The conundrum of $\sqrt{-1} = i$ and the use of the square root-operation with complex numbers:

A guided discussion for physicist

F. Vismarra

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Fatti non foste a viver come bruti, ma per seguir virtute e canoscenza

1 Introduction

The operation $\sqrt{\cdot}$ in $\mathcal{C}$, has not the same meaning as $\sqrt{\cdot}$ in $\mathcal{R}$.
While ${}^{\mathcal{R}}\sqrt{a}$, with $a > 0$, $a \in \mathcal{R}$ typically admits one solution; ${}^{\mathcal{C}}\sqrt{z}$, with $z \in \mathcal{C}$, generally, has two. This is true unless we decide to reduce the solution space to just one sub-set. This procedure, referred to as **principal value**[1] of square roots in $\mathcal{C}$, has several benefits as it allows for temporary reconciliation between the operation ${}^{\mathcal{C}}\sqrt{}$ and ${}^{\mathcal{R}}\sqrt{}$.
The principal-value approach is particularly convenient in the framework of algebra, where we want to treat the square root as a function:

$$\left[{}^{(p)}\sqrt{z} \to f(z)\right] : z \Rightarrow f(z) \tag{1}$$

i.e., ${}^{(p)}\sqrt{z}$ is an operation that sets a univocal relation between two spaces.
In the framework of the principal value, the quantity $\sqrt{-1}$ can be associated univocally with the number i. Consequently, ${}^{(p)}\sqrt{z}$ has always one solution.
As discussed in later sections, however, restraining the operation ${}^{\mathcal{C}}\sqrt{z}$ to its principal value comes at a price. Indeed, while simplifying the operation, this approach introduces significant discontinuities. Furthermore, this simplification affects some properties of the square root operation, which instead holds for the square root of real numbers, ${}^{\mathcal{R}}\sqrt{}$. The most notable example is the lack of a homomorphism[2] for ${}^{(p)}\sqrt{z}$, i.e.:

$${}^{(p)}\sqrt{z_1 \times z_2\}} \neq {}^{(p)}\sqrt{z_1} \times {}^{(p)}\sqrt{z_2} \tag{2}$$

Consequently, as shown later, if we want to recover this property, so that:

$${}^{\mathcal{C}}\sqrt{z_1 \times z_2} = {}^{\mathcal{C}}\sqrt{z_1} \times {}^{\mathcal{C}}\sqrt{z_2} \tag{3}$$

a generalized approach must be followed.
Within this framework, ${}^{\mathcal{C}}\sqrt{z}$ ceases to be a single-valued function and yields two distinct solutions when $(z \neq 0)$. As a result, in this more comprehensive context, the first historical definition of the imaginary unit, $i = \sqrt{-1}$, becomes ambiguous and potentially misleading. Overlooking these facts in the square-root operation of complex number can lead to erroneous interpretations, such as the folkloristic mathematical claim: $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \times i = -1$. This statement, though popular, is a fallacy and does not hold under rigorous mathematical meaning.

[1] PrincipaleValueWiki
[2] HomomorphismWiki

2 Complex numbers and square roots

Following the modern definition suggested by Hamilton in the 19th century, [3], a complex number z is identified as a couple of real numbers, (a, b), for this couple the following properties hold:

$$i)\,(a,b)+(c,d)=(a+c,b+d) \tag{4}$$

$$ii)\,(a,b)\times(c,d)=(ac-bd,bc+ad) \tag{5}$$

Following these relations, the complex number z can be consistently written in the following ways:

$$z=a+ib=r(\cos(\phi)+i\sin(\phi))=re^{i\phi} \tag{6}$$

where ϕ is a phase, expressed in radians, while r is a positive real-number. Together, they form the polar representation of the complex number, as depicted in Fig.1.

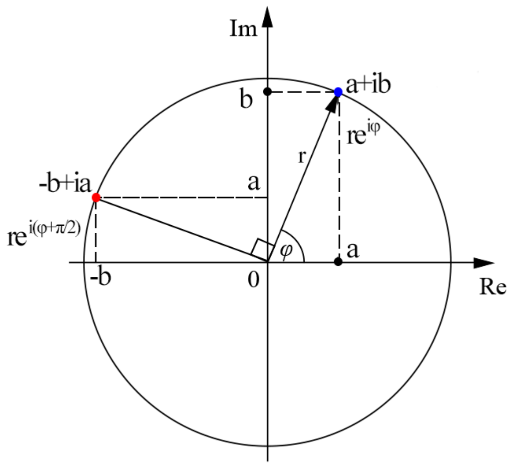

Figure 1: Representation on the complex plane of the three equivalent definition of the complex number z, (6). The blue doth is $z_1 = a + ib$, the red dot is $z_2 = -b + ia$.

