COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

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AMS subject classifications. n.a.

1. Introduction. We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators form a 3D domain, Ω, to 1D manifold, Λ, and vice versa. This is a non trivial objective because the standard trace operator form a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω.

2. Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex open set. Let $\Omega_\ominus$ be a generalized cylinder embedded into Ω and be $\Omega_\oplus = \Omega \setminus \overline{\Omega}_\ominus$ be the complementary set of the cylinder. We also introduce the set Λ, a 1D manifold that is the centerline of $\Omega_\ominus$. We define the arc-length coordinate along Λ, denoted by $s \in (0, S)$. We denote with $\mathcal{D}(s)$ and $\partial\mathcal{D}(s)$ a cross section of $\Omega_\ominus$ and its boundary, respectively. We also assume that $\Omega_\ominus$ crosses Ω from side to side and we call Γ the lateral (cylindrical) surface of $\Omega_\ominus$, while the upper and lower side faces of $\Omega_\ominus$ belong to $\partial\Omega$. We refer to Figure 2.1 for an illustration of the notation.

We consider the problem arising from *Dirichlet-Neumann* conditions. It consists to find $u_\oplus, u_\ominus$ s.t.:

$$
\begin{aligned}
&(2.1a) && -\Delta u_\oplus + u_\oplus = f && \text{in } \Omega_\oplus,\\
&(2.1b) && -\Delta u_\ominus + u_\ominus = g && \text{in } \Omega_\ominus,\\
&(2.1c) && -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus && \text{on } \Gamma,\\
&(2.1d) && u_\ominus = u_\oplus && \text{on } \Gamma,\\
&(2.1e) && u_\oplus = 0 && \text{on } \partial\Omega\,.
\end{aligned}
$$

The objective of this work is to derive and analyze a simplified version of problem (2.1), where the domain $\Omega_\ominus$ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely $\mathcal{D}$. This new problem setting will be called the *reduced* problem. Form the mathematical standpoint it is more challenging than (2.1), because it involves the coupling of 3D/1D elliptic problems. For the model reduction process, we decompose integrals as follows, for any sufficiently regular function w,

$$
\int_{\Omega_\ominus} w d\omega = \int_\Lambda \int_{\mathcal{D}} w d\sigma ds = \int_\Lambda |\mathcal{D}| \overline{\overline{w}} ds\,, \quad \int_\Gamma w d\sigma = \int_\Lambda \int_{\partial\mathcal{D}} w d\gamma ds = \int_\Lambda |\partial\mathcal{D}| \overline{w} ds\,,
$$

where $\overline{\overline{w}}$, $\overline{w}$ denote the following mean values respectively,

$$
\overline{\overline{w}} = |\mathcal{D}|^{-1} \int_{\mathcal{D}} w d\sigma\,, \quad \overline{w} = |\partial\mathcal{D}|^{-1} \int_{\partial\mathcal{D}} w d\gamma\,.
$$

We apply the model reduction approach at the level of the variational formulation. We start from the variational formulation of problem (2.1), that is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ u_\ominus \in H^1_{\partial\Omega_\ominus \setminus \Gamma}(\Omega_\ominus),\ \lambda \in H^{-\frac{1}{2}}(\partial\Omega_\ominus)$ s.t.

$$
\begin{aligned}
&(2.2a) && (u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Omega_\ominus)} + \langle v_\oplus - v_\ominus, \lambda \rangle_{H^{-\frac{1}{2}}(\Gamma)}\\
&&& \qquad = (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Omega_\ominus)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Omega_\ominus \setminus \Gamma}(\Omega_\ominus)\\
&(2.2b) && \langle u_\oplus - u_\ominus, \mu \rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,,
\end{aligned}
$$

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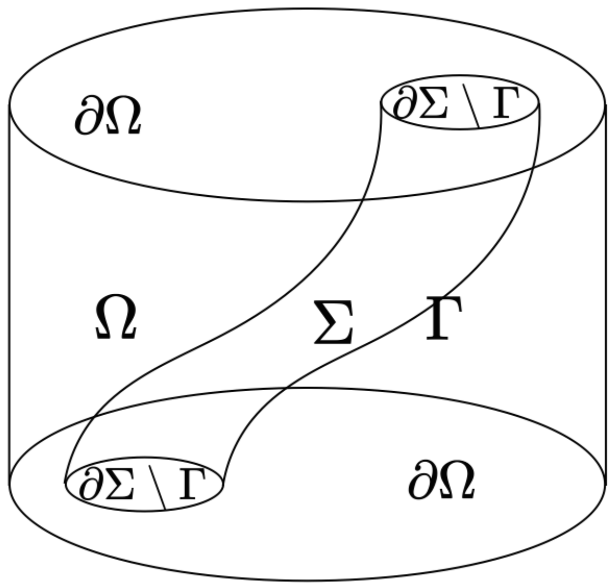

FIGURE 2.1. *Geometrical setting of the problem*

where $\langle v, \mu \rangle_{H^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. In this case, the additional variable λ is equivalent to $\lambda = \nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

Using a model reduction approach based on averaging, we end up with two different formulations of a reduced problem for the unknown u defined on the entire 3D domain Ω, coupled with the unknown $u_\odot$, defined on the 1D manifold Λ and a Lagrange multiplier defined either on Γ (problem 1) or on Λ (problem 2). The scope of this work is to compare them, with the aim to determine which is the most suitable to set up a computational model for 3D-1D coupled PDEs.

2.1. Topological model reduction.

Model reduction of the problem on $\Omega_\ominus$. We apply the averaging technique to equation (2.1b). In particular, we consider an arbitrary portion $\mathcal{P}$ of the cylinder $\Omega_\ominus$, with lateral surface $\Gamma_\mathcal{P}$ and bounded by two perpendicular sections to Λ, namely $\mathcal{D}(s_1)$, $\mathcal{D}(s_2)$ with $s_1 < s_2$. We have,

$$\int_\mathcal{P} -\Delta u_\ominus + u_\ominus d\omega = -\int_{\partial\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus \, d\sigma + \int_\mathcal{P} u_\ominus d\omega = \\ \int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma - \int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma + \int_\mathcal{P} u_\ominus d\omega$$

By the fundamental theorem of integral calculus combined with the Reynolds transport Theorem, being ν the normal deformation of the boundary along $(0, S)$, we have,

$$\begin{aligned} \int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma &= -\int_{s_1}^{s_2} d_s \int_{\mathcal{D}(s)} \partial_s u_\ominus d\sigma ds \\ &= -\int_{s_1}^{s_2} d_{ss}^2 \int_{\mathcal{D}(s)} u_\ominus \, d\sigma \, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus \, d\gamma \right) ds \,, \end{aligned}$$

and assuming that $\mathcal{D}(s)$ can not change shape, we have

$$-\int_{s_1}^{s_2} d_{ss}^2 \int_{\mathcal{D}(s)} u_\ominus \, d\sigma \, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus \, d\gamma \right) ds = -\int_{s_1}^{s_2} \left[d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) - d_s(\nu |\partial\mathcal{D}(s)| \overline{u}_\ominus) \right] ds$$
$$= -\int_{s_1}^{s_2} \left[d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) - d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) \right] ds.$$

Moreover, we have

$$\int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma = \int_{\Gamma_\mathcal{P}} \lambda \, d\sigma = \int_{s_1}^{s_2} \int_{\partial\mathcal{D}(s)} \lambda d\gamma \, ds = \int_{s_1}^{s_2} |\partial\mathcal{D}| \overline{\lambda} \, ds \, .$$

From the combination of all the above terms with the right hand side, we obtain that the solution $u_\ominus$ of (2.1b) satisfies,

$$\int_{s_1}^{s_2} \left[-d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) + d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) + |\mathcal{D}(s)| \overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)| \overline{\lambda} \right] ds = \int_{s_1}^{s_2} |\mathcal{D}(s)| \overline{\overline{g}} \, ds \, .$$

Since the choice of the points s_1, s_2 is arbitrary, we conclude that the following equation holds true,

$$-d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) + d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) + |\mathcal{D}(s)| \overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)| \overline{\lambda} = |\mathcal{D}(s)| \overline{\overline{g}} \quad \text{on } \Lambda \, , \tag{2.3}$$

which is complemented by the following conditions at the boundary of Λ,

$$|\mathcal{D}(s)| d_s \overline{\overline{u}}_\ominus = 0, \quad d_s |\mathcal{D}(s)| = 0, \quad \text{on} \quad s = 0, S. \tag{2.4}$$

