Dimensions of Diversification

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Smart beta is often framed as an active investment decision; it is said to imply belief in the systematic mispricing of securities whether the weighting scheme deviates from a benchmark because of explicitly targeting certain types of security or simply because it uses an alternative weighting method. For instance, equal weighting an equity universe clearly has a bias toward small-capitalization stocks when compared with capitalization weighting. It will outperform when smaller companies outperform larger companies, and to buy one over the other implies a belief in just such a "size effect." There is, however, a different reason to eschew market capitalization weighting that is focused on risk rather than return. Diversification-based portfolio construction methods aim to improve risk-adjusted returns not by targeting assets that are expected to outperform but by diversifying the risks of the constituents better than the average market participant. It can be challenging to disentangle the impact of these two perspectives on smart beta as both can contribute to the risk-adjusted return profile (for an example of work testing the attribution of smart beta to factor tilts, see Amenc, Goltz, and Lodh [2016]).

The motivation for diversification-based portfolio construction can be found in the historical behavior of cap-weighted indexes. At the height of the tech bubble of the late 1990s, more than 40% of the S&P 500 Index was made up of technology and telecommunications companies. Microsoft and Cisco each alone comprised over 4% of the index. The fortunes of index investors (and correspondingly of the average active investor) were dominated by the impact of a single phenomenon: the burgeoning technology industry. Although extreme, this scenario illustrates the purpose of diversification-based portfolio construction: A portfolio allocating equal risk to each sector would not have experienced as much volatility as the index and would have lessened the drawdown as valuations fell back from their peak. When investors are concerned about shocks affecting specific market segments, they can lessen their exposure by diversifying across those segments.

Concentration of risk may be viewed in terms of the contribution of factors to returns. As this view of risk has grown in significance, so too has grown the number of risk factors alleged to contribute to the cross-section of asset returns and, hence, to portfolio performance. The elements of the "factor zoo" often conflict or intersect with each other, making it challenging to identify which factors are relevant to a given portfolio. It is clear, however, that when the industry talks about diversification, it is now referring to not only the division of capital between a large number of securities but also the avoidance of risk concentration in any of these factors.

Major index providers now routinely offer so-called smart beta indexes in addition to the traditional market-cap-weighted representations of segments of the equity universe (see, for example, Cheema [2015]). The strong historical performance of non-cap-weighted portfolios may be attributed to at least two complementary sources. First, these portfolios may capture implicitly some exposure to known risk premia, as outlined in Chow et al. [2011], for example. Second, they may allow superior diversification, which in turn reduces their exposure to downside risks and improves their long-term compounding returns (Amenc, Goltz, and Lodh [2012]). When seeking such diversification, the first hurdle is to decide along which lines a portfolio should be diversified. Allocating equal risk by asset class is a widespread idea, implemented in various forms as "risk parity" (Chaves et al. [2011]). In a similar way, one could allocate equal risk to equities along sector, country, or style lines. A prudent portfolio manager ought to be aware of risk concentration in any of these dimensions. For systematic portfolio construction to compete with manager-driven solutions, it must account for the multidimensional nature of risk.

In this article, we present a way to measure and hence maximize diversification simultaneously across multiple dimensions. We base our work on analysis of categorical properties of holdings rather than their return time series. In particular, we look at the distribution of stand-alone risk across each set of categories: the dimensions of diversification. This has the advantage of being applicable where returns are not available or are untrustworthy. It is also avoids the trap of dependence on recent history and leads to more robust portfolios—ones that are less sensitive to estimation error and that may also have lower turnover. We construct an objective function that can be applied to both the discrete problem of security selection and the continuous problem of position sizing. This gives rise to an optimal diversification methodology with a more explicit multidimensional aspect than existing methods. It also suggests a comparative statistic for assessing the relative diversification of existing portfolios. We show the practical value of this approach by using it to backtest equity portfolios.

DIVERSIFICATION MEASURES

At the most basic level, diversification can be thought of in terms of the number of securities held in a portfolio. Indeed, in studies of individual investor behavior, this is often viewed as the first-order measure of diversification (e.g., Goetzmann and Kumar [2008]) and is often sufficient to report apparently irrational behaviors in the case of individuals. More color can be added by calculating notional diversification measures, such as the sum of squared portfolio weights. This incorporates the information about the concentration of positions in notional terms as well as the number of positions but has no intuitive interpretation. A further level of detail is given by the structure of the joint distribution of returns. One simple summary of this is the average pairwise correlation; portfolios holding assets with lower average correlation are, all else equal, better diversified.

