

COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

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AMS subject classifications. n.a.

1. Introduction. We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators from a 3D domain, Ω , to 1D manifold, Λ , and vice versa. This is a non trivial objective because the standard trace operator from a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω .

2. Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex open set. Let Σ be a generalized cylinder embedded into Ω and be $\Omega_{\oplus} = \Omega \setminus \bar{\Sigma}$ be the complementary set of the cylinder. We also introduce the set Λ , a 1D manifold that is the centerline of Σ . We define the arc-length coordinate along Λ , denoted by $s \in (0, S)$. We denote with $\mathcal{D}(s)$ and $\partial\mathcal{D}(s)$ a cross section of Σ and its boundary, respectively. In what follows, we assume for simplicity of notation that Σ has a constant cross section, but this is not a restriction of the approach. We also assume that Σ crosses Ω from side to side and we call Γ the lateral (cylindrical) surface of Σ , while the upper and lower side faces of Σ belong to $\partial\Omega$. We refer to Figure 2.1 for an illustration of the notation.

We consider the problem arising from *Dirichlet-Neumann* conditions. It consists to find u_{\oplus}, u_{\ominus} s.t.:

$$\begin{aligned} (2.1a) \quad & -\Delta u_{\oplus} + u_{\oplus} = f && \text{in } \Omega_{\oplus}, \\ (2.1b) \quad & -\Delta u_{\ominus} + u_{\ominus} = g && \text{in } \Sigma, \\ (2.1c) \quad & -\nabla u_{\ominus} \cdot \mathbf{n}_{\ominus} = -\nabla u_{\oplus} \cdot \mathbf{n}_{\ominus} && \text{on } \Gamma, \\ (2.1d) \quad & u_{\ominus} = u_{\oplus} && \text{on } \Gamma, \\ (2.1e) \quad & u_{\oplus} = 0 && \text{on } \partial\Omega. \end{aligned}$$

The objective of this work is to derive and analyze a simplified version of problem (2.1), where the domain Σ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely \mathcal{D} . This new problem setting will be called the *reduced* problem. From the mathematical standpoint it is more challenging than (2.1), because it involves the coupling of 3D/1D elliptic problems. For the model reduction process, we decompose integrals as follows, for any sufficiently regular function w ,

$$\int_{\Sigma} w d\omega = \int_{\Lambda} \int_{\mathcal{D}} w d\sigma ds = \int_{\Lambda} |\mathcal{D}| \bar{\bar{w}} ds, \quad \int_{\Gamma} w d\sigma = \int_{\Lambda} \int_{\partial\mathcal{D}} w d\gamma ds = \int_{\Lambda} |\partial\mathcal{D}| \bar{w} ds,$$

where $\bar{\bar{w}}$, \bar{w} denote the following mean values respectively,

$$\bar{\bar{w}} = |\mathcal{D}|^{-1} \int_{\mathcal{D}} w d\sigma, \quad \bar{w} = |\partial\mathcal{D}|^{-1} \int_{\partial\mathcal{D}} w d\gamma.$$

We apply the model reduction approach at the level of the variational formulation. We start from the variational formulation of problem (2.1), that is to find $u_{\oplus} \in H_{\partial\Omega}^1(\Omega_{\oplus})$, $u_{\ominus} \in H_{\partial\Sigma \setminus \Gamma}^1(\Sigma)$, $\lambda \in H^{-\frac{1}{2}}(\partial\Sigma)$ s.t.

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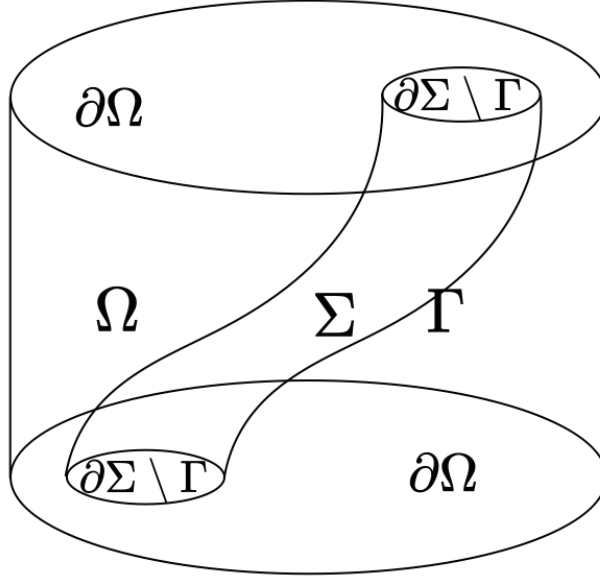


FIGURE 2.1. *Geometrical setting of the problem*

$$\begin{aligned}
 (2.2a) \quad & (u_{\oplus}, v_{\oplus})_{H^1(\Omega_{\oplus})} + (u_{\ominus}, v_{\ominus})_{H^1(\Sigma)} + \langle v_{\ominus} - v_{\oplus}, \lambda \rangle_{H^{-\frac{1}{2}}(\Gamma)} \\
 & = (f, v_{\oplus})_{L^2(\Omega_{\oplus})} + (g, v_{\ominus})_{L^2(\Sigma)} \quad \forall v_{\oplus} \in H_{\partial\Omega}^1(\Omega_{\oplus}), v_{\ominus} \in H_{\partial\Sigma \setminus \Gamma}^1(\Sigma) \\
 (2.2b) \quad & \langle u_{\ominus} - u_{\oplus}, \mu \rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma),
 \end{aligned}$$

where $\langle v, \mu \rangle_{H^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. In this case, the additional variable λ is equivalent to $\lambda = -\nabla u_{\ominus} \cdot \mathbf{n}_{\ominus}$.

Using the averaging tools for the model reduction approach, we end up with a reduced problem for the unknown u defined on the entire 3D domain Ω , coupled with the unknown u_{\ominus} , defined on the 1D manifold Λ and a Lagrange multiplier L defined on Λ . In the reduced problem the multiplier assumes the following interpretation,

$$L = -\frac{1}{|\partial\mathcal{D}|} \int_{\partial\mathcal{D}} \nabla u_{\ominus} \cdot \mathbf{n}_{\ominus} = -\frac{1}{|\partial\mathcal{D}|} \int_{\partial\mathcal{D}} \lambda.$$

We consider two alternative formulations. The scope of this work is to compare them, with the aim to determine which is the most suitable as a computational model based on 3D-1D coupled PDEs.

