}})^{s_2}(\widetilde{x}^+)^p(\widetilde{L}^+_{\{r'})^{s_1-s}(\widetilde{L}^+_{t'})^{s_2}T^{(s)}_{s+2\epsilon_0;(s);r'(s)\}}\Big)_{\mathfrak{D}}, \tag{7.7.28}$$

and

$$N^{+(s)}_{m,n;m',n'}((s_1,s_2)|(s_1',s_2')) \;=\; \delta_{m,m'}\delta_{n,n'}\delta_{m+n,s_1+s_2-s}\delta_{s_1,s_1'}\delta_{s_2,s_2'}N^{+(s)}(p,q;(s_1,s_2))\ , \tag{7.7.29}$$

where $p=\frac{m-n+s-s_1-s_2}{2}$ and $q=\frac{n-m+s-s_1-s_2}{2}$, and we note that $n=0$ for $p>0$ and $m=0$ for $q>0$, in order to avoid redundancy, and

$$N^{+(s)}(p,q;(s_1,s_2))^{r'(s_1),t'(s_2)}_{r(s_1),t(s_2)}$$
$$=\begin{cases}\Big(T^{(s)}_{\pm;\{r(s)},\Big((\widetilde{L}^-)^m(\widetilde{L}^+)^n\Big)_{r(s_1-s),t(s_2)\}}\Big((\widetilde{L}^+)^m(\widetilde{L}^-)^n\Big)_{\{r'(s_1-s),t'(s_2)}T^{(s)}_{\pm;r'(s)\}}\Big)_{\mathcal{M}} & \text{for } p,q\leq 0\\ \Big(T^{(s)}_{\pm;\{r(s)},(\widetilde{L}^-)^m_{r(s_1-s),t(s_2)\}}(\widetilde{L}^+)^m_{\{r'(s_1-s),t'(s_2)}T^{(s)}_{\pm;r'(s)\}}\Big)_{\mathcal{M}} & \text{for } p>0\\ \Big(T^{(s)}_{\pm;\{r(s)},(\widetilde{L}^+)^n_{r(s_1-s),t(s_2)\}}(\widetilde{L}^-)^n_{\{r'(s_1-s),t'(s_2)}T^{(s)}_{\pm;r'(s)\}}\Big)_{\mathcal{M}} & \text{for } q>0\end{cases} \tag{7.7.30}$$

the above inner product matrices do not depend on the signature. The matrices $M^{(s)}$ are positive definite[10], and from (7.7.22) and (7.7.24) it follows that $(\cdot,\cdot)_{\mathfrak{D}}$ is positive definite (only) in the signature $\mathfrak{so}(D-1,2)$. We conjecture that also the matrices $N^{\pm(s)}$ are positive definite, so that also $(\cdot,\cdot)_{\mathcal{M}}$ is positive definite on $\mathcal{W}^{\pm}$ in the signature $\mathfrak{so}(D-1,2)$. For

[10]The complex space $\mathfrak{D}^+(e_0;(s))\oplus\mathfrak{D}^-(e_0;(s))$, where $e_0=2\epsilon_0+s$, can be equipped with the Hilbert-space inner product

$$\widehat{M}(e;(s_1,s_2)|e';(s_1',s_2')) \;=\; (|e;(s_1,s_2)\rangle)^\dagger\,|e';(s_1',s_2')\rangle\ , \tag{7.7.31}$$

example, in the case of $N^{+(0)}$ in $D=4$, the elements in $\mathcal{W}^{=}_{(0)}$ have $s_2=0$ and $p,q\leq 0$, and thus

$$D=4\quad:\quad N^{+(0)}_{m,n;m',n'}((s_1)|(s_1'))\;=\;\delta_{m,m'}\delta_{n,n'}\delta_{m+n,s_1}\delta_{s_1,s_1'}N^{+(0)}(p,q;(m+n))\tag{7.7.36}$$

where the matrix elements

$$\begin{aligned} N^{+(0)}(p,q;(m+n))^{r'(m+n)}_{r(m+n)} &= \left(T^{(0)}_{+},\left((\widetilde{L}^{-})^{m}(\widetilde{L}^{+})^{n}\right)_{\{r(m+n)\}}\left((\widetilde{L}^{+})^{m}(\widetilde{L}^{-})^{n}\right)^{\{r'(m+n)\}}T^{(0)}_{\pm}\right)_{\mathcal{M}} \\ &= \delta^{\{r'(m+n)\}}_{\{r(m+n)\}}N_{m,n}\ , \end{aligned}\tag{7.7.37}$$

for coefficients

$$N_{m,n}\;=\;\frac{1}{\dim(m)}\left(T^{(0)}_{+},\left((\widetilde{L}^{-})^{m}(\widetilde{L}^{+})^{n}\right)_{\{r(m+n)\}}\left((\widetilde{L}^{+})^{m}(\widetilde{L}^{-})^{n}\right)^{\{r(m+n)\}}T^{(0)}_{\pm}\right)_{\mathcal{M}}\tag{7.7.38}$$

with $\dim(m)=\delta^{\{r(m)\}}_{\{r(m)\}}=2m+1$ being the dimension of the type (m) irrep of $\mathfrak{so}(3;\mathbb{C})$. Positive definiteness amounts to that $N_{m,n}$ are strictly positive for all m and n. Using (7.7.6) and

$$\widetilde{L}^{+}_{r}\widetilde{L}^{-}_{r}T^{(0)}_{0;(0)}\;=\;\widetilde{L}^{-}_{r}\widetilde{L}^{+}_{r}T^{(0)}_{0;(0)}\;=\;2T^{(0)}_{0;(0)}\ ,\tag{7.7.39}$$

we have found that this is indeed the case for the following low-lying levels:

$$N_{1,0}\;=\;\frac{4}{3}\ ,\qquad N_{2,0}\;=\;8\ ,\qquad N_{1,1}\;=\;\frac{32}{15}\ ,\tag{7.7.40}$$

although we have no proof to all levels.

assuming the following forms in various signatures:

$$\mathfrak{so}(D-1,2)\supset\mathfrak{so}(D-1,1)\quad:\quad\widehat{M}\;=\;\begin{bmatrix} M & 0\\ 0 & M\end{bmatrix}\ ,\tag{7.7.32}$$

$$\mathfrak{so}(D,1)\supset\mathfrak{so}(D-1,1)\quad:\quad\widehat{M}\;=\;\begin{bmatrix} 0 & M\\ M & 0\end{bmatrix}\ ,\tag{7.7.33}$$

$$\mathfrak{so}(D,1)\supset\mathfrak{so}(D)\quad:\quad\widehat{M}\;=\;\begin{bmatrix} 0 & (-1)^{-(e-e_0)}M\\ (-1)^{e-e_0}M & 0\end{bmatrix}\ ,\tag{7.7.34}$$

$$\mathfrak{so}(D+1)\supset\mathfrak{so}(D)\quad:\quad\widehat{M}\;=\;\begin{bmatrix} (-1)^{e-e_0}M & 0\\ 0 & (-1)^{-(e-e_0)}M\end{bmatrix}\ ,\ ,\tag{7.7.35}$$

where M is given in (7.7.27), and $\pm(e-e_0)$ is the total number of $L^{\pm}_{r}$ operators applied to the ground states $|\pm e_0;(s)\rangle^{\pm}$. We notice that $\widehat{M}$ couples states with the same energy in signatures $\mathfrak{so}(D-1,2)$ and $\mathfrak{so}(D+1)$ (where $(|\pm e_0;(s)\rangle^{\pm})^{\dagger}={}^{\pm}\langle\pm e_0;(s)|$), and with the opposite energy in signature $\mathfrak{so}(D,1)$ (where $(|\pm e_0;(s)\rangle^{\pm})^{\dagger}={}^{\mp}\langle\mp e_0;(s)|$), and that $\widehat{M}$ is positive definite only in signature $\mathfrak{so}(D-1,2)$.

7.8 Supersingleton and Oscillator Realization in $D = 4$

A particularly simple realization of a composite reflector can be given, in four dimensions, in terms of the states belonging to the scalar and spinor singletons. In other words, one can make use of the 4D oscillator realization (3.3.35) defined in Subsection 3.3.2 and define the composite reflector state $|1\!\!1\rangle_{\mathfrak{D}_0\oplus\mathfrak{D}_{1/2}}$ as the reflection of the identity operator on the Fock space (3.3.38), that contains both the scalar and the spinor singleton.

We begin by defining the action of the reflection map on the vacuum $|0\rangle \equiv |1/2, 0\rangle$ and on the oscillators:

$$R(|0\rangle) \;=\; \langle 0| \;, \qquad R(|0\rangle^-) \;=\; {}^-\langle 0| \;, \tag{7.8.1}$$

$$R(a^{\dagger\, i}) \;=\; i a^i \;, \qquad R(a^i) \;= i a^{\dagger\, i} \;. \tag{7.8.2}$$

One can check that R is an antiautomorphism of the oscillator algebra (3.3.35). On a state $|n\rangle = a^{\dagger\, i_1}...a^{\dagger\, i_n}|0\rangle$ it acts as

$$R(|n\rangle) \;=\; i^n \langle 0| a^{i_1}...a^{i_n} \;, \tag{7.8.3}$$

and similarly for anti-singleton states

$$R(|n\rangle^-) \;= i^{n\,-}\langle 0| a^{\dagger\, i_1}...a^{\dagger\, i_n} \;. \tag{7.8.4}$$

We note that the variable z, defined in (3.1.41), is

$$z \;=\; 2L_r^+ L_r^- \;=\; 2\frac{i^2}{4}(\sigma_r)_{ij}(\sigma_r)^{kl} a^{\dagger\, i} a^{\dagger\, j} a_k a_l \;=:\; (a^{\dagger\, i} a_i)^2 \;:\; . \tag{7.8.5}$$

Thus, from the Fock-space point of view, eqs. (7.5.49-7.5.55) can be rewritten as

$$1\!\!1_{\mathfrak{D}_0} \;=\; 1\!\!1_{\mathcal{F}_{even}} \;=\; \sum_{\text{n even}} |n\rangle\langle n| \;=\; \sum_{\text{n even}} \frac{1}{n!} a^{\dagger\, i_1}\dots a^{\dagger\, i_n}|0\rangle\langle 0| a_{i_1}\dots a_{i_n} \;, \tag{7.8.6}$$

or

$$1\!\!1_{\mathcal{F}_{even}} \;=:\; \sum_{\text{n even}} \frac{1}{n!} N^n |0\rangle\langle 0| \;:\, =:\; \cosh N |0\rangle\langle 0| \;:\; , \tag{7.8.7}$$

where $N = a^{\dagger\, i} a_i = \sqrt{z}$. Analogously, the decomposition of the reflector over spinor singleton states, that leads to (7.5.76), can be expressed as

$$1\!\!1_{\mathfrak{D}_{1/2}} \;=\; 1\!\!1_{\mathcal{F}_{odd}} \;=:\; \sum_{\text{n odd}} \frac{1}{n!} N^n |0\rangle\langle 0| \;:\, =:\; \sinh N |0\rangle\langle 0| \;:\; . \tag{7.8.8}$$

One can now combine these two into the identity over the entire Fock module, obtaining

$$\mathbb{1}_{\mathfrak{D}} \;=\; {}^{\times}_{\times}\, e^{\sqrt{z}}|1/2,0\rangle\langle 1/2,0|\,{}^{\times}_{\times}\;, \tag{7.8.9}$$

that can be reinterpreted simply as the completeness of the Fock-space basis of states,

$$\mathbb{1}_{\mathfrak{D}} \;=\; \mathbb{1}_{\mathcal{F}} \;=\; \sum_{n=0}^{\infty}|n\rangle\langle n| \;=:\; \sum_{n=0}^{\infty}\frac{1}{n!}N^n|0\rangle\langle 0| \;:=:\; e^N\,|0\rangle\langle 0| \;:\;. \tag{7.8.10}$$

The latter equations indeed simply amounts to the fact that the Fock-space vacuum-to-vacuum projector $|0\rangle\langle 0|$ admits the realization $:\, e^{-N}\, :$. On the other hand, (7.8.7) and (7.8.8) give the definitions of the projectors onto $\mathcal{F}_{even}$ and $\mathcal{F}_{odd}$, respectively,

$$\mathbb{1}_{\mathcal{F}_{even}} = P_+ \;=\; \frac{1}{2}(1+\Gamma)\;, \qquad \mathbb{1}_{\mathcal{F}_{odd}} = P_- \;=\; \frac{1}{2}(1-\Gamma)\;, \tag{7.8.11}$$

$$P_\pm \star P_\pm \;=\; P_\pm\;, \tag{7.8.12}$$

where

$$\Gamma \;=:\; e^{-2N}\;:\;, \qquad \Gamma\star\Gamma \;=\; 1 \tag{7.8.13}$$

(see Appendix I). The reflection of (7.8.9) gives the combination of scalar excitations in the doubleton basis $\mathfrak{D}_0^{\otimes 2}\oplus\mathfrak{D}_{1/2}^{\otimes 2}$ onto which the identity operator in the twisted adjoint representation can be mapped,

$$|\mathbb{1}\rangle \;=\; e^{iy}|1/2,0\rangle_1|1/2,0\rangle_2\;, \tag{7.8.14}$$

where we have defined $y\equiv\sqrt{x}=a_i^\dagger(1)a^{\dagger\, i}(2)$.

The standard inner product on the Fock space induces the trace operation Tr. The important feature of such realization is that one can normalize the reflector appearing in the last equation. As reviewed in Appendix I for the simpler case of a single oscillator, to any consistent inner-product law for the Fock module is associated a corresponding trace operation in the space of operators f acting on that module. Now, there are two possible trace operations which are consistent with the oscillator algebra, namely

$$\mathrm{Tr}_\pm(f) \;=\; \mathrm{Tr}_{\mathfrak{D}_0}(f)\pm\mathrm{Tr}_{\mathfrak{D}_{1/2}}(f) \tag{7.8.15}$$

(note that these definitions are analogous to those given in Appendix I), where

$$\mathrm{Tr}_{\mathfrak{D}_0}(f) \;=\; \sum_{k=0}^{\infty}\langle 2k|f|2k\rangle\;, \tag{7.8.16}$$

denotes the trace on the scalar singleton states, and

$$\mathrm{Tr}_{\mathfrak{D}_{1/2}}(f) \;=\; \sum_{k=0}^{\infty}\langle 2k+1|f|2k+1\rangle\;, \tag{7.8.17}$$

the trace on the spinor singleton states. This means that the trace of the identity operator amounts to a sum of the multiplicities of each state $|n\rangle$ (that has spin $s = n/2$), *i.e.*

$$\begin{aligned} \mathrm{Tr}_{\mathfrak{D}_0}(\mathbb{1}) &= \sum_{k=0}^{\infty}(2k+1)\ , \\ \mathrm{Tr}_{\mathfrak{D}_{1/2}}(\mathbb{1}) &= \sum_{k=0}^{\infty}(2k+1+1)\ . \end{aligned} \tag{7.8.18}$$

This can be summarized as

$$\mathrm{Tr}_{\pm}(\mathbb{1}) \ = \ \sum_{n=0}^{\infty}(\pm 1)^n\,(n+1) \ = \ \begin{cases} \infty\ , & \text{for } \mathrm{Tr}_+ \\ \frac{1}{4} & \text{for } \mathrm{Tr}_- \end{cases}\ ,$$

Notice that only Tr_- leads to a finite result, and for this to happen is also necessary to compute Tr_- over the full Fock space $\mathfrak{D}_0 \oplus \mathfrak{D}_{1/2}$, while the traces on $\mathfrak{D}_0$ and $\mathfrak{D}_{1/2}$ only are necessarily divergent. Moreover, as shown in Appendix I, for an arbitrary element of the oscillator algebra $f = f(a_i, a^{\dagger\, i})$, $\mathrm{Tr}_-(f)$ actually coincides with $\mathrm{Tr}_+((-1)^N_\star \star f)$, and the latter in its turn coincides with the supertrace operation $\mathrm{Str}(f) = f(0,0)$ defined in [63], up to an overall factor. With two oscillators, the precise relation is

$$\mathrm{Tr}_-(f) \ = \frac{1}{4}\,\mathrm{Str}(f) \ = \ \frac{1}{4}\,f(0,0)\ . \tag{7.8.19}$$

7.9 Divergencies in the Perturbative Expansion and a Proposal for Their Regularization

As we have already stressed, the map constructed above establishes a correspondence between the master zero-form at a point in space-time and the physical fluctuation fields with definite energy and $\mathfrak{so}(3)$-spin. In doing so, we have mapped fluctuations in field strengths and scalar fields to nonpolynomial combinations of twisted adjoint elements. At this point, however, a potential subtlety arises. As remarked in Section 5.2, the $\star$-product of nonpolynomial generators will in general give rise to divergencies.

