The polarized degree of irrationality of $K3$ surfaces

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Abstract

Given a polarized variety (X, L) we use an elementary observation to produce projections of low degree $X \subset \mathbb{P}(H^0(L^\vee)) \dashrightarrow \mathbb{P}^r$ via vector bundles. In the case of surfaces this observation can be used to show that maps of the degree at most d move in families. In the case of polarized $K3$ surfaces we study the most special projections up to genus 6 and we give a new upper bound for higher genera. As a different application of our construction we exhibit new rational map of low degree for some hyper-Kähler varieties, abelian varieties and Gushel-Mukai threefolds.

1 Introduction

Given a variety X, the most natural invariant to measure how far X is from being rational is the degree of irrationality

$$\mathrm{irr}(X) = \min\{\deg(\varphi) \mid \varphi : X \dashrightarrow \mathbb{P}^{\dim(X)} \text{ is a rational, generically finite map}\}.$$

The case when X is a curve leads to the classical notion of gonality, whereas for $\dim(X) \geq 2$ very little is known in general and the problem has recently received considerable attention. The main results in this direction are for hypersurfaces [5, 6, 9, 20, 24] or abelian varieties [2, 8, 11, 15, 16, 19, 21, 22, 25] or hyper-Kähler varieties [23] or more specific examples [1, 12]. Giving a sharp lower bound is in general a very difficult problem. This led people to introduce other measures of irrationality. For $K3$ surfaces the most interesting invariants are the fibering genus and the fibering gonality. The fibering genus has been estimated in [13] and there are relations between the fibering gonality and the degree of irrationality, see [3].

In this paper we introduce and study a natural invariant very close to the degree of irrationality one may attach to a polarized variety. Let (X, L) be a polarized projective variety of dimension $m \geq 2$, for any $V \subset H^0(L)$ of dimension $m+1$ we get a rational map $\varphi_V : X \dashrightarrow \mathbb{P}^{\dim(X)}$, we may define

$$\mathrm{irr}_L(X) = \min\{\deg(\varphi_V) \mid V \in \mathrm{Gr}(m+1, H^0(L)),\ \text{defined in codimension 2, generically finite}\}.$$

Notice that $\mathrm{irr}(X) = \min_{L \in \mathrm{Pic}(X)} \mathrm{irr}_L X$. For a general curve of genus g and a general line bundle $L \in \mathrm{Pic}^d(C)$ of degree $d \geq g+1$ one has to drop the assumption that φ_V is defined in codimension 2 in order to get a non trivial notion. One can compute $\mathrm{irr}_L(C) = \max\{2g+2-d, \left\lfloor \frac{g+3}{2} \right\rfloor\}$. For a surface S and a primitive class L in $\mathrm{Pic}(S)$ the rational map $\varphi_V : S \dashrightarrow \mathbb{P}^2$ is defined in codimension 2 and generically finite for any $V \in \mathrm{Gr}(3, H^0(L))$.

We will mainly focus on the case of polarized $K3$ surfaces.

Theorem 1.1. *Let (S, L) be a $K3$ surface of genus $g = 2+2n(n+1)+k$ with $\mathrm{Pic}(S) = \mathbb{Z}\cdot L$, then*

$$\mathrm{irr}_L(S) \leq 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

Moreover, if $g = 3, 4, 6,$ then $\mathrm{irr}_L(S) = 3$. If $g = 5$ then $\mathrm{irr}_L(S) = 4$.

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The upper bounds for $\mathrm{irr}_L(S)$ given in the theorem above are better then the ones given in [20]. In low genus the upper bounds given are very likely to be optimal and, in principle, they may be optimal for $g = 2 + 2n(n + 1)$. The technique is based on an elementary observation that is worth mentioning here since it has other potential applications. It can be seen as natural variation of the Lazarsfeld-Mukai bundles introduced for studying linear series on curves on $K3$ surfaces ([14]). For a variety X of dimension m and a linear system $|L \otimes \mathcal{I}|$, inducing a rational map $\varphi_{|L\otimes\mathcal{I}|} : X \dashrightarrow \mathbb{P}^m$ one may construct the kernel reflexive sheaf (also called syzygy bundle)