Then, we consider variational formulation of the averaged equation (2.3). After multiplication by a test function $v_\odot \in H^1(\Lambda)$, integration on Λ and suitable application of integration by parts, we obtain,

$$\int_\Lambda d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) d_s v_\odot \, ds - d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) v_\odot |_{s=0}^{s=S} - \int_\Lambda (d_s |\mathcal{D}(s)|) \overline{u}_\ominus d_s v_\odot \, ds + (d_s |\mathcal{D}(s)|) \overline{u}_\ominus v_\odot |_{s=0}^{s=S}$$
$$+ \int_\Lambda |\mathcal{D}(s)| \overline{\overline{u_\ominus}} v_\odot - \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda} v_\odot \, ds = \int_\Lambda |\mathcal{D}(s)| \overline{\overline{g}} V \, ds \, .$$

Using boundary conditions, the identity $d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) = |\mathcal{D}(s)| d_s \overline{\overline{u}}_\ominus + d_s(|\mathcal{D}(s)|) \overline{\overline{u}}_\ominus$ and reminding that $d_s |\mathcal{D}(s)|) / |\partial\mathcal{D}(s)| = \nu$, we obtain,

$$(d_s \overline{\overline{u}}_\ominus, d_s v_\odot)_{\Lambda, |\mathcal{D}|} + (\nu (\overline{\overline{u}}_\ominus - \overline{u}_\ominus), d_s v_\odot)_{\Lambda, |\partial\mathcal{D}|} + (\overline{\overline{u_\ominus}}, v_\odot)_{\Lambda, |\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda, |\partial\mathcal{D}|} = (\overline{\overline{g}}, V)_{\Lambda, |\mathcal{D}|} \, . \tag{2.5}$$

where we have introduced the following weighted inner product notation,

$$(u_\odot, v_\odot)_{\Lambda, w} = \int_0^S w(s) u_\odot(s) v_\odot(s) ds \, .$$

Let us now formulate the modelling assumption that allows us to reduce equation (2.5) to a solvable one-dimensional (1D) model. More precisely, we assume that:

A1 the function $u_\ominus$ has a *uniform profile* on each cross section $\mathcal{D}(s)$, namely $u_\ominus(r, s, t) = u_\odot(s)$.

Therefore, observing that $u_\odot = \overline{u}_\ominus = \overline{\overline{u}}_\ominus$, problem (2.5) consists to find $u_\odot \in H^1(\Lambda)$ such that

$$(d_s u_\odot, d_s v_\odot)_{\Lambda, |\mathcal{D}|} + (u_\odot, v_\odot)_{\Lambda, |\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda, |\partial\mathcal{D}|} = (\overline{\overline{g}}, v_\odot)_{\Lambda, |\mathcal{D}|} \quad \forall v_\odot \in H^1(\Lambda) \, . \tag{2.6}$$

Topological model reduction of the problem on $\Omega_\oplus$. We focus here on the subproblem of (2.1a) related to $\Omega_\oplus$. We multiply both sides of (2.1a) by a test function $v \in H_0^1(\Omega)$ and integrate on $\Omega_\oplus$. Integrating by parts and using boundary and interface conditions, we obtain

$$\int_{\Omega_\oplus} f v \, d\omega = \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\omega - \int_{\partial\Omega_\oplus} \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v \, d\sigma + \int_{\Omega_\oplus} u_\oplus v$$
$$= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\Omega - \int_\Gamma \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v + \int_{\Omega_\oplus} u_\oplus v$$
$$= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\Omega + \int_\Gamma \lambda v + \int_{\Omega_\oplus} u_\oplus v.$$

Then, we make the following modelling assumptions:

A2 we identify the domain $\Omega_\oplus$ with the entire Ω, and we correspondingly omit the subscript $\oplus$ to the functions defined on $\Omega_\oplus$, namely

$$\int_{\Omega_\oplus} u_\oplus \, d\omega \simeq \int_\Omega u \, d\omega \, .$$

Therefore, we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\lambda, v)_\Gamma = (f, v)_\Omega$$

and combining with (2.6) we obtain the first formulation of the reduced problem.

Problem 1 (3D-1D-3D). Let $\langle \cdot, \cdot \rangle_\Gamma$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. The problem consists to find $u \in H^1_0(\Omega)$, $u_\odot \in H^1_0(\Lambda)$, $\lambda \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned} &(u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda), |\mathcal{D}|} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \lambda \rangle_\Gamma \\ &\qquad = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda), |\mathcal{D}|} \quad \forall v \in H^1_0(\Omega), \ v_\odot \in H^1(\Lambda) \end{aligned} \tag{2.7a}$$

$$\langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, \mu \rangle_\Gamma = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma) \, . \tag{2.7b}$$

Here, $\mathcal{T}_\Gamma : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ denotes the trace operator from Ω to Γ and $\mathcal{E}_\Gamma : H^1_0(\Lambda) \to H^1_0(\Gamma)$ denotes the uniform extension from Λ to Γ.

Now, we apply a topological model reduction of the interface conditions, namely we go from a 3D-1D-2D formulation involving sub-problems on Ω and Λ and coupling operators defined on Γ to a 3D-1D-1D formulation where the coupling terms are set on Λ. To this purpose, let us write the Lagrange multiplier and the test functions on every cross section $\partial\mathcal{D}(s)$ as their average plus some fluctuation,

$$\lambda = \overline{\lambda} + \tilde{\lambda}, \qquad v = \overline{v} + \tilde{v}, \quad \text{on } \partial\mathcal{D}(s) \, ,$$

where $\overline{\tilde{\lambda}} = \overline{\tilde{v}} = 0$. Therefore, the coupling term on Γ can be decomposed as,

$$\int_\Gamma \lambda v \, d\sigma = \int_\Lambda \int_{\partial\mathcal{D}(s)} (\overline{\lambda} + \tilde{\lambda})(\overline{v} + \tilde{v}) d\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda}\overline{v} \, ds + \int_\Lambda \int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma ds \, .$$

Then, we make the following modelling assumptions:

A3 we assume that the product of fluctuations is small, namely

$$\int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma \simeq 0$$

and we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\overline{\lambda}, \overline{v})_{\Lambda, |\partial\mathcal{D}|} = (f, v)_\Omega,$$

which, combined with (2.6) leads to the second formulation of the reduced problem.

2.2. Problem 2 (3D-1D-1D). Let $\langle \cdot, \cdot \rangle_\Lambda$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. The problem requires to find $u \in H^1_0(\Omega)$, $u_\odot \in H^1_0(\Lambda)$, $\lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned} &(u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda), |\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \lambda_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} \\ &\qquad = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, V)_{L^2(\Lambda), |\mathcal{D}|} \quad \forall v \in H^1_0(\Omega), \ v_\odot \in H^1_0(\Lambda) \, , \end{aligned} \tag{2.8a}$$

$$\langle \overline{\mathcal{T}}_\Lambda u - u_\odot, \mu_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} = 0 \quad \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda) \, , \tag{2.8b}$$

where $\overline{\mathcal{T}}_\Lambda$ denotes the composition of operators $\mathcal{T}_\Gamma \circ \overline{(\cdot)}$. We notice that all the integrals of the reduced problem are well defined because $\overline{\mathcal{T}}_\Lambda : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Lambda)$ as shown in the following lemma.

LEMMA 2.1. *Let Γ be a tensor product domain, $\Gamma = (0,X) \times (0,Y)$. For any regular $u(x,y)$ in Γ, let $\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x,y)\,dy$. Then, for any $u \in H^{\frac{1}{2}}_{00}(\Gamma)$, then $\overline{u}(x) \in H^{\frac{1}{2}}_{00}((0,X))$. Moreover, if $u(x,y) \in H^{\frac{1}{2}}_{00}(\Gamma)$ is constant with respect to y, namely $u(x,y) = u(x)$, then*

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = Y\|u\|_{H^{\frac{1}{2}}_{00}(0,X)}.$$

Proof. Let us consider the eigenvalue problems for the Laplace operator on Γ with homogeneous Dirichlet conditions at $x = 0, X$ and periodic boundary conditions at $y = 0, Y$ and on $(0,X)$ with homogeneous Dirichlet conditions. Let us denote as ϕ_{ij} and ρ_{ij}, for $i = 1,2,\ldots$, $j = 0,1,\ldots$, the eigenfunctions and the eigenvalues of the laplacian on Γ, and with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the laplacian on $(0,X)$. In particular,

$$\begin{aligned}
\phi_{ij}(x,y) = \sin\left(\frac{i\pi x}{X}\right)\left(cos\left(\frac{j2\pi y}{Y}\right) + sin\left(\frac{j2\pi y}{Y}\right)\right),\\
\rho_{ij} = \left(\frac{i\pi}{X}\right)^2 + \left(\frac{j2\pi}{Y}\right)^2,\\
\phi_i(x) = \sin\left(\frac{i\pi x}{X}\right),\\
\rho_i = \left(\frac{i\pi}{X}\right)^2.
\end{aligned}$$

