COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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1. Introduction.

We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators form a 3D domain, Ω, to 1D manifold, Λ, and vice versa. This is a non trivial objective because the standard trace operator form a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω.

2. Problem setting.

In this section we fist provide the main notation and definitions to set up the problem. In particular, we address the geometry of the domain, which is equivalent to the one addressed in previous works of the authors [8], and we define the functional spaces and related operators necessary for a rigorous problem formulation and analysis. Finally, the section ends with the formal derivation of the problem, based on averaging techniques applied to a fully three-dimensional model in the geometrical setting defined before.

2.1. Geometry of the problem.

The domain is denoted as Ω is convex and composed by two parts, $\Omega_\ominus$ and $\Omega_\oplus := \Omega \setminus \overline{\Omega}_\ominus$. Let $\Omega_\ominus$ be a *generalized cylinder*, that is the swept volume of a two dimensional set moved along a curve in the three-dimensional space, see for example [6]. We also assume that $\Omega_\ominus$ crosses Ω from side to side, and we call Γ the lateral surface of $\Omega_\ominus$, while the upper and lower faces of $\Omega_\ominus$ belong to $\partial\Omega$. Let $\boldsymbol{\lambda}(s) = [\xi(s), \tau(s), \zeta(s)],\ s \in (0, S)$ be a $\mathcal{C}^2$-regular curve in the three-dimensional space. For simplicity, let us assume that $\|\boldsymbol{\lambda}'(s)\| = 1$ such that the arc-length and the coordinate s coincide. Let $\Lambda = \{\boldsymbol{\lambda}(s),\ s \in (0, S)\}$ be the centerline of the cylinder. According to the geometrical setting, we will denote with v, $v_\oplus$, $v_\ominus$, $v_\odot$, functions defined on Ω, $\Omega_\oplus$, $\Omega_\ominus$, Λ, respectively. We refer to Figure 2.1 for a sketch of the domains.

Let $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ be the Frenet frame related to the curve. Let $\mathcal{D}(s) = [x(r,t), y(r,t)] : (0, R(s)) \times (0, T(s)) \to \mathbb{R}^2$ be a parametrization of the cross section. Let us assume that $\mathcal{D}(s)$ is convex for any $s \in (0, S)$. The cross section can change size along Λ but not shape. Let us also parametrize the boundary of the cross section as $\partial\mathcal{D}(s) = [\partial x(r,t), \partial y(r,t)] : (0, R(s)) \times (0, T(s)) \to \mathbb{R}^2$, and let us assume that $\partial\mathcal{D}(s)$ is a piecewise $\mathcal{C}^2$-regular curve. Then, the generalized cylinder $\Omega_\ominus$ can be defined as follows

$$\Omega_\ominus = \{\boldsymbol{\lambda}(s) + x(r,t)\boldsymbol{N}(s) + y(r,t)\boldsymbol{B}(s),\ r \in (0, R(s)),\ s \in (0, S),\ t \in (0, T(s))\},$$

and the lateral boundary is

$$\Gamma = \{\boldsymbol{\lambda}(s) + \partial x(r,t)\boldsymbol{N}(s) + \partial y(r,t)\boldsymbol{B}(s),\ r \in (0, R(s)),\ s \in (0, S),\ t \in (0, T(s))\}.$$

Let $|\cdot|$ denote the Lebesgue measure of a set. If $|\mathcal{D}(0)|,\ |\mathcal{D}(S)| > 0$ then $\Omega_\ominus$ has *top* and *bottom* boundaries, which belong to $\partial\Omega$.

Thanks to the geometrical structure of the generalized cylinder, for any sufficiently regular function w we decompose integrals as follows and we define the corresponding averages $\overline{w}$, $\overline{\overline{w}}$,

$$\int_{\Omega_\ominus} w d\omega = \int_\Lambda \int_{\mathcal{D}(s)} w d\sigma ds = \int_\Lambda |\mathcal{D}(s)| \overline{\overline{w}}(s) ds\,, \text{ where } \overline{\overline{w}}(s) = |\mathcal{D}(s)|^{-1} \int_{\mathcal{D}(s)} w d\sigma\,,$$

$$\int_{\partial\Omega_\ominus} w d\sigma = \int_\Lambda \int_{\partial\mathcal{D}(s)} w d\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)| \overline{w}(s) ds\,, \text{ where } \overline{w}(s) = |\partial\mathcal{D}(s)|^{-1} \int_{\partial\mathcal{D}(s)} w d\gamma\,,$$

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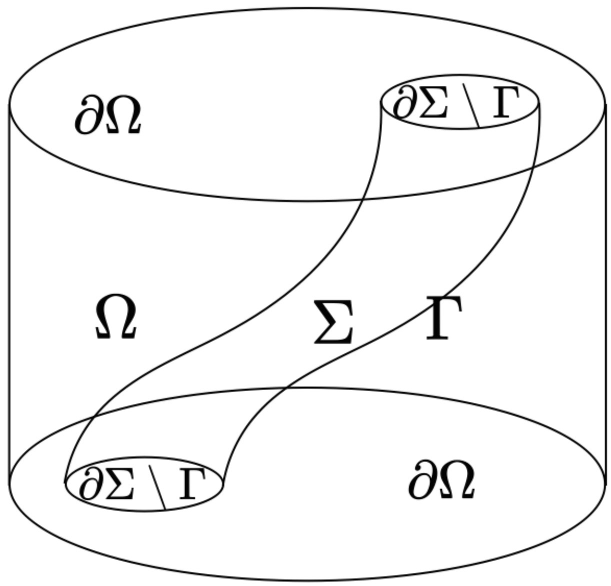


FIGURE 2.1. *Geometrical setting of the problem*

being $d\omega$, $d\sigma$, $d\gamma$ the generic volume, surface and curvilinear Lebesgue measures. Analogously, for functions defined on Λ and $\Omega_\ominus$ respectively, we define as d_s and ∂_s the ordinary and partial derivative with respect to the arclength.

This work is based on the assumption that *the transversal diameter of* $\Omega_\ominus$ *is small compared to the diameter of* Ω. Let $d = \max_{s\in(0,S)} \mathrm{diam}(\mathcal{D}(s))$ be the largest diameter of the cross sections of $\Omega_\ominus$, that is the transversal diameter of $\Omega_\ominus$. The central assumption of this work is that $d \ll \mathrm{diam}(\Omega)$. Let us now scale the domains Ω and $\Omega_\ominus$. Let $\chi_\Omega(\boldsymbol{x}) = \boldsymbol{x}/\mathrm{diam}(\Omega)$ be a scaling function and let be $\Omega_\chi = \chi_\Omega(\Omega)$, $\Omega_{\ominus\chi} = \chi_\Omega(\Omega_\ominus)$ be the scaled domains. The previous assumption implies that for the scaled domains $\epsilon = d_\chi = d/\mathrm{diam}(\Omega)$ is such that $0 < \epsilon \ll 1$. For simplicity of notation, and without loss of generality, form now on we will implicitly refer to the scaled domains dropping the subindex χ.

2.2. Functional setting and preliminary results. Let D be a generic regular bounded domain in $\mathbb{R}^3$. Let X be a Hilbert space defined on D. Let $(\cdot,\cdot)_X$ denote the inner product of X. Let $(\cdot,\cdot)_L^2(D)$, $(\cdot,\cdot)_D$ or simply $(\cdot,\cdot)$ be the $L^2(D)$ inner product on D. For the domain Λ we will use the following notation for the weighted inner product,

$$(u_\odot, v_\odot)_{\Lambda,w} = \int_0^S w(s) u_\odot(s) v_\odot(s) ds\,.$$

Given an Hilbert space X and its dual X^*, the duality pairing between the two is denoted as $\langle\cdot,\cdot\rangle$. When needed, the integration weights will be also applied to the duality product.

We will use the standard notation $H^q(D)$ to denote the Sobolev space of functions on D with all derivatives up to the order q in $L^2(D)$. The corresponding norm is $\|\cdot\|_{H^q(D)}$ and the seminorm is $|\cdot|_{H^q(D)}$. The space $H_0^q(D)$ represents the closure in $H^q(D)$ of smooth functions with compact support in D.

Let Σ be a Lipschitz co-dimension one subset of D. We denote with $\mathcal{T} : H^q(D) \to H^{q-\frac{1}{2}}(\Sigma)$ the trace operator from D to Σ. For the purpose of this work we will extensively use traces on Γ of functions defined on Ω. Such operator will be denoted with $\mathcal{T}_\Gamma$. The operator obtained form the combination of the average operator $\overline{(\cdot)}$ with the trace on Γ will be denoted with $\overline{\mathcal{T}}_\Lambda = \overline{(\cdot)} \circ \mathcal{T}_\Gamma$, as it maps functions on Ω to functions on Λ.

Fractional Sobolev spaces arising from the application of the trace operator are particularly important in this work. For example the trace space on Γ of $H_0^1(\Omega)$ will be be extensively used for this study. We denote this space as $H^{\frac{1}{2}}_{00}(\Gamma) = \mathcal{T}_\Gamma H^1_0(\Omega)$. It can be interpreted as the subspace of $H^{\frac{1}{2}}(\Gamma)$ of functions with continuous extension by zero outside Γ. We remark that $(H^{\frac{1}{2}}_{00}(D))^* = H^{-\frac{1}{2}}(D)$ with $D = \Gamma, \Lambda$. We will see later that functions belonging to $H^{\frac{1}{2}}_{00}(\Lambda)$ are generated from the application of $\overline{\mathcal{T}}_\Lambda$ to $H^1_0(\Omega)$.

In what follows we will also use the uniform extension to $\Omega_\ominus$ of functions defined on Λ. Namely, given $v_\odot(s)$ we define $\mathcal{E}_{\Omega_\ominus}$ as $\mathcal{E}_{\Omega_\ominus} v_\odot(s,r,t) = v_\odot(s)$. Then, taking the trace on Γ of the extended function we obtain an extension map from Λ to Γ, denoted as $\mathcal{E}_\Gamma$. We observe that $\mathcal{E}_\Gamma : H^{\frac{1}{2}}_0(\Lambda) \to H^{\frac{1}{2}}_{00}(\Gamma)$ satisfies the following property

$$\langle \overline{\mathcal{T}}_\Lambda u, v_\odot \rangle_{\Lambda,|\partial\mathcal{D}|} = \int_\Lambda |\partial\mathcal{D}| \left(\frac{1}{|\partial\mathcal{D}|} \int_{\partial\mathcal{D}} \mathcal{T}_\Gamma u \, d\gamma \right) v_\odot \, ds = \langle \mathcal{T}_\Gamma u, \mathcal{E}_\Gamma v_\odot \rangle_\Gamma \,, \tag{2.1}$$

which shows that the average and the uniform extension operator commute.

From the theory of interpolation of Sobolev spaces we define the fractional norms $\|\cdot\|_{H^{\frac{1}{2}}_{00}(\Gamma)}$ and $\|\cdot\|_{H^{\frac{1}{2}}_{00}(\Lambda)}$ as functions of the eigenvalues and eigenfunctions of the Laplacian on Γ and Λ respectively, see for example [9, 11]. We denote as ϕ_{ij} and ρ_{ij}, for $i = 1, 2, \dots,$ $j = 0, 1, \dots,$ the eigenfunctions and the eigenvalues of the Laplacian on Γ with homogeneous Dirichlet condition at the boundary. Then, the norms of $H^{\frac{1}{2}}_{00}(\Gamma)$, $H^{\frac{1}{2}}_{00}(\Lambda)$ and the corresponding dual norms are defined as

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \left(\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (1+\rho_{ij})^{\frac{1}{2}} |a_{ij}|^2 \right)^{\frac{1}{2}} \text{ with } a_{ij} = (u, \phi_{ij})_\Gamma \,; \quad \|v\|_{H^{-\frac{1}{2}}(\Gamma)} = \sup_{u \in H^{\frac{1}{2}}_{00}(\Gamma)} \frac{\langle u, v \rangle_\Gamma}{\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} \,. \tag{2.2}$$

Similarly, denoting with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the Laplacian on Λ with homogenous Dirichlet boundary conditions, if $u \in H^{\frac{1}{2}}(\Lambda)$ we have

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)} = \left(\sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}} |a_i|^2 \right)^{\frac{1}{2}} \text{ with } a_i = (u, \phi_i)_\Lambda \,; \quad \|v\|_{H^{-\frac{1}{2}}(\Lambda)} = \sup_{u \in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle u, v \rangle_\Lambda}{\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)}} \,. \tag{2.3}$$

We define the equivalent weighted norm in $H^{\frac{1}{2}}_{00}(\Lambda)$ in the same way,

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} = \left(\sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}} |a_i|^2 \right)^{\frac{1}{2}}, \text{ with} a_i = (u, \phi_i)_{\Lambda,|\partial\mathcal{D}|} \,; \quad \|v\|_{H^{-\frac{1}{2}}(\Lambda)} = \sup_{u \in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle u, v \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}} \,. \tag{2.4}$$

We conclude this section with the analysis of a fundamental property for the problem formulation that we will address. It consists in the characterization of the regularity of the operator $\overline{\mathcal{T}}_\Lambda$. More precisely we aim to show that $\overline{\mathcal{T}}_\Lambda : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Lambda)$. This is a consequence of the following lemma.

