The problem investigated in the HPC course:

Parallel resolution of a Poisson equation

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Poisson equation

The Poisson equation reads: $\Delta u = f$

Also written as: $\nabla^2 u = f$

In 2D, the it is written out component-wise as follows:

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = f(x, y)$$

It has a unique solution, e.g. with Dirichlet boundaries:

$$u(x, y) = b(x, y) \quad \text{on the boundary} \quad \partial\Omega$$

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Poisson equation

$$\Delta u = f$$

The Poisson equation describes

- The distribution of heat in a conductive solid.
- The distribution of pressure in an incompressible fluid.
- The distribution chemical species in a diffusion process.

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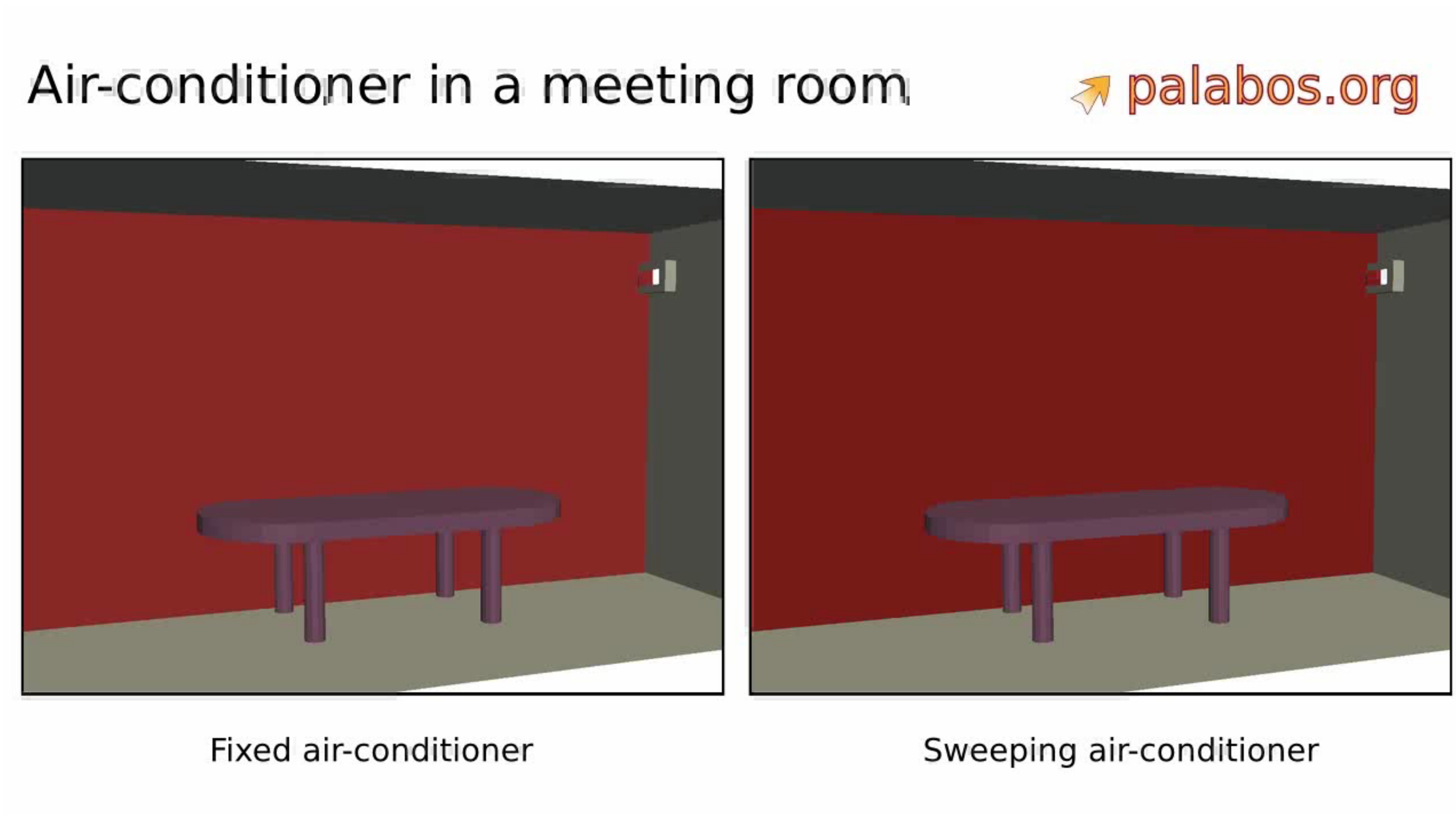
Example: heat transport



