1 Numerical integration on the Lorenz system

1. Integrate numerically the system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - xz - y \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

   up to t=60, with a time step $dt = 0.005$ using the Euler forward method, with the following sets of values for the parameters

   - Set A: $\sigma, b, r = (10, 8/3, 28)$
   - Set B: $\sigma, b, r = (10, 8/3, 9)$

   and the initial condition $L_0 \equiv (x_0, y_0, z_0) = (9, 10, 20)$. Plot trajectories $L(t, L_0)$ in the (x, z) space and describe the result (max 100 words).

2. Repeat 1. with $L'_0 = (x'_0, y'_0, z'_0) = (9 + \epsilon, 10, 20)$ with $\epsilon = 1 \times 10^{-10}$. Plot $x(t)$ for the two solutions $L(t, L_0)$ and $L(t, L'_0)$ and their difference. Describe the result (max 100 words).

3. For the set of parameters A, treat $L(t, L_0)$ as *an observed state* and call it $L_{true} = (x_{true}, y_{true}, z_{true})$ and $L(t, L'_0) = (x', y', z')$ as an integration affected by an error ϵ in the initial condition. Compute the *root-mean-square error* (RMSE)

$$R(t) = \sqrt{(x_{true}(t) - x'(t))^2 + (y_{true}(t) - y'(t))^2 + (z_{true}(t) - z'(t))^2}$$

   as function of lead time and plot it on a linear and semilog scale. Describe the result (max 150 words). Repeat with $\epsilon = 1 \times 10^{-5}$, $\epsilon = 1 \times 10^{-3}$ and $\epsilon = 10^{-1}$ and fill a table with the values of ϵ and the first time t_a such that $R(t) > a$ with $a = 0.5$. What can we say about the predictability time and how it varies with the error ϵ? Discuss in no more than 200 words.

4. Generate an *ensemble of initial conditions* by perturbing the initial condition L_0 with a random perturbation $E_k \equiv (\epsilon_k, 0, 0)$ for $k = 1, ..., N$ and N=100. Suppose for simplicity that ϵ_k is uniformly distributed between -0.75 and 0.75. Note that numpy.random.random() generates a random real number between 0 and 1 sampling from a uniform distribution. For each member of the ensemble $L'_k(t, L_0 + E_k)$ integrate the system up to t=4 and compute the root mean square error

$$R_k(t)$$

   and the root mean square error of the average $\langle L \rangle$, where

$$\langle \dots \rangle = \frac{1}{N} \sum_{k=1}^{N} (\dots).$$

Compare the latter with the average mean square error $\langle R \rangle = \dfrac{1}{N}\sum_{k=1}^{N} R_k$ by plotting them as function of lead time. Describe and discuss the result.

5. (OPTIONAL) With the ensemble generated in 4, compute

$$S = \sqrt{\langle (x_k - \langle x_k \rangle)^2 \rangle}$$

Analyse S up to t=4 for different initial conditions and compare it with the RMSE of the ensemble mean. Discuss the interpretation of the result.

6. (OPTIONAL) Integrate numerically the system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - xz - y + f(t) \\ \dot{z} = xy - bz \end{cases} \tag{2}$$

up to t=500, with $f(t) = 0$ and with an ensemble of 50 initial conditions with a uniform distribution of x between -20 and -20, y between -20 and 20, and z between 0 and 30. For each member k, consider subsets of the time series each 5000 time step long, from $1 + \tau$ to $5000 + \tau$ with $\tau = 1, 2, ..., M$ (choose the maximum M given the length of the time series). For each member, sample the probability p_x that $x < 0$ as the number of time steps with $x < 0$ divided by the total number of time steps and estimate it in chunks of 5000 time steps, compute this value for each τ and assign it to an array $P_k(\tau)$. Plot $P_k(\tau)$ for each member and plot the mean $\langle P_k \rangle$, all in the same panel (ax.plot(...,alpha=0.5,...) can be used to plot semi-transparent lines). Repeat all these tasks with $f(t) = \frac{1}{3}\sqrt{t}$. Plot also $f(t)$. Describe the result and discuss what we can conclude about the interpretation of the response of a chaotic system to a forcing f (max 200 words).

7. Write a short report (max 4 pages), including a brief introduction, figures and text produced during the session (points 1-4 and 5-6 where applicable) and a conclusive comment.