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Práctica 6.- Matemática 3.

1) • $E(\bar{X}_1) = E\left(\frac{1}{n-1}\sum_{i=1}^{n} x_i\right) = \frac{1}{n-1} E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n-1}\sum_{i=1}^{n} E(x_i)$

$= \frac{1}{n-1}\sum_{i=1}^{n}\mu = \frac{1}{n-1} n\mu = \frac{\mu n}{n-1}$

$V(\bar{X}_1) = V\left(\frac{1}{n-1}\sum_{i=1}^{n} x_i\right) = \left(\frac{1}{n-1}\right)^2 \cdot V\left(\sum_{i=1}^{n} x_i\right)$

$= \frac{1^2}{(n-1)^2} \cdot \sum_{i=1}^{n} \cdot V(x_i) = \frac{1}{(n-1)^2}\sum_{i=1}^{n}\sigma^2$

$= \frac{1}{(n-1)^2} \cdot n\sigma^2 = \frac{n\sigma^2}{n-1}$

$$\boxed{\begin{aligned} ECM(\bar{X}_1) &= \{E[\bar{X}_1 - E(\bar{X}_1)]\}^2 + V(\bar{X}_1) \\ &= (\mu-\mu)^2 + \frac{n\sigma^2}{n-1} = \frac{n\sigma^2}{n-1}\end{aligned}}$$

• $E(\bar{x}_2) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} E(x_i) = \frac{1}{n} \cdot n \cdot \mu$

$= \mu$

$V(\bar{X}_2) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} V(X_i) = \frac{1}{n} n V(x_i)$

$= \frac{1}{n} n\sigma^2 = \sigma^2$

$$\boxed{\begin{aligned} ECM(\bar{X}_2) &= \{E[\bar{X}_2 - E(\bar{X}_2)]\}^2 + V(\bar{X}_2) \\ &= (\mu-\mu)^2 + \sigma^2 = \sigma^2\end{aligned}}$$

$\therefore \bar{X}_2$ es el mejor estimador porque tiene menor ECM.

2) a) $E(\theta_1) = E\left(\frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7}{7}\right) =$ Por linealidad de la esperanza $=$

$= \frac{E(x_1+x_2+x_3+x_4+x_5+x_6+x_7)}{7} =$ Por linealidad de la esp. $=$

$$= \frac{E(x_1) + E(x_2) + E(x_3) + E(x_4) + E(x_5) + E(x_6) + E(x_7)}{7}$$

$$= \frac{7E(x_1)}{7} = E(x_1) = \mu$$

$$E(\theta_2) = E\left(\frac{2x_1 - x_6 + x_4}{2}\right) = \frac{E(2x_1 - x_6 + x_4)}{2} =$$

$$= \text{Por linealidad} = \frac{E(2x_1) - E(x_6) + E(x_4)}{2} =$$

$$= \frac{2E(x_1) - E(x_6) + E(x_4)}{2} = \frac{2\mu}{2} = \mu$$

$$E(\theta_3) = E\left(\frac{2x_1 - x_7 + x_3}{3}\right) = \frac{E(2x_1 - x_7 + x_3)}{3} =$$

$$= \text{por linealidad} = \frac{2E(x_1) - E(x_7) + E(x_3)}{3} =$$

$$= \frac{2\mu}{3} = \frac{2}{3}\mu$$

$\therefore$ θ_1 y θ_2 son insesgados.

b) $$V(\theta_1) = V\left(\frac{x_1 + x_2 + \ldots + x_7}{7}\right) = \left(\frac{1}{7}\right)^2 V(x_1 + x_2 + \ldots + x_7) =$$

$$= \text{por independencia} = \frac{1}{49}\left[V(x_1) + V(x_2) + \ldots + V(x_7)\right] =$$

$$= \frac{1}{49} \cdot 7 \cdot \sigma^2 = \frac{1}{7}\sigma^2$$

$$V(\theta_2) = V\left(\frac{2x_1 - x_6 + x_4}{2}\right) \underset{\text{por ind.}}{=} \left(\frac{1}{2}\right)^2 \left[V(2x_1) - V(x_6) + V(x_4)\right] =$$

$$= \frac{1}{4}\left[2V(X_1) - V(X_6) + V(X_4)\right] = \frac{1}{4}\cdot\left[2\sigma^2 - \sigma^2 + \sigma^2\right] =$$

$$= \frac{2\sigma^2}{4} = \frac{1}{2}\sigma^2$$

$$V(\theta_3) = V\left(\frac{2X_1 - X_7 + X_3}{3}\right) = \left(\frac{1}{3}\right)^2\left[V(2X_1) - V(X_7) + V(X_3)\right] =$$

por ind.

