COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

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1. Introduction. We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators form a 3D domain, Ω, to 1D manifold, Λ, and vice versa. This is a non trivial objective because the standard trace operator form a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω.

2. Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex open set. Let $\Omega_\ominus$ be a generalized cylinder embedded into Ω and be $\Omega_\oplus = \Omega \setminus \overline{\Omega_\ominus}$ be the complementary set of the cylinder. We also introduce the set Λ, a 1D manifold that is the centerline of $\Omega_\ominus$. We define the arc-length coordinate along Λ, denoted by $s \in (0, S)$. We denote with $\mathcal{D}(s)$ and $\partial\mathcal{D}(s)$ a cross section of $\Omega_\ominus$ and its boundary, respectively. We also assume that $\Omega_\ominus$ crosses Ω from side to side and we call Γ the lateral (cylindrical) surface of $\Omega_\ominus$, while the upper and lower side faces of $\Omega_\ominus$ belong to $\partial\Omega$. We refer to Figure 2.1 for an illustration of the notation.

We consider the problem arising from *Dirichlet-Neumann* conditions. It consists to find $u_\oplus, u_\ominus$ s.t.:

$$
\begin{aligned}
&(2.1a) && -\Delta u_\oplus + u_\oplus = f && \text{in } \Omega_\oplus,\\
&(2.1b) && -\Delta u_\ominus + u_\ominus = g && \text{in } \Omega_\ominus,\\
&(2.1c) && -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus && \text{on } \Gamma,\\
&(2.1d) && u_\ominus = u_\oplus && \text{on } \Gamma,\\
&(2.1e) && u_\oplus = 0 && \text{on } \partial\Omega\,.
\end{aligned}
$$

The objective of this work is to derive and alalyze a simplified version of problem (2.1), where the domain $\Omega_\ominus$ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely $\mathcal{D}$. This new problem setting will be called the *reduced* problem. Form the mathematical standpoint it is more challenging than (2.1), because it involves the coupling of 3D/1D elliptic problems. For the model reduction process, we decompose integrals as follows, for any sufficiently regular function w,

$$\int_{\Omega_\ominus} w d\omega = \int_\Lambda \int_{\mathcal{D}} w d\sigma ds = \int_\Lambda |\mathcal{D}| \overline{\overline{w}} ds\,, \quad \int_\Gamma w d\sigma = \int_\Lambda \int_{\partial\mathcal{D}} w d\gamma ds = \int_\Lambda |\partial\mathcal{D}| \overline{w} ds\,,$$

where $\overline{\overline{w}}$, $\overline{w}$ denote the following mean values respectively,

$$\overline{\overline{w}} = |\mathcal{D}|^{-1} \int_{\mathcal{D}} w d\sigma\,, \quad \overline{w} = |\partial\mathcal{D}|^{-1} \int_{\partial\mathcal{D}} w d\gamma\,.$$

We apply the model reduction approach at the level of the variational formulation. We start from the variational formulation of problem (2.1), that is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ u_\ominus \in H^1_{\partial\Omega_\ominus \setminus \Gamma}(\Omega_\ominus),\ \lambda \in H^{-\frac{1}{2}}(\partial\Omega_\ominus)$ s.t.

$$
\begin{aligned}
&(2.2a) && (u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Omega_\ominus)} + \langle v_\oplus - v_\ominus, \lambda \rangle_{H^{-\frac{1}{2}}(\Gamma)}\\
& && \quad = (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Omega_\ominus)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Omega_\ominus \setminus \Gamma}(\Omega_\ominus)\\
&(2.2b) && \langle u_\oplus - u_\ominus, \mu \rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,,
\end{aligned}
$$

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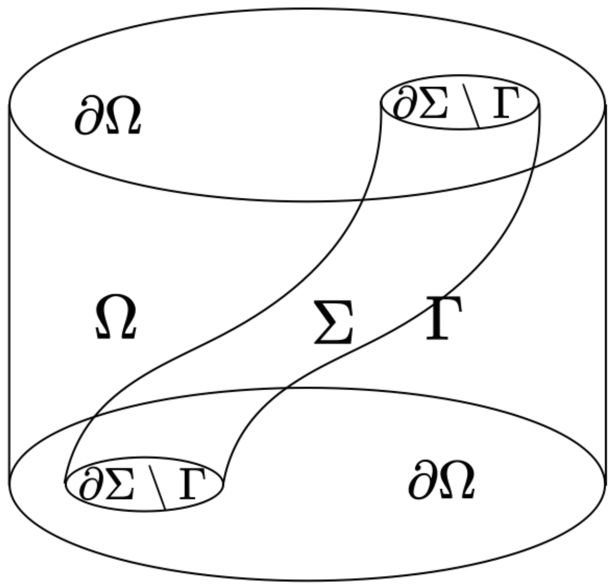

FIGURE 2.1. *Geometrical setting of the problem*

where $\langle v, \mu \rangle_{H^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. In this case, the additional variable λ is equivalent to $\lambda = \nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

Using the averaging tools for the model reduction approach, we end up with two different formulations of a reduced problem for the unknown u defined on the entire 3D domain Ω, coupled with the unknown $u_\odot$, defined on the 1D manifold Λ and a Lagrange multiplier defined either on Γ (problem 1) or on Λ (problem 2).

The scope of this work is to compare them, with the aim to determine which is the most suitable as a computational model based on 3D-1D coupled PDEs.

2.1. Topological model reduction.

Model reduction of the problem on $\Omega_\ominus$. We apply the averaging technique to equation (2.1b). In particular, we consider an arbitrary portion $\mathcal{P}$ of the cylinder $\Omega_\ominus$, with lateral surface $\Gamma_\mathcal{P}$ and bounded by two perpendicular sections to Λ, namely $\mathcal{D}(s_1)$, $\mathcal{D}(s_2)$ with $s_1 < s_2$. We have,

$$\int_\mathcal{P} -\Delta u_\ominus + u_\ominus d\omega = -\int_{\partial\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus \, d\sigma + \int_\mathcal{P} u_\ominus d\omega =$$
$$\int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma - \int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma + \int_\mathcal{P} u_\ominus d\omega$$

By the fundamental theorem of integral calculus combined with the Reynolds transport Theorem, being ν the normal component of the velocity of the boundary, we have,

$$\begin{aligned}\int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma &= -\int_{s_1}^{s_2} d_s \int_{\mathcal{D}(s)} \partial_s u_\ominus d\sigma ds \\ &= -\int_{s_1}^{s_2} d^2_{ss} \int_{\mathcal{D}(s)} u_\ominus \, d\sigma \, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus \, d\gamma \right) ds \,,\end{aligned}$$

and assuming that $\mathcal{D}(s)$ can not change shape, we have

$$-\int_{s_1}^{s_2} d^2_{ss}\int_{\mathcal{D}(s)} u_\ominus \, d\sigma\, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus \, d\gamma\right) ds = -\int_{s_1}^{s_2} \left[d^2_{ss}(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus) - d_s(\nu|\partial\mathcal{D}(s)|\overline{u}_\ominus)\right] ds$$
$$= -\int_{s_1}^{s_2} \left[d^2_{ss}(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus) - d_s\left(d_s(|\mathcal{D}(s)|)\overline{u}_\ominus\right)\right] ds.$$

Moreover, we have

$$\int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma = \int_{\Gamma_\mathcal{P}} \lambda \, d\sigma = \int_{s_1}^{s_2}\int_{\partial\mathcal{D}(s)} \lambda d\gamma\, ds = \int_{s_1}^{s_2} |\partial\mathcal{D}|\overline{\lambda}\, ds\,.$$

From the combination of all the above terms with the right hand side, we obtain that the solution $u_\ominus$ of (2.1b) satisfies,

$$\int_{s_1}^{s_2} \left[-d^2_{ss}(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus) + d_s\left(d_s(|\mathcal{D}(s)|)\overline{u}_\ominus\right) + |\mathcal{D}(s)|\overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)|\overline{\lambda}\right] ds = \int_{s_1}^{s_2} |\mathcal{D}(s)|\overline{\overline{g}}\, ds\,.$$