Portfolio diversification can also be viewed through the lens of total portfolio performance. The aim of a diversified portfolio is, after all, to exhibit better risk-adjusted returns. Using risk measures as an assessment of diversification can be confounded by portfolios having large positions in less risky assets, which can exhibit lower risk while being less diversified. Some correction for the average risk of assets in the portfolio gives rise to an effective measure of the portfolio diversification. Choueifaty and Coignard [2008] used an example of this type of measure. Alternatively, diversification can be assessed by the impact on compounding of returns, as in Booth and Fama [1992].

DIVERSIFIED PORTFOLIO CONSTRUCTION

Portfolio construction with the explicit aim of maximizing diversification, known sometimes as risk budgeting, is an idea growing in popularity. Allocation of equal risk, risk minimization, and diversification maximization are positioned as competitors to so-called efficient portfolio construction; they are vaunted for their robustness to estimation error and for their simplicity (Duchin and Levy [2009]). They are also favorably compared with market-cap weighting, which is said to be over-concentrated (Malevergne, Santa-Clara, and Sornette [2009]) and to overweight overpriced securities (Treynor [2005]). Noteworthy, diversification-based methods of portfolio construction include minimum variance (Clarke, de Silva, and Thorley [2006]); risk parity or volatility parity, where each allocation has equal stand-alone volatility (Asness, Frazzini, and Pedersen [2012]); equal risk contribution (ERC), where the marginal contribution to risk of each allocation is the same (Maillard, Roncalli, and Teïletche

[2010]); and maximum diversification (Choueifaty and Coignard [2008]). All of these are used in practice, most prominently in smart beta products (see Exhibit 1). Also widely used are equal notional weighting (also called *1/N* weighting), which requires no parameter estimation at all, and fundamental weighting, which bases index weights on some fundamental property of companies, such as sales or book value of equity.

Portfolio construction methods may be placed on a spectrum between optimal and robust. At one extreme are the highly parameterized, high-fidelity models that seek to find portfolios with certain optimality properties under strict assumptions about the model of the joint return distribution. The most famous of these is mean–variance optimization, but Black–Litterman methods and more general utility maximization also rely on similarly normative approaches. Practitioners often seek to move down the spectrum to reduce the risk of estimation error and increase robustness to regime changes in asset price comovement. Robustification can be achieved using, among other things, parameter shrinkage (Ledoit and Wolf [2003]), penalized objective functions in optimization (Ceria and Stubbs [2006]), or portfolio resampling (Michaud and Michaud [2008]). The resulting portfolios avoid the sometimes overcomplicated solutions uncovered by optimization and, as a result, can be more intuitive as well as delivering more consistent risk properties and lower turnover. Taking this idea to its extreme leads to risk budgeting. To varying degrees, risk budgeting methods are less dependent on input parameters. Indeed, the ultimate robust method, equal notional weighting, is completely free from inputs.

The problem with optimized risk budgeting, such as ERC or maximum diversification, is that these methods assume knowledge of the joint distribution of asset returns. This is, in a sense, contrary to the aims of robust portfolio construction. It requires prediction of parameters that cannot be known a-priori and is thus subject to estimation error. Robust parameter estimation can help to protect against this, but a better approach to diversification might make use of additional information outside of asset price comovement. In particular, assets can often be categorized into segments of the investable universe between which cross-correlation is expected to be lower. This information can be more robust and persistent than features of price action; a utility company is unlikely to become a technology company overnight. Using the weights allocated to each sub-segment, diversification can be calculated along each of a number of intersecting dimensions, giving a broader view of the diversification of the portfolio. An aggregate function of these dimensions can be used as an objective for optimization, leading to a portfolio diversified as well as possible along each categorical dimension.

WHICH DIMENSIONS MATTER?

The decision of allocation across asset classes is a very important one (see Xiong et al. [2010]). Here, because the empirical assessment will use equities, we focus on the dimensions relevant to equity portfolios. For different asset classes some, but not all, of these dimensions will still apply.

1. *Sector.* Diversification across sectors gives exposure to the different economic drivers of sectors. Categorization for some companies can be contentious,

EXHIBIT 1

A Selection of Diversification-Based Funds Illustrating the Adoption of Risk Budgeting Ideas in the Smart Beta Portion of the Investment Industry

Fund	Weighting Scheme	AUM (billions)
IShares MSCI USA Minimum Volatility ETF	Constrained volatility minimization	US$ 10.59
Most Diversified Portfolio SICAV–TOBAM Anti-Benchmark US Equity Fund	Maximum diversification	US$ 0.74
Lyxor ETF MSCI Europe ERC	Equal risk contribution	€ 0.25
Powershares FTSE RAFI US 1000 ETF	Fundamental	US$ 3.97
Guggenheim S&P Equal Weight ETF	Equal notional	US$ 8.76

Source: Bloomberg, as of March 2016.

but so long as each sector is sufficiently broad, these concerns will be marginal.

