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Práctica 6 .- Matemática 3.

1) • $E(\overline{X}_1) = E\left(\frac{1}{n-1}\sum_{i=1}^{n} x_i\right) = \frac{1}{n-1} E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n-1}\sum_{i=1}^{n} E(x_i)$

$= \frac{1}{n-1}\sum_{i=1}^{n}\mu = \frac{1}{n-1} n\mu = \frac{\mu n}{n-1}$

$V(\overline{X}_1) = V\left(\frac{1}{n-1}\sum_{i=1}^{n} x_i\right) = \left(\frac{1}{n-1}\right)^2 \cdot V\left(\sum_{i=1}^{n} x_i\right)$

$= \frac{1^2}{(n-1)^2} \cdot \sum_{i=1}^{n} \cdot V(x_i) = \frac{1}{(n-1)^2}\sum_{i=1}^{n}\sigma^2$

$= \frac{1}{(n-1)^2} \cdot n\sigma^2 = \frac{n\sigma^2}{n-1}$

$$\boxed{\begin{aligned} ECM(\overline{X}_1) &= \{E[\overline{X}_1 - E(\overline{X}_1)]\}^2 + V(\overline{X}_1) \\ &= (\mu-\mu)^2 + \frac{n\sigma^2}{n-1} = \frac{n\sigma^2}{n-1}\end{aligned}}$$

• $E(\overline{x}_2) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} E(x_i) = \frac{1}{n} \cdot n \cdot \mu$

$= \mu$

$V(\overline{X}_2) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} V(X_i) = \frac{1}{n} n V(x_i)$

$= \frac{1}{n} n\sigma^2 = \sigma^2$

$$\boxed{\begin{aligned} ECM(\overline{X}_2) &= \{E[\overline{X}_2 - E(\overline{X}_2)]\}^2 + V(\overline{X}_2) \\ &= (\mu-\mu)^2 + \sigma^2 = \sigma^2\end{aligned}}$$

$\therefore$ $\overline{X}_2$ es el mejor estimador porque tiene menor ECM.

2) a) $E(\theta_1) = E\left(\frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7}{7}\right) =$ Por linealidad de la esperanza $=$

$= \frac{E(x_1+x_2+x_3+x_4+x_5+x_6+x_7)}{7} =$ Por linealidad de la esp. $=$

$$= \frac{E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(X_6) + E(X_7)}{7}$$

$$= \frac{7E(X_1)}{7} = E(X_1) = \mu$$

$$E(\theta_2) = E\left(\frac{2X_1 - X_6 + X_4}{2}\right) = \frac{E(2X_1 - X_6 + X_4)}{2} =$$

$$= \text{Por linealidad} = \frac{E(2X_1) - E(X_6) + E(X_4)}{2} =$$

$$= \frac{2E(X_1) - E(X_6) + E(X_4)}{2} = \frac{2\mu}{2} = \mu$$

$$E(\theta_3) = E\left(\frac{2X_1 - X_7 + X_3}{3}\right) = \frac{E(2X_1 - X_7 + X_3)}{3} =$$

$$= \text{por linealidad} = \frac{2E(X_1) - E(X_7) + E(X_3)}{3} =$$

$$= \frac{2\mu}{3} = \frac{2}{3}\mu$$

$\therefore$ θ_1 y θ_2 son insesgados.

b) $$V(\theta_1) = V\left(\frac{X_1 + X_2 + \ldots + X_7}{7}\right) = \left(\frac{1}{7}\right)^2 V(X_1 + X_2 + \ldots + X_7) =$$

$$= \text{por independencia} = \frac{1}{49}\left[V(X_1) + V(X_2) + \ldots + V(X_7)\right] =$$

$$= \frac{1}{49} \cdot 7 \cdot \sigma^2 = \frac{1}{7}\sigma^2$$

$$V(\theta_2) = V\left(\frac{2X_1 - X_6 + X_4}{2}\right) = \left(\frac{1}{2}\right)^2 \left[V(2X_1) - V(X_6) + V(X_4)\right] =$$

por ind.

$$= \frac{1}{4}\left[2V(X_1) - V(X_6) + V(X_4)\right] = \frac{1}{4}\cdot\left[2\sigma^2 - \sigma^2 + \sigma^2\right] =$$

$$= \frac{2\sigma^2}{4} = \frac{1}{2}\sigma^2$$

$$V(\theta_3) = V\left(\frac{2X_1 - X_7 + X_3}{3}\right) \underset{\text{por ind.}}{=} \left(\frac{1}{3}\right)^2\left[V(2X_1) - V(X_7) + V(X_3)\right] =$$

$$= \frac{1}{9}\left[2\sigma^2 - \sigma^2 + \sigma^2\right] = \frac{2}{9}\sigma^2$$

