Classical Electrodynamics: The Dipole Radiation

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Contents

Abstract

In this paper, I am going to present and discuss a well-known topic in classical electrodynamics: **the electromagnetic field irradiated by an electric dipole in the far field region**.

This problem, despite its appearance, is not trivial. Therefore, to be fully understood, these pages require fundamental knowledge in classical physics, electromagnetism, a basic understanding of mathematical tools, like the Fourier transform and complex analysis, and well-established skills in calculus and algebra.

The methodology followed is purely ab initio, hence, no preliminary knowledge on the topic are required.

This discussion has been mainly inspired by:

- J.Jackson, *Classical Electrodynamics*, Wiley 1962, 1st edition
- P.Dennery A. Krzywicki, *Mathematics for Physicist*, Dover 1996

Introduction: let there be light

As in any well posed problem, let us start from postulates, that are what we assume to be a rule of nature, in our case Maxwell equations[1].

Given $\left(\mathbf{E}(\mathbf{r},t),\mathbf{B}(\mathbf{r},t)\right)$ electromagnetic field defined in time, t, and space, $\mathbf{r}$, given $\rho(\mathbf{r},t)$ density of charges and $\mathbf{J}(\mathbf{r},t)$ density of current, then in the S.I. units the Maxwell equations are written as:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{B} = \frac{1}{c_0^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

where ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability and c_0 is the speed of light.

Let us now imagine being in complete vacuum with nothing but just two point-like oppositely charged particles which are moving with their own law of motion. We can model them as Dirac deltas $\delta(\mathbf{r}_1(t))$ and $\delta(\mathbf{r}_2(t))$, then:

$$\rho(\mathbf{r},t) = e\delta(\mathbf{r} - \mathbf{r}_1(t)) - e\delta(\mathbf{r} - \mathbf{r}_2(t)) \tag{1}$$

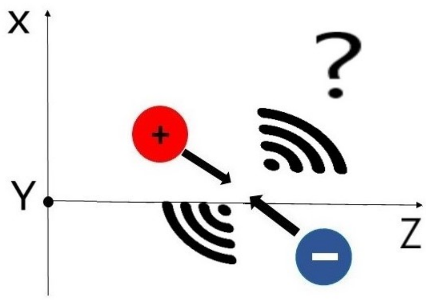


Figure 1: *Hideous representation of two charges in free space that moves. Ceci n'est pas...*

Now our starting question is the following:

What will happen to the space if we have two moving charges?

The answer has been demonstrated many times and can be resumed by saying that, in general, when charges moves in space they produce a time-varying electromagnetic field.

Then we can move on a more refined question:

Can we generate a well-defined propagating electromagnetic radiation by moving a generic charges distribution in a particular way?

So we must study in details the problem with an ab initio approach. Thus, we will plug in Maxwell equations some moving sources, and we will admire the consequences.

This apparently innocent problem will require a long journey full of math and smart tricks, so be ready for this challenge.

[1] These "rules" for the classical interpretation of light have been proved through experiments and technologies, literally, billions of times in the last centuries. Therefore, I think they do not need any further justifications.

1 Electromagnetic four-potential: the problem framework

First things first, the $(\mathbf{E}, \mathbf{B})$ formulation is not so comfortable, i.e. it is fine but it does not allow writing "easy" equations. Therefore, a very good practice is to introduce the vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\phi(\mathbf{r}, t)$, they are known as the **electromagnetic four-potential** $(\mathbf{A}, \phi)$. This four-potential that we are going to introduce in few lines, is directly derived from Maxwell equations.

Indeed, from $\nabla \cdot \mathbf{B} = 0$ we can exploit the vectorial identity $\nabla \cdot (\nabla \times \mathbf{V}) = 0$, where $\mathbf{V}$ is a generic vectorial field, and then write:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Whereas, by exploiting another vectorial identity $\nabla \times (\nabla v) = 0$, where v is a generic scalar field and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, we can write:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \nabla \times \mathbf{A}}{\partial t} \\ \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) &= 0 \\ \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} &= -\nabla \phi \end{aligned}$$

Therefore we can consistently introduce this new formulation of the electromagnetic field:

$$\begin{cases} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \end{cases} \tag{2}$$

where for simplicity we have given for granted the space and time dependent of the object other discussion.

Let us complete the characterization of this four potential by plunging it in the so far unused Maxwell equations. Indeed, two equations have been used for the definition of this object $(\mathbf{A}, \phi)$ the remaining ones, instead, define its evolution in space and time.
From:

$$\begin{cases} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{B} &= \frac{1}{c_0^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \end{cases} \tag{3}$$

we have:

$$\nabla \cdot (-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}) = \frac{\rho}{\epsilon_0} \tag{4}$$

$$\nabla \times \nabla \times \mathbf{A} = \frac{1}{c_0^2} \frac{\partial}{\partial t}(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}) + \mu_0 \mathbf{J} \tag{5}$$

where we have just substituted (2) in (3).

It is interesting to observe that (4) and (5), together with the definition of $(\mathbf{A}, \phi)$, can be seen as the new Maxwell equations, and thus provide a full description of the electromagnetic phenomena[2].
Let us now rewrite (4) in a more handy way.

[2] At least in classical regime

We can exploit the trivial identity:

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi$$

then,

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{6}$$

Let us do the same for (5):

As a matter of fact the following identity holds:

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

so then,

$$\nabla^2 \mathbf{A} - \frac{1}{c_0^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c_0^2}\frac{\partial \phi}{\partial t}) = -\mu_0 \mathbf{J} \tag{7}$$

In synthesis:

$$\begin{cases} \nabla^2 \phi + \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \dfrac{1}{c_0^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \dfrac{1}{c_0^2}\dfrac{\partial \phi}{\partial t}) = -\mu_0 \mathbf{J} \end{cases} \tag{8}$$

This two new equations are the starting point of our challenge and of many other problems in classical electrodynamics.
Since many theoretical physicists are lazy[3], we want to find better looking equations to face easy problems, like two charges moving.

Then, by restarting from the definition of **A** and exploiting the already used vectorial identity, where this time v is substituted by an equivalently arbitrary scalar function χ,

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad\qquad \nabla \times (\nabla \chi) = 0 \tag{9}$$

we can smartly observe that:

$$\mathbf{A} = \mathbf{A'} + \nabla \chi \tag{10}$$

it is then possible to equivalently use **A** or **A'** without causing and change in the magnetic field **B**. In clearer terms:

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A'}$$

If we propagate this ambiguity in the **A** definition to **E**, then we have:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi = -\frac{\partial \mathbf{A'}}{\partial t} - \frac{\partial}{\partial t}\nabla \chi - \nabla \phi$$

Therefore,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi = \frac{\partial \mathbf{A'}}{\partial t} - \nabla \phi'$$

where

$$\phi = \phi' - \frac{\partial \chi}{\partial t} \tag{11}$$

[3]And fortunately they are! Otherwise, it would be impossible to create technology and do experimental Science

The equations (10) and (11) show that we can freely choose a scalar function χ, changing, as a consequence, the proprieties of the four-potential, without modifying the physical description of the system under investigation. Therefore, this freedom in the choice of χ describes an invariance, **gauge invariance**, of the physically relevant quantities $(\mathbf{E}, \mathbf{B})$.