It is easy to verify that

$$\int_0^Y \phi_{ij}(x,y) = 0 \quad \forall j > 0, \forall i \tag{2.9}$$

$$\int_0^Y \phi_{ij}(x,y) = Y\,\sin\left(\frac{i\pi x}{X}\right) \quad \text{if } j = 0, \forall i. \tag{2.10}$$

(2.11)

Moreover we recall that $\phi_{i,j}(x,y)$ and $\phi_i(x)$ are orthogonal basis of $L^2(\Gamma)$ and $L^2(0,X)$ respectively. Therefore,

$$\begin{aligned}
\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x,y)\,dy = \frac{1}{Y}\int_0^Y \sum_{i,j} a_{i,j}\phi_{i,j}(x,y)\,dy\\
= \frac{1}{Y}\sum_{i,j} a_{i,j}\int_0^Y \phi_{i,j}(x,y)\,dy = \sum_i a_{i,0}\phi_i(x).
\end{aligned}$$

From [4, Lemma 4.11] we have

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}\,|a_{ij}|^2, \text{ with } a_{ij} = \int_0^X\int_0^Y u(x,y)\phi_{ij}(x,y)dydx. \tag{2.12}$$

and

$$\|\overline{u}\|^2_{H^{\frac{1}{2}}(0,X)} = \sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}\,|\overline{a}_i|^2, \text{ with } \overline{a}_i = \int_0^X \overline{u}(x)\phi_i(x)dx.$$

Therefore, we have

$$
\begin{aligned}
\|\overline{u}\|^2_{H^{\frac{1}{2}}(0,X)} &= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X \overline{u}(x)\sin\left(\frac{i\pi x}{X}\right)\,dx\right)^2 \\
&= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty} a_{j,0}\int_0^1 \sin\left(\frac{j\pi x}{X}\right)\sin\left(\frac{i\pi x}{X}\right)\,dx\right)^2 \\
&= \sum_{i=1}^{\infty}\frac{X^2}{4}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}} a_{i,0}^2 \\
&\le \frac{X^2}{4}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}|a_{i,j}|^2 = \frac{X^2}{4}\|u\|^2_{H^{\frac{1}{2}}(\Gamma)},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
&\int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)\,dx = 0 \quad \text{if } i\neq j \\
&\int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)\,dx = \frac{X}{2} \quad \text{if } i=j.
\end{aligned}
$$

Moreover, in the case in which u is constant with respect to y, from (2.12) we have

$$
\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}|a_{ij}|^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X\int_0^Y u(x,y)\phi_{ij}(x,y)\,dy\,dx\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X u(x)\int_0^Y \phi_{ij}(x,y)\,dydx\right)^2,
\end{aligned}
$$

and using (2.9) and (2.10), we obtain

$$
\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} &= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X Yu(x)\,\sin\left(\frac{i\pi x}{X}\right)dx\right)^2 \\
&= Y^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}|a_i|^2 = Y^2\|u\|^2_{H^{\frac{1}{2}}(0,X)}. \quad \square
\end{aligned}
$$

COROLLARY 2.2. *Let Γ be the lateral surface of a prism with N faces Γ_i, $i=1,\dots,N$, with $\Gamma_i = (0,X_i)\times(0,S)$. For any $u\in H^{\frac{1}{2}}_{00}(\Gamma)$, let $\overline{u}(s) = \frac{1}{\sum_i |X_i|}\sum_{i=1}^N \int_0^{X_i} u(x,s)\,dx$. Then $\overline{u}\in H^{\frac{1}{2}}_{00}(0,S)$.*

COROLLARY 2.3. *Let Γ be the lateral surface of a cylinder of radius R. Let $u\in H^{\frac{1}{2}}_{00}(\Gamma)$ and let $\overline{u}(s) = \frac{1}{2\pi R}\int_0^{2\pi} u(s,\theta)R\,d\theta$. Then $\overline{u}\in H^{\frac{1}{2}}_{00}(0,S)$.*

It is apparent that problems (2.2) and (**??**) share the same mathematical structure. For this reason, the well-posedness of (**??**) can be studied in the framework of the classical theory of saddle point problems.

3. Saddle-point problem analysis. Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bounded bilinear forms. Let us consider the general saddle point problem of the form: find $u \in X$, $\lambda \in Q$ s.t.

$$\begin{cases} a(u,v) + b(v,\lambda) = c(v) & \forall v \in X \\ b(u,\mu) = d(\mu) & \forall \mu \in Q. \end{cases} \tag{3.1}$$

We denote with A and B the operators associated to the bilinear forms a and b, namely $A : X \longrightarrow X'$ with $\langle Au, v\rangle_{X',X} = a(u,v)$ and $\langle Bv, \mu\rangle_{X',Q} = b(v,\mu)$. Problem (3.1) embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem.

Theorem 3.1. *[5, Theorem 2.34] Problem (3.1) is well posed iff*

$$\begin{cases} \exists \alpha > 0 : \inf_{u \in ker(B)} \sup_{v \in ker(B)} \frac{a(u,v)}{\|u\|_X \|v\|_X} \geq \alpha \\ \forall v \in ker(B),\ (\forall u \in ker(B),\ a(u,v) = 0) \implies v = 0. \end{cases} \tag{3.2}$$

and

$$\exists \beta > 0 : \inf_{\mu \in Q} \sup_{v \in X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Q} \geq \beta. \tag{3.3}$$

Notice that if a is coercive on $ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1. It consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ \lambda \in H^{-\frac{1}{2}}(\Gamma)$, solutions of (3.1), where

$$a([u,u_\odot],[v,v_\odot]) = (u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|}$$

$$b([v,v_\odot],\mu) = \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma$$

$$c([v,v_\odot]) = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|}$$

$$d(\mu) = 0$$

We prove that the hypothesis of 3.1 are fullfilled choosing $X = H^1_0(\Omega) \times H^1_0(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$, where X is equipped with the norm $|||[u,u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + \|u_\odot\|^2_{H^1(\Lambda),|\mathcal{D}|}$ and Q equipped with the norm

$$\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} := \sup_{q \in H^{\frac{1}{2}}(\Gamma)} \frac{\langle q, \mu\rangle_\Gamma}{\|q\|_{H^{\frac{1}{2}}(\Gamma)}}$$

.

Lemma 3.2. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The bilinear form $a(\cdot\ ,\ \cdot)$ is clearly bounded since

$$a([u,u_\odot],[v,v_\odot]) \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + \|u_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|} \leq 2|||[u,u_\odot]|||\,|||[v,v_\odot]|||.$$

Concerning the bilinear form $b(\cdot\ ,\ \cdot)$ we have

$$\begin{aligned} b([v,v_\odot],\mu) &= \langle \mathcal{T}_\Gamma v - \mathcal{U}_E v_\odot, \mu\rangle_\Gamma \leq \|\mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot\|_{H^{\frac{1}{2}}(\Gamma)} \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^{\frac{1}{2}}(\Gamma)}\right)\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \left(C_T\|v\|_{H^1(\Omega)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^1(\Gamma)}\right)\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(C_T\|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\right)\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim |||[v,v_\odot]|||\,\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \square \end{aligned}$$

LEMMA 3.3. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive .*

Proof. Indeed, we have,

$$a([u,u_\odot],[u,u_\odot]) = (u,u)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot,u_\odot)_{H^1(\Lambda)} = |||[u,u_\odot]|||^2\,.$$

□

LEMMA 3.4. *The inf-sup inequality (3.3) is fulfilled, namely $\exists\beta_1 > 0$ such that $\forall\mu \in H^{-\frac{1}{2}}(\Gamma)$:*

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v,v_\odot]|||} \geq \beta_1 \sup_{q\in H^{\frac{1}{2}}(\Gamma)} \frac{\langle q,\mu\rangle_\Gamma}{\|q\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

Proof. We choose $v_\odot \in H_0^1(\Lambda)$ such that $\mathcal{E}_\Gamma v_\odot = 0$. Therefore,

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v,v_\odot]|||} \geq \sup_{v\in H_0^1(\Omega)} \frac{\langle \mathcal{T}_\Gamma v,\mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