$$= \frac{1}{9}\left[2\sigma^2 - \sigma^2 + \sigma^2\right] = \frac{2}{9}\sigma^2$$

$$ECM(\theta_1) = E\left(\theta_1 - E(\theta_1)\right)^2 + V(\theta_1) = E\left(\theta_1 - \mu\right)^2 + \frac{1}{7}\sigma^2 =$$

$$= \text{por linealidad} = \left(E(\theta_1) - E(\mu)\right)^2 + \frac{1}{7}\sigma^2 =$$

$$= (\mu - \mu)^2 + \frac{1}{7}\sigma^2 = \frac{1}{7}\sigma^2$$

$$ECM(\theta_2) = E\left(\theta_2 - E(\theta_2)\right)^2 + V(\theta_2) = E\left(\theta_2 - \mu\right)^2 + \frac{1}{2}\sigma^2 =$$

$$= \times\ \text{linealidad} = (\mu - \mu)^2 + \frac{1}{2}\sigma^2 = \frac{1}{2}\sigma^2$$

$$ECM(\theta_3) = E\left(\theta_3 - E(\theta_3)\right)^2 + V(\theta_3) = E\left(\theta_3 - \frac{2}{3}\mu\right)^2 + \frac{2}{9}\sigma^2 =$$

$$= \times\ \text{linealidad} = \left(\frac{3}{3}\mu - \frac{2}{3}\mu\right)^2 + \frac{2}{9}\sigma^2 =$$

$$= \frac{2}{9}\sigma^2$$

c) ∴ Mejor $\left[ECM(\theta_1), ECM(\theta_2), ECM(\theta_3)\right] = ECM(\theta_1) =$

$$= \frac{1}{7}\sigma^2$$

3) a) $E(X_i) = \mu$; $V(X_i) = \sigma^2$, $V(\bar{X}) = \frac{\sigma^2}{n}$

$\hookrightarrow E(\bar{X}^2) = V(\bar{X}) + (E(\bar{X}))^2 = V\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \left(E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right)^2 =$

$$= \frac{V(n x_i)}{n^2} + \left(\frac{E(n x_i)}{n}\right)^2 = \frac{n V(x_i)}{n^2} + \left(\frac{n E(x_i)}{n}\right)^2$$

por linealidad

$$= \frac{n \sigma^2}{n^2} + \mu^2 = \underbrace{\frac{\sigma^2}{n} + \mu^2}$$

$\bar{X}^2$ es un estimador sesgado de μ^2 ya que $\mu^2 \neq \frac{\sigma^2}{n} + \mu^2$.

b) $b(\bar{X}^2) = E(\bar{X}^2) - \mu^2$

$$= \frac{\sigma^2}{n} + \mu^2 - \mu^2$$

$$= \frac{\sigma^2}{n}$$

c) $\lim_{n \to \infty} \left(\frac{\sigma^2}{n}\right) = 0$

4) 'nº diario de desconexiones accidentales de un servidor' $\sim P(\lambda)$. $= N \sim P(\lambda)$

a) EMV(λ): $P(N = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$, $k = 0, 1, \ldots$

$$\longrightarrow L(\lambda) = P(N_1 = k_1) \cdot P(N_2 = k_2) \cdot \ldots \cdot P(N_n = k_n)$$

$$= \prod_{i=1}^{n} \left(\frac{e^{-\lambda} \cdot \lambda^{k_i}}{k_i!}\right) \underset{\text{distribuyo}}{=} \frac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}}{\prod_{i=1}^{n} k_i!}$$

$$= \ln\left(\frac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}}{\prod_{i=1}^{n} k_i!}\right) = \text{prop. de división de logaritmos.}$$

$$= \ln\left(e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}\right) - \ln\left(\prod_{i=1}^{n}(k_i!)\right) = \text{prop. de productos del log} =$$

