Práctica 6 .- Matemática 3.

1) $\cdot E(\bar{X}_1) = E\left(\frac{1}{n-1}\sum_{i=1}^{m}X_i\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{m}x_i\right) = \frac{1}{n-1}\sum_{i=1}^{m}E(x_i)$

$$= \frac{1}{n-1}\sum_{i=1}^{m}\mu = \frac{1}{n-1}m\mu = \frac{\mu n}{n-1}$$

$\cdot V(\bar{X}_1) = V\left(\frac{1}{n-1}\sum_{i=1}^{m}x_i\right) = \left(\frac{1}{n-1}\right)^2 \cdot V\left(\sum_{i=1}^{m}x_i\right)$

$$= \frac{1^2}{(n-1)^2}\sum_{i=1}^{m} \cdot V(x_i) = \frac{1}{(n-1)^2}\sum_{i=1}^{m}\sigma^2$$

$$= \frac{1}{(n-1)^2} \cdot n\sigma^2 = \frac{n\sigma^2}{n-1}$$

$$\boxed{\begin{array}{l}ECM(\bar{X}_1) = \left\{E[\bar{X}_1 - E(\bar{X}_1)]\right\}^2 + V(\bar{X}_1) \\ = (\mu-\mu)^2 + \frac{n\sigma^2}{n-1} = \frac{n\sigma^2}{n-1}\end{array}}$$

$\cdot E(\bar{X}_2) = E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) = \frac{1}{n} \cdot \sum_{i=1}^{n}E(x_i) = \frac{1}{n} \cdot n \cdot \mu$

$$= \mu$$

$\cdot V(\bar{X}_2) = V\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) = \frac{1}{n}\sum_{i=1}^{n}V(X_i) = \frac{1}{n}nV(x_i)$

$$= \frac{1}{n}n\sigma^2 = \sigma^2$$

$$\boxed{\begin{array}{l}ECM(\bar{X}_2) = \left\{E[\bar{X}_2 - E(\bar{X}_2)]\right\}^2 + V(\bar{X}_2) \\ = (\mu-\mu)^2 + \sigma^2 = \sigma^2\end{array}}$$

$\therefore \bar{X}_2$ es el mejor estimador porque tiene menor ECM.

2)a) $E(\theta_1) = E\left(\frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7}{7}\right) = \text{Por linealidad} = \text{de la esperanza}$

$$= \frac{E(x_1+x_2+x_3+x_4+x_5+x_6+x_7)}{7} = \text{Por linealidad} = \text{de la esp.}$$

$$= \frac{E(x_1) + E(x_2) + E(x_3) + E(x_4) + E(x_5) + E(x_6) + E(x_7)}{7}$$

$$= \frac{7E(x_1)}{7} = E(x_1) = \mu$$

~~θ_2~~

$$E(\theta_2) = E\left(\frac{2x_1 - x_6 + x_4}{2}\right) = \frac{E(2x_1 - x_6 + x_4)}{2} =$$

$$= \text{Por linealidad} = \frac{E(2x_1) - E(x_6) + E(x_4)}{2} =$$

$$= \frac{2E(x_1) - E(x_6) + E(x_4)}{2} = \frac{2\mu}{2} = \mu$$

$$E(\theta_3) = E\left(\frac{2x_1 - x_7 + x_3}{3}\right) = \frac{E(2x_1 - x_7 + x_3)}{3} =$$

$$= \text{por linealidad} = \frac{2E(x_1) - E(x_7) + E(x_3)}{3} =$$

$$= \frac{2\mu}{3} = \frac{2}{3}\mu$$

$\therefore \ \theta_1 \ y \ \theta_2 \ \text{son insesgados}.$

b) $V(\theta_1) = V\left(\frac{x_1 + x_2 + \dots + x_7}{7}\right) = \left(\frac{1}{7}\right)^2 V(x_1 + x_2 + \dots + x_7) =$

$$= \text{por independencia} = \frac{1}{49}\left[V(x_1) + V(x_2) + \dots + V(x_7)\right] =$$

$$= \frac{1}{49} \cdot 7 \cdot \sigma^2 = \frac{1}{7}\sigma^2$$

$$V(\theta_2) = V\left(\frac{2x_1 - x_6 + x_4}{2}\right) = \left(\frac{1}{2}\right)^2\left[V(2x_1) - V(x_6) + V(x_4)\right] =$$

$$\downarrow$$

$$\text{por ind.}$$

$$= \frac{1}{4} \left[2V(x_1) - V(x_6) + V(x_4) \right] = \frac{1}{4} \cdot \left[2\sigma^2 - \sigma^2 + \sigma^2 \right] =$$

$$= \frac{2\sigma^2}{4} = \frac{1}{2} \sigma^2$$

$$V(\theta_3) = V\left(\frac{2x_1 - x_7 + x_3}{3} \right) = \left(\frac{1}{3} \right)^2 \left[V(2x_1) - V(x_7) + V(x_3) \right] =$$

por ind.