To recap:
One can solve

$$\begin{cases} \nabla^2\phi + \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2\mathbf{A} - \dfrac{1}{c_0^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \dfrac{1}{c_0^2}\dfrac{\partial \phi}{\partial t}) = -\mu_0 \mathbf{J} \end{cases} \tag{12}$$

or

$$\begin{cases} \nabla^2\phi' + \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A'}) = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2\mathbf{A'} - \dfrac{1}{c_0^2}\dfrac{\partial^2 \mathbf{A'}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A'} + \dfrac{1}{c_0^2}\dfrac{\partial \phi'}{\partial t}) = -\mu_0 \mathbf{J} \end{cases} \tag{13}$$

without any difference in the physics of the problem and still describing the same electromagnetic field.
Therefore, we can make a particular choice of χ in

$$\begin{cases} \mathbf{A} = \mathbf{A'} + \nabla\chi \\ \phi = \phi' - \dfrac{\partial \chi}{\partial t} \end{cases} \tag{14}$$

so that:

$$\nabla \cdot \mathbf{A} + \frac{1}{c_0^2}\frac{\partial \phi}{\partial t} = 0 \tag{15}$$

this is the **Lorentz condition**, which leads to:

$$\begin{cases} \nabla^2\phi - \dfrac{1}{c_0^2}\dfrac{\partial^2 \phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2\mathbf{A} - \dfrac{1}{c_0^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \end{cases} \tag{16}$$

In the Lorentz condition the four-potentials are decoupled, and they can be solved much more easily.

The gauge functions that satisfy the Lorentz condition are given by:

$$\nabla^2\chi - \frac{1}{c_0^2}\frac{\partial^2 \chi}{\partial t^2} = -(\nabla \cdot \mathbf{A'} + \frac{1}{c_0^2}\frac{\partial \phi'}{\partial t})$$

For the sake of completeness, it is more common to find an even more stringent condition, known as the **Lorentz gauge**, where the gauge function must satisfy:

$$\nabla^2\chi - \frac{1}{c_0^2}\frac{\partial^2 \chi}{\partial t^2} = 0$$

The Lorentz gauge is therefore a sub-space of solution for the Lorentz condition.

In any case our final framework is (16), this is the problem, the challenge we want to solve in mathematical terms. It is a system of uncoupled partial differential equations that will be faced exploiting some complex analysis, Fourier transform and Green function.

2 Classical electrodynamics: the problem resolution

2.1 Green Function

Math is beautiful because it de-contextualizes any physical problem, and not only. Math traces a clear and coherent path and the role of physicist is to follow it and interpret it. However, usually, this is not enough. Indeed, in order to practically solve the problem, it is often necessary to apply some tricks and reformulates the problem in easier terms by using a strong physical intuition of what is going on. In fact, one should not forget that our final purpose[4] is to understand nature!

Therefore, let us start considering the following problem:
Given $\psi(\mathbf{r}, t)$ unknown real and scalar field and $f(\mathbf{r}, t)$ known real and scalar field, then we want to find solution for:

$$\nabla^2\psi - \frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2} = -\mu_0 f(\mathbf{r}, t) \tag{17}$$

It is pointless to observe that solving (17) it gives us the solution of (16). With these connections in mind we call f the source term and ψ as a scalar wave.

An incredibly powerful way of solving the problem is based on the **Green's function** method. In order to solve (17) is useful to find solution for:

$$\nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r'}, t, t') - \frac{1}{c_0^2}\frac{\partial^2 G(\mathbf{r}, \mathbf{r'}, t, t')}{\partial t^2} = -\mu_0 \delta(\mathbf{r} - \mathbf{r'})\delta(t - t') \tag{18}$$

where $G(\mathbf{r}, \mathbf{r'}, t, t')$ is the scalar wave generated by a source that is localized in space in **r'** and exists only at t', and $\nabla^2_{\mathbf{r}}$ is the laplacian operator[5]. Once one has found G then ψ can be reconstructed as:

$$\psi(\mathbf{r}, t) = \int_{\mathcal{R}^4} G(\mathbf{r}, \mathbf{r'}, t, t') f(\mathbf{r'}, t') d^3r' dt' \tag{19}$$

which corresponds[6] to a infinitely dense sum, an integral, over all the Green's functions, obtained for different deltas, weighted by f. For the sake of simplicity we will use the dalambertian operator that we define as:

$$\Box_{\mathbf{r},t} = \nabla^2_{\mathbf{r}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} \tag{20}$$

So (18) become:

$$\Box_{\mathbf{r},t} G(\mathbf{r}, \mathbf{r'}, t, t') = -\mu_0 \delta(\mathbf{r} - \mathbf{r'})\delta(t - t') \tag{21}$$

As it happens many times in life, in order to solve a problem, e.g. (17), we must solve first another one, (21).

To do so, let us use the Fourier transform tool[7]. Then, fixed the $+i(\omega t - \mathbf{k} \cdot \mathbf{r})$ convention[8] we have:

$$G(\mathbf{r}, \mathbf{r'}, t, t') = \int d^3k \int d\omega g(\mathbf{k}, \omega) e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})} e^{+i\omega(t-t')} \tag{22}$$

[4] As physicist, of course!

[5] Note: it is highlighted the variable on which it operates, in our case is **r**.

[6] An intuitive and basic proof is reported in Appendix A

[7] In doing so, we are forcing our solution to belong to a certain space of functions. So we may wonder: *can we describe all possible solutions?*. The answer is: for sure we can describe all the objects, all the function, that can be decomposed as "plane waves" following Fourier formula. As you will see later on, the previous question is crucial.

[8] In many excellent book, like in *Jackson*, the convention used is the opposite, $+i(-\omega t + \mathbf{k} \cdot \mathbf{r})$ but, luckily, the final outcome is the same.

Physically speaking, we have written our Green's function as a superposition of plane waves, with **k** spatial wave vector and ω the (angular) frequency. Furthermore, by the deltas definition, we know:

$$\delta(\mathbf{r}-\mathbf{r'})\delta(t-t') = \frac{1}{(2\pi)^4}\int d^3k \int d\omega e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')}$$

Thus from equation (21),

$$\Box_{\mathbf{r},t}\int d^3k \int d\omega g(\mathbf{k},\omega) e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')} = -\frac{\mu_0}{(2\pi)^4}\int d^3k \int d\omega e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')} \tag{23}$$

The dalambertian operator commutes with the integral in **k** and ω and, hence:

$$\int d^3k \int d\omega g(\mathbf{k},\omega)\Box_{\mathbf{r},t}(e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')}) = -\frac{\mu_0}{(2\pi)^4}\int d^3k \int d\omega e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')}$$

$$g(\mathbf{k},\omega)(-k^2+\frac{\omega^2}{c_0^2}) = -\frac{\mu_0}{(2\pi)^4}$$

where $k = |\mathbf{k}|$, then

$$g(\mathbf{k},\omega) = \frac{\mu_0}{(2\pi)^4}\frac{1}{(k^2-\frac{\omega^2}{c_0^2})} \tag{24}$$

(24) is the Fourier transform of the green function that solves (21).

$$G(\mathbf{r},\mathbf{r'},t,t') = \int d^3k \int d\omega \frac{\mu_0}{(2\pi)^4}\frac{1}{(k^2-\frac{\omega^2}{c_0^2})}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')} \tag{25}$$

Finally, we have

$$\psi(\mathbf{r},t) = \int d^3r'dt'\Big[\int d^3k d\omega \frac{\mu_0}{(2\pi)^4}\frac{1}{(k^2-\frac{\omega^2}{c_0^2})}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')}\Big]f(\mathbf{r'},t') \tag{26}$$