We notice that the trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. Indeed, $\forall\xi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we can find v solution of

$$\begin{aligned} -\Delta v &= 0 \quad \text{in } \Omega \\ v &= 0 \ \text{ on } \partial\Omega \\ v &= \xi \ \text{ on } \Gamma. \end{aligned}$$

We denote with $\mathcal{E}_\Omega$ the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|\mathcal{E}_\Omega\| \in \mathbb{R}$ such that $v = \mathcal{E}_\Omega(\xi)$ and $\|v\|_{H^1(\Omega)} \leq \|\mathcal{E}_\Omega\|\|\xi\|_{H^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$\sup_{v\in H_0^1(\Omega)} \frac{\langle \mathcal{T}_\Gamma v,\mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi\in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle \xi,\mu\rangle_\Gamma}{\|\mathcal{E}_\Omega\|\|\xi\|_{H^{\frac{1}{2}}(\Gamma)}} = \|\mathcal{E}_\Omega\|^{-1}\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)}, \tag{3.4}$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H_{00}^{\frac{1}{2}}(\Gamma))^*$. □

3.2. Problem 2. This problem requires to find $u \in H_0^1(\Omega),\ u_\odot \in H_0^1(\Lambda),\ \lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, solution of (3.1) with

$$a([u,u_\odot],[v,v_\odot]) = (u,v)_{H^1(\Omega)} + (u_\odot,v_\odot)_{H^1(\Lambda),|\mathcal{D}|}$$

$$b([v,v_\odot],\mu_\odot) = \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}$$

$$c([v,v_\odot]) = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}},v_\odot)_{L^2(\Lambda),|\mathcal{D}|}$$

$$d(\mu_\odot) = 0$$

We prove that the hypotesis of Theorem 3.1 are fulfilled with the following spaces $X = H_0^1(\Omega)\times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped again with the norm $|||[\cdot,\cdot]|||$ and Q equipped with the norm $\|\cdot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}$, defined as

$$\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} := \sup_{q\in H^{\frac{1}{2}}(\Lambda)} \frac{\langle q,\mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}}$$

Then, we have the following lemmas.

LEMMA 3.5. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The boundedness of $a(\cdot\ ,\ \cdot)$ can be proved as in Lemma 3.2. Concerning $b(\cdot\ ,\ \cdot)$, we have

$$
\begin{aligned}
b([v,v_\odot],\mu_\odot) = \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} &\le \|\overline{\mathcal{T}}_\Lambda v - v_\odot\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(\|\overline{\mathcal{T}}_\Lambda v\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} + \|v_\odot\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} + \|v_\odot\|_{H^{1}(\Lambda),|\partial\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(C_T\|v\|_{H^1(\Omega)} + \left(\frac{\max|\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\|v_\odot\|_{H^{1}(\Lambda),|\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\lesssim |||[v,v_\odot]|||\,\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}. \qquad \square
\end{aligned}
$$

LEMMA 3.6. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive.*

LEMMA 3.7. *The inf-sup inequality (3.3) holds, namely $\exists\beta_2 > 0$ such that $\forall\mu_\odot \in H^{-\frac{1}{2}}(\Lambda)$, :*

$$
\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v,v_\odot]|||} \ge \beta_2\|\mu_\odot\|_{H^{\frac{1}{2}}(\Lambda)}.
$$

We choose $v_\odot = 0$ and we obtain

$$
\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v,v_\odot]|||} \ge \sup_{v\in H^1_0(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}}.
$$

For any $q \in H^{\frac{1}{2}}_{00}(\Lambda)$, we consider its uniform extension to Γ $\mathcal{E}_\Gamma q$ and then we consider the harmonic extension $v = \mathcal{E}_\Omega\mathcal{E}_\Gamma q \in H^1_0(\Omega)$. It follows that $\overline{\mathcal{T}}_\Lambda v = q$. Therefore,

$$
\sup_{v\in H^1_0(\Omega)} \langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \ge \sup_{q\in H^{\frac{1}{2}}_{00}(\Lambda)} \langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}\,.
$$

Moreover, using Lemma 2.1 we obtain

$$
\|v\|_{H^1_0(\Omega)} \le \|\mathcal{E}_\Omega\|\|\mathcal{E}_\Gamma q\|_{H^{\frac{1}{2}}(\Gamma)} = \|\mathcal{E}_\Omega\|\|q\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.
$$

Therefore,

$$
\begin{aligned}
\sup_{v\in H^1_0(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} \ge \sup_{q\in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} &\ge \|\mathcal{E}_\Omega\|^{-1} \sup_{q\in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}}\\
&= \|\mathcal{E}_\Omega\|^{-1}\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.
\end{aligned}
$$

4. Finite element approximation. The discrete equivalent of (3.1) reads as finding $u_h \in X_h \subset X$, $\lambda_h \in Q_h \subset Q$ s.t.

$$
\begin{cases} a(u_h,v_h) + b(v_h,\lambda_h) = c(v_h) & \forall v_h \in X_h\\ b(u_h,\mu_h) = d(\mu_h) & \forall \mu_h \in Q_h. \end{cases} \tag{4.1}
$$

Define Bh, etc

The well posedness of such problem is governed by the classical inf-sup theory in Banach spaces. The main result is reported below.

Corollary 4.1. *[5, Theorem 2.42] Problem (4.1) is well-posed if and only if*

$$\exists \alpha_h > 0 : \inf_{u_h \in ker(B_h)} \sup_{v_h \in ker(B_h)} \frac{a(u_h, v_h)}{\|u_h\|_X \|v_h\|_X} \geq \alpha_h \tag{4.2}$$

and

$$\exists \beta_h > 0 : \inf_{\mu_h \in Q_h} \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X \|\mu_h\|_Q} \geq \beta_h. \tag{4.3}$$

This corollary is the discrete counterpart of Theorem 3.1 where at the discrete level condition (4.2) implies both of (3.2). Conversely, (4.3) does not follow from the conformity of the finte element spaces and it must be analysed independetly of (3.3). Let us notice that for both problem 1 and problem 2 the bilinear form $a(\cdot,\cdot)$ is coercive as stated in Lemmas (3.3) and (3.6). Consequently, (4.2) is automatically satisfied, being α_h the coercivity constant.

Let us introduce a shape-regular triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_h^\Lambda$ of Λ. We analyze two different cases: the one in which the 3D mesh is conforming to the interface Γ, namely the set of the intersections of the 3D elements of $\mathcal{T}_h^\Omega$ with Γ is constituted by facets of such elements, and the non conforming case, namely the interface Γ cuts the mesh arbitrarly.

4.1. $\mathcal{T}_h^\Omega$ conforming to Γ.

4.1.1. Problem 1. We denote by $X^0_{h,k}(\Omega) \subset H^1_0(\Omega)$ the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satisfying homogeneous Dirichlet conditions on the boundary and by $X^0_{h,k}(\Lambda) \subset H^1_0(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ, satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. Problem 1 consists to find $u_h \in X^0_{h,k}(\Omega),\ u_{\odot h} \in X^0_{h,k}(\Lambda), \lambda_h \in Q_h(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned} &(u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle T v_h - \mathcal{U}_E v_{\odot h}, \lambda_h \rangle_\Gamma \\ &\qquad = (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X^0_{h,k}(\Omega),\ v_{\odot h} \in X^0_{h,k}(\Lambda) \end{aligned} \tag{4.4a}$$

$$\langle T u_h - \mathcal{U} u_{\odot h}, \mu_h \rangle_\Gamma = 0 \quad \forall \mu_h \in Q_h(\Gamma)\,, \tag{4.4b}$$

The space $Q_h(\Gamma)$ must be suitably chosen such that (4.3) holds. For this purpose we define conformity conditios between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$ with Γ. MOre precisely we require that the intersection of $\mathcal{T}_h^\Omega$ and Γ is amde of entire faces of elemets $K \in \mathcal{T}_h^\Omega$. Furthermore we also set a restriction between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$. We want that the intersection of Γ with any orthogonal plane to Λ that crosses Λ at the nodes of $\mathcal{T}_h^\Lambda$, consists of entire edges of $\mathcal{T}_h^\Omega$. Namely the intersection of Γ with orthogonal planes to Λ is conformal with $\mathcal{T}_h^\Lambda$. Let us denote with $W^0_{h,k}(\Gamma) \subset H^{\frac{1}{2}}_{00}(\Gamma)$ the trace space of functions running in $X^0_{h,k}(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$. We choose $Q_h(\Gamma) = W^0_{h,k}(\Gamma)$, therefore we impose homogeneous Dirichlet boundary condition on $\partial\Omega$ also for the Lagrange multiplier. For this choice of $Q_h(\Gamma)$ we can prove the well-posedness of the discrete problem, as shown in the following.