2.1. Problem 1. The idea is to couple a 3D PDE with a 1D one, using a Lagrange multiplier space defined on a 2D surface that surrounds the 1D manifold. Let $\langle \cdot, \cdot \rangle_{\Gamma}$ denote the duality pairing between $H_{00}^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. The problem consists to find $u \in H_0^1(\Omega)$, $u_{\ominus} \in H_0^1(\Lambda)$, $L \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned}
 (2.3a) \quad & (u, v)_{H^1(\Omega)} + |\mathcal{D}|(u_{\ominus}, v_{\ominus})_{H^1(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_{\ominus}, L \rangle_{\Gamma} \\
 & = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_{\ominus})_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), v_{\ominus} \in H^1(\Lambda)
 \end{aligned}$$

$$(2.3b) \quad \langle \Pi_1 u - \Pi_2 u_{\ominus}, M \rangle_{\Gamma} = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Gamma).$$

Here, $\Pi_1 : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$ and $\Pi_2 : H_0^1(\Lambda) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$ and we remark that Σ may be considered as a virtual surface not necessarily of the same size as the underlying physical structure that is modeled. The Π_1 and Π_2 operators may be defined in terms of the averaging operators above, but may also be realized in terms

of e.g. Green functions (I don't know if this is a good idea). Furthermore, Γ may be discretized in terms of facets neighboring Λ and may as such not be represented as a separate structure in the implementation.

2.2. Problem 2. Another form of the reduced problem uses Lagrange multipliers defined directly on the 1D manifold. In this case, $\langle \cdot, \cdot \rangle_\Lambda$ denotes the duality pairing between $H_{00}^{\frac{1}{2}}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. The problem requires to find $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $L \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$(2.4a) \quad \begin{aligned} (u, v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_\odot, L \rangle_{H^{-\frac{1}{2}}(\Lambda)} \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), v_\odot \in H_0^1(\Lambda) \end{aligned}$$

$$(2.4b) \quad |\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_\odot, M \rangle_{H^{-\frac{1}{2}}(\Lambda)} = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Lambda).$$

We notice that all the integrals of the reduced problem are well defined because $u, v \in H_0^1(\Omega)$, $u, v|_\Gamma \in H_{00}^{\frac{1}{2}}(\Gamma)$ and thus $\bar{u}, \bar{v} \in H_{00}^{\frac{1}{2}}(\Lambda)$. More precisely, $\Pi_1 : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$ such that $\Pi_1 u = \overline{(u|_\Gamma)}$ is the combination between the trace on Γ and the average on $\partial\mathcal{D}$. The operator $\Pi_2 : H_0^1(\Lambda) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$ is the injection from $H_0^1(\Lambda)$ and $H_{00}^{\frac{1}{2}}(\Lambda)$. It is apparent that problems (2.2) and (4.1) share the same mathematical structure. For this reason, the well-posedness of (4.1) can be studied in the framework of the classical theory of saddle point problems.

3. Saddle-point problem analysis. Let us consider the general saddle point problem of the form: find $u \in X$, $p \in Q$ s.t.

$$(3.1) \quad \begin{cases} a(u, v) + b(v, p) = f(v) & \forall v \in X \\ b(u, q) = g(q) & \forall q \in M \end{cases}$$

which embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem. We denote with A and B the operators associated to the bilinear forms a and b , namely $A : X \rightarrow X'$ with $\langle Au, v \rangle_{X', X} = a(u, v)$ and $\langle Bv, q \rangle_{X', Q} = b(v, q)$.

THEOREM 3.1 (theorem 2.34 Ern-Guermond). *Problem (3.1) is well posed iff*

$$(3.2) \quad \begin{cases} \exists \alpha > 0 : \inf_{u \in \ker(B)} \sup_{v \in \ker(B)} \frac{a(u, v)}{\|u\|_X \|v\|_X} \geq \alpha \\ \forall v \in \ker(B), (\forall u \in \ker(B), a(u, v) = 0) \implies v = 0. \end{cases}$$

and

$$(3.3) \quad \exists \beta > 0 : \inf_{q \in Q} \sup_{v \in X} \frac{b(v, q)}{\|v\|_X \|q\|_Q} \geq \beta.$$

Notice that if a is coercive on $\ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1. It consists to find $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $L \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$(3.4a) \quad \begin{aligned} (u, v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L \rangle_\Gamma \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), v_\odot \in H^1(\Lambda) \end{aligned}$$

$$(3.4b) \quad \langle \Pi_1 u - \Pi_2 u_\odot, M \rangle_\Gamma = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Gamma),$$

Here, $\Pi_1 : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$ is the trace operator while Π_2 is the uniform extension from $H_0^1(\Lambda)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. We notice that the trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. We apply theorem 3.1 in the following spaces $X = H_0^1(\Omega) \times H^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$ and we prove that:

- a coercive \implies (3.2) is fulfilled

- We have to prove that $\forall M \in H^{-\frac{1}{2}}(\Gamma)$, $\exists \beta > 0$:

$$\sup_{v \in H_0^1(\Omega), v_\odot \in H_0^1(\Lambda)} \frac{\langle \Pi_1 v - \Pi_2 v_\odot, M \rangle_\Gamma}{\sqrt{\|v\|_{H^1(\Omega)}^2 + \|v_\odot\|_{H^1(\Lambda)}^2}} \geq \beta \sup_{q \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle q, M \rangle}{\|q\|_{H_{00}^{\frac{1}{2}}(\Gamma)}}.$$

We choose $v_\odot \in H_0^1(\Lambda)$ such that $\Pi_2 v_\odot = 0$. Therefore,

$$\sup_{v \in H_0^1(\Omega), v_\odot \in H_0^1(\Lambda)} \frac{\langle \Pi_1 v - \Pi_2 v_\odot, M \rangle_\Gamma}{\sqrt{\|v\|_{H^1(\Omega)}^2 + \|v_\odot\|_{H^1(\Lambda)}^2}} \geq \sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M \rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

The trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. Consequently, $\forall \xi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we find v solution of

$$\begin{aligned} -\Delta v &= 0 & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega \\ v &= \xi & \text{on } \Gamma. \end{aligned}$$

We denote with E the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|E\| \in \mathbb{R}$ such that $v = E\xi$ and $\|v\|_{H^1(\Omega)} \leq \|E\| \|\xi\|_{H_{00}^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$(3.5) \quad \sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M \rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle \xi, M \rangle_\Gamma}{\|E\| \|\xi\|_{H_{00}^{\frac{1}{2}}(\Gamma)}} = \|E\|^{-1} \|M\|_{H^{-\frac{1}{2}}(\Gamma)},$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H_{00}^{\frac{1}{2}}(\Gamma))^*$.