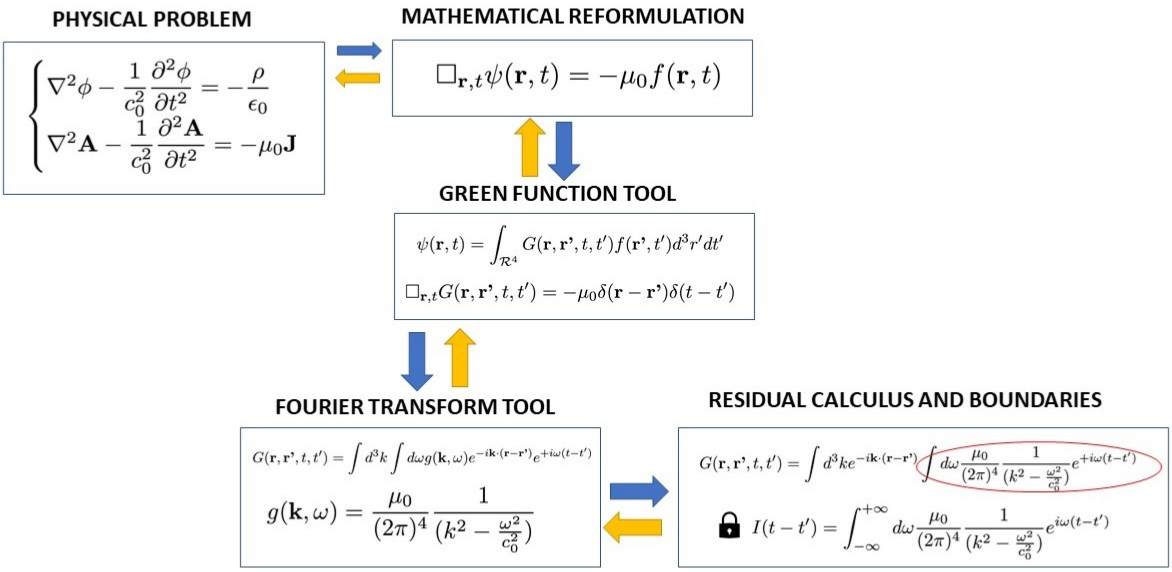
(26) is the most general solution of our problem. So, in principle, provided that we know how to integrate the quantities, equation (17) is solved. Math, once again, has done its job greatly.
Provided that we know is the right phrase, indeed, in general we don't.
Therefore, if we want to go further and practically use (26), we need to solve the integrals with some physical intuition and some other mathematical tools.
Now, allow me to spoil what it is coming next:

First: we are trying to solve the (25), in doing so some physical boundaries will be applied and, then, residual calculus will be exploited. In this way we are going to achieve a final and user-friendly form for G.

Second: we will re-contextualize the problem selecting a proper source function and making some physically reasonable approximations.

Last: we will go back to our initial task, and we will evaluate the electromagnetic field irradiated by a dipole of charge.

At the moment we are still stuck at the first step. So let us recap our short mathematical discussion in a figure:



If we are able to solve this very last problem, then we can go back and carry out the four-vector solution for a generic source and as a consequence the electromagnetic field!

2.2 Residual calculus and "Physicist tricks"

Thus, our new challenge is to solve this integral,

$$G(\mathbf{r},\mathbf{r'},t,t') = \int_{\mathcal{R}^3} d^3k \int_{-\infty}^{+\infty} d\omega \frac{\mu_0}{(2\pi)^4}\frac{1}{(k^2-\frac{\omega^2}{c_0^2})}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r'})}e^{+i\omega(t-t')} \tag{27}$$

at this point we have to add some physical reasoning. G is the response of a delta-source which exists only for $t = t'$, for the **principle of causality** this kind of element can have effect in space only for $t > t'$. Thus, we have a problem! In fact, from a preliminary look, equation (27) it will be hardly zero for $t < 0$. Our mathematical description seems to be not compatible with our physical needs: the description of an outgoing wave from a source. In addition, the integration interval for ω contains point where the function goes to infinity, $\omega = +kc$ and $\omega = -kc$. This is clearly bad.

We are completely blocked and our conventional rules of integration does not work. A reasonable physical problem turn out to be not conventionally solvable.

Where is the problem? Why and how did we end up here?

The answer is not straightforward. A reasonable guess can be made starting from the analysis of the mathematical tools that we have used.

- The first thing we have done, was to introduce the Green function as a method to retrieve a solution for a physically meaningful equation (17), the ψ equation. We didn't question that G is, indeed, the solution of a differential equation with objects, the Dirac deltas, that are not traditional functions, they are more mathematical artefacts than real physical objects.

- The "worst thing" we have done was to represent an object, G, that should start to be different from zero at some point in time, t', as a weighted integral of monochromatic waves[9], that instead exist from $-\infty$ to $+\infty$. This is not forbidden, but one has to accept that some problem may arise and less conventional tricks must be **invented** to find a solution, think, for instance, to the Fourier transform of a step function.

In mathematical terms our path is not that strange and everything is crystal clear. Indeed, in our discussion we are jumping without too much care between different Hilbert space[10] so we must be careful.

Let us now focus only on the resolution, and accept that the following discussion could seem a bit abstract. Here we need to understand the fundamental of **Residual calculus**, this discussion is postponed in *Appendix B*.

Our new problem is the following:

$$I(t-t') = \int_{-\infty}^{+\infty} \frac{e^{i\omega(t-t')}}{k^2 - \frac{\omega^2}{c_0^2}} d\omega \tag{28}$$

$$I(\tau) = -\int_{-\infty}^{+\infty} \frac{e^{i\omega\tau}}{(\omega - c_0 k)(\omega + c_0 k)} c_0^2 d\omega \tag{29}$$

where $\tau = t - t'$.
This integral can have four solutions, but only one is meaningful for us, i.e. only one satisfies the $I(\tau) = 0$ for $\tau < 0$, our boundary condition.

In residual terminology: we have two singularities on the real ω axis.

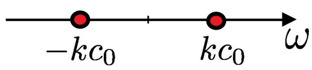


Figure 2: *Representation of the singularities, poles, on the real-axis in* ω

As we have seen in Appendix B, the result of the integration depends on the particular path followed. Therefore, using the complex analysis theory, we move the poles of a small quantity, ϵ, above the real ω-axis[11]. Then, we can identify two paths, as shown in **Figure 3**:

Path 1: For $\tau > 0$, in this case the $I(\tau)$ goes to zero for value above the real axis, in fact $e^{i(i\omega)\tau}$ goes to zero for $\omega \to \infty$. Therefore, shifting the pole we can apply the residual calculus and $I(\tau) \neq 0$.

Path 2: For $\tau < 0$, in this case the $I(\tau)$ goes to zero for value below the real axis, in fact $e^{i(i\omega)\tau}$ goes to zero for $\omega \to -\infty$. Therefore, for the same pole shifting this time $I(\tau) = 0$.

Then,

$$I(\tau) = \lim_{\epsilon \to 0} \begin{cases} -\int_1 \frac{e^{i\omega\tau}}{(\omega + kc_0 - i\epsilon)(\omega - kc_0 - i\epsilon)} c_0^2 d\omega & \tau > 0 \\ -\int_2 \frac{e^{i\omega\tau}}{(\omega + kc_0 - i\epsilon)(\omega - kc_0 - i\epsilon)} c_0^2 d\omega = 0 & \tau < 0 \end{cases} \tag{30}$$

[9] a.k.a Fourier transform
[10] L^2 to D, distribution space
[11] The other 3 way we could have moved the pole corresponds to different "boundaries" conditions, in which the integral (28) assume a different value that is, however, not compatible with the request: $I(\tau) = 0$ for $\tau < 0$.

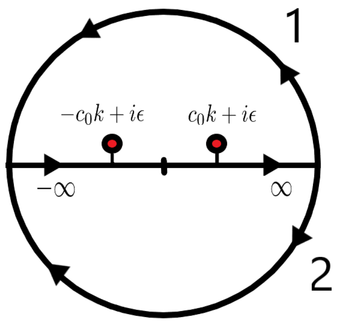


Figure 3: *Two possible paths corresponding to $\tau > 0$, "1", and $\tau < 0$, "2". Depending on the path, we have different values for different time interval.*

Let us then solve for $\tau > 0$:

$$\begin{aligned} I(\tau) =& \lim_{\epsilon\to 0}\int_1 -\frac{e^{i\omega\tau}}{(\omega + kc_0 - i\epsilon)(\omega - kc_0 - i\epsilon)}c_0^2 d\omega \\ =& - c_0^2 \lim_{\epsilon\to 0} 2\pi i\Big[\frac{e^{i(-kc_0+i\epsilon)\tau}}{-2kc_0} + \frac{e^{i(kc_0+i\epsilon)\tau}}{2kc_0}\Big] \\ =& - c_0 2\pi i[\frac{e^{ikc_0} - e^{-ikc_0}}{2k}] \end{aligned}$$

So,

$$I(t-t') = \int_{-\infty}^{+\infty} \frac{e^{i\omega(t-t')}}{k^2 - \frac{\omega^2}{c_0^2}} d\omega = c_0 2\pi \frac{\sin(kc_0(t-t'))}{k} \tag{31}$$

We are, finally, ready to solve for G the Green's function.