Since the choice of the points s_1, s_2 is arbitrary, we conclude that the following equation holds true,

$$-d^2_{ss}(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus) + d_s\left(d_s(|\mathcal{D}(s)|)\overline{u}_\ominus\right) + |\mathcal{D}(s)|\overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)|\overline{\lambda} = |\mathcal{D}(s)|\overline{\overline{g}} \quad \text{on } \Lambda\,, \tag{2.3}$$

which is complemented by the following conditions at the boundary of Λ,

$$|\mathcal{D}(s)|d_s\overline{\overline{u}}_\ominus = 0, \quad d_s|\mathcal{D}(s)| = 0, \quad \text{on} \quad s = 0, S. \tag{2.4}$$

Then, we consider variational formulation of the averaged equation (2.3). After multiplication by a test function $v_\odot \in H^1(\Lambda)$, integration on Λ and suitable application of integration by parts, we obtain,

$$\int_\Lambda d_s(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus)d_s v_\odot\, ds - d_s(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus)v_\odot|_{s=0}^{s=S} - \int_\Lambda (d_s|\mathcal{D}(s)|)\overline{u}_\ominus d_s v_\odot\, ds + (d_s|\mathcal{D}(s)|)\overline{u}_\ominus v_\odot|_{s=0}^{s=S}$$
$$+ \int_\Lambda |\mathcal{D}(s)|\overline{\overline{u_\ominus}}v_\odot - \int_\Lambda |\partial\mathcal{D}(s)|\overline{\lambda}v_\odot\, ds = \int_\Lambda |\mathcal{D}(s)|\overline{\overline{g}}V\, ds\,.$$

Using boundary conditions, the identity $d_s(|\mathcal{D}(s)|\overline{\overline{u}}_\ominus) = |\mathcal{D}(s)|d_s\overline{\overline{u}}_\ominus + d_s(|\mathcal{D}(s)|)\overline{\overline{u}}_\ominus$ and reminding that $d_s|\mathcal{D}(s)|)/|\partial\mathcal{D}(s)| = \nu$, we obtain,

$$(d_s\overline{\overline{u}}_\ominus, d_s v_\odot)_{\Lambda,|\mathcal{D}|} + (\nu(\overline{\overline{u}}_\ominus - \overline{u}_\ominus), d_s v_\odot)_{\Lambda,|\partial\mathcal{D}|} + (\overline{\overline{u_\ominus}}, v_\odot)_{\Lambda,|\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}}, V)_{\Lambda,|\mathcal{D}|}\,. \tag{2.5}$$

where we have introduced the following weighted inner product notation,

$$(u_\odot, v_\odot)_{\Lambda,w} = \int_0^S w(s)u_\odot(s)v_\odot(s)ds\,.$$

Let us now formulate the modelling assumption that allows us to reduce equation (2.5) to a solvable one-dimensional (1D) model. More precisely, we assume that:

A1 the function $u_\ominus$ has a *uniform profile* on each cross section $\mathcal{D}(s)$, namely $u_\ominus(r,s,t) = u_\odot(s)$.

Therefore, observing that $u_\odot = \overline{u}_\ominus = \overline{\overline{u}}_\ominus$, problem (2.5) consists to find $u_\odot \in H^1(\Lambda)$ such that

$$(d_s u_\odot, d_s u_\odot)_{\Lambda,|\mathcal{D}|} + (u_\odot, v_\odot)_{\Lambda,|\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}}, v_\odot)_{\Lambda,|\mathcal{D}|} \quad \forall v_\odot \in H^1(\Lambda)\,. \tag{2.6}$$

Topological model reduction of the problem on $\Omega_\oplus$. We focus here on the subproblem of (2.1a) related to $\Omega_\oplus$. We multiply both sides of (2.1a) by a test function $v \in H^1_0(\Omega)$ and integrate on $\Omega_\oplus$. Integrating by parts and using boundary and interface conditions, we obtain

$$\int_{\Omega_\oplus} fv\, d\omega = \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v\, d\omega - \int_{\partial\Omega_\oplus} \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v\, d\sigma + \int_{\Omega_\oplus} u_\oplus v$$
$$= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v\, d\Omega - \int_\Gamma \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v + \int_{\Omega_\oplus} u_\oplus v$$
$$= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v\, d\Omega + \int_\Gamma \lambda v + \int_{\Omega_\oplus} u_\oplus v.$$

Then, we make the following modelling assumptions:

A2 we identify the domain $\Omega_\oplus$ with the entire Ω, and we correspondingly omit the subscript $\oplus$ to the functions defined on $\Omega_\oplus$, namely

$$\int_{\Omega_\oplus} u_\oplus \, d\omega \simeq \int_\Omega u \, d\omega \, .$$

Therefore, we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\lambda, v)_\Gamma = (f, v)_\Omega$$

and combining with (2.6) we obtain the first formulation of the reduced problem.

Problem 1 (3D-1D-3D). Let $\langle \cdot, \cdot \rangle_\Gamma$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. The problem consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ \lambda \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$(u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle Tv - \mathcal{U}_E v_\odot, \lambda \rangle_\Gamma \tag{2.7a}$$
$$= (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda)$$

$$\langle Tu - \mathcal{U}_E u_\odot, \mu \rangle_\Gamma = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma) \, . \tag{2.7b}$$

Here, $T : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ denotes the trace operator on Γ and $\mathcal{U}_E : H^1_0(\Lambda) \to H^1_0(\Gamma)$ denotes the uniform extension from Λ to Γ. The idea is then to couple a 3D PDE with a 1D one, using a Lagrange multiplier space defined on a 2D surface that surrounds the 1D manifold.

Now, we apply a topological model reduction of the interface conditions, namely we go from a 3D-1D-3D to a 3D-1D-1D formulation. To this purpose, let us write the lagrange multiplier and the test functions on every cross section $\partial\mathcal{D}(s)$ as their average plus some fluctuation,

$$\lambda = \overline{\lambda} + \tilde{\lambda}, \qquad v = \overline{v} + \tilde{v}, \quad \text{on } \partial\mathcal{D}(s) \, ,$$

where $\overline{\tilde{\lambda}} = \overline{\tilde{v}} = 0$. Therefore, using the coordinates system (r, s, t) on Γ, we have

$$\int_\Gamma \lambda v \, d\sigma = \int_\Lambda \int_{\partial\mathcal{D}(s)} (\overline{\lambda} + \tilde{\lambda})(\overline{v} + \tilde{v}) d\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda}\overline{v} \, ds + \int_\Lambda \int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma ds \, .$$

Then, we make the following modelling assumptions:

A3 we assume that the product of fluctuations is small, namely

$$\int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma \simeq 0$$

and we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\overline{\lambda}, \overline{v})_{\Lambda,|\partial\mathcal{D}|} = (f, v)_\Omega,$$

which, combined with (2.6) leads to the second formulation of the reduced problem.

2.2. Problem 2 (3D-1D-1D). Let $\langle \cdot, \cdot \rangle_\Lambda$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. The problem requires to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ \lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$(u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{Tv} - v_\odot, \lambda_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \tag{2.8a}$$
$$= (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, V)_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1_0(\Lambda)$$

$$\langle \overline{Tu} - u_\odot, \mu_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} = 0 \quad \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda) \, . \tag{2.8b}$$

We notice that all the integrals of the reduced problem are well defined because $u, v \in H^1_0(\Omega)$, $Tu,\ Tv \in H^{\frac{1}{2}}_{00}(\Gamma)$ and thus $\overline{Tu}, \overline{Tv} \in H^{\frac{1}{2}}_{00}(\Lambda)$, as shown in the following lemma.