Lemma 2.1. *Let Γ be a tensor product domain, $\Gamma = (0,X) \times (0,Y)$. For any regular $u(x,y)$ in Γ, let $\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x,y)\,dy$. Then, for any $u \in H^{\frac{1}{2}}_{00}(\Gamma)$, then $\overline{u}(x) \in H^{\frac{1}{2}}_{00}((0,X))$. Moreover, if $u(x,y) \in H^{\frac{1}{2}}_{00}(\Gamma)$ is constant with respect to y, namely $u(x,y) = u(x)$, then*

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = Y \|u\|_{H^{\frac{1}{2}}_{00}(0,X)}.$$

Proof. In order to apply (2.2) and (2.3), let us consider the eigenvalue problems for the Laplace operator on Γ with homogeneous Dirichlet conditions at $x = 0$, X and periodic boundary conditions at $y = 0$, Y. Let us also consider the Laplace eigenproblem on $(0,X)$ with homogeneous Dirichlet conditions. Let us denote as $\phi_{ij}(x,y)$ and ρ_{ij}, for $i = 1, 2, \dots,$ $j = 0, 1, \dots,$ the eigenfunctions and the eigenvalues of the Laplacian on

Γ, and with $\phi_i(x)$ and ρ_i the eigenfunctions and the eigenvalues of the laplacian on $(0, X)$. In particular,

$$\phi_{ij}(x,y) = \sin\left(\frac{i\pi x}{X}\right)\left(\cos\left(\frac{j2\pi y}{Y}\right) + \sin\left(\frac{j2\pi y}{Y}\right)\right), \qquad \rho_{ij} = \left(\frac{i\pi}{X}\right)^2 + \left(\frac{j2\pi}{Y}\right)^2,$$

$$\phi_i(x) = \sin\left(\frac{i\pi x}{X}\right), \qquad \rho_i = \left(\frac{i\pi}{X}\right)^2.$$

It is easy to verify that

$$\text{(2.5)} \qquad \int_0^Y \phi_{ij}(x,y) = 0 \quad \forall j > 0, \forall i\,, \qquad \int_0^Y \phi_{ij}(x,y) = Y \sin\left(\frac{i\pi x}{X}\right) \quad \text{if } j = 0, \forall i.$$

Moreover we recall that $\phi_{i,j}(x,y)$ and $\phi_i(x)$ form an orthogonal basis of $L^2(\Gamma)$ and $L^2(0,X)$ respectively. Therefore,

$$\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x,y)\,dy = \frac{1}{Y}\int_0^Y \sum_{i,j} a_{i,j}\phi_{i,j}(x,y)\,dy = \frac{1}{Y}\sum_{i,j} a_{i,j}\int_0^Y \phi_{i,j}(x,y)\,dy = \sum_i a_{i,0}\phi_i(x).$$

From (2.3) we have

$$\begin{aligned}
\|\overline{u}\|^2_{H^{\frac{1}{2}}_{00}(0,X)} &= \sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}\,a_i^2 = \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X \overline{u}(x)\sin\left(\frac{i\pi x}{X}\right)\,dx\right)^2 \\
&= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty} a_{j,0}\int_0^X \sin\left(\frac{j\pi x}{X}\right)\sin\left(\frac{i\pi x}{X}\right)\,dx\right)^2 = \sum_{i=1}^{\infty}\frac{X^2}{4}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}} a_{i,0}^2 \\
&\leq \frac{X^2}{4}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}|a_{i,j}|^2 = \frac{X^2}{4}\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)},
\end{aligned}$$

where we have used the fact that

$$\int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)\,dx = 0 \quad \text{if } i \neq j \qquad \int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)\,dx = \frac{X}{2} \quad \text{if } i = j$$

and we have applied (2.2) in the last equality. Moreover, in the case in which u is constant with respect to y, we have

$$\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}\,|a_{ij}|^2 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X\int_0^Y u(x,y)\phi_{ij}(x,y)\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X u(x)\int_0^Y \phi_{ij}(x,y)\right)^2,
\end{aligned}$$

and using (2.5) we obtain

$$\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} = \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X Y u(x)\sin\left(\frac{i\pi x}{X}\right)\right)^2 = Y^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}\,|a_i|^2 = Y^2\|u\|^2_{H^{\frac{1}{2}}_{00}(0,X)}.$$

□

Under the geometric assumptions stated above for Ω, Γ, Λ, Lemma 2.1 implies the following result.

Corollary 2.2. *If $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ then $\overline{u} \in H^{\frac{1}{2}}_{00}(\Lambda)$ and there exists a constant C_Γ such that*

$$\|\overline{u}\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} \leq C_\Gamma\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)}.$$

2.3. Problem formulation. We consider the problem arising from *Dirichlet-Neumann* conditions. Find $u_\oplus, u_\ominus$ s.t.:

$$-\Delta u_\oplus + u_\oplus = f \qquad \text{in } \Omega_\oplus, \tag{2.6a}$$
$$-\Delta u_\ominus + u_\ominus = g \qquad \text{in } \Omega_\ominus, \tag{2.6b}$$
$$-\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus \qquad \text{on } \Gamma, \tag{2.6c}$$
$$u_\ominus = u_\oplus \qquad \text{on } \Gamma, \tag{2.6d}$$
$$u_\oplus = 0 \qquad \text{on } \partial\Omega\,. \tag{2.6e}$$

We observe that the coupling constraints defined on Γ involve *essential*-type conditions. To address such conditions at the level of the variational formulation, we adopt the method of Lagrange multipliers, which has been widely used for this purpose since the origin of the finite element method, [2]. We remark that this case is substantially different from the one previously addressed in [8].

The variational formulation of problem (2.6) is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus)$, $u_\ominus \in H^1_{\partial\Omega_\ominus\setminus\Gamma}(\Omega_\ominus)$, $\lambda \in H^{-\frac{1}{2}}(\partial\Omega_\ominus)$ s.t.

$$\begin{aligned}(u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Omega_\ominus)} + \langle v_\oplus - v_\ominus, \lambda\rangle_\Gamma \\ = (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Omega_\ominus)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Omega_\ominus\setminus\Gamma}(\Omega_\ominus)\end{aligned} \tag{2.7a}$$

$$\langle u_\oplus - u_\ominus, \mu\rangle_\Gamma = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,, \tag{2.7b}$$

In this case, λ is the Lagrange multiplier and it is equivalent to $\nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

The objective of this work is to derive and analyze a simplified version of problem (2.7), where the domain $\Omega_\ominus$ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely $\mathcal{D}$. This new problem setting will be called the *reduced* problem. Form the mathematical standpoint it is more challenging than (2.7), because it involves the coupling of 3D/1D elliptic problems. We end up with two different formulations of a reduced problem for the unknown u defined on the entire 3D domain Ω, coupled with the unknown $u_\odot$, defined on the 1D manifold Λ and a Lagrange multiplier defined either on Γ (problem 1) or on Λ (problem 2). The scope of this work is to analyze and compare the two alternatives, with the aim to determine which is the most suitable to set up a computational model for 3D-1D PDEs coupled with Dirichlet-Neumann constraint.

2.4. Topological model reduction.

Model reduction of the problem on $\Omega_\ominus$. We apply the averaging technique to equation (2.6b). In particular, we consider an arbitrary portion $\mathcal{P}$ of the cylinder $\Omega_\ominus$, with lateral surface $\Gamma_\mathcal{P}$ and bounded by two perpendicular sections to Λ, namely $\mathcal{D}(s_1)$, $\mathcal{D}(s_2)$ with $s_1 < s_2$. We have,

$$\begin{aligned}\int_\mathcal{P} -\Delta u_\ominus + u_\ominus d\omega = -\int_{\partial\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus\, d\sigma + \int_\mathcal{P} u_\ominus d\omega = \\ \int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma - \int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma + \int_\mathcal{P} u_\ominus d\omega\end{aligned}$$

By the fundamental theorem of integral calculus combined with the Reynolds transport Theorem, being ν the normal deformation of the boundary along $(0, S)$, we have,

$$\begin{aligned}\int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma &= -\int_{s_1}^{s_2} d_s \int_{\mathcal{D}(s)} \partial_s u_\ominus d\sigma ds \\ &= -\int_{s_1}^{s_2} d^2_{ss} \int_{\mathcal{D}(s)} u_\ominus\, d\sigma\, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus\, d\gamma\right) ds\,,\end{aligned}$$

and assuming that $\mathcal{D}(s)$ can not change shape, we have

$$
\begin{aligned}
-\int_{s_1}^{s_2} d_{ss}^2 \int_{\mathcal{D}(s)} u_\ominus \, d\sigma \, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus \, d\gamma \right) ds &= -\int_{s_1}^{s_2} \left[d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) - d_s(\nu |\partial\mathcal{D}(s)| \overline{u}_\ominus) \right] ds \\
&= -\int_{s_1}^{s_2} \left[d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) - d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) \right] ds.
\end{aligned}
$$

Moreover, we have

$$
\int_{\Gamma_{\mathcal{P}}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma = \int_{\Gamma_{\mathcal{P}}} \lambda \, d\sigma = \int_{s_1}^{s_2} \int_{\partial\mathcal{D}(s)} \lambda d\gamma \, ds = \int_{s_1}^{s_2} |\partial\mathcal{D}| \overline{\lambda} \, ds \, .
$$

From the combination of all the above terms with the right hand side, we obtain that the solution $u_\ominus$ of (2.6b) satisfies,

$$
\int_{s_1}^{s_2} \left[-d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) + d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) + |\mathcal{D}(s)| \overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)| \overline{\lambda} \right] ds = \int_{s_1}^{s_2} |\mathcal{D}(s)| \overline{\overline{g}} \, ds \, .
$$

Since the choice of the points s_1, s_2 is arbitrary, we conclude that the following equation holds true,

$$
- d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) + d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) + |\mathcal{D}(s)| \overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)| \overline{\lambda} = |\mathcal{D}(s)| \overline{\overline{g}} \quad \text{on } \Lambda \, , \tag{2.8}
$$

which is complemented by the following conditions at the boundary of Λ,

$$
|\mathcal{D}(s)| d_s \overline{\overline{u}}_\ominus = 0, \quad d_s |\mathcal{D}(s)| = 0, \quad \text{on} \quad s = 0, S. \tag{2.9}
$$

Then, we consider variational formulation of the averaged equation (2.8). After multiplication by a test function $v_\odot \in H^1(\Lambda)$, integration on Λ and suitable application of integration by parts, we obtain,

$$
\begin{aligned}
\int_\Lambda d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) d_s v_\odot \, ds - d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) v_\odot |_{s=0}^{s=S} - \int_\Lambda (d_s |\mathcal{D}(s)|) \overline{u}_\ominus d_s v_\odot \, ds + (d_s |\mathcal{D}(s)|) \overline{u}_\ominus v_\odot |_{s=0}^{s=S} \\
+ \int_\Lambda |\mathcal{D}(s)| \overline{\overline{u}}_\ominus v_\odot - \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda} v_\odot \, ds = \int_\Lambda |\mathcal{D}(s)| \overline{\overline{g}} V \, ds \, .
\end{aligned}
$$