By simply substituting the previous result in (27):

$$G(\mathbf{r}, \mathbf{r'}, \tau) = \int_{\mathcal{R}^3} d^3k \frac{\mu_0 c_0}{(2\pi)^3} \frac{\sin(c_0 k\tau)}{k} e^{-i\mathbf{k}\cdot\mathbf{R}}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r'}$, $R = |\mathbf{R}|$.

Let us go forward and solve this spatial integral:

$$G(R,\tau) = \frac{\mu_0 c_0}{(2\pi)^3}\int_0^{\pi}\int_0^{+\infty} k\sin(c_0 k\tau)e^{-ikR\cos\theta} 2\pi \sin\theta d\theta dk$$

where we have substituted $d^3k = k^2 \sin\theta d\theta d\varphi$. Note that φ is the azimuth and θ is the angle between $\mathbf{k}$ and the k_z axis.
Then:

$$G = \frac{\mu_0 c_0}{(2\pi)^2}\int_0^{+\infty} dk k\sin(c_0 k\tau)\int_0^{\pi} e^{-ikR\cos\theta}\sin\theta d\theta$$

by simple variable substitution $-cos\theta = X$, $dX = +\sin\theta d\theta$, and solving the X-integral, we have:

$$G(R,\tau) = \frac{\mu_0 c_0}{2\pi^2 R}\int_0^{+\infty} sin(kR)sin(c_0 k\tau)dk$$

The integrand is even in k, so after onther variable substitution $Y = c_0 k$

$$G(R,\tau) = \frac{\mu_0}{4\pi^2 R}\int_{-\infty}^{+\infty} sin(Y\frac{R}{c_0})sin(Y\tau)dY$$

Exploiting the fact:

$$\sin a \sin b = \frac{\cos(a-b) - cos(a+b)}{2}$$

then,

$$G(R,\tau) = \frac{\mu_0}{4\pi^2 R}\frac{1}{4}\int_{-\infty}^{+\infty}(e^{i(\tau-\frac{R}{c_0})Y} + e^{-i(\tau-\frac{R}{c_0})Y} - e^{-i(\tau+\frac{R}{c_0})Y} - e^{+i(\tau+\frac{R}{c_0})Y})dY$$

Due to symmetry of the integrands, we have a sum of two equivalent integrals.
Therefore,

$$G(R,\tau) = \frac{\mu_0}{4\pi^2 R}\frac{2}{4}\int_{-\infty}^{+\infty}(e^{i(\tau-\frac{R}{c_0})Y} - e^{+i(\tau+\frac{R}{c_0})Y})dY$$

From the definition of delta:

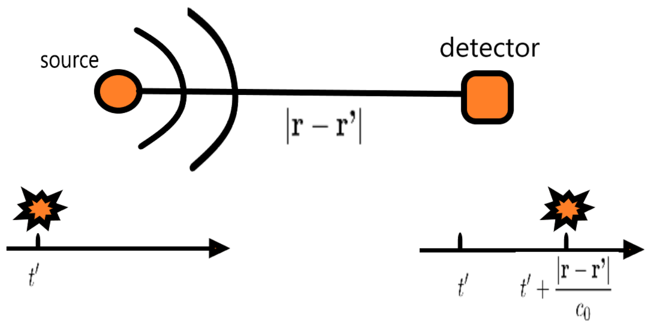
$$\delta(\tau - \frac{R}{c_0}) = \frac{1}{2\pi}\int dY e^{-i(\tau-\frac{R}{c_0})Y}$$

$$G(R,\tau) = \frac{\mu_0}{4\pi R}\Big(\delta(\tau - \frac{R}{c_0}) + \delta(\tau + \frac{R}{c_0})\Big)$$

Now, $R > 0$, since it is the modulus of **R**, $\tau > 0$ is, instead, our working condition, then only the first delta is different from zero somewhere.
As a result, for $t - t' > 0$:

$$\boxed{G(\mathbf{r},\mathbf{r'},t,t') = \frac{\mu_0}{4\pi}\frac{\delta(t - \frac{|\mathbf{r}-\mathbf{r'}|}{c_0} - t')}{|\mathbf{r}-\mathbf{r'}|}} \tag{32}$$

This is a milestone of physics known as **retarded Green's function**, because it describes an object generated by a space-time localized source that propagates in space at the speed of light. In other word, fixed a position in space **r**, you observe the effect of a source in $(t', \mathbf{r'})$ after a delay $t - t' = +\dfrac{|\mathbf{r}-\mathbf{r'}|}{c_0}$.



Propagating back the result we are finally able to write down the solution of our initial mathematical task (17)!

$$\nabla^2\psi - \frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2} = -\mu_0 f(\mathbf{r}, t)$$

$$\psi(\mathbf{r}, t) = \int \frac{\mu_0}{4\pi}\frac{\delta(t - \frac{|\mathbf{r}-\mathbf{r'}|}{c_0} - t')}{|\mathbf{r}-\mathbf{r'}|} f(\mathbf{r'}, t')dt'd^3r'$$

Or,

$$\psi(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{\mathcal{R}^3} \frac{f(\mathbf{r'}, t - \frac{|\mathbf{r}-\mathbf{r'}|}{c_0})}{|\mathbf{r}-\mathbf{r'}|} d^3r' \tag{33}$$

Adding physical interpretation: equation (33) is telling us that the wave, perceived at time t, is due to the effect of the source that occurred at a previous time, $t - \frac{|\mathbf{r}-\mathbf{r'}|}{c_0}$. This delay of course depends on how far the source is from the detection place.

After a (very) long mathematical journey, we are ready to go back to our physical problem and solve for the four-potential for a generic source $(\rho, \mathbf{J})$.

Then, after few more steps, we will find out the irradiating field of the electric dipole.

3 The oscillating dipole radiation

We had previously derived:

$$\begin{cases} \nabla^2\phi - \dfrac{1}{c_0^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2\mathbf{A} - \dfrac{1}{c_0^2}\dfrac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} \end{cases}$$

Without any loss of generality, we can consider sinusoidally oscillating sources:

$$\rho(\mathbf{r},t) = \rho(\mathbf{r})e^{i\omega t} + c.c. \tag{34}$$

$$\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})e^{i\omega t} + c.c. \tag{35}$$

For simplicity, we consider for both only the first term. Of course, at the end one must remember to take the real part.

So, exploiting the result of equation (33), we can write down the solution for **A** as:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{\mathcal{R}^3} \frac{\mathbf{J}(\mathbf{r'}, t - \frac{|\mathbf{r}-\mathbf{r'}|}{c_0})}{|\mathbf{r}-\mathbf{r'}|} d^3r'$$

It follows:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{\mathcal{R}^3} \frac{\mathbf{J}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} e^{i\omega(t-\frac{|\mathbf{r}-\mathbf{r'}|)}{c_0}} d^3r'$$

Therefore,

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{+i\omega t} \tag{36}$$

This result allows to exploit the Fourier decomposition and, at last, to generalize to a generic source.

$$\mathbf{J}(\mathbf{r},t) = \frac{1}{2\pi}\int \mathbf{J}(\mathbf{r},\omega)e^{+i\omega t}d\omega \tag{37}$$

And hence,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\pi}\int \mathbf{A}(\mathbf{r},\omega)e^{+i\omega t}d\omega \tag{38}$$

To be fully correct, we should say that we are performing an analysis of the field in the **time-harmonic domain**.

