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Boundary Subspaces for the Finite Element Method With Lagrange Multipliers*

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Summary. The paper is concerned with the problem of constructing compatible interior and boundary subspaces for finite element methods with Lagrange multipliers to approximately solve Dirichlet problems for second-order elliptic equations. A new stability condition relating the interior and boundary subspaces is first derived, which is easier to check in practice than the condition known so far. Using the new condition, compatible boundary subspaces are constructed for quasiuniform triangular and rectangular interior meshes in two dimensions. The stability and optimal-order convergence of the finite element methods based on the constructed subspaces are proved.

Subject Classifications: AMS(MOS) 65-66, 65-80; CR 5-11, 5-17.

1. Introduction

Let Ω be a bounded domain of R^2 with a smooth boundary $\partial\Omega$. We consider the second-order elliptic model problem

$$\begin{aligned} -\Delta u+u&=f \quad \text{in} \quad \Omega \\ u&=g \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}$$

It is well known (c.f. [1, 2, 3]) that by introducing a Lagrange multiplier at the boundary, one can derive finite element approximations to problem (1.1) without requiring the fulfillment of the boundary condition in the subspaces. The essence of such methods is to select two subspaces, one for the interior of Ω and one for the boundary, and to combine the subspaces with a saddlepoint variational principle [1].

The Lagrange multiplier method is known to be generally stable and

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convergent provided that a certain abstract condition relating the two subspaces is satisfied [2, 4]. Counterexamples are also known showing that the failure of the stability condition may easily occur in practice and result in an unacceptably slow convergence rate [2]. Unfortunately to check the condition for any given finite element subspaces is a non-trivial task itself. So far the only practical criterion available in the literature for the situation considered is stated as follows [1]. Let the subspaces be associated with quasiuniform finite element meshes, so that h and k are the mesh parameters in the interior of Ω and on the boundary, respectively. Then the stability and optimal-order convergence of the method are assured provided that

$$k \geqq Ch, \tag{1.2}$$

where C is a given constant depending on Ω.

Condition (1.2) still leaves practical problems, as C depends on Ω in a complicated manner [1]. Also, if C happens to be large, the fulfillment of Eq. (1.2) may not be possible in practice. Fortunately, condition (1.2) is sufficient rather than necessary, and it is the purpose of this paper to show that stability and convergence of optimal order can be achieved also under conditions that are less severe than Eq. (1.2).

In fact, we find a variety of situations where, by appropriately defining the subspaces, the constant C in Eq. (1.2) can be allowed a value close to unity.

2. Preliminaries

The bounded domain $\Omega \subset R^2$ is said to be smooth if the boundary curve is locally representable by an infinitely differentiable function. The Sobolev spaces $H^m(\Omega)$ of real functions, m integer, $m \geqq 0$, are defined in the ordinary manner; the inner product and norm in $H^m(\Omega)$ are denoted by $(\cdot,\cdot)_{H^m(\Omega)}$ and $\|\cdot\|_{H^m(\Omega)}$, respectively. As usual, $H^0(\Omega) = L_2(\Omega)$. For non-integral s, $s > 0$, one defines the space $H^s(\Omega)$ by interpolation. For $s < 0$, $H^s(\Omega)$ is defined as the dual of $H^{-s}(\Omega)$ [6].

The Sobolev spaces $H^s(\partial\Omega)$ for the boundary of Ω are introduced in a similar manner. As is well known, the space $H^{1/2}(\partial\Omega)$ can also be interpreted as the trace space of functions in $H^1(\Omega)$. Denoting the trace operator by T, the norm of $H^{1/2}(\partial\Omega)$ can be defined as

$$\|\phi\|_{1/2,\partial\Omega} = \inf_{u \in H^1(\Omega)} \{\|u\|_{1,\Omega} : Tu = \phi\}.$$

The space $H^{-1/2}(\partial\Omega)$ is of special importance in the Lagrange multiplier method. By $H^{-1/2}(\partial\Omega)$ we mean the closure of $H^0(\partial\Omega) = L_2(\partial\Omega)$ with respect to the norm

$$\|\psi\|_{-1/2,\partial\Omega} = \sup_{\phi \in H^{1/2}(\partial\Omega)} \frac{\int_{\partial\Omega} \psi\phi\, ds}{\|\phi\|_{1/2,\partial\Omega}}.$$

In what follows we will use the abbreviation

$$\langle\psi, \phi\rangle = \int_{\partial\Omega} \psi\,\phi\, ds, \qquad \psi, \phi \in H^0(\partial\Omega).$$

We state below a well-known existence and regularity result for the solution of the problem (1.1). For the proof, see [1, 2, 6].

Theorem 2.1. *Let $f \in H^s(\Omega)$, $s \geqq 0$ and $g \in H^r(\partial\Omega)$, $r \geqq 1/2$. Then there exists a unique weak solution u_0 to problem (1.1) in $H^1(\Omega)$; the trace of the normal derivative $\partial u_0/\partial n$ exists in $H^{-1/2}(\partial\Omega)$, and we have the a priori estimates*

$$\|u_0\|_{1+\varepsilon,\Omega} + \left\|\frac{\partial u_0}{\partial n}\right\|_{-1/2+\varepsilon,\partial\Omega} \leqq C\left[\|f\|_{s,\Omega} + \|g\|_{r,\partial\Omega}\right]$$

where

$$\varepsilon = \min\{s+1,\ r-\tfrac{1}{2}\}.$$

Let us now describe the Lagrange multipliers method for the solution of the problem (1.1). Let $H = H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ with the norm $\|(u,\lambda)\|_H^2 = \|u\|_{1,\Omega}^2 + \|\lambda\|_{-1/2,\partial\Omega}^2$, and define on $H \times H$ the bilinear form

$$\mathscr{B}(u, \lambda; v, \mu) = (u, v)_{H^1(\Omega)} + \langle\mu, Tu\rangle - \langle\lambda, Tv\rangle.$$

Then if u_0 is the solution of the problem (1.1), it can be shown (see [1]) that $(u_0, \partial u_0/\partial n)$ is the unique solution of the problem

$$\mathscr{B}(u, \lambda; v, \mu) = (f, v)_{H^0(\Omega)} + \langle\mu, g\rangle \qquad \text{for all } (v, \mu) \in H;\ (u, \lambda) \in H. \tag{2.1}$$

Let M_1^h and M_2^h be finite-dimensional subspaces of $H^1(\Omega)$ and $H^{-1/2}(\partial\Omega)$, respectively. Then one can define the approximate solution of the problem (2.1) in the subspace $M^h = M_1^h \times M_2^h \subset H$ as the element $(u_h, \lambda_h) \in M^h$ which satisfies

$$\mathscr{B}(u_h, \lambda_h; v, \mu) = (f, v)_{H^0(\Omega)} + \langle\mu, g\rangle \qquad \text{for all } (v, \mu) \in M^h;\ (u_h, \lambda_h) \in M^h. \tag{2.2}$$

We state below the basic theorem of stability and convergence of the abstract approximate method based on Eq. (2.2). For the proof, see [1, 3 or 4].