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I} \longrightarrow 0.$$

Then, via the identification $\mathrm{Gr}(\dim(X), V) = \mathbb{P}V^\vee \subset \mathbb{P}(H^0(E))$ the fibers of the map may be seen as zero loci of sections of E. Conversely if one is given a sufficiently general vector bundle E of rank m and at least $m + 1$ global sections generating E in codimension 2, then one may produce rational maps of degree $\leq c_m(E)$. Moreover, if the vector bundle has many global sections one can impose base points on them and make the degree drop. The basic example one should keep in mind is that of a $K3$ surface of genus 6. In this case there is a vector bundle of rank 2 with $h^0(E) = 5$ and $c_2(E) = 4$, then for any $P \in S$ the map given by $V = H^0(E \otimes m_P)^\vee \subset H^0(L)$ is of degree $c_2(E) - 1 = 3$.

Let us mention here some of the potential applications. For surfaces the technique can be used to give a scheme structure to the locus of special maps. We get a variational theory giving the structure of an algebraic variety to the locus

$$W^r_d(S, L) = \{V \in \mathrm{Gr}(r + 1, H^0(L)) \quad | \deg(\varphi_V) \leq d\}.$$

Some explicit computation are carried out:

Theorem 1.2. *If (S, L) is a general $K3$ surface of genus 5 the Brill-Noether locus $W^2_3(S, L)$ is empty whereas $W^2_4(S, L)$ has two components one birational to S and the other one birational to a $\mathbb{P}^3$-bundle over $\mathcal{M}$, the moduli space of stable rank 2 vector bundles with $c_2 = 4$. If (S, L) is a general $K3$ of genus 6 it turns out that $W^2_3(S, L) = S$.*

In a joint work with Andrés Rojas a similar theorem for $K3$ surfaces up to genus 14 is being developed. Other kind of polarized surfaces are interesting to study from this point of view, for instance $(S, 3\Theta)$ for a principally polarized abelian surface (S, Θ).

For higher dimensional varieties there are other potential applications, let us mention here the most surprising ones. Recall that a Gushel-Mukai threefold is a Fano threefold of genus 6 and index 1.

Theorem 1.3. *The following estimates hold:*

- *if X is a general Gushel-Mukai threefold, then $\mathrm{irr}(X) \leq 3$;*
- *if (S, L) is a general $(1, 6)$ abelian surface then $\mathrm{irr}_L(S) = 3$;*
- *if (S, L) is a general $K3$ surface of genus 6 then $\mathrm{irr}(\mathrm{Hilb}^2(S)) \leq 6$;*
- *if (A, L) is a general abelian threefold of type $(1, 3, 12)$ or $(1, 6, 6)$ then $\mathrm{irr}_L(A) \leq 8$;*
- *if (S, L) is a general $K3$ surface of genus 10 then $\mathrm{irr}(\mathrm{Hilb}^3(S)) \leq 20$.*

Let us remark that the case of the $(1, 6)$ abelian surface settles one of the three remaining open cases for abelian surfaces. Indeed $\mathrm{irr}(S) = 4$ if (S, L) is a general $(1, d)$ abelian surface with $d \neq 1, 2, 3, 6$ (c.f. [8, 16]) and $\mathrm{irr}(S) = 3$ for $d = 2$ (c.f. [25]). For the very general abelian threefold the best lower bound known is 5 (c.f. [11]).

Another interesting situation would be studying the case when $E = \Omega^1_X$ or $E = \Omega^{m-1}_X$. Notice that in case $h^0(\Omega^1_X) = m + 1$ then one gets a canonically defined map $X \dashrightarrow \mathbb{P}^m$

which is nothing but the Gauss map. We don´t know whether the canonical map one gets if $h^0(\Omega_X^{m-1}) = m+1$ has been studied in the literature.

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2 Towards a Brill-Noether theory for surfaces

We fix a surface S, and a primitive line bundle $L \in \mathrm{Pic}(S)$. We show that the locus of special projections

$$W_d^r(S, L) = \{V \in \mathrm{Gr}(r+1, H^0(L)) \quad | \deg(\varphi_V) \leq d\}$$

is an algebraic variety. Notice that the definition of a general member of W_d^r is different then the one for curves, in the sense that in the above definition d may drop at special points. We should remark here that Mendes-Lopes, Pardini and Pirola defined and studied Brill-Noether loci for irregular varieties in [17] as jump cohomology loci in the $\mathrm{Pic}^0 S$ (where $h^i(L \otimes \eta)$ jumps for $\eta \in \mathrm{Pic}^0(S)$), we will refer to them as special subvarieties. Our definition is completely different, if the variety is irregular the Brill-Noether loci defined subsequently may be glued over the special subvarieties they construct giving rise to a more refined theory.