3.2. Problem 2. This problem requires to find $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $L \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$(3.6a) \quad \begin{aligned} (u, v)_{H^1(\Omega)} + |\mathcal{D}|(U, V)_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 V - \Pi_2 v, L \rangle_\Lambda \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), V \in H^1(\Lambda) \end{aligned}$$

$$(3.6b) \quad |\partial\mathcal{D}|\langle \Pi_1 U - \Pi_2 u, M \rangle_\Lambda = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Lambda).$$

Here, $\Pi_1 : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$ is the immersion operator and $\Pi_2 : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$ is defined as the composition of the trace operator $T_\Gamma : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$ and the average operator $(\bar{\cdot}) : H_{00}^{\frac{1}{2}}(\Gamma) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$, namely $\Pi_2 = (\bar{\cdot}) \circ T_\Gamma$. First of all we prove that if $v \in H_0^1(\Omega)$, then $\Pi_2 v \in H_{00}^{\frac{1}{2}}(\Lambda)$. In particular, from standard trace theory, we have that $T_\Gamma v \in H_{00}^{\frac{1}{2}}(\Gamma)$, therefore we have to prove that if $v \in H^{\frac{1}{2}}(\Gamma)$ then $\bar{v} \in H^{\frac{1}{2}}(\Lambda)$.

LEMMA 3.2. *When Γ is a cylinder, if $u \in H_{00}^{\frac{1}{2}}(\Gamma)$, then $\bar{u} \in H_{00}^{\frac{1}{2}}(\Lambda)$.*

Proof. Let us denote as ϕ_{ij} and ρ_{ij} , for $i = 1, 2, \dots, j = 0, 1, \dots$, the eigenfunctions and the eigenvalues of the laplacian on Γ , and with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the laplacian on Λ . In particular,

$$\begin{aligned} \phi_{ij}(s, \theta) &= \sin(i\pi s) (\cos(j\theta) + \sin(j\theta)), \\ \rho_{ij} &= i\pi^2 + \frac{j^2}{R^2}, \\ \phi_i(s) &= \sin(i\pi s), \\ \rho_i &= i\pi^2. \end{aligned}$$

It is easy to verify that

$$(3.7) \quad \int_0^{2\pi} \phi_{ij}(s, \theta) d\theta = 0 \quad \forall j > 0, \forall i$$

$$(3.8) \quad \int_0^{2\pi} \phi_{ij}(s, \theta) d\theta = 2\pi R \sin(i\pi s) \quad \text{if } j = 0, \forall i.$$

$$(3.9)$$

Moreover we recall that $\phi_{i,j}(s, \theta)$ and $\phi_i(s)$ are orthogonal basis of $L^2(\Gamma)$ and $L^2(\Lambda)$ respectively. Therefore,

$$\begin{aligned} \bar{u}(s) &= \frac{1}{2\pi R} \int_0^{2\pi} u(s, \theta) R d\theta = \frac{1}{2\pi R} \int_0^{2\pi} \sum_{i,j} a_{i,j} \phi_{i,j}(s, \theta) R d\theta \\ &= \frac{1}{2\pi R} \sum_{i,j} a_{i,j} \int_0^{2\pi} \phi_{i,j}(s, \theta) R d\theta = \sum_i a_{i,0} \phi_i(s). \end{aligned}$$

From [3, Lemma 4.11] we have

$$(3.10) \quad \|u\|_{H^{\frac{1}{2}}(\Gamma)}^2 = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (1 + \rho_{ij})^{\frac{1}{2}} |a_{ij}|^2, \quad \text{with } a_{ij} = \int_0^1 \int_0^{2\pi} u(s, \theta) \phi_{ij} R d\theta ds.$$

and

$$\|\bar{u}\|_{H^{\frac{1}{2}}(\Lambda)}^2 = \sum_{i=1}^{\infty} (1 + \rho_i)^{\frac{1}{2}} |\bar{a}_i|^2, \quad \text{with } \bar{a}_i = \int_0^1 \bar{u}(s) \phi_i(s) ds.$$

Therefore, we have

$$\begin{aligned} \|\bar{u}\|_{H^{\frac{1}{2}}(\Lambda)}^2 &= \sum_{i=1}^{\infty} (1 + i^2 \pi^2)^{\frac{1}{2}} \left(\int_0^1 \bar{u}(s) \sin(i\pi s) ds \right)^2 \\ &= \sum_{i=1}^{\infty} (1 + i^2 \pi^2)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} a_{j,0} \int_0^1 \sin(j\pi s) \sin(i\pi s) ds \right)^2 \\ &= \sum_{i=1}^{\infty} \frac{1}{4} (1 + i^2 \pi^2)^{\frac{1}{2}} a_{i,0}^2 \\ &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(1 + i^2 \pi^2 + \frac{j^2}{R^2} \right)^{\frac{1}{2}} |a_{i,j}|^2 = \|u\|_{H^{\frac{1}{2}}(\Gamma)}^2, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \int_0^1 \sin(i\pi s) \sin(j\pi s) ds &= 0 \quad \text{if } i \neq j \\ \int_0^1 \sin(i\pi s) \sin(j\pi s) ds &= \frac{1}{2} \quad \text{if } i = j. \end{aligned}$$

c.v.d. \square

LEMMA 3.3. *If Σ is a straight cylinder, if $u \in H_{00}^{\frac{1}{2}}(\Gamma)$ is constant on each cross section, namely $u(s, \theta) = u(s)$, then*

$$\|u\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = 2\pi R \|u\|_{H_{00}^{\frac{1}{2}}(\Lambda)}.$$

Proof. From (3.10),

$$\begin{aligned}\|u\|_{H^{\frac{1}{2}}(\Gamma)}^2 &= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (1 + \rho_{ij})^{\frac{1}{2}} |a_{ij}|^2 = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(1 + i\pi^2 + \frac{j^2}{R^2}\right)^{\frac{1}{2}} \left(\int_0^1 \int_0^{2\pi} u(s, \theta) \phi_{ij} R d\theta ds\right)^2 \\ &= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(1 + i\pi^2 + \frac{j^2}{R^2}\right)^{\frac{1}{2}} \left(\int_0^1 u(s) \int_0^{2\pi} \phi_{ij} R d\theta ds\right)^2,\end{aligned}$$

and using (3.7) and (3.8), we obtain

$$\begin{aligned}\|u\|_{H^{\frac{1}{2}}(\Gamma)}^2 &= \sum_{i=1}^{\infty} (1 + i\pi^2)^{\frac{1}{2}} \left(\int_0^1 u(s) \sin(i\pi s) 2\pi R ds\right)^2 \\ &= 4\pi^2 R^2 \sum_{i=1}^{\infty} (1 + \rho_i)^{\frac{1}{2}} |a_i|^2 = 4\pi^2 R^2 \|u\|_{H^{\frac{1}{2}}(\Lambda)}^2. \quad \square\end{aligned}$$