Lemma 2.1. *TO DO: generalize to not uniform $\partial\mathcal{D}$ When Γ is a cylinder, if $u \in H^{\frac{1}{2}}_{00}(\Gamma)$, then $\overline{u} \in H^{\frac{1}{2}}_{00}(\Lambda)$. Moreover, if $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ is constant on each cross section, namely $u(s,\theta) = u(s)$, then*

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = 2\pi R \|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)}.$$

Proof. Let us denote as ϕ_{ij} and ρ_{ij}, for $i = 1, 2, \dots$, $j = 0, 1, \dots$, the eigenfunctions and the eigenvalues of the laplacian on Γ, and with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the laplacian on Λ. In particular,

$$\begin{aligned} \phi_{ij}(s,\theta) &= sin(i\pi s)\left(cos(j\theta) + sin(j\theta)\right), \\ \rho_{ij} &= i\pi^2 + \frac{j^2}{R^2}, \\ \phi_i(s) &= sin(i\pi s), \\ \rho_i &= i\pi^2. \end{aligned}$$

It is easy to verify that

$$\int_0^{2\pi} \phi_{ij}(s,\theta) = 0 \quad \forall j > 0, \forall i \tag{2.9}$$

$$\int_0^{2\pi} \phi_{ij}(s,\theta) = 2\pi R \, \sin(i\pi s) \quad \text{if } j = 0, \forall i. \tag{2.10}$$

(2.11)

Moreover we recall that $\phi_{i,j}(s,\theta)$ and $\phi_i(s)$ are orthogonal basis of $L^2(\Gamma)$ and $L^2(\Lambda)$ respectively. Therefore,

$$\begin{aligned} \overline{u}(s) = \frac{1}{2\pi R}\int_0^{2\pi} u(s,\theta) R\, d\theta = \frac{1}{2\pi R}\int_0^{2\pi} \sum_{i,j} a_{i,j}\phi_{i,j}(s,\theta) R\, d\theta \\ = \frac{1}{2\pi R}\sum_{i,j} a_{i,j}\int_0^{2\pi} \phi_{i,j}(s,\theta) R\, d\theta = \sum_i a_{i,0}\phi_i(s). \end{aligned}$$

From [4, Lemma 4.11] we have

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} (1+\rho_{ij})^{\frac{1}{2}} |a_{ij}|^2, \text{ with } a_{ij} = \int_0^1 \int_0^{2\pi} u(s,\theta)\phi_{ij}\, Rd\theta ds. \tag{2.12}$$

and

$$\|\overline{u}\|^2_{H^{\frac{1}{2}}(\Lambda)} = \sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}} |\overline{a}_i|^2, \text{ with } \overline{a}_i = \int_0^1 \overline{u}(s)\phi_i(s) ds.$$

Therefore, we have

$$\begin{aligned} \|\overline{u}\|^2_{H^{\frac{1}{2}}(\Lambda)} &= \sum_{i=1}^{\infty} \left(1+i^2\pi^2\right)^{\frac{1}{2}} \left(\int_0^1 \overline{u}(s) sin(i\pi s)\, ds\right)^2 \\ &= \sum_{i=1}^{\infty} \left(1+i^2\pi^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} a_{j,0}\int_0^1 \sin(j\pi s)\sin(i\pi s)\, ds\right)^2 \\ &= \sum_{i=1}^{\infty} \frac{1}{4}\left(1+i^2\pi^2\right)^{\frac{1}{2}} a_{i,0}^2 \\ &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left(1 + i^2\pi^2 + \frac{j^2}{R^2}\right)^{\frac{1}{2}} |a_{i,j}|^2 = \|u\|^2_{H^{\frac{1}{2}}(\Gamma)}, \end{aligned}$$

where we have used the fact that

$$\int_0^1 \sin(i\pi s)\sin(j\pi s)\,ds = 0 \quad \text{if } i \neq j$$
$$\int_0^1 \sin(i\pi s)\sin(j\pi s)\,ds = \frac{1}{2} \quad \text{if } i = j.$$

Moreover, in the case in which u is constant on each cross section, from (2.12) we have

$$\begin{aligned}\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}|a_{ij}|^2 &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+i\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}\left(\int_0^1\int_0^{2\pi}u(s,\theta)\phi_{ij}\,Rd\theta ds\right)^2\\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+i\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}\left(\int_0^1 u(s)\int_0^{2\pi}\phi_{ij}\,Rd\theta ds\right)^2,\end{aligned}$$

and using (2.9) and (2.10), we obtain

$$\begin{aligned}\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} &= \sum_{i=1}^{\infty}\left(1+i\pi^2\right)^{\frac{1}{2}}\left(\int_0^1 u(s)sin(i\pi s)2\pi Rds\right)^2\\ &= 4\pi^2R^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}|a_i|^2 = 4\pi^2R^2\|u\|^2_{H^{\frac{1}{2}}(\Lambda)}.\end{aligned}$$

c.v.d. □

REMARK 2.1. *The results of (2.1) can be generalized to the case of a different geometry of Γ, for example a parallelepiped.*

It is apparent that problems (2.2) and (**??**) share the same mathematical structure. For this reason, the well-posedness of (**??**) can be studied in the framework of the classical theory of saddle point problems.

3. Saddle-point problem analysis. Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bounded bilinear forms. Let us consider the general saddle point problem of the form: find $u \in X$, $\lambda \in Q$ s.t.

$$\begin{cases} a(u,v) + b(v,\lambda) = c(v) & \forall v \in X \\ b(u,\mu) = d(\mu) & \forall \mu \in Q. \end{cases} \tag{3.1}$$

We denote with A and B the operators associated to the bilinear forms a and b, namely $A : X \longrightarrow X'$ with $\langle Au, v\rangle_{X',X} = a(u,v)$ and $\langle Bv, \mu\rangle_{X',Q} = b(v,\mu)$. Problem (3.1) embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem.

THEOREM 3.1 (theorem 2.34 Ern-Guermond). *Problem (3.1) is well posed iff*

$$\begin{cases} \exists\alpha > 0 : \inf_{u\in ker(B)}\sup_{v\in ker(B)}\frac{a(u,v)}{\|u\|_X\|v\|_X} \geq \alpha \\ \forall v \in ker(B),\ (\forall u \in ker(B),\ a(u,v) = 0) \implies v = 0. \end{cases} \tag{3.2}$$

and

$$\exists\beta > 0 : \inf_{\mu\in Q}\sup_{v\in X}\frac{b(v,\mu)}{\|v\|_X\|\mu\|_Q} \geq \beta. \tag{3.3}$$

Notice that if a is coercive on $ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1. It consists to find $u \in H_0^1(\Omega),\ u_\odot \in H_0^1(\Lambda),\ \lambda \in H^{-\frac{1}{2}}(\Gamma)$, solutions of (3.1), where

$$a([u, u_\odot], [v, v_\odot]) = (u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda), |\mathcal{D}|}$$

$$b([v, v_\odot], \mu) = \langle Tv - \mathcal{U}_E v_\odot, \mu \rangle_\Gamma$$

$$c([v, v_\odot]) = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda), |\mathcal{D}|}$$

$$d(\mu) = 0$$

We prove that the hypothesis of 3.1 are fullfilled choosing $X = H_0^1(\Omega) \times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$, where X is equipped with the norm $|||[u, u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + \|u_\odot\|^2_{H^1(\Lambda), |\mathcal{D}|}$ and Q equipped with the norm

$$\|\mu_\odot\|_{H^{-\frac{1}{2}}} := \sup_{q \in H^{\frac{1}{2}}(\Lambda)} \frac{\langle q, \mu_\odot \rangle}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}$$

.

Lemma 3.2. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The bilinear form $a(\cdot\ ,\ \cdot)$ is clearly bounded since

$$a([u, u_\odot], [v, v_\odot]) \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|u_\odot\|_{H^1(\Lambda), |\mathcal{D}|} \|v_\odot\|_{H^1(\Lambda), |\mathcal{D}|} \leq 2 |||[u, u_\odot]||| \, |||[v, v_\odot]|||.$$

Concerning the bilinear form $b(\cdot\ ,\ \cdot)$ we have

$$\begin{aligned} b([v, v_\odot], \mu) &= \langle Tv - \mathcal{U}_E v_\odot, \mu \rangle_\Gamma \leq \|Tv - \mathcal{U}_E v_\odot\|_{H^{\frac{1}{2}}(\Gamma)} \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(\|Tv\|_{H^{\frac{1}{2}}(\Gamma)} + \|\mathcal{U}_E v_\odot\|_{H^{\frac{1}{2}}(\Gamma)} \right) \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \left(C_T \|v\|_{H^1(\Omega)} + \|\mathcal{U}_E v_\odot\|_{H^1(\Gamma)} \right) \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \left(C_T \|v\|_{H^1(\Omega)} + \left(\frac{\max |\partial \mathcal{D}|}{\min |\mathcal{D}|} \right)^{\frac{1}{2}} \|v_\odot\|_{H^1(\Lambda), |\mathcal{D}|} \right) \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim |||[v, v_\odot]||| \, \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \end{aligned}$$ □

Lemma 3.3. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive .*

Proof. Indeed, we have,

□

$$a([u, u_\odot], [u, u_\odot]) = (u, u)_{H^1(\Omega)} + |\mathcal{D}| (u_\odot, u_\odot)_{H^1(\Lambda)} = |||[u, u_\odot]|||^2 .$$