Theorem 2.2. *Let*

$$\gamma_1 = \inf_{\mu \in M_2^h} \sup_{v \in M_1^h} \frac{\langle\mu, Tv\rangle}{\|\mu\|_{-1/2,\partial\Omega}\|v\|_{1,\Omega}}.$$

Then if $\gamma_1 > 0$, Eq. (2.2) defines (u_n, λ_n) uniquely. Moreover, the following error bound holds:

$$\left\|\left(u_0, \frac{\partial u_0}{\partial n}\right) - (u_h, \lambda_h)\right\|_H \leqq K\gamma_1^{-1} \inf_{(v,\mu)\in M^h} \left\|\left(u_0, \frac{\partial u_0}{\partial n}\right) - (v, \mu)\right\|_H,$$

where K is a numerical constant.

For a sequence of finite element subspaces $\{M_1^h, M_2^h\}$, $h \to 0$, the uniform stability condition is stated as

$$\gamma_1(h) \geqq \gamma_0 > 0. \tag{2.3}$$

The weaker condition

$$\gamma_1(h) > 0, \qquad h > 0, \tag{2.4}$$

which merely assures that the approximate solution is unique, is referred to in the literature as the rank condition. Conditions of the same character arise or are hidden also in many other situations (c.f. [3, 4, 7]).

We point out that the rank condition (2.4) is certainly satisfied, if M_2^h is in the trace space of M_1^h, i.e.,

$$M_2^h \subset T(M_1^h). \tag{2.5}$$

In the sequel we have the choice (2.5) mainly in mind. Also keeping in mind that the choice $M_2^h = T(M_1^h)$ generally results in disaster [2], the problem is to find out what kind of proper subspaces of $T(M_1^h)$ are allowable.

3. Another Stability Condition

We begin by associating to the subspaces M_1^h and M_2^h another parameter γ_2 defined as

$$\gamma_2 = \inf_{\mu \in M_2^h} \sup_{v \in M_1^h} \frac{\langle \mu, Tv \rangle}{h^{1/2} \|\mu\|_{0,\partial\Omega} \|v\|_{1,\Omega}}.$$

The purpose of this section is to show that the condition

$$\gamma_2(h) \geqq \gamma_0 > 0 \tag{3.1}$$

is sufficient for the condition (2.3) under rather general assumptions on the subspaces. Obviously the condition (3.1) is more easy to check in practice as the norms involved in γ_2 can be evaluated locally.

Let us associate to each $\mu \in M_2^h$ a function $z(\mu) \in H^1(\Omega)$, defined as the solution to the Neumann problem

$$(z(\mu), v)_{H^1(\Omega)} = \langle \mu, Tv \rangle \qquad \text{for all } v \in H^1(\Omega). \tag{3.2}$$

Function $z(\mu)$ is uniquely defined for each μ. Moreover, we have:

Lemma 3.1.

$$\|z(\mu)\|_{1,\Omega}^2 = \langle \mu, Tz(\mu) \rangle = \|\mu\|_{-1/2,\partial\Omega}^2, \tag{3.3}$$

and

$$\|z(\mu)\|_{3/2,\Omega} \leqq K_1 \|\mu\|_{0,\partial\Omega}. \tag{3.4}$$

where K_1 depends only on Ω.

For the proof, see [3] and [6].

We will need the following additional constants, which generally depend on the subspaces, but are assumed to be bounded independently of h:

$$K_2 = \sup_{\mu \in M_2^h} \inf_{v \in M_1^h} \frac{\|z(\mu) - v\|_{1,\Omega}}{h^{1/2} \|z(\mu)\|_{3/2,\Omega}}, \tag{3.5}$$

$$K_3 = \sup_{\mu \in M_2^h} \frac{h^{1/2} \|\mu\|_{0,\partial\Omega}}{\|\mu\|_{-1/2,\partial\Omega}}. \tag{3.6}$$

Constant K_2 is obviously independent of h whenever the space M_1^h has the sufficient approximability properties. Constant K_3 is related to an inverse assumption; if the finite element mesh at the boundary is quasiuniform with mesh lengths proportional to h, then K_3 is bounded independently of h.

Relations between the various constants introduced above are established by the next Theorem.

Theorem 3.2. *The following inequalities hold:*

$$\tfrac{1}{2} \min\{1, (K_1 K_2)^{-1} \gamma_2\} \leqq \gamma_1 \leqq K_3^{-1} \gamma_2, \tag{3.7}$$

and

$$\gamma_2 K_3 \leqq 1. \tag{3.8}$$

Proof. Equation (3.8) and the second inequality in Eq. (3.7) follow directly from the definition of the constant K_3 and from the definitions of the norms in $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. So, let us prove only the first inequality in Eq. (3.7). For this, let $\mu \in M_2^h$ be given and $z = z(\mu)$ be defined according to Eq. (3.2). Further, define $z_h \in M_1^h$ so that

$$(z_h, v)_{H^1(\Omega)} = (z, v)_{H^1(\Omega)} \qquad \text{for all } v \in M_1^h.$$

Then z_h is the best approximation to z in M_1^h, in the norm of $H^1(\Omega)$. So, by Eq. (3.3),

$$\|z_h\|_{1,\Omega} \leqq \|z\|_{1,\Omega} = \|\mu\|_{-1/2,\partial\Omega},$$

and in view of Eqs. (3.5) and (3.4),

$$\|z - z_h\|_{1,\Omega} \leqq K_2 h^{1/2} \|z\|_{3/2,\Omega} \leqq K_1 K_2 h^{1/2} \|\mu\|_{0,\partial\Omega}.$$