Then, we apply Theorem 3.1 with the following spaces $X = H_0^1(\Omega) \times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped with the norm $\| [u, u_{\odot}] \|^2 = \|u\|_{H^1(\Omega)}^2 + |\mathcal{D}| \|u_{\odot}\|_{H^1(\Lambda)}^2$, Q equipped with the norm

$$\|M\|_{H^{-\frac{1}{2}}} := \sup_{q \in H^{\frac{1}{2}}(\Lambda)} \frac{\langle q, M \rangle}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}$$

Then, the following properties hold true:

- the form $a([u, U], [v, V]) = (u, v)_{H^1(\Omega)} + |\mathcal{D}| (U, V)_{H^1(\Lambda)}$ is coercive \implies (3.2) is fulfilled. Indeed, we have,

$$a([u, u_{\odot}], [u, u_{\odot}]) = (u, u)_{H^1(\Omega)} + |\mathcal{D}| (u_{\odot}, u_{\odot})_{H^1(\Lambda)} = \| [u, u_{\odot}] \|^2.$$

- We have that $\forall M \in H^{-\frac{1}{2}}(\Lambda)$, $\exists \beta > 0$:

$$\sup_{v \in H_0^1(\Omega), V \in H_0^1(\Lambda)} \frac{\langle \Pi_1 V - \Pi_2 v, M \rangle_{\Lambda}}{\sqrt{\|v\|_{H^1(\Omega)}^2 + |\mathcal{D}| \|V\|_{H^1(\Lambda)}^2}} \geq \beta \sup_{q \in H^{\frac{1}{2}}(\Lambda)} \frac{\langle q, M \rangle_{\Lambda}}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}.$$

We choose $V = 0$ and we obtain

$$\sup_{v \in H_0^1(\Omega), V \in H_0^1(\Lambda)} \frac{\langle \Pi_1 V - \Pi_2 v, M \rangle_{\Lambda}}{\sqrt{\|v\|_{H^1(\Omega)}^2 + |\mathcal{D}| \|V\|_{H^1(\Lambda)}^2}} \geq \sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_2 v, M \rangle_{\Lambda}}{\|v\|_{H^1(\Omega)}}.$$

For any $q \in H_{00}^{\frac{1}{2}}(\Lambda)$, we consider its uniform extension to Γ and then we consider the harmonic extension $v = E(q) \in H_0^1(\Omega)$. It follows that $\Pi_2 v = q$. Therefore,

$$\sup_{v \in H_0^1(\Omega)} \langle \Pi_2 v, M \rangle_{\Lambda} \geq \sup_{q \in H_{00}^{\frac{1}{2}}(\Lambda)} \langle q, M \rangle_{\Lambda}.$$

Moreover, using Lemma 3.3 we obtain

$$\|v\|_{H_0^1(\Omega)} \leq \|E\| \|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)} = |\partial \mathcal{D}| \|E\| \|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)}.$$

Therefore,

$$\begin{aligned}\sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_2 v, M \rangle_{\Lambda}}{\|v\|_{H^1(\Omega)}} &\geq \sup_{q \in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q, M \rangle_{\Lambda}}{\|v\|_{H^1(\Omega)}} \geq |\partial \mathcal{D}|^{-1} \|E\|^{-1} \sup_{q \in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q, M \rangle_{\Lambda}}{\|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)}} \\ &= |\partial \mathcal{D}|^{-1} \|E\|^{-1} \|M\|_{H^{-\frac{1}{2}}(\Lambda)}.\end{aligned}$$

REMARK 3.1. The results of (3.2) and (3.3) can be generalized to the case of a different geometry of Γ , for example a parallelepiped.

4. Finite element approximation (Different meshes for solution and Lagrange multiplier).

Let us introduce an admissible triangulation \mathcal{T}_h^Ω of Ω and an admissible partition \mathcal{T}_h^Λ of Λ . We denote by $X_h^0(\Omega) \subset H_0^1(\Omega)$ the conforming finite element space of continuous piecewise linear functions defined on Ω and by $X_h(\Lambda) \subset H_0^1(\Lambda)$ the space of continuous piecewise linear functions defined on Λ . Moreover, Q_H denotes a suitable trial space for the lagrange multiplier L_H , defined on a different triangulation of Γ with mesh size H . In particular, $Q_H \subset H^{-\frac{1}{2}}(\Gamma)$ in the case of Problem 1 and $Q_H \subset H^{-\frac{1}{2}}(\Lambda)$ in the case of Problem 2.

4.1. Problem 1. It consists to find $u_h \in X_h(\Omega)$, $u_{\odot h} \in X_h(\Lambda)$, $L_H \in Q_H(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$(4.1a) \quad \begin{aligned} (u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + \langle \Pi_1 v_h - \Pi_2 v_{\odot h}, L_H \rangle_\Gamma \\ = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega), v_{\odot h} \in X_h(\Lambda) \end{aligned}$$

$$(4.1b) \quad \langle \Pi_1 u_h - \Pi_2 u_{\odot h}, M_H \rangle_\Gamma = 0 \quad \forall M_H \in Q_H(\Gamma),$$

THEOREM 4.1. $\exists \gamma_1 > 0$ s.t.

$$(4.2) \quad \inf_{M_H \in Q_H(\Gamma)} \sup_{\substack{v_h \in X_h(\Omega), \\ v_{\odot h} \in X_h(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_H \rangle_\Gamma}{\| [v_h, v_{\odot h}] \|} \| M_H \|_{H^{-\frac{1}{2}}(\Gamma)} \geq \gamma_1.$$

Proof. Let $M_H \in Q_H(\Gamma)$. As in the continuous case, let us choose $v_{\odot h} = 0$, therefore

$$\sup_{\substack{v_h \in X_h(\Omega), \\ v_{\odot h} \in X_h(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_H \rangle_\Gamma}{\| [v_h, v_{\odot h}] \|} \geq \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_1 v_h, M_H \rangle_\Gamma}{\| v_h \|_{H^1(\Omega)}}.$$

Following [5, Theorem 11.5], it can be shown under the following assumptions

- the mesh size h of the trial space $X_h(\Omega)$ is sufficiently small compared to the mesh size H of $Q_H(\Gamma)$, i.e. $h \leq c_0 H$ with $c_0 < 1$, and
- a global inverse inequality for the trial space $Q_H(\Gamma)$ holds,

that exists a positive constant c_S

$$(4.3) \quad c_S \| M_H \|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\| w_h \|_{H^{\frac{1}{2}}(\Gamma)}} \quad \forall M_H \in Q_H(\Gamma),$$

being $W_h(\Gamma)$ the trace space of the functions in $X_h(\Omega)$, namely the space of the restrictions of the functions in $X_h(\Omega)$ to Γ . Using the boundedness of the extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H_0^1(\Omega)$ introduced in the previous section, we have

$$\sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\| w_h \|_{H^{\frac{1}{2}}(\Gamma)}} \leq \| E \| \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\| E w_h \|_{H^1(\Omega)}}.$$