Using boundary conditions, the identity $d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) = |\mathcal{D}(s)| d_s \overline{\overline{u}}_\ominus + d_s(|\mathcal{D}(s)|) \overline{\overline{u}}_\ominus$ and reminding that $d_s |\mathcal{D}(s)|) / |\partial\mathcal{D}(s)| = \nu$, we obtain,

$$
(d_s \overline{\overline{u}}_\ominus, d_s v_\odot)_{\Lambda, |\mathcal{D}|} + (\nu (\overline{\overline{u}}_\ominus - \overline{u}_\ominus), d_s v_\odot)_{\Lambda, |\partial\mathcal{D}|} + (\overline{\overline{u}}_\ominus, v_\odot)_{\Lambda, |\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda, |\partial\mathcal{D}|} = (\overline{\overline{g}}, V)_{\Lambda, |\mathcal{D}|} \, . \tag{2.10}
$$

Let us now formulate the modelling assumption that allows us to reduce equation (2.10) to a solvable one-dimensional (1D) model. More precisely, we assume that the function $u_\ominus$ has a *uniform profile* on each cross section $\mathcal{D}(s)$, namely $u_\ominus(r, s, t) = u_\odot(s)$. Therefore, observing that $u_\odot = \overline{u}_\ominus = \overline{\overline{u}}_\ominus$, problem (2.10) turns out to find $u_\odot \in H^1(\Lambda)$ such that

$$
(d_s u_\odot, d_s v_\odot)_{\Lambda, |\mathcal{D}|} + (u_\odot, v_\odot)_{\Lambda, |\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda, |\partial\mathcal{D}|} = (\overline{\overline{g}}, v_\odot)_{\Lambda, |\mathcal{D}|} \quad \forall v_\odot \in H^1(\Lambda) \, . \tag{2.11}
$$

Topological model reduction of the problem on $\Omega_\oplus$. We focus here on the subproblem of (2.6a) related to $\Omega_\oplus$. We multiply both sides of (2.6a) by a test function $v \in H_0^1(\Omega)$ and integrate on $\Omega_\oplus$. Integrating by parts and using boundary and interface conditions, we obtain

$$
\begin{aligned}
\int_{\Omega_\oplus} f v \, d\omega &= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\omega - \int_{\partial\Omega_\oplus} \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v \, d\sigma + \int_{\Omega_\oplus} u_\oplus v \\
&= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\Omega - \int_\Gamma \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v + \int_{\Omega_\oplus} u_\oplus v \\
&= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\Omega + \int_\Gamma \lambda v + \int_{\Omega_\oplus} u_\oplus v.
\end{aligned}
$$

Then, we make the following modelling assumption: we identify the domain $\Omega_\oplus$ with the entire Ω, and we correspondingly omit the subscript $\oplus$ to the functions defined on $\Omega_\oplus$, namely

$$\int_{\Omega_\oplus} u_\oplus \, d\omega \simeq \int_\Omega u \, d\omega \, .$$

Therefore, we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\lambda, v)_\Gamma = (f, v)_\Omega$$

and combining with (2.11) we obtain the first formulation of the reduced problem.

Problem 1 (3D-2D-1D). Find $u \in H_0^1(\Omega),\ \lambda \in H^{-\frac{1}{2}}(\Gamma),\ u_\odot \in H_0^1(\Lambda)$, such that

$$\begin{aligned}
&(2.12a) && (u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \lambda \rangle_\Gamma \\
& && \quad = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|} && \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda)\, ,\\
&(2.12b) && \langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, \mu \rangle_\Gamma = 0 && \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\, .
\end{aligned}$$

This coupled problem is classifies as 3D-2D-1D because the unknowns $u, \lambda, u_\odot$ belong to $\Omega \subset \mathbb{R}^3$, $\Gamma \subset \mathbb{R}^2$, $\Lambda \subset \mathbb{R}$, respectively. Then, we apply a topological model reduction of the interface conditions, namely we go from a 3D-2D-1D formulation involving sub-problems on Ω and Λ and coupling operators defined on Γ to a 3D-1D-1D formulation where the coupling terms are set on Λ. To this purpose, let us write the Lagrange multiplier and the test functions on every cross section $\partial\mathcal{D}(s)$ as their average plus some fluctuation,

$$\lambda = \overline{\lambda} + \tilde{\lambda}, \qquad v = \overline{v} + \tilde{v}, \quad \text{on } \partial\mathcal{D}(s)\, ,$$

where $\overline{\tilde{\lambda}} = \overline{\tilde{v}} = 0$. Therefore, the coupling term on Γ can be decomposed as,

$$\int_\Gamma \lambda v \, d\sigma = \int_\Lambda \int_{\partial\mathcal{D}(s)} (\overline{\lambda} + \tilde{\lambda})(\overline{v} + \tilde{v}) d\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda}\overline{v} \, ds + \int_\Lambda \int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma ds \, .$$

Thanks to the additional assumption that the product of fluctuations is small, namely

$$\int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma \simeq 0$$

the term $(\mathcal{T}_\Gamma v, \lambda)_\Gamma$ becomes $\left(\overline{\mathcal{T}}_\Lambda v, \overline{\lambda}\right)_{\Lambda,|\partial\mathcal{D}|}$, where $\overline{\mathcal{T}}_\Lambda$ denotes the composition of operators $\mathcal{T}_\Gamma \circ \overline{(\cdot)}$. Combined with (2.11), this leads to the second formulation of the reduced problem.

Problem 2 (3D-1D-1D). Find $u \in H_0^1(\Omega),\ u_\odot \in H_0^1(\Lambda),\ \lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}
&(2.13a) && (u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \lambda_\odot \rangle_{\Lambda,|\partial\mathcal{D}|} \\
& && \quad = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, V)_{L^2(\Lambda),|\mathcal{D}|} && \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\, ,\\
&(2.13b) && \langle \overline{\mathcal{T}}_\Lambda u - u_\odot, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|} = 0 && \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda)\, ,
\end{aligned}$$

We notice that all the integrals of the reduced problem are well defined because $\overline{\mathcal{T}}_\Lambda : H_0^1(\Omega) \to H_{00}^{\frac{1}{2}}(\Lambda)$ as shown in Corollary 2.2.

The well-posedness of (2.12) and (2.13) can be studied in the framework of the classical theory of saddle point problems as shown in the following.

3. Saddle-point problem analysis. Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bounded bilinear forms. Let us consider the general saddle point problem of the form: find $u \in X$, $\lambda \in Q$ s.t.

$$(3.1) \qquad \begin{cases} a(u, v) + b(v, \lambda) = c(v) & \forall v \in X \\ b(u, \mu) = d(\mu) & \forall \mu \in Q. \end{cases}$$

We denote with A and B the operators associated to the bilinear forms a and b, namely $A : X \longrightarrow X'$ with $\langle Au, v\rangle_{X',X} = a(u, v)$ and $B : X \longrightarrow Q'$ with $\langle Bv, \mu\rangle_{Q',Q} = b(v, \mu)$. Problem (3.1) embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem, see for example [5, Theorem 2.34] and [3] for the seminal work on this topic.

THEOREM 3.1. *Problem (3.1) is well posed iff*

$$\begin{cases} \exists\alpha > 0 :\ \inf_{u\in ker(B)} \sup_{v\in ker(B)} \frac{a(u,v)}{\|u\|_X\|v\|_X} \geq \alpha \\ \forall v \in ker(B),\ (\forall u \in ker(B),\ a(u, v) = 0) \implies v = 0. \end{cases} \tag{3.2}$$

$$\exists\beta > 0 :\ \inf_{\mu\in Q} \sup_{v\in X} \frac{b(v, \mu)}{\|v\|_X\|\mu\|_Q} \geq \beta. \tag{3.3}$$

Notice that if a is coercive on $ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1.

We aim to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ \lambda \in H^{-\frac{1}{2}}(\Gamma)$, solutions of (3.1), where

$$\begin{aligned} a([u, u_\odot], [v, v_\odot]) &= (u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} \\ b([v, v_\odot], \mu) &= \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma \\ c([v, v_\odot]) &= (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|} \\ d(\mu) &= 0 \end{aligned}$$

We prove that the hypothesis of 3.1 are fullfilled choosing $X = H^1_0(\Omega) \times H^1_0(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$, where X is equipped with the norm $|||[u, u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + \|u_\odot\|^2_{H^1(\Lambda),|\mathcal{D}|}$.

LEMMA 3.2. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The bilinear form $a(\cdot\ ,\ \cdot)$ is clearly bounded since

$$a([u, u_\odot], [v, v_\odot]) \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + \|u_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|} \leq 2|||[u, u_\odot]||| |||[v, v_\odot]|||.$$

Concerning the bilinear form $b(\cdot\ ,\ \cdot)$ we have

$$\begin{aligned} b([v, v_\odot], \mu) &= \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma \leq \|\mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot\|_{H^{\frac{1}{2}}(\Gamma)} \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^{\frac{1}{2}}(\Gamma)}\right) \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \left(C_T\|v\|_{H^1(\Omega)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^1(\Gamma)}\right) \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(C_T\|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}} \|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\right) \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\right) |||[v, v_\odot]||| \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \end{aligned}$$

∎

LEMMA 3.3. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive .*

Proof. Indeed, we have,

$$a([u, u_\odot], [u, u_\odot]) = (u, u)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, u_\odot)_{H^1(\Lambda)} = |||[u, u_\odot]|||^2 .$$

∎

LEMMA 3.4. *The inf-sup inequality (3.3) is fulfilled, namely $\exists\beta_1 > 0$ such that $\forall\mu \in H^{-\frac{1}{2}}(\Gamma)$:*

$$\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v, v_\odot]|||} \geq \beta_1 \sup_{q\in H^{\frac{1}{2}}_{00}(\Gamma)} \frac{\langle q, \mu\rangle_\Gamma}{\|q\|_{H^{\frac{1}{2}}_{00}(\Gamma)}}.$$

Proof. We choose $v_\odot \in H^1_0(\Lambda)$ such that $\mathcal{E}_\Gamma v_\odot = 0$. Therefore,

$$\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v, v_\odot]|||} \geq \sup_{v\in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

We notice that the trace operator is surjective from $H^1_0(\Omega)$ to $H^{\frac{1}{2}}_{00}(\Gamma)$. Indeed, $\forall \xi \in H^{\frac{1}{2}}_{00}(\Gamma)$, we can find v solution of

$$-\Delta v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \quad v = \xi \text{ on } \Gamma.$$

We denote with $\mathcal{E}_\Omega$ the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|\mathcal{E}_\Omega\| \in \mathbb{R}$ such that $v = \mathcal{E}_\Omega(\xi)$ and $\|v\|_{H^1(\Omega)} \leq \|\mathcal{E}_\Omega\|\|\xi\|_{H^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$\sup_{v\in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi\in H^{\frac{1}{2}}_{00}(\Gamma)} \frac{\langle \xi, \mu\rangle_\Gamma}{\|\mathcal{E}_\Omega\|\|\xi\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} = \|\mathcal{E}_\Omega\|^{-1}\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)}, \tag{3.4}$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}_{00}(\Gamma))^*$. ∎

3.2. Problem 2. We aim to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ \lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, solution of (3.1) with

$$\begin{aligned}
a([u, u_\odot], [v, v_\odot]) &= (u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|}\\
b([v, v_\odot], \mu_\odot) &= \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}\\
c([v, v_\odot]) &= (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|}\\
d(\mu_\odot) &= 0
\end{aligned}$$

We prove that the hypotesis of Theorem 3.1 are fulfilled with the following spaces $X = H^1_0(\Omega) \times H^1_0(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped again with the norm $|||[\cdot, \cdot]|||$ and Q equipped with the norm $\|\cdot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}$. Then, we have the following lemmas.