As shown below, the sole coherent method to **define** the imaginary unit i, in alignment with Eq. (6) and the properties of Eqs. (4,5), is: $\mathbf{i}^2 = -1$.
To prove this, let us consider the multiplication between two complex numbers, e.g. the number z and the imaginary unit i.

$$z\times i=(a+ib)\times i=ai+i^2b=-b+ia \tag{7}$$

This operation acquires a different taste when observed in polar coordinates.

$$z\times i=re^{i\varphi}\times e^{i\pi/2}=re^{i\varphi+i\pi/2} \tag{8}$$

[3]In the literature, instead, you can find plenty of nice discussion on the history of complex numbers. The historical perspective, as always, is different from the modern reformulation of complex number theory. The most comprehensive one is Florian Cajori, "A History of Mathematical Notations" (1928 - Dover reprint), Vol II. Around pages 128-138 of this book, you will find the following insightful discussion.
Euler was the first to use the letter i for $\sqrt{-1}$ to solve the problem $x^2 + 1 = 0$, even though the origin of the problem is antecedent. Euler gave this definition in a memoir presented in 1777 to the Academy at St. Petersburg, and entitled "De formulis differentialibus etc.," but it was not published until 1794 after the death of Euler. As far as is now known, the symbol i for $\sqrt{-1}$, did not again appear in print for seven years, until 1801. In that year, Gauss began to make systematic use of it and introduced much of the notation on complex numbers we use today. Subsequently, William Rowan Hamilton (1805-65) provided a definitive framework for complex numbers, which is still in use. In a memoir from 1831, he articulated complex numbers as ordered pairs of real numbers (a, b). He defined operations such as addition and multiplication for these pairs as: $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, bc + ad)$. This operative definition allowed us to introduce the quantity $i^2 = -1$ as the proper and consistent definition of the imaginary quantity. Following these developments, and as discussed in various texts, the notation $\sqrt{-1} = i$ underwent a serious reevaluation. Augustin-Louis Cauchy, a key figure in the establishment of complex function theory, critiqued the use of this notation to introduce imaginary quantity. In an 1847 paper, he suggested that $\sqrt{-1} = i$ could be entirely discarded, stating: " $\sqrt{-1}$ *we can repudiate completely and which we can abandon without regret because one does not know what this pretended sign signifies nor what sense one ought to attribute to it*". See also, ShortHistoryOfComplexNumbers

Therefore, the multiplication by i is equivalent to the rotation by 90 degrees of the complex number z, identified in the complex plane as a vector.
These two approaches lead to the same results only because we have imposed $i^2 = -1$, which is nothing more than a consequence of the peculiar multiplication rules defined in Eq.(5).
This modern definition of the complex number z and of the imaginary unity, i, allows for the definition of any linear operation between complex numbers.

2.1 The problem of square-rooting in $\mathcal{C}$

The square root of a certain number z, is a number w for which $w^2 = z$. This is the most general definition of the operation: $\sqrt{z}$. This definition holds for complex numbers, as well; in this case, the goal is to find the values x and y of $w = x + iy$.

$$\begin{aligned}
&w^2 = x^2 + i^2 y + 2ixy = a + ib \\
&i)\, a = x^2 - y^2 \\
&ii)\, b = 2xy
\end{aligned}$$

We have a non-linear system of two-equation, which, in principle, can have more than one solution. It is easy to show that its solution is:

$$x = \pm\sqrt{(\sqrt{a^2 + b^2} + a)/2} \tag{9}$$

$$y = \pm\sqrt{(\sqrt{a^2 + b^2} - a)/2} \tag{10}$$

Without particular restrictions, $\sqrt{z}$ admits two solutions.
For consistency, can also find the solution of $\sqrt{z}$ by exploiting Euler's notation and the so-called De-Moivre formula:

$$\begin{aligned}
w &= Re^{i\Phi} \\
z &= re^{i\phi + 2k\pi} \\
w^2 &= R^2 e^{i2\Phi}
\end{aligned}$$

where k is a discrete integer, which takes into account the periodicity in the definition of the complex exponential z.[4] Consequently,

$$R = {}^{\mathcal{R}}\sqrt{r} \tag{11}$$

the square root of r, here regarded as ${}^{\mathcal{R}}\sqrt{}$ is univocally defined since $R > 0$. On the other hand,

$$\Phi = \phi/2 + k\pi \tag{12}$$

The periodicity in the phase Φ, is equivalent to a rotation of +180 degrees in the complex plane. Therefore, as shown in Fig. 2, in consistency with Eqs. (9,10), both $\pm w = \pm\sqrt{r}e^{i\phi/2}$ are solutions of $\sqrt{z}$.
The presented solution is the generalization of the operation: $\sqrt{z} = w$ to the complex numbers domain; hence, we can indicate it as ${}^{\mathcal{C}}\sqrt{z}$. The existence of the $-w$ solution can be explained as a consequence of the rotational equivalence of the multiplication between complex numbers, discussed for the case of $z \times i$. In other words, $-w \times -w$, implies a peculiar transformation (rotation+stretching) of the vector $-w$, which leads to the final vector z.