Let us recall some notation. Consider a $K3$ surface and a rational map $\varphi : S \dashrightarrow \mathbb{P}^r$. We say that φ is associated with $(V, L) = (V^\vee, E)$ where

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I} \longrightarrow 0$$

and $V = H^0(L \otimes \mathcal{I})$ is the linear system associated with the rational map. Notice that $V^\vee \otimes \mathcal{O}_S \to E$ generate E outside a finite number of points. Since L is primitive we can deduce that E is stable, hence the map $(V, L) \to (V^\vee, E)$ is a bijection (in case E has endomorphisms one has to be careful, but we rule out this case from the beginning).

The construction is based on the following observation. An alternative proof can be found in the last section, lemma 2.9.

Lemma 2.1. *Let (S, L) be a $K3$ surface of genus g with Picard group generated by L. Let $V \in \mathrm{Gr}(r+1, H^0(L))$, with associated kernel bundle $E^\vee$ and base locus $\mathrm{Bl} = \mathrm{Spec}\ \mathcal{O}_S/\mathcal{I}$, i.e. we have an exact sequence*

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I} \longrightarrow 0.$$

Consider the identification $\mathrm{Gr}(r-1, V^\vee) = \mathrm{Gr}(2, V)$, where $V^\vee \subset H^0(E)$ so that we have a map $W \to T_W$, then for general W

$$Z(W) \cap (S - \mathrm{Bl}) = Z_{\mathrm{cycle}}(\mathcal{I}/(\mathcal{J}_{Z(T_W)})) \cap (S - \mathrm{Bl}),$$

where $Z(T_W) = \cap_{t \in T_W} Z(t)$ and $Z(W) = Z_{\mathrm{cycle}}(\bigwedge^{r-1} W)$.

Proof. The proof is a local computation and follows from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^\vee & \longrightarrow & T_W \otimes \mathcal{O}_S & \longrightarrow & L \otimes \mathcal{J}_{Z(T_W)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E^\vee & \longrightarrow & V \otimes \mathcal{O}_S & \longrightarrow & L \otimes \mathcal{I} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F & \longrightarrow & W^\vee \otimes \mathcal{O}_S & \longrightarrow & \mathcal{I}/(\mathcal{J}_{Z(T_W)}) & \longrightarrow & 0.
\end{array}$$

It is immediate to check that set theoretically the degeneracy locus $W \to E$ coincides with $Z(\mathcal{I}/(\mathcal{J}(T_W)))$ outside of the base locus. Hence if they are both reduced away from the base locus, they coincide outside of the base locus (this happens for general W). □

The above lemma allows us to give a nice dual interpretation of a map $\varphi : S \dashrightarrow \mathbb{P}^r$ via the sections of its kernel bundle.

Corollary 2.2. *Let $\varphi_V : S \dashrightarrow \mathbb{P}^r$ be a rational map with kernel bundle E and base locus* Spec $\mathcal{O}_S/\mathcal{I}$*, then the linear series of $r-2$ linear spaces is given by the zero cycles associated to $W \in \mathrm{Gr}(r-1, V^\vee)$. In particular*

$$\deg(\varphi_V) = \deg Z(W_0) - \deg\Big(\bigcap_{W \in \mathrm{Gr}(r-1,V^\vee)} Z_{\text{cycle}}(W)\Big),$$

where $W_0 \in \mathrm{Gr}(r-1, V^\vee)$ and the intersection is intended as zero cycles (i.e. we are taking the degree of the general element minus the degree of the fixed part as zero cycles).