Hence, we have

$$\begin{aligned} \langle \mu, T z_h \rangle &= \langle \mu, T z \rangle + \langle \mu, T(z_h - z) \rangle \\ &\geqq \|\mu\|_{-1/2,\partial\Omega}^2 - \|\mu\|_{-1/2,\partial\Omega} \|z_h - z\|_{1,\Omega} \\ &\geqq \left[1 - K_1 K_2 \frac{h^{1/2} \|\mu\|_{0,\partial\Omega}}{\|\mu\|_{-1/2,\partial\Omega}}\right] \|\mu\|_{-1/2,\partial\Omega}^2. \end{aligned}$$

Now if μ is such that

$$\|\mu\|_{0,\partial\Omega} \leqq (2 K_1 K_2)^{-1} h^{-1/2} \|\mu\|_{-1/2,\partial\Omega}, \tag{3.9}$$

then, by the above inequality,

$$\sup_{v\in M_1^h} \frac{\langle \mu, Tv\rangle}{\|\mu\|_{-1/2,\partial\Omega}\|v\|_{1,\Omega}} \geqq \frac{\langle \mu, Tz_h\rangle}{\|\mu\|_{-1/2,\partial\Omega}\|z_h\|_{1,\Omega}} \geqq \frac{1}{2}.$$

On the other hand, if Eq. (3.9) is not satisfied, i.e.,

$$\|\mu\|_{-1/2,\partial\Omega} \leqq 2K_1K_2h^{1/2}\|\mu\|_{0,\partial\Omega},$$

then

$$\sup_{v\in M_1^h} \frac{\langle \mu, Tv\rangle}{\|\mu\|_{-1/2,\partial\Omega}\|v\|_{1,\Omega}} \geqq (2K_1K_2)^{-1} \sup_{v\in M_1^h} \frac{\langle \mu, Tv\rangle}{h^{1/2}\|\mu\|_{0,\partial\Omega}\|v\|_{1,\Omega}}$$
$$\geqq (2K_1K_2)^{-1}\gamma_2.$$

Combining the two inequalities we have

$$\sup_{v\in M_1^h} \frac{\langle \mu, Tv\rangle}{\|\mu\|_{-1/2,\partial\Omega}\|v\|_{1,\Omega}} \geqq \frac{1}{2}\min\{1, (K_\cdot K_2)^{-1}\gamma_2\}$$

for all $\mu\in M_2^h$. This completes the proof. ∎

Remark 1. To prove the sufficiency of condition (3.1) for the stability, one needs only to assume that K_1 and K_2 are bounded independently of h. However, Eq. (3.8) indicates that condition (3.1) cannot hold unless K_3 in Eq. (3.6) is also bounded independently of h.

Remark 2. Another way to fulfill condition (2.3) is as follows. Let k be a parameter associated to the boundary subspace so that the constant

$$K_3' = \sup_{\mu\in M_2^h} \frac{k^{1/2}\|\mu\|_{0,\partial\Omega}}{\|\mu\|_{-1/2,\partial\Omega}}$$

is bounded independently of k. Then the assumption

$$\frac{k}{h} \geqq (2K_1K_2K_3')^2$$

forces the inequality (3.9), so that (2.3) follows. This is the essence of the proof given in [1] to show that condition (1.2) is sufficient for stability when constant C is sufficiently large.

The actual verification of the new stability condition (3.1) can be somewhat eased using the following simple Lemma.

Lemma 3.2. *Let $\{\psi_1, \ldots, \psi_N\}$ be a basis for M_2^h, and assume that for each $\mu\in M_2^h$, $\mu = \sum_{i=1}^{N} \alpha_i\psi_i$, one can find a function $v_\mu \in M_1^h$ so that*

$$C_1 h \sum_{i=1}^{N} |\alpha_i|^2 \leqq \langle \mu, Tv_\mu\rangle \leqq C_2 h \sum_{i=1}^{N} |\alpha_i|^2$$

and

$$\|v_\mu\|_{1,\Omega}^2 \leqq C_3 \sum_{i=1}^{N} |\alpha_i|^2.$$

Then

$$\gamma_2 \geqq C_1 [C_2 C_3]^{-1/2}.$$

4. Examples of Boundary Subspaces

4.1. Triangular Interior Mesh

For $0<h<1$, $\Omega \subset R^2$, $\partial\Omega$ smooth, let S_h be a finite set of points of $\partial\Omega$ defined so that $C_1 h \leqq |z_1 - z_2| \leqq C_2 h$ for any two adjacent points $z_1, z_2 \in S_h$. Let Ω_h be a polygonal domain obtained by connecting these adjacent points of S_h by straight lines. We introduce a quasiuniform triangular grid on Ω_h so that the set of nodes of the grid on $\partial\Omega_h$ equals S_h. By quasiuniformity we mean that the angles of the triangles are bounded away from zero uniformly in h (the minimal angle condition) and that the diameters of all the triangles are proportional to h. We will also assume that each triangle has at most one edge on $\partial\Omega_h$.

Assuming that the straight line segments on $\partial\Omega_h$ are deformed so as to fit locally to $\partial\Omega$, one obtains a partition of Ω into subregions, either triangles or deformed triangles with one curved edge. This set of subregions is denoted by $\{\Delta_i\}$, $i=1,\dots,N_h$.

Let m be a fixed integer, $m \geqq 1$, and denote by P_m the space of polynomials on R^2 of degree at most m. We define the space M_1^h as the subspace of $H^1(\Omega)$ of the highest possible dimension within the restrictions that

$$M^h_{1|\Delta_i} = P_{m|\Delta_i}, \qquad i=1,\dots,N_h.$$

To define the boundary subspace M_2^h, let $\{\Delta_i\}_{i\in\Lambda}$ be the set of subregions having one edge on $\partial\Omega$. For $i\in\Lambda$, let $\Gamma_i = \partial\Omega \cap \bar{\Delta}_i$, and let $\{x_1^{(i)}, x_2^{(i)}\}$ be a local coordinate system defined so that the line $x_2^{(i)}=0$ passes through the endpoints of Γ_i (see Fig. 1). Now denoting by $P_m(x_1^{(i)})$ the set of polynomials of degree at most m in the variable $x_1^{(i)}$, we require that

$$M^h_{2|\Gamma_i} = P_m(x_1^{(i)})|_{\Gamma_i}. \tag{4.1}$$

The space M_2^h is defined as the subspace of $H^1(\partial\Omega)$ of the highest possible dimension, so that Eq. (4.1) holds for all $i\in\Lambda$.