So:

$$\rho_\omega(\mathbf{r},t) = \rho(\mathbf{r},\omega)e^{i\omega t} + c.c.$$
$$\mathbf{J}_\omega(\mathbf{r},t) = \mathbf{J}(\mathbf{r},\omega)e^{i\omega t} + c.c.$$

and,

$$\mathbf{A}_\omega(\mathbf{r},t) = \mathbf{A}(\mathbf{r},\omega)e^{+i\omega t}$$

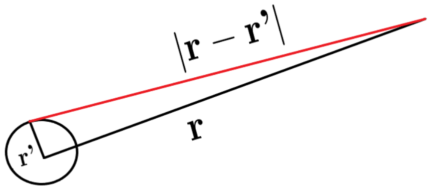
Let us now continue with our analysis in the harmonic domain, forgetting about $\mathbf{A}_\omega$ and just using the simpler notation, $\mathbf{A}(\mathbf{r})$.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{R}^3} \frac{\mathbf{J}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} e^{-i\frac{\omega}{c_0}|\mathbf{r}-\mathbf{r'}|} d^3r' \tag{39}$$

Therefore, given a certain distribution of charge that oscillates in time, we can calculate the generated vector potential in space-time.

Usually, the profile of equation (39) is pretty complex, thus many approximations are usually involved. One of the most used approximation is to look at the field generated in a region far with respect the source characteristic dimension.

Let us assume that the source is small with respect the distance at which we are looking its effects. This approximation, practically speaking, implies that $kR >> 1$, where $k = \frac{\omega}{c_0}$, this is the so called **far region**.



At this point we can approximate, since $k|\mathbf{r} - \mathbf{r'}| >> 1$:

$$|\mathbf{r} - \mathbf{r'}| \approx r - \mathbf{n} \cdot \mathbf{r'}$$

where $\mathbf{n} = \dfrac{\mathbf{r}}{r}$.
Then,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e^{-ikr}}{r} \int_{\mathcal{R}^3} \mathbf{J}(\mathbf{r'}) e^{-ik\mathbf{n}\cdot\mathbf{r'}} d^3r' \tag{40}$$

If we now make an expansion of the exponential term inside the integral:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e^{-ikr}}{r} \sum_n \frac{(-ik)^n}{n!} \int_{\mathcal{R}^3} \mathbf{J}(\mathbf{r'}) (\mathbf{n} \cdot \mathbf{r'})^n d^3r' \tag{41}$$

Stopping at n=0 means that we are assuming $|\mathbf{r} - \mathbf{r'}| = r$. This is the **electric dipole approximation** for the source.

We are almost there, indeed, in order to find out the electric field generated by a certain source, we just have to solve the following integral:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e^{-ikr}}{r} \int_{\mathcal{R}^3} \mathbf{J}(\mathbf{r'}) d^3r' \tag{42}$$

Exploiting the integration by part:

$$\int_{\mathcal{R}^3} \mathbf{J}(\mathbf{r'}) d^3r' = \begin{bmatrix} \int_{\mathcal{R}^3} \mathbf{J} \cdot \nabla x \\ \int_{\mathcal{R}^3} \mathbf{J} \cdot \nabla y \\ \int_{\mathcal{R}^3} \mathbf{J} \cdot \nabla z \end{bmatrix} = \begin{bmatrix} -\int_{\mathcal{R}^3} x \nabla \cdot \mathbf{J} + \int_{\Gamma} x\mathbf{J} \cdot \mathbf{n} \\ -\int_{\mathcal{R}^3} y \nabla \cdot \mathbf{J} + \int_{\Gamma} y\mathbf{J} \cdot \mathbf{n} \\ -\int_{\mathcal{R}^3} z \nabla \cdot \mathbf{J} + \int_{\Gamma} z\mathbf{J} \cdot \mathbf{n} \end{bmatrix} \tag{43}$$

where Γ is the external surface of the integration domain, $\mathbf{n}$ is the normal vector orthogonal to the surface. Since the source is limited in space, it will be zero on the boundaries of a generic external surface, as a consequence:

$$\int_{\mathcal{R}^3} \mathbf{J}(\mathbf{r'}) d^3r' = \begin{bmatrix} -\int_{\mathcal{R}^3} x \nabla \cdot \mathbf{J} \\ -\int_{\mathcal{R}^3} y \nabla \cdot \mathbf{J} \\ -\int_{\mathcal{R}^3} z \nabla \cdot \mathbf{J} \end{bmatrix} = -\int_{\mathcal{R}^3} \mathbf{r} (\nabla \cdot \mathbf{J}(\mathbf{r'})) d^3r' \tag{44}$$

but in the time-harmonic domain,

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = -i\omega\rho(\mathbf{r}) \tag{45}$$

so,

$$\mathbf{A}(\mathbf{r}) = \frac{i\omega\mu_0}{4\pi r} e^{-ikr} \int_{\mathcal{R}^3} \mathbf{r}\rho(\mathbf{r}) d^3r' \tag{46}$$

At this point, we define **electric dipole** as :

$$\boxed{\mathbf{d} = e^{i\omega t}\mathbf{d}_0 = e^{i\omega t} \int_{\mathcal{R}^3} \mathbf{r}\rho(\mathbf{r}) d^3r'} \tag{47}$$

Finally, we obtain the vector-potential generated by an electric dipole source with a certain spatial distribution $\rho(\mathbf{r})$ that oscillates sinusoidally.

$$\boxed{\mathbf{A}(\mathbf{r}, t) = \frac{i\omega\mu_0}{4\pi r} e^{i(\omega t - kr)} \mathbf{d}_0} \tag{48}$$

At this point we should reflect on the meaning of what we have retrieved, in particular let us focus on equation (47):

Our journey has started with equation (1).
We wanted originally to describe two charges that move in space with a generic law.
Equation (1) is a model that could represent classical particles in motion. However, we can imagine that also with (47) we can model this kind of phenomenon. The difference between the two models is that in the first case, equation (1), the two charges are localized, while in equation (47), we have a charge distribution that change periodically sign, as if the two charges after half period have switched their position.

Without entering into details on the difference shades of different models, we can imagine, if things are done properly, that we are still describing particles moving $\mathbf{d}_0 = e\mathbf{R}_0$, where $\mathbf{R}_0$ is the maximum distance between charges, i.e. the oriented dipole maximum displacement.

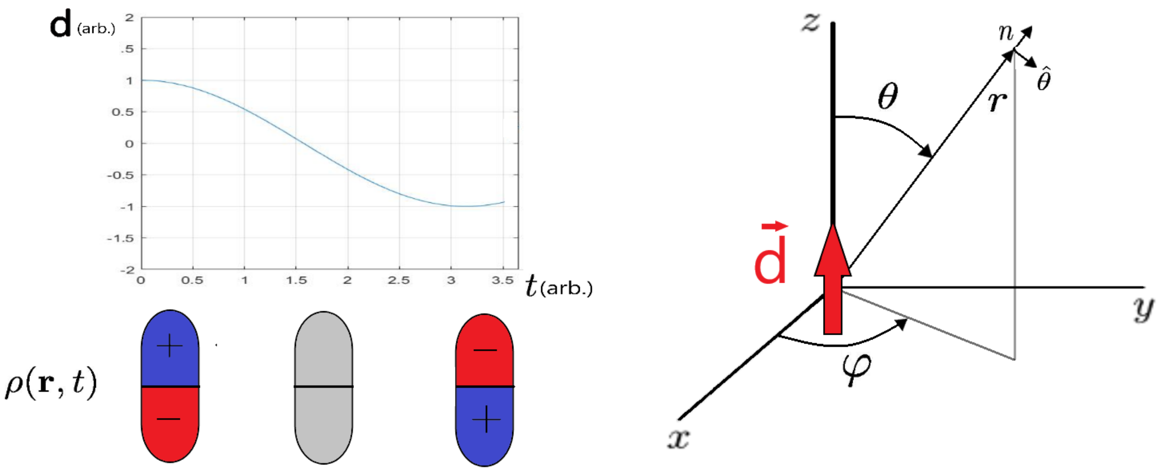


Figure 4: *On the left panel, representation of the dipole oscillation in time(upper left panel) and in spatial charge distribution(lower left panel)*