Proof. We have a map $\varphi_V : (S - \mathrm{Bl}) \to \mathbb{P}^r$, where Bl denotes the base locus of the original map. By the above theorem if I take two hyperplanes and I pull them back in $\xi = \varphi_V^{-1}(Z(H_1) \cap Z(H_2))$, the residual cycle $\mathcal{I}/\mathcal{J}_\xi$ is given by the zero cycle associated to some $T \in \mathrm{Gr}(r-1, V^\vee)$ outside of the base locus (i.e. $\xi = Z(T) \cap (S - \mathrm{Bl})$). The proof is complete (one should observe that if all the sections of $V^\vee$ vanishes at P then $P \in \mathrm{Bl}$). □

Notice that in the case $r = 2$ the above corollary reads that the preimages of points can be read by the zero cycle of sections of $V^\vee \subset H^0(E)$. Moreover, the fixed part of the cycles associated to the sections of $V^\vee$ is supported at $\mathrm{Spec}(\mathcal{O}_S/\mathcal{I})$. Now we start the construction the Brill-Noether loci. We construct it over $\bigcup_{c_2} \mathcal{M}_{c_1,c_2,r}$, as a subscheme of the relative grassmanian. Let us point out that c_2 here stands for the second Chern class and not for the second Chern character (it is more natural for our construction). Let us consider the grassmanian bundle $\pi : \mathcal{G} \to \mathrm{Gr}(r+1, H^0(E))$, (i.e. the fiber over $V^\vee \in \mathrm{Gr}(r+1, H^0(E))$ is $\mathrm{Gr}(r-1, V^\vee)$). We have canonical cycle maps $Z_c : \mathcal{G} \to \mathrm{Sym}^{c_2(E)}(S)$, $Z_\mathcal{I} : \mathcal{G} \to \mathrm{Sym}^{L^2 - c_2(E)}(S)$ to the symmetric product of the surface S, where $Z_c(W, V^\vee)$ is the zero cycle associated to

$$L/(\mathrm{coker}(W \otimes \mathcal{O}_S \to E)$$

and $Z_\mathcal{I}(W, V^\vee)$ is the zero cycle associated to

$$L/(\mathrm{coker}(E^\vee \to V^\vee \otimes \mathcal{O}_S)).$$

Let us consider a sort of incidence variety $\Delta_k =\subset \mathcal{G}$. More precisely, Δ_k is the pullback of the following locus in $\mathrm{Sym}^{c_2(E)}(S) \times \mathrm{Sym}^{L^2 - c_2(E)}(S)$

$$\{(P_1 + \cdots + P_{c_2(E)}, Q_1 + \cdots + Q_{L^2 - c_2(E)}) \ \mid P_1 = Q_{i_1}, \ldots, P_k = Q_{i_k}\}.$$

Let us remark that in principle $i_{j_1} = i_{j_2}$ so the above locus is really assuring that there are at least $k-$points counted with multiplicity of the first cycle belonging to the support of the second one.

Now by the above corollary the following scheme describes precisely the maps $\varphi : S \dashrightarrow \mathbb{P}^r$ of degree $\leq d$ with kernel bundle E

$$W^r_d(S, L)_E = \{V \in \mathrm{Gr}(r+1, H^0(E)) \quad | \, \pi^{-1}(V) \subset \Delta_{c_2(E)-d}\}.$$

This loci can be patched together and obtain a scheme

$$W^r_d(S, L) \to \bigcup_{c_2} \mathcal{M}_{c_1,c_2,r},$$

whose points corresponds to maps $S \dashrightarrow \mathbb{P}^r$ of degree $\leq d$ given by a linear subsystem of $|L|$. Moreover, there is a canonical injection $W^r_d(S, L) \to \mathrm{Gr}(r+1, H^0(L))$ and for each fixed c_2 there is a canonical map $W^r_d(S, L)_{c_2} \to \mathrm{Hilb}^{L^2-c_2}(\mathrm{S})$. For a $K3$ or an abelian surface we got the following:

Theorem 2.3. *Let (S, L) be a primitively polarized surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$. Then the locus*

$$W^r_d(S, L) = \{V \in \mathrm{Gr}(r+1, H^0(L))\} \quad | \, \deg(\varphi_V) \leq d\},$$

can be given a scheme structure. Moreover each of his component $\mathcal{W}$ has a map $\mathcal{W} \to \mathrm{Hilb}^l(S)$. These maps induce a stratification

$$\cdots \subset W^r_d(S, L)_{c_2} \subset \cdots \subset W^r_{c_2}(S, L)_{c_2} \subset \mathrm{Hilb}^{L^2-c_2}(S),$$

The last term being a Grassman bundle over $\mathcal{M}_{c_1,c_2,r}$.