Lemma 3.5. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The boundedness of $a(\cdot\ ,\ \cdot)$ can be proved as in Lemma 3.2. Concerning $b(\cdot\ ,\ \cdot)$, we have

$$\begin{aligned}
b([v, v_\odot], \mu_\odot) &= \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \leq \|\overline{\mathcal{T}}_\Lambda v - v_\odot\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\leq \left(\|\overline{\mathcal{T}}_\Lambda v\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} + \|v_\odot\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\leq \left(C_\Gamma\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}_{00}(\Gamma)} + \|v_\odot\|_{H^1(\Lambda),|\partial\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\leq \left(C_\Gamma C_T\|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\leq \left(C_\Gamma C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\right)|||[v, v_\odot]|||\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.
\end{aligned}$$ ∎

Lemma 3.6. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive.*

Lemma 3.7. *The inf-sup inequality (3.3) holds, namely $\exists\beta_2 > 0$ such that $\forall\mu_\odot \in H^{-\frac{1}{2}}(\Lambda)$, :*

$$\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v, v_\odot]|||} \geq \beta_2\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

We choose $v_\odot = 0$ and we obtain

$$\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v,v_\odot]|||} \geq \sup_{v\in H^1_0(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}}.$$

For any $q \in H^{\frac{1}{2}}_{00}(\Lambda)$, we consider its uniform extension to Γ named as $\mathcal{E}_\Gamma q$ and then we consider the harmonic extension $v = \mathcal{E}_\Omega\mathcal{E}_\Gamma q \in H^1_0(\Omega)$. It follows that $\overline{\mathcal{T}}_\Lambda v = q$. Therefore,

$$\sup_{v\in H^1_0(\Omega)} \langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \geq \sup_{q\in H^{\frac{1}{2}}_{00}(\Lambda)} \langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}\,.$$

Moreover, using Lemma 2.1 we obtain

$$\|v\|_{H^1_0(\Omega)} \leq \|\mathcal{E}_\Omega\|\|\mathcal{E}_\Gamma q\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \|\mathcal{E}_\Omega\|\|q\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}.$$

Therefore,

$$\begin{aligned}\sup_{v\in H^1_0(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} \geq \sup_{q\in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} &\geq \|\mathcal{E}_\Omega\|^{-1} \sup_{q\in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}} \\ &= \|\mathcal{E}_\Omega\|^{-1}\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.\end{aligned}$$

4. Finite element approximation. In this section we consider the discretization of Problem 1 and 2 by means of the finite element method. We address two main challenges; first we aim to identify a suitable approximation space for the Lagrange multiplier and to analyze the stability of the discrete saddle point problem; second we aim to derive a stable discretization method that uses indepent and conforming computational meshes for Ω, Γ and Λ. Let us introduce a shape-regular triangulation $\mathcal{T}^\Omega_h$ of Ω and an admissible partition $\mathcal{T}^\Lambda_h$ of Λ. We analyze two different cases: the one in which the 3D mesh is conforming to the interface Γ, namely the set of the intersections of the 3D elements of $\mathcal{T}^\Omega_h$ with Γ is constituted by facets of such elements, and the non conforming case, namely the interface Γ cuts the mesh arbitrarly. The discrete equivalent of (3.1) reads as finding $u_h \in X_h \subset X$, $\lambda_h \in Q_h \subset Q$ s.t.

$$\begin{cases} a(u_h, v_h) + b(v_h, \lambda_h) = c(v_h) & \forall v_h \in X_h \\ b(u_h, \mu_h) = d(\mu_h) & \forall \mu_h \in Q_h. \end{cases} \tag{4.1}$$

Let $B_h : Q'_h \longrightarrow Q_h$ be the operator induced by b such that $\langle B_h v_h, \mu_h\rangle_{Q'_h,Q_h} = b(v_h, \mu_h)$. The well posedness of such problem is governed by the classical inf-sup theory in Banach spaces. The main result is reported below.

Corollary 4.1. *[5, Theorem 2.42]Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be continuous bilinear forms. Problem (4.1) is well-posed if and only if*

$$\exists \alpha_h > 0 : \inf_{u_h\in ker(B_h)} \sup_{v_h\in ker(B_h)} \frac{a(u_h, v_h)}{\|u_h\|_X\|v_h\|_X} \geq \alpha_h \tag{4.2}$$

and

$$\exists \beta_h > 0 : \inf_{\mu_h\in Q_h} \sup_{v_h\in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X\|\mu_h\|_Q} \geq \beta_h. \tag{4.3}$$

This corollary is the discrete counterpart of Theorem 3.1 where at the discrete level condition (4.2) implies both of (3.2). Conversely, (4.3) does not follow from the conformity of the finte element spaces and it must be analysed independetly of (3.3). Let us notice that for both problem 1 and problem 2 the bilinear form $a(\cdot,\cdot)$ is coercive as stated in Lemmas (3.3) and (3.6). Consequently, (4.2) is automatically satisfied, being α_h the coercivity constant.

4.1. $\mathcal{T}_h^\Omega$ conforming to Γ. We first analyze the case in which the 3D mesh is conforming to the interface Γ. With this aim, we define conformity conditions between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$ with Γ. More precisely we require that the intersection of $\mathcal{T}_h^\Omega$ and Γ is made of entire faces of elemets $K \in \mathcal{T}_h^\Omega$. Furthermore, we also set a restriction between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$. We assume that Λ is a piecewise linear manifold. We want that the intersection of Γ with any orthogonal plane to Λ that crosses Λ at the internal nodes of $\mathcal{T}_h^\Lambda$, consists of entire edges of $\mathcal{T}_h^\Omega$. Namely the intersection of Γ with orthogonal planes to Λ is conformal with $\mathcal{T}_h^\Lambda$.

4.1.1. Problem 1. We denote by $X_{h,0}^k(\Omega) \subset H_0^1(\Omega)$, with $k > 0$, the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satisfying homogeneous Dirichlet conditions on the boundary and by $X_{h,0}^k(\Lambda) \subset H_0^1(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ, satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. Problem 1 consists to find $u_h \in X_{h,0}^k(\Omega)$, $u_{\odot h} \in X_{h,0}^k(\Lambda)$, $\lambda_h \in Q_h \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned}
&(u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Lambda v_{\odot h}, \lambda_h \rangle_\Gamma \\
&\qquad = (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X_{h,0}^k(\Omega),\ v_{\odot h} \in X_{h,0}^k(\Lambda)
\end{aligned} \tag{4.4a}$$

$$\langle \mathcal{T}_\Gamma u_h - \mathcal{E}_\Lambda u_{\odot h}, \mu_h \rangle_\Gamma = 0 \quad \forall \mu_h \in Q_h\,, \tag{4.4b}$$

The space Q_h must be suitably chosen such that (4.3) holds. Let Q_h be the trace space of functions running in $X_{h,0}^k(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$.As a result, $Q_h = X_{h,0}^k(\Gamma)$. Therefore we impose homogeneous Dirichlet boundary condition on $\partial\Omega$ also for the Lagrange multiplier. For this choice of Q_h we can prove the well-posedness of the discrete problem, as shown in the following.

LEMMA 4.2. *Let $P_h : H_{00}^{\frac{1}{2}}(\Gamma) \longrightarrow Q_h$ be the orthogonal projection operator defined for any $v \in H_{00}^{\frac{1}{2}}(\Gamma)$ by*

$$(P_h v, \psi_h)_\Gamma = (v, \psi_h)_\Gamma \qquad \forall \psi_h \in Q_h.$$

Then, P_h is continuous on $H_{00}^{\frac{1}{2}}(\Gamma)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|v\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{4.5}$$

where C is a positive constant independent of h.

Proof. We prove that P_h is continuous on $L^2(\Gamma)$ and on $H_0^1(\Gamma)$ following [5, Section 1.6.3]. Then, the inequality (4.5) can be dirived by Hilbertian interpolation. For the L^2-continuity, we exploit the fact that, from the definition of P_h,

$$(v - P_h v, P_h v)_\Gamma = 0.$$

Therefore, by Pythagoras identity,

$$\|v\|_{L^2(\Gamma)}^2 = \|v - P_h v\|_{L^2(\Gamma)}^2 + \|P_h v\|_{L^2(\Gamma)}^2 \geq \|P_h v\|_{L^2(\Gamma)}^2.$$

Let us now consider $v \in H_0^1(\Gamma)$.The Scott-Zhang interpolation operator SZ_h from $H_0^1(\Gamma)$ to Q_h satisfies the following inequalities,

$$\|SZ_h v\|_{H^1(\Gamma)} \leq C_1 \|v\|_{H^1(\Gamma)} \tag{4.6}$$

$$\|v - SZ_h v\|_{L^2(\Gamma)} \leq C_2 h \|v\|_{H^1(\Gamma)}. \tag{4.7}$$

Therefore, using (4.6),

$$\begin{aligned}
\|\nabla P_h v\|_{L^2(\Gamma)} &\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + \|\nabla SZ_h v\|_{L^2(\Gamma)} \\
&\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)}
\end{aligned}$$

and by using the inverse inequality we obtain

$$
\begin{aligned}
\|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} &\leq \frac{C_3}{h}\|P_h v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&= \frac{C_3}{h}\|P_h(v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq \ \text{(Stability of } P_h \text{ in } L^2\text{)}\ \frac{C_3}{h}\|v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq \ \text{(using (4.7))}\ \frac{C_3}{h} C_2 h\|v\|_{H^1(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq (C_2C_3 + C_1)\|v\|_{H^1(\Gamma)},
\end{aligned}
\qquad \square
$$

from which we obtain the continuity in $H^1_0(\Gamma)$.

LEMMA 4.3. *There exists a constant $\gamma > 0$ such that for any $\mu_h \in Q_h$*

$$
\sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \geq \gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}.
$$

Proof. Let μ_h be in Q_h. From the continuous case, in particular from (3.4), we have

$$
\|\mathcal{E}_\Omega\|^{-1}\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h\rangle}{\|v\|_{H^1(\Omega)}}
$$

and by the trace inequality $\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_T\|v\|_{H^1(\Omega)}$ (see [1, 7.56]), we obtain

$$
\sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h\rangle}{\|v\|_{H^1(\Omega)}} \leq C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h\rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}}.
$$

By the definition of P_h and (4.5)

$$
\begin{aligned}
C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h\rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} &= C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle P_h(\mathcal{T}_\Gamma v), \mu_h\rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} \\
&\leq C_T C \sup_{v \in H^1_0(\Omega)} \frac{\langle P_h(\mathcal{T}_\Gamma v), \mu_h\rangle}{\|P_h(\mathcal{T}_\Gamma v)\|_{H^{\frac{1}{2}}(\Gamma)}} \\
&= C_T C \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}}.
\end{aligned}
\qquad \square
$$

THEOREM 4.4 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists \beta_{h,1} > 0$ s.t.*

$$
\inf_{\mu_h \in Q_h} \sup_{\substack{v_h \in X^k_{h,0}(\Omega),\\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Gamma v_{\odot h}, \mu_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||\,\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \beta_{h,1}. \tag{4.8}
$$

Proof. Let $\mu_h \in Q_h$. As in the continuos case, we choose $v_{\odot h} = 0$ and we have

$$
\sup_{\substack{v_h \in X^k_{h,0}(\Omega),\\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Gamma v_{\odot h}, \mu_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.
$$

Therefore, we want to prove that there exists $\beta_{h,1}$ such that

$$
\sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,1}\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall \mu_h \in Q_h.
$$

Using Lemma 4.3 and the boundedness of the armonic extension operator $\mathcal{E}_\Omega$ from $H^{\frac{1}{2}}_{00}(\Gamma)$ to $H^1_0(\Omega)$ introduced in the previous section, we have

$$\gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}_\Omega\| \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}}.$$

Let $R_h : H^1_0(\Omega) \to X^k_{h,0}(\Omega)$ be a quasi interpolation operator (such as the Scott-Zhang operator) satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq C_R\|v\|_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega).$$

Therefore, we obtain

$$\|\mathcal{E}_\Omega\| \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}} \leq \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}}$$

and we have

$$\begin{aligned}(4.9)\quad \gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}_\Omega q_h\|_{H^1(\Gamma)}} \\ &= \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle \mathcal{T}_\Gamma R_h\mathcal{E}_\Omega q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}} \leq \|\mathcal{E}_\Omega\| C_R \sup_{v_h\in X_{h,k}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h,\mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.\end{aligned}$$

Therefore the inf-sup condition (4.8) holds with $\beta_{h,1} = \gamma\|\mathcal{E}_\Omega\|^{-1}C_R^{-1}$. We notice that in (4.9) we exploit the fact that the operator $\mathcal{T}_\Gamma R_h\mathcal{E}_\Omega$ coincides with the identity on the space Q_h, thanks to the conformity of $\mathcal{T}^\Omega_h$ to the interface Γ. ◻

4.1.2. Problem 2. This problem requires to find $u_h \in X^k_{h,0}(\Omega)$, $u_{\odot h} \in X^k_{h,0}(\Lambda)$, $\lambda_{\odot h} \in Q_h \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}(4.10a)\qquad & (u_h,v_h)_{H^1(\Omega)} + (u_{\odot h},v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle\overline{\mathcal{T}}_\Lambda v_h - v_{\odot h},\lambda_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} \\ &\quad = (f,v_h)_{L^2(\Omega)} + (\overline{\overline{g}},v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h}\in X_h(\Lambda) \\ (4.10b)\qquad & \langle\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h},\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} = 0 \quad \forall \mu_{\odot h} \in Q_h\,.\end{aligned}$$

We choose $Q_h = X^k_{h,0}(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda\cap\partial\Omega$ also for the Lagrange multiplier. With this choice for Q_h, we can prove the well-posedness of the discrete problem. In particular, following the same steps as for Problem 1, we can prove the following results.