$$ECM(\theta_1) = E\left(\theta_1 - E(\theta_1)\right)^2 + V(\theta_1) = E\left(\theta_1 - \mu\right)^2 + \frac{1}{7}\sigma^2 =$$

$$= \text{por linealidad} = \left(E(\theta_1) - E(\mu)\right)^2 + \frac{1}{7}\sigma^2 =$$

$$= (\mu - \mu)^2 + \frac{1}{7}\sigma^2 = \frac{1}{7}\sigma^2$$

$$ECM(\theta_2) = E\left(\theta_2 - E(\theta_2)\right)^2 + V(\theta_2) = E\left(\theta_2 - \mu\right)^2 + \frac{1}{2}\sigma^2 =$$

$$= \times\ \text{linealidad} = (\mu - \mu)^2 + \frac{1}{2}\sigma^2 = \frac{1}{2}\sigma^2$$

$$ECM(\theta_3) = E\left(\theta_3 - E(\theta_3)\right)^2 + V(\theta_3) = E\left(\theta_3 - \frac{2}{3}\mu\right)^2 + \frac{2}{9}\sigma^2 =$$

$$= \times\ \text{linealidad} = \left(\frac{3}{3}\mu - \frac{2}{3}\mu\right)^2 + \frac{2}{9}\sigma^2 =$$

$$= \frac{2}{9}\sigma^2$$

c) ∴ Mejor $\left[ECM(\theta_1), ECM(\theta_2), ECM(\theta_3)\right] = ECM(\theta_1) =$

$$= \frac{1}{7}\sigma^2$$

3) a) $E(x_i) = \mu$; $V(x_i) = \sigma^2$, $V(\bar{x}) = \frac{\sigma^2}{n}$

$\hookrightarrow E(\bar{x}^2) = V(\bar{x}) + (E(\bar{x}))^2 = V\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \left(E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right)^2 =$

$= \frac{V(n x_i)}{n^2} + \left(\frac{E(n x_i)}{n}\right)^2 = \frac{n V(x_i)}{n^2} + \left(\frac{n E(x_i)}{n}\right)^2$

por linealidad

$= \frac{n\sigma^2}{n^2} + \mu^2 = \underbrace{\frac{\sigma^2}{n} + \mu^2}$

$\bar{x}^2$ es un estimador sesgado de μ^2 ya que $\mu^2 \neq \frac{\sigma^2}{n} + \mu^2$.

b) $b(\bar{x}^2) = E(\bar{x}^2) - \mu^2$

$= \frac{\sigma^2}{n} + \mu^2 - \mu^2$

$= \frac{\sigma^2}{n}$

c) $\lim_{n \to \infty} \left(\frac{\sigma^2}{n}\right) = 0$

4) 'nº diario de desconexiones accidentales de un servidor' $\sim P(\lambda)$. $= N \sim P(\lambda)$

a) $EMV(\lambda)$: $P(N = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$, $k = 0, 1, \ldots$

$\longrightarrow L(\lambda) = P(N_1 = k_1) \cdot P(N_2 = k_2) \cdot \ldots \cdot P(N_n = k_n)$

$= \prod_{i=1}^{n} \left(\frac{e^{-\lambda} \cdot \lambda^{k_i}}{k_i!}\right) = \frac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}}{\prod_{i=1}^{n} k_i!}$

distribuyo

$= \ln\left(\frac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}}{\prod_{i=1}^{n} k_i!}\right) =$ prop. de división de logaritmos.

$= \ln(e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n}(k_i)}) - \ln\left(\prod_{i=1}^{n}(k_i!)\right) =$ prop. de productos del log. $=$

$= \ln(e^{-\lambda n}) + \ln(\lambda^{\sum_{i=1}^{n}(k_i)}) - \ln(\prod_{i=1}^{n}(k_i!)) =$ prop. de potencia del log. $=$

$= -\lambda n \cdot \cancel{\ln(e)} + \sum_{i=1}^{n}(k_i) \cdot \ln(\lambda) - \ln(\prod_{i=1}^{n}(k_i!)) =$

$= -\lambda n \cdot 1 + \sum_{i=1}^{n}(k_i)\ln(\lambda) - \underbrace{\sum_{i=1}^{n}(k_i!)}_{\text{prop. de productos del log.}}$

derivando e igualando a 0 para maximizar:

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n}(k_i)}{\lambda} - 0 = 0$$

$$\frac{\sum_{i=1}^{n}(k_i)}{\lambda} = n$$

media muestral: $\frac{\sum_{i=1}^{n}(k_i)}{n}$

$$\sum_{i=1}^{n}(k_i) = n\lambda = \bar{X}$$

Sumando las observaciones $\underbrace{(2,5,3,3,7)}_{n=5 \text{ observaciones}}$: $\frac{20}{5} = 4$

b

$E(\bar{X}) \overset{?}{=} \lambda \rightarrow E\left(\frac{\sum_{i=1}^{n}(k_i)}{n}\right) = \frac{E(\sum_{i=1}^{n}(k_i))}{n} =$

por linealidad de la esp.