Lemma 3.4. *The inf-sup inequality (3.3) is fulfilled, namely $\exists \beta_1 > 0$ such that $\forall \mu \in H^{-\frac{1}{2}}(\Gamma)$:*

$$\sup_{\substack{v \in H_0^1(\Omega), \\ v_\odot \in H_0^1(\Lambda)}} \frac{\langle Tv - \mathcal{U}_E v_\odot, \mu \rangle_\Gamma}{|||[v, v_\odot]|||} \geq \beta_1 \sup_{q \in H^{\frac{1}{2}}(\Gamma)} \frac{\langle q, \mu \rangle}{\|q\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

Proof. We choose $v_\odot \in H_0^1(\Lambda)$ such that $\mathcal{U}_E v_\odot = 0$. Therefore,

$$\sup_{\substack{v \in H_0^1(\Omega), \\ v_\odot \in H_0^1(\Lambda)}} \frac{\langle Tv - \mathcal{U}_E v_\odot, \mu \rangle_\Gamma}{|||[v, v_\odot]|||} \geq \sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu \rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

We notice that the trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. Indeed, $\forall \xi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we can find v solution of

$$\begin{aligned} -\Delta v &= 0 \quad \text{in } \Omega \\ v &= 0 \;\; \text{on } \partial\Omega \\ v &= \xi \;\; \text{on } \Gamma. \end{aligned}$$

We denote with $\mathcal{E}$ the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|\mathcal{E}\| \in \mathbb{R}$ such that $v = \mathcal{E}(\xi)$ and $\|v\|_{H^1(\Omega)} \leq \|\mathcal{E}\| \|\xi\|_{H^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$\sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu \rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle \xi, \mu \rangle_\Gamma}{\|\mathcal{E}\| \|\xi\|_{H^{\frac{1}{2}}(\Gamma)}} = \|\mathcal{E}\|^{-1} \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)}, \tag{3.4}$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H_{00}^{\frac{1}{2}}(\Gamma))^*$. ◻

3.2. Problem 2. This problem requires to find $u \in H_0^1(\Omega),\ u_\odot \in H_0^1(\Lambda),\ \lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, solution of (3.1) with

$$a([u, u_\odot], [v, v_\odot]) = (u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda), |\mathcal{D}|}$$

$$b([v, v_\odot], \mu_\odot) = \langle \overline{Tv} - v_\odot, \mu_\odot \rangle_{\Lambda, |\partial\mathcal{D}|}$$

$$c([v, v_\odot]) = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda), |\mathcal{D}|}$$

$$d(\mu_\odot) = 0$$

We prove that the hypotesis of Theorem 3.1 are fulfilled with the following spaces $X = H_0^1(\Omega) \times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped again with the norm $|||[\cdot, \cdot]|||$ and Q equipped with the norm $\|\cdot\|_{H^{-\frac{1}{2}}}$. Then, we have the following lemmas.

Lemma 3.5. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The boundedness of $a(\cdot\ ,\ \cdot)$ can be proved as in Lemma 3.2. Concerning $b(\cdot\ ,\ \cdot)$, we have

$$\begin{aligned} b([v, v_\odot], \mu_\odot) = \langle \overline{Tv} - v_\odot, \mu_\odot \rangle_{\Lambda, |\partial\mathcal{D}|} &\leq \|\overline{Tv} - v_\odot\|_{H^{\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)} \\ &\leq \left(\|\overline{Tv}\|_{H^{\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} + \|v_\odot\|_{H^{\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} \right) \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)} \\ &\leq \left(\|Tv\|_{H^{\frac{1}{2}}(\Gamma)} + \|v_\odot\|_{H^1(\Lambda), |\partial\mathcal{D}|} \right) \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)} \\ &\leq \left(C_T \|v\|_{H^1(\Omega)} + \left(\frac{\max |\mathcal{D}|}{\min |\mathcal{D}|} \right)^{\frac{1}{2}} \|v_\odot\|_{H^1(\Lambda), |\mathcal{D}|} \right) \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)} \\ &\lesssim |||[v, v_\odot]||| \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)} \end{aligned}$$

check $\| |\partial\mathcal{D}| \overline{Tv} \|_{H^{\frac{1}{2}}(\Lambda)} \leq \|Tv\|_{H^{\frac{1}{2}}(\Gamma)}$ ◻

Lemma 3.6. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive.*

Lemma 3.7. *The inf-sup inequality (3.3) holds, namely $\exists \beta_2 > 0$ such that $\forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda)$, :*

$$\sup_{\substack{v \in H_0^1(\Omega), \\ v_\odot \in H_0^1(\Lambda)}} \frac{\langle \overline{Tv} - v_\odot, \mu_\odot \rangle_{\Lambda, |\partial\mathcal{D}|}}{|||[v, v_\odot]|||} \geq \beta_2 \|\mu_\odot\|_{H^{\frac{1}{2}}(\Lambda)}.$$

We choose $v_\odot = 0$ and we obtain

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \overline{Tv} - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v, v_\odot]|||} \geq \sup_{v\in H_0^1(\Omega)} \frac{\langle \overline{Tv}, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}}.$$

For any $q \in H_{00}^{\frac{1}{2}}(\Lambda)$, we consider its uniform extension to Γ $\mathcal{U}_E q$ and then we consider the harmonic extension $v = \mathcal{E}(\mathcal{U}_E q) \in H_0^1(\Omega)$. It follows that $\overline{Tv} = q$. Therefore,

$$\sup_{v\in H_0^1(\Omega)} \langle \overline{Tv}, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \gtrsim \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \langle q, \mu_\odot\rangle_\Lambda\,.$$

Moreover, using Lemma 2.1 we obtain

$$\|v\|_{H_0^1(\Omega)} \leq \|\mathcal{E}\|\|\mathcal{U}_E q\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim \|\mathcal{E}\|\|q\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Therefore,

$$\sup_{v\in H_0^1(\Omega)} \frac{\langle \overline{Tv}, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} \gtrsim \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_\Lambda}{\|v\|_{H^1(\Omega)}} \gtrsim \|\mathcal{E}\|^{-1} \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_\Lambda}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}$$
$$= \|\mathcal{E}\|^{-1}\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda)}$$

and the constants in the inequalities depend on $\min_{s\in(0,S)} |\partial\mathcal{D}(s)|$ and $\max_{s\in(0,S)} |\partial\mathcal{D}(s)|$ which we suppose to be strictly positive.
What if we use also for the multiplier a weighted norm?

4. Finite element approximation.

The discrete equivalent of (3.1) reads as finding $u_h \in X_h \subset X$, $\lambda_h \in Q_h \subset Q$ s.t.

$$\begin{cases} a(u_h, v_h) + b(v_h, \lambda_h) = c(v_h) & \forall v_h \in X_h \\ b(u_h, \mu_h) = d(\mu_h) & \forall \mu_h \in Q_h. \end{cases} \tag{4.1}$$

Define Bh, etc

Theorem 4.1 (Ern-Guermond 2.42). *Problem (4.1) is well-posed if and only if*

$$\exists \alpha_h > 0 : \inf_{u_h\in ker(B_h)} \sup_{v_h\in ker(B_h)} \frac{a(u_h, v_h)}{\|u_h\|_X\|v_h\|_X} \geq \alpha_h \tag{4.2}$$

and

$$\exists \beta_h > 0 : \inf_{\mu_h\in Q_h} \sup_{v_h\in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X\|\mu_h\|_Q} \geq \beta_h. \tag{4.3}$$

Let us notice that for both problem 1 and problem 2 the bilinear form $a(\cdot,\cdot)$ is coercive as stated in Lemmas (3.3) and (3.6). Consequently, (4.2) is automatically satisfied, being α_h the coercivity constant.

Let us introduce a shape-regular triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_h^\Lambda$ of Λ. We analyze two different cases: the one in which the 3D mesh is conforming to the interface Γ, namely the set of the intersections of the 3D elements of $\mathcal{T}_h^\Omega$ with Γ is constituted by facets of such elements, and the non conforming case, namely the 3D elements of $\mathcal{T}_h^\Omega$ 'cut' the interface Γ.

4.1. $\mathcal{T}_h^\Omega$ conforming to Γ.