Lemma 4.2. *Let $P_h : H^{\frac{1}{2}}_{00}(\Gamma) \longrightarrow W^0_{h,k}(\Gamma)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Gamma)$ by*

$$(P_h v, \psi_h)_\Gamma = (v, \psi_h)_\Gamma \qquad \forall \psi_h \in W^0_{h,k}(\Gamma).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Gamma)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|v\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{4.5}$$

where C is a positive constant independent of h.

Proof. We prove that P_h is continuous on $L^2(\Gamma)$ and on $H_0^1(\Gamma)$ following [5, Section 1.6.3]. Then, the inequality (4.5) can be dirived by Hilbertian interpolation. For the L^2-continuity, we exploit the fact that, from the definition of P_h,

$$(v - P_h v, P_h v)_\Gamma = 0.$$

Therefore, by Pythagoras identity,

$$\|v\|^2_{L^2(\Gamma)} = \|v - P_h v\|^2_{L^2(\Gamma)} + \|P_h v\|^2_{L^2(\Gamma)} \geq \|P_h v\|^2_{L^2(\Gamma)}.$$

Let us now consider $v \in H_0^1(\Gamma)$.The Scott-Zhang interpolation operator SZ_h from $H_0^1(\Gamma)$ to $W^0_{h,k}(\Gamma)$ satisfies the following inequalities,

$$\|SZ_h v\|_{H^1(\Gamma)} \leq C_1 \|v\|_{H^1(\Gamma)} \tag{4.6}$$

$$\|v - SZ_h v\|_{L^2(\Gamma)} \leq C_2 h \|v\|_{H^1(\Gamma)}. \tag{4.7}$$

Therefore, using (4.6),

$$\begin{aligned}
\|\nabla P_h v\|_{L^2(\Gamma)} &\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + \|\nabla SZ_h v\|_{L^2(\Gamma)} \\
&\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)}
\end{aligned}$$

and by using the inverse inequality we obtain

$$\begin{aligned}
\|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} &\leq \frac{C_3}{h}\|P_h v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&= \frac{C_3}{h}\|P_h(v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq \ \text{(Stability of } P_h \text{ in } L^2\text{)}\ \frac{C_3}{h}\|v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq \ \text{(using (4.7))}\ \frac{C_3}{h} C_2 h \|v\|_{H^1(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq (C_2 C_3 + C_1)\|v\|_{H^1(\Gamma)},
\end{aligned}$$

from which we obtain the continuity in $H_0^1(\Gamma)$.
I would skip this proof and just leave the citation ∎

LEMMA 4.3. *There exists a constant $\gamma > 0$ such that for any $\mu_h \in Q_h$*

$$\sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \geq \gamma \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. Let μ_h be in $Q_h(\Gamma)$. From the continuous case, in particular from (3.4), we have

$$\|\mathcal{E}\|^{-1}\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h \rangle}{\|v\|_{H^1(\Omega)}}$$

and by the trace inequality $\|Tv\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_T \|v\|_{H^1(\Omega)}$ (see [1, 7.56]), we obtain

$$\sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h \rangle}{\|v\|_{H^1(\Omega)}} \leq C_T \sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h \rangle}{\|Tv\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

By the definition of P_h and (4.5)

$$C_T \sup_{v\in H_0^1(\Omega)} \frac{\langle Tv, \mu_h\rangle}{\|Tv\|_{H^{\frac{1}{2}}(\Gamma)}} = C_T \sup_{v\in H_0^1(\Omega)} \frac{\langle P_h(Tv), \mu_h\rangle}{\|Tv\|_{H^{\frac{1}{2}}(\Gamma)}}$$
$$\leq C_T C \sup_{v\in H_0^1(\Omega)} \frac{\langle P_h(Tv), \mu_h\rangle}{\|P_h(Tv)\|_{H^{\frac{1}{2}}(\Gamma)}}$$
$$= C_T C \sup_{q_h\in W_{h,k}^0(\Gamma)} \frac{\langle q_h, \mu_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

∎

THEOREM 4.4 (Discrete inf-sup). *The inequality* (4.3) *holds, namely* $\exists\beta_{h,1} > 0$ *s.t.*

$$\inf_{\mu_h\in Q_h(\Gamma)} \sup_{\substack{v_h\in X_{h,k}^0(\Omega),\\ v_{\odot h}\in X_{h,k}^0(\Lambda)}} \frac{\langle Tv_h - \mathcal{U}_E v_{\odot h}, \mu_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||\,\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \beta_{h,1}. \tag{4.8}$$

Proof. Let $\mu_h \in Q_h(\Gamma)$. As in the continuos case, we choose $v_{\odot h} = 0$ and we have

$$\sup_{\substack{v_h\in X_{h,k}^0(\Omega),\\ v_{\odot h}\in X_{h,k}^0(\Lambda)}} \frac{\langle Tv_h - \mathcal{U}_E v_{\odot h}, \mu_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h\in X_{h,k}^0(\Omega)} \frac{\langle Tv_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Therefore, we want to prove that there exists $\beta_{h,1}$ such that

$$\sup_{v_h\in X_{h,k}^0(\Omega)} \frac{\langle Tv_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,1}\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall\mu_h \in Q_h(\Gamma).$$

Using Lemma 4.3 and the boundedness of the armonic extension operator $\mathcal{E}$ from $H_{00}^{\frac{1}{2}}(\Gamma)$ to $H_0^1(\Omega)$ introduced in the previous section, we have

$$\gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h, \mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}\| \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h, \mu_h\rangle_\Gamma}{\|\mathcal{E}q_h\|_{H^1(\Omega)}}.$$

Let $R_h : H_0^1(\Omega) \to X_{h,k}^0(\Omega)$ be a quasi interpolation operator satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq C_R\|v\|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Therefore, we obtain

$$\|\mathcal{E}\| \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h, \mu_h\rangle_\Gamma}{\|\mathcal{E}q_h\|_{H^1(\Omega)}} \leq \|\mathcal{E}\|C_R \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h, \mu_h\rangle_\Gamma}{\|R_h\mathcal{E}q_h\|_{H^1(\Omega)}}$$

and we have

$$\begin{aligned}\gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h, \mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}\|C_R \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h, \mu_h\rangle_\Gamma}{\|R_h\mathcal{E}q_h\|_{H^1(\Gamma)}} \\ &= \|\mathcal{E}\|C_R \sup_{q_h\in Q_h(\Gamma)} \frac{\langle T\ R_h E q_h, \mu_h\rangle_\Gamma}{\|R_h\mathcal{E}q_h\|_{H^1(\Omega)}} \leq \|\mathcal{E}\|C_R \sup_{v_h\in X_{h,k}(\Omega)} \frac{\langle Tv_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.\end{aligned} \tag{4.9}$$

Therefore the inf-sup condition (4.8) holds with $\beta_{h,1} = \gamma\|\mathcal{E}\|^{-1}C_R^{-1}$. ∎

REMARK 4.1. *We notice that to prove the result in Lemma 4.3 (and then the discrete inf-sup condition) basically we need a projection operator* $P_h : H_{00}^{\frac{1}{2}} \longrightarrow W_{h,k}^0(\Gamma)$ *orthogonal in the multiplier space* $Q_h(\Gamma)$*,* *namely such that $\langle P_h v, \mu_h\rangle = \langle v, \mu_h\rangle, \forall \mu_h \in Q_h(\Gamma)$, and continuous in $H^{\frac{1}{2}}(\Gamma)$. Therefore, in principle different choices than $Q_h(\Lambda) = W^0_{h,k}(\Gamma)$ could be considered if we can build an operator P_h satisfying these properties. In [2] such operator P_h is built for a particular choice of $Q_h(\Gamma)$ but it is not clear how they prove the H^1-stability inequality (and consequently the $H^{\frac{1}{2}}$-stability) with a constant independent of the mesh size h...*

4.1.2. Problem 2. This problem requires to find $u_h \in X^0_{h,k}(\Omega),\ u_{\odot h} \in X^0_{h,k}(\Lambda),\ \lambda_{\odot h} \in Q_h(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$(u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{Tv_h} - v_{\odot h}, \lambda_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} \\ = (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h} \in X_h(\Lambda) \tag{4.10a}$$

$$\langle \overline{Tu_h} - u_{\odot h}, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} = 0 \quad \forall \mu_{\odot h} \in Q_h(\Lambda)\,. \tag{4.10b}$$

We introduce the space $W^0_{h,k}(\Lambda) \subset H^{\frac{1}{2}}_{00}(\Lambda)$, which is the averaged trace space of functions running in $H^1_0(\Omega)$. It coincides with the space of continuous piecewise polynomials of degree k defined on Λ and satisfying homogeneous Dirichlet boundary condition. (Add assumptions..) We choose $Q_h(\Lambda) = W^0_{h,k}(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda \cap \partial\Omega$ also for the Lagrange multiplier. With this choice for $Q_h(\Lambda)$, we can prove the well-posedness of the discrete problem. In particular, following the same steps as for Problem 1, we can prove the following results.