2. *Country.* Diversification across countries reduces exposure to the fortunes of any given economy. With the increasing integration of global markets, the diversification benefit is thought to have shrunk, but not vanished (Goetzmann, Li, and Rouwenhorst [2005]).
3. *Style.* By style we refer to the segments of the equity universe identifiable using simple, measurable properties and related to established risk premia. Diversification across the capitalization spectrum and between value and growth, for instance, would be the most prominent examples of this. Ideally, these dimensions would be orthogonal. This is often not the case because equity styles sometimes imply a sector bias. For example, value indexes often overweight utility companies if not constructed in a sector-neutral fashion.
4. *Company.* All else equal, the greater the dispersion across individual names, the better the portfolio diversification. Much has been made of the number of stocks required to diversify (see, for example, Campbell et al. [2001]), but it is widely accepted that concentration in single stocks is a bad thing.

MEASURING DIVERSIFICATION IN A SINGLE DIMENSION AND IN AGGREGATE

Diversification measures can be constructed using the weight allocated to each bucket of a given dimension. This allocation can be in either notional terms or in terms of a measure of risk contribution, such as stand-alone volatility. It should be a concave function, because a weighted combination of two portfolios will always have equal or lower risk than the weighted average of their stand-alone volatilities. For stand-alone risk weights p_i, one possible measure of diversification is the entropy function in Equation (1). This is, in a sense, related to the information theoretical interpretation of the function: the Shannon entropy, used as a measure of information content (or equivalently disorder) in a probability distribution. Higher dispersion in portfolio allocation implies less confidence in the relative performance of assets: an investor with certain knowledge of which asset will have the greatest return should allocate 100% to that stock (minimum entropy) and an investor with less confidence should invest in a more diversified, higher entropy portfolio.

$$\mathrm{H(p)} = -\sum_i p_i \log p_i \tag{1}$$

In the absence of constraints, the entropy is maximized when equal weight is allocated to each bucket. The maximum entropy portfolio is the portfolio of least conviction. Using notional weight, this can be used to motivate naïvely diversified ($1/n$) portfolios and using volatility or variance weights to motivate equal risk portfolios.

Entropy has been used as a measure of diversification (see, for example, Staines [2016]) and indeed proposed as an objective in portfolio optimization (see Bera and Park [2008] or Neukirch [2008]). The concavity of the entropy problem makes maximization generally easy even in the presence of constraints. Some additional thought is required when any asset has zero weight, because the function is not defined in that case. The limit of the entropy discussed earlier gives a contribution of zero from a zero-weighted asset. Because a contribution of zero is also intuitive, we adopt this convention.

Another benefit of entropy is that the joint entropy provides a natural extension to simultaneously considering multi dimensions. Assuming independence of the dimensions this is given by Equation (2), where p_{ij} is the stand-alone risk weight allocated to category j in dimension i.

$$\mathrm{H(p)} = -\sum_i \left(\prod_j p_{ij}\right) \log\left(\prod_j p_{ij}\right) = -\sum_{i,j} p_{ij} \log p_{ij} \tag{2}$$

The contributions from each dimension can be weighted to reflect views on the relative importance of diversification of each dimension. As a starting point, each dimension can be weighted by the multiplicative inverse of the logarithm of the number of buckets over which it is distributed. In this case, each dimension contributes equally if the portfolio is perfectly diversified across all dimensions.

We refer to the portfolio that optimizes this reweighted objective (the continuous problem) as the multi-dimensional optimized portfolio (MDOP). This is simply the portfolio, out of all possible 100% invested, long-only portfolios, with highest portfolio entropy. This may lead to undesired concentration in individual assets under certain circumstances unless the assets themselves are added as an additional dimension (i.e., with each category containing only one asset). Another option is to select a subset of the universe using categorical diversification and equal weight these. We call the solution to this combinatorial problem the multi-dimensional selected portfolio (MDSP). This is the portfolio out of all possible $1/N$ portfolios with fixed N, with highest portfolio entropy.

$$w_{MDOP} = \underset{w}{\operatorname{argmax}}\ H\left(p(w)\right) \sum_n w_n = 1,\ w_n \geq 0 \quad (3)$$

$$w_{MDSP} = \underset{w}{\operatorname{argmax}}\ H\left(p(w)\right) \sum_n w_n = 1,\ w \in \left[0, \frac{1}{N}\right]^N \quad (4)$$

The MDOP is easily found using a gradient-following method, so long as any constraints are convex. The MDSP can be more challenging, but stochastic optimization such as simulated annealing can be employed, and even greedy local searches will give rise to acceptable approximate solutions.