Lemma 4.5. *Let $P_h : H^{\frac{1}{2}}_{00}(\Lambda) \longrightarrow Q_h$ be the orthogonal projection operator defined for any $v\in H^{\frac{1}{2}}_{00}(\Lambda)$ by*

$$(P_h v,\psi)_{\Lambda,|\partial\mathcal{D}|} = (v,\psi)_{\Lambda,|\partial\mathcal{D}|} \qquad \forall\psi\in Q_h.$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \leq C\|v\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|},$$

where C is a positive constant independent of h.

Lemma 4.6. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h\in Q_h} \frac{\langle q_h,\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \geq \gamma\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall\mu_{\odot h}\in Q_h.$$

The proofs are equivalent ot the ones of Lemmas 4.5 and 4.3.

THEOREM 4.7 (Discrete inf-sup). *The inequality* (4.3) *holds, namely* $\exists \beta_{h,2} > 0$ *s.t.*

$$\inf_{\mu_h \in Q_h} \sup_{\substack{v_h \in X^k_{h,0}(\Omega), \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{|||[v_h, v_{\odot h}]||| \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \beta_{h,2}. \tag{4.11}$$

Proof. Let $\mu_{\odot h}$ be arbitrarly chosen in Q_h. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists $\beta_{h,2}$ such that

$$\sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,2} \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_{\odot h} \in Q_h.$$

From Lemma 2.1 and its corollaries, for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{E}_\Gamma w\|_{H^{\frac{1}{2}}(\Gamma)} = \|w\|_{H^{\frac{1}{2}}(\Lambda), |\partial \mathcal{D}|}.$$

Consequently, from Lemma 4.6, using again the extension operator $\mathcal{E}_\Omega$ from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^k_{h,0}(\Omega)$, we obtain

$$\begin{aligned} \gamma \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda), |\partial \mathcal{D}|}} \\ &= \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|\mathcal{E}_\Gamma q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}_\Omega\| \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|\mathcal{E}_\Omega \mathcal{E}_\Gamma q_h\|_{H^1(\Omega)}} \\ &\leq \|\mathcal{E}_\Omega\| C_R \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|R_h \mathcal{E}_\Omega \mathcal{E}_\Gamma q_h\|_{H^1(\Omega)}} \\ &= \|\mathcal{E}_\Omega\| C_R \sup_{q_h \in Q_h} \frac{\langle \overline{\mathcal{T}}_\Lambda R_h \mathcal{E}_\Omega \mathcal{E}_\Gamma q_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|R_h \mathcal{E}_\Omega \mathcal{E}_\Gamma w_h\|_{H^1(\Omega)}} \\ &\leq \|\mathcal{E}_\Omega\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}}. \end{aligned} \tag{4.12}$$

Also in this case to prove the disrete inf-sup condition we exploit the conformity of the meshes on Ω, Γ and Λ and the fact that the operator $\overline{\mathcal{T}}_\Lambda R_h \mathcal{E}_\Omega \mathcal{E}_\Gamma$ coincides with the identity if applied to functions in Q_h. ∎

4.2. $\mathcal{T}^\Omega_h$ non conforming to Γ. We analyze now the case in which the elements of the 3D mesh $\mathcal{T}^\Omega_h$ cut the interface Γ. The formulation of Problem 2 is more suitable for this purpose. Therefore we focus on the analysis of Problem 2.

4.2.1. Problem 2. We consider for the solutions u_h and $u_{\odot h}$ the spaces $X^1_{h,0}(\Omega)$ and $X^1_{h,0}(\Lambda)$. Notice that in this case we assume that the mesh sizes of the 3D mesh $\mathcal{T}^\Omega_h$ and the 1D mesh $\mathcal{T}^\Lambda_{h'}$ are different, in particular we assume the 1D mesh is finer. Concerning the multiplier space, let $\mathcal{G}_h = \{K \in \mathcal{T}^\Omega_h : K \cap \Lambda \neq \emptyset\}$, namely $\mathcal{G}_h$ is the set of the 3D elements which intersect Λ. Then we define $Q_h = \{\lambda_{\odot h} : \lambda_{\odot h} \in P^0(K) \, \forall K \in \mathcal{G}_h\}$. We notice that we are extending the multiplier to the 3D elements so that we do not need to consider the 1D mesh given by the intersection of the elements of $\mathcal{T}^\Omega_h$ with $\mathcal{T}^\Lambda_{h'}$. With this choice of multipliers the problem is not inf-sup stable, therefore the idea is to add a stabilization term $s(\lambda_{\odot h}, \mu_{\odot h})$ to (4.10a) following the approach introduced in [4]. The stabilization operator is based on a new multiplier space L_H for which the discrete inf-sup condition is fulfilled and a projection operator $\pi_L : Q_h \to L_H$ such that for any $[v, v_\odot] \in X$

$$b([v, v_\odot], \lambda_{\odot h} - \pi_L \lambda_{\odot h}) \lesssim |||[v, v_\odot]||| \|\lambda_{\odot h} - \pi_L \lambda_{\odot h}\|_{L_H}, \tag{4.13}$$

where for X and $|||[\cdot, \cdot]|||$ we use the definitions of section 3.2 and $\|\cdot\|_{L_H}$ denotes a suitable discrete norm on L_H. Then, the following relaxed form of the inf-sup condition is satisfied, Lemma 4.8. *[4, Lemma2.1] Under the previous assumption,* $\forall \lambda_{\odot h} \in Q_h$,

$$\|\lambda_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \lesssim \sup_{\substack{v_h \in X^1_{h,0}(\Omega)\\ v_{\odot h} \in X^1_{h,0}(\Lambda)}} \frac{b([v_h, v_{\odot h}], \lambda_{\odot h})}{|||[v_h, v_{\odot h}]|||} + \|\lambda_{\odot h} - \pi_L \lambda_{\odot h}\|_{L_H}.$$

Based on this projection operator, we build the stabilization term $s(\lambda_{\odot h}, \mu_{\odot h})$ satisfying

$$\|\lambda_{\odot h} - \pi_L \lambda_{\odot h}\|_{L_H} \lesssim s(\lambda_{\odot h}, \lambda_{\odot h})$$

and prove that $\forall [u_h, u_{\odot h}]$, there exists a function $\xi_h \in Q_h$ depending on $[u_h, u_{\odot h}]$, namely $\xi_h = \xi_h([u_h, u_{\odot h}])$, s.t.

$$a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h) \geq \alpha_\xi |||[u_h, u_{\odot h}]|||_{X^1_{h,0}(\Omega) \times X^1_{h,0}(\Lambda)}, \tag{4.14}$$

$$(s(\xi_h, \xi_h))^{\frac{1}{2}} \leq c_s |||[u_h, u_{\odot h}]|||_{X^1_{h,0}(\Omega) \times X^1_{h,0}(\Lambda)}, \tag{4.15}$$

being $|||[\cdot\,,\,\cdot]|||_{X^1_{h,0}(\Omega) \times X^1_{h,0}(\Lambda)}$ a suitable discrete norm. Then, the final objective of this section is to prove that the stabilized version of the 3D-1D-1D problem, precisely,

$$\begin{aligned} a([u_h, u_{\odot h}], [v_h, v_{\odot h}]) + b([v_h, v_{\odot h}], \lambda_{\odot h}) + \\ b([u_h, u_{\odot h}], \mu_{\odot h}) - s(\lambda_{\odot h}, \mu_{\odot h}) = c(v_h) + d(\mu_{\odot h}) \\ \forall [v_h, v_{\odot h}] \in X^1_{h,0}(\Omega) \times X^1_{h,0}(\Lambda),\ \forall \mu_{\odot_h} \in Q_h \end{aligned} \tag{4.16}$$

is well posed. In what follows we discuss the definition of the space L_H, the projector π_L and the stabilization operator such that lemmas 2.1 and 2.3 of [4] can be applied to problem (4.16). We recall that in the case of Problem 2,

$$b([u_h, u_{\odot h}], \lambda_{\odot h}) = \left(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \lambda_{\odot h}\right)_{\Lambda, |\partial \mathcal{D}|}.$$

The construction of the inf-sup stable space L_H is based on macro elements of diameter H, where H is sufficiently large. In particular, we assume that there exists positive constants c_h and c_H such that $c_h h \leq H \leq c_H^{-1} h$. The space is constructed assembling the 3D elements of $\mathcal{G}_h$ into macro patches ω_j such that $H \leq |\omega_j \cap \Lambda| \leq cH$, with $c \geq 1$. Let M_j be the number of elements of the patch ω_j, namely, $\omega_j = \cup_{i=0}^{M_j} K_i$, where $K_i \in \mathcal{G}_h$, M_j is uniformly bounded in j by some $M \in \mathbb{N}$ and $H = \min_j |\omega_j \cap \Lambda|$. We assume that the interiors of the patches ω_j are disjoint. We define

$$L_H = \left\{ l_{\odot H} : l_{\odot H} \in P^0(\omega_j) \, \forall j \right\}.$$

Moreover, we associate to each patch ω_j a shape-regular extended patch, still denoted by ω_j for notational simplicity, which is built adding to ω_j a sufficient number of elements of $\mathcal{T}_h^\Omega$ and we assume that the interiors of the new extended patches ω_j are still disjoint (see Figure 4.1). Here we are using the classical definition of shape-regularity, see for example [5], namely there exist a constant $C > 0$ such that for any ω_j, $\frac{\tilde{\rho}_j}{\bar{\rho}_j} \leq C$, being $\tilde{\rho}_j$ the diameter of ω_j and $\bar{\rho}_j$ the diameter of the largest ball that can be inscribed in ω_j. The extended patches ω_j are built such that they fulfill the conditions $\mathrm{meas}(\omega_j) = \mathcal{O}(H^3)$ and $\mathrm{diam}(\Gamma_{\omega_j \cap \Lambda} \cap \omega_j) = \mathcal{O}(H)$, where $\Gamma_{\omega_j \cap \Lambda}$ is the portion of Γ with centerline $\omega_j \cap \Lambda$. See Figure 4.2 for a representation in the simple case in which ω_j is composed just by one tetrahedron. The latter assumption is required to ensure that the intersection of $\Gamma_{\omega_j \cap \Lambda}$ and ω_j is not too small and it will be needed later on to prove the inf-sup stability of the space L_H in Lemma 4.9. We introduce the following discrete norms

$$\|\, \lambda \,\|_{\pm \frac{1}{2}, h, \Lambda} = \|h^{\mp \frac{1}{2}} \lambda\|_{L^2(\Lambda)},$$

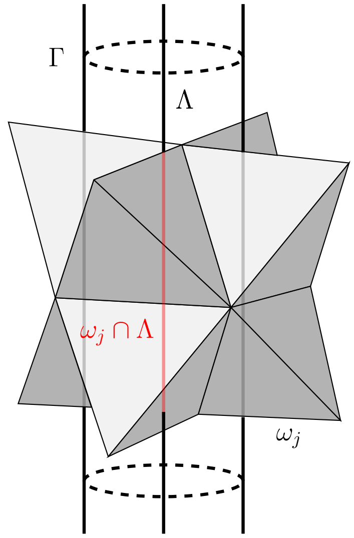


FIGURE 4.1. *Extended patches ω_j.*

recalling that h is the mesh size of $\mathcal{T}_h^\Omega$. We equipe the space $X_h(\Omega) \times X_{h'}(\Lambda)$ with the discrete norm

$$|||[u_h, u_{\odot h}]|||^2_{X_h(\Omega)\times X_{h'}(\Lambda)} = \|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|},$$

and the space L_H with the norm

$$\|l_{\odot H}\|_{L_H} = \|l_{\odot H}\|_{-\frac{1}{2},h,\Lambda}.$$

As shown in [4, Section III], we can always choose π_L as the L_2 orthogonal projection operator from Λ_h to L_H in order to satisfy (4.13) and then in practice replace it with any interpolation $\tilde{\pi}_L$ of Λ_h in L_H. In particular, we define $\forall \lambda_h \in \Lambda_h$,

$$\tilde{\pi}_L \lambda_{\odot_{|\omega_j}} = M^{-1} \sum_{i:\ K_i \in \mathcal{G}_h,\ K_i \cap \omega_j \neq \emptyset} \lambda_{\odot|K_i} \qquad \text{for all } \omega_j,$$

being M the cardinality of the set $\{i : \ K_i \in \mathcal{G}_h,\ K_i \cap \omega_j \neq \emptyset\}$. These choices lead to the following stabilization

$$s(\lambda_{\odot h}, \mu_{\odot h}) = \sum_{K\in\mathcal{G}_h} \int_{\partial K\setminus\partial\mathcal{G}_h} h[\![\lambda_{\odot h}]\!][\![\mu_{\odot h}]\!], \tag{4.17}$$

being $[\![\lambda_{\odot h}]\!]$ the jump of $\lambda_{\odot h}$ across the internal faces of $\mathcal{G}_h$.