Indeed, the component-field expansions of the master fields A and Φ are formal sums that are not subject to any convergence criteria. Once the master fields are constrained on-shell (in the context of the weak-field expansion) the component fields $A_{\mu,a(s-1),b(t)}$ and $\Phi_{a(s+k),b(s)}$ become identified with various *non-linear and higher-derivative* constructs built from the physical scalar ϕ, metric $g_{\mu\nu}$ and higher-spin gauge tensor gauge fields $\varphi_{\mu_1\ldots\mu_s}$ $(s = 4, 6, \ldots)$ defined in (3.1.87), (3.1.78) and (3.1.79), respectively. There is nothing, in principle, that prevents the resulting (full) master fields A and Φ from having potentially divergent $\star$-product compositions[11]. Such divergencies do indeed arise in the composition $\pi(\Phi)\star\Phi$ of linearized solutions (given by harmonic expansions), possibly jeopardizing the weak-field expansion. However, due to the detailed nature of Vasiliev's equations (having to do with doubling of oscillators) it is possible that these divergencies do not affect the actual higher-order corrections to the full master fields. In other words, there is some evidence that the weak-field expansion of Vasiliev's equations gives rise to well-defined classical interactions vertices among states (one-particle as well as static states) with fixed compact quantum numbers.

To understand why this can indeed be the case, we will use as a prototype example the self-interaction of the lowest weight element $T^{(0)}_{(2\epsilon_0);(0)}$ in $D=4$, that has the simple form (7.3.15), *i.e.* $\Phi' \equiv \Phi|_p = e^{-4E}$. At the second order in perturbation expansion, the Z-dependence of the zero-form master field is given by

$$\partial_\alpha \widehat{\Phi}'^{(2)} \;=\; -\widehat{A}'^{(1)}_\alpha \star \Phi' + \Phi' \star \pi(\widehat{A}'^{(1)}_\alpha) \equiv j^{(2)}_\alpha(z) \; . \tag{7.9.1}$$

The source term is of second order in Φ', since

$$\widehat{A}'^{(1)}_\alpha \;=\; z_\alpha \int_0^1 dt\, \Phi'(-zt,\bar{y}) e^{ity^\alpha z_\alpha} \; , \tag{7.9.2}$$

and in particular, substituting $\Phi'(y,\bar{y}) = \exp((\sigma_0)^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}})$ in the equation below, one finds

$$\begin{aligned} \widehat{A}'^{(1)}_\alpha \star \Phi' \;&=\; \frac{\partial}{\partial\rho^\alpha}\int_0^1 dt\, \left(e^{ity^\alpha z_\alpha + \rho^\alpha z_\alpha}\Phi'(-zt,\bar{y})\star\Phi(y,\bar{y})\right) \\ &\sim\; \sum_{n=0}^{\infty}(n+\frac{1}{2})t^n + \ldots \;=\; \frac{1}{2}\frac{1+t}{(1-t)^2} + \ldots \end{aligned} \tag{7.9.3}$$

where we only write the coefficient of the identity. This shows a divergency in the upper bound of the integration domain, $t=1$. However, being this an isolated singularity, one can apply the method of regularization proposed in [97], and obtain a well-defined weak-field expansion by circumventing the singularities that arise in the perturbative expansion using a closed integration contour γ as

$$\widehat{\Phi}'^{(n)} \;=\; \oint_\gamma \frac{dt}{2\pi i} \ln\left(\frac{t}{1-t}\right) z^\alpha j^{(n)}_\alpha(zt) \; , \tag{7.9.4}$$

[11] Divergencies were encountered recently in [99], where exact solutions of the non-minimal model based on $\mathfrak{ho}_1(D+1;\mathbb{C})$ were constructed using projectors $P\in\mathcal{A}$ obeying $P\star P = P$. These solutions would admit a consistent truncation to the minimal model provided $\tau(P)=P$. However, the projectors used in [99], built using Fock-space methods, are not τ-invariant. The reason is that the naive attempt to impose the τ-condition by replacing P by $P+\tau(P)$ leads to divergencies residing in $P\star\tau(P)$ that remain unresolved at the level of the full Vasiliev equations (although, mathematically speaking, they are related to the those arising in the harmonic expansion).

where γ encircles the branch cut from $t = 0$ to $t = 1$ (similarly for $A'^{(n)}_{\alpha}$, see [97])[12]. This more general presentation of the weak-field expansion allows for singular initial data for the evolution along Z. In the case of regular initial data, on the other hand, the closed-contour can be collapsed onto the branch cut as to reproduce the open-contour presentation.

An important subtlety however still needs to be resolved. Indeed, at every order in the perturbative expansion one has to make sure that the source terms $j^{(n)}_{\alpha}(z)$ are finite, and this request fixes γ. But at higher orders, the isolated singularity is pushed further and further away from the origin of the complex t-plane, and the necessity of encircling it may lead to losing the associativity of the $\star$-product composition, which is in its turn crucial for the integrability of the equations. We hope this issue admits a resolution, which we shall elaborate further on elsewhere.

[12]Notice that the presence of the t-dependent exponential $e^{ity^{\alpha}z_{\alpha}}$ in the sources leads to an essential singularity at $t = \infty$, which always prevents the closed-contour integrals from being trivial.

Chapter 8

Conclusions and Outlook

In this Thesis, I have reviewed in some detail the main features of interacting Higher-Spin Gauge Theories focusing on the Vasiliev equations, and presented some original results concerning the structure of the gauge algebra on which they are based and some new exact solutions. As it can be appreciated from the review part of this work, the study of HS gauge fields has made some very significant progress in the last twenty years, and at the same time the importance of a better understanding of their dynamics has grown considerably also due to the developments in different research fields, and most notably in String Theory. As a result, HSGT as we know them today involve a number of physical and formal tools and concepts that are not only fascinating and promising in themselves, but also of potential interest to other fields at the forefront of research in High-Energy Theoretical Physics.

The Vasiliev equations are arguably the most important achievement in HSGT obtained so far, and are in fact the only known consistent set of equations encoding the full dynamics of massless HS fields (at least in four dimensions, while for higher dimensions this statement is so far limited to totally symmetric tensors). As we have seen, they encode a very complicate dynamics into a few elegant curvature constraints according to the unfolded formulation. The latter has been the key to overcome the main obstacles to the formulation of a consistent nonlinear theory of HS fields, and also enables a uniform treatment of higher-derivative couplings together with, importantly, a background-independent description. On the other hand, a conventional action principle from which the Vasiliev equation descend is not known, at present, and in the unfolded scheme it might not be easy to recover certain results that are instead more readily within reach in the known low-spin Lagrangian Field Theory. Moreover, making contact with the known lower-spin gauge theories is also not easy, at present: this is due to the fact that no consistent truncation of the equations down to the lower-spin sector is known, as lower-spin fields serve as sources for higher-spin fields, and some mechanism of spontaneous breaking of the full HS symmetry must be known before one can turn off the couplings to HS fields consistently. However, as we hope we have made clear in this Thesis, the Vasiliev system offers other important windows on the peculiar features of HS dynamics, and we believe one should exploit such possibilities as much as possible.

For example, although the full equations in space-time are a system of formidable complexity, and indeed are not even known in closed form at present, we can nonetheless find exact solutions by exploiting their relatively simple form in the extended space (x, Z). Indeed, the most important feature of the unfolded approach is that, roughly speaking, it enables a trading of the space-time evolution for the fiber evolution, and this makes it possible not only to write the field equations in a background-independent way, but also to solve them by means of purely algebraic methods. For example, as we have explicitly shown in Chapter 6, imposing symmetries on the zero-form, may simplify a lot the form of the the fiber equations. Moreover, a large class of solutions to the latter can be found by using projectors of the gauge algebra. The methods developed for the construction of the exact solutions found so far [97, 99, 98] are likely to be useful to find new ones, and in general the homotopy invariance of Vasiliev's equations gives some hope in this sense. Of particular interest is the research of a spherically symmetric solution, as well as of black hole solutions. For example, the exact solution that describes the embedding of a BTZ black hole in the three-dimensional HSGT (in which, however, higher-spin fields do not carry local degrees of freedom), found in [98], might be elevated to the dynamically more interesting case of $D = 4$.
The importance of such issues actually goes beyond the realm of higher spins. Since any system admits an unfolded reformulation, one may even speculate that such algebraic methods can be of use also in ordinary gravity, and that solutions could be obtained *algebraically* starting from the knowledge of the Weyl zero-form at a point in space-time.

In this Thesis, starting from HS gauge theories in four dimensions based on infinite dimensional extensions of $SO(5;\mathbb{C})$, we have determined their real forms in spacetimes with Euclidean $(4, 0)$ and Kleinian $(2, 2)$ signature, in addition to the usual Lorentzian $(3, 1)$ signature. We have then found three new types of solutions in addition to the maximally symmetric ones.
Type 1 solutions, which are invariant under an infinite dimensional extension of $SO(4-p, p)$, give us a nontrivial deformation of the maximally symmetric solutions, and depend on a continuous real parameter as well as on an infinite set of discrete parameters. Interestingly, a particular choice of the discrete parameters, in the limit of vanishing continuous parameter, gives rise to a degenerate, indeed rank one, metric. Given that degenerate metrics are known to play an important role in topology change in quantum gravity [94], it is remarkable that such metrics emerge naturally in HS gauge theory.
Type 2 solutions, which provide another kind of deformation of the maximally symmetric solutions, have a non-vanishing spinorial master one-form.
Type 3 solutions are particularly remarkable because all the higher spin fields are non-vanishing, and the corresponding Weyl tensors furnish a higher spin generalization of Type

D gravitational instantons. It would be interesting to apply the framework we have used in this paper to finding pp-wave, black hole and domain wall solutions with non-vanishing HS fields.
We stress that our models in Euclidean and Kleinian signatures are formulated using the 4D spinor-oscillator formulation. It would be interesting to compare these models to the vector-oscillator formulation [72, 52], which exists in any dimension and signature, and relies on the gauging of an internal $\mathfrak{sp}(2)$ gauge symmetry (as briefly discussed in Chapter 3). At the full level, the vector-oscillator master field equations, in any dimension and signature, are formulated using a *single* $\mathfrak{sp}(2)$-doublet Z-oscillator, leaving, apparently, no room for parity violating interactions. The precise relation between the spinor and vector-oscillator formulations in $D = 4$ therefore deserve further study.
In the context of supersymmetric field theories, including supergravity, the non-Lorentzian signature typically presents obstacles, since the spinor properties are sensitive to the space-time signature. Here, however, we have considered bosonic HS gauge theories in which the spinor oscillators play an auxiliary role, and we have formulated the non-Lorentzian signature theories with suitable definition of the spinors without having to face such obstacles. Remarkably, non-supersymmetric 4D theories in Kleinian signature describing self-dual gravity arise in worldsheet $N = 2$ supersymmetric string theories, known as $N = 2$ strings. For reasons mentioned in the Introduction to this Chapter, it is an interesting open problem to find a niche for Kleinian HS gauge theory in a variant of an $N = 2$ string.

There are several other open problems that deserve investigation. To begin with, we have not determined the symmetries of Type 2 and Type 3 solutions. moreover, as we have seen, some of the solutions found so far admit an interesting cosmological interpretation. However, the latter is not straightforward and, at present, cannot be carried out in a HS covariant fashion. Indeed, there is no HS-invariant line element to describe the geodesic motion of test particles and, consequently, to define the notions of horizons and singularities in a sensible way. In order to do this, and to be able to characterize the solutions physically, it would be of extreme importance to extend the set of invariants under the infinitely many symmetries of HSGT. To date, a partial set of invariants (6.3.118) has been constructed [97] only in terms of the master zero-form, while no observable that involves the master one-form has been found (although it is clear how to to extend the construction to the case of one-forms). Moreover, while it may be useful in its own right to determine whether our Type 3 solutions support a complex, possibly Kähler, structure up to a conformal scaling, such results may be limited in shedding light to the geometry associated with infinitely many gauge fields present in HS gauge theory. A proper formulation of the HS geometry would also provide a framework for constructing the above-mentioned invariants that could distinguish the gauge inequivalent classes of exact solutions.
It would also be interesting to study the fluctuations about our exact solutions, and ex-

plore their potential application in quantum gravity and cosmology. Similarities between the frameworks for studying instanton and soliton solutions of the noncommutative field theories (see, for example, [130]), and in particular open string field theory, are also worth investigating.

As we have stressed, there are many reasons why it would be desirable to gain a better understanding of the unfolded formulation and of the structure of the HS algebras. With such motivations in mind, in Chapter 7 we have elaborated further on them, and on the representations that are of relevance to the Vasiliev equations. In particular, an analysis of the physical content of the Vasiliev system was developed, which is valid for arbitrary signature, and is somehow in the spirit of the unfolded formulation. Indeed, as the study of the chiral model in Chapter 6 puts in greater evidence, the field-strengths are the natural place where to look for the physical degrees of freedom, and this raises the issue of examining carefully how the local data is encoded in the zero-form master-field. This is investigated in Chapter 7 in some detail and in a general way, and normalizable and non-normalizable fluctuation fields carrying definite energy and spin quantum numbers are shown to emerge, *a priori*, from the unfolding of the local data. In particular, some nonperturbative solutions to the linearized equations originally found in [97] are nicely seen to come out from this analysis. Although such states are nonunitary in the standard inner product in AdS [133], they appear to be unitary in the inner product $(.,.)_{\mathcal{M}}$ based on the trace operation defined in Chapter 7. It is however too early to understand what this means in a quantum theory.

Moreover, the mapping developed in Chapter 7 might admit simple extensions to the case of other interesting representations that we have not treated here, namely massive and partially massless fields in maximally symmetric space-times with nonvanishing cosmological constant. A lagrangian formulation is known for the free massive case of arbitrary spin, and massive fields are known to be related to the tensor product of three or more singletons (although a complete classification is not available, at present). An appropriate generalization of the map in [134] to multipletons might give an indication on the relevant master-fields that would enter an unfolded formulation of massive fields. Moreover, in the limit in which the mass of fields with spin $s \geq 2$ becomes proportional to the cosmological constant, with some precise real or imaginary factors, the theory acquires a partial gauge invariance under transformation that have a lower rank parameter, compared to the massless case, and at least two derivatives. Such fields are called "partially massless", and are peculiar to space-times with nonvanishing cosmological constant. Although both a lagrangian and an unfolded description of such fields are available (see, for example, [137] and references therein), to the best of our knowledge a group-theoretical one is still unclear. The study carried on in Chapter 7 might shed some light on the corresponding representations of the background isometry algebra and on the way they can be related to its irreducible representations and their negative-energy counterparts. For example, it might be the case that partially massless representations arise from the decomposition of the tensor product of the p-linetons presented in Chapter 7, through some analog of the Flato-Fronsdal formula. We plan to study such issue in a future work.