DIMENSIONS OF DIVERSIFICATION IN PRACTICE: EQUITY PORTFOLIOS

For long-only equity portfolios in the absence of active views, the aim should be to diversify along each of the dimensions described previously. For this case study, we build diversified portfolios (both MDSP and MDOP) of large-cap U.S. equities with the S&P 500 making up the opportunity set. We attempt to diversify using dimensions based on sectors (using the Bloomberg Industry Classification level 1) and quintiles of the universe in terms of market cap and price-to-book ratio. For MDSP, we select portfolios of 50 stocks. Whenever possible, at each rebalance we retain any stocks whose characteristics match those needed to minimize turnover. Each stock is equally weighted in terms of stand-alone volatility. For MDOP, we constrain the portfolio to 100% invested and prevent shorting, also adding the company diversification dimension explicitly into the objective so that no single stock is allocated too great a weight. For both entropy-based constructions, the dimensions are weighted $1/\log(n_i)$ where n_i is the number of categories in dimension i.

For comparison, we also assess the performance of a suite of risk budgeting approaches, as well as a cap-weighted benchmark. All are subject to the same transaction cost assumptions (20 basis points), contain only stocks for which all relevant data (including sector allocation and price to book ratio) are available, and are rebalanced at the end of each month. Equal risk refers to equal stand-alone volatility, and in the equal risk per-sector construction, every stock in a given sector is allotted the same volatility. ERC is constructed as in Maillard, Roncalli, and Teïletche [2010], and for both this and minimum volatility, the long-only and 100% invested constraints apply. Exhibit 2 shows the relative performance of the various risk budgeting methodologies.

All of the diversification methods show improvements in risk-adjusted return and risk metrics against a cap-weighted benchmark. MDOP is among the best performers across all measures, and MDSP performs remarkably well given that its diversification is inherently limited by holding just 50 stocks. One drawback is that it does incur significant turnover to maintain diversification with a smaller number of assets. Exhibit 3 shows that the most concentrated position in MDSP doesn't materially exceed that of the index, while MDOP has a far lower maximum concentration on average.

EXHIBIT 2
Performance of Weighting Schemes for U.S. Large-Cap Equity, October 1994–March 2016

	MDSP	MDOP	Cap Weighted	Equal Risk per Stock	Equal Risk per Sector	ERC	Minimum Volatility
Mean return	11.0%	13.6%	9.0%	13.8%	13.5%	**14.2%**	12.8%
Standard deviation	15.6%	15.4%	15.2%	15.4%	15.3%	15.4%	**13.2%**
Sharpe ratio	0.51	0.69	0.39	0.70	0.68	0.73	**0.75**
Max drawdown	52.4%	53.4%	54.6%	53.9%	**53.2%**	54.7%	54.9%
Monthly VaR 5%	6.1%	**5.7%**	7.3%	5.9%	5.9%	6.1%	5.8%
Annualized turnover	209%	69.6%	**9.6%**	64.4%	64.7%	65.7%	220%

Notes: Annualized statistics using monthly performance and monthly rebalancing. The drawdown figures use daily close data.

EXHIBIT 3
Maximum Position Sizes through Time for MDOP and MDSP Compared with the S&P 500