Regarding the validity of the inclusion (2.5) for the above subspaces, note first that for any given $\psi \in M_2^h$ there exist the polynomials $p_i(x)$, $x\in R^2$, of degree m such that $p_i = \psi$ on $\Gamma_i = \bar{\Delta}_i \cap \partial\Omega$, $i\in\Lambda$. Now if the sets Δ_i, $i\in\Lambda$, only touch each other on $\partial\Omega$, then there exists a function $v\in M_1^h$ such that $v_{|\Delta_i} = p_i$, $i\in\Lambda$. Thus, for (2.5) to be valid, each node at the boundary should be shared by at least three sets Δ_i, $i\in\{1,\dots,N_h\}$.

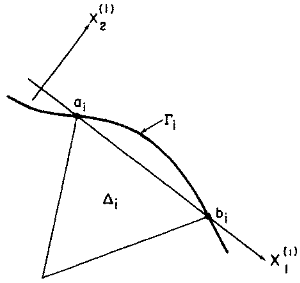

Fig. 1

The next theorem shows that for the above subspaces the stability condition (3.1) is satisfied when h is sufficiently small.

Theorem 4.1. *Assuming the above subspaces, there exist the positive constants C_1 and C_2, independent of h, so that*

$$\gamma_2 \geqq C_1 - C_2 h.$$

Proof. Consider a given subregion Δ_i, $i \in \Lambda$, with $\Gamma_i = \partial\Omega \cap \bar{\Delta}_i$. Since $\partial\Omega$ is smooth, it follows that, for h sufficiently small, we have the local representation (Fig. 1)

$$\Gamma_i = \{(x_1^{(i)}, x_2^{(i)});\ a_i \leqq x_1^{(i)} \leqq b_i,\ x_2^{(i)} = \phi_i(x_1^{(i)})\},$$

where

$$\phi_i \in C^\infty[a_i, b_i], \qquad \phi_i(a_i) = \phi_i(b_i) = 0. \tag{4.2}$$

We have

$$C_1 h \leqq |a_i - b_i| \leqq C_2 h, \tag{4.3}$$

which together with Eq. (4.2) implies that

$$|\phi_i(x_1^{(i)})| \leqq C h^2, \qquad x_1^{(i)} \in [a_i, b_i], \tag{4.4}$$

for some constant C independent of h.

Now let $\mu \in M_2^h$ be given, and let

$$\mu_{|\Gamma_i} = \mu_i(x_1^{(i)}) \in P_m(x_1^{(i)}), \qquad i \in \Lambda.$$

Define the set of $m+1$ discrete points on $[a_i, b_i]$ as

$$t_j^{(i)} = a_i + \frac{j-1}{m}(b_i - a_i), \qquad j = 1, \ldots, m+1,$$

and let

$$\alpha_j^{(i)} = \mu_i(t_j^{(i)}), \qquad j = 1, \ldots, m+1.$$

From the construction of the space M_1^h it is clear that to each $v \in M_1^h$ there corresponds a unique function $P_h v \in H^1(\Omega_h)$ such that $P_h v = v$ on $\Omega \cap \Omega_h$ and $P_h v$ is piecewise polynomial in the set of triangles composing Ω_h. Consider then functions $P_h v$, $v \in M_1^h$, which satisfy, in the above local coordinate systems,

$$P_h v(t_j^{(i)}, 0) = \mu_i(t_j^{(i)}) = \alpha_j^{(i)}, \quad i \in \Lambda, \; j = 1, \dots, m+1. \tag{4.5}$$

Among the functions $v \in M_1^h$ which satisfy Eq. (4.5), let v_μ be the one which has the smallest support. To each $\mu \in M_2^h$ there obviously corresponds a unique $v_\mu \in M_1^h$ with support contained in a strip of width of order h at the boundary.

Using locally supported nodal basis functions to represent v_μ, it can be verified directly that

$$\|v_\mu\|_{1,\Omega}^2 \leqq C_1 \sum_{i \in \Lambda} \sum_{j=1}^{m+1} |\alpha_j^{(i)}|^2 \tag{4.6}$$

and

$$\langle \mu, T v_\mu \rangle \leqq C_2 h \sum_{i \in \Lambda} \sum_{j=1}^{m+1} |\alpha_j^{(i)}|^2, \tag{4.7}$$

where C_1 and C_2 are independent of h.

Finally, we use Eqs. (4.3) and (4.4) to obtain the lower bound

$$\begin{aligned} \langle \mu, T v_\mu \rangle &= \sum_{i \in \Lambda} \int_{\Gamma_i} \mu v_\mu \, ds \\ &\geqq (1 - C h) \sum_{i \in \Lambda} \int_{a_i}^{b_i} |\mu_i(x_1^{(i)})|^2 \, dx_1^{(i)} \\ &\geqq C_3 h (1 - C h) \sum_{i \in \Lambda} \sum_{j=1}^{m+1} |\alpha_j^{(i)}|^2, \end{aligned} \tag{4.8}$$

with C_3 and C positive and independent of h. Upon combining Eqs. (4.6) through (4.8) with Lemma 3.2, the assertion of Theorem 4.1 follows. ■

To obtain the convergence rate of the finite element method based on the above subspaces, one should establish the approximability properties of M_1^h and M_2^h. For the space M_1^h the standard results of approximation theory are valid, as can be verified by the usual local considerations (c.f. [5]). In particular, one can state that

$$\inf_{v \in M_1^h} \|u - v\|_{1,\Omega} \leqq C h^{s-1} \|u\|_{s,\Omega}, \quad u \in H^s(\Omega), \; 1 \leqq s \leqq m+1. \tag{4.9}$$