4.1.1. Problem 1. We denote by $X^0_{h,k}(\Omega) \subset H^1_0(\Omega)$ the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satisfying homogeneous Dirichlet conditions on the boundary and by $X^0_{h,k}(\Lambda) \subset H^1_0(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ, satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. Problem 1 consists to find $u_h \in X^0_{h,k}(\Omega),\ u_{\odot h} \in X^0_{h,k}(\Lambda), \lambda_h \in Q_h(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$(u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle T v_h - \mathcal{U}_E v_{\odot h}, \lambda_h \rangle_\Gamma = (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X^0_{h,k}(\Omega),\ v_{\odot h} \in X^0_{h,k}(\Lambda) \tag{4.4a}$$

$$\langle T u_h - \mathcal{U} u_{\odot h}, \mu_h \rangle_\Gamma = 0 \quad \forall \mu_h \in Q_h(\Gamma)\,, \tag{4.4b}$$

The space $Q_h(\Gamma)$ must be suitably chosen such that (4.3) holds. Let us denote with $W^0_{h,k}(\Gamma) \subset H^{\frac{1}{2}}_{00}(\Gamma)$ the trace space of functions running in $X^0_{h,k}(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$. We choose $Q_h(\Gamma) = W^0_{h,k}(\Gamma)$, therefore we impose homogeneous Dirichlet boundary condition on $\partial\Omega$ also for the Lagrange multiplier. For this choice of $Q_h(\Gamma)$ we can prove the well-posedness of the discrete problem, as shown in the following.

LEMMA 4.2. *Let $P_h : H^{\frac{1}{2}}_{00}(\Gamma) \longrightarrow W^0_{h,k}(\Gamma)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Gamma)$ by*

$$(P_h v, \psi)_\Gamma = (v, \psi)_\Gamma \qquad \forall \psi \in W^0_{h,k}(\Gamma).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Gamma)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|v\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{4.5}$$

where C is a positive constant independent of h.

Proof. We prove that P_h is continuous on $L^2(\Gamma)$ and on $H^1_0(\Gamma)$ following [5, Section 1.6.3]. Then, the inequality (4.5) can be dirived by Hilbertian interpolation. For the L^2-continuity, we exploit the fact that, from the definition of P_h,

$$(v - P_h v, P_h v)_\Gamma = 0.$$

Therefore, by Pythagoras identity,

$$\|v\|^2_{L^2(\Gamma)} = \|v - P_h v\|^2_{L^2(\Gamma)} + \|P_h v\|^2_{L^2(\Gamma)} \geq \|P_h v\|_{L^2(\Gamma)}.$$

Let us now consider $v \in H^1_0(\Gamma)$.The Scott-Zhang interpolation operator SZ_h from $H^1_0(\Gamma)$ to $W^0_{h,k}(\Gamma)$ satisfies the following inequalities,

$$\|SZ_h v\|_{H^1(\Gamma)} \leq C_1 \|v\|_{H^1(\Gamma)} \tag{4.6}$$

$$\|v - SZ_h v\|_{L^2(\Gamma)} \leq C_2 h \|v\|_{H^1(\Gamma)}. \tag{4.7}$$

Therefore,

$$\begin{aligned}\|\nabla P_h v\|_{L^2(\Gamma)} &\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + \|\nabla SZ_h v\|_{L^2(\Gamma)} \\ &\leq \text{ (using (4.6)) } \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1 \|v\|_{H^1(\Gamma)}\end{aligned}$$

and by using the inverse inequality we obtain

$$\begin{aligned}
\|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} &\leq \frac{C_3}{h}\|P_h v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&= \frac{C_3}{h}\|P_h(v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq \text{ (Stability of } P_h \text{ in } L^2) \ \frac{C_3}{h}\|v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq \text{ (using (4.7)) } \frac{C_3}{h} C_2 h\|v\|_{H^1(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\
&\leq (C_2C_3 + C_1)\|v\|_{H^1(\Gamma)},
\end{aligned}$$

from which we obtain the continuity in $H_0^1(\Gamma)$.

I would skip this proof and just leave the citation ∎

LEMMA 4.3. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, \mu_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \geq \gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. Let μ_h be in $Q_h(\Gamma)$. From the continuous case, in particular from (3.4), we have

$$\|\mathcal{E}\|^{-1}\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h\rangle}{\|v\|_{H^1(\Omega)}}$$

and by the trace inequality $\|Tv\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_T\|v\|_{H^1(\Omega)}$ (see [1, 7.56]), we obtain

$$\sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h\rangle}{\|v\|_{H^1(\Omega)}} \leq C_T \sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h\rangle}{\|Tv\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

By the definition of P_h and (4.5)

$$\begin{aligned}
C_T \sup_{v \in H_0^1(\Omega)} \frac{\langle Tv, \mu_h\rangle}{\|Tv\|_{H^{\frac{1}{2}}(\Gamma)}} &= C_T \sup_{v \in H_0^1(\Omega)} \frac{\langle P_h(Tv), \mu_h\rangle}{\|Tv\|_{H^{\frac{1}{2}}(\Gamma)}} \\
&\leq C_T C \sup_{v \in H_0^1(\Omega)} \frac{\langle P_h(Tv), \mu_h\rangle}{\|P_h(Tv)\|_{H^{\frac{1}{2}}(\Gamma)}} \\
&= C_T C \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, \mu_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}}.
\end{aligned}$$ ∎

THEOREM 4.4 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists \beta_{h,1} > 0$ s.t.*

$$\inf_{\mu_h \in Q_h(\Gamma)} \sup_{\substack{v_h \in X^0_{h,k}(\Omega), \\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle Tv_h - \mathcal{U}_E v_{\odot h}, \mu_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \beta_{h,1}. \tag{4.8}$$

Proof. Let $\mu_h \in Q_h(\Gamma)$. As in the continuos case, we choose $v_{\odot h} = 0$ and we have

$$\sup_{\substack{v_h \in X^0_{h,k}(\Omega), \\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle Tv_h - \mathcal{U}_E v_{\odot h}, \mu_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle Tv_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Therefore, we want to prove that there exists $\beta_{h,1}$ such that

$$\sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle Tv_h, \mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,1}\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall \mu_h \in Q_h(\Gamma).$$

Using Lemma 4.3 and the boundedness of the armonic extension operator $\mathcal{E}$ from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ introduced in the previous section, we have

$$\gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}\| \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|\mathcal{E}q_h\|_{H^1(\Omega)}}.$$

Let $R_h : H^1_0(\Omega) \to X^0_{h,k}(\Omega)$ be a quasi interpolation operator satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq C_R\|v\|_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega).$$

Therefore, we obtain

$$\|\mathcal{E}\| \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|\mathcal{E}q_h\|_{H^1(\Omega)}} \leq \|\mathcal{E}\|C_R \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}q_h\|_{H^1(\Omega)}}$$

and we have

$$\begin{aligned}(4.9)\quad \gamma\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}\|C_R \sup_{q_h\in Q_h(\Gamma)} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}q_h\|_{H^1(\Gamma)}} \\ &= \|\mathcal{E}\|C_R \sup_{q_h\in Q_h(\Gamma)} \frac{\langle T\ R_hEq_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}q_h\|_{H^1(\Omega)}} \leq \|\mathcal{E}\|C_R \sup_{v_h\in X_{h,k}(\Omega)} \frac{\langle Tv_h,\mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.\end{aligned}$$

Therefore the inf-sup condition (4.8) holds with $\beta_{h,1} = \gamma\|\mathcal{E}\|^{-1}C_R^{-1}$. ∎

Remark 4.1. *We notice that to prove the result in Lemma 4.3 (and then the discrete inf-sup condition) basically we need a projection operator $P_h : H^{\frac{1}{2}}_{00} \longrightarrow W^0_{h,k}(\Gamma)$ orthogonal in the multiplier space $Q_h(\Gamma)$, namely such that $\langle P_h v,\mu_h\rangle = \langle v,\mu_h\rangle$, $\forall\mu_h \in Q_h(\Gamma)$, and continuous in $H^{\frac{1}{2}}(\Gamma)$. Therefore, in principle different choices than $Q_h(\Lambda) = W^0_{h,k}(\Gamma)$ could be considered if we can build an operator P_h satisfying these properties. In [2] such operator P_h is built for a particular choice of $Q_h(\Gamma)$ but it is not clear how they prove the H^1-stability inequality (and consequently the $H^{\frac{1}{2}}$-stability) with a constant independent of the mesh size h...*