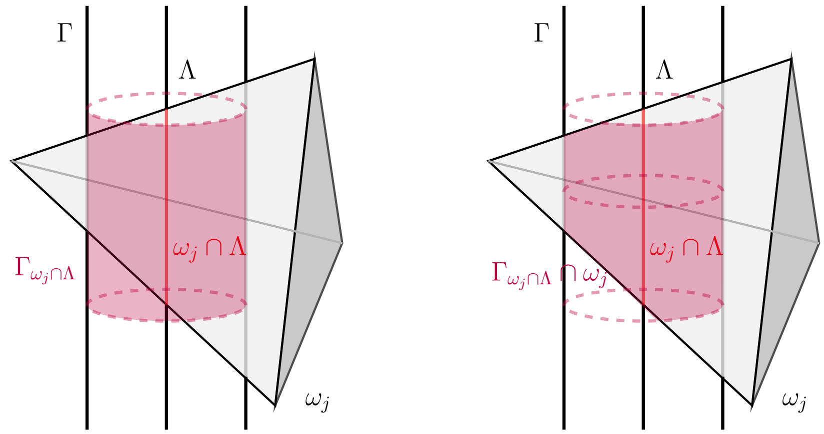
Thanks to the shape regularity of these extended patches, we have that the following discrete trace and Poincar-type inequalites hold. More precisely, for any function $v \in H^1(\omega_j)$,

$$\|\mathcal{T}_\Gamma v\|_{L^2(\Gamma\cap\omega_j)} \lesssim H^{-\frac{1}{2}} \|v\|_{L^2(\omega_j)} \tag{4.18}$$

$$\|v - \mathcal{E}_{\omega_j} \pi_L \overline{\mathcal{T}}_\Lambda v\|_{L^2(\omega_j)} \leq c_P H \|\nabla v\|_{L^2(\omega_j)}. \tag{4.19}$$

For any $w \in L^2(\Lambda)$ we have that $\mathcal{E}_{\omega_j}\pi_L w \in L_H$, where with a little abuse of notation we denote as π_L the operator

$$\pi_L w_{|\omega_j\cap\Lambda} = \frac{1}{|\Gamma_{\omega_j\cap\Lambda}|} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| w \quad \forall j, \tag{4.20}$$



(a) $\Gamma_{\omega_j\cap\Lambda}$ (b) $\Gamma_{\omega_j\cap\Lambda} \cap \omega_j$

FIGURE 4.2. *$\Gamma_{\omega_j\cap\Lambda}$, the portion of Γ generated by $\omega_j \cap \Lambda$ in (a) and the intersection between $\Gamma_{\omega_j\cap\Lambda}$ and ω_j in (b). Here for simplicity ω_j is represented as a single tetrahedron but actually it is a collection of tetrahedra as shown in Figure 4.1.*

whereas the operator $\mathcal{E}_{\omega_j}$ simply extends the constant $\pi_L \overline{\mathcal{T}}_\Lambda v_{|\omega_j\cap\Lambda}$ to ω_j. Moreover $\forall u_h \in X^1_{h,0}(\Omega)$ we have the following average inequality which is a consequence of the definition of $\overline{\mathcal{T}}_\Lambda$ and Jensen inequality,

$$\sum_j \|\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} = \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda u_h^2 = \int_\Lambda |\partial\mathcal{D}| \left(\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}} \mathcal{T}_\Gamma u_h\right)^2 \leq \int_\Lambda \int_{\partial\mathcal{D}} (\mathcal{T}_\Gamma u_h)^2$$
$$= \int_\Gamma (\mathcal{T}_\Gamma u_h)^2 = \sum_j \int_{\omega_j\cap\Gamma} (\mathcal{T}_\Gamma u_h)^2 = \sum_j \|\mathcal{T}_\Gamma u_h\|^2_{L^2(\omega_j\cap\Gamma)}. \tag{4.21}$$

To prove that (4.16) is well posed, we proceed through three main steps. The first step consists of showing that the space L_H is inf-sup stable. This result is addressed in the following lemma.

LEMMA 4.9. *The space L_H is inf-sup stable, namely $\forall l_{\odot H} \in L_H$, $\exists \beta > 0$ s.t.*

$$\sup_{\substack{v_h \in X^1_{h,0}(\Omega),\\ v_{\odot h} \in X^1_{h',0}(\Lambda)}} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, l_{\odot H}\right)_{\Lambda,|\partial\mathcal{D}|}}{\|\|[v_h, v_{\odot h}]\|\|} \geq \beta \|l_{\odot H}\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

Proof. As in the continuous case, we can choose $v_{\odot h} = 0$ and we prove that

$$\sup_{v_h \in X^1_{h,0}(\Omega)} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h, l_{\odot H}\right)_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta \|l_{\odot H}\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

Proving the last inequality it is equivalent to find the Fortin operator $\pi_F : H^1_0(\Omega) \to X^1_{h,0}(\Omega)$, such that

$$\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, l_{\odot H}\right)_{\Lambda,|\partial\mathcal{D}|} = 0, \quad \forall v \in H^1_0(\Omega),\, l_{\odot H} \in L_H$$

and

$$\|\pi_F v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}. \tag{4.22}$$

We define

$$\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j}$$

where $I_h : H^1(\Omega) \to X^1_{h,0}$ denotes an $H^1(\Omega)$-stable interpolant and $\varphi_j \in X^1_{h,0}(\Omega)$ is such that $\mathrm{supp}(\varphi_j) \subset \omega_j$, $\mathrm{supp}(\mathcal{T}_\Gamma \varphi_j) \subset \Gamma_{\omega_j \cap \Lambda} \cap \omega_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j = \mathcal{O}(H) \text{ and } \|\nabla \varphi_j\|_{L^2(\omega_j)} = \mathcal{O}(1). \tag{4.23}$$

We notice that $\mathrm{supp}(\mathcal{T}_\Gamma \varphi_j) \subset \Gamma_{\omega_j \cap \Lambda} \cap \omega_j$ ensures that $\overline{\mathcal{T}}_\Lambda \varphi_j \subset \omega_j \cap \Lambda$. Therefore, since for construction the interiors of $\omega_j \cap \Lambda$ are disjoint and $\varphi_j = 0$ on $\partial\omega_j$, the functions $\overline{\mathcal{T}}_\Lambda \varphi_j \,\forall j$ have all disjoint supports. This construction is always possible since $\mathrm{meas}(\omega_j) = \mathcal{O}(H^3)$ and $\mathrm{diam}(\Gamma_{\omega_j \cap \Lambda} \cap \omega_j) = \mathcal{O}(H)$, provided H is sufficiently larger that h. Indeed, this guarantees that the functions φ_j and their traces $\mathcal{T}_\Gamma \varphi_j$ have a sufficiently large support so that they can be built in order to satisfy (4.23). Then we have

$$\begin{aligned}
\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, l_{\odot H}\right)_{\Lambda, |\partial\mathcal{D}|} &= \sum_j \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \sum_i \alpha_i \overline{\mathcal{T}}_\Lambda \varphi_i\right] l_{\odot H} \\
(\mathrm{supp}(\overline{\mathcal{T}}_\Lambda \varphi_i) \subset \omega_i \cap \Lambda \,\forall i) &= \sum_j \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \alpha_j \overline{\mathcal{T}}_\Lambda \varphi_j\right] l_{\odot H} \\
= \sum_j \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v) l_{\odot H} &- \frac{\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j} \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j l_{\odot H} \\
&(\text{using } l_{\odot H} \text{ constant on } \omega_j \cap \Lambda) = 0.
\end{aligned}$$

Concerning the continuity of π_F, we exploit the assumptions that the interiors of ω_j are disjoint and $\mathrm{supp}(\varphi_j) \subset \omega_j$ and we have

$$\begin{aligned}
\|\nabla \pi_F v\|_{L^2(\Omega)} &\le \|\nabla I_h v\|_{L^2(\Omega)} + \left(\sum_j \alpha_j^2 \|\nabla \varphi_j\|^2_{L^2(\omega_j)}\right)^{\frac{1}{2}} \\
(\text{stability of } I_h) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \left(\sum_j \alpha_j^2 \|\nabla \varphi_j\|^2_{L^2(\omega_j)}\right)^{\frac{1}{2}}
\end{aligned}$$

and for the second term

$$
\begin{aligned}
&\sum_j \alpha_j^2 \|\nabla\varphi_j\|^2_{L^2(\omega_j)} \leq \\
&\big(\text{using } \|\nabla\varphi_j\|_{L^2(\omega_j)} = \mathcal{O}(1)\big) \lesssim \sum_j \frac{\Big(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\Big)^2}{\Big(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j\Big)^2} \\
&\left(\text{since } \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda\varphi_j = O(H)\right) \lesssim \frac{1}{H^2}\sum_j \left(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\right)^2 \\
&(\text{Jensen}) \lesssim \frac{1}{H^2}\sum_j |\omega_j\cap\Lambda| \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|^2(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)^2 \\
&(\text{being } |\omega_j\cap\Lambda| \leq cH) \lesssim \frac{1}{H}\sum_j \|\overline{\mathcal{T}}_\Lambda(v - I_h v)\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&(\text{average inequality (4.21)}) \lesssim \frac{1}{H}\sum_j \|\mathcal{T}_\Gamma(v - I_h v)\|^2_{L^2(\omega_j\cap\Gamma)} \\
&(\text{trace inequality (4.18)}) \lesssim \frac{1}{H^2}\sum_j \|v - I_h v\|^2_{L^2(\omega_j)} \lesssim \frac{1}{H^2}\|v - I_h v\|^2_{L^2(\Omega)} \\
&(\text{approximation properties of } I_h) \lesssim \|\nabla v\|^2_{L^2(\Omega)}
\end{aligned}
$$

and the continuity of π_F follows. We notice that the constant in the inequality (4.22) is independent of how Λ is cut by the elements of the mesh $\mathcal{T}_h^\Omega$. ∎

The second step of the stability analysis of the stabilized 3D-1D-1D scheme, namely (4.16), consists of showing that (4.14) and (4.15) are satisfied. In particular, we consider the projection operator defined in (4.20) and the stabilization operator (4.17). Also, the function $\xi_h([u_h, u_{\odot h}])$ is defined as follows

$$
\xi_{h|\omega_j\cap\Lambda} = \delta\frac{1}{H}\pi_L(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})_{|\omega_j\cap\Lambda}.
$$

Then the following result holds true.

