Unfitted Finite Element methods and Multimodel coupling (Burman, 2013) One domain Ω decomposed in two domains Ω_1 and Ω_2 separated by interface Γ. The problem is to solve

$$\nabla \cdot \sigma_i(u_i) = f, \quad \text{in } \