4.1.2. Problem 2. This problem requires to find $u_h \in X^0_{h,k}(\Omega)$, $u_{\odot h} \in X^0_{h,k}(\Lambda)$, $\lambda_{\odot h} \in Q_h(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}(4.10a)\qquad & (u_h,v_h)_{H^1(\Omega)} + (u_{\odot h},v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle\overline{Tv_h} - v_{\odot h},\lambda_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} \\ &\quad = (f,v_h)_{L^2(\Omega)} + (\overline{\overline{g}},v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h} \in X_h(\Lambda)\end{aligned}$$

$$(4.10b)\qquad \langle\overline{Tu_h} - u_{\odot h},\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} = 0 \quad \forall\mu_{\odot h} \in Q_h(\Lambda)\,.$$

We introduce the space $W^0_{h,k}(\Lambda) \subset H^{\frac{1}{2}}_{00}(\Lambda)$, which is the averaged trace space of functions running in $H^1_0(\Omega)$. It coincides with the space of continuous piecewise polynomials of degree k defined on Λ and satisfying homogeneous Dirichlet boundary condition. (Add assumptions..) We choose $Q_h(\Lambda) = W^0_{h,k}(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda\cap\partial\Omega$ also for the Lagrange multiplier. With this choice for $Q_h(\Lambda)$, we can prove the well-posedness of the discrete problem. In particular, following the same steps as for Problem 1, we can prove the following results.

Lemma 4.5. *Let $P_h : H^{\frac{1}{2}}_{00}(\Lambda) \longrightarrow W^0_{h,k}(\Lambda)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Lambda)$ by*

$$(P_h v,\psi)_\Lambda = (v,\psi)_\Lambda \qquad \forall\psi \in W^0_{h,k}(\Lambda).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}_{00}(\Lambda)} \leq C\|v\|_{H^{\frac{1}{2}}_{00}(\Lambda)},$$

where C is a positive constant independent of h.

LEMMA 4.6. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \geq \gamma \|\mu_h\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_h \in W^0_{h,k}(\Lambda).$$

THEOREM 4.7 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists \beta_{h,2} > 0$ s.t.*

$$\inf_{\mu_h \in Q_h(\Lambda)} \sup_{\substack{v_h \in X^0_{h,k}(\Omega), \\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle \overline{Tv_h} - v_{\odot h}, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{|||[v_h, v_{\odot h}]||| \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \beta_{h,2}. \tag{4.11}$$

Proof. Let $\mu_{\odot h}$ be arbitrarly chosen in $Q_h(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists $\beta_{h,2}$ such that

$$\sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle \overline{Tv_h}, \mu_{\odot h} \rangle_{\Lambda, |\partial \mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,2} \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_{\odot h} \in Q_h(\Lambda).$$

Let us denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ. Using Lemma 2.12, we easily have for any $w \in H^{\frac{1}{2}}(\Lambda)$,
TO DO: generalize to non constant $\partial \mathcal{D}$

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial \mathcal{D}| \|w\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Consequently, from Lemma 4.9, using again the extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^0_{h,k}(\Omega)$, we obtain

$$\begin{aligned} \gamma \|\mu_h\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h \rangle_\Lambda}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\ &= |\partial \mathcal{D}| \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h \rangle_\Lambda}{\|\mathcal{U}_E q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq |\partial \mathcal{D}| \|E\| \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h \rangle_\Lambda}{\|E \mathcal{U}_E q_h\|_{H^1(\Omega)}} \\ &\leq |\partial \mathcal{D}| \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h \rangle_\Lambda}{\|R_h E \mathcal{U}_E q_h\|_{H^1(\Omega)}} \\ &= |\partial \mathcal{D}| \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle \Pi_1 R_h E \mathcal{U}_E q_h, \mu_h \rangle_\Lambda}{\|R_h E \mathcal{U}_E w_h\|_{H^1(\Omega)}} \\ &\leq |\partial \mathcal{D}| \|E\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, \mu_h \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}. \end{aligned} \tag{4.12}$$

∎

*******************extension to non uniform $|\partial \mathcal{D}|$

LEMMA 4.8. *Let $P_h : H^{\frac{1}{2}}_{00}(\Lambda) \longrightarrow W^0_{h,k}(\Lambda)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Lambda)$ by*

$$(P_h v, \psi)_{\Lambda, |\partial \mathcal{D}|} = (v, \psi)_{\Lambda, |\partial \mathcal{D}|} \qquad \forall \psi \in W^0_{h,k}(\Lambda).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Lambda), |\partial \mathcal{D}|} \leq C \|v\|_{H^{\frac{1}{2}}(\Lambda), |\partial \mathcal{D}|},$$

where C is a positive constant independent of h.

LEMMA 4.9. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \geq \gamma \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_{\odot h} \in W^0_{h,k}(\Lambda).$$

THEOREM 4.10 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists \beta_{h,2} > 0$ s.t.*

$$\inf_{\mu_h \in Q_h(\Lambda)} \sup_{\substack{v_h \in X^0_{h,k}(\Omega),\\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle \overline{Tv_h} - v_{\odot h}, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]||| \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \beta_{h,2}. \tag{4.13}$$

Proof. Let $\mu_{\odot h}$ be arbitrarly chosen in $Q_h(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists $\beta_{h,2}$ such that

$$\sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle \overline{Tv_h}, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,2} \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall \mu_{\odot h} \in Q_h(\Lambda).$$

Let us denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ. We notice that for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = \|w\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.$$

Consequently, from Lemma 4.9, using again the extension operator $\mathcal{E}$ from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^0_{h,k}(\Omega)$, we obtain

$$\begin{aligned}
\gamma \|\mu_h\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \\
&= \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{U}_E q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}\| \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{E}\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&\leq \|\mathcal{E}\| C_R \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, \mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|R_h E\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&= \|\mathcal{E}\| C_R \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle \overline{TR_h E\mathcal{U}_E q_h}, \mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq \|\mathcal{E}\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \overline{Tv_h}, \mu_h\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}}. \qquad \square
\end{aligned} \tag{4.14}$$

4.2. $\mathcal{T}^\Omega_h$ non conforming to Γ. We analyze now the case in which the elements of the 3D mesh $\mathcal{T}^\Omega_h$ cut the interface Γ. It is easy to understand that the formulation of Problem 2 is more suitable. Add more details, we can refer also to cutFEM (Burman, Massing etc.) explaining the limitation of that approach (network case for example).
Therefore we focus on the analysis of Problem 2.

4.2.1. Problem 2. We consider for the solutions u_h and $u_{\odot h}$ the spaces $X^0_{h,1}(\Omega)$ and $X^0_{h,1}(\Lambda)$, see the previous subsection for the definition. Concerning the multiplier space, we make the following choice, $Q_h(\Lambda) = \{\lambda_{\odot h} : \lambda_{\odot h} \in P^0(K) \forall K \in \mathcal{T}^\Lambda_{h'}\}$, namely the multiplier lives on the same mesh used for the 1D solution $u_{\odot h}$. Notice that in this case we suppose that the mesh sizes of the 3D mesh $\mathcal{T}^\Omega_h$ and the 1D mesh $\mathcal{T}^\Lambda_{h'}$ are different, in particular we suppose the 1D mesh is finer. With this choice the problem is not inf-sup stable, therefore the idea is to add a stabilization term $s(\lambda_{\odot h}, \mu_{\odot h})$ to (4.10a) following the approach introduce in [3]. In particular, we build a new multiplier space $L_h(\Lambda)$ for which the discrete inf-sup condition is fulfilled and we build a projection operator $\pi_L : Q_h(\Lambda) \to L_h(\Lambda)$. Based on this projection operator, we build the stabilization term $s(\lambda_{\odot h}, \mu_{\odot h})$ and prove that $\forall [u_h, u_{\odot h}]$, there exists $\xi_h([u_h, u_{\odot h}]) \in Q_h(\Lambda)$ s.t.

$$a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \geq \alpha_\xi |||[u_h, u_{\odot h}]|||_{X^0_{h,1}(\Omega)\times X^0_{h,1}(\Lambda)}, \tag{4.15}$$

$$(s(\xi_h, \xi_h))^{\frac{1}{2}} \leq c_s |||[u_h, u_{\odot h}]|||_{X^0_{h,1}(\Omega)\times X^0_{h,1}(\Lambda)}, \tag{4.16}$$

being $|||[\cdot\,,\,\cdot]|||_{X^0_{h,1}(\Omega)\times X^0_{h,1}(\Lambda)}$ a suitable discrete norm.