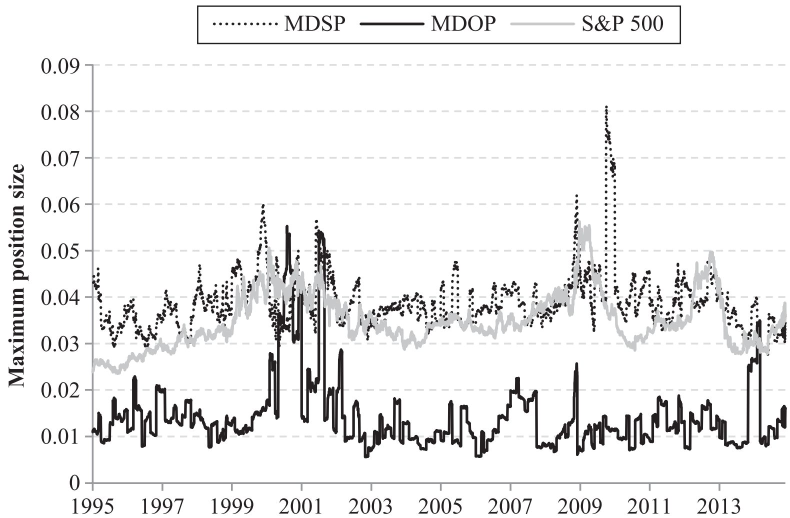


Exhibit 4 shows a breakdown of the contributions across each of the dimensions of diversification. Both MDOP and MDSP deliver their aim of more even distribution than the index. Particularly noteworthy are the avoidance of risk concentration through the tech bubble and the overweighting of small-cap stocks throughout the backtest. It happens that near-perfect distribution across each dimension has been possible at each rebalance, although this is not guaranteed in any environment.

Examining the sensitivity of the portfolio returns to Fama–French factors, the performance gains of both MDOP and MDSP can be shown to be largely explained by the historical performance of these factors. As evidenced by Exhibit 5, this is also the case for other risk budgeting approaches. The same reasoning that motivates the use of other risk budgeting techniques applies equally to multi-dimensional diversification. The differences in performance between the methods have limited significance. The most noteworthy difference between multi-dimensional diversified portfolios and ERC or minimum volatility is multi-dimensional diversified portfolios' greater sensitivity to the value factor, as outlined in Exhibit 5. This is surprising, because both MDOP and MDSP aim to be diversified across the valuation spectrum, but finding may be due to the compounding effect of overweighting both low-valued sectors and the low-valued companies within those sectors. If the intersectional properties of dimensions are not desired by a portfolio manager, then they could still apply multi-dimensional diversification using a more orthogonal definition of the relevant dimensions. Otherwise, this can be seen as simply an efficient way to capture desirable factors.