Let $R_h : H_0^1(\Omega) \rightarrow X_h(\Omega)$ be a quasi interpolation operator satisfying

$$\| R_h v \|_{H^1(\Omega)} \leq C_R \| v \|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Therefore, we obtain

$$\| E \| \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\| E w_h \|_{H^1(\Omega)}} \leq \| E \| C_R \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\| R_h E w_h \|_{H^1(\Omega)}}$$

and using (4.3), we have

$$\begin{aligned}
(4.4) \quad c_S \|M_H\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|E\| C_R \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H \rangle_\Gamma}{\|E w_h\|_{H^1(\Gamma)}} \\
&= \|E\| C_R \sup_{w_h \in W_h(\Gamma)} \frac{\langle \Pi_1 R_h E w_h, M_H \rangle_\Gamma}{\|R_h E w_h\|_{H^1(\Omega)}} \leq \|E\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_1 v_h, M_H \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned}$$

Therefore the inf-sup condition (5.5) holds with $\gamma_1 = c_S \|E\|^{-1} C_R^{-1}$. \square

4.2. Problem 2. This problem requires to find $u_h \in X_h(\Omega)$, $u_{\odot h} \in X_h(\Lambda)$, $L_H \in Q_H(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}
&(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + |\partial\mathcal{D}| \langle \Pi_1 v_{\odot h} - \Pi_2 v_h, L_H \rangle_\Lambda \\
&\quad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega), v_{\odot h} \in X_h(\Lambda) \\
&|\partial\mathcal{D}| \langle \Pi_1 u_{\odot h} - \Pi_2 u_h, M_H \rangle_\Lambda = 0 \quad \forall M_H \in Q_H(\Lambda).
\end{aligned}$$

THEOREM 4.2. $\exists \gamma_2 > 0$ s.t.

$$(4.6) \quad \inf_{M_H \in Q_H(\Lambda)} \sup_{\substack{v_h \in X_h(\Omega), \\ v_{\odot h} \in X_h(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_H \rangle_\Lambda}{\| [v_h, v_{\odot h}] \| \|M_H\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \gamma_2.$$

Proof. Let M_H be arbitrarily chosen in $Q_H(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that (4.6) reduces to prove

$$\gamma_2 \|M_H\|_{H^{\frac{1}{2}}(\Lambda)} \leq \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, M_H \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}} \quad \forall M_H \in Q_H(\Lambda).$$

Assume that

- the mesh size h of the trial space $X_h(\Omega)$ is sufficiently small compared to the mesh size H of $Q_H(\Lambda)$, i.e. $h \leq c_1 H$ with $c_1 < 1$,
- a global inverse inequality for the trial space $Q_H(\Lambda)$ holds and
- the space $W_h(\Lambda)$, defined as the space of the restrictions on Γ of the functions in $X_h(\Omega)$ averaged on the cross section, has the approximation property, namely if Q_h^σ denotes the projection from $H^\sigma(\Lambda)$ to $W_h(\Lambda)$, we have

$$\|w - Q_h^\sigma w\|_{H^\sigma(\Lambda)} \leq c_A h^{s-\sigma} |w|_{H^s(\Lambda)} \quad \forall w \in H^s(\Lambda).$$

(Actually $W_h(\Lambda)$ coincides with the space of piecewise linear continuous polynomials on Λ).

Under the previous assumptions, an inequality similar to (4.3) holds also for $W_h(\Lambda)$. In particular, following the same proof in Steinbach, we can prove

$$(4.7) \quad c_{S_2} \|M_H\|_{H^{-\frac{1}{2}}(\Lambda)} \leq \sup_{w_h \in W_h(\Lambda)} \frac{\langle w_h, M_H \rangle_\Lambda}{\|w_h\|_{H^{\frac{1}{2}}(\Lambda)}} \quad \forall M_H \in Q_H(\Lambda).$$

If we denote with \mathcal{U}_E the uniform extension operator from Λ to Γ , using Lemma 3.3, we easily have for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}| \|w\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Consequently, using again the extension operator E from $H^{\frac{1}{2}}(\Omega)$ to $H_0^1(\Omega)$ and the quasi interpolation

operator R_h from $H_0^1(\Omega)$ to $X_h(\Omega)$, we obtain

$$\begin{aligned}
(4.8) \quad c_{S_2} \|M_H\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{w_h \in W_h(\Lambda)} \frac{\langle w_h, M_H \rangle_\Lambda}{\|w_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\
&\leq |\partial\mathcal{D}| \sup_{w_h \in W_h(\Lambda)} \frac{\langle w_h, M_H \rangle_\Lambda}{\|\mathcal{U}_E w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq |\partial\mathcal{D}| \|E\| \sup_{w_h \in W_h(\Lambda)} \frac{\langle w_h, M_H \rangle_\Lambda}{\|E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}| \|E\| C_R \sup_{w_h \in W_h(\Lambda)} \frac{\langle w_h, M_H \rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&= |\partial\mathcal{D}| \|E\| C_R \sup_{w_h \in W_h(\Lambda)} \frac{\langle \Pi_1 R_h E\mathcal{U}_E w_h, M_H \rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}| \|E\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, M_H \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned}$$

Therefore, (4.6) holds with $\gamma_2 = c_{S_2} |\partial\mathcal{D}|^{-1} \|E\|^{-1} C_R^{-1}$ \square

5. Finite element approximation (Same mesh for solution and Lagrange multiplier). Let us introduce a shape-regular triangulation \mathcal{T}_h^Ω of Ω and an admissible partition \mathcal{T}_h^Λ of Λ . We denote by $X_{h,k}^0(\Omega) \subset H_0^1(\Omega)$ the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satisfying homogeneous Dirichlet conditions on the boundary and by $X_{h,k}^0(\Lambda) \subset H_0^1(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ , satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. Moreover, Q_h denotes a suitable trial space for the lagrange multiplier L_h . In particular, $Q_h \subset H^{-\frac{1}{2}}(\Gamma)$ in the case of Problem 1 and $Q_h \subset H^{-\frac{1}{2}}(\Lambda)$ in the case of Problem 2.