Let us calculate the electric field irradiated by this kind of source, that could be a reasonable model for an atomic dipole. As you will see, by calculating the magnetic field, first, it won't be necessary to evaluate the scalar potential ϕ to retrieve **E**. Since we are in the time-harmonic regime, we can write:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{49}$$

$$\frac{1}{c_0^2}(i\omega\mathbf{E}) = \nabla \times \mathbf{B} \tag{50}$$

Therefore, starting again from:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r} e^{-ikr} i\omega \mathbf{d}_0 \tag{51}$$

we have:

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) =& \frac{i\mu_0\omega}{4\pi}\Big[\nabla \times (\frac{e^{-ikr}}{r}\mathbf{d}_0)\Big] \\
=& \frac{i\mu_0\omega}{4\pi}\Big[-\mathbf{d}_0 \times \nabla(\frac{e^{-ikr}}{r})\Big] \\
=& \frac{i\mu_0\omega}{4\pi}\Big[-\mathbf{d}_0 \times \mathbf{n}(-ik\frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r^2})\Big] \\
=& \frac{\mu_0}{4\pi} i\omega(\mathbf{d}_0 \times \mathbf{n})\frac{e^{-ikr}}{r}(-ik - \frac{1}{r})
\end{aligned}$$

where we have used:

$$\nabla \times (\mathbf{A}f) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f$$

with **A** vectorial field, f scalar field. In our case the $\nabla \times \mathbf{d}_0 = 0$ since $\mathbf{d}_0$ is a constant.

Then, in the far zone $kr >> 1$ we can neglect $\frac{1}{r}$, so

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\omega k e^{-ikr}}{r}\mathbf{n} \times \mathbf{d}_0 \tag{52}$$

Using (50), and the previous vectorial identity:

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) =& \frac{1}{i\omega} c_0^2 \nabla \times \mathbf{B}(\mathbf{r}) = \frac{1}{i\omega} c_0^2 \nabla \times (\frac{\mu_0}{4\pi}\frac{\omega k e^{-ikr}}{r}\mathbf{n} \times \mathbf{d}_0) \\
=& -\frac{\mu_0}{i4\pi} c_0^2 k \Big(\mathbf{n} \times \mathbf{d}_0 \times \nabla(\frac{e^{-ikr}}{r})\Big)
\end{aligned}$$

where we have already neglected $\frac{1}{r^2}$ term. Then,

$$\mathbf{E}(\mathbf{r}) = -\frac{\mu_0}{i4\pi\omega} c_0^2 k \Big(\mathbf{n} \times \mathbf{d}_0 \times \mathbf{n}\Big)(-\frac{-ike^{-ikr}}{r})$$

Finally,

$$\mathbf{E}(\mathbf{r}) = \frac{\mu_0 c_0^2 k^2}{4\pi}\mathbf{n} \times \mathbf{d}_0 \times \mathbf{n}\frac{e^{-ikr}}{r} = \frac{\omega^2\mu_0}{4\pi}\mathbf{n} \times \mathbf{d}_0 \times \mathbf{n}\frac{e^{-ikr}}{r}$$

At last,

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{\omega^2\mu_0}{4\pi r}\mathbf{n} \times \mathbf{d}_0 \times \mathbf{n} e^{i(\omega t - kr)}} \tag{53}$$

Assume now, that $\mathbf{d}_0 = d_0\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the normal vector oriented in the z-direction, as in **Figure 4**.
Then, it is a matter of fact that

$$\begin{aligned}
\mathbf{n} &= \cos\varphi\sin\theta\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\mathbf{n}\times\hat{\mathbf{z}} &= -\cos\theta\sin\theta\hat{\mathbf{y}} + \sin\theta\sin\varphi\hat{\mathbf{x}} \\
\mathbf{n}\times\hat{\mathbf{z}}\times\mathbf{n} &= \cos^2\varphi\sin^2\theta\hat{\mathbf{z}} - \cos\varphi\cos\theta\sin\theta\hat{\mathbf{x}} + \sin^2\theta\sin^2\varphi\hat{\mathbf{z}} - \cos\theta\sin\theta\sin\varphi\hat{\mathbf{y}}
\end{aligned}$$

Thus,

$$\mathbf{n}\times\hat{\mathbf{z}}\times\mathbf{n} = -\sin\theta(\cos\varphi\cos\theta\hat{\mathbf{x}} + \sin\varphi\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}) = -\sin\theta\hat{\boldsymbol{\theta}} \tag{54}$$

where $\hat{\boldsymbol{\theta}}$ is the normalized vector indicated in **Figure 4**.

In conclusion,
given a source that moves as an oscillating electric dipole $\mathbf{d} = \mathbf{d}_0 e^{+i\omega t}$, we have proved that the electric field generates in the far region is a **propagating electromagnetic field**:

$$\boxed{\mathbf{E}(\mathbf{r},t) = -\frac{k^2 d_0}{4\pi\epsilon_0 r}\sin\theta\cos(+(\omega t - kr))\hat{\boldsymbol{\theta}}} \tag{55}$$

where we have substituted $\mu_0 = \dfrac{1}{c_0^2\epsilon_0}$, and taken the real part of (53).

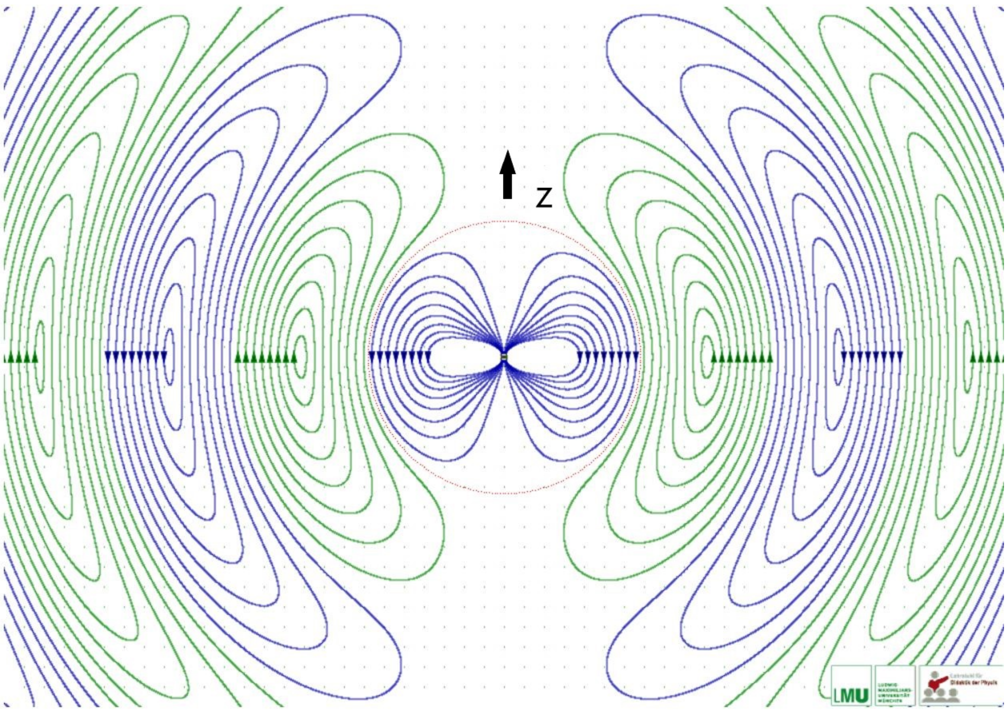


Figure 5: *Electric field generated by the oscillating dipole. You can observe the in the far field we almost have a spherical wave.*

4 Conclusion

This was just a generic, but detailed, overview of the electromagnetic wave generated by an electric dipole.