Finally, let us mention that some other partial results, obtained in collaboration with A. Sagnotti and P. Sundell [138], seem to point towards the possible existence of a non-standard action principle for the Vasiliev equations. Although of BF type (an action of this form was first considered in [79]), it seems that it can encode the correct local degrees of freedom by virtue of the properties of unfolded systems. Moreover, the first attempts at a quantization of a simple field theoretical model formulated in this way have given some positive results. Although many related issues are unclear to us at present, a partial analysis of the features of this "unfolded action" is encouraging, and shows once more that a deeper understanding of the unfolded formulation would be of relevance also for lower-spin gauge theories.

Appendix A

Gauging space-time symmetries

The usual Einstein-Hilbert action $S[g]$ is invariant under diffeomorphisms. The same is true for $S[e, \omega]$, defined by (2.2.4), since everything is written in terms of differential forms. The action (2.2.4) is also manifestly invariant under local Lorentz transformations $\delta\omega = d\epsilon + [\omega, \epsilon]$ with gauge parameter $\epsilon = \epsilon^{ab} M_{ab}$, because $\epsilon_{a_1 \ldots a_d}$ is an invariant tensor of $SO(D-1,1)$. The gauge formulation of gravity shares many common features with a Yang-Mills theory formulated in terms of a connection ω taking values in the Poincaré algebra.

However, gravity is actually *not* a Yang-Mills theory with Poincaré as (internal) gauge group. The aim of this section is to express precisely the distinction between internal and space-time gauge symmetries.

To warm up, let us mention several obvious differences between Einstein-Cartan's gravity and Yang-Mills theory. First of all, the Poincaré algebra $\mathfrak{iso}(D-1,1)$ is not semisimple (since it is not a *direct* sum of simple Lie algebras, containing a nontrivial ideal spanned by translations). Secondly, the action (2.2.4) cannot be written in a Yang-Mills form $\int Tr[F^* F]$. Thirdly, the action (2.2.4) is not invariant under the gauge transformations $\delta\omega = d\epsilon + [\omega, \epsilon]$ generated by *all* Poincaré algebra generators, *i.e.* with gauge parameter $\epsilon(x) = \epsilon^a(x) P_a + \epsilon^{ab}(x) M_{ab}$. For $D > 3$, the action (2.2.4) is invariant only when $\epsilon^a = 0$. (For $D = 3$, the action (2.2.4) describes a genuine Chern-Simons theory with local $ISO(2,1)$ symmetries.)

This latter fact is not in contradiction with the fact that one actually gauges the Poincaré group in gravity. Indeed, the torsion constraint allows one to relate the local translation parameter ϵ^a to the infinitesimal change of coordinates parameter ξ^μ. Indeed, the infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ acts as the Lie derivative

$$\delta_\xi = \mathcal{L}_\xi \equiv i_\xi d + d i_\xi \,,$$

where the inner product i is defined by

$$i_\xi \equiv \xi^\mu \frac{\partial}{\partial(dx^\mu)}\,,$$

where the derivative is understood to act from the left. Any coordinate transformation of the frame field can be written as

$$\delta_\xi e^a = i_\xi(de^a) + d(i_\xi e^a) = i_\xi T^a + \underbrace{\epsilon^a{}_b e^b + D^L \epsilon^a}_{=\delta_\epsilon\, e^a}\,,$$

where the Poincaré gauge parameter is given by $\epsilon = i_\xi \omega$. Therefore, when T^a vanishes any coordinate transformation of the frame field can be interpreted as a local Poincaré transformation of the frame field, and reciprocally.

To summarize, the Einstein-Cartan formulation of gravity is indeed a fibre bundle construction where the Poincaré algebra $\mathfrak{iso}(D-1,1)$ is the fiber, ω the connection and R the curvature, but, unlike for Yang-Mills theories, the equations of motion imposes some constraints on the curvature ($T^a = 0$), and some fields are auxiliary (ω^{ab}). A fully covariant formulation is achieved in the AdS case with the aid of compensator formalism as explained in Section 2.2.1.

Appendix B

Details of the Procedure of Factoring out the Ideal $\mathcal{I}[V]$

In this Appendix we present some of the details of the procedure of factoring out the ideal $\mathcal{I}[V]$ defined in (3.1.9) from the enveloping algebra $\mathcal{U}$ of $\mathfrak{so}(D+1;\mathbb{C})$ defined in (3.1.4).

In general, suppose that $\mathcal{I}[V] = V \star \mathcal{U}$ for an $\mathfrak{so}(D+1;\mathbb{C})$ irreducible element $V = \lambda^{A_1,\dots,A_n} V_{A_1,\dots,A_n}$, where $\lambda^{A_1,\dots,A_n} \in \mathbb{C}$ and $V_{A_1,\dots,A_n}$ is a (Young projected) monomial built from M_{AB} and η_{AB}, thus obeying

$$[M_{BC}, V_{A_1,\dots,A_n}]_\star \;=\; 2i\eta_{A_1[C}V_{B],A_2,\dots,A_n} + \cdots + 2i\eta_{A_n[C|}V_{A_1,\dots,A_{n-1},|B]}\ . \qquad \text{(B.0.1)}$$

It follows that if $X \in \mathcal{U}$, then $X \star V = V \star X'$ for some $X' \in \mathcal{U}$, so that the space $\mathcal{I}[V]$ is a both a left and right ideal in $\mathcal{U}$, and it is equivalent to the right ideal generated by right-multiplication by V, *i.e.* $\mathcal{I}[V] = V \star \mathcal{U} = \mathcal{U} \star V$. Thus, one may use the notation

$$X \star V \star X' \;\simeq\; 0 \qquad \text{for all } X, X' \in \mathcal{U}\ . \qquad \text{(B.0.2)}$$

The above considerations can be extended straightforwardly to the case that V is reducible.

Turning to the specific case of V given by (3.1.10) and (3.1.11), let us we show their equivalence to (3.1.49). First, by decomposing $V_{AB} \simeq 0$ and $V_{ABCD} \simeq 0$ under $\mathfrak{so}(D;\mathbb{C})$ one immediately arrives at (3.1.44)-(3.1.47). Formally, the $\mathfrak{so}(D+1;\mathbb{C})$-irreducibility of $V_{AB} \simeq 0$ implies that $V_{0'a} \simeq 0$ and $V_{ab} \simeq 0$ follow from $V_{0'0'} \simeq 0$. Similarly, the constraint $V_{abcd} \simeq 0$ follows from $V_{0'abc} \simeq 0$. Explicitly, from the algebra and $V_{0'0'} \simeq 0$, and using the fact that μ^2 defined by (3.1.17) is a commuting element, it follows that

$$P^a \star M_{ab} \simeq M_{ba} \star P^a \simeq i(\epsilon_0 + 1)P_b\ . \qquad \text{(B.0.3)}$$

The constraint (3.1.45) then follows immediately. Alternatively, one may compute $V_{0'a} = -\frac{i}{4}[P_a, P^b \star P_b]_\star \simeq -\frac{i}{4}[P_a, \mu^2]_\star = 0$. Next, the algebra, eq. (B.0.3) and $P^a \star P_a \simeq \mu^2$ imply

$$M_{(a}{}^c \star M_{b)c} \;\simeq\; -P_{(a} \star P_{b)} + \mu^2\eta_{ab}\ , \qquad \text{(B.0.4)}$$

that is, $V_{ab} \simeq 0$. Similarly, the algebra implies that $[P_{[a}, V_{0'bcd]}]_\star$ is proportional to V_{abcd}, so that $V_{0'abc} \simeq 0$, *i.e.* $P_{[a} \star P_b \star P_{c]} \simeq 0$, implies $V_{abcd} \simeq 0$, *i.e.* $M_{[ab} \star M_{cd]} \simeq 0$. Finally, the value of μ^2 is fixed from (3.1.48).

Next, let us use the contraction rules (3.1.49), (3.1.50) and (3.1.51) to derive the lemma (3.1.55), that is, compute the coefficient

$$\kappa_n \equiv \kappa_{n,0;1} \ . \tag{B.0.5}$$

To do so, we demand that the right-hand side of (I.0.11) is traceless. To this end, we first expand

$$\begin{aligned} \eta^{bc} P_{(b} \star P_c \star P_{a_1} \cdots \star P_{a_{n-2})} &\simeq \epsilon_0 P_{(a_1} \cdots \star P_{a_{n-2})} \\ + \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} P_{a_1} \cdots \star P_{a_{i-1}} \star [P_b, P_{a_i} \star \cdots \star P_{a_{j-2}}]_\star \star P^b \star P_{a_{j-1}} \star \cdots \star P_{a_{n-2})} \ , \end{aligned}$$

where we have used (3.1.49), and proceed by calculating the terms in the sum up trace parts, which only affect the higher traces in (I.0.11). Thus, for $i=1$ and $j=n$, and using also (3.1.50), we find that

$$\begin{aligned} & [P_b, P_{a_1} \star \cdots \star P_{a_{n-2}}]_\star \star P^b \\ = \ & iM_{ba_1} \star P_{a_2} \star \cdots \star P_{a_{n-2}} \star P^b + P_{a_1} \star (iM_{ba_2}) \star \cdots \star P_{a_{n-2}} \star P^b + \cdots \\ \simeq \ & (\epsilon_0+1) P_{a_1} \star \cdots \star P_{a_{n-2}} + \eta_{ba_2} P_{a_1} \star P_{a_3} \star \cdots \star P_{a_{n-2}} \star P^b \\ & + P_{a_2} \star (\eta_{ba_3} P_{a_2}) \star P_{a_4} \star \cdots \star P_{a_{n-2}} \star P^b + P_{a_2} \star P_{a_3} \star (\eta_{ba_4} P_{a_3}) \star P_{a_5} \star \cdots \star P_{a_{n-2}} \star P^b + \cdots \\ & + (\epsilon_0+1) P_{a_1} \star \cdots \star P_{a_{n-2}} + P_{a_2} \star (\eta_{ba_3} P_{a_2}) \star P_{a_4} \star \cdots \star P_{a_{n-2}} \star P^b + \cdots \\ & + \cdots \\ & + \mathcal{O}(\eta) \\ = \ & \left((n-2)(\epsilon_0+1) + \tfrac{1}{2}(n-2)(n-3)\right) P_{a_1} \star \cdots \star P_{a_{n-2}} + \mathcal{O}(\eta) \ , \end{aligned} \tag{B.0.6}$$

where the symmetrization on $a_1 \cdots a_{n-2}$ has been suppressed. The contributions from the terms with $j=n$ and $i=1+k$ for $k=0,\dots,n-2$ are obtained by letting $n \to n-k$ in (B.0.6), and their sum is given by

$$\sum_{k=0}^{n-2} k(\epsilon_0 + 1 + \tfrac{1}{2}(k-1)) P_{(a_1} \star \cdots \star P_{a_{n-2})} = \tfrac{1}{6}(n-1)(n-2)(n+3\epsilon_0) P_{(a_1} \star \cdots \star P_{a_{n-2})} \tag{B.0.7}$$

plus trace parts. The contributions from the the terms with $j = n-k$ for $k=0,\dots,n-2$ are obtained by letting $n \to n-k$ in (B.0.7), and their sum is given by $P_{(a_1} \star \cdots \star P_{a_{n-2})}$ times the numerical factor

$$\tfrac{1}{6} \sum_{k=0}^{n-2} (n-1-k)(n-2-k)(n-k+3\epsilon_0) = \tfrac{1}{24} n(n-1)(n-2)(n+4\epsilon_0+1) \tag{B.0.8}$$

Hence,

$$\begin{aligned}\eta^{bc} P_{(b} \star P_c \star P_{a_1} \cdots \star P_{a_{n-2})} &\simeq \left(\epsilon_0 + \tfrac{1}{12}(n-2)(n+4\epsilon_0+1)\right) P_{(a_1} \star \cdots \star P_{a_{n-2})} + \mathcal{O}(\eta) \\ &= \tfrac{1}{12}(n+1)(n+4\epsilon_0-2) P_{(a_1} \star \cdots \star P_{a_{n-2})} \ . \end{aligned} \tag{B.0.9}$$

On the other hand, tracing the second term on the right hand side of (I.0.11), we find κ_n times

$$\eta^{bc}\eta_{(bc} P_{a_1} \star \cdots \star P_{a_{n-2})} = \frac{2(2(n-2)+2\epsilon_0+3)}{n(n-1)} P_{(a_1} \star \cdots \star P_{a_{n-2})} + \mathcal{O}(\eta) \tag{B.0.10}$$

The tracelessness of the right hand side of (I.0.11) thus requires

$$\tfrac{1}{12}(n+1)(n+4\epsilon_0-2) + \frac{2(2n+2\epsilon_0-1)}{n(n-1)}\kappa_n = 0 \ , \tag{B.0.11}$$

which is equivalent to (3.1.56).

Finally, let us show (3.1.58) in the case of $s=0$, where it reads

$$\mathrm{Ac}_{P_a} T_{b(n)} = \{P_a, T_{b(n)}\}_\star = 2T_{ab(n)} + 2 l_n^{(0)} \eta_{a\{b_1} T_{b(n-1)\}} \ , \tag{B.0.12}$$

where, in the second term, the symmetric and traceless projection

$$\eta_{a\{b_1} T_{b(n-1)\}} \equiv \eta_{a(b_1} T_{b(n-1))} + \alpha_n \eta_{(b_1 b_2} T_{b(n-2))a} \ , \qquad \alpha_n = \alpha_{n,0} = -\frac{n-1}{2(n+\epsilon_0-\frac{1}{2})} \ , \tag{B.0.13}$$

and the coefficient

$$\lambda_n^{(0)} = \frac{n(n+2\epsilon_0-1)(n+1)}{8(n+\epsilon_0+\frac{1}{2})} \ . \tag{B.0.14}$$

Let us first compute this coefficient by imposing the trace conditions on (B.0.12) using the contraction rules (3.1.49), (3.1.50) and (3.1.51). Thus, we contract (B.0.12) by η^{ab_n}, which yields

$$\{P^a, T_{ab(n-1)}\}_\star = \lambda_n^{(0)} \eta^{ac} \eta_{a\{b_1} T_{b(n-2)c\}} \ . \tag{B.0.15}$$

On the right hand side, we use (B.0.13), and calculate

$$\begin{aligned}\eta^{ac}\eta_{a\{b_1} T_{b(n-2)c\}} &= \frac{(n+2\epsilon_0+2)(n+\epsilon_0-\frac{1}{2}) - n + 1}{n(n+\epsilon_0-\frac{1}{2})} T_{b(n-1)} \\ &= \frac{(n+\epsilon_0+\frac{1}{2})(n+2\epsilon_0)}{n(n+\epsilon_0-\frac{1}{2})} T_{b(n-1)} \ , \end{aligned} \tag{B.0.16}$$

so that

$$\{P^a, T_{ab(n-1)}\}_\star \;=\; \lambda_n^{(0)} \frac{(n+\epsilon_0+\frac{1}{2})(n+2\epsilon_0)}{n(n+\epsilon_0-\frac{1}{2})} T_{b(n-1)} \;. \tag{B.0.17}$$

On the left hand side, we first use $P^a \star T_{ab(n-1)} = T_{ab(n-1)} \star P^a$, which can be shown using the anti-automorphism τ in (5.2.15). To calculate $P^a \star T_{ab(n-1)}$ we use the lemma (I.0.11) and the ideal contraction rules (3.1.49) and (3.1.50):

$$\begin{aligned}
P^a \star T_{ab(n-1)} &= \frac{1}{n} P^a \star \sum_{i=1}^{n} P_{(b_1} \star \cdots P_{b_{i-1}|} \star P_a \star P_{|b_i} \star \cdots \star P_{b_{n-1})} \\
&\quad + \frac{2\kappa_n}{n} P_{(b_1} \star \cdots \star P_{b_{n-1})} + \mathcal{O}(\eta) \\
&\simeq \left(\epsilon_0 + \tfrac{1}{2}(\epsilon_0+1)(n-1) + \tfrac{1}{6}(n-1)(n-2) + \frac{2\kappa_n}{n}\right) P_{(b_1} \star \cdots \star P_{b_{n-1})} + \mathcal{O}(\eta) \\
&= \frac{(n+2\epsilon_0)(n+2\epsilon_0-1)(n+1)}{8(n+\epsilon_0-\frac{1}{2})} P_{(b_1} \star \cdots \star P_{b_{n-1})} + \mathcal{O}(\eta) \\
&= \frac{(n+2\epsilon_0)(n+2\epsilon_0-1)(n+1)}{8(n+\epsilon_0-\frac{1}{2})} T_{b(n-1)} + \mathcal{O}(\eta) \;,
\end{aligned} \tag{B.0.18}$$

where we note that the trace parts (which are irrelevant for our calculations) must cancel among themselves. Substituting back into (B.0.17), we finally obtain (B.0.14).