5.1. Problem 1. It consists to find $u_h \in X_{h,k}^0(\Omega)$, $u_{\odot h} \in X_{h,k}^0(\Lambda)$, $L_h \in Q_h(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned}
(5.1a) \quad (u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}| (u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + \langle \Pi_1 v_h - \Pi_2 v_{\odot h}, L_h \rangle_\Gamma \\
= (f, v_h)_{L^2(\Omega)} + |\mathcal{D}| (\bar{g}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_{h,k}^0(\Omega), \quad v_{\odot h} \in X_{h,k}^0(\Lambda)
\end{aligned}$$

$$(5.1b) \quad \langle \Pi_1 u_h - \Pi_2 u_{\odot h}, M_h \rangle_\Gamma = 0 \quad \forall M_h \in Q_h(\Gamma),$$

Let us denote with $W_{h,k}^0(\Gamma) \subset H_{00}^{\frac{1}{2}}(\Gamma)$ the trace space of functions running in $X_{h,k}^0(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$. We choose $Q_h(\Gamma) = W_{h,k}^0(\Gamma)$, therefore we impose homogeneous Dirichlet boundary condition on $\partial\Omega$ also for the Lagrange multiplier. For this choice of $Q_h(\Gamma)$ we can prove the well-posedness of the discrete problem, as shown in the following.

LEMMA 5.1. *Let $P_h : H_{00}^{\frac{1}{2}} \longrightarrow W_{h,k}^0(\Gamma)$ be the orthogonal projection operator defined for any $v \in H_{00}^{\frac{1}{2}}(\Gamma)$ by*

$$(P_h v, \psi)_\Gamma = (v, \psi)_\Gamma \quad \forall \psi \in W_{h,k}^0(\Gamma).$$

Then, P_h is continuous on $H_{00}^{\frac{1}{2}}(\Gamma)$, namely

$$(5.2) \quad \|P_h v\|_{H_{00}^{\frac{1}{2}}(\Gamma)} \leq C \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma)},$$

where C is a positive constant independent of h .

Proof. We prove that P_h is continuous on $L^2(\Gamma)$ and on $H_0^1(\Gamma)$ following [4, Section 1.6.3]. Then, the inequality (5.2) can be derived by Hilbertian interpolation. For the L^2 -continuity, we exploit the fact that, from the definition of P_h ,

$$(v - P_h v, P_h v)_\Gamma = 0.$$

Therefore, by Pythagoras identity,

$$\|v\|_{L^2(\Gamma)}^2 = \|v - P_h v\|_{L^2(\Gamma)}^2 + \|P_h v\|_{L^2(\Gamma)}^2 \geq \|P_h v\|_{L^2(\Gamma)}^2.$$

Let us now consider $v \in H_0^1(\Gamma)$. The Scott-Zhang interpolation operator SZ_h from $H_0^1(\Gamma)$ to $W_{h,k}^0(\Gamma)$ satisfies the following inequalities,

$$(5.3) \quad \|SZ_h v\|_{H^1(\Gamma)} \leq C_1 \|v\|_{H^1(\Gamma)}$$

$$(5.4) \quad \|v - SZ_h v\|_{L^2(\Gamma)} \leq C_2 h \|v\|_{H^1(\Gamma)}.$$

Therefore,

$$\begin{aligned} \|\nabla P_h v\|_{L^2(\Gamma)} &\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + \|\nabla SZ_h v\|_{L^2(\Gamma)} \\ &\leq (\text{using (5.3)}) \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)} \end{aligned}$$

and by using the inverse inequality we obtain

$$\begin{aligned} \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)} &\leq \frac{C_3}{h} \|P_h v - SZ_h v\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)} \\ &= \frac{C_3}{h} \|P_h(v - SZ_h v)\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)} \\ &\leq (\text{Stability of } P_h \text{ in } L^2) \frac{C_3}{h} \|v - SZ_h v\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)} \\ &\leq (\text{using (5.4)}) \frac{C_3}{h} C_2 h \|v\|_{H^1(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)} \\ &\leq (C_2 C_3 + C_1) \|v\|_{H^1(\Gamma)}, \end{aligned}$$

from which we obtain the continuity in $H_0^1(\Gamma)$. □

LEMMA 5.2. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \geq \gamma \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. Let M_h be in $W_{h,k}^0(\Gamma)$. From the continuous case, in particular from (3.5), we have

$$\|E\|^{-1} \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M_h \rangle}{\|v\|_{H^1(\Omega)}} = \sup_{v \in H_0^1(\Omega)} \frac{\langle T_\Gamma v, M_h \rangle}{\|v\|_{H^1(\Omega)}}$$

and by the trace inequality $\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \leq K_1 \|v\|_{H_0^1(\Omega)}$ (see [1, 7.56]), we obtain

$$\sup_{v \in H_0^1(\Omega)} \frac{\langle T_\Gamma v, M_h \rangle}{\|v\|_{H^1(\Omega)}} \leq K_1 \sup_{v \in H_0^1(\Omega)} \frac{\langle T_\Gamma v, M_h \rangle}{\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

By the definition of P_h and (5.2)

$$\begin{aligned} K_1 \sup_{v \in H_0^1(\Omega)} \frac{\langle T_\Gamma v, M_h \rangle}{\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} &= K_1 \sup_{v \in H_0^1(\Omega)} \frac{\langle P_h(T_\Gamma v), M_h \rangle}{\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} \\ &\leq K_1 C \sup_{v \in H_0^1(\Omega)} \frac{\langle P_h(T_\Gamma v), M_h \rangle}{\|P_h(T_\Gamma v)\|_{H^{\frac{1}{2}}(\Gamma)}} \\ &= K_1 C \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}}. \end{aligned}$$

□

THEOREM 5.3 (Discrete inf-sup). $\exists \gamma_1 > 0$ s.t.

$$(5.5) \quad \inf_{M_h \in W_{h,k}^0(\Gamma)} \sup_{\substack{v_h \in X_{h,k}^0(\Omega), \\ v_{\odot h} \in X_{h,k}^0(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_h \rangle_\Gamma}{\| [v_h, v_{\odot h}] \| \| M_h \|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \gamma_1.$$

Proof. Let $M_h \in W_{h,k}^0$. As in the continuous case, we choose $v_{\odot h} = 0$ and we have

$$\sup_{\substack{v_h \in X_{h,k}^0(\Omega), \\ v_{\odot h} \in X_{h,k}^0(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_h \rangle_\Gamma}{\| [v_h, v_{\odot h}] \|} \geq \sup_{v_h \in X_{h,k}^0(\Omega)} \frac{\langle \Pi_1 v_h, M_h \rangle_\Gamma}{\| v_h \|_{H^1(\Omega)}}.$$