Lemma 4.5. *Let $P_h : H^{\frac{1}{2}}_{00}(\Lambda) \longrightarrow W^0_{h,k}(\Lambda)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Lambda)$ by*

$$(P_h v, \psi)_\Lambda = (v, \psi)_\Lambda \qquad \forall \psi \in W^0_{h,k}(\Lambda).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}_{00}(\Lambda)} \le C\|v\|_{H^{\frac{1}{2}}_{00}(\Lambda)},$$

where C is a positive constant independent of h.

Lemma 4.6. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \ge \gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_h \in W^0_{h,k}(\Lambda).$$

Theorem 4.7 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists \beta_{h,2} > 0$ s.t.*

$$\inf_{\mu_h \in Q_h(\Lambda)} \sup_{\substack{v_h \in X^0_{h,k}(\Omega),\\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle \overline{Tv_h} - v_{\odot h}, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]|||\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}} \ge \beta_{h,2}. \tag{4.11}$$

Proof. Let $\mu_{\odot h}$ be arbitrarly chosen in $Q_h(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists $\beta_{h,2}$ such that

$$\sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle \overline{Tv_h}, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \ge \beta_{h,2}\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_{\odot h} \in Q_h(\Lambda).$$

Let us denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ. Using Lemma 2.12, we easily have for any $w \in H^{\frac{1}{2}}(\Lambda)$,
TO DO: generalize to non constant $\partial\mathcal{D}$

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}|\|w\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Consequently, from Lemma 4.9, using again the extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^0_{h,k}(\Omega)$, we obtain

$$
\begin{aligned}
\gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_\Lambda}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\
&= |\partial\mathcal{D}| \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_\Lambda}{\|\mathcal{U}_E q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq |\partial\mathcal{D}|\|E\| \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_\Lambda}{\|E\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}|\|E\|C_R \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_\Lambda}{\|R_h E\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&= |\partial\mathcal{D}|\|E\|C_R \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle \Pi_1 R_h E\mathcal{U}_E q_h,\mu_h\rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}|\|E\|C_R \sup_{v_h\in X_h(\Omega)} \frac{\langle \Pi_2 v_h,\mu_h\rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}. \qquad \square
\end{aligned} \tag{4.12}
$$

******************extension to non uniform $|\partial\mathcal{D}|$

Lemma 4.8. *Let $P_h : H^{\frac{1}{2}}_{00}(\Lambda) \longrightarrow W^0_{h,k}(\Lambda)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Lambda)$ by*

$$(P_h v,\psi)_{\Lambda,|\partial\mathcal{D}|} = (v,\psi)_{\Lambda,|\partial\mathcal{D}|} \qquad \forall\psi \in W^0_{h,k}(\Lambda).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \leq C\|v\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|},$$

where C is a positive constant independent of h.

Lemma 4.9. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \geq \gamma\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall\mu_{\odot h} \in W^0_{h,k}(\Lambda).$$

Theorem 4.10 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists\beta_{h,2} > 0$ s.t.*

$$\inf_{\mu_h\in Q_h(\Lambda)} \sup_{\substack{v_h\in X^0_{h,k}(\Omega),\\ v_{\odot h}\in X^0_{h,k}(\Lambda)}} \frac{\langle \overline{Tv_h} - v_{\odot h},\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h,v_{\odot h}]|||\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \beta_{h,2}. \tag{4.13}$$

Proof. Let $\mu_{\odot h}$ be arbitrarly chosen in $Q_h(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists $\beta_{h,2}$ such that

$$\sup_{v_h\in X^0_{h,k}(\Omega)} \frac{\langle \overline{Tv_h},\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,2}\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall\mu_{\odot h} \in Q_h(\Lambda).$$

Let us denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ. We notice that for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = \|w\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.$$

Consequently, from Lemma 4.9, using again the extension operator $\mathcal{E}$ from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^0_{h,k}(\Omega)$, we obtain

$$
\begin{aligned}
(4.14)\quad \gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Lambda)} &\le \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \\
&= \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{U}_E q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \le \|\mathcal{E}\| \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{E}\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&\le \|\mathcal{E}\| C_R \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle q_h,\mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|R_h E\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&= \|\mathcal{E}\| C_R \sup_{q_h\in W^0_{h,k}(\Lambda)} \frac{\langle \overline{TR_h E\mathcal{U}_E q_h},\mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\le \|\mathcal{E}\| C_R \sup_{v_h\in X_h(\Omega)} \frac{\langle \overline{Tv_h},\mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}}. \quad \square
\end{aligned}
$$

4.2. $\mathcal{T}_h^\Omega$ non conforming to Γ. We analyze now the case in which the elements of the 3D mesh $\mathcal{T}_h^\Omega$ cut the interface Γ. It is easy to understand that the formulation of Problem 2 is more suitable. Add more details, we can refer also to cutFEM (Burman, Massing etc.) explaining the limitation of that approach (network case for example).
Therefore we focus on the analysis of Problem 2.

4.2.1. Problem 2. We consider for the solutions u_h and $u_{\odot h}$ the spaces $X^0_{h,1}(\Omega)$ and $X^0_{h,1}(\Lambda)$, see the previous subsection for the definition. Concerning the multiplier space, we make the following choice, $Q_h(\Lambda) = \{\lambda_{\odot h} : \lambda_{\odot h} \in P^0(K)\forall K \in \mathcal{T}^\Lambda_{h'}\}$, namely the multiplier lives on the same mesh used for the 1D solution $u_{\odot h}$. Notice that in this case we suppose that the mesh sizes of the 3D mesh $\mathcal{T}_h^\Omega$ and the 1D mesh $\mathcal{T}^\Lambda_{h'}$ are different, in particular we suppose the 1D mesh is finer. With this choice the problem is not inf-sup stable, therefore the idea is to add a stabilization term $s(\lambda_{\odot h},\mu_{\odot h})$ to (4.10a) following the approach introduce in [3]. In particular, we build a new multiplier space $L_h(\Lambda)$ for which the discrete inf-sup condition is fulfilled and we build a projection operator $\pi_L : Q_h(\Lambda) \to L_h(\Lambda)$. Based on this projection operator, we build the stabilization term $s(\lambda_{\odot h},\mu_{\odot h})$ and prove that $\forall[u_h,u_{\odot h}]$, there exists $\xi_h([u_h,u_{\odot h}]) \in Q_h(\Lambda)$ s.t.

$$
(4.15)\qquad a([u_h,u_{\odot h}],[u_h,u_{\odot h}]) + b([u_h,u_{\odot h}],\xi_h([u_h,u_{\odot h}])) \ge \alpha_\xi |||[u_h,u_{\odot h}]|||_{X^0_{h,1}(\Omega)\times X^0_{h,1}(\Lambda)},
$$

$$
(4.16)\qquad (s(\xi_h,\xi_h))^{\frac{1}{2}} \le c_s |||[u_h,u_{\odot h}]|||_{X^0_{h,1}(\Omega)\times X^0_{h,1}(\Lambda)},
$$

being $|||[\cdot\,,\,\cdot]|||_{X^0_{h,1}(\Omega)\times X^0_{h,1}(\Lambda)}$ a suitable discrete norm.