In the past century, this derivation[12] has been used as a starting point of many discussions. However, for some reason that I ignore, many times this topic is given for granted and therefore skipped.

This is a real pity and I encourage to never underestimate the power and the beauty of classical electrodynamics, since it will always provide us a preliminary[13] but useful insight on the physics of classical light-matter interaction.

[12] probably a more complete version

[13] Indeed, you would need QM to have a more fundamental comprehension

A Appendix A: An intuitive "proof" of Green Function

Given $G(\mathbf{r}, \mathbf{r'}, t, t')$ parametric scalar field, with **r'** and t' parameters, given ψ and f scalar field. Let us proof that, the solution for ψ of:

$$\nabla^2\psi - \frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2} = -\mu_0 f(\mathbf{r}, t) \tag{56}$$

is given by the resolution of

$$\nabla_r^2 G(\mathbf{r}, \mathbf{r'}, t, t') - \frac{1}{c_0^2}\frac{\partial^2 G(\mathbf{r}, \mathbf{r'}, t, t')}{\partial t^2} = -\mu_0\delta(\mathbf{r} - \mathbf{r'})\delta(t - t') \tag{57}$$

together with

$$\psi(\mathbf{r}, t) = \int_{\mathcal{R}^4} G(\mathbf{r}, \mathbf{r'}, t, t') f(\mathbf{r'}, t') d^3r' dt' \tag{58}$$

For the sake of simplicity, let us rewrite the equations using the d'Alembert operator[14]:

$$\Box_{\mathbf{r},t} = \nabla_{\mathbf{r}}^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} \tag{59}$$

Then given,

$$\Box_{\mathbf{r},t}\psi = -\mu_0 f(\mathbf{r}, t) \tag{60}$$

we make the following ansatz, i.e. we are assuming that our solution can be written in this form:

$$\psi = \int G(\mathbf{r}, \mathbf{r'}, t, t') A(\mathbf{r'}, t') dt' d^3r' \tag{61}$$

Now, as a matter of fact f can be written as:

$$f(\mathbf{r}, t) = \int f(\mathbf{r'}, t')\delta(\mathbf{r} - \mathbf{r'})\delta(t - t') d^3r' dt' \tag{62}$$

Therefore,

$$\Box_{\mathbf{r},t}\int G(\mathbf{r}, \mathbf{r'}, t, t') A(\mathbf{r'}, t') dt' d^3r' = -\mu_0 \int f(\mathbf{r'}, t')\delta(\mathbf{r} - \mathbf{r'})\delta(t - t') d^3r' dt'$$

The operator $\Box_{\mathbf{r},t}$ commutes with the integral operator, i.e. the operator and the integral act on different variable, so we can switch them.

Thus,

$$\int A(\mathbf{r'}, t')\Box_{\mathbf{r},t} G(\mathbf{r}, \mathbf{r'}, t, t') dt' d^3r' = -\mu_0 \int f(\mathbf{r'}, t')\delta(\mathbf{r} - \mathbf{r'})\delta(t - t') d^3r' dt'$$

Now we equate the integrand, and we obtain:

$$\begin{cases} \Box_{\mathbf{r},t} G(\mathbf{r}, \mathbf{r'}, t, t') = -\mu_0\delta(\mathbf{r} - \mathbf{r'})\delta(t - t') \\ A(\mathbf{r'}, t') = f(\mathbf{r'}, t') \end{cases} \tag{63}$$

where we have collected in the first equation all the terms that depends on (**r**,t) in the two integrands, in the second equation only the terms that depend only on $(\mathbf{r'}, t')$.

[14]To be honest the d'Alembert operator is defined as $\Box = c_0^2\nabla^2 - \frac{\partial^2}{\partial t^2}$, but I am sure the reader will forgive me if I use it in this way, provided that I am doing it consistently!

B Appendix B: Residue calculus

Given $f(x)$ analytic function, which means it can be written as a weighted polynomial series. Then let us introduce,

$$\Delta = \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx \tag{64}$$

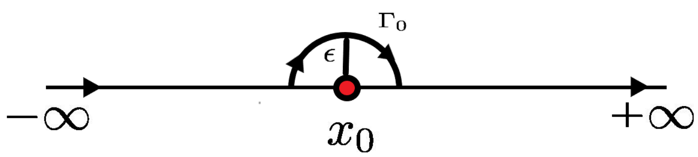


Figure 6: *Path in the complex plane, ϵ is the ray of the Γ_0 path*

We can see (64) as a limit:

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx = \lim_{\epsilon \to 0} \Big[\int_{-\infty}^{x_0 - \epsilon} \frac{f(x)}{x - x_0} dx + \int_{x_0 + \epsilon}^{\infty} \frac{f(x)}{x - x_0} dx + \int_{\Gamma_0} \frac{f(z)}{z - x_0} dz \Big] \tag{65}$$

where the last integral is a complex integral made on the path Γ_0 a semi-circle of ray ϵ in the complex plane z .

We define principal value:

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx = \lim_{\epsilon \to 0} \Big[\int_{-\infty}^{x_0 - \epsilon} \frac{f(x)}{x - x_0} dx + \int_{x_0 + \epsilon}^{\infty} \frac{f(x)}{x - x_0} dx \Big] \tag{66}$$

We will, instead prove that:

$$\int_{\Gamma_0} \frac{f(z)}{z - x_0} dz = -i\pi f(x_0) \tag{67}$$

PROOF:

$f(z)$ is analytic, therefore:

$$f(z) = f(x_0) + \sum_{n=1}^{\infty} \frac{d^n f}{dz^n}\Big|_{z=x_0} \frac{(z - x_0)^n}{n!} \tag{68}$$

then,

$$\int_{\Gamma_0} \frac{f(z)}{z - x_0} dz = \lim_{\epsilon \to 0} \int_{semiC_\epsilon} \Big(f(x_0) + \sum_{n=1}^{\infty} \frac{d^n f}{dz^n}\Big|_{z=x_0} \frac{(z - x_0)^n}{n!} \Big) \frac{1}{z - x_0} dz$$

making the variable substitution $z - x_0 = \epsilon e^{i\theta}$, with θ that goes from the angle π to 0, while ϵ is fixed. As a consequence $dz = \epsilon i e^{i\theta} d\theta$.
So,

$$\int_{\Gamma_0} \frac{f(z)}{z - x_0} dz = \lim_{\epsilon \to 0} \Big[\int_{\pi}^{0} f(x_0) \frac{1}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta + \int_{\pi}^{0} \Big(\sum_{n=1}^{\infty} \frac{d^n f}{dz^n}\Big|_{z=x_0} \frac{(\epsilon)^n e^{in\theta}}{n!} \Big) i\epsilon e^{i\theta} \frac{1}{\epsilon e^{i\theta}} d\theta \Big]$$

$$\int_{\Gamma_0} \frac{f(z)}{z-x_0}dz = \lim_{\epsilon\to 0}\Big[if(x_0)\int_{\pi}^{0} d\theta + \int_{\pi}^{0}\Big(\sum_{n=1}^{\infty}\frac{d^n f}{dz^n}\Big|_{z=x_0}\frac{(\epsilon)^n e^{in\theta}}{n!}\Big)id\theta\Big]$$

It is easy to observe that the first integral has lost its ϵ dependence, while the second goes as a power of ϵ^n for $n = 1, 2, 3...$, thus its limit is 0.