To verify that M_2^h also has the approximability properties to be expected, let us take ψ to be a smooth function on $\partial\Omega$ and define, in the above local coordinate systems, the functions $\psi_i(x_1^{(i)})$, $i \in \Lambda$, so that

$$\psi_i(x_1^{(i)}) = \psi(x_1^{(i)}, x_2^{(i)}), \quad (x_1^{(i)}, x_2^{(i)}) \in \Gamma_i, \; i \in \Lambda.$$

Let $\psi_{m,i}(x_1^{(i)})$ be the m-th order Lagrange interpolant of ψ_i on the interval $[a_i, b_i]$, and define $\psi_h \in M_2^h$ so that

$$\psi_h(x_1^{(i)}, x_2^{(i)}) = \psi_{m,i}(x_1^{(i)}), \quad (x_1^{(i)}, x_2^{(i)}) \in \Gamma_i, \; i \in \Lambda.$$

Now if k and ℓ are such that $0 \leqq k \leqq \ell \leqq m+1$, we have

$$\begin{aligned}\|\psi-\psi_h\|_{H^k(\partial\Omega)} &\leqq C_1 \Big[\sum_{i\in A} \|\psi_i-\psi_{m,i}\|^2_{H^k[a_i,b_i]}\Big]^{1/2} \\ &\leqq C_2 h^{\ell-k} \Big[\sum_{i\in A} \|\psi_i\|^2_{H^\ell[a_i,b_i]}\Big]^{1/2} \\ &\leqq C_3 h^{\ell-k} \|\psi\|_{H^\ell(\partial\Omega)}.\end{aligned}$$

Hence, the estimate

$$\inf_{\mu\in M_2^h} \|\psi-\mu\|_{H^k(\partial\Omega)} \leqq C h^{\ell-k} \|\psi\|_{H^\ell(\partial\Omega)}$$

is true for any smooth ψ and accordingly for any $\psi\in H^\ell(\partial\Omega)$. It now suffices to use standard interpolation and duality arguments (c.f. [2]) to conclude that

$$\inf_{\mu\in M_2^h} \|\psi-\mu\|_{-1/2,\partial\Omega} \leqq C h^q \|\psi\|_{r,\partial\Omega}, \qquad \psi\in H^r(\partial\Omega),\ r\geqq -\tfrac{1}{2}$$
$$q=\min\{m+\tfrac{3}{2},\ r+\tfrac{1}{2}\}, \tag{4.10}$$

as expected.

Combining Eqs. (4.9) and (4.10) with Theorem 2.1 we obtain for the convergence of the Lagrange multiplier method the following.

Theorem 4.2. *Let u_0 be the solution to the problem* (1.1) *where $f\in H^s(\Omega)$, $s\geqq 0$ and $g\in H^r(\partial\Omega)$, $r\geqq 1/2$. Then the approximate solution derived from Eq.* (2.2) *and from the above subspaces satisfies the error estimates*

$$\|u_0-u_h\|_{1,\Omega}+\left\|\frac{\partial u_0}{\partial n}-\lambda_h\right\|_{-1/2,\partial\Omega} \leqq C\{h^{q_1}\|f\|_{s,\Omega}+h^{q_2}\|g\|_{r,\partial\Omega}\}, \tag{4.11}$$

where

$$q_1=\min\{m+1,\ s+1\}, \qquad q_2=\min\{m+1,\ r-\tfrac{1}{2}\}$$

and m is the degree of polynomials in M_1^h.

4.2. Rectangular Interior Mesh

In this section we attempt to define boundary subspaces compatible with a rectangular interior mesh. We need first some notation related to the orientation of the domain Ω in a fixed global coordinate system.

For each $x\in\partial\Omega$, define $n(x)$ as the outward pointing unit normal of $\partial\Omega$ at x, with components $n_i(x)$, $i=1,2$, along the coordinate axes in some fixed coordinate system. Let θ be a fixed real number satisfying $1/\sqrt{2}<\theta<1$ and define a partition of $\partial\Omega$ into a finite number of disjoint arcs Γ_{ij}, $i=1,2$, $j=1,\dots,N_i$, so that the following is true:

$$|n_i(x)|<\theta,\ x\in\Gamma_{ij};\ i=1,2,\ j=1,\dots,N_i. \tag{4.12}$$

We will define

$$\partial\Omega_i = \bigcup_{j=1}^{N_i} \Gamma_{ij}, \qquad i=1,2.$$

To define the interior subspace M_1^h, let $\mathcal{T}_h^{(i)}$, $i=1,2$, be two sets of discrete points of R^1,

$$\mathcal{T}_h^{(i)} = \{t_j^{(i)}\}, \qquad j=\pm 1, \pm 2, \ldots, i=1,2,$$

constructed so that the following conditions are satisfied:

i) $C_1 h \leqq t_j^{(i)} - t_{j-1}^{(i)} \leqq C_2 h$, $t_{j-1}^{(i)}, t_j^{(i)} \in \mathcal{T}_h^{(i)}$,
$j = \pm 1, \pm 2, \ldots, i=1,2,$

ii) $\overline{\partial\Omega_1} \cap \overline{\partial\Omega_2} \subset \mathcal{T}_h^{(1)} \times \mathcal{T}_h^{(2)}$,

and

iii) for each $z \in \overline{\partial\Omega_1} \cap \overline{\partial\Omega_2}$ there exists an open rectangle $E_z \subset \Omega$ so that E_z has all corners in $\mathcal{T}_h^{(1)} \times \mathcal{T}_h^{(2)}$ and one corner at z.

It is not difficult to verify that for any smooth domain one can find families $\{\mathcal{T}_h^{(i)}\}_{0<h<1}$, $i=1,2$, so that the conditions i) through iii) are satisfied for sufficiently small h.