2.1 The upper bound

We use the contruction of the previous section to prove the existence of rational maps of low degree from $K3$ surfaces of genus g. Our construction improves asymptotically by a factor between 3 and 6 the best bound previously known, due to Stapleton (depending on the genus), see [20]. We report a tabular of the upper bound we get in low genus

g	6	8	10	12	14	16	18	20	22	24	26	42	62	$2+2n(n+1)$
$\mathrm{irr}(S)$	3	4	4	5	4	5	5	6	6	7	5	6	7	$2+n$

First we deal with the case $g = 2 + 2n(n+1)$, where our construction is optimal. The second Chern class of the minimal rank 2 vector bundle E is $c_2(E) = 2 + n(n+1)$. We will prove that a $K3$ surface S with these invariants has degree of irrationality $\mathrm{irr}(S) \leq 2 + \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = 2 + n \sim \frac{1}{\sqrt{2}}\sqrt{g}$.

The theorem is an easy consequence of the construction in the previous section.

Theorem 2.4. *Let (S, L) be a $K3$ surface of genus $g = 2+2n(n+1)$, then $\mathrm{irr}(S) \leq 2+n$.*

Proof. Consider the minimal rank 2 vector bundle E with invariants

$$c_1(E) = L, \quad c_2(E) = 2 + n(n+1), \quad h^0(E) = 3 + n(n+1).$$

Let $P \in S$ be any point then there exists a vector space $V_P^\vee \subset H^0(E \otimes m_P^n)$ of dimension 3. Any section of V_P vanishes at P with order n^2. By corollary 2.2 (or lemma 2.9)

$$\deg(\varphi_{V_P}) \leq c_2(E) - n^2 = 2 + n,$$

where φ_{V_P} is the map $S \dashrightarrow \mathbb{P}^2$ induced by $V_P \subset H^0(L)$. $\square$

The general case is an easy consequence

Corollary 2.5. *Let (S, L) be a $K3$ surface of genus $g = 2+2n(n+1)+k < 2+2(n+1)(n+2)$ (i.e. $k < 4n + 4$), then*

$$\operatorname{irr}(S) \leq 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

Proof. A minimal rank 2 vector bundle E has invariants

$$c_1(E) = L, \quad c_2(E) = 2 + n(n+1) + \left\lceil \frac{k}{2} \right\rceil, \quad h^0(E) = 3 + n(n+1) + \left\lceil \frac{k}{2} \right\rceil.$$

Let

$$V^\vee \in \operatorname{Gr}(3, H^0(E \otimes m_P^n \otimes m_{Q_1} \otimes \cdots \otimes m_{Q_{\lfloor \frac{k}{4} \rfloor}})),$$

then φ_V is of degree

$$\deg(\varphi_V) \leq c_2(E) - n^2 - \left\lfloor \frac{k}{4} \right\rfloor = 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

□

2.2 Brill-Noether theory for a $K3$ surface of genus 5

The following is possibly well known, we were unable to find a reference.

Lemma 2.6. *Let (S, L) be a general $K3$ surface of genus 5 and E be a rank 2 vector bundle such that $h^0(E) = 4, c_1(E) = L, c_2(E) = 4$. Then E is globally generated.*

Proof. If $H^1(E \otimes m_P) = 1$ then there exists a non trivial stable extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow F \longrightarrow E \otimes m_P \longrightarrow 0.$$

Now observe that $(E \otimes m_P) \otimes k(P) = 4$. Hence F is not a vector bundle locally at P, this means that $c_2(F^{\vee\vee}) < c_2(F) = c_2(E) + 2 = 6$. We get a contradiction because there does not exist any non trivial stable rank 3 vector bundle with $c_2 < 6$.

□

The above lemma allows us to give a lower bound on the degree of rational maps $S \dashrightarrow \mathbb{P}^2$ arising from linear systems contained in the primitive one, in other words maps arising from projections $S \subset \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$.