Lemma 4.10. *The inequalities* (4.14) *and* (4.15) *are satisfied (under the previous assumptions).*

Proof. Concerning the coercivity property (4.14), we have to show that $\forall[u_h, u_{\odot h}]$, there esists $\xi_h \in Q_h$ s.t.

$$
(u_h, u_h)_{H^1(\Omega)} + (u_{\odot h}, u_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \xi_h)_{\Lambda,|\partial\mathcal{D}|} \geq \alpha_\xi |||[u_h, u_{\odot h}]|||^2_{X_h(\Omega)\times X_{h'}(\Lambda)}.
$$

Using the definitions of π_L and $\xi_h([u_h, u_{\odot h}])$ previously presented and recalling that $\mathcal{E}_{\omega_j}\xi_h \in L_H \subset Q_h$, we obtain

$$
\begin{aligned}
\left(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \xi_h\right)_{\Lambda,|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\xi_h \\
(\text{definition of } \xi_h) &= \delta\frac{1}{H}\sum_j \pi_L(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}) \\
\left(|\Gamma_{\omega_j\cap\Lambda}| = \int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|\right) &= \delta\frac{1}{H}\sum_j\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\pi_L(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}))^2 \\
(\text{orthogonality of } \pi_L) &= -\delta\frac{1}{H}\|(\pi_L-\mathcal{I})(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} + \delta\frac{1}{H}\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\geq -2\delta\frac{1}{H}\sum_j\|(\pi_L-\mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} - 2\delta\frac{1}{H}\sum_j\|(\pi_L-\mathcal{I})u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\quad + \delta\frac{1}{H}\sum_j\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|}.
\end{aligned}
$$

For the first term, we introduce the uniform extension operator $\mathcal{E}_\Gamma$ from $\omega_j\cap\Lambda$ to the subset of Γ with centerline $\omega_j\cap\Lambda$ and we have

$$
\begin{aligned}
\sum_j\|(\pi_L-\mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\pi_L\overline{\mathcal{T}}_\Lambda u_h - \overline{\mathcal{T}}_\Lambda u_h)^2 \\
(\text{Average inequality (4.21)}) &\leq \sum_j\int_{\omega_j\cap\Gamma}(\mathcal{E}_\Gamma\pi_L\overline{\mathcal{T}}_\Lambda u_h - \mathcal{T}_\Gamma u_h)^2 \\
(\text{trace inequality (4.18)}) &\leq \sum_j\frac{1}{H}\int_{\omega_j}(\mathcal{E}_{\omega_j}\pi_L\overline{\mathcal{T}}_\Lambda u_h - u_h)^2 \\
(\text{Poincare, see [5, Corollary B.65]}) &\leq \sum_j Hc_P^2\|\nabla u_h\|^2_{L^2(\omega_j)}.
\end{aligned}
$$

For the second term we have

$$
\begin{aligned}
\sum_j\|(\pi_L-\mathcal{I})u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\pi_L u_{\odot h} - u_{\odot h})^2 \\
(\text{Poincare, [5, Corollary B.65]}) &\lesssim \sum_j H^2c_P^2\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\nabla u_{\odot h})^2 \\
(\text{since } H \text{ is fixed, we can find a constant s.t. } H|\partial\mathcal{D}|\lesssim|\mathcal{D}|) &\lesssim \sum_j Hc_P^2\int_{\omega_j\cap\Lambda}|\mathcal{D}|(\nabla u_{\odot h})^2 \\
&\lesssim \sum_j Hc_P^2\|\nabla u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\mathcal{D}|}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&a([u_h,u_{\odot h}],[u_h,u_{\odot h}]) + b([u_h,u_{\odot h}],\xi_h([u_h,u_{\odot h}])) \geq \\
&\qquad (1-2\delta c_P^2)\|\nabla u_h\|^2_{L^2(\Omega)} + (1-2\delta c_P^2)\|\nabla u_{\odot h}\|^2_{L^2(\Lambda),|\mathcal{D}|} + \delta c_H\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}
\end{aligned}
$$

and choosing $\delta = \frac{1}{4c_P^2}$ we obtain the desired inequality.

Concerning the stability inequality (4.15), the proof is analogous to the one in [4]. ∎

REMARK 4.1. *We notice that if we choose $Q_h = X^1_{h',0}(\Lambda)$, the constant in the inf-sup inequality (4.3) depends on the mesh size h'. Indeed,*

$$(4.24)\qquad \sup_{\substack{v_h\in X^1_{h,0}(\Omega),\\ v_{\odot h}\in X^1_{h',0}(\Lambda)}} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, l_{\odot H}\right)_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_{\odot h}\in X^1_{h',0}(\Lambda)} \frac{(-v_{\odot h}, l_{\odot H})_{\Lambda,|\partial\mathcal{D}|}}{\|v_{\odot h}\|_{H^1(\Lambda)}} \geq \frac{\|l_{\odot H}\|^2_{L^2(\Lambda)}}{\|v_{\odot h}\|_{H^1(\Lambda)}}$$

$$(\textit{inverse inequality}) \geq \frac{{h'}^2}{c_I}\|l_{\odot H}\|_{L^2(\Lambda)} \geq \frac{{h'}^2}{c_I}\|l_{\odot H}\|_{H^{-\frac{1}{2}},(\Lambda)}$$

being c_I the constant in the inverse inequality

$$\|l_{\odot H}\|_{H^1(\Lambda)} \leq \frac{c_I}{{h'}^2}\|l_{\odot H}\|_{L^2(\Lambda)}.$$

5. A benchmark problem with analytical solution. Let $\Omega = [0,1]^3$, $\Lambda = \{x = \frac{1}{2}\}\times\{y = \frac{1}{2}\}\times[0,1]$ and $\Sigma = [\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4},\frac{3}{4}]\times[0,1]$. Finally we let $\partial\mathcal{D}$ be the cross section of the virtual interface $\Gamma = \partial\Sigma$. As a benchmark for the two formulations we consider the following coupled problems

$$(5.1\text{a})\qquad -\Delta u = f \quad \text{in } \Omega$$

$$(5.1\text{b})\qquad -d^2_{zz}u_\odot = g \quad \text{on } \Lambda$$

$$(5.1\text{c})\qquad u = u_b \quad \text{on } \partial\Omega,$$

where for formulation (2.12) the mix-dimensional coupling constraint reads

$$(5.2)\qquad \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1 \quad \text{on } \Gamma,$$

while for (2.13) we set

$$(5.3)\qquad \overline{u} - u_\odot = q_2 \quad \text{on } \Lambda.$$

In (5.1)-(5.3) the right-hand sides shall be defined as

$$f = 8\pi^2\sin(2\pi x)\sin(2\pi y),\qquad g = \pi^2\sin(\pi z),\quad u_b = \sin(2\pi x)\sin(2\pi y),$$
$$q_1 = \sin(2\pi x)\sin(2\pi y) - \sin(\pi z),\quad q_2 = -\sin(\pi z).$$

The exact solution of (5.1), regardless of the coupling constraint, is given by

$$(5.4)\qquad u = \sin(2\pi x)\sin(2\pi y)$$

$$(5.5)\qquad u_\odot = \sin(\pi z).$$

Let us notice that $u_\odot$ satisfies homogeneous Dirichlet conditions at the boundary of Λ. Moreover, the solution (5.4)-(5.5) satisfies on Γ the relation

$$(5.6)\qquad L = \nabla u\cdot\mathbf{n}_\oplus = d_z u_\odot n_{\oplus,z} = 0,$$

with $n_{\oplus,z}$ the $z-$component of the normal unit vector to Γ.

We prove that (5.1) is solution of (2.13) in the simplified case in which the starting 3D-3D problem is

$$(5.7\text{a})\qquad -\Delta u_\oplus = f \qquad \text{in } \Omega_\oplus,$$

$$(5.7\text{b})\qquad -\Delta u_\ominus = g \qquad \text{in } \Sigma,$$

$$(5.7\text{c})\qquad -\nabla u_\ominus\cdot\boldsymbol{n}_\ominus = -\nabla u_\oplus\cdot\boldsymbol{n}_\ominus \qquad \text{on } \Gamma,$$

$$(5.7\text{d})\qquad u_\ominus - u_\oplus = q_i \qquad \text{on } \Gamma,$$

$$(5.7\text{e})\qquad u_\oplus = h \qquad \text{on } \partial\Omega.$$

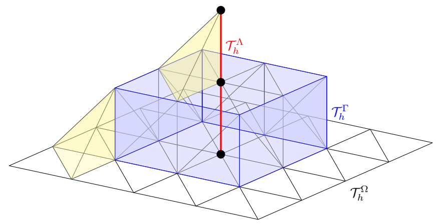


FIGURE 5.1. Λ *and* Γ *conforming discretization of* Ω *used for* (5.8) *and* (5.9).

h^{-1}	$\lVert u - u_h \rVert_{1,\Omega}$	$\lVert u_\odot - u_{\odot h} \rVert_{1,\Lambda}$	$\lVert \lambda - \lambda_h \rVert_{-\frac{1}{2},\Gamma}$	$\lVert \lambda - \lambda_h \rVert_{0,\Gamma}$
4	3.4E0(−)	5.3E-1(−)	2.9E0(−)	8.7E0(−)
8	1.7E0(0.99)	2.6E-1(1.06)	6.1E-1(2.25)	1.9E0(2.21)
16	8.7E-1(0.99)	1.3E-1(1.02)	1.4E-1(2.13)	4.7E-1(1.99)
32	4.4E-1(1.00)	6.3E-2(1.00)	3.4E-2(2.03)	1.3E-1(1.80)
64	2.2E-1(1.00)	3.1E-2(1.00)	8.6E-3(2.00)	4.2E-2(1.68)

h^{-1}	$\lVert u - u_h \rVert_{1,\Omega}$	$\lVert u_\odot - u_{\odot,h'} \rVert_{1,\Lambda}$	$\lVert \lambda_\odot - \lambda_{\odot h} \rVert_{-\frac{1}{2},\Lambda}$	$\lVert \lambda_\odot - \lambda_{\odot h} \rVert_{0,\Lambda}$
4	3.1E0(−)	5.4E-1(−)	4.4E-2(−)	7.8E-2(−)
8	1.7E0(0.87)	2.6E-1(1.06)	1.1E-2(2.01)	1.9E-2(2.01)
16	8.6E-1(0.96)	1.3E-1(1.02)	2.7E-3(2.01)	4.8E-3(2.02)
32	4.4E-1(0.99)	6.3E-2(1.00)	6.7E-4(2.01)	1.2E-3(2.01)
64	2.2E-1(1.00)	3.1E-2(1.00)	1.7E-4(2.01)	3.0E-4(2.01)
128	1.1E-1(1.00)	1.6E-2(1.00)	4.1E-5(2.01)	7.4E-5(2.00)

TABLE 5.1

Error convergence of (5.8) *and* (5.9) *on a benchmark problem* (5.1). *Continuous linear Lagrange elements are used.*

instead of (2.6). Therefore the reduced problem in (2.12) and (2.13) become respectively

$$\begin{aligned}(5.8\text{a})\qquad & (\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \lambda \rangle_\Gamma \\ &\quad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\end{aligned}$$

$$(5.8\text{b})\qquad \langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, \mu \rangle_\Gamma = \langle q_1, \mu \rangle_\Gamma \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,.$$

and

$$\begin{aligned}(5.9\text{a})\qquad & (\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \overline{v} - v_\odot, \lambda_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda)} \\ &\quad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\end{aligned}$$

$$(5.9\text{b})\qquad |\partial\mathcal{D}|\langle \overline{u} - u_\odot, \mu_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \overline{q_2}, \mu_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda)\,.$$

Let us prove that (5.4)-(5.5) is solution of (5.9). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (5.9a) we have

$$\begin{aligned}& -(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d_{ss}^2 u_\odot, v_\odot)_{L^2(\Lambda)} + |\mathcal{D}|\langle \overline{v} - v_\odot, \lambda_\odot \rangle_\Lambda \\ &\quad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), v_\odot \in H^1(\Lambda).\end{aligned}$$

Since $\lambda_\odot = \overline{L} = 0$ and (5.4) satisfies (5.1a) and (5.5) satisfies (5.1b), we have that

$$\begin{aligned}-(\Delta u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} \\ -|\partial\mathcal{D}|(d_{ss}^2 u_\odot, v_\odot)_{L^2(\Lambda)} &= |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)},\end{aligned}$$

Thus (5.4)-(5.5) satisfy (5.9a). The fact that the solution satisfy (5.9b) follows from (5.3).

We can prove in a similar way that (5.4)-(5.5), with $\lambda = L = 0$ satisfy (5.8). Note in particular that q_1 is such that $\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1$ on Γ.

5.1. Numerical experiments. $\mathcal{T}_h^\Omega$ conforming to Γ. Using the benchmark problem (5.1) we now investigate convergence properties of the two formulations. To this end we consider a *uniform* tessilation of $\mathcal{T}_h^\Omega$ of Ω consisting of tetrahedra with diameter h. Further, the discretization shall be geometrically *conforming* to both Λ and Γ such that the tessilations $\mathcal{T}_h^\Gamma$, $\mathcal{T}_h^\Lambda$ are made up of facets and edges of $\mathcal{T}_h^\Omega$ respectively, cf. Figure 5.1 for illustration.