$$= \ln(e^{-\lambda n}) + \ln(\lambda^{\sum_{i=1}^{n}(k_i)}) - \ln(\prod_{i=1}^{n}(k_i!)) = \text{prop. de potencia del log.} =$$

$$= -\lambda n \cdot \cancel{\ln(e)} + \sum_{i=1}^{n}(k_i) \cdot \ln(\lambda) - \ln(\prod_{i=1}^{n}(k_i!)) =$$

$$= -\lambda n \cdot 1 + \sum_{i=1}^{n}(k_i)\ln(\lambda) - \underbrace{\sum_{i=1}^{n}(k_i!)}_{\text{prop. de productos del log.}}$$

derivando e igualando a 0 para maximizar:

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n}(k_i)}{\lambda} - 0 = 0$$

$$\frac{\sum_{i=1}^{n}(k_i)}{\lambda} = n$$

medio muestral: $\frac{\sum_{i=1}^{n}(k_i)}{n}$

$$\sum_{i=1}^{n}(k_i) = n\lambda = \bar{X}$$

Sumando las observaciones $\underbrace{(2,5,3,3,7)}_{n=5 \text{ observaciones}}$: $\frac{20}{5} = 4$ → (b)

$$E(\bar{X}) \overset{?}{=} \lambda \to E\left(\frac{\sum_{i=1}^{n}(k_i)}{n}\right) = \frac{E(\sum_{i=1}^{n}(k_i))}{n} =$$

por linealidad de la esp.

$$= \frac{nE(k_i)}{n} = \frac{nE(\lambda)}{n} = E(\lambda) = \lambda$$

sustituyendo por λ

∴ Es un estimador insesgado.

¿ $\lim_{n \to \infty} E(\bar{x}) \overset{?}{=} \lambda$ y $\lim_{n \to \infty} V(\bar{x}) \overset{?}{=} 0$?

$\Rightarrow \lim_{n \to \infty} \frac{\cancel{n} E(\lambda)}{\cancel{n}} = \lim_{n \to \infty} E(\lambda) = \lambda$ ✓

$\Rightarrow \lim_{n \to \infty} (V(\bar{x})) =$

$\lim_{n \to \infty} V\left(\frac{\sum_{i=1}^{n}(k_i)}{n}\right)$

$= \lim_{n \to \infty} \frac{V(\sum_{i=1}^{n}(k_i))}{n^2}$

por ind.

$= \lim_{n \to \infty} \frac{V(n k_i)}{n^2} =$

$= \lim_{n \to \infty} \frac{n V(k_i)}{n^2} =$

$= \lim_{n \to \infty} \frac{n\lambda}{n^2} = \lim_{n \to \infty} \frac{\lambda}{n}$

$= 0$ ✓

$\Rightarrow \therefore$ Sí, es consistente el estimador.

b) Hecho.

c) $P(N \geq 3) = 1 - P(N < 3) = 1 - (P(N=0) + P(N=1) + P(N=2)) =$

$= 1 - \left(\frac{e^{-\lambda} \cdot \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} + \frac{e^{-\lambda} \cdot \lambda^2}{2!}\right) = 1 - \left(e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)\right)$

Factor común

$\Rightarrow \hat{P}_{EMV}(\geq 3 \mid \hat{\lambda}_{EMV}) = 1 - e^{-\hat{\lambda}_{EMV}}\left(1 + \hat{\lambda}_{EMV} + \frac{\hat{\lambda}_{EMV}^2}{2}\right)$

$= 1 - e^{-4}(1 + 4 + 8) = 1 - 0{,}238 = 0{,}762$.

5) a) $X \sim B(1,p)$.

$$P(X=k) = \binom{1}{k} p^k (1-p)^{1-k} \quad , \quad k=0,1,..$$

$$\underbrace{L(k_1, k_2, \dots, k_n; p)}_{} = p^{\sum_{i=1}^n (k_i)} (1-p)^{n-\sum_{i=1}^n (k_i)}$$

$$\ln L(p) = \sum_{i=1}^n (k_i) \ln(p) + \left(n - \sum_{i=1}^n (k_i)\right) \ln(1-p)$$

aplicando prop. del log.