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Numerical solution: finite differences

We approximate the Laplacian by centered finite differences:

$$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) - f(x,y) = 0$$

$$\frac{u(x+h,y) + u(x-h,y) - 2*u(x,y)}{h^2} \qquad \frac{u(x,y+h) + u(x,y-h) - 2*u(x,y)}{h^2}$$

Leads to the linear system:

$$\boxed{u(x,y) = \frac{1}{4}\left(u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - h^2 f(x,y)\right)}$$

where $h = \frac{1}{N}$ Is the discrete grid spacing.

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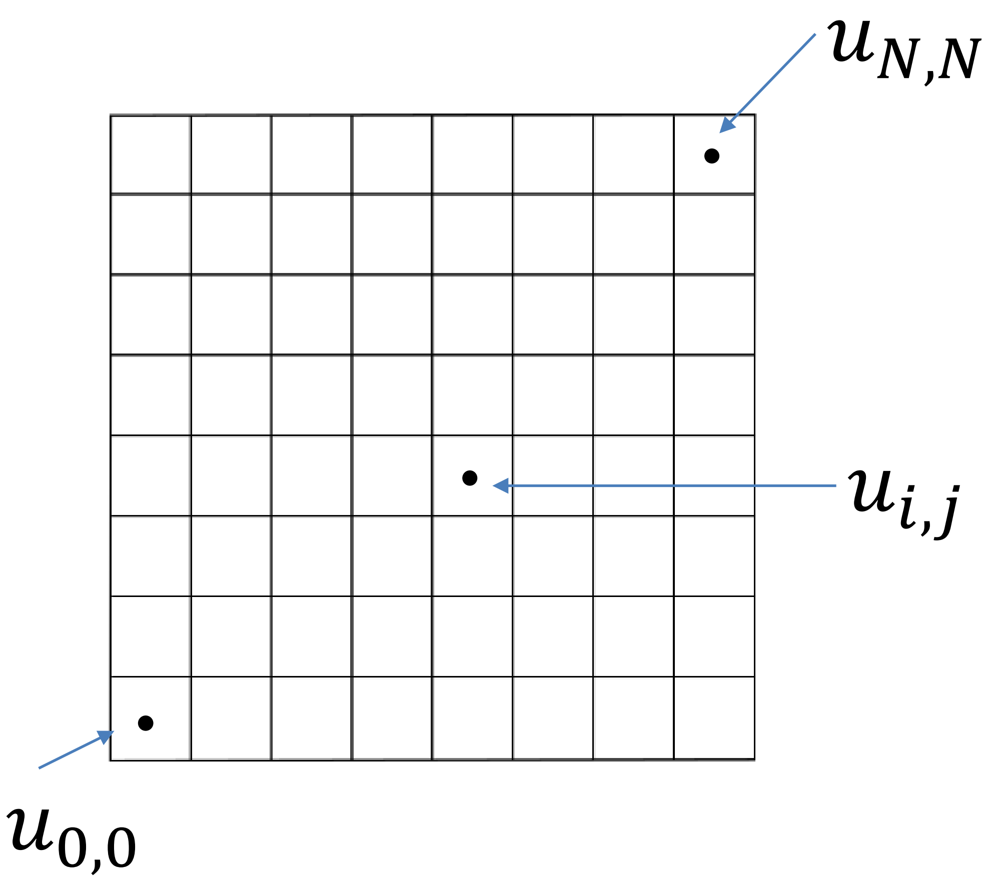
Discrete representation of space

We use a homogeneous discretization of space: a matrix



$$\boxed{u(x,y) \equiv u_{i,j}}$$

Iterative solution: Jacobi iterations

The system

$$\boxed{u(x,y) = \frac{1}{4}(u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - h^2 f(x,y))}$$

Becomes

$$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j.1} - h^2 f_{i,j}\right)}$$

Which can be solved iteratively, for successive iterations *k*:

$$\boxed{u_{i,j}^{k+1} = \frac{1}{4}\left(u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - h^2 f_{i,j}\right)}$$

Your turn…

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