Now, it is easy to prove that ${}^{\mathcal{C}}\sqrt{}$ has homomorphism.
Given $a = r_1 e^{i\phi_1 + 2k_1\pi}$ and $b = r_2 e^{i\phi_2 + 2k_2\pi}$, then,

$${}^{\mathcal{C}}\sqrt{ab} = {}^{\mathcal{C}}\sqrt{r_1 r_2} e^{i(\phi_1/2 + \phi_2/2 + k\pi)}$$

where $k = k_1 + k_2$, and:

$${}^{\mathcal{C}}\sqrt{a}\,{}^{\mathcal{C}}\sqrt{b} = \sqrt{r_1}\sqrt{r_2} e^{i(\phi_1/2 + k_1\pi} e^{+\phi_2/2 + k_2\pi)} = \sqrt{r_1 r_2} e^{i(\phi_1/2 + \phi_2/2 + k\pi)} = {}^{\mathcal{C}}\sqrt{ab}$$

Following these results, we observe that, if $b \to 0$, objects like $\sqrt{-1}$ would lead to $\sqrt{-1} = \pm i$, as well as $\sqrt{1} = \pm 1$. This

[4] This periodicity, of course, derives from the periodicity of sin and cos. Please note, the periodicity, $+2k\pi$, should be expressed only for z, since Φ is what we want to find, given the periodicity of the angle of z.

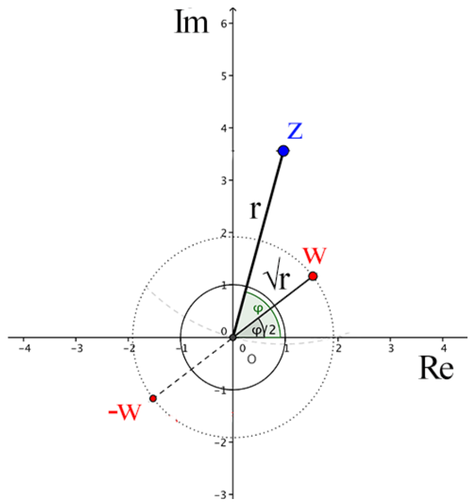

Figure 2: Representation of the operation of square root as a homomorphism in $\mathcal{C}$.

2.2 The principal value and its discontinuity

To establish a single-valued function for the square root of complex numbers, it is common practice to use its principal value. In this context, the principal value of $\sqrt{z}$ is defined as: ${}^{(p)}\sqrt{z} = w = Re^{i\Phi}$ where the phase Φ of the complex number z is constrained to the interval $(-\pi, \pi]$. For example, it is straightforward to prove that according to this definition, the square root of $-1 + \epsilon i$, where ϵ is small number approaches $+i$, and for $-1-$, approaches $-i$. Although we have established now a non-ambiguous definition of the square root operation, a minor variation in the imaginary component of z near the negative axis, in this case around -1, results in a substantial change in the imaginary part of $\sqrt{z}$. In this framework, the square root of -1 is defined as i, considering it as the limit from the positive imaginary direction. The presented discontinuity, which also causes the lack of homomorphism, is the price we pay for limiting our subset of solutions to the one with phasis between $(-\pi, \pi]$.
The principal value convention is the same adopted in computational methods for calculating $\sqrt{z}$ numerically.
This convention has significant practical implications in various fields where the square root of a complex number is routinely calculated. In domains like fluid dynamics, circuit design, and control theory, complex numbers are employed to represent vectorial quantities. Therefore, a consistent and clear convention, such as $\sqrt{-1} = i$ for handling complex square roots, is essential.
Moreover, in ambiguous situations, validation against calculations using actual numbers is advisable to ensure that only meaningful results are retained.
However, it is important to recognize that near the negative real axis, minor numerical inaccuracies in the imaginary part of a complex number can cause major variations in the selection of the appropriate square root (${}^{(p)}\sqrt{}$). This problem is not present for ${}^{\mathcal{C}}\sqrt{}$.
In more complex domains, such as quantum mechanics complex numbers do not correspond directly to vectorial quantities. In these scenarios, particularly where ψ^2 appears in the equation (for instance, in quantum field theory or models for Bose-Einstein condensates), extra caution is warranted, and the homomorphic definition of the square root of a complex number should be utilized.

Similar considerations apply to the complex logarithm function. It is generally agreed that for z not being a negative real number (nor zero), $\log(z)$ refers to the complex solution of $e^u = z$ with an imaginary part within the interval $-\pi$ to $+\pi$, excluding these endpoints. For negative real numbers z, $\log(z)$ is chosen to have an imaginary part of $+\pi$. It is crucial to note that the equality $log(u \cdot v) = \log(u) + \log(v)$ holds only up to an imaginary multiple of 2π.

3 Conclusion

As discussed in this short commentary, the question:
"Is $\sqrt{-1}$ *univocally equal to* i?"
is, generally, an ill-posed question.
Indeed, one should first wonder what is the meaning of the square root operation in such a context.
In other terms, the addressee of the question should always try to reframe it as:
"When you ask for the square root, do you mean as the homomorphism operation or its principal value?"
However, it should be noted by the brave reader that if you wish to get laid... it is strongly advisable to avoid discussing the subject directly and, instead, leave it unspoken.