$= \frac{nE(k_i)}{n} = \frac{nE(\lambda)}{n} = E(\lambda) = \lambda$ $\therefore$ Es un estimador insesgado.

sustituyendo por λ

$$¿\lim_{n\to\infty} E(\bar{x}) \overset{?}{=} \lambda \quad y \quad \lim_{n\to\infty} V(\bar{x}) \overset{?}{=} 0 \, ?$$

$$\to \lim_{n\to\infty} \frac{\cancel{n} E(\lambda)}{\cancel{n}} = \lim_{n\to\infty} E(\lambda) = \lambda \checkmark$$

$$\to \lim_{n\to\infty} (V(\bar{x})) = \lim_{n\to\infty} V\left(\frac{\sum_{i=1}^{n}(k_i)}{n}\right)$$

$$= \lim_{n\to\infty} \frac{V(\sum_{i=1}^{n}(k_i))}{n^2}$$

por ind.

$$= \lim_{n\to\infty} \frac{V(n k_i)}{n^2} = \lim_{n\to\infty} \frac{n V(k_i)}{n^2} = \lim_{n\to\infty} \frac{n\lambda}{n^2} = \lim_{n\to\infty} \frac{\lambda}{n} = 0 \checkmark$$

$\to \therefore$ Sí, es consistente el estimador.

b) Hecho.

c) $P(N \geq 3) = 1 - P(N<3) = 1 - (P(N=0) + P(N=1) + P(N=2)) =$

$$= 1 - \left(\frac{e^{-\lambda}\cdot\lambda^0}{0!} + \frac{e^{-\lambda}\cdot\lambda^1}{1!} + \frac{e^{-\lambda}\cdot\lambda^2}{2!}\right) = 1 - \left(e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)\right)$$

Factor común

$$\Rightarrow \hat{P}_{EMV}(\geq 3 \mid \hat{\lambda}_{EMV}) = 1 - e^{-\hat{\lambda}_{EMV}}\left(1 + \hat{\lambda}_{EMV} + \frac{\hat{\lambda}_{EMV}^2}{2}\right)$$

$$= 1 - e^{-4}(1+4+8) = 1 - 0{,}238 = 0{,}762.$$

5) a) $X \sim B(1,p)$.

$$P(X=k) = \binom{1}{k} p^k (1-p)^{1-k} \quad , \quad k=0,1,..$$

$$L(k_1,k_2,\dots,k_n; p) = p^{\sum_{i=1}^{n}(k_i)} (1-p)^{n-\sum_{i=1}^{n}(k_i)}$$

$$\ln L(p) = \sum_{i=1}^{n}(k_i)\ln(p) + \left(n-\sum_{i=1}^{n}(k_i)\right)\ln(1-p)$$

aplicando prop. del log.

derivando: $\dfrac{\partial \ln L(p)}{\partial p} = 0$

$$\frac{\sum_{i=1}^{n}(k_i)}{p} - \frac{n-\sum_{i=1}^{n}(k_i)}{1-p} = 0$$

$$\frac{\sum_{i=1}^{n}(k_i)}{p} = \frac{n-\sum_{i=1}^{n}(k_i)}{1-p}$$

$$\frac{1}{p} = \frac{n}{\sum_{i=1}^{n}(k_i)}$$

$$\hat{p}_{EMV} = \frac{\sum_{i=1}^{n}(k_i)}{n}$$

b) i) $\hat{p}_{EMV} = \dfrac{5}{100} = 0{,}05$

ii) $\hat{p}_{EMV} = (1-\hat{p}_{EMV})^6 = (1-0{,}05)^6 = 0{,}95^6 \approx 0{,}735$

6) a) $\mu = \int_{-\infty}^{+\infty} x \cdot (2\theta+1) x^{2\theta} \, dx = \int_0^1 (2\theta+1) x^{2\theta+1} \, dx$

$$= (2\theta+1)\left(\frac{x^{2\theta+2}}{2\theta+2}\right)\Bigg|_0^1 = (2\theta+1)\left(\frac{1}{2\theta+2} - 0\right) = \frac{2\theta+1}{2\theta+2}$$

$n = \overline{\text{observaciones}} = 10 . \longrightarrow \bar{X} = \dfrac{\sum(\text{observaciones})}{n}$

$$= \frac{7{,}8}{10} = 0{,}78$$

$$\frac{2\theta+1}{2\theta+2} = 0{,}78$$

$$2\theta+1 = (2\theta+2)0{,}78$$

$$2\theta - 1{,}56\theta = 1{,}56 - 1$$

$$\theta_{MM} = \frac{14}{11} \approx 1{,}2727$$