Considering inf-sup stable discretization in terms of continuous linear Lagrange (P_1) elements (for all the spaces), Table 5.1 lists the errors of formulations (5.8) and (5.9) on the benchmark problem. It can be seen the error in u and $u_\odot$ in H^1 norm converges linearly (as can be expected due to P_1 element discretization). Moreover, the error of the Lagrange multiplier approximation in $H^{-1/2}$ norm decreases quadratically. In

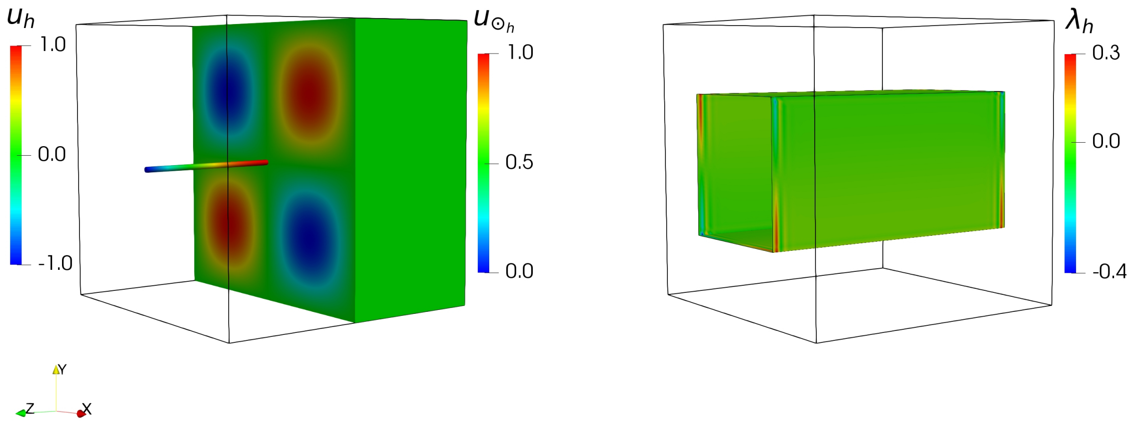


FIGURE 5.2. *Numerical solution of problem* (5.8)*: functions* u_h *and* $u_{\odot h}$ *on the left and the Lagrance multiplier* λ_h *on the right.*

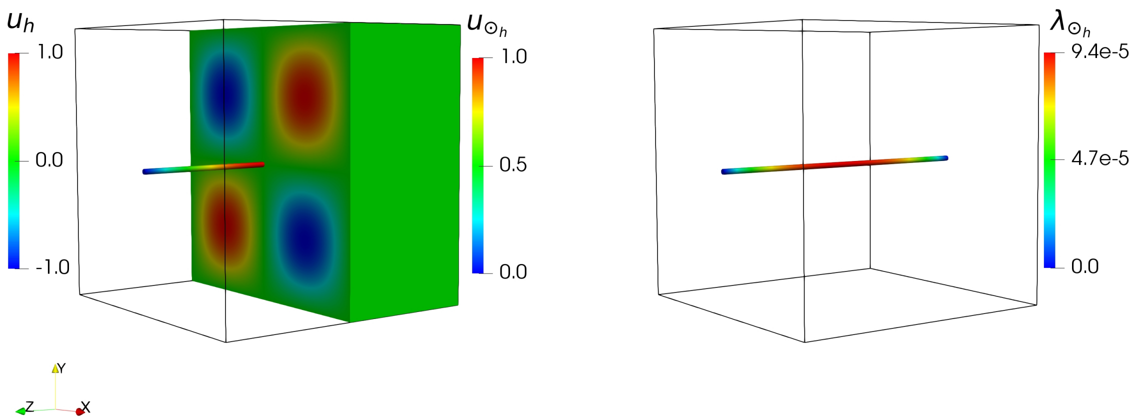


FIGURE 5.3. *Numerical solution of problem* (5.9)*: functions* u_h *and* $u_{\odot h}$ *on the left and the Lagrance multiplier* $\lambda_{\odot h}$ *on the right.*

the light of P_1 discretization this rate appears superconvergent. We speculate that the result is due to the fact that the exact solution is particularly simple, $\lambda = \lambda_\odot = 0$. We remark that for u and $u_\odot$ the error is interpolated into the finite element space of piecewise quadratic *discontinous* functions. For (5.9) we evaluate the fractional norm and interpolate the error using piecewise continuous cubic functions. This is due to the fact that evaluating the fractional norm in higher order spaces for on Γ is prohibitively costly. For the sake of comparison with non-conforming formulation of (2.13) from §4.2 Table 5.1 also lists the error of the Lagrange multiplier in the L^2 norm. Here, quadratic convergence is observed for (5.9). For (5.8) the rate between 1.5 and 2.

We plot the numerical solution of problem (5.8) and (5.9) in Figure 5.2 and 5.3, respectively.

5.2. Numerical experiments. $\mathcal{T}_h^\Omega$ non-conforming to Γ. Using benchmark problem (5.1) we consider (2.13) in the setting of §4.2. To this end we let $\mathcal{T}_h^\Omega$ be a uniform tessilation of Ω such that no cell $\mathcal{T}_h^\Omega$ has any edge lying on Λ. Further we let $h' = h/3$ in $\mathcal{T}_{h'}^\Lambda$, cf. Figure 5.4.

Using discretization in terms of P_1-P_1-P_0 element Table 5.2 lists the error of the stabilized formulation of (2.13). A linear convergence in the H^1 norm can be observed in the error of u and $u_\odot$. We remark that the norms were computed as in §5.1. For simplicity the convergence of the multiplier is measured in the L^2 norm rather then the $H^{-1/2}(\Gamma)$ norm used in the analysis. Then, convergence exceeding order 1.5 can be observed, however, the rates are rather unstable.

The solution is plotted in ..... I guess there should be some zoom in on the cut cells.

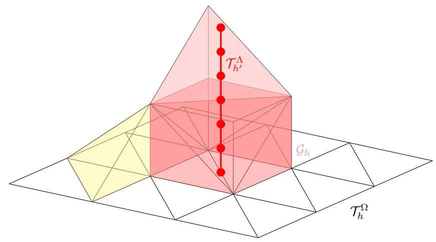


FIGURE 5.4. *Sample discretization of the benchmark geometry in the non-conforming case.*

h^{-1}	$\Vert u - u_h \Vert_{1,\Omega}$	$\Vert u_\odot - u_{\odot h} \Vert_{1,\Lambda}$	$\Vert \lambda_\odot - \lambda_{\odot h} \Vert_{0,\Lambda}$
5	2.6E0(−)	2.3E-1(−)	1.7E-1(−)
9	1.5E0(0.84)	9.4E-2(1.42)	7.1E-2(1.36)
17	8.1E-1(0.94)	4.3E-2(1.18)	2.9E-2(1.37)
33	4.2E-1(0.98)	2.1E-2(1.06)	7.9E-3(1.91)
65	2.1E-1(0.99)	1.1E-2(1.02)	2.6E-3(1.64)
129	1.1E-1(1.00)	5.2E-3(1.01)	8.5E-4(1.61)

TABLE 5.2
Error convergence of (5.9) on a benchmark problem (5.1) in case $\mathcal{T}_h^\Omega$ does not conform to Λ.

5.3. Comparison. In Tables 5.1, 5.2 one can observe that all the formulations yield practically identically accurate approximations of u. Futher, compared to the conforming case, the stabilized formulation (2.13) results in a greater accuracy of $u_{\cdot,h}$ as the underlying mesh $\mathcal{T}_h^\Lambda$ is here finer. Due to the different definitions in the three formulations, comparision of the Lagrange multiplier convergence is not straightforward. We therefore limit ourselves to a comment that in the L^2 norm all the formulations yield faster than linear convergence.

In order to discuss solution cost of the formulations we consider the resulting preconditioned linear systems. In particular, we shall compare spectral condition numbers and the time to convergence of the preconditioned minimal residual (MinRes) solver with the with stopping criterion requiring the relative preconditioned residual norm to be less than 10^{-8}. We remark that we shall ignore the setup cost of the preconditioner.

Following operator preconditioning technique [10] we propose as preconditioners for (2.12) and (2.13) in the conforming case the (approximate) Riesz mapping with respect to the inner products of the spaces in which the two formulations were proved to be well posed. In particular, the preconditioner for the Lagrange multiplier relies on (the inverse of) the fractional Laplacian $-\Delta^{-1/2}$ on Γ for (5.8) and Λ for (5.9). A detailed analysis of the preconditioners will be presented in a separate work. We remark that in both cases the fractional Laplacian was here realized by spectral decomposition [7].

For the unfitted stabilized (2.13) the Lagrange multiplier preconditioner uses a Riesz map with respect to the inner product due to $L^2(\mathcal{G}_h)$ and the stabilization (4.17), i.e.

$$(\lambda_{\odot h}, \mu_{\odot h}) \mapsto \sum_{K\in\mathcal{G}_h} \int_K \lambda_{\odot h}\mu_{\odot h} + \sum_{K\in\mathcal{G}_h} \int_{\partial K\setminus\partial\mathcal{G}_h} h[\![\lambda_{\odot h}]\!][\![\mu_{\odot h}]\!].$$

This simple choice does not yield bounded iterations. However, establishing a robust precondtioner in this case is beyond the scope of the paper and shall be pursued in the future works.

In Table 5.3 we compare solution time, number of iterations and condition numbers of the (linear systems due to the) three formulations. Let us first note that the proposed preconditioners for (2.12) and (2.13) in the conforming case seem robust with respect to discretization parameter as the iteration counts and condition numbers are bounded in h. We then see that the solution time for (5.8) is about 2 times longer compared to (5.9) which is about 4 times more expensive than the solution of the Poisson problem (5.1a). This is in addition to the higher setup costs of the preconditioner which in our implementation involve solving an eigenvalue problem for the fractional Laplacian. Therefore it is advantageous to keep the multiplier space as small as possible. We remark that the missing results for (5.8) in Table 5.3 and 5.1 are due to the memory limitations which we encounter when solving the eigenvalue problem for the Laplacian, which for finest mesh involves cca 32 thousand eigenvalues, cf. Appendix A.

Due to the missing proper preconditioner for the Lagrange multiplier block the number of iterations in the third, unfitted formulation can be seen to approximatly double on refinement.

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l	(2.12)			(2.13)			Stabilized (2.13)			(5.1a)	
	#	T [s]	κ	#	T [s]	κ	#	T [s]	κ	#	T [s]
1	20	0.03	15.56	9	0.02	3.04	21	0.01	9.70	3	< 0.01
2	35	0.06	16.28	17	0.03	4.67	31	0.03	15.87	4	< 0.01
3	38	0.14	16.64	22	0.06	6.25	53	0.15	32.93	5	0.01
4	39	1.70	16.75	24	0.89	7.03	110	4.54	61.48	5	0.12
5	38	12.04	16.78	20	5.21	5.02	232	59.43	94.25	5	0.90
6	–	–	–	17	28.77	–	507	832.90	–	6	7.75

TABLE 5.3

Cost comparison of the formulations across refinement levels l. Number of Krylov (preconditioned conjugate gradient for (5.1a), MinRes otherwise) iterations and the conditioned number of the preconditioned problem is denoted by # and κ respectively. Time till convergence of the iterative solver (excluding the setup) is shown as T.

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Appendix A. System sizes in benchmark formulations. Below we list dimensions of the finite element spaces used to discretize formualations (2.12), (2.13) and stabilized (2.13) on different levels of refinement.

l	(2.12)			(2.13)			Stabilized (2.13)		
	$\lvert X^1_{h,0}(\Omega)\rvert$	$\lvert X^1_{h,0}(\Lambda)\rvert$	$\lvert Q_h(\Gamma)\rvert$	$\lvert X^1_{h,0}(\Omega)\rvert$	$\lvert X^1_{h,0}(\Lambda)\rvert$	$\lvert Q_h(\Lambda)\rvert$	$\lvert X^1_{h,0}(\Omega)\rvert$	$\lvert X^1_{h',0}(\Lambda)\rvert$	$\lvert Q_h(\mathcal{G}_h)\rvert$
1	125	5	40	125	5	5	180	13	24
2	729	9	144	729	9	9	900	25	48
3	4913	17	544	4913	17	17	5508	49	96
4	35937	33	2112	35937	33	33	38148	97	192
5	274625	65	8320	274625	65	65	283140	193	384
6	–	–	-	2146689	129	129	2180100	385	768