As a check of (B.0.12), one may verify the closure relation

$$[\mathrm{Ac}_{P_a}, \mathrm{Ac}_{P_b}] T_{c(n)} \;=\; i\widetilde{\mathrm{Ad}}_{M_{ab}} T_{c(n)} \;. \tag{B.0.19}$$

Using (B.0.12), the left hand side can be expanded as

$$\begin{aligned}
& 2\widetilde{\mathrm{Ad}}_{P_a} T_{bc(n)} + 2\lambda_n^{(0)} \eta_{b\{c_1|} \widetilde{\mathrm{Ad}}_{P_a} T_{|c(n-1)\}} - (a \leftrightarrow b) \\
=\;& 4\lambda_{n+1}^{(0)} \eta_{a\{b} T_{c(n)\}} + 4\lambda_n^{(0)} \eta_{b\{c_1} T_{c(n-1)\}a} + 4\lambda_n^{(0)} \lambda_{n-1}^{(0)} \eta_{b\{c_1|} \eta_{a|c_2} T_{|c(n-2)\}} - (a \leftrightarrow b)
\end{aligned} \tag{B.0.20}$$

The last term cancel upon the anti-symmetrization in a and b, as can be seen by expanding it explicitly using (B.0.13):

$$\begin{aligned}
& \eta_{b\{c_1|} \eta_{a|c_2} T_{|c(n-2)\}} - (a \leftrightarrow b) \\
=\;& \eta_{bc_1} \left(\eta_{ac_2} T_{c(n-2)} + \alpha_{n-1} \eta_{c_2c_3} T_{c(n-3)a}\right) + \alpha_n \eta_{c_1c_2} \left(\eta_{a(c_3} T_{c(n-3)b)} + \alpha_{n-1} \eta_{(c_3c_4} T_{c(n-4)b)a}\right) - (a \leftrightarrow b) \\
=\;& \left(\alpha_{n-1} - \frac{n-2}{n-1}\alpha_n + \frac{2\alpha_n\alpha_{n-1}}{n-1}\right) \eta_{bc_1} \eta_{c_2c_3} T_{c(n-3)a} \;=\; 0 \;,
\end{aligned} \tag{B.0.21}$$

where the symmetrization on $c_1, \ldots, c_n$ has been suppressed. Expanding the two first terms in (B.0.20) using (B.0.13) and (B.0.14), one finds

$$\begin{aligned}
&4\lambda^{(0)}_{n+1}\left(\eta_{a(b}T_{c(n))} + \alpha_{n+1}\eta_{(bc_1}T_{c(n-1))a}\right) + 4\lambda^{(0)}_{n}\left(\eta_{b(c_1}T_{c(n-1))a} + \alpha_n\eta_{(c_1c_2}T_{c(n-2))ba}\right) - (a \leftrightarrow b) \\
= \;& 8\left(\frac{n\lambda^{(0)}_{n+1}}{n+1} - \frac{2\alpha_{n+1}\lambda^{(0)}_{n+1}}{n+1} - \lambda^{(0)}_n\right)\eta_{c_1[a}T_{b]c(n-1)} \\
= \;& 2n\eta_{c_1[a}T_{b]c(n-1)}\ , \qquad \text{(B.0.22)}
\end{aligned}$$

which one identifies as the right hand side of (B.0.19).

Appendix C

Quadratic and Quartic Casimir Operators

The quadratic and and quartic Casimir operators of $\mathfrak{so}(D+1;\mathbb{C})$ are defined by

$$C_2[\mathfrak{so}(D+1;\mathbb{C})] = \tfrac{1}{2} M^{AB} \star M_{AB} \ , \tag{C.0.1}$$

$$C_4[\mathfrak{so}(D+1;\mathbb{C})] = \tfrac{1}{2} M_A{}^B \star M_B{}^C \star M_C{}^D \star M_D{}^A \ . \tag{C.0.2}$$

Acting on lowest and highest weight states $|e_0;\mathbf{s}_0\rangle^{\pm}$, which are annihilated by $L_r^{\mp}$, they can be rewritten using (3.2.12)-(3.2.15), and the resulting values are

$$C_2[\mathfrak{so}(D+1;\mathbb{C})]|e_0;\mathbf{s}_0] = e_0(e_0 \mp (D-1)) + C_2[\mathfrak{so}(D-1;\mathbb{C})|\mathbf{s}_0] \tag{C.0.3}$$

$$\begin{aligned} C_4[\mathfrak{so}(D+1;\mathbb{C})|e_0;\mathbf{s}_0]| &= e_0(e_0 \mp (D-1))\big(e_0(e_0 \mp (D-1)) + \tfrac{1}{2}(D-1)(D-2)\big) \\ &\quad + C_4[\mathfrak{so}(D-1;\mathbb{C})|\mathbf{s}_0] - C_2[\mathfrak{so}(D-1;\mathbb{C})|\mathbf{s}_0] \ , \end{aligned} \tag{C.0.4}$$

where the Casimir operators that are quadratic and quartic in angular momenta are defined by

$$C_2[\mathfrak{so}(D-1;\mathbb{C})] = \tfrac{1}{2} M^{rs} M_{rs} \ , \tag{C.0.5}$$

$$C_4[\mathfrak{so}(D-1;\mathbb{C})] = \tfrac{1}{2} M_r{}^s \star M_s{}^t \star M_t{}^u \star M_u{}^r \ , \tag{C.0.6}$$

and given in the $\mathfrak{so}(D-1;\mathbb{C})$ irrep with highest weight $\mathbf{s}_0 = (m_1, \ldots, m_{\nu-1})$ by

$$C_2[\mathfrak{so}(D-1;\mathbb{C})|\mathbf{s}_0] = \sum_{k=1}^{\nu-1} m_k(m_k + D_k) \ , \tag{C.0.7}$$

$$C_4[\mathfrak{so}(D-1;\mathbb{C})|\mathbf{s}_0] = \sum_{k=1}^{\nu-1} m_k(m_k + D_k)\big(m_k(m_k + D_k) + \tfrac{1}{2}(D_k)(D_k - 1) + 1 - k\big) \ , \tag{C.0.8}$$

where $D_k = D - 1 - 2k$. The latter two equations follow by recursive use of (**??**) and (**??**) in the case of a highest weight state.

To evaluate the Casimir operators in ℓth level of the adjoint representation, defined in (3.1.71), we use (C.0.3) and (C.0.3) with the highest weight $(2\ell+1, 2\ell+1)$, which immediately gives (7.1.47) and (7.1.48) (with $s = 2\ell+2$). The same values follow for the massless lowest weight and highest weight spaces $\mathfrak{D}^{\pm}(\pm(s+2\epsilon_0);(s))$. In the twisted adjoint representation, we first rewrite the π-twisted commutators in terms of commutators plus terms that can be calculated directly using the contraction rule (3.1.49). In the case of the quadratic Casimir we find

$$\begin{aligned}\widetilde{\mathrm{Ad}}_{C_2[\mathfrak{so}(D+1;\mathbb{C})]}(S) &= \mathrm{Ad}_{C_2[\mathfrak{so}(D;\mathbb{C})]}(S) - \{P^a, \{P_a, S\}_\star\}_\star \\ &= \mathrm{Ad}_{C_2[\mathfrak{so}(D;\mathbb{C})]}(S) - 2\epsilon_0 S - 2P^a \star S \star P_a \\ &= \mathrm{Ad}_{C_2[\mathfrak{so}(D+1;\mathbb{C})]}(S_\ell) + 4P^a \star S \star P_a \ ,\end{aligned} \qquad \text{(C.0.9)}$$

from which $P^a \star S \star P_a$ can be eliminated, which yields the explicit formula

$$T(C_2[\mathfrak{so}(D+1;\mathbb{C})])(S) = 2\mathrm{Ad}_{C_2[\mathfrak{so}(D;\mathbb{C})]}(S) - \mathrm{Ad}_{C_2[\mathfrak{so}(D+1;\mathbb{C})]}(S_\ell) - 4\epsilon_0 S \ . \qquad \text{(C.0.10)}$$

At the ℓth level, defined by (3.1.83), the elements S_ℓ carry highest weights $(s+k, s)$ and $(s+k, s+k)$ with $s = 2\ell+2$ and $k = 0, 1, \ldots,$ of (the adjoint actions) of $\mathfrak{so}(D;\mathbb{C})$ and $\mathfrak{so}(D+1;\mathbb{C})$, respectively, and we find that for all k

$$\begin{aligned}T(C_2[\mathfrak{so}(D+1;\mathbb{C})])(S_\ell) &= (2C_2[\mathfrak{so}(D;\mathbb{C})|(s+k,s)] - C_2[\mathfrak{so}(D+1;\mathbb{C})|(s+k,s+k)] - 4\epsilon_0)\, S_\ell \\ &= (2C_2[\mathfrak{so}(D;\mathbb{C})|(s,s)] - C_2[\mathfrak{so}(D+1;\mathbb{C})|(s,s)] - 4\epsilon_0)\, S_\ell \\ &= C_2[\mathfrak{so}(D+1;\mathbb{C})|\ell] S_\ell \ .\end{aligned} \qquad \text{(C.0.11)}$$

Similarly, in the case of the quartic Casimir,

$$\begin{aligned}\widetilde{\mathrm{Ad}}_{C_4[\mathfrak{so}(D+1;\mathbb{C})]}(S) &= \mathrm{Ad}_{C_4[\mathfrak{so}(D;\mathbb{C})]}(S) + \\ &\quad +\tfrac{1}{2}[M_a{}^b, [M_b{}^c, \{P_c, \{P^a, S\}_\star\}_\star]_\star]_\star + \tfrac{1}{2}[M_a{}^b, \{P_b, \{P^c, [M_c{}^a, S]_\star\}_\star\}_\star]_\star \\ &\quad +\tfrac{1}{2}\{P_a, \{P^b, [M_b{}^c, [M_c{}^a, S]_\star]_\star\}_\star\}_\star + +\tfrac{1}{2}\{P^a, [M_a{}^b, [M_b{}^c, \{P_c, S\}_\star]_\star]_\star\}_\star \\ &\quad +\tfrac{1}{2}\{P_a, \{P^b, \{P_b, \{P^a, S\}_\star\}_\star\}_\star\}_\star + +\tfrac{1}{2}\{P^a, \{P_b, \{P^b, \{P_b, S\}_\star\}_\star\}_\star\}_\star \\ &= \mathrm{Ad}_{C_4[\mathfrak{so}(D;\mathbb{C})]}(S) + C_+(S) + C_-(S) \\ &= \mathrm{Ad}_{C_4[\mathfrak{so}(D+1;\mathbb{C})]}(S) + 2C_-(S) \ ,\end{aligned} \qquad \text{(C.0.12)}$$

where $C_+(S)$ and $C_-(S)$ are the terms with an even and odd number of translation generators standing to the right of S, respectively. Eliminating $C_-(S)$ leads to

$$\widetilde{\mathrm{Ad}}_{C_4[\mathfrak{so}(D+1;\mathbb{C})]}(S) = 2\mathrm{Ad}_{C_4[\mathfrak{so}(D;\mathbb{C})]}(S) - \mathrm{Ad}_{C_4[\mathfrak{so}(D+1;\mathbb{C})]}(S) + 2C_+(S) \ . \qquad \text{(C.0.13)}$$

The quantity $C_+(S)$ can be calculated using of (3.1.50), (B.0.4), and $M_{ab}\star S\star M^{ab} = -\mathrm{Ad}_{C_2[\mathfrak{so}(D;\mathbb{C})]}(S)+\{M^{ab}\star M_{ab},S\}_\star$, and one finds

$$C_+(S) \;=\; \mathrm{Ad}_{C_2[\mathfrak{so}(D;\mathbb{C})]}(S)-2\epsilon_0(2\epsilon_0^2-\epsilon_0+1)S\ . \tag{C.0.14}$$

Thus, using the above assignments of highest weights for S_ℓ, we find that

$$\begin{aligned}\widetilde{\mathrm{Ad}}_{C_4[\mathfrak{so}(D+1;\mathbb{C})]}(S_\ell) \;&=\; \left(C_4[\mathfrak{so}(D;\mathbb{C})|(s+k,s)]-C_4[\mathfrak{so}(D+1;\mathbb{C})|(s+k,s+k)]\right)S_\ell\\ &\quad+\Big(C_2[\mathfrak{so}(D;\mathbb{C})|(s+k,s)]-4\epsilon_0(2\epsilon_0^2-\epsilon_0+1)\Big)\,S_\ell\\ &=\; C_4[\mathfrak{so}(D+1;\mathbb{C})|\ell]S_\ell\ .\end{aligned} \tag{C.0.15}$$

Appendix D

Computing $\mathrm{Ac}_{P_a} T_{b(n),c(m)}$ from the Mass Formula

In this Appendix we shall derive the expression (3.1.64) for the coefficient $\lambda_k^{(s)}$ in (3.1.58) using the Casimir relation (7.4.12), or, equivalently, the linearized zero-form constraint (7.4.6) and the Weyl-tensor mass formula (7.4.10).

Let us begin with the case of $s = 0$, where (3.1.58) reduces to (B.0.12). Starting from the linearized zero-form master constraint (7.4.5), *i.e.* $\nabla\Phi_{(0)} - ie^a\{P_a, \Phi_{(0)}\}_\star = 0$, and expanding $\Phi_{(0)}$ using (7.6.16), we obtain the component form (7.4.6) of the constraint for $s = 0$, that is

$$\nabla_b\Phi_{a(n)} - 2n\eta_{b\{a_1}\Phi_{a(n-1)\}} + \frac{2\lambda_n^{(0)}}{n+1}\Phi_{ba(n)} \quad = \quad 0 \ , \tag{D.0.1}$$

where we recall the definition (B.0.13) of the symmetric and traceless projection $\eta_{b\{a_1}\Phi_{a(n-1)}$. The symmetric traceless part of (D.0.1) immediately gives (7.4.7) for $s = 0$, while the trace part of (D.0.1) can be used to derive the masses (7.4.10) for $s = 0$. to this end, one first contracts (D.0.1) with ∇^b, which yields

$$\nabla^2\Phi_{a(n)} - 2n\eta^{bc}\left(\eta_{b(a_1|}\nabla_c\Phi_{|a(n-1))} + \alpha_n\eta_{(a_1a_2|}\nabla_c\Phi_{|a(n-2))b}\right) + \frac{2\lambda_{n+1}^{(0)}}{n+1}\nabla^b\Phi_{ba(n)} \quad = (\text{D}.0.2)$$

One then substitutes $\nabla_c\Phi_{a(n-1)}$ and $\nabla_c\Phi_{a(n1)}$ using (7.4.7), and takes the symmetric and traceless projection in $a(n)$, which leads to

$$\nabla^2\Phi_{a(n)} + 4\lambda_n^{(0)}\Phi_{a(n)} + 4\lambda_{n+1}^{(0)}\eta^{bc}\left(\eta_{c(b}\Phi_{a(n))} + \alpha_{n+1}\eta_{(ba_1}\Phi_{a(n-1))c}\right) \quad = \quad 0 \ . \tag{D.0.3}$$

Performing the traces, one ends up with the following expression for the mass:

$$\begin{aligned} m^2_{0,n} &= -4\lambda^{(0)}_n - 4\lambda^{(0)}_{n+1}\frac{1}{n+1}\left(n+2\epsilon_0+3-\frac{n}{n+\epsilon_0+\frac{1}{2}}\right) \\ &= -4l^{(0)}_n - 4l^{(0)}_{n+1}\frac{(n+2\epsilon_0+1)(n+\epsilon_0+\frac{3}{2})}{(n+1)(n+\epsilon_0+\frac{1}{2})} \ . \end{aligned} \tag{D.0.4}$$

Inserting (B.0.14), this expressions can be simplified, with the result

$$m^2_{0,n} = -(n^2+(2\epsilon_0+1)n+4\epsilon_0) \ , \tag{D.0.5}$$

in agreement with (7.4.10) for $s=0$.