We recall that in the case of Problem 2,

$$
b([u_h,u_{\odot h}],\lambda_{\odot h}) = \left(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h},\lambda_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}.
$$

The construction of the inf-sup stable space $L_h(\Lambda)$ is based on macro elements of diameter H, where H is sufficiently larger than h. In particular, we assume that there exists positive constants c_h and c_H such that $c_h h \le H \le c_H^{-1} h$, with c_h is sufficiently large. The space is constructed assembling the elements of the 3D mesh $\mathcal{T}_h^\Omega$ intersecting the 1D manifold Λ into macro patches ω_j. These patches are such that and $H \le |\omega_j \cap \Lambda| \le H + h$. Namely, $\omega_j = \cup_{i=0}^{M_j} K_i$, where $K_i \in \mathcal{T}_h^\Omega$ and M_j is uniformly bounded in j by some $M_F \in \mathbb{N}$. We define

$$
L_h(\Lambda) = \left\{ l_{\odot h} : l_{\odot h} \in P^0(\omega_j \cap \Lambda)\,\forall j \right\}.
$$

Moreover, we associate to each patch ω_j a shape regular extended patch which is still denoted by ω_j for notational simplicity, which is built adding to ω_j a sufficient number of elements of $\mathcal{T}_h^\Omega$. In addition to the shape regularity assumption, we assume also that interiors of the extended patches ω_j are disjoint and they satisfy the conditions that $\Gamma \subset \cup_j \omega_j$ and $\text{meas}(\Gamma_{\omega_j\cap\Lambda} \cap \omega_j) = O(H^2)$, where $\Gamma_{\omega_j\cap\Lambda}$ is the portion of Γ with centerline $\omega_j \cap \Lambda$. The latter assumption is required to make sure that the intersection of $\Gamma_{\omega_j\cap\Lambda}$ and ω_j is not too small and it will be needed later on to prove the inf-sup stability of the space L_h. Thanks to the shape regularity of these extended patches, we have that the following discrete trace and Poincarè-type inequalites hold. More precisely, for every function $v \in H^1(\omega_j)$,

$$\|\mathcal{T}_\Gamma v\|_{L^2(\Gamma\cap\omega_j)} \lesssim H^{-\frac{1}{2}} \|v\|_{L^2(\omega_j)} \tag{4.17}$$

$$\|v - \pi_L \overline{\mathcal{T}}_\Lambda v\|_{L^2(\omega_j)} \le c_P H \|\nabla v\|_{L^2(\omega_j)}, \tag{4.18}$$

where for any $w \in L^2(\Lambda)$, $\pi_L w \in L_h$ and it is defined as

$$\pi_L w_{|\omega_j\cap\Lambda} = \frac{1}{|\Gamma_{\omega_j\cap\Lambda}|} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| w.$$

Moreover $\forall u_h \in X_h^\Omega$ we have the following average inequality

$$\sum_j \|\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \le \sum_j \|\mathcal{T}_\Gamma T u_h\|^2_{L^2(\omega_j\cap\Gamma)}.$$

I think this inequality is valid but only globally. Indeed locally it is not guaranteed that the portion of Γ with centerline $\omega_j \cap \Lambda$, namely $\Gamma_{\omega_j\cap\Lambda}$ is contained in $\omega_j \cap \Gamma$.
These choices lead to the following stabilization

$$s(\lambda_{\odot h}, \mu_{\odot h}) = \sum_{K\in\mathcal{T}^\Lambda_{h'}} \int_{\partial K} h[\![\lambda_{\odot h}]\!][\![\mu_{\odot h}]\!],$$

being $[\![\lambda_{\odot h}]\!]$ the jump of $\lambda_{\odot h}$ across the internal faces of $\mathcal{T}^\Lambda_{h'}$.

Lemma 4.11. *The space L_h is inf-sup stable, namely $\forall l_{\odot h} \in L_h(\Lambda)$, $\exists \beta > 0$ s.t.*

$$\sup_{\substack{v_h\in X^0_{h,1}(\Omega),\\ v_{\odot h}\in X^0_{h',1}(\Lambda)}} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{|\!|\!|[v_h, v_{\odot h}]|\!|\!|} \ge \beta \|l_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

and the constant is independent of the cuts.

Proof. As in the continuous case, we can choose $v_{\odot h} = 0$ and we prove that

$$\sup_{v_h\in X^0_{h,1}(\Omega)} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \ge \beta \|l_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

Proving the last inequality it is equivalent to find the Fortin operator $\pi_F : H^1_0(\Omega) \to X^0_{h,1}(\Omega)$, such that

$$\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|} = 0, \quad \forall v \in H^1_0(\Omega),\, l_{\odot h} \in L_h(\Lambda)$$

and

$$\|\pi_F v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.$$

We define

$$\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j}$$

where $I_h : H^1(\Omega) \to X^0_{h,1}$ denotes an $H^1(\Omega)$-stable interpolant and $\varphi_j \in X^0_{h,1}(\Omega)$ is such that $\mathrm{supp}(\varphi_j) \subset \bar{\omega}_j$, $\mathrm{supp}(\mathcal{T}_\Gamma \varphi) \subset \Gamma_{\omega_j \cap \Lambda} \cap \omega_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j = O(H) \text{ and } \|\nabla\varphi\|_{L^2(\omega_j)} = O(1).$$

We notice that $\mathrm{supp}(\mathcal{T}_\Gamma \varphi) \subset \Gamma_{\omega_j \cap \Lambda} \cap \omega_j$ ensures that $\overline{\mathcal{T}}_\Lambda \varphi \subset \omega_j \cap \Lambda$. This construction is always possible provided H is sufficiently larger that h. Then we have

$$\begin{aligned}
(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, l_{\odot h})_{\Lambda, |\partial\mathcal{D}|} &= \sum_j \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \sum_i \alpha_i \overline{\mathcal{T}}_\Lambda \varphi_i \right] l_{\odot h} \\
&= (\mathrm{supp}(\overline{\mathcal{T}}_\Lambda \varphi) \subset \omega_j \cap \Lambda) \sum_j \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \alpha_j \overline{\mathcal{T}}_\Lambda \varphi_j \right] l_{\odot h} \\
&= \sum_j \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v) l_{\odot h} - \frac{\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j} \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j l_{\odot h} \\
&= (\text{using } l_h \text{ constant on } \omega_j \cap \Lambda)\, 0.
\end{aligned}$$

Concerning the continuity of π_F, we have

$$\begin{aligned}
\|\nabla \pi_F v\|_{L^2(\Omega)} &\le \|\nabla I_h v\|_{L^2(\Omega)} + \left(\sum_j |\alpha_j|^2 \|\nabla\varphi_j\|^2_{L^2(\bar{\omega}_j)} \right)^{\frac{1}{2}} \\
&(\text{stability of } I_h) \lesssim \|\nabla v\|_{L^2(\Omega)} + \left(\sum_j |\alpha_j|^2 \|\nabla\varphi_j\|^2_{L^2(\bar{\omega}_j)} \right)^{\frac{1}{2}}
\end{aligned}$$

and for the second term we have

$$\begin{aligned}
&\sum_j |\alpha_j|^2 \|\nabla\varphi_j\|^2_{L^2(\bar{\omega}_j)} \le \\
&(\text{using } \|\nabla\varphi_j\| = O(1)) \lesssim \sum_j \frac{\left(\left| \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v) \right| \right)^2}{\left(\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j \right)^2} \\
&\left(\text{since } \left| \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j \right| = O(H) \right) \lesssim \frac{1}{H^2} \sum_j \left(\left| \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v) \right| \right)^2 \\
&(\text{Jensen}) \lesssim \frac{1}{H^2} \sum_j |\omega_j \cap \Lambda| \int_{\omega_j \cap \Lambda} |\partial\mathcal{D}|^2 (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)^2 \\
&(\text{being } |\omega_j \cap \Lambda| \le H + h) \lesssim \frac{1}{H} \sum_j \|\overline{\mathcal{T}}_\Lambda V - I_h v\|^2_{L^2(\omega_j \cap \Lambda), |\partial\mathcal{D}|} \\
&(\text{average inequality}) \lesssim \frac{1}{H} \sum_j \|\mathcal{T}_\Gamma (v - I_h v)\|^2_{L^2(\omega_j \cap \Gamma)} \\
&(\text{trace inequality}) \lesssim \frac{1}{H^2} \sum_j \|v - I_h v\|^2_{L^2(\omega_j)} \lesssim \frac{1}{H^2} \|v - I_h v\|^2_{L^2(\Omega)} \\
&(\text{approximation properties of } I_h) \lesssim \|\nabla v\|^2_{L^2(\Omega)}
\end{aligned}$$

and the continuity of π_F follows. □

We choose the following discrete norm

$$|||[u_h, u_{\odot h}]|||^2_{X_h(\Omega)\times X_{h'}(\Lambda)} = \|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|},$$

where $\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|} = \|h^{\frac{1}{2}}(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\|^2_{L^2(\Lambda),|\partial\mathcal{D}|}$. Then, we have the following lemma.