Theorem 2.7. *Let (S, L) be a $K3$ surface genus $g = 5$ whose Nèron-Severi group is of rank 1. Then, there are no projections $S \subset \mathbb{P}(H^0(L)^\vee) \dashrightarrow \mathbb{P}^2$ of degree ≤ 3.*

Proof. Consider S a $K3$ of genus 5 and suppose we have a rational map $\varphi_{|L\otimes\mathcal{I}|} : S \dashrightarrow \mathbb{P}^2$ of degree 3. Consider the associated kernel bundle E. We may suppose $h^0(E) \geq 3$ and $h^1(E) = 0$. This means we only have to analyze the cases $c_2(E) = 4, c_2(E) = 5$. In the first case by the previous lemma for any P the grassmannian $\operatorname{Gr}(3, H^0(E \otimes m_P))$ is empty. Hence for any $V \in \operatorname{Gr}(3, H^0(E))$ the map φ_V is of degree 4 by corollary 2.2. If $c_2(E) = 5$ the problem is even easier because then $\dim(\mathcal{O}_S/\mathcal{I}) = 3$, hence the sections of E may have at most one common zero (in that case $\mathcal{I} = m_P^2$). We get $\deg(\varphi_{|L\otimes\mathcal{I}|}) \geq c_2(E) - 1 = 4$. The proof is complete.

□

We can also describe the Brill-Noether locus $W_4^2(S)$ in this case.

Corollary 2.8. *The locus $W_4^2(S) \subset \mathrm{Gr}(3, H^0(L))$ consists of two disjoint components birational to S and to a $\mathrm{Gr}(3,4)$-bundle over $\mathcal{M}$, the moduli space of stable rank 2 vector bundles with $c_2 = 4$.*

Proof. The component birational to S is given by

$$P \to H^0(L \otimes \mathcal{I}_P^2) \in W_4^2(S) \subset \mathrm{Gr}(3, H^0(L)).$$

For the other component, consider $\mathcal{M}$ the moduli space of stable vector bundles with $c_2(E) = 4$ and $c_1(E) = L$. Let $\mathcal{G} \to \mathcal{M}$ be the relative Grassmanian of hyperplanes (i.e. $\mathcal{G}_E = \mathrm{Gr}(3, H^0(E))$). By duality we get a morphism $\mathcal{G} \to W_4^2(S)$ given by

$$(E, V) \to V^\vee \in W_4^2(S) \subset \mathrm{Gr}(3, H^0(L)).$$

The fact that they are disjoint is a consequence of the previous Theorem. The proof is complete. $\square$

2.3 An application to higher dimensional varieties

In this section we generalize the construction of the previous chapter to higher dimensional variety. We give a tool to construct maps of low degree. Lemma 2.1 can be easily generalized to the following

Lemma 2.9. *Let X be a smooth projective variety of dimension n. Let E be a rank n vector bundle with first Chern class L together with $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$ such that the determinant map $\bigwedge^n V^\vee \to V \subset H^0(L)$ is an isomorphism. Then*

$$\deg(\varphi_V) = c_n(E) - \deg(Z(V^\vee)),$$

where $Z(V^\vee) = \mathrm{Bl} \cap Z_{\mathrm{cycle}}(s)$ is the intersection of the zero loci of a general section of $s \in V^\vee$ with the base locus of the rational map (i.e. we are taking the degree of the general element minus the degree of the fixed part as zero cycles).

Proof. We give an alternative proof to Lemma 2.1. One may just observe that outside of the base locus

$$E = \varphi_V^*(T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)),$$

more precisely we have a well defined pullback morphism $\varphi_V^* : H^0(T\mathbb{P}^n(-1)) \to H^0(E)$ since the base locus is of codimension (at least) 2. The general section of $T\mathbb{P}^n(-1)$ vanishes in exactly one point. Hence the degree of the map may be computed by $Z(\varphi_V^* s) \cap (X - \mathrm{Bl})$, for general $s \in H^0(T\mathbb{P}^n(-1))$, Bl denotes the base locus of φ_V and $\varphi_V^* s \in H^0(E)$ (it is well defined in codimension 2 hence it extends to the whole E). Indeed, $Z(\varphi_V^* s) \cap (X - \mathrm{Bl})$ is the fiber of $\varphi_V : X - \mathrm{Bl} \to \mathbb{P}^n$ over $p = Z(s) \in \mathbb{P}^n$. Notice that if all the sections of $V^\vee$ vanishes at P then $P \in \mathrm{Bl}$. $\square$

This yields the following. The proof is just sketched.