Turning to the case of general s, we use (3.1.58) and the expansion (7.6.16) in the linearized zero-form master constraint (7.4.5):

$$\begin{aligned} 0 &= \nabla_c\Phi - i\{P_c,\Phi\}_\star \\ &= \sum_{s,n}\frac{i^n}{n!}T^{a(s+n),b(s)}\nabla_c\Phi_{a(s+n),b(s)} - \sum_{s,n}\frac{i^{n+1}}{n!}2\Delta_{n+s,s}T_c{}^{a(s+n),b(s)}\Phi_{a(s+n),b(s)} \\ &\quad -\sum_{s,n}\frac{i^{n+1}}{n!}2\lambda^{(s)}_n\eta_{c\langle a}T_{a(s+n-1),b(s)\rangle}\Phi^{a(s+n),b(s)} \ . \end{aligned} \tag{D.0.6}$$

To read off the corresponding component equations, we rewrite the middle term using

$$T_c{}^{a(s+n),b(s)}\Phi_{a(s+n),b(s)} = T^{a(s+n+1),b(s)}\eta_{c(a}\Phi_{a(s+n)),b(s)} = T^{a(s+n+1),b(s)}\eta_{c\{a}\Phi_{a(s+n),b(s)\}} \ , \tag{D.0.7}$$

and the last term using

$$\begin{aligned} \eta_{c\{a}T_{a(s+n-1),b(s)\}}\Phi^{a(s+n),b(s)} &= \left(\eta_{c(a}T_{a(s+n-1)),b(s)} + (\eta_{aa}\text{ and }\eta_{ab}\text{ traces})\right)\Phi^{a(s+n),b(s)} \\ = \eta_{c(a}T_{a(s+n-1)),b(s)}\Phi^{a(s+n),b(s)} &= T_{a(s+n-1),b(s)}\Phi_c{}^{\{a(s+n-1),b(s)\}} \ , \end{aligned} \tag{D.0.8}$$

after which we arrive at (7.4.6). Contracting by ∇^c and we get

$$\nabla^2\Phi_{a(s+n),b(s)} = 2n\Delta_{s+n-1,s}\eta^{cd}\nabla_d\eta_{c\{a}\Phi_{a(s+n-1),b(s)\}} - \frac{2\lambda^{(s)}_{n+1}}{n+1}\eta^{cd}\nabla_d\Phi_{c\{a(s+n),b(s)\}} \tag{D.0.9}$$

where we recall that $\eta_{c\{a}\Phi_{a(s+n-1),b(s)\}}$ is given by (3.1.59). The next step is to use (7.4.6) to substitute the gradients on the right hand side. In the first term we obtain

$$\begin{aligned} &-4\Delta_{s+n-1,s}\lambda^{(s)}_n\eta^{cd}\left(\eta_{ca}\Phi_{d\langle a(s+n-1),b(s)\rangle} + \alpha_{s+n,s}\eta_{a(2)}\Phi_{d\langle a(s+n-2)c,b(s)\rangle}\right. \\ &\left.+\beta_{s+n,s}\eta_{a(2)}\Phi_{d\langle a(n+s-2)b,cb(s-1)\rangle} + \gamma_{s+n,s}\eta_{ab}\Phi_{d\langle a(s+n-1),cb(s-1)\rangle}\right) \\ = &-4\Delta_{s+n-1,s}\lambda^{(s)}_n\Phi_{a\langle a(s+n-1),b(s)\rangle} \\ = &-4\Delta_{s+n-1,s}\lambda^{(s)}_n\widetilde{\Delta}_{s+n,s}\Phi_{a(s+n),b(s)} \ , \end{aligned} \tag{D.0.10}$$

where the intermediate $\langle\cdots\rangle$ Young projections imposed *prior* to the final symmetrization on a and b indices, and the coefficient $\widetilde{\Delta}_{s+n,s}$ is defined by

$$\Phi_{a\langle a(s+n-1),b(s)\rangle} \;=\; \widetilde{\Delta}_{s+n,s}\Phi_{a(s+n),b(s)}\ . \tag{D.0.11}$$

To compute this coefficient, we expand the left hand side using the definition of the Young projector:

$$\Phi_{a\langle a(s+n-1),b(s)\rangle} \;=\; \frac{(s+n-1)!s!}{(s+n)\cdots(n+1)(n-1)\cdots 1\times s!}\sum_{k=0}^{s}(-1)^k\binom{s}{k}\Phi_{a(s+n-k)b(k),a(k)b(s-k)}\ . \tag{D.0.12}$$

In the kth term, we cycle the b-indices back to their original position, using $\Phi_{(a(s-k)b(k),a_1)a(k-1)b(s-k)}=0$, which yields

$$\Phi_{a(s-k)b(k),a(k)b(s-k)} \;=\; -\frac{k}{s+n-k+1}\Phi_{a(s-k+1)b(k-1),a(k-1)b(s-k+1)} \;=\; \frac{(-1)^k}{\binom{s+n}{k}}\Phi_{a(s),b(s)}\ . \tag{D.0.13}$$

Thus,

$$\widetilde{\Delta}_{s+n,s} \;=\; \frac{n}{s+n}\sum_{k=0}^{s}\frac{\binom{s}{k}}{\binom{s+n}{k}} \;=\; \frac{n}{s+n}\frac{1}{(s+1)_n}\sum_{k=0}^{s}(s+1-k)_n \;=\; \frac{n}{s+n}\frac{s+n+1}{n+1}\ . \tag{D.0.14}$$

Using also (3.1.60), the first term on the right-hand side of (D.0.9) is found to be

$$-4\Delta_{s+n-1,s}\lambda_n^{(s)}\widetilde{\Delta}_{s+n,s} \;=\; -4\lambda_n^{(s)}\ . \tag{D.0.15}$$

Turning to the second term, substitution of the gradient yields

$$\begin{aligned}
&-4\lambda_{n+1}^{(s)}\Delta_{s+n,s}\eta^{cd}\eta_{c\{d}\Phi_{\{a(s+n),b(s)\}\}}\\
=\;&-4\lambda_{n+1}^{(s)}\Delta_{s+n,s}\frac{1}{n+s+1}\left(s+n+2\epsilon_0+3+2\alpha_{s+n+1,s}-\frac{2\beta_{s+n+1,s}}{s+n}+\gamma_{s+n+1,s}\right)\Phi_{a(s+n),b(s)}\\
=\;&-4\lambda_{n+1}^{(s)}\Delta_{s+n,s}\frac{(n+s+2\epsilon_0)(n+s+\epsilon_0+\frac{3}{2})(n+2s+2\epsilon_0+1)}{(n+s+1)(n+2s+2\epsilon_0)(n+s+\epsilon_0+\frac{1}{2})}\Phi_{a(s+n),b(s)}\ .
\end{aligned} \tag{D.0.16}$$

Thus, upon adding the two contributions, we find

$$m^2_{s,n} \;=\; -4\lambda_n^{(s)}-4\lambda_{n+1}^{(s)}\Delta_{s+n,s}\frac{(n+s+2\epsilon_0)(n+s+\epsilon_0+\frac{3}{2})(n+2s+2\epsilon_0+1)}{(n+s+1)(n+2s+2\epsilon_0)(n+s+\epsilon_0+\frac{1}{2})}\ , \tag{D.0.17}$$

where $m^2_{s,n}=-4\epsilon_0-2s-(n+2s+2\epsilon_0+1)n$, which serves as a recursion relation for determining $\lambda_n^{(s)}$, given the initial datum $\lambda_0^{(s)}=0$, with solution given by (3.1.64).

Appendix E

Two-Component Spinor and Curvature Conventions

We use conventions in which the generators of the various real forms of $SO(5;\mathbb{C})$ obey

$$[M_{AB}, M_{CD}] \ = \ 4i\eta_{[C|[B}M_{A]|D]} \ , \qquad (M_{AB})^\dagger \ = \ \sigma(M_{AB}) \ , \tag{E.0.1}$$

with $\eta_{AB} = (\eta_{ab}; -\lambda^2)$, where η_{ab} specifies the signature of the tangent space. Under $M_{AB} \to (M_{ab}, P_a)$, the commutation relations decompose into

$$[M_{ab}, M_{cd}]_\star \ = \ 4i\eta_{[c|[b}M_{a]|d]} \ , \qquad [M_{ab}, P_c]_\star \ = \ 2i\eta_{c[b}P_{a]} \ , \qquad [P_a, P_b]_\star \ = \ i\lambda^2 M_{ab} \ . \tag{E.0.2}$$

The corresponding oscillator realization is taken to be (3.3.23), which we repeat here for convenience

$$M_{ab} \ = \ -\frac{1}{8}\left[\, (\sigma_{ab})^{\alpha\beta}y_\alpha y_\beta + (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\tilde{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} \, \right] \ , \qquad P_a \ = \ \frac{\lambda}{4}(\sigma_a)^{\alpha\dot{\beta}}y_\alpha\bar{y}_{\dot{\beta}} \ . \tag{E.0.3}$$

Here, the van der Waerden symbols obey

$$(\sigma^a)_\alpha{}^{\dot{\alpha}}(\bar{\sigma}^b)_{\dot{\alpha}}{}^\beta \ = \ \eta^{ab}\delta_\alpha^\beta \ + \ (\sigma^{ab})_\alpha{}^\beta \ , \qquad (\bar{\sigma}^a)_{\dot{\alpha}}{}^\alpha(\sigma^b)_\alpha{}^{\dot{\beta}} \ = \ \eta^{ab}\delta_{\dot{\alpha}}^{\dot{\beta}} \ + \ (\bar{\sigma}^{ab})_{\dot{\alpha}}{}^{\dot{\beta}} \ , \tag{E.0.4}$$

$$\tfrac{1}{2}\epsilon_{abcd}(\sigma^{cd})_{\alpha\beta} \ = \ \begin{cases} (\sigma_{ab})_{\alpha\beta} \ , & (4,0) \text{ and } (2,2) \text{ signature} \ , \\ i(\sigma_{ab})_{\alpha\beta} \ , & (3,1) \text{ signature} \ , \end{cases} \tag{E.0.5}$$

and reality conditions

$$((\sigma^a)_{\alpha\dot{\beta}})^\dagger \ = \ \begin{cases} -(\bar{\sigma}^a)^{\dot{\beta}\alpha} \ = \ -(\sigma^a)^{\alpha\dot{\beta}} & \text{for } SU(2) \ , \\ (\bar{\sigma}^a)_{\dot{\alpha}\beta} \ = \ (\sigma^a)_{\beta\dot{\alpha}} & \text{for } SL(2,\mathbb{C}) \ , \\ (\bar{\sigma}^a)_{\dot{\beta}\alpha} \ = \ (\sigma^a)_{\alpha\dot{\beta}} & \text{for } Sp(2) \ , \end{cases} \tag{E.0.6}$$

and

$$((\sigma^{ab})_{\alpha\beta})^\dagger \;=\; \begin{cases} (\sigma^{ab})^{\alpha\beta} & \text{for } SU(2)\ , \\ (\bar\sigma^{ab})_{\dot\alpha\dot\beta} & \text{for } SL(2,\mathbb{C})\ , \\ (\sigma^{ab})_{\alpha\beta} & \text{for } Sp(2)\ . \end{cases} \tag{E.0.7}$$

The reality conditions on spinor oscillators are given in (6.2.25), (6.2.26) and (6.2.27). Spinor indices are raised and lowered according to the following conventions, $A^\alpha = \epsilon^{\alpha\beta}A_\beta$, $A_\alpha = A^\beta\epsilon_{\beta\alpha}$, where

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\delta} \;=\; 2\delta^{\alpha\beta}_{\gamma\delta}\ , \qquad \epsilon^{\alpha\beta}\epsilon_{\alpha\gamma} \;=\; \delta^\beta_\gamma\ , \tag{E.0.8}$$

and

$$(\epsilon_{\alpha\beta})^\dagger \;=\; \begin{cases} \epsilon^{\alpha\beta} & \text{for } SU(2)\ , \\ \epsilon_{\dot\alpha\dot\beta} & \text{for } SL(2,\mathbb{C})\ , \\ \epsilon_{\alpha\beta} & \text{for } Sp(2)\ . \end{cases} \tag{E.0.9}$$

One may use the following convenient representations:

$$SU(2)\ :\qquad \sigma^a = (i,\sigma^i)\ ,\qquad \bar\sigma^a = (-i,\sigma^i)\ ,\qquad \epsilon = i\sigma^2\ ; \tag{E.0.10}$$

$$SL(2,\mathbb{C})\ :\qquad \sigma^a = (-i\sigma^2,-i\sigma^i\sigma^2)\ ,\qquad \bar\sigma^a = (-i\sigma^2,i\sigma^2\sigma^i)\ ,\qquad \epsilon = i\sigma^2, \tag{E.0.11}$$

$$Sp(2)\ :\qquad \sigma^a = (1,\tilde\sigma^i)\ ,\qquad \bar\sigma^a = (-1,\tilde\sigma^i)\ ,\qquad \epsilon = i\sigma^2\ , \tag{E.0.12}$$

where in the last case $\tilde\sigma^i = (\sigma^1, i\sigma^2, \sigma^3)$. The real form of the $\mathfrak{so}(5;\mathbb{C})$-valued connection Ω can be expressed, using (6.2.57) and (E.0.3), as

$$\Omega \;=\; \frac{1}{4i}dx^\mu\left[\omega_\mu^{\alpha\beta}\ y_\alpha y_\beta + \overline{\omega}_\mu{}^{\dot\alpha\dot\beta}\ \bar y_{\dot\alpha}\bar y_{\dot\beta} + 2e_\mu^{\alpha\dot\beta}\ y_\alpha\bar y_{\dot\beta}\right]\ , \tag{E.0.13}$$

where

$$\omega^{\alpha\beta} \;=\; -\tfrac14(\sigma_{ab})^{\alpha\beta}\ \omega^{ab}\ ,\quad \bar\omega^{\dot\alpha\dot\beta} \;=\; -\tfrac14(\bar\sigma_{ab})^{\dot\alpha\dot\beta}\ \omega^{ab}\ ,\quad e^{\alpha\dot\alpha} \;=\; \tfrac{\lambda}{2}(\sigma_a)^{\alpha\dot\alpha}\ e^a\ . \tag{E.0.14}$$