Therefore, we want to prove that there exists γ_1 such that

$$\sup_{v_h \in X_{h,k}^0(\Omega)} \frac{\langle \Pi_1 v_h, M_h \rangle_\Gamma}{\| v_h \|_{H^1(\Omega)}} = \sup_{v_h \in X_{h,k}^0(\Omega)} \frac{\langle T_\Gamma v_h, M_h \rangle}{\| v_h \|_{H^1(\Omega)}} \geq \gamma_1 \| M_h \|_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall M_h \in W_{h,k}^0.$$

Using Lemma 5.2 and the boundedness of the harmonic extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H_0^1(\Omega)$ introduced in the previous section, we have

$$\gamma \| M_h \|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle_\Gamma}{\| q_h \|_{H^{\frac{1}{2}}(\Gamma)}} \leq \| E \| \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle_\Gamma}{\| E q_h \|_{H^1(\Omega)}}.$$

Let $R_h : H_0^1(\Omega) \rightarrow X_{h,k}^0(\Omega)$ be a quasi interpolation operator satisfying

$$\| R_h v \|_{H^1(\Omega)} \leq C_R \| v \|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Therefore, we obtain

$$\| E \| \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle_\Gamma}{\| E q_h \|_{H^1(\Omega)}} \leq \| E \| C_R \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle_\Gamma}{\| R_h E q_h \|_{H^1(\Omega)}}$$

and we have

$$(5.6) \quad \begin{aligned} \gamma \| M_h \|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle_\Gamma}{\| q_h \|_{H^{\frac{1}{2}}(\Gamma)}} \leq \| E \| C_R \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle q_h, M_h \rangle_\Gamma}{\| R_h E q_h \|_{H^1(\Gamma)}} \\ &= \| E \| C_R \sup_{q_h \in W_{h,k}^0(\Gamma)} \frac{\langle \Pi_1 R_h E q_h, M_h \rangle_\Gamma}{\| R_h E q_h \|_{H^1(\Omega)}} \leq \| E \| C_R \sup_{v_h \in X_{h,k}(\Omega)} \frac{\langle \Pi_1 v_h, M_h \rangle_\Gamma}{\| v_h \|_{H^1(\Omega)}}. \end{aligned}$$

Therefore the inf-sup condition (5.5) holds with $\gamma_1 = \gamma \| E \|^{-1} C_R^{-1}$. \square

REMARK 5.1. We notice that to prove the result in Lemma 5.2 (and then the discrete inf-sup condition) basically we need a projection operator $P_h : H_{00}^{\frac{1}{2}} \rightarrow W_{h,k}^0(\Gamma)$ orthogonal in the multiplier space $Q_h(\Gamma)$, namely such that $\langle P_h v, M_h \rangle = \langle v, M_h \rangle$, $\forall M_h \in Q_h(\Gamma)$, and continuous in $H^{\frac{1}{2}}(\Gamma)$. Therefore, in principle different choices than $Q_h(\Lambda) = W_{h,k}^0(\Gamma)$ could be considered if we can build an operator P_h satisfying these properties. In [2] such operator P_h is built for a particular choice of $Q_h(\Gamma)$ but it is not clear how they prove the H^1 -stability inequality (and consequently the $H^{\frac{1}{2}}$ -stability) with a constant independent of the mesh size h ...

5.2. Problem 2. This problem requires to find $u_h \in X_{h,k}^0(\Omega)$, $u_{\odot h} \in X_{h,k}^0(\Lambda)$, $L_h \in Q_h(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned} &(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + |\partial \mathcal{D}| \langle \Pi_1 v_{\odot h} - \Pi_2 v_h, L_h \rangle_\Lambda \\ &= (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega), v_{\odot h} \in X_h(\Lambda) \\ &|\partial \mathcal{D}| \langle \Pi_1 u_{\odot h} - \Pi_2 u_h, M_h \rangle_\Lambda = 0 \quad \forall M_h \in Q_h(\Lambda). \end{aligned}$$

We introduce the space $W_{h,k}^0(\Lambda) \subset H_{00}^{\frac{1}{2}}(\Lambda)$, which is the averaged trace space of functions running in $H_0^1(\Omega)$. It coincides with the space of continuous piecewise polynomials of degree k defined on Λ and satisfying homogeneous Dirichlet boundary condition. We choose $Q_h(\Lambda) = W_{h,k}^0(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda \cap \partial\Omega$ also for the Lagrange multiplier. With this choice for $Q_h(\Lambda)$, we can prove the well-posedness of the discrete problem. In particular, following the same steps as for Problem 1, we can prove the following results.

LEMMA 5.4. *Let $P_h : H_{00}^{\frac{1}{2}}(\Lambda) \rightarrow W_{h,k}^0(\Lambda)$ be the orthogonal projection operator defined for any $v \in H_{00}^{\frac{1}{2}}(\Lambda)$ by*

$$(P_h v, \psi)_\Lambda = (v, \psi)_\Lambda \quad \forall \psi \in W_{h,k}^0(\Lambda).$$

Then, P_h is continuous on $H_{00}^{\frac{1}{2}}(\Lambda)$, namely

$$\|P_h v\|_{H_{00}^{\frac{1}{2}}(\Lambda)} \leq C \|v\|_{H_{00}^{\frac{1}{2}}(\Lambda)},$$

where C is a positive constant independent of h .

LEMMA 5.5. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W_{h,k}^0(\Lambda)} \frac{\langle q_h, M_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \geq \gamma \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M_h \in W_{h,k}^0(\Lambda).$$

THEOREM 5.6 (Discrete inf-sup). $\exists \gamma_2 > 0$ s.t.

$$(5.8) \quad \inf_{M_h \in Q_h(\Lambda)} \sup_{\substack{v_h \in X_{h,k}^0(\Omega), \\ v_{\odot h} \in X_{h,k}^0(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_h \rangle_\Lambda}{\| [v_h, v_{\odot h}] \| \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \gamma_2.$$

Proof. Let M_h be arbitrarily chosen in $W_{h,k}^0(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists γ_2 s.t.

$$\gamma_2 \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)} \leq \sup_{v_h \in X_{h,k}^0(\Omega)} \frac{\langle \Pi_2 v_h, M_h \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}} \quad \forall M_h \in W_{h,k}^0(\Lambda).$$