$$= \frac{1}{9} \left[2\sigma^2 - \sigma^2 + \sigma^2 \right] = \frac{2}{9} \sigma^2$$

$$ECM(\theta_1) = E\left(\theta_1 - E(\theta_1) \right)^2 + V(\theta_1) = E\left(\theta_1 - \mu \right)^2 + \frac{1}{7} \sigma^2 =$$
$$= por \ linealidad = \left(E(\theta_1) - E(\mu) \right)^2 + \frac{1}{7} \sigma^2 =$$

$$= \left(\mu - \mu \right)^2 + \frac{1}{7} \sigma^2 = \frac{1}{7} \sigma^2$$

$$ECM(\theta_2) = E\left(\theta_2 - E(\theta_2) \right)^2 + V(\theta_2) = E\left(\theta_2 - \mu \right)^2 + \frac{1}{2} \sigma^2 =$$

$$= x \ linealidad = \left(\mu - \mu \right)^2 + \frac{1}{2} \sigma^2 = \frac{1}{2} \sigma^2$$

$$ECM(\theta_3) = E\left(\theta_3 - E(\theta_3) \right)^2 + V(\theta_3) = E\left(\theta_3 - \frac{2}{3} \mu \right)^2 + \frac{2}{9} \sigma^2 =$$

$$= x \ linealidad = \left(\frac{2}{3}\mu - \frac{2}{3}\mu \right)^2 + \frac{2}{9} \sigma^2 =$$

$$= \frac{2}{9} \sigma^2$$

c) ∴ Mejor $\left[ECM(\theta_1), ECM(\theta_2), ECM(\theta_3) \right] = ECM(\theta_1) =$

$$= \frac{1}{7} \sigma^2$$

3) a) $E(x_i) = \mu$; $V(x_i) = \sigma^2$, $V(\bar{x}) = \dfrac{\sigma^2}{n}$

$\hookrightarrow E(\bar{x}^2) = V(\bar{x}) + (E(\bar{x}))^2 = V\left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right) + \left(E\left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right)\right)^2 =$

$\qquad = \dfrac{V(n x_i)}{n^2} + \left(\dfrac{E(n x_i)}{n}\right)^2 = \dfrac{n V(x_i)}{n^2} + \left(\dfrac{n E(x_i)}{n}\right)^2$

$\qquad = \dfrac{n \sigma^2}{n^2} + \mu^2 = \underbrace{\dfrac{\sigma^2}{n} + \mu^2}$ por linealidad

b) $b(\bar{x}^2) = E(\bar{x}^2) - \mu^2$

$\qquad = \dfrac{\sigma^2}{n} + \mu^2 - \mu^2$

$\qquad = \dfrac{\sigma^2}{n}$

4 $\bar{x}^2$ es un estimador resgado de μ^2 ya que

$\mu^2 \neq \dfrac{\sigma^2}{n} + \mu^2$.

c) $\displaystyle\lim_{n \to \infty} \left(\dfrac{\sigma^2}{n}\right) = 0$

4) 'nº diaria de desconexiones occidentales de un servidor'

$\sim P(\lambda).$ $= N \sim P(\lambda)$

a) EMV(λ): $P(N=k) = \dfrac{e^{-\lambda} \cdot \lambda^k}{k!}$, $k = 0, 1, \dots$

$\longrightarrow L(\lambda) = P(N_1 = k_1) \cdot P(N_2 = k_2) \cdot \dots \cdot P(N_n = k_n)$

$\qquad = \displaystyle\prod_{i=1}^{n} \left(\dfrac{e^{-\lambda} \cdot \lambda^{k_i}}{k_i!}\right) \underset{\text{distribuye}}{=} \dfrac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}}{\prod_{i=1}^{n} k_i!}$

$= \ln\left(\dfrac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}}{\prod_{i=1}^{n} k_i!}\right)$ $\underset{\text{de logaritmos}}{\overset{\text{prop. de división}}{=}}$