Likewise, for the curvature $\mathcal{R} = d\Omega + \Omega\star\Omega$ one finds

$$\mathcal{R}_{\alpha\beta} \;=\; d\omega_{\alpha\beta} + \omega_{\alpha\gamma}\wedge\omega_\beta{}^\gamma + e_{\alpha\dot\delta}\wedge e_\beta{}^{\dot\delta}\ , \tag{E.0.15}$$

$$\overline{\mathcal{R}}_{\dot\alpha\dot\beta} \;=\; d\overline{\omega}_{\dot\alpha\dot\beta} + \overline{\omega}_{\dot\alpha\dot\gamma}\wedge\overline{\omega}_{\dot\beta}{}^{\dot\gamma} + e_{\delta\dot\alpha}\wedge e^\delta{}_{\dot\beta}\ , \tag{E.0.16}$$

$$\mathcal{R}_{\alpha\dot\beta} \;=\; de_{\alpha\dot\beta} + \omega_{\alpha\gamma}\wedge e^\gamma{}_{\dot\beta} + \overline{\omega}_{\dot\beta\dot\delta}\wedge e_\alpha{}^{\dot\delta}\ , \tag{E.0.17}$$

and

$$\mathcal{R}^{ab} \;=\; d\omega^{ab} + \omega^a{}_c\wedge\omega^{cb} + \lambda^2 e^a\wedge e^b\ ,\qquad \mathcal{R}^a \;=\; de^a + \omega^a{}_b\wedge e^b\ . \tag{E.0.18}$$

Setting $\mathcal{R} = 0$ gives the Riemann tensor

$$R_{\mu\nu,\rho\sigma} \;=\; -\lambda^2 \left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right) \ , \tag{E.0.19}$$

corresponding to the maximally symmetric vacuum solution of gravity with action

$$S \;=\; \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}(R-2\Lambda) \ , \qquad \Lambda \;=\; -3\lambda^2 \ . \tag{E.0.20}$$

Appendix F

Further Notation Used for the Solutions

The gauge function $L(x; y, \bar{y})$ defined in (6.3.19) can be written as

$$L = \frac{2h}{1+h} \exp(-iya\bar{y}) \ , \tag{F.0.1}$$

where

$$a_{\alpha\dot{\alpha}} = \frac{\lambda x_{\alpha\dot{\alpha}}}{1+h} \ , \qquad x_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}} x_a \ , \tag{F.0.2}$$

$$x^2 = \eta_{ab} x^a x^b \ , \qquad h = \sqrt{1-\lambda^2 x^2} \ . \tag{F.0.3}$$

Useful relations that follow from these definitions are

$$a^2 = \frac{1-h}{1+h} \ , \qquad h = \frac{1-a^2}{1+a^2} \ . \tag{F.0.4}$$

The Maurer-Cartan form based on L defined in (6.3.19) yields the the vierbein and Lorentz connection

$$e_{(0)}{}^{\alpha\dot{\alpha}} = -\frac{\lambda(\sigma^a)^{\alpha\dot{\alpha}} dx_a}{h^2} \ , \qquad \omega_{(0)}{}^{\alpha\beta} = -\frac{\lambda^2(\sigma^{ab})^{\alpha\beta} dx_a x_b}{h^2} \ , \tag{F.0.5}$$

with Riemann tensor given by

$$R_{(0)\mu\nu,\rho\sigma} = -\lambda^2 \left(g_{(0)\mu\rho} g_{(0)\nu\sigma} - g_{(0)\nu\rho} g_{(0)\mu\sigma} \right) \ . \tag{F.0.6}$$

In the case of type 3 solutions, a useful definition is

$$b_{\alpha\beta} = 2\lambda_{(\alpha}\mu_{\beta)} \ , \qquad \lambda^\alpha \mu_\alpha = \tfrac{i}{2} \ . \tag{F.0.7}$$

It obeys the relation $(b^2)_\alpha{}^\beta = -\frac{1}{4}\delta_\alpha^\beta$ and it defines an almost complex structure via the relations (see, for example, [132])

$$b_{\alpha\beta} = \frac{1}{8}(\sigma^{ab})_{\alpha\beta}\, J_{ab}\ , \qquad J_{ab} = (\sigma_{ab})^{\alpha\beta}\, b_{\alpha\beta}\ , \qquad J_a{}^c J_c{}^b = -\delta_a^b\ . \tag{F.0.8}$$

Similarly, using the definition

$$\tilde{b}_{\alpha\beta} = a^{-2}(a\bar{b}\bar{a})_{\alpha\beta}\ , \tag{F.0.9}$$

we have the relations

$$\tilde{b}_{\alpha\beta} = \frac{1}{8}(\sigma^{ab})_{\alpha\beta}\, \tilde{J}_{ab}\ , \qquad \tilde{J}_{ab} = (\sigma_{ab})^{\alpha\beta}\, \tilde{b}_{\alpha\beta}\ , \qquad \tilde{J}_a{}^c \tilde{J}_c{}^b = -\delta_a^b\ . \tag{F.0.10}$$

Finally, we have the following definition for spinors used in describing a Type 3 solution:

$$U_{\dot\alpha} = \frac{x^a}{\sqrt{x^2}}\,(\bar{\sigma}_a\lambda)_{\dot\alpha}\ , \qquad V_{\dot\alpha} = \frac{x^a}{\sqrt{x^2}}\,(\bar{\sigma}_a\mu)_{\dot\alpha}\ . \tag{F.0.11}$$

Appendix G

Weyl-ordered Projectors

Weyl-ordered projectors $P(y, \bar{y})$ can be constructed by recombining $(y, \bar{y})$ into a pair of Heisenberg oscillators (a_i, b^j) $(i, j = 1, 2)$ obeying

$$[a_i, b^j]_\star \;=\; \delta_i^j \;. \tag{G.0.1}$$

For example, one can take

$$a_1 \;=\; u \;=\; \lambda^\alpha y_\alpha \;, \qquad b^1 \;=\; v \;=\; \mu^\alpha y_\alpha \;, \tag{G.0.2}$$

$$a_2 \;=\; \bar{u} \;=\; \bar{\lambda}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}} \;, \qquad b^2 \;=\; \bar{v} \;=\; \bar{\mu}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}} \;, \tag{G.0.3}$$

where the constant spinors are normalized as

$$\lambda^\alpha \mu_\alpha \;=\; \tfrac{i}{2} \;, \qquad \bar{\lambda}^{\dot{\alpha}} \bar{\mu}_{\dot{\alpha}} \;=\; \tfrac{i}{2} \;. \tag{G.0.4}$$

The projectors, obeying the appropriate reality conditions, take the form

$$P \;=\; \sum_{n_1, n_2 \in \;\mathbb{Z} + \frac{1}{2}} \theta_{n_1,n_2} P_{n_1,n_2} \;, \qquad \bar{P} \;=\; \sum_{n_1, n_2 \in \;\mathbb{Z} + \frac{1}{2}} \bar{\theta}_{n_1,n_2} P_{n_1,n_2} \;, \tag{G.0.5}$$

where $\theta_{n_1,n_2} \in \{0, 1\}$ and $\bar{\theta}_{n_1,n_2} \in \{0, 1\}$, with

$$(3,1) \text{ signature} \;:\; \theta_{n_1,n_2} \;=\; \bar{\theta}_{n_1,n_2} \;, \tag{G.0.6}$$

$$(4,0) \text{ and } (2,2) \text{ signatures} \;:\; \theta_{n_1,n_2} \;, \bar{\theta}_{n_1,n_2} \quad \text{independent} \;, \tag{G.0.7}$$

and

$$P_{n_1,n_2} \;=\; 4(-1)^{n_1+n_2-\frac{\epsilon_1+\epsilon_2}{2}} e^{-2\sum_i \epsilon_i w_i} L_{n_1-\frac{\epsilon_1}{2}}(4\epsilon_1 w_1) L_{n_2-\frac{\epsilon_2}{2}}(4\epsilon_2 w_2) \;, \tag{G.0.8}$$

$$w_i \;=\; b^i a_i \;=\; b^i \star a_i + \tfrac{1}{2} \;=\; a_i \star b^i - \tfrac{1}{2} \qquad (\text{no sum}) \;, \tag{G.0.9}$$

with $\epsilon_i = n_i/|n_i|$ and $L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}(e^{-x}x^n)$ are the Laguerre polynomials. The projector property of P and $\bar{P}$ follows from

$$P_{m_1,m_2} \star P_{n_1,n_2} = \delta_{m_1 n_1}\delta_{m_2 n_2} P_{n_1,n_2} \ , \tag{G.0.10}$$

$$(w_i - n_i) \star P_{n_1,n_2} = 0 \ , \tag{G.0.11}$$

$$\tau(P_{n_1,n_2}) = P_{-n_1,-n_2} \ . \tag{G.0.12}$$

Here, $w_i - \frac{1}{2}$ is the Weyl-ordered form of the number operator, and

$$2(-1)^{n_i - \frac{\epsilon_i}{2}} e^{-2\epsilon_i w_i} L_{n_i - \frac{\epsilon_i}{2}}(4\epsilon_i w_i) = \begin{cases} |n_i\rangle\langle n_i| & \text{for } n_i > 0 \\ (-1)^{-n_i - \frac{1}{2}} |n_i\rangle\langle n_i| & \text{for } n_i < 0 \end{cases} \tag{G.0.13}$$

where $|n_i\rangle = \frac{(b^i)^{n_i - \frac{1}{2}}}{\sqrt{(n_i - \frac{1}{2})!}}|0\rangle$ with $n_i > 0$ belongs to the standard Fock space, built by acting with b^i on the ground state $|0\rangle$ obeying $a_i|0\rangle = 0$, while $|n_i\rangle = \frac{(a_i)^{-n_i - \frac{1}{2}}}{\sqrt{(-n_i - \frac{1}{2})!}}|\tilde{0}\rangle$ for $n_i < 0$ are anti-Fock space states, built by acting with a_i on the anti-ground state $|\tilde{0}\rangle = 0$ obeying $b^i|\tilde{0}\rangle = 0$. Equation (G.0.10) holds formally, since the inner product between a Fock space state and an anti-Fock space state vanishes. However, the corresponding Weyl-ordered projectors have divergent $\star$-products, as can be seen from the lemma

$$e^{suv} \star e^{tuv} = \frac{1}{1 + \frac{st}{4}} \exp\left(\frac{s+t}{1 + \frac{st}{4}}\, uv\right) \ . \tag{G.0.14}$$

Thus, lacking, at present, a suitable regularization scheme that does not violate associativity and other basic properties of the $\star$-product algebra, we shall restrict our attention to projectors that are constructed in either the Fock space or the anti-Fock space, *i.e.*

$$\theta_{n_1,n_2} = 1 \quad \text{only if } (n_1, n_2) \in Q \ , \tag{G.0.15}$$

where Q is anyone of the four quadrants in the (n_1, n_2) plane. From (G.0.12), it follows that these projectors are not invariant under the τ map, and therefore the master fields Type 2 and Type 3 solutions will be those of the non-minimal model, where the τ conditions are relaxed to $\pi\bar{\pi}$ conditions, which are certainly satisfied.

We also note that in order to solve the higher-spin equations it is essential that

$$[P, \bar{P}]_\star = \sum_{n_1,n_2} (\theta_{n_1,n_2}\bar{\theta}_{n_1,n_2} - \bar{\theta}_{n_1,n_2}\theta_{n_1,n_2}) P_{n_1,n_2} = 0 \ , \tag{G.0.16}$$

which holds for independent θ_{n_1,n_2} and $\bar{\theta}_{n_1,n_2}$ parameters (in the Euclidean and Kleinian signatures). Moreover, one can work with a reduced set of oscillators, say $a_1 = u$ and

$b^1 = v$, by taking $\theta_{n_1,n_2} = \theta(\pm n_2)\theta_{n_1}$, where $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$, and summing over all values of n_2 using

$$\sum_{k=0}^{\infty} t^k L_k(x) \quad = \quad (1-t)^{-1}\exp(-xt(1-t)^{-1}) \ . \tag{G.0.17}$$

Setting $n = n_1$ and $\epsilon = \epsilon_1$, this leads to

$$P \quad = \sum_{n\in\ \mathbb{Z}+\frac{1}{2}} \theta_n P_n \ , \qquad \bar{P} \ = \sum_{n\in\ \mathbb{Z}+\frac{1}{2}} \bar{\theta}_n P_n \tag{G.0.18}$$

$$P_n \quad = \quad 2(-1)^{n-\frac{\epsilon}{2}} e^{-2\epsilon uv} L_n(4\epsilon uv) \ , \tag{G.0.19}$$

with suitable reality conditions on the θ_n parameters. Finally, setting $\theta_n = \theta(\pm n)$, and using (G.0.17) once more, one finds that setting all θ-parameters equal to 1 gives $P = 1$.

Appendix H

Calculation of $V = L^{-1} \star P \star L$

In this Appendix we compute $V = L^{-1} \star P \star L$ where L is the gauge function given in (F.0.1) and P is a projector of the form given in (G.0.5). Let us begin by considering the case of $P = P_{\frac{1}{2}} = 2e^{-2uv}$, *i.e.*

$$V \;=\; \frac{8h^2}{(1+h)^2} e^{iya\bar{y}} \star e^{yby} \star e^{-iya\bar{y}} \;, \tag{H.0.1}$$

where $ya\bar{y} = y^\alpha a_\alpha{}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}}$ and $yby = y^\alpha b_\alpha{}^\beta y_\beta$, with $a_{\alpha\dot{\alpha}}$ and $b_{\alpha\beta}$ given by (F.0.2) and (F.0.7). The first $\star$-product can be performed treating the integration variables $(\xi_\alpha, \eta_\alpha)$ and $(\bar{\xi}_{\dot{\alpha}}, \bar{\eta}_{\dot{\alpha}})$ as separate real variables. Using the formulae (B.1) provided in [97], we find

$$V \;=\; \frac{8h^2}{(1+h)^2} e^{iya\bar{y}+(y-\bar{y}a)b(y+a\bar{y})} \star e^{-iya\bar{y}} \;. \tag{H.0.2}$$

The remaining $\star$-product leads to the Gaussian integral

$$V \;=\; \frac{8h^2}{(1+h)^2} \int \frac{d^4\xi d^4\eta}{(2\pi)^4} e^{\frac{1}{2}\Xi^I M_I{}^J \Xi_J + \Xi^I N_I + (y-\bar{y}a)b(y+a\bar{y})} \;, \tag{H.0.3}$$

where $\Xi^I = (\xi^\alpha, \bar{\xi}^{\dot{\alpha}}; \eta^\alpha, \bar{\eta}^{\dot{\alpha}})$ and $\Xi_I = (\xi_\alpha, \bar{\xi}_{\dot{\alpha}}; \eta_\alpha, \bar{\eta}_{\dot{\alpha}}) = \Xi^J \Omega_{JI}$, with block-diagonal symplectic metric $\Omega = \epsilon \oplus \bar{\epsilon} \oplus \epsilon \oplus \bar{\epsilon}$, and

$$M \;=\; \begin{bmatrix} A & -i \\ i & B \end{bmatrix} \;, \tag{H.0.4}$$

$$A \;=\; \begin{bmatrix} 2b & ia + 2ba \\ ia - 2ba & -2\bar{a}ba \end{bmatrix} \;, \qquad B \;=\; \begin{bmatrix} 0 & -ia \\ -i\bar{a} & 0 \end{bmatrix} \;, \tag{H.0.5}$$

$$N \;=\; \begin{bmatrix} i(1-2ib)a\bar{y} + 2by \\ -2\bar{a}ba\bar{y} + i\bar{a}(1+2ib)y \\ -ia\bar{y} \\ -i\bar{a}y \end{bmatrix} \;. \tag{H.0.6}$$