Corollary 2.10. *The following estimates hold:*

- *if (X, H) is a general Gushel-Mukai threefold (i.e. a Fano threefold of genus 6 and index 1) then $\mathrm{irr}_H(X) \leq 3$;*
- *if (S, L) is a general $(1, 6)$ abelian surface then $\mathrm{irr}_L(S) = 3$;*
- *if (S, L) is a general $K3$ surface of genus 6 then $\mathrm{irr}(\mathrm{Hilb}^2(S)) \leq 6$;*
- *if (A, L) is a general abelian threefold of type $(1, 3, 12)$ or $(1, 6, 6)$ then $\mathrm{irr}_L(A) \leq 8$;*

- *if* (S, L) *is a general* $K3$ *surface of genus* 10 *then* $\mathrm{irr}(\mathrm{Hilb}^3(S)) \leq 20$.

Proof.
- For the Gushel-Mukai threefold one may just fix a pencil in $|H|$ giving a map $X \dashrightarrow \mathbb{P}^1$. Then observe that the minimal rank 2 vector bundle on a $K3$ in the pencil comes from X (it is the tautological vector bundle on the Grasmannian), and use the map $X \dashrightarrow \mathbb{P}^2$ given by $\mathrm{coker}(E \to H^0(E \otimes m_p))$, where p is a base point of the pencil. Then, the map $X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is of degree 3.

  Alternatively we may just lift the linear system of degree 3 from an hyperplane section (which is a $K3$ surface). More precisely, consider a general section $s \in H^0(X, H)$ defining a smooth $K3$ of genus 6 denoted by S. We have a surjection $H^0(\mathcal{O}_X(H)) \to H^0(\mathcal{O}_S(H))$. By Theorem 3.4 such a $K3$ has a rational map $S \dashrightarrow \mathbb{P}^2$ of degree 3 defined by a subspace $V \in \mathrm{Gr}(3, H^0(\mathcal{O}_S(H)))$. Then $\langle s, V\rangle \in \mathrm{Gr}(4, H^0(\mathcal{O}_X(H)))$ defines a map $X \to \mathbb{P}^3$ of degree 3. Indeed the base locus is of degree 6 and there is at least one point in the base locus which is not of local complete intersection (it is scheme theoretically the same of the $K3$), this makes the degree drop by one (the Hilbert-Samuel multiplicity is one more then the local length at that point).

- For the $(1, 6)$ abelian surface consider the vector bundle of rank 2 with $c_2(E) = 3$ and $h^0(E) = 3$. Let $L \otimes \mathcal{I} = \mathrm{coker}(E^\vee \to H^0(E)^\vee \otimes \mathcal{O}_S)$, then by lemma 2.9 $\deg \varphi_{|L\otimes\mathcal{I}|} \leq 3$. There are many ways to see that such a vector bundle exists, for instance via Mukai´s theory of semi-homogeneous vector bundles (c.f. [18, Theorem 7.11]) or as the Lazarsfeld-Mukai bundle associated to a g^1_3 on a trigonal curve in $|L|$.

- For $\mathrm{Hilb}^2(S)$ consider the minimal vector bundle E of rank 2 on S (i.e. $c_1(E) = L, c_2(E) = 4, h^0(E) = 5$). This vector bundle induces a tautological rank 4 vector bundle $\mathcal{E}$ on $\mathrm{Hilb}^2(S)$ with top Chern class $c_4(\mathcal{E}) = 6$ and $H^0(E) = H^0(\mathcal{E})$. By lemma 2.9 the map induced by the linear system $\mathrm{coker}(\mathcal{E}^\vee \to H^0(\mathcal{E})^\vee \otimes \mathcal{O}_{\mathrm{Hilb}^2(S)})$ is of degree ≤ 6.

- The remaining cases are similar and based on the construction of a suitable vector bundle: for the abelian threefold one should use Mukai´s theory of semi-homogeneous vector bundles (c.f. [18, Theorem 7.11]), for the hyper-Kähler sixfold a suitable tautological vector bundle. All these results have analogues for other polarizations and higher dimensions.

$\square$

Remark. *If* S *is an abelian surface of type* $(1, 2)$ *in [7] the authors considered the Galois closure in* S^3 *of a degree* 3 *map* $S \dashrightarrow \mathbb{P}^2$ *to construct a Lagrangian surface in* S^3 *(with respect to the invariant* 2*-form). It is likely that a similar construction can be carried out for the* $(1, 6)$ *abelian surface.*

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