We recall that in the case of Problem 2,

$$b([u_h, u_{\odot h}], \lambda_{\odot h}) = \left(\overline{Tu_h} - u_{\odot h}, \lambda_{\odot h}\right)_{\Lambda, |\partial\mathcal{D}|}.$$

The construction of the inf-sup stable space $L_h(\Lambda)$ is based on assembling the elements of the 3D mesh $\mathcal{T}^\Omega_h$ intersecting the 1D manifold Λ into macro patches $\{F_j\}_j$. These patches are such that and $H \leq |F_j \cap \Lambda| \leq H + h$, where H is sufficiently larger than h. Moreover, we assume there exist constants c_h and c_H such that $c_h h \leq H \leq c_H^{-1} h$. We define the space $L_h(\Lambda)$ as the space of functions which are P^0 on each intersection $F_j \cap \Lambda$. Moreover, we associate to each patch F_j a shape regular macro elements ω_j, which is built adding to F_j a sufficient number of elements of $\mathcal{T}^\Omega_h$. We make the following technical assumption: $\Gamma \subset \bigcup_j \omega_j$. Thanks to the shape regularity of these macro elements, we have that the discrete trace and Poincarè inequalites hold. More precisely, for every function $v \in H^1(\omega_j)$,

$$\|Tv\|_{\Gamma\cap\omega_j} \lesssim H^{-\frac{1}{2}} \|v\|_{L^2(\omega_j)} \tag{4.17}$$

$$\|v - \pi_L v\|_{L^2(\omega_j)} \leq c_P H \|\nabla v\|_{L^2(\omega_j)}, \tag{4.18}$$

where π_L is defined as the projection onto piecewise constant functions on $F_j \cap \Lambda$. Moreover $\forall u_h \in X^\Omega_h$ we have the following average inequality

$$\sum_j \|\overline{Tu_h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \leq \sum_j \|Tu_h\|^2_{L^2(\omega_j\cap\Gamma)}.$$

I think this inequality is valid but only globally. Indeed locally it is not guaranteed that the portion of Γ corresponding to $F_j \cap \Lambda$ is contained in $\omega_j \cap \Gamma$.

These choices lead to the following stabilization

$$s(\lambda_{\odot h}, \mu_{\odot h}) = \sum_{K\in\mathcal{T}^\Lambda_{h'}} \int_{\partial K} h [\![\lambda_{\odot h}]\!] [\![\mu_{\odot h}]\!],$$

being $[\![\lambda_{\odot h}]\!]$ the jump of $\lambda_{\odot h}$ across the internal faces of $\mathcal{T}^\Lambda_{h'}$.

Lemma 4.11. *The space L_h inf-sup stable, namely $\forall l_{\odot h} \in L_h(\Lambda)$, $\exists \beta > 0$ s.t.*

$$\sup_{\substack{v_h\in X^0_{h,1}(\Omega),\\ v_{\odot h}\in X^0_{h',1}(\Lambda)}} \frac{\left(\overline{Tv_h} - v_{\odot h}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]|||} \geq \beta \|l_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

and the constant is independent of the cuts.

Proof. As in the continuous case, we can choose $v_{\odot h} = 0$ and we prove that

$$\sup_{v_h\in X^0_{h,1}(\Omega)} \frac{\left(\overline{Tv_h}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta \|l_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

Proving the last inequality it is equivalent to find the Fortin operator $\pi_F : H^1_0(\Omega) \to X^0_{h,1}(\Omega)$, such that

$$\left(\overline{Tv} - \overline{T\pi_F v}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|} = 0, \quad \forall v \in H^1_0(\Omega),\, l_{\odot h} \in L_h(\Lambda)$$

and

$$\|\pi_F v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.$$

We define

$$\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})}{\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j}}$$

and $\varphi_j \in X^0_{h,1}(\Omega)$ s.t. $\mathrm{supp}(\varphi_j) \subset \bar{\omega}_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j} = O(H) \text{ and } \|\nabla\varphi\|_{L^2(\omega_j)} = O(1).$$

This construction is always possible provided H is sufficiently larger that h. Then we have

$$\begin{aligned}
\left(\overline{Tv} - \overline{T\pi_F v}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|} &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}| \left[\overline{Tv} - \overline{TI_h v} - \sum_i \alpha_i \overline{T\varphi_i}\right] l_{\odot h} \\
&= (\mathrm{supp}\varphi \subset \omega_j) \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}| \left[\overline{Tv} - \overline{TI_h v} - \alpha_j \overline{T\varphi_j}\right] l_{\odot h} \\
&= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v}) l_{\odot h} - \frac{\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})}{\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j}} \int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j} l_{\odot h} \\
&= (\text{using } l_h \text{ constant on } F_j \cap \Lambda)\, 0.
\end{aligned}$$

Concerning the continuity of π_F, we have

$$\begin{aligned}
\|\nabla\pi_F v\|_{L^2(\Omega)} &\leq \|\nabla I_h v\|_{L^2(\Omega)} + \left(\sum_j |\alpha_j|^2 \|\nabla\varphi_j\|^2_{L^2(\bar{\omega}_j)}\right)^{\frac{1}{2}} \\
(\text{stability of } I_h) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \left(\sum_j |\alpha_j|^2 \|\nabla\varphi_j\|^2_{L^2(\bar{\omega}_j)}\right)^{\frac{1}{2}}
\end{aligned}$$

and for the second term we have

$$
\begin{aligned}
&\sum_j |\alpha_j|^2 \|\nabla\varphi_j\|^2_{L^2(\bar{\omega}_j)} \leq \\
&\qquad \text{(using } \|\nabla\varphi_j\| = O(1)) \lesssim \sum_j \frac{\left(\left|\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})\right|\right)^2}{\left(\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j}\right)^2} \\
&\qquad \left(\text{since } \left|\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j}\right| = O(H)\right) \lesssim \frac{1}{H^2}\sum_j \left(\left|\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})\right|\right)^2 \\
&\qquad \text{(Jensen)} \lesssim \frac{1}{H^2}\sum_j |F_j\cap\Lambda| \int_{F_j\cap\Lambda} |\partial\mathcal{D}|^2(\overline{Tv} - \overline{TI_h v})^2 \\
&\qquad \text{(being } |F_j\cap\Lambda| \leq H + h) \lesssim \frac{1}{H}\sum_j \|\overline{Tv} - \overline{TI_h v}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\qquad \text{(average inequality)} \lesssim \frac{1}{H}\sum_j \|T(v - I_h v)\|^2_{L^2(\omega_j\cap\Gamma)} \\
&\qquad \text{(trace inequality)} \lesssim \frac{1}{H^2}\sum_j \|v - I_h v\|^2_{L^2(\omega_j)} \lesssim \frac{1}{H^2}\|v - I_h v\|^2_{L^2(\Omega)} \\
&\qquad \text{(approximation properties of } I_h) \lesssim \|\nabla v\|^2_{L^2(\Omega)}
\end{aligned}
$$

and the continuity of π_F follows. ∎

We choose the following discrete norm

$$
|||[u_h, u_{\odot h}]|||^2_{X_h(\Omega)\times X_{h'}(\Lambda)} = \|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|},
$$

where $\|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|} = \|h^{\frac{1}{2}}(\overline{Tu_h} - u_{\odot h})\|^2_{L^2(\Lambda),|\partial\mathcal{D}|}$. Then, we have the following lemma.