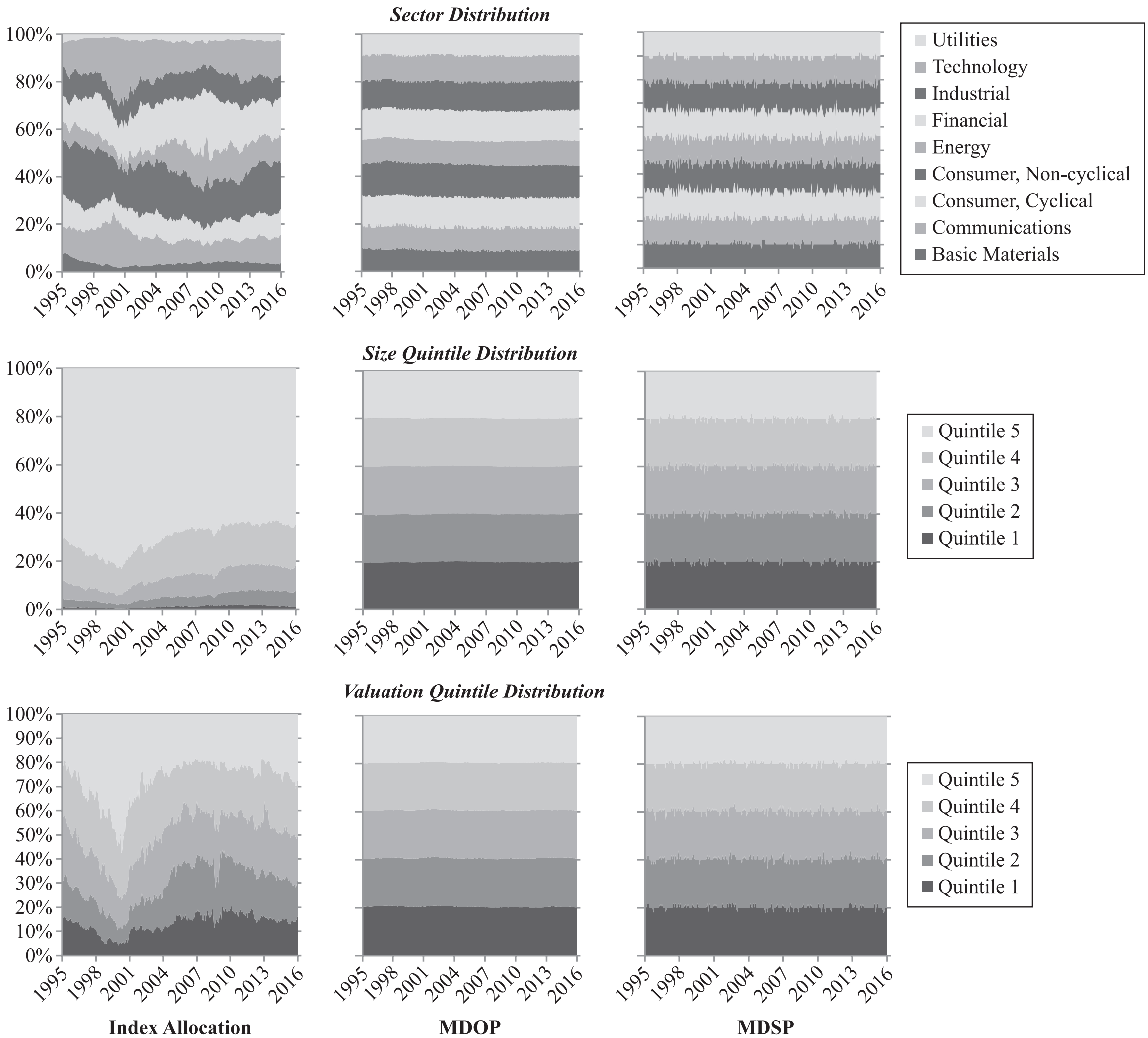
CONCLUSIONS

The construction of diversified portfolios remains an area for development. A weakness of existing methods, such as ERC or minimum variance, is their dependence on predicting features of the distribution of asset returns. This article shows that portfolios with similar properties can be constructed with less reliance on predicting correlations by categorizing assets along a set of measurable, economically meaningful dimensions. Measuring diversification along multiple dimensions better reflects the intersecting nature of the risk of a portfolio and permits the use of qualitative distinctions between assets to better understand the likely future correlation structure. Portfolio construction using this diversification measure to either select assets or to determine weighting can both result in improvements in risk-adjusted return and minimization of exposures to tail events.

One significant advantage of this type of diversification methodology is that it can be applied to assets for which there is insufficient price history to determine correlation. Key areas where this would be especially useful include fixed income, where the construction

EXHIBIT 4

Stand-Alone Risk Allocations by Sector, Size Quintile, and Valuation Quintile



Note: The leftmost column shows the index allocations; the middle, MDOP; and the rightmost, MDSP.

EXHIBIT 5

The Sensitivities of Diversified Portfolios in Multiple Regression

	MDSP	MDOP	ERC	Minimum Volatility
Residual Alpha	–0.28% (0.82)	0.66% (0.55)	0.15% (0.71)	0.81% (0.40)
Market	0.93 (<0.01)	0.86 (<0.01)	1.02 (<0.01)	0.77 (<0.01)
Value (low minus high)	0.30 (<0.01)	0.43 (<0.01)	0.29 (<0.01)	0.25 (<0.01)
Size (small minus big)	0.22 (<0.01)	0.17 (<0.01)	0.21 (<0.01)	0.15 (<0.01)
R^2	90%	89%	93%	88%

Notes: As with other diversification methods, the performance of the methodology described in this article is attributable to exposures to known premia. Significance values are shown in brackets. The value and size factors are top minus bottom quartiles of the S&P 500 by price-to-book ratio and market cap, respectively. The market factor is the total return of the S&P 500.

of a correlation matrix can be challenging because of issuance and maturity, and illiquid assets, which are only marked to market on a monthly basis. As with Barra-type analyses, this work recognizes the multi-dimensionality of risk but also accurately reflects the intersection between the dimensions in the case of non-orthogonality. Although multi-dimensional views of risk and diversification are well understood qualitatively, it is important that they be systemized for the investment community to be able to assess their value and truly understand their meaning.

Although the cap-weighted S&P is a very strong candidate as a benchmark for U.S. large-cap equity funds, it doesn't necessarily follow that any investor without a view should build a fully passive portfolio and invest in it without modification. Deviations from cap weighting toward risk budgeting require only the premise that an investor shouldn't bear concentrations of risk generated by the market as a whole. Because the majority of returns are explainable by the cap-weighted market factor, gains from such deviations are necessarily modest but can outweigh the cost of accessing them. The growth of smart beta and the types of methodology discussed in this article give investors an opportunity to access thoughtfully built portfolios without being forced to rely on (and pay for) the predictive capacity of active managers.

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