The Gaussian integration gives

$$V = \frac{8h^2}{(1+h)^2\sqrt{\det M}}\, e^{\frac{1}{2}N^I(M^{-1})_I{}^J N_J + (y-\bar{y}a)b(y+a\bar{y})} \ . \tag{H.0.7}$$

From $\det M = \det(1+AB)$, and noting that the matrices defined as

$$C \equiv \frac{BA-a^2}{2i} = \begin{bmatrix} a^2 b & a^2 ba \\ -\bar{a}b & -\bar{a}ba \end{bmatrix} , \qquad \tilde{C} \equiv \frac{AB-a^2}{2i} = \begin{bmatrix} -a^2 b & -ba \\ \bar{a}ba^2 & \bar{a}ba \end{bmatrix} , \tag{H.0.8}$$

are nilpotent, i.e. $C^2 = \tilde{C}^2 = 0$, one finds

$$\det M = (1-a^2)^4 , \tag{H.0.9}$$

and, using $1-a^2 = 2h/(1+h)$, the pre-factor in V is thus given by

$$\frac{8h^2}{(1+h)^2\sqrt{\det M}} = 2 \ . \tag{H.0.10}$$

Next, using geometric series expansions, one finds

$$M^{-1} = \frac{i}{(1-a^2)} \begin{bmatrix} i(1-a^2)B + 2B\tilde{C} & -(1-a^2) - 2iC \\ 1-a^2+2i\tilde{C} & i(1-a^2)A + 2AC \end{bmatrix} , \tag{H.0.11}$$

and

$$\tfrac{1}{2}N^I(M^{-1})_I{}^J N_J = \frac{4a^2 yby + 2(1+4a^2-a^4)yba\bar{y} - (3-a^2)(1+a^2)\bar{y}\bar{a}ba\bar{y}}{(1-a^2)^2} \tag{H.0.12}$$

Adding the classical term in the exponent in (H.0.3) yields the final result

$$V = 2\exp\left(-\frac{[2\bar{y}\bar{a} - (1+a^2)y]\, b\, [2a\bar{y} + (1+a^2)y]}{(1-a^2)^2}\right) \ . \tag{H.0.13}$$

The projector property $V \star V = V$ follows manifestly from

$$V = 2\exp(-2\tilde{u}\tilde{v}) , \qquad [\tilde{u},\tilde{v}]_\star = 1 , \tag{H.0.14}$$

where

$$\tilde{u} = \lambda^\alpha \eta_\alpha , \qquad \tilde{v} = \mu^\alpha \eta_\alpha , \tag{H.0.15}$$

with

$$\eta_\alpha = \frac{[(1+a^2)y + 2a\bar{y}]_\alpha}{1-a^2} , \qquad [\eta_\alpha, \eta_\beta]_\star = 2i\epsilon_{\alpha\beta} \ . \tag{H.0.16}$$

Thus, the net effect of rotating the projector $P_{\frac{1}{2}}(u,v)$ given in (G.0.19) is to replace the oscillators u and v by their rotated dittos $\tilde{u}$ and $\tilde{v}$. We claim, without proof, that this generalizes to any n, *viz.*

$$L^{-1} \star P_n(u,v) \star L \;\; = \;\; P_n(\tilde{u},\tilde{v}) \; . \tag{H.0.17}$$

Similarly, for $P_n(\bar{u},\bar{v})$ we have

$$L^{-1} \star P_n(\bar{u},\bar{v}) \star L \;\; = \;\; P_n(\tilde{\bar{u}},\tilde{\bar{v}}) \; , \tag{H.0.18}$$

where

$$\tilde{\bar{u}} \;\; = \;\; \bar{\lambda}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} \; , \qquad \tilde{\bar{v}} \; = \; \bar{\mu}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} \; , \tag{H.0.19}$$

with

$$\bar{\eta}_{\dot{\alpha}} \;\; = \;\; \frac{[(1+a^2)\bar{y} + 2\bar{a}y]_{\dot{\alpha}}}{1-a^2} \; , \qquad [\bar{\eta}_{\dot{\alpha}}, \bar{\eta}_{\dot{\beta}}]_\star \; = \; 2i\epsilon_{\dot{\alpha}\dot{\beta}} \; . \tag{H.0.20}$$

Finally, using $[\eta_\alpha, \bar{\eta}_{\dot{\alpha}}]_\star = 0$, we deduce that

$$V \;\; = \;\; L^{-1} \star P \star L \; = \; \sum_{n_1,n_2} \theta_{n_1,n_2} P_{n_1,n_2}(\tilde{u},\tilde{v};\tilde{\bar{u}},\tilde{\bar{v}}) \; . \tag{H.0.21}$$

Appendix I

Traces and Projectors in Oscillator Algebras

In this appendix we collect some basic properties of the representation theory of a single oscillator of importance to the doublet-oscillator realization of $\mathfrak{so}(5;\mathbb{C})$ in Section 7.8.

Trace and Supertrace

We start from the complexified Heisenberg algebra

$$u \star v - v \star u \;\; = \;\; 1 \; , \tag{I.0.1}$$

generates an associative $\star$-product algebra of Weyl-ordered functions $f(u,v)$ with product

$$f \star g \;\; = \;\; \int_{\mathbb{C}\times\mathbb{C}} \frac{d\xi d\bar{\xi} d\eta d\bar{\eta}}{\pi^2} \; e^{2i(\bar{\xi}\eta+\bar{\eta}\xi)} f(u+\xi, v+\bar{\xi}) g(u+i\eta, v-i\bar{\eta}) \; , \tag{I.0.2}$$

where $d\xi d\bar{\xi} = 2d(\mathrm{Re}\xi)d(\mathrm{Im}\xi)$. The algebra admits two inequivalent hermitian conjugation rules,

$$u^\dagger \;\; = \;\; v \; , \qquad v^\dagger \; = u \; , \tag{I.0.3}$$

$$u^\ddagger \;\; = \;\; -v \; , \qquad v^\ddagger \;\; = \;\; -u \; . \tag{I.0.4}$$

It also admits two inequivalent traces, namely, the cyclic trace

$$\mathrm{Tr}_+(f) \;\; = \;\; \int_{\mathbb{C}} \frac{du d\bar{u}}{2\pi} f(u,\bar{u}) \; , \tag{I.0.5}$$

obeying

$$\mathrm{Tr}_+(f \star g) \;\; = \;\; \mathrm{Tr}_+(fg) \;\; = \;\; \mathrm{Tr}_+(g \star f) \; , \tag{I.0.6}$$

up to boundary terms, and the graded-cyclic trace

$$\mathrm{Tr}_-(f) \;=\; \frac{f(0,0)}{2}\;, \tag{I.0.7}$$

obeying

$$\mathrm{Tr}_-(f\star g) \;=\; (-1)^{\epsilon(f)\epsilon(g)}\mathrm{Tr}_-(g\star f)\;, \tag{I.0.8}$$

for functions f and g with definite parity, $f(-u,-v) = (-1)^{\epsilon(f)}f(u,v)$ *idem* g. Note that the functions f and g in the argument of the traces are always supposed to be Weyl-ordered.

Let us show that

$$\mathrm{Tr}_\pm(f) \;=\; \mathrm{Tr}_\mp((-1)^N_\star\star f)\;, \qquad N \;=\; v\star u\;, \tag{I.0.9}$$

where we use the notation

$$x^A_\star \;=\; \exp_\star(A\ln x)\;, \qquad \exp_\star A \;=\; \sum_{n=0}^\infty \frac{A^{\star n}}{n!}\;, \qquad A^{\star n} \;=\; \underbrace{A\star\cdots\star A}_{n\ \text{times}}\;. \tag{I.0.10}$$

To this end, we begin by deriving the lemma

$$\exp_\star(\alpha w) \;=\; \frac{\exp(2w\tanh\frac{\alpha}{2})}{\cosh\frac{\alpha}{2}}\;, \qquad w \;=\; N+\tfrac12 \;=\; uv\;, \tag{I.0.11}$$

with $\alpha\in\mathbb{C}\backslash\{\pm i\pi,\pm 3i\pi,\dots\}$. This follows by acting on both sides of (I.0.11) with $\partial/\partial\alpha$ and using

$$w\star f(w) \;=\; \left(w-\frac14\frac{\partial}{\partial w}-\frac14 w\frac{\partial^2}{\partial w^2}\right)f(w)\;. \tag{I.0.12}$$

Thus, setting $\exp_\star\alpha w = r(\alpha)\exp(s(\alpha)w)$, one finds $r' = -rs/4$ and $s' = 1-s^2/4$, subject to $r(0)=1$ and $s(0)=0$, with the solution $r^{-1}=\cosh(\alpha/2)$ and $s = 2\tanh(\alpha/2)$. To proceed with the proof of (I.0.11), we need to examine the nature of $\exp_\star(i(\pi+\epsilon)N)$ more carefully in the singular limit $\eta = -\sin(\epsilon/2)\to 0$. Here,

$$\exp_\star(i(\pi+\epsilon)N) \sim -i\frac{\exp\frac{2iuv}{\eta}}{\eta}\;, \tag{I.0.13}$$

so that, using (I.0.6),

$$\lim_{\epsilon\to 0}\mathrm{Tr}_+(\exp_\star(i(\pi+\epsilon)N)\star f) \;=\; -i\lim_{\eta\to 0}\int_{\mathbb{C}}\frac{dud\bar u}{2\pi}\frac{\exp\frac{2iu\bar u}{\eta}}{\eta}f(u,\bar u) \;=\; \frac{f(0,0)}{2}\;. \tag{I.0.14}$$

Moreover, from $\exp_\star(\alpha N) \star \exp_\star(\beta N) = \exp_\star((\alpha+\beta)N)$, it follows, using the regularized Weyl-ordered form, that

$$\lim_{\epsilon\to 0} [\exp_\star(i(\pi+\epsilon)N)]^{\star 2} = \lim_{\epsilon\to 0} \exp_\star(2i(\pi+\epsilon)N) \tag{I.0.15}$$

$$= \lim_{\epsilon\to 0} \frac{\exp(2\tanh(i(\pi+\epsilon))N - i(\pi+\epsilon))}{\cosh i(\pi+\epsilon)} = 1 \ , \tag{I.0.16}$$

in agreement with $(-1)^N_\star \star (-1)^N_\star = (-1)^{2N}_\star = 1$.

Inner Products and Projectors

The two inequivalent traces $\mathrm{Tr}_\pm$ are related to two inequivalent inner products ${}_\pm\langle\Psi|\Psi'\rangle \equiv I_\pm(|\Psi\rangle, |\Psi'\rangle)$ on the Fock space

$$\mathcal{F} = \mathrm{span}_{\mathbb{C}} \left\{ |n\rangle = \frac{v^n}{\sqrt{n!}} |0\rangle \right\}_{n=0}^{\infty} \ , \qquad u|0\rangle = 0 \ , \tag{I.0.17}$$

defined by

$$I_\pm(\mu|m\rangle, \nu|n\rangle) = \bar{\mu}\nu(\pm 1)^m \delta_{mn} \ , \qquad \mu,\nu \in \mathbb{C} \ . \tag{I.0.18}$$

The relation is

$$I_\pm(|m\rangle, |n\rangle) = \mathrm{Tr}_\pm(P_{n,m}) \ , \tag{I.0.19}$$

where

$$P_{n,m} = \frac{1}{\sqrt{m!n!}} v^n \star P_{0,0} \star u^m \ , \qquad P_{0,0} = 2e^{-2w} \ , \tag{I.0.20}$$

can be identified as

$$P_{n,m} = |n\rangle_+\langle m| = |n\rangle\langle m| \ , \tag{I.0.21}$$

where we use the convention that $\langle\Psi| = {}_+\langle\Psi|$. To show (I.0.19), we use the cyclicity properties of $\mathrm{Tr}_\pm$ and

$$u \star P_{0,0} = P_{0,0} \star v = 0 \ , \tag{I.0.22}$$

to derive $\mathrm{Tr}_\pm(P_{n,m}) = (\pm 1)^n \delta_{mn} \mathrm{Tr}_\pm(P_{0,0})$ where $\mathrm{Tr}_\pm(P_{0,0}) = 1$, which yields the desired result. As a byproduct of (I.0.22) it follows that

$$P_{m,n} \star P_{p,q} = \delta_{np} P_{m,q} \ . \tag{I.0.23}$$

The relation (I.0.9), between the traces, implies

$$ {}_{\pm}\langle\Psi|\Psi'\rangle \;=\; {}_{\mp}\langle\Psi|(-1)^N_\star|\Psi'\rangle \;, \tag{I.0.24}$$

inducing the hermitian conjugation rules (I.0.3) and (I.0.61), as follows

$$ I_\pm(f|\Psi\rangle,|\Psi'\rangle) \;=\; I_\pm(|\Psi\rangle,g|\Psi'\rangle)\;, \qquad g \;=\; \begin{cases} f^\dagger & \text{for } I_+ \\ f^\ddagger & \text{for } I_- \end{cases} \tag{I.0.25}$$

The operator $(-1)^N_\star$ has a finite normal-ordered representation,

$$ (-1)^N_\star \;=\; :e^{-2N}:\;, \tag{I.0.26}$$

as can be seen by computing the projectors on even and odd states,

$$ P_{(\pm)} \;=\; \sum_{n=0}^{\infty}\tfrac{1}{2}(1\pm(-1)^n)|n\rangle\langle n| \;=\; \tfrac{1}{2}:(e^N\pm e^{-N})|0\rangle\langle 0|:\;, \tag{I.0.27}$$

adding them, *viz.* $P_{(+)}+P_{(-)} =: e^N|0\rangle\langle 0| := 1$, to obtain $|0\rangle\langle 0| =: e^{-N}:$, from which it follows that

$$ P_{(\pm)} \;=\; \tfrac{1}{2}(1\pm :e^{-2N}:)\;. \tag{I.0.28}$$

Finally, as an explicit check of $\mathrm{Tr}_\pm(P_{n,n})=(\pm 1)^n$ one can compute explicitly

$$ P_{n,n} \;=\; |n\rangle\langle n| \;=\; 2(-1)^n e^{-2w}L_n(4w)\;, \tag{I.0.29}$$

either by direct evaluation of the $\star$-products in (I.0.20), or, by starting from (I.0.21) and using

$$:e^{au+bv}: \;=\; e^{au}_\star\star e^{bv}_\star \;=\; e_\star^{au+bv+\frac{1}{2}ab} \;=\; e^{au+bv+\frac{1}{2}ab}\;, \tag{I.0.30}$$

and $|0\rangle\langle 0| =: \exp(-uv):$, together with Fourier transformation techniques. This enables one to calculate

$$\begin{aligned} P_{n,n} \;&=\; |n\rangle\langle n| \;=\; \frac{1}{n!}:v^n e^{-vu}u^n: \\ &=\; \int\frac{dkd\bar{k}}{2\pi}:e^{-i(\bar{k}u+kv)-\bar{k}k}:L_n(\bar{k}k) \\ &=\; \sum_{p=0}^{n}\binom{n}{n-p}\frac{1}{p!}(\partial_u\partial_v)^p\int\frac{dkd\bar{k}}{2\pi}e^{-i(\bar{k}u+kv)-\frac{1}{2}\bar{k}k} \\ &=\; 2\sum_{p=0}^{n}\binom{n}{n-p}(-2)^p e^{-2w}L_p(2w) \\ &=\; 2(-1)^n e^{-2w}L_n(4w)\;, \end{aligned} \tag{I.0.31}$$

where

$$L_n(x) \;=\; \frac{1}{n!}e^x\frac{d^n}{dx^n}(e^{-x}x^n) \;=\; \sum_{p=0}^{n}\binom{n}{n-p}\frac{(-1)^p}{p!}x^p\ , \tag{I.0.32}$$

and the last step in the calculation follows from the identities

$$\sum_{n=0}^{\infty}(-z)^n L_n(2x) \;=\; \frac{e^{\frac{2xz}{1+z}}}{1+z} \;=\; \sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{n-p}(-2)^p L_p(x) z^n\ . \tag{I.0.33}$$

Anti-Fock Space and Discrete Maps

Next, we introduce the anti-Fock space

$$\mathcal{F}^- \;=\; \left\{|n\rangle^- \;=\; \frac{u^n}{\sqrt{n!}}\ |0\rangle^-\right\}_{n=0}^{\infty}\ , \qquad v|0\rangle^- \;=\; 0\ , \tag{I.0.34}$$

and define its two inequivalent duals $\mathcal{F}^{\star -}_{\pm}$ by

$${}^-_{\pm}\langle m|n\rangle^- \;=\; (\mp 1)^m\delta_{mn}\ . \tag{I.0.35}$$

One can identify ${}^-_{\pm}\langle m|n\rangle^- = \mathrm{Tr}_{\pm}(P^-_{n,m})$ with

$$P^-_{n,m} \;=\; |n\rangle^{--}\langle m| \;=\; \frac{1}{\sqrt{n!m!}}\ u^n\star P^-_{0,0}\star v^m\ , \qquad P^-_{0,0} \;=\; 2e^{2w}\ , \tag{I.0.36}$$

where we note that $P^-_{m,n}\star P^-_{p,q} = (-1)^n\delta_{np}P^-_{m,q}$ and

$$P^-_{n,n} \;=\; 2e^{2w}L_n(-4w)\ . \tag{I.0.37}$$

We also set $\mathcal{F}^+ = \mathcal{F}$ and $P^+_{n,m} = P_{n,m}$.