derivando: $\dfrac{\partial \ln L(p)}{\partial p} = 0$

$$\frac{\sum_{i=1}^n (k_i)}{p} - \frac{n - \sum_{i=1}^n (k_i)}{1-p} = 0$$

$$\frac{\sum_{i=1}^n (k_i)}{p} = \frac{n - \sum_{i=1}^n (k_i)}{1-p}$$

$$\frac{1}{p} = \frac{n}{\sum_{i=1}^n (k_i)}$$

$$\hat{p}_{EMV} = \frac{\sum_{i=1}^n (k_i)}{n}$$

b) i) $\hat{p}_{EMV} = \dfrac{5}{100} = 0{,}05$

ii) $\hat{p}_{EMV} = (1-\hat{p}_{EMV})^6 = (1-0{,}05)^6 = 0{,}95^6 \approx 0{,}735$

6) a) $\mu = \int_{-\infty}^{+\infty} x \cdot (2\theta+1) x^{2\theta} \, dx = \int_0^1 (2\theta+1) x^{2\theta+1} \, dx$

$$= (2\theta+1)\left(\frac{x^{2\theta+2}}{2\theta+2}\right)\Bigg|_0^1 = (2\theta+1)\left(\frac{1}{2\theta+2} - 0\right) = \frac{2\theta+1}{2\theta+2}$$

$n = \overline{\text{observaciones}} = 10 . \longrightarrow \bar{X} = \frac{\sum(\text{observaciones})}{n}$

$$= \frac{7,8}{10} = 0,78$$

$$\frac{2\theta+1}{2\theta+2} = 0,78$$

$$2\theta+1 = (2\theta+2)0,78$$

$$2\theta - 1,56\theta = 1,56 - 1$$

$$\theta_{MM} = \frac{14}{11} \approx 1,2727$$

b) $\underbrace{EMV(\theta_{MM})}_{L(\theta)} = \prod_{i=1}^{n} \left((2\theta+1) x_i^{2\theta}\right) = (2\theta+1)^n \cdot \prod_{i=1}^{n} (x_i^{2\theta})$

$\ln L(\theta) = \ln\left((2\theta+1)^n\right) + \ln\left(\prod_{i=1}^{n} (x_i^{2\theta})\right) =$ prop. log. (producto) $=$

$= n \cdot (\ln(2\theta+1)) + \sum_{i=1}^{n} \left(\ln(x_i^{2\theta})\right) =$ prop. log. (pot.) $=$

$= n \cdot \ln(2\theta+1) + \sum_{i=1}^{n} \left(2\theta \ln(x_i)\right) =$ prop. de la sumatoria $=$

$= n \ln(2\theta+1) + 2\theta \sum_{i=1}^{n} (\ln(x_i))$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \longrightarrow \frac{\partial\left(n \cdot \ln(2\theta+1) + 2\theta \sum_{i=1}^{n} (\ln(x_i))\right)}{\partial \theta} = 0 \rightarrow$$

$$\rightarrow \frac{\left(n \cdot \ln(2\theta+1) + n \cdot \frac{1}{2\theta+1}\right) + 2\sum_{i=1}^{n} \ln(x_i)}{\theta} = 0 \rightarrow$$

$$\rightarrow \frac{2n}{2\theta+1} + 2\sum_{i=1}^{n} (\ln x_i) = 0 \longrightarrow \frac{2n}{2\theta+1} = -2\sum_{i=1}^{n} \ln(x_i)$$

$$\rightarrow \frac{n}{2\theta+1} = -\sum_{i=1}^{n} \ln(x_i) \rightarrow n = (2\theta+1)\left(-\sum_{i=1}^{n} \ln(x_i)\right)$$

$$\rightarrow \frac{n}{-\sum_{i=1}^{n} \ln(x_i)} = 2\theta+1 \rightarrow \frac{1}{2}\left(\frac{n}{\left(-\sum_{i=1}^{n} \ln(x_i)\right)} - 1\right) = \theta$$

$$\hat{\theta}_{EMV} = \frac{1}{2} \cdot \left(\frac{-10}{((-0{,}08)+(-0{,}24)+(-0{,}11)+(-0{,}43)+(-0{,}15)+(-0{,}76)+(-0{,}31)+(-0{,}03)+(-0{,}06)+(-0{,}26))} - 1 \right)$$

$$= \frac{1}{2}\left(\frac{-10}{-2{,}42} - 1\right) \approx \approx 1{,}55761$$