Let m be a fixed integer, $m \geqq 1$, and define $L_{m,h}$ as the m:th-order Lagrange interpolation space associated to the mesh $\mathcal{T}_h^{(1)} \times \mathcal{T}_h^{(2)}$, i.e., $L_{m,h}$ is the space of continuous functions on R^2 so that the restriction of $L_{m,h}$ to any rectangle $\Delta_{j\ell}$ of the type

$$\Delta_{j\ell} = \{(x_1, x_2);\ t_{j-1}^{(1)} < x_1 < t_j^{(1)},\ t_{\ell-1}^{(2)} < x_2 < t_\ell^{(2)}\},$$
$$t_{j-1}^{(1)}, t_j^{(1)} \in \mathcal{T}_h^{(1)},\ t_{\ell-1}^{(2)}, t_\ell^{(2)} \in \mathcal{T}_h^{(2)},$$

is the space of polynomials of degree at most m in each variable x_i, $i=1,2$. The interior subspace M_1^h is now defined as

$$M_1^h = L_{m,h|\Omega}.$$

To define the boundary subspace, let $L_{m,h}^{(i)}$, $i=1,2$, be the subspaces of $L_{m,h}$ consisting of functions that depend only on the variable x_i, i.e., $L_{m,h}^{(i)}$ consists of functions $\phi(x_1, x_2) \equiv \psi(x_i)$ where ψ is piecewise polynomial to a degree m in the mesh $\mathcal{T}_h^{(i)}$. Referring to the above partition of $\partial\Omega$ into the subsets Γ_{ij} we now define M_2^h as the space of continuous functions on $\partial\Omega$, of the highest possible dimension within the restrictions

$$M_{2|\Gamma_{ij}}^h = L_{m,h|\Gamma_{ij}}^{(i)}, \qquad i=1,2, j=1,\ldots,N_i.$$

One can verify from the assumptions made that the inclusion (2.5) is true for the above subspaces. Let us now prove that also the stability condition (3.1) is satisfied. To this end, consider a given subset $\Gamma_{ij} \subset \partial\Omega$. By the above construction

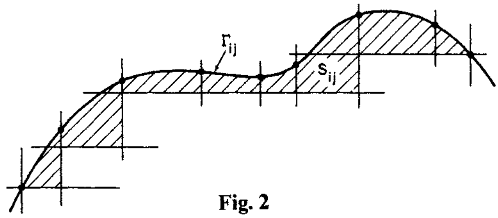

Fig. 2

there exists an open interval $\Pi_{ij} \subset R^1$, with endpoints in $\mathscr{T}_h^{(i)}$, so that Γ_{ij} has the representation

$$\begin{aligned}\Gamma_{ij} &= \{(x_1, x_2);\ x_1 \in \Pi_{ij},\ x_2 = \phi_{ij}(x_1)\}, \qquad i=1\\ &= \{(x_1, x_2);\ x_2 \in \Pi_{ij},\ x_1 = \phi_{ij}(x_2)\}, \qquad i=2\end{aligned}$$

where ϕ_{ij} is infinitely differentiable on $\bar{\Pi}_{ij}$. Let k_{ij} and K_{ij} be integers so that

$$\bar{\Pi}_{ij} \cap \mathscr{T}_h^{(i)} = \{t_k^{(i)};\ k = k_{ij}, k_{ij}+1, \dots, K_{ij}\}.$$

For each k, $k = k_{ij}, k_{ij}+1, \dots, K_{ij}-1$, let E_{ijk} be the open square of the smallest diameter which has its corners in $\mathscr{T}_h^{(1)} \times \mathscr{T}_h^{(2)}$ and satisfies

$$E_{ijk} \cap \partial\Omega = \{(x_1, x_2) \in \Gamma_{ij},\ t_k^{(i)} < x_i < t_{k+1}^{(i)}\}. \tag{4.13}$$

Then define (see Fig. 2)

$$S_{ij} = \Omega \cap \left(\bigcup_{k=k_{ij}}^{K_{ij}-1} E_{ijk} \right). \tag{4.14}$$

Let $\mu \in M_2^h$ be given. Then $\mu_{|\Gamma_{ij}} = \mu_{ij}(x_i)$, where μ_{ij} is a piecewise polynomial function on Π_{ij}. Define a function v_{ij} on the set S_{ij} as

$$v_{ij}(x_1, x_2) \equiv \mu_{ij}(x_i), \quad (x_1, x_2) \in S_{ij}. \tag{4.15}$$

By the assumption iii), the sets S_{ij} are disjoint. Therefore, we may define a function $v_\mu \in M_1^h$ as the one having the smallest possible support within the restrictions

$$v_{\mu|S_{ij}} = v_{ij}, \qquad i=1,2, j=1,\dots,N_i.$$

Obviously, v_μ has again its support in a boundary zone of width proportional to h. Moreover, the construction of v_μ implies that $Tv_\mu = \mu$. So, we have

$$\begin{aligned}\langle \mu, Tv_\mu \rangle &= \sum_{i=1}^{2} \sum_{j=1}^{N_i} \int_{\Gamma_{ij}} |\mu|^2\, ds\\ &= \sum_{i=1}^{2} \sum_{j=1}^{N_i} \int_{\Pi_{ij}} (1 + |\phi_{ij}'(x_i)|^2)^{1/2}\, |\mu_{ij}(x_i)|^2\, dx_i.\end{aligned}$$

Upon introducing the standard basis on Π_{ij} to represent μ_{ij} in terms of its nodal values $\alpha_{ij\ell}$, it is easy to verify from the above expression that

$$C_1 h \sum_{i,j,\ell} |\alpha_{ij\ell}|^2 \leqq \langle \mu, T v_\mu \rangle \leqq C_2 h \sum_{i,j,\ell} |\alpha_{ij\ell}|^2,$$

where C_1 and C_2 are independent of h. A similar calculation yields for v_μ the estimate

$$\|v_\mu\|_{1,\Omega}^2 \leqq C_3 \sum_{i,j,\ell} |\alpha_{ij\ell}|^2.$$

The validity of the stability condition (3.1) now follows immediately from Lemma 3.2.

Finally, the approximability properties of the subspaces should be studied. This part of the analysis, however, is practically identical with that of Sect. 4.1, so it will be omitted.

We have ended up in the following theorem.

Theorem 4.3. *Let M_1^h and M_2^h be constructed as above so that the assumptions* i) *through* iii) *are satisfied. Then the finite element method based on Eq.* (2.2) *is stable and the approximate solution satisfies the error estimate* (4.11).