$= \ln\left(e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}\right) - \ln\left(\prod_{i=1}^{n}(k_i!)\right) = $ prop. de producto del log$=$

$$= \ln(e^{-\lambda n}) + \ln\left(\lambda^{\sum_{i=1}^{n}(k_i)}\right) - \ln\left(\prod_{i=1}^{n}(k_i!)\right) = \begin{array}{l}\text{prop. de}\\ \text{potencia}\\ \text{del log.}\end{array} =$$

$$= -\lambda n \cdot \ln(e) + \sum_{i=1}^{n}(k_i) \cdot \ln(\lambda) - \ln\left(\prod_{i=1}^{n}(k_i!)\right) =$$

$$= -\lambda n \cdot 1 + \sum_{i=1}^{n}(k_i) \ln(\lambda) - \underbrace{\sum_{i=1}^{n}\ln(k_i!)}_{}$$

prop. de
producto.
del log.

derivando e igualando a 0 para maximizar:

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n}(k_i)}{\lambda} - 0 = 0$$

$$\frac{\sum_{i=1}^{n}(k_i)}{\lambda} = n \qquad \begin{array}{l}\text{media muestral:}\\ \dfrac{\sum_{i=1}^{n}(k_i)}{n}\end{array}$$

$$\sum_{i=1}^{n}(k_i) = n\lambda = \bar{X}$$

Sumando las observaciones $(2,5,3,3,7)$: $\quad \dfrac{20}{5} = 4$

$n = 5$ observaciones

(b)

$$E(\bar{X}) \overset{?}{=} \lambda \longrightarrow E\left(\frac{\sum_{i=1}^{n}(k_i)}{n}\right) = \frac{E(\sum_{i=1}^{n}(k_i))}{n} =$$

por linealidad
de la esp.

$$= \frac{n E(k_i)}{n} = \frac{n E(\lambda)}{n} = E(\lambda) = \lambda \qquad \therefore \text{ Es un estimador insesgado.}$$

sustituyendo
por λ

$\cdot$ ¿ $\lim\limits_{n \to \infty} E(\bar{x}) \overset{?}{=} \lambda$ y $\lim\limits_{n \to \infty} V(\bar{x}) \overset{?}{=} 0$?

$\longrightarrow \lim\limits_{n \to \infty} \dfrac{n \partial E(\lambda)}{\lambda} = \lim\limits_{n \to \infty} E(\lambda) = $ $\longrightarrow \lim\limits_{n \to \infty} (V(\bar{x})) = $

$= \lambda \checkmark$

$\qquad\qquad\qquad\qquad \lim\limits_{n \to \infty} V\left(\dfrac{\sum_{i=1}^{n}(k_i)}{n}\right)$

$\qquad\qquad\qquad\qquad \underset{\downarrow}{=} \lim\limits_{n \to \infty} \dfrac{V\left(\sum_{i=1}^{n}(k_i)\right)}{n^2}$

$\qquad\qquad\qquad\qquad$ por ind.

$\qquad\qquad\qquad\qquad = \lim\limits_{n \to \infty} \dfrac{V(n k_i)}{n^2} =$

$\qquad\qquad\qquad\qquad = \lim\limits_{n \to \infty} \dfrac{n V(k_i)}{n^2} =$

$\qquad\qquad\qquad\qquad = \lim\limits_{n \to \infty} \dfrac{n \lambda}{n^2} = \lim\limits_{n \to \infty} \dfrac{\lambda}{n}$

$\qquad\qquad\qquad\qquad = 0 \checkmark$

$\longrightarrow \therefore$ Sí, es consistente el estimador.

b) Hecho.

c) $P(N \geqslant 3) = 1 - P(N < 3) = 1 - \left(P(N=0) + P(N=1) + P(N=2)\right) =$

$= 1 - \left(\dfrac{e^{-\lambda} \cdot \lambda^0}{0!} + \dfrac{e^{-\lambda} \cdot \lambda^1}{1!} + \dfrac{e^{-\lambda} \cdot \lambda^2}{2!}\right) = 1 - \left(e^{-\lambda}\left(1 + \lambda + \dfrac{\lambda^2}{2}\right)\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{\downarrow}{}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Factor
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Común

$\Longrightarrow \hat{P}_{EMV}\left(\geqslant 3 \mid \hat{\lambda}_{EMV}\right) = 1 - e^{-\hat{\lambda}_{EMV}}\left(1 + \hat{\lambda}_{EMV} + \dfrac{\hat{\lambda}_{EMV}^2}{2}\right)$

$\qquad = 1 - e^{-4}(1 + 4 + 8) = 1 - 0,238 = 0,762.$

5) a) $X \sim B(1, p)$.

$$P(x = k) = \binom{1}{k} p^k (1-p)^{1-k} \quad , \quad k = 0, 1, \ldots$$

$$L(k_1, k_2, \ldots, k_n \, ; \, p) = p^{\sum_{i=1}^{n} (k_i)} (1-p)^{1 - \sum_{i=1}^{n} (k_i)}$$

$$\ln L(p) = \sum_{i=1}^{n} (k_i) \ln(p) + \left(n - \sum_{i=1}^{n} (k_i)\right) \ln(1-p)$$