As a consequence:

$$\int_{\Gamma_0} \frac{f(z)}{z-x_0}dz = -i\pi f(x_0) \tag{69}$$

C.V.D

So,

$$\Delta = \Delta_0 = \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx - i\pi f(x_0) \tag{70}$$

It should surprise you that, in general, **the solution** of Δ, (72), **depends on the path chosen**! This "ambiguity" is extremely relevant when we need boundaries condition in our physical problems.

Indeed, we could have reformulated the problem by choosing the semi-circle below the x-axis, Γ_1, in that case everything would have been unchanged except the angle of integration.

$$\int_{\Gamma_1} \frac{f(z)}{z-x_0}dz = \lim_{\epsilon\to 0}\Big[if(x_0)\int_{\pi}^{2\pi} d\theta + \int_{\pi}^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{d^n f}{dz^n}\Big|_{z=x_0}\frac{(\epsilon)^n e^{in\theta}}{n!}\Big)id\theta\Big]$$

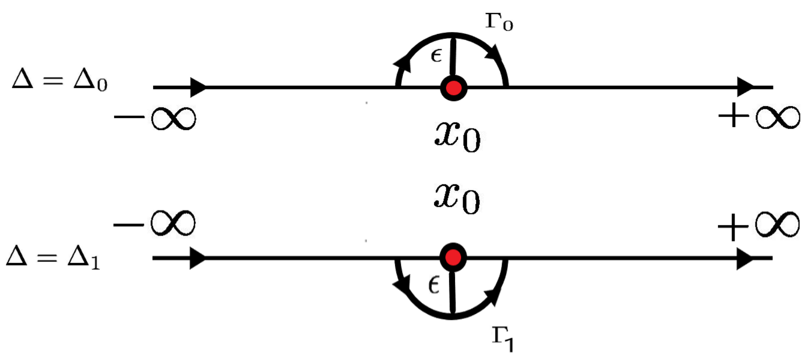


Figure 7: *Path in the complex plane, ϵ is the ray of the Γ_0 path*

Therefore,

$$\int_{\Gamma_1} \frac{f(z)}{z-x_0}dz = i\pi f(x_0) \tag{71}$$

and hence,

$$\Delta = \Delta_1 = \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx + i\pi f(x_0) \tag{72}$$

What is curious is that, while the integral Δ depends on the path, the principal value is independent on the path and, therefore, is always the same.

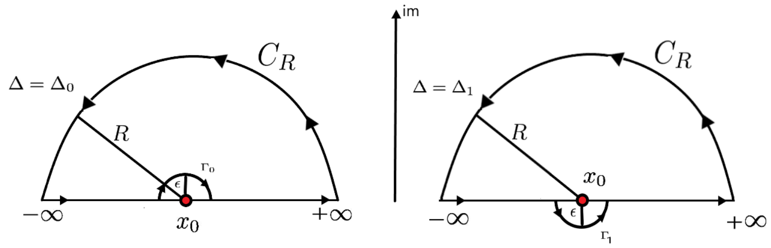


Figure 8: *Paths in the complex plane, ϵ is the ray of the Γ_0 path, R the ray of the external semicircle C_R*

Let us now focus on a particular case, meaningful in many physical discussion. We assume that $f(iR)$ goes to zero for $R \to \infty$.

$$\lim_{z \to iR} f(z) = 0 \tag{73}$$

It is clear that the two path in figure can be written as:

$$\Delta = \Delta_0 = \lim_{R \to +\infty, \epsilon \to 0} \int_0 \frac{f(z)}{x - x_0} dz = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx + \lim_{R \to +\infty} \int_{C_R} \frac{f(z)}{z - x_0} dz - i\pi f(x_0)$$
$$\Delta = \Delta_1 = \lim_{R \to +\infty, \epsilon \to 0} \int_1 \frac{f(z)}{z - x_0} dz = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx + \lim_{R \to +\infty} \int_{C_R} \frac{f(z)}{z - x_0} dz + i\pi f(x_0)$$

It is not hard to imagine that the integral over the external semi-circle will go to zero,

$$\lim_{R \to +\infty} \int_{C_R} \frac{f(z)}{z - x_0} = 0$$

Let us now recall the residual theorem, without proving it:

$$\int_{\Gamma} g(z) dz = 2\pi i \sum_{j=1}^{M} Res(g(z_j)) Ind_{\Gamma}(z_j) \tag{74}$$

where j is the index of the pole, M is the number of pole, Res is the residual which is calculated as

$$Res(g(z_j)) = \lim_{z \to z_j} g(z)(z - z_j)$$

and Ind_{Γ} instead counts the number of time the path Γ wrap the j-th pole, with sign "$+$" if it does so anti-clockwise, "$-$" if clockwise.

Thus, considering the two different paths:

$$\Delta = \Delta_0 = \lim_{R \to +\infty, \epsilon \to 0} \int_0 \frac{f(z)}{x - x_0} dz = 0$$
$$\Delta = \Delta_1 = \lim_{R \to +\infty, \epsilon \to 0} \int_1 \frac{f(z)}{z - x_0} dz = 2\pi i f(x_0)$$

In fact, while the first path avoid the pole, the second wrap it. We conclude that:

$$\Delta = \Delta_0 = 0 = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx - i\pi f(x_0)$$
$$\Delta = \Delta_1 = 2\pi i f(x_0) = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx + i\pi f(x_0)$$

As a consequence,

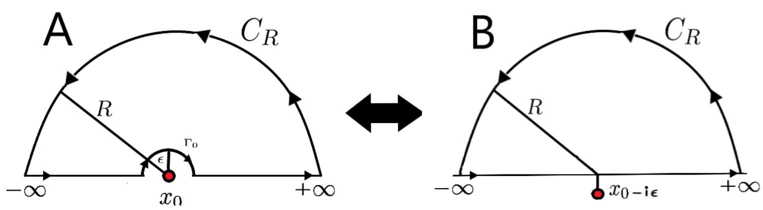
$$\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx = +i\pi f(x_0) \tag{75}$$

To recap:
In complex analysis one integral, like Δ, can have different results depending on the "way" we are integrating it. What remains unchanged is its **principal value**! The tricks lay in slightly changing, as a limit, the path to circumnavigate the pole on the x-axis.

It would have been equivalent if $f(-iR)$ had gone to 0 for $R \to \infty$, in that case the path to be considered would have been a semi-circle of ray R that goes below the x-axis.

Now that we know how to integrate Δ, let us focus on another interesting and extremely useful result.

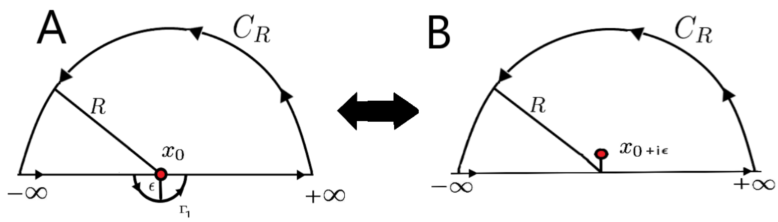
A totally equivalent way of solving integrals like Δ is to change the pole position instead of the path, as in the following figure:



in other words:

$$\Delta_0 = \lim_{\epsilon\to 0}\int_A \frac{f(z)}{z-x_0}dz = \lim_{\epsilon\to 0^+}\int_B \frac{f(z)}{z-x_0+i\epsilon}dz \tag{76}$$

The ambiguity of the final result remains in the freedom we have to move up or down the pole. Indeed,



or

$$\Delta_1 = \lim_{\epsilon\to 0}\int_A \frac{f(z)}{z-x_0}dz = \lim_{\epsilon\to 0^+}\int_B \frac{f(z)}{z-x_0-i\epsilon}dz \tag{77}$$

this result is known as **Plemelj theorem**, and it will be used in our main context to solve our classical electrodynamics problem.