The above construction of the subspaces raises the natural question, whether one could actually drop the assumptions ii) and iii) so that the interior mesh could be defined independently of the domain Ω. As an example of how the boundary subspaces could be defined in such more general case, we consider piecewise bilinear approximation on a uniform mesh, i.e., we take $m=1$ and replace assumption i) by

i)* $t_j^{(i)} - t_{j-1}^{(i)} = h,\ t_{j-1}^{(i)}, t_j^{(i)} \in \mathscr{T}_h^{(i)}, \quad j = \pm 1, \pm 2 \ldots, i = 1, 2.$

Assume that the subsets Γ_{ij} of $\partial\Omega$ are defined as above. We introduce for each $i=1,2$, $j=1,\ldots,N_i$, another subset Γ_{ij}^* defined so that

$$\Gamma_{ij}^* = \{(x_1, x_2) \in \Gamma_{ij};\ x_i \in \Pi_{ij}^*\} \tag{4.16}$$

where Π_{ij}^* is an open interval with endpoints in $\mathscr{T}_h^{(i)}$; each Π_{ij}^* is chosen within the additional restrictions that

$$C h \geqq \operatorname{dist}(\Gamma_{ij}^*, \Gamma_{k\ell}^*) \geqq \varepsilon h, \qquad \Gamma_{ij}^* \neq \Gamma_{k\ell}^*, \tag{4.17}$$

where C and ε are some positive constants independent of h.

As above, let $L_{1,h}^{(i)}$ be the space of functions on R^2 which are continuous piecewise linear in the variable x_i on the mesh $\mathscr{T}_h^{(i)}$ and constant in the other variable. We define the space M_1^h in the same way as above, and the space M_2^h so as to satisfy

1) $M_{2|\Gamma_{ij}^*}^h = L_{1,h|\Gamma_{ij}^*}^{(i)}, \quad i=1,2, j=1,\ldots,N_i,$

2) for any connected arc $\Gamma_k \subset \partial\Omega - \bigcup_{i,j} \Gamma_{ij}^*$, either $M^h_{2|\Gamma_k} = L^{(1)}_{1,h|\Gamma_k}$ or $M^h_{2|\Gamma_k} = L^{(2)}_{1,h|\Gamma_k}$,

3) $M_2^h \subset H^1(\partial\Omega)$.

Let us mention that the inclusion (2.5) is not valid for the above subspaces. Nevertheless, we can establish the following.

Theorem 4.4. *Let M_1^h and M_2^h be defined as above. Then one can choose the constants θ and ε in Eqs.* (4.12) *and* (4.17) *so that Eq.* (3.1) *is true for sufficiently small h; the condition*

$$\varepsilon(1-\theta^2)^{1/2} > \tfrac{1}{12} \tag{4.18}$$

is sufficient.

Proof. Let $\mu \in M_2^h$ be given. We will first construct a function $v_\mu \in M_1^h$ so that Tv_μ is close to μ. For this, consider a given Γ_{ij}^* as defined by Eq. (4.16) with Π_{ij}^* given by

$$\Pi_{ij}^* = \{t_k^{(i)} < x_i < t_\ell^{(i)}\}, \qquad t_k^{(i)}, t_\ell^{(i)} \in \mathscr{T}_h^{(i)}.$$

For sufficiently small h we may assume that Π_{ij}^* consists of at least two intervals of $\mathscr{T}_h^{(i)}$. Then define

$$\Pi_{ij}^{**} = \{t_{k+1}^{(i)} < x_i < t_{\ell-1}^{(i)}\},$$

and

$$\Gamma_{ij}^{**} = \{(x_1, x_2) \in \Gamma_{ij}^*;\ x_i \in \Pi_{ij}^{**}\}.$$

Now for each Γ_{ij}^{**}, $i=1,2, j=1,\dots,N_i$, defined so as above, one can define the set S_{ij} and the function v_{ij} in the same manner as in Eqs. (4.13) through (4.15). Assuming this is done, it remains to fill the gaps between the different Γ_{ij}^{**}. Consider a representant Γ_k of such a gap, defined, say, as

$$\Gamma_k = \{(x_1, x_2) \in \partial\Omega,\ x_1 > t_j^{(1)},\ x_2 > t_\ell^{(2)}\},$$

where $t_j^{(1)} \in \mathscr{T}_h^{(1)}$ and $t_\ell^{(2)} \in \mathscr{T}_h^{(2)}$. Let $(t_j^{(1)}, \eta_1)$ and $(\xi_1, t_\ell^{(2)})$ be the endpoints of Γ_k. By our construction, there exist $\xi_2 < \xi_1$ and $\eta_2 < \eta_1$ so that the points $(t_{j+1}^{(1)}, \eta_2)$ and $(\xi_2, t_{\ell+1}^{(2)})$ also lie on Γ_k (see Fig. 3). Moreover, the assumptions made above imply that

$$C_1 h < \xi_2 - t_{j+1}^{(1)} < C_2 h, \qquad C_1 h < \eta_2 - t_{\ell+1}^{(2)} < C_2 h,$$

where C_1 and C_2 are positive and depend only on the constants θ and ε in Eqs. (4.12) and (4.17).

In the above notation, let us define the subset $S_k \subset \Omega$ as (see Fig. 3)

$$\begin{aligned} S_k = \{&(x_1, x_2) \in \Omega;\ x_1 > t_j^{(1)}, x_2 > t_{\ell+1}^{(2)}\} \\ \cup \{&(x_1, x_2) \in \Omega;\ x_1 > t_{j+1}^{(1)}, x_2 > t_\ell^{(2)}\}. \end{aligned}$$

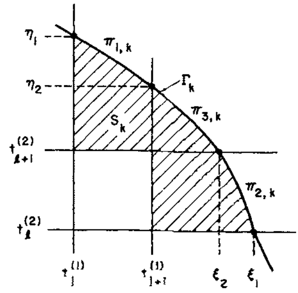

Fig. 3

Then the following conditions define uniquely a function v_k on S_k:

1) v_k is a bilinear polynomial in the regions $\{(x_1\, x_2)\in S_k;$

$t_j^{(1)}<x_1<t_{j+1}^{(1)}\}$ and $\{(x_1, x_2)\in S_k;\ t_\ell^{(2)}<x_2<t_{\ell+1}^{(2)}\}$,