The $\star$-product algebra admits the automorphism π and anti-automorphism τ given by

$$\pi(f(u,v)) \;=\; f(iv,iu)\ , \qquad \tau(f(u,v)) \;=\; f(iu,iv)\ . \tag{I.0.38}$$

These exchange the $P^{\pm}$ projectors as follows

$$\pi(P^{\pm}_{n,m}) \;=\; i^{m+n}P^{\mp}_{n,m}\ , \qquad \tau(P^{\pm}_{n,m}) \;=\; i^{m+n}P^{\mp}_{m,n}\ . \tag{I.0.39}$$

Their self-compositions are given by

$$\pi\circ\pi \;=\; \tau\circ\tau \;=\; \mathrm{Ad}_{(-1)^{N'}_{\star}}\ , \qquad N' \;=\; \sum_{n=0}^{\infty}\left((w-\tfrac{1}{2})\star P_{n,n} + (w+\tfrac{1}{2})\star P^-_{n,n}\right) \tag{I.0.40}$$

where $\mathrm{Ad}_X(Y) = X \star Y \star X^{-1}$, while their mutual composition defines the reflector

$$R \;=\; \pi \circ \tau \;, \qquad R(f(u,v)) = \; f(-v,-u) \;, \qquad R \circ R \;=\; \mathrm{Id} \;. \tag{I.0.41}$$

We note that $(w \mp 1/2)|n\rangle^{\pm} = \pm n|n\rangle^{\pm}$ and ${}^{\pm}\langle n|(w \mp 1/2) = \pm n {}^{\pm}\langle n|$, while $w|n\rangle^{\pm} = \pm(n+\frac{1}{2})|n\rangle^{\pm}$ and ${}^{\pm}\langle n|w = \pm(n+\frac{1}{2}){}^{\pm}\langle n|$. The above maps extend naturally to

$$\pi \;:\; \mathcal{F}^{\pm} \oplus \mathcal{F}^{\star\pm} \to \mathcal{F}^{\mp} \oplus \mathcal{F}^{\star\mp} \;, \tag{I.0.42}$$

$$\tau \;:\; \mathcal{F}^{\pm} \to \mathcal{F}^{\star\mp} \;, \tag{I.0.43}$$

$$R \;:\; \mathcal{F}^{\pm} \to \mathcal{F}^{\star\pm} \;, \tag{I.0.44}$$

by defining

$$\pi(|0\rangle^{\pm}) \;=\; |0\rangle^{\mp} \;, \qquad \pi({}^{\pm}\langle 0|) \;=\; {}^{\mp}\langle 0| \;, \tag{I.0.45}$$

$$\tau(|0\rangle^{\pm}) \;=\; {}^{\mp}\langle 0| \;, \qquad \tau({}^{\pm}\langle 0|) \;=\; |0\rangle^{\pm} \;, \tag{I.0.46}$$

such that (I.0.40) hold in the generalized sense that $\mathrm{Ad}_X(|\Psi\rangle) = X|\Psi\rangle$ and $\mathrm{Ad}_X(\langle\Psi|) = \langle\Psi|X^{-1}$. It follows that

$$R(|0\rangle^{\pm}) \;=\; {}^{\pm}\langle 0| \;, \qquad R({}^{\pm}\langle 0|) \;=\; |0\rangle^{\pm} \;. \tag{I.0.47}$$

We see that the reflector acts in the real basis $\{|n\rangle^{\pm}, {}^{\pm}\langle n|\}$ as the hermitian conjugation (I.0.61). This is to be compared with the case of doublet-oscillators where the reflector is modified by the requirement thst it should preserve $SU(2)$ quantum numbers (spatial spins) while flipping $U(1)$ quantum numbers (the energy).

Projectors and w-Invariants

The projectors $P^{\pm}_{n,n}$ are special cases of the more general functions $M_\kappa = M_\kappa(w)$ obeying the w-invariance condition

$$(w-\kappa) \star M_\kappa \;\;=\;\; 0 \tag{I.0.48}$$

for $\kappa \in \mathbb{C}$. These functions can be written as

$$M_\kappa \;\;=\;\; \mathcal{N}_\kappa \oint_C \frac{d\alpha}{2\pi i} g^{(\kappa)}(\alpha) \;, \tag{I.0.49}$$

where $\mathcal{N}_\kappa \in \mathbb{C}$;

$$g^{(\kappa)}(\alpha) \;\;=\;\; e_\star^{\alpha(w-\kappa)} \;\;=\;\; (1+\tfrac{s}{2})^{\frac{1}{2}-\kappa}(1-\tfrac{s}{2})^{\frac{1}{2}+\kappa} e^{sw} \;, \qquad s \;=\; 2\tanh\tfrac{\alpha}{2} \;; \tag{I.0.50}$$

and C a contour encircling $i\pi$ clockwise, so that its image $\Gamma = s(C)$ encircles $[-2,2]$ counterclockwise. Taking C to be a small circle, Γ becomes a large contour. Enlarging C to the "box" $-C = \{i\epsilon + x : -L \leq x \leq L\} \cup \{L + ix : \epsilon \leq x \leq 2\pi - \epsilon\} \cup \{i(2\pi - \epsilon) - x : -L \leq x \leq L\} \cup \{-L + i(2\pi - x) : \epsilon \leq x \leq 2\pi - \epsilon\}$, Γ becomes a "dogbone" encircling $[-2,2]$. The normal-ordered form of the group elements reads

$$g^{(\kappa)}(\alpha) \;=\; (1+\lambda)^{\frac{1}{2}-\kappa} : e^{\lambda w} : \,, \qquad \lambda \;=\; \frac{s}{1-\frac{s}{2}} \;=\; e^{\alpha} - 1 \,. \tag{I.0.51}$$

The composition rule

$$e^{sw} \star e^{s'w} \;=\; \frac{1}{1+\frac{ss'}{4}} e^{s''w} \,, \qquad s'' \;=\; \frac{s+s'}{1+\frac{ss'}{4}} \,, \tag{I.0.52}$$

$$: e^{\lambda w} : \star : e^{\lambda' w} \;=\; : e^{\lambda'' w} : \,, \qquad \lambda'' \;=\; \lambda + \lambda' + \lambda\lambda' \,, \tag{I.0.53}$$

implies

$$g^{(\kappa)}(\alpha) \star g^{(\kappa)}(\alpha') \;=\; g^{(\kappa)}(\alpha + \alpha') \;=\; g^{(\kappa)}(\alpha(s'')) \;=\; g^{(\kappa)}(\alpha(\lambda'')) \,. \tag{I.0.54}$$

Changing variables, one finds

$$M_\kappa \;=\; \mathcal{N}_\kappa \oint_\Gamma \frac{ds}{2\pi i(1-\frac{s^2}{4})} g^{(\kappa)}(\alpha(s)) \;=\; \mathcal{N}_\kappa \oint_{\Gamma'} \frac{d\lambda}{2\pi i(1+\lambda)} g^{(\kappa)}(\alpha(\lambda)) \,. \tag{I.0.55}$$

Using the Weyl-ordered form and (I.0.12), one can then easily verify (I.0.48). There exist other choices of contour C leading to w-invariants, such as $C = i[-\pi,\pi]$ and $C = \mathbb{R}$ leading to integrals over $U(1)$ and $GL(1;\mathbb{R})$, respectively, but we shall not consider them further here.

For $\kappa = \pm(n+1/2)$, Γ collapses to a circle around the pole at $s = \mp 2$, so that

$$M_{\pm(n+\frac{1}{2})} \;=\; 2(\mp 1)^n e^{\mp 2w} L_n(\pm 4w) \;=\; P^{\pm}_{n,n} \quad \text{for } \mathcal{N}_{n+\frac{1}{2}} = 1,\, \mathcal{N}_{-n-\frac{1}{2}} = (-1)^{n+1} \tag{I.0.56}$$

Using the change of variables found in [77], and making use of analytical continuation and deformations of contours, one can calculate

$$(M_{\pm(n+\frac{1}{2})})^{\star 2} \;=\; \mathcal{N}_{\pm(n+\frac{1}{2})} \oint_{\mp 2} \frac{ds}{2\pi i(1-\frac{s^2}{4})} M_{\pm(n+\frac{1}{2})} = \pm \mathcal{N}_{\pm(n+\frac{1}{2})} M_{\pm(n+\frac{1}{2})} = (-1)^n M_{\pm(n+\frac{1}{2})} \,, \tag{I.0.57}$$

in agreement with $(P^{\pm}_{n,n})^{\star 2} = (-1)^n P^{\pm}_{n,n}$. For $\kappa \notin (\mathbb{Z} + 1/2)$, the branch cut prevents Γ from collapsing, so that deforming back the contour it now encircles both $s = -2$ and $s = 2$, with the result that

$$M_\kappa^{\star 2} \;=\; \mathcal{N}_\kappa \oint_\Gamma \frac{ds}{2\pi i(1-\frac{s^2}{4})} M_\kappa \;=\; 0 \,. \tag{I.0.58}$$

Interestingly, the case of $\kappa = 0$ is closely related to the dressing function of the 5D higher-spin gauge theory based on spinor oscillators, introduced in [89] and later analyzed in more detail in [77]. The above analysis suggests that a suitable regularization of the dressing function has a well-defined *vanishing* self-composition, which would greatly simplify the perturbative weak-field expansion.

Fermionic Oscillators

It is also interesting to look along similar lines at the complexified Clifford algebra

$$\{\gamma,\delta\}_\star \;\equiv\; \gamma\star\delta+\delta\star\gamma \;=\; 1\ , \tag{I.0.59}$$

with Weyl-ordering defined by $\gamma\star\delta = \gamma\delta + 1/2$ where $\gamma\delta = [\gamma,\delta]_\star/2 = -\delta\gamma$. The corresponding Fock space consists of a vacuum state $|0\rangle$ obeying $\gamma|0\rangle = 0$ and an excited state $|1\rangle = \delta|0\rangle$. The algebra (I.0.59) admits two inequivalent hermitian conjugation rules,

$$\gamma^\dagger \;=\; \delta\ ,\qquad \delta^\dagger \;=\; \gamma\ , \tag{I.0.60}$$

$$\gamma^\ddagger \;=\; -\delta\ ,\qquad \delta^\ddagger \;=\; -\gamma\ . \tag{I.0.61}$$

corresponding to the inner products

$$I_\pm(f\star|\Psi\rangle,|\Psi'\rangle) \;=\; I_\pm(|\Psi\rangle,g\star|\Psi'\rangle)\ ,\qquad g \;=\; \left\{\begin{array}{ll} f^\dagger & \text{for } I_+ \\ f^\ddagger & \text{for } I_- \end{array}\right.\ , \tag{I.0.62}$$

related by

$$_\pm\langle\Psi|\Psi'\rangle \;=\; _\mp\langle\Psi|\star(-1)^F_\star\star\Psi'\rangle\ ,\qquad F \;=\; \delta\star\gamma\ . \tag{I.0.63}$$

We note that $_\pm\langle 1|1\rangle = \pm_\pm\langle 0|0\rangle$, and we choose $_\pm\langle 0|0\rangle = 1$. The description in terms of states and inner products can be replaced by the dual $\star$-algebra picture that makes use of the projectors

$$P_0 \;=\; |0\rangle\langle 0| \;=\; 1-\delta\star\gamma \;=\; \tfrac{1}{2}(1-2\delta\gamma)\ , \tag{I.0.64}$$

$$P_1 \;=\; |1\rangle\langle 1| \;=\; \delta\star\gamma \;=\; \tfrac{1}{2}(1+2\delta\gamma)\ , \tag{I.0.65}$$

and the trace operations

$$\mathrm{Tr}_+(f) \;=\; 2\,f(0,0)\ , \tag{I.0.66}$$

$$\mathrm{Tr}_-(f) \;=\; -\int d\gamma d\bar\gamma\, f(\gamma,\bar\gamma)\ , \tag{I.0.67}$$

where $f = f(\gamma,\delta)$ is Weyl-ordered and we use the Berezin integration rule $\int d\gamma d\bar\gamma\;\bar\gamma\gamma = 1$. Interestingly enough, comparing to bosons, the definitions of the traces Tr_+ and Tr_- are interchanged. For example, we have $\mathrm{Tr}_+(P_0) = \mathrm{Tr}_+(P_1) = 1$ while $\mathrm{Tr}_-(P_0) = -\mathrm{Tr}_-(P_1) = 1$. Finally, one can show that

$$\mathrm{Tr}_\pm((-1)^F_\star \star f) \;=\; \mathrm{Tr}_\mp(f)\ . \tag{I.0.68}$$

To this end, we first simplify

$$\int d\gamma d\bar\gamma\, f \star g \;=\; \int d\gamma d\bar\gamma\, fg\ , \tag{I.0.69}$$

using the fact that the terms in $f \star g$ involving at least one contraction do not survive the integration, as can be seen directly by expanding $f = f_0 + f_1\gamma + f_2\bar\gamma + f_3\bar\gamma\gamma$, *idem* g. Then, from $F = P_1$ it follows that

$$(-1)^F_\star \;=\; \exp_\star(i\pi F) \;=\; 1 + \sum_{n=1}^{\infty} \frac{(i\pi)^n}{n!} P_1 \;=\; 1 + (e^{i\pi} - 1)P_1 \;=\; 1 - 2P_1 \;=\; -2\bar\gamma\gamma \tag{I.0.70}$$

so that, using (I.0.69),

$$\mathrm{Tr}_-((-1)^F_\star \star f) \;=\; -\int d\gamma d\bar\gamma\,(-2\bar\gamma\gamma) f(\gamma,\bar\gamma) \;=\; 2f(0,0) \;=\; \mathrm{Tr}_+(f)\ \ , \tag{I.0.71}$$

where integration is performed using $\bar\gamma\gamma = \delta(\bar\gamma)\delta(\gamma)$. The converse, *i.e.* $\mathrm{Tr}_+((-1)^F_\star \star f) = \mathrm{Tr}_-(f)$, then follows by letting $f \to (-1)^F_\star \star f$ and using $(-1)^F_\star \star (-1)^F_\star = (-2\bar\gamma\gamma) \star (-2\bar\gamma\gamma) = 1$.

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