Let us denote with \mathcal{U}_E the uniform extension operator from Λ to Γ . Using Lemma 3.3, we easily have for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}| \|w\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Consequently, from Lemma 5.5, using again the extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H_0^1(\Omega)$ and the quasi interpolation operator R_h from $H_0^1(\Omega)$ to $X_{h,k}^0(\Omega)$, we obtain

$$(5.9) \quad \begin{aligned} \gamma \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h \in W_{h,k}^0(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\ &= |\partial\mathcal{D}| \sup_{q_h \in W_{h,k}^0(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|\mathcal{U}_E q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq |\partial\mathcal{D}| \|E\| \sup_{q_h \in W_{h,k}^0(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|E \mathcal{U}_E q_h\|_{H^1(\Omega)}} \\ &\leq |\partial\mathcal{D}| \|E\| C_R \sup_{q_h \in W_{h,k}^0(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|R_h E \mathcal{U}_E q_h\|_{H^1(\Omega)}} \\ &= |\partial\mathcal{D}| \|E\| C_R \sup_{q_h \in W_{h,k}^0(\Lambda)} \frac{\langle \Pi_1 R_h E \mathcal{U}_E q_h, M_h \rangle_\Lambda}{\|R_h E \mathcal{U}_E w_h\|_{H^1(\Omega)}} \\ &\leq |\partial\mathcal{D}| \|E\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, M_h \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}. \quad \square \end{aligned}$$

6. A benchmark problem with analytical solution. We consider the following 3D-1D coupled problem,

$$\begin{aligned}
(6.1a) \quad & -\Delta u = f \quad \text{in } \Omega \\
(6.1b) \quad & -d_{zz}^2 u_{\odot} = g \quad \text{on } \Lambda \\
(6.1c) \quad & u = 0 \quad \text{on } \partial\Omega \\
(6.1d) \quad & u_{\odot} - \bar{u} = q \quad \text{on } \Lambda
\end{aligned}$$

where $\Omega = [0, 1] \times [0, 1] \times [0, H]$, $\Lambda = \{x = 0.5\} \times \{y = 0.5\} \times [0, H]$ and

$$\begin{aligned}
f &= 8\pi^2 \sin(2\pi x) \sin(2\pi y) \\
g &= \frac{\pi^2}{H^2} \sin\left(\frac{\pi z}{H}\right) \\
q &= \sin\left(\frac{\pi z}{H}\right).
\end{aligned}$$

In this case the z coordinate coincides with the axial coordinate along Λ . We define $\Sigma = [0.25, 0.75] \times [0.25, 0.75] \times [0, H]$. The average of the 3D solution \bar{u} in (6.1d) is computed on the cross section $\partial\mathcal{D}$ of the virtual interface $\Gamma = \partial\Sigma$. The exact solution of (6.1) is given by

$$\begin{aligned}
(6.2) \quad & u = \sin(2\pi x) \sin(2\pi y) \\
(6.3) \quad & u_{\odot} = \sin\left(\frac{\pi z}{H}\right)
\end{aligned}$$

Let us notice that u_{\odot} satisfies homogeneous Dirichlet conditions at the boundary of Λ . Moreover, the solution (6.2)-(6.3) satisfies on Γ the relation

$$(6.4) \quad \lambda = \nabla u \cdot \mathbf{n}_{\oplus} = d_z u_{\odot} n_{\oplus, z} = 0,$$

being $n_{\oplus, z}$ the z -component of the normal unit vector to Γ .

We prove that (6.2)-(6.3) is solution of (2.3) and (2.4) in the simplified case in which the starting 3D-3D problem is

$$\begin{aligned}
(6.5a) \quad & -\Delta u_{\oplus} = f && \text{in } \Omega_{\oplus}, \\
(6.5b) \quad & -\Delta u_{\ominus} = g && \text{in } \Sigma, \\
(6.5c) \quad & -\nabla u_{\ominus} \cdot \mathbf{n}_{\ominus} = -\nabla u_{\oplus} \cdot \mathbf{n}_{\ominus} && \text{on } \Gamma, \\
(6.5d) \quad & u_{\ominus} - u_{\oplus} = q && \text{on } \Gamma, \\
(6.5e) \quad & u_{\oplus} = 0 && \text{on } \partial\Omega.
\end{aligned}$$

instead of (2.1). Therefore the reduced problems in the two different formulations (2.3) and (2.4) become respectively

$$\begin{aligned}
(6.6a) \quad & (\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_{\odot}, d_s v_{\odot})_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_{\odot}, L \rangle_{\Gamma} \\
& = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_{\odot})_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), \quad v_{\odot} \in H^1(\Lambda)
\end{aligned}$$

$$(6.6b) \quad \langle \Pi_1 u - \Pi_2 u_{\odot}, M \rangle_{\Gamma} = \langle q, M \rangle_{\Gamma} \quad \forall M \in H^{-\frac{1}{2}}(\Gamma).$$

and

$$\begin{aligned}
(6.7a) \quad & (\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_{\odot}, d_s v_{\odot})_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_{\odot}, L \rangle_{H^{-\frac{1}{2}}(\Lambda)} \\
& = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), \quad v_{\odot} \in H_0^1(\Lambda)
\end{aligned}$$

$$(6.7b) \quad |\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_{\odot}, M \rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \bar{q}, M \rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M \in H^{-\frac{1}{2}}(\Lambda).$$

Let us prove that (6.2)-(6.3) is solution of (6.6). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ , from (6.6a) we have

$$\begin{aligned} & -(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d_{ss}^2 u_{\odot}, v_{\odot})_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_{\odot}, L \rangle_{\Gamma} \\ & = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\bar{g}, v_{\odot})_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega), v_{\odot} \in H^1(\Lambda). \end{aligned}$$

Clearly, since (6.2) satisfies (6.1a) and (6.3) satisfies (6.1b), we have that

$$\begin{aligned} & -(\Delta u, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \\ & -|\mathcal{D}|(d_{ss}^2 u_{\odot}, v_{\odot})_{L^2(\Lambda)} = |\mathcal{D}|(\bar{g}, v_{\odot})_{L^2(\Lambda)} \end{aligned}$$

and being $L = \lambda = 0$, we can conclude that (6.2)-(6.3) satisfy (6.6a). The fact that the solution satisfy (6.6b) follows from (6.1d). We can prove in the same way that (6.2)-(6.3) is solution of (6.7), exploiting the fact that in this case $L = \bar{\lambda} = 0$.

REMARK 6.1. *Let us notice that the 3D solution (6.2) is such that $\bar{u} = 0$. Therefore in (6.1) it is like we are solving two separated problems, one in Ω and the other on Λ .*

REMARK 6.2. *It would be interesting to make a comparison between the solution of the fully coupled 3D-3D problem (2.1) (also in the simplified case of type (6.5)) and the solution of the reduced problems (2.3) and (2.4). Therefore, we could set the values of the data of the problem such that the reduced formulation becomes non-trivial and fully coupled. Then, we will solve both the original and reduced problem to observe the differences in the solutions and the values of the Lagrange multiplier.*

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