2) in the region $x_1\geqq t_{j+1}^{(1)}, x_2\geqq t_{\ell+1}^{(2)}, (x_1, x_2)\in\Omega$, v_k is a polynomial of degree 1 with $v_k(t_{j+1}^{(1)}, t_{\ell+1}^{(2)})=0$

and

3)
$$\begin{aligned}
&v_k(t_j^{(1)}, x_2)\equiv\mu(t_j^{(1)}, \eta_1)=\beta_1,\ x_2\geqq t_{\ell+1}^{(2)}\\
&v_k(t_{j+1}^{(1)}, \eta_2)=\mu(t_{j+1}^{(1)}, \eta_2)=\beta_2\\
&v_k(x_1, t_\ell^{(2)})\equiv\mu(\xi_1, t_\ell^{(2)})=\delta_1,\ x_1\geqq t_{j+1}^{(1)}\\
&v_k(\xi_2, t_{\ell+1}^{(2)})=\mu(\xi_2, t_{\ell+1}^{(2)})=\delta_2.
\end{aligned}\tag{4.19}$$

Assuming that the above construction of functions v_{ij} and v_k is repeated for each Γ_{ij}^{**} and for each connected arc $\Gamma_k\subset\partial\Omega-\bigcup_{i,j}\Gamma_{ij}^{**}$, we can finally define a function $v_\mu\in M_1^h$ as having the minimal support within the restrictions

$$v_{\mu|S_{ij}}=v_{ij},\qquad i=1,2,\ j=1,\dots,N_i,$$

and

$$v_{\mu|S_k}=v_k,\qquad k=1,\dots,N_1+N_2.$$

Denoting the nodal values of μ at the boundary by α_j, it is easy to verify that

$$\|v_\mu\|_{1,\Omega}^2\leqq C_1\sum_j|\alpha_j|^2,\tag{4.20}$$

and that

$$\langle\mu, Tv_\mu\rangle\leqq C_2\, h\sum_j|\alpha_j|^2.\tag{4.21}$$

In order to get a lower bound for $\langle\mu, Tv_\mu\rangle$, consider the piece Γ_k of the boundary as defined above. Since Γ_k is an arc of a smooth curve, it deviates from a straight line passing through its endpoints to an amount of order $0(h^2)$ only. So, let us assume, as a first approximation, that Γ_k is a segment of a straight line.

Recall from the definition of M_2^h that μ is linear in x_1 on the set

$$\Pi_{1,k}=\{(x_1,x_2)\in\Gamma_k;\ t_j^{(1)}<x_1<t_{j+1}^{(1)}\}.$$

Similarly, μ is linear in x_2 on the set

$$\Pi_{2,k}=\{(x_1,x_2)\in\Gamma_k;\ t_\ell^{(2)}<x_2<t_{\ell+1}^{(2)}\},$$

and linear in either of the two variables on the set

$$\Pi_{3,k}=\Gamma_k-(\Pi_{1,k}\cup\Pi_{2,k}).$$

By the definition of the function v_μ we know that if Γ_k is a straight line segment, then

$$Tv_{\mu|\Pi_{3,k}}=\mu_{|\Pi_{3,k}}. \tag{4.22}$$

By our assumptions, the diameter of each $\Pi_{i,k}$, $i=1,2,3$, is of the order h. Therefore, in the notation of Eq. (4.19), Eq. (4.22) implies that

$$\int_{\Pi_{3,k}} v_\mu\,\mu\,ds\geqq C\,h(\beta_2^2+\delta_2^2),\qquad C>0. \tag{4.23}$$

To obtain similar estimates for the remaining pieces $\Pi_{1,k}$ and $\Pi_{2,k}$, we need the following Lemma, which can be verified by a direct calculation.

Lemma 4.1. *Let A be a 2×2-matrix with elements*

$$a_{ij}=\tfrac{1}{2}\int_0^1(\phi_i\psi_j+\phi_j\psi_i)\,dx,\qquad i,j=1,2,$$

where $\phi_1(x)=1-x$, $\psi_1(x)=1-x+c\,x(1-x)$, and $\phi_2(x)=\psi_2(x)=x$.

Then the eigenvalues of A are positive provided that $-4<c<12$.

Now a simple calculation gives

$$\int_{\Pi_{2,k}} v_\mu\,\mu\,ds=[h^2+(\xi_2-\xi_1)^2]^{1/2}\sum_{i=1}^{2}\sum_{j=1}^{2}a_{ij}\,\delta_i\,\delta_j,$$

with the constant c given by

$$c=(\xi_1-\xi_2)(\xi_2-t_{j+1}^{(1)})^{-1}.$$

One can verify by simple geometric considerations that if condition (4.18) is satisfied, then $0<c<12$. By scaling and using Lemma 4.1 we then get the lower bound

$$\int_{\Pi_{2,k}} v_\mu\,\mu\,ds\geqq C\,h(\delta_1^2+\delta_2^2),\qquad C>0. \tag{4.24}$$

By symmetry we also have

$$\int_{\Pi_{1,k}} v_\mu \mu \, ds \geqq C h(\beta_1^2+\beta_2^2), \qquad C>0, \tag{4.25}$$

where β_1 and β_2 refer to Eq. (4.19).

The above inequalities were derived under the assumption that Γ_k is a segment of a straight line. It is now easy to verify that the actual curvature only amounts to a correction of order $O(h)$ in the constant C in Eqs. (4.23) through (4.25). Hence, for sufficiently small h, the inequalities are still valid, and we have

$$\int_{\Gamma_k} v_\mu \mu \, ds \geqq C h(\beta_1^2+\beta_2^2+\delta_1^2+\delta_2^2), \qquad C>0.$$

The above arguments apply to each connected arc Γ_k in $\partial\Omega - \bigcup_{i,j} \Gamma_{ij}^{**}$. For pieces Γ_{ij}^{**}, we can apply the same arguments as in the proof of Theorem 4.3. Upon summing the inequalities and returning to the global indexing of the nodal values of μ we then obtain

$$\begin{aligned}\langle \mu, T v_\mu \rangle &= \sum_{i,j} \int_{\Gamma_{ij}^{**}} |\mu|^2 \, ds + \sum_k \int_{\Gamma_k} \mu v_\mu \, ds \\ &\geqq C h \sum_j |\alpha_j|^2, \qquad C>0.\end{aligned}$$

Together with Eqs. (4.20) and (4.21) and Lemma 3.2, this completes the proof of Theorem 4.4. ■

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