Lemma 4.12. *The inequalities* (4.15) *and* (4.16) *hold.*

Proof. Concerning the coercivity property (4.15), we have to show that $\forall [u_h, u_{\odot h}]$, there esists $\xi_h \in Q_h(\Lambda)$ s.t.

$$\begin{aligned}(u_h, u_h)_{H^1(\Omega)} + (u_{\odot h}, u_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \xi_h)_{\Lambda,|\partial\mathcal{D}|} \\ \geq \alpha_\xi(\|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}.\end{aligned}$$

We choose

$$\xi_{h|\omega_j\cap\Lambda} = \delta\frac{1}{H}\pi_L(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})_{|\omega_j\cap\Lambda}$$

and we recall that

$$\pi_L(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})_{|\omega_j\cap\Lambda} = \frac{1}{|\Gamma_{\omega_j\cap\Lambda}|}\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}).$$

Actually, $\xi_h \in L_h(\Lambda) \subset Q_h(\Lambda)$. Then,

$$\begin{aligned}\big(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \xi_h\big)_{\Lambda,|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\xi_h \\ &= \delta\frac{1}{H}\sum_j\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\pi_L(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}))^2 \\ (\text{orthogonality of } \pi_L) &= \delta\frac{1}{H}\|(\pi_L - \mathcal{I})(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} + \delta\frac{1}{H}\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\ &\geq -\delta\frac{1}{H}\sum_j\|(\pi_L - \mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} - \delta\frac{1}{H}\sum_j\|(\pi_L - \mathcal{I})u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\ &\quad + \delta\frac{1}{H}\sum_j\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|}.\end{aligned}$$

For the first term we have

$$\begin{aligned}\sum_j\|(\pi_L - \mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j\int_{\omega_j\cap\Lambda}|\partial\mathcal{D}|(\pi_L\overline{\mathcal{T}}_\Lambda u_h - \overline{\mathcal{T}}_\Lambda u_h)^2 \\ (\text{Average inequality}) &\leq \sum_j\int_{\omega_j\cap\Gamma}(\pi_L\overline{\mathcal{T}}_\Lambda u_h - \mathcal{T}_\Gamma u_h)^2 \\ (\text{trace inequality}) &\leq \sum_j\frac{1}{H}\int_{\omega_j}(\pi_L\overline{\mathcal{T}}_\Lambda u_h - u_h)^2 \\ (\text{Poincare, see [5, Corollary B.65]}) &\leq \sum_j Hc_P^2\|\nabla u_h\|^2_{L^2(\omega_j)}.\end{aligned}$$

For the second term we have

$$\sum_j \|(\pi_L - \mathcal{I})u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} = \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L u_{\odot h} - u_{\odot h})^2$$

$$\text{(Poincare, [5, Corollary B.65])} \lesssim \sum_j H^2 c_P^2 \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\nabla u_{\odot h})^2$$

$$\text{(since } H \text{ is fixed, we can find a constant s.t. } H|\partial\mathcal{D}| \lesssim |\mathcal{D}|) \lesssim \sum_j H c_P^2 \int_{\omega_j\cap\Lambda} |\mathcal{D}|(\nabla u_{\odot h})^2$$

$$\lesssim \sum_j H c_P^2 \|\nabla u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\mathcal{D}|}.$$

N.B. we are using a kind of weigthed Poincare inequality, check... I think it should work because I can do something like this

$$\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|u^2 \le max|\partial\mathcal{D}| \int_{\omega_j\cap\Lambda} u^2 \le max|\partial\mathcal{D}| \int_{\omega_j\cap\Lambda} (\nabla u)^2 = \frac{max|\partial\mathcal{D}|}{min|\partial\mathcal{D}|} min|\partial\mathcal{D}| \int_{\omega_j\cap\Lambda} (\nabla u)^2 \le \frac{max|\partial\mathcal{D}|}{min|\partial\mathcal{D}|} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\nabla u)^2$$

Therefore, we obtain

$$a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \ge (1-\delta c_P^2)\|\nabla u_h\|^2_{L^2(\Omega)} + (1-\delta c_P^2)\|\nabla u_{\odot h}\|^2_{L^2(\Lambda),|\mathcal{D}|} + \delta c_H \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}$$

and choosing $\delta = \frac{1}{2c_P^2}$ we obtain the coercivity inequality.
Concerning the stability inequality (4.16), the proof is analogous to the one in [3]. ◻

5. A benchmark problem with analytical solution. We consider the following 3D-1D coupled problem,

$$-\Delta u = f \quad \text{in } \Omega \tag{5.1a}$$
$$-d^2_{zz} u_\odot = g \quad \text{on } \Lambda \tag{5.1b}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{5.1c}$$
$$u_\odot - \overline{u} = q \quad \text{on } \Lambda \tag{5.1d}$$

where $\Omega = [0,1]\times[0,1]\times[0,H]$, $\Lambda = \{x = 0.5\}\times\{y = 0.5\}\times[0,H]$ and

$$f = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$$
$$g = \frac{\pi^2}{H^2}\sin\left(\frac{\pi z}{H}\right)$$
$$q = \sin\left(\frac{\pi z}{H}\right).$$

In this case the z coordinate coincides with the axial coordinate along Λ. We define $\Sigma = [0.25, 0.75]\times[0.25, 0.75]\times[0,H]$. The average of the 3D solution $\overline{u}$ in (5.1d) is computed on the cross section $\partial\mathcal{D}$ of the virtual interface $\Gamma = \partial\Sigma$. The exact solution of (5.1) is given by

$$u = \sin(2\pi x)\sin(2\pi y) \tag{5.2}$$
$$u_\odot = \sin\left(\frac{\pi z}{H}\right) \tag{5.3}$$

Let us notice that $u_\odot$ satisfies homogeneous Dirichlet conditions at the boundary of Λ. Moreover, the solution (5.2)-(5.3) satisfies on Γ the relation

$$\lambda = \nabla u \cdot \mathbf{n}_\oplus = d_z u_\odot n_{\oplus,z} = 0, \tag{5.4}$$

being $n_{\oplus,z}$ the $z-$component of the normal unit vector to Γ.

We prove that (5.2)-(5.3) is solution of (2.7) and (2.8) in the simplified case in which the starting 3D-3D problem is

$$\begin{aligned}
&(5.5a) && -\Delta u_\oplus = f && \text{in } \Omega_\oplus,\\
&(5.5b) && -\Delta u_\ominus = g && \text{in } \Sigma,\\
&(5.5c) && -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus && \text{on } \Gamma,\\
&(5.5d) && u_\ominus - u_\oplus = q && \text{on } \Gamma,\\
&(5.5e) && u_\oplus = 0 && \text{on } \partial\Omega\,.
\end{aligned}$$

instead of (2.1). Therefore the reduced problems in the two different formulations (2.7) and (2.8) become respectively

$$\begin{aligned}
&(5.6a) && (\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma\\
& && \quad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda)\\
&(5.6b) && \langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = \langle q, M\rangle_\Gamma \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,.
\end{aligned}$$

and

$$\begin{aligned}
&(5.7a) && (\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)}\\
& && \quad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\\
&(5.7b) && |\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \overline{q}, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,.
\end{aligned}$$

Let us prove that (5.2)-(5.3) is solution of (5.6). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (5.6a) we have

$$\begin{aligned}
&-(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d_{ss}^2 u_\odot, v_\odot)_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma\\
&\quad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda).
\end{aligned}$$

Clearly, since (5.2) satisfies (5.1a) and (5.3) satisfies (5.1b), we have that

$$\begin{aligned}
-(\Delta u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)}\\
-|\mathcal{D}|(d_{ss}^2 u_\odot, v_\odot)_{L^2(\Lambda)} &= |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)}
\end{aligned}$$

and being $L = \lambda = 0$, we can conclude that (5.2)-(5.3) satisfy (5.6a). The fact that the solution satisfy (5.6b) follows from (5.1d). We can prove in the same way that (5.2)-(5.3) is solution of (5.7), exploiting the fact that in this case $L = \overline{\lambda} = 0$.

Remark 5.1. *Let us notice that the 3D solution* (5.2) *is such that* $\overline{u} = 0$. *Therefore in* (5.1) *it is like we are solving two separated problems, one in* Ω *and the other on* Λ.

Remark 5.2. *It would be interesting to make a comparison between the solution of the fully coupled 3D-3D problem* (2.1) *(also in the simplified case of type* (5.5)*) and the solution of the reduced problems* (2.7) *and* (2.8). *Therefore, we could set the values of the data of the problem such that the reduced formulation becomes non-trivial and fully coupled. Then, we will solve both the original and reduced problem to observe the differences in the solutions and the values of the Lagrange multiplier.*

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