↓

aplicando prop.
del log.

derivando: $\dfrac{\partial \ln L(p)}{\partial p} = 0$

$$\frac{\sum_{i=1}^{n} (k_i)}{p} - \frac{n - \sum_{i=1}^{n} (k_i)}{1-p} = 0$$

$$\frac{\sum_{i=1}^{n} (k_i)}{p} = \frac{n - \sum_{i=1}^{n} (k_i)}{1-p}$$

$$\frac{1}{p} = \frac{n}{\sum_{i=1}^{n} (k_i)}$$

$$\hat{p}_{EMV} = \frac{\sum_{i=1}^{n} (k_i)}{n}$$

b) i) $\hat{p}_{EMV} = \dfrac{5}{100} = 0{,}05$

ii) $\hat{p}_{EMV} = \left(1 - \hat{p}_{EMV}\right)^6 = \left(1 - 0{,}05\right)^6 = 0{,}95^6 \approx 0{,}735$

6) a)
$$\mu = \int_{-\infty}^{+\infty} x \cdot (2\theta + 1) x^{2\theta} \, dx = \int_{0}^{1} (2\theta + 1) x^{2\theta + 1} \, dx$$

$$= (2\theta + 1) \left(\frac{x^{2\theta + 2}}{2\theta + 2} \right) \Big|_{0}^{1} = (2\theta + 1) \left(\frac{1}{2\theta + 2} - 0 \right) = \frac{2\theta + 1}{2\theta + 2}$$

$$n = \overline{\text{observación}} = 10. \quad \longrightarrow \overline{X} = \frac{\sum (\text{observaciones})}{n}$$

$$= \frac{7,8}{10} = 0,78$$

$$\frac{2\theta + 1}{2\theta + 2} = 0,78$$

$$2\theta + 1 = (2\theta + 2) \, 0,78$$

$$2\theta - 1,56\theta = 1,56 - 1$$

$$\theta_{MM} = \frac{14}{11} \approx 1,2727$$

b)
$$\underbrace{EMV(\theta_{MM})}_{L(\theta)} = \prod_{i=1}^{n} \left((2\theta + 1) x_i^{2\theta} \right) = (2\theta + 1)^n \cdot \prod_{i=1}^{n} (x_i^{2\theta})$$

$$\ln L(\theta) = \ln \left((2\theta + 1)^n \right) + \ln \left(\prod_{i=1}^{n} (x_i^{2\theta}) \right) = \text{prop. log. (producto)} =$$

$$= n \cdot (\ln (2\theta + 1)) + \sum_{i=1}^{n} \left(\ln(x_i^{2\theta}) \right) = \text{prop. log. (pot.)} =$$

$$= n \cdot \ln (2\theta + 1) + \sum_{i=1}^{n} \left(2\theta \ln(x_i) \right) = \text{prop. de la sumatoria} =$$

$$= n \ln(2\theta + 1) + 2\theta \sum_{i=1}^{n} (\ln(x_i))$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \longrightarrow \frac{\partial \left(n \cdot \ln (2\theta + 1) + 2\theta \sum_{i=1}^{n} (\ln(x_i)) \right)}{\partial \theta} \overset{=0}{\longrightarrow}$$

$$\longrightarrow \frac{\left(n \cdot \ln (2\theta + 1) + n \cdot \frac{1}{2\theta + 1} \right) + 2 \sum_{i=1}^{n} \ln(x_i)}{\theta} = 0 \longrightarrow$$

$$\longrightarrow \frac{2n}{2\theta + 1} + 2 \sum_{i=1}^{n} (\ln x_i) = 0 \longrightarrow \frac{2n}{2\theta + 1} = -2 \sum_{i=1}^{n} \ln(x_i)$$

$$\rightarrow \frac{n}{2\theta+1} = -\sum_{i=1}^{n} \ln(x_i) \longrightarrow n = (2\theta+1)\left(-\sum_{i=1}^{n} \ln(x_i)\right)$$

$$\rightarrow \frac{n}{-\sum_{i=1}^{n} \ln(x_i)} = 2\theta+1 \longrightarrow \frac{n}{2\left(-\sum_{i=1}^{n} \ln(x_i)\right)} - 1 = \theta$$

$$\hat{\theta}_{EMV} = \frac{-10}{-2\left((-0,08)+(-0,24)+(-0,11)+(-0,43)+(-0,15)+(-0,76)+ \atop + (-0,31)+(-0,03)+(-0,06)+(-0,26)\right)} - 1$$

$$= \frac{-10}{5,32} - 1 = -2,879\ldots$$