Lemma 4.12. *The inequalities* (4.15) *and* (4.16) *hold.*

Proof. Concerning the coercivity property (4.15), we have to show that $\forall [u_h, u_{\odot h}]$, there esists $\xi_h \in Q_h(\Lambda)$ s.t.

$$
\begin{aligned}
&(u_h, u_h)_{H^1(\Omega)} + (u_{\odot h}, u_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + (\overline{Tu_h} - u_{\odot h}, \xi_h)_{\Lambda,|\partial\mathcal{D}|} \\
&\qquad\qquad \geq \alpha_\xi(\|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}.
\end{aligned}
$$

We choose

$$
\xi_{h|F_j\cap\Lambda} = \delta\frac{1}{H}\pi_L(\overline{Tu_h} - u_{\odot h}) \qquad \text{with } \pi_L(\overline{Tu_h} - u_{\odot h}) = \frac{1}{|\Gamma_{F_j\cap\Lambda}|}\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tu_h} - u_{\odot h}),
$$

being $\Gamma_{F_j\cap\Lambda}$ the portion of Γ with centerline $F_j\cap\Lambda$. Actually, $\xi_h \in L_h(\Lambda) \subset Q_h(\Lambda)$. Then,

$$
\begin{aligned}
\big(\overline{Tu_h} - u_{\odot h}, \xi_h\big)_{\Lambda,|\partial\mathcal{D}|} &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tu_h} - u_{\odot h})\xi_h \\
&= \delta\frac{1}{H}\sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L(\overline{Tu_h} - u_{\odot h}))^2 \\
\text{(orthogonality of } \pi_L) &= \delta\frac{1}{H}\|(\pi_L - \mathcal{I})(\overline{Tu_h} - u_{\odot h})\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} + \delta\frac{1}{H}\|\overline{Tu_h} - u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\geq -\delta\frac{1}{H}\sum_j \|(\pi_L - \mathcal{I})\overline{Tu_h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} - \delta\frac{1}{H}\sum_j \|(\pi_L - \mathcal{I})u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\qquad + \delta\frac{1}{H}\sum_j \|\overline{Tu_h} - u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|}.
\end{aligned}
$$

For the first term we have

$$
\begin{aligned}
\sum_j \|(\pi_L - \mathcal{I})\overline{Tu_h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L\overline{Tu_h} - \overline{Tu_h})^2 \\
\text{(Average inequality)} &\leq \sum_j \int_{\omega_j\cap\Gamma} (\pi_L\overline{Tu_h} - Tu_h)^2 \\
\text{(trace inequality)} &\leq \sum_j \frac{1}{H}\int_{\omega_j} (\pi_L\overline{Tu_h} - u_h)^2 \\
\text{(Poincare, see [5, Corollary B.65])} &\leq \sum_j Hc_P^2\|\nabla u_h\|^2_{L^2(\omega_j)}.
\end{aligned}
$$

For the second term we have

$$
\begin{aligned}
\sum_j \|(\pi_L - \mathcal{I})u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L u_{\odot h} - u_{\odot h})^2 \\
\text{(Poincare, [5, Corollary B.65])} &\lesssim \sum_j H^2c_P^2\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\nabla u_{\odot h})^2 \\
\text{(since } H \text{ is fixed, we can find a constant s.t. } H|\partial\mathcal{D}| \lesssim |\mathcal{D}|) &\lesssim \sum_j Hc_P^2\int_{F_j\cap\Lambda} |\mathcal{D}|(\nabla u_{\odot h})^2 \\
&\lesssim \sum_j Hc_P^2\|\nabla u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\mathcal{D}|}.
\end{aligned}
$$

N.B. we are using a kind of weigthed Poincare inequality, check... I think it should work because I can do something like this

$$
\int_{F_j\cap\Lambda} |\partial\mathcal{D}|u^2 \leq max|\partial\mathcal{D}|\int_{F_j\cap\Lambda} u^2 \leq max|\partial\mathcal{D}|\int_{F_j\cap\Lambda} (\nabla u)^2 = \frac{max|\partial\mathcal{D}|}{min|\partial\mathcal{D}|}min|\partial\mathcal{D}|\int_{F_j\cap\Lambda} u^2 \leq \frac{max|\partial\mathcal{D}|}{min|\partial\mathcal{D}|}\int_{F_j\cap\Lambda} |\partial\mathcal{D}|u^2
$$

Therefore, we obtain

$$
\begin{aligned}
&a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \geq \\
&\qquad (1 - \delta c_P^2)\|\nabla u_h\|^2_{L^2(\Omega)} + (1 - \delta c_P^2)\|\nabla u_{\odot h}\|^2_{L^2(\Lambda),|\mathcal{D}|} + \delta c_H\|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}
\end{aligned}
$$

and choosing $\delta = \frac{1}{2c_P^2}$ we obtain the coercivity inequality.
Concerning the stability inequality (4.16), the proof is analogous to the one in [3]. $\square$

5. A benchmark problem with analytical solution. We consider the following 3D-1D coupled problem,

$$-\Delta u = f \quad \text{in } \Omega \tag{5.1a}$$

$$-d_{zz}^2 u_\odot = g \quad \text{on } \Lambda \tag{5.1b}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{5.1c}$$

$$u_\odot - \overline{u} = q \quad \text{on } \Lambda \tag{5.1d}$$

where $\Omega = [0,1]\times[0,1]\times[0,H]$, $\Lambda = \{x = 0.5\}\times\{y = 0.5\}\times[0,H]$ and

$$f = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$$
$$g = \frac{\pi^2}{H^2}\sin\left(\frac{\pi z}{H}\right)$$
$$q = \sin\left(\frac{\pi z}{H}\right).$$

In this case the z coordinate coincides with the axial coordinate along Λ. We define $\Sigma = [0.25, 0.75]\times[0.25, 0.75]\times[0,H]$. The average of the 3D solution $\overline{u}$ in (5.1d) is computed on the cross section $\partial\mathcal{D}$ of the virtual interface $\Gamma = \partial\Sigma$. The exact solution of (5.1) is given by

$$u = \sin(2\pi x)\sin(2\pi y) \tag{5.2}$$

$$u_\odot = \sin\left(\frac{\pi z}{H}\right) \tag{5.3}$$

Let us notice that $u_\odot$ satisfies homogeneous Dirichlet conditions at the boundary of Λ. Moreover, the solution (5.2)-(5.3) satisfies on Γ the relation

$$\lambda = \nabla u \cdot \mathbf{n}_\oplus = d_z u_\odot n_{\oplus,z} = 0, \tag{5.4}$$

being $n_{\oplus,z}$ the $z-$component of the normal unit vector to Γ.

We prove that (5.2)-(5.3) is solution of (2.7) and (2.8) in the simplified case in which the starting 3D-3D problem is

$$-\Delta u_\oplus = f \qquad \text{in } \Omega_\oplus, \tag{5.5a}$$

$$-\Delta u_\ominus = g \qquad \text{in } \Sigma, \tag{5.5b}$$

$$-\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus \qquad \text{on } \Gamma, \tag{5.5c}$$

$$u_\ominus - u_\oplus = q \qquad \text{on } \Gamma, \tag{5.5d}$$

$$u_\oplus = 0 \qquad \text{on } \partial\Omega\,. \tag{5.5e}$$

instead of (2.1). Therefore the reduced problems in the two different formulations (2.7) and (2.8) become respectively

$$\begin{aligned}(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda)\end{aligned} \tag{5.6a}$$

$$\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = \langle q, M\rangle_\Gamma \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{5.6b}$$

and

$$\begin{aligned}(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)} \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\end{aligned} \tag{5.7a}$$

$$|\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \overline{q}, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{5.7b}$$

Let us prove that (5.2)-(5.3) is solution of (5.6). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (5.6a) we have

$$
\begin{aligned}
&-(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d^2_{ss} u_\odot, v_\odot)_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\
&\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda).
\end{aligned}
$$

Clearly, since (5.2) satisfies (5.1a) and (5.3) satisfies (5.1b), we have that

$$
\begin{aligned}
-(\Delta u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} \\
-|\mathcal{D}|(d^2_{ss} u_\odot, v_\odot)_{L^2(\Lambda)} &= |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)}
\end{aligned}
$$

and being $L = \lambda = 0$, we can conclude that (5.2)-(5.3) satisfy (5.6a). The fact that the solution satisfy (5.6b) follows from (5.1d). We can prove in the same way that (5.2)-(5.3) is solution of (5.7), exploiting the fact that in this case $L = \overline{\lambda} = 0$.

Remark 5.1. *Let us notice that the 3D solution* (5.2) *is such that* $\overline{u} = 0$. *Therefore in* (5.1) *it is like we are solving two separated problems, one in* Ω *and the other on* Λ.

Remark 5.2. *It would be interesting to make a comparison between the solution of the fully coupled 3D-3D problem* (2.1) *(also in the simplified case of type* (5.5)*) and the solution of the reduced problems* (2.7) *and* (2.8). *Therefore, we could set the values of the data of the problem such that the reduced formulation becomes non-trivial and fully coupled. Then, we will solve both the original and reduced problem to observe the differences in the solutions and the values of the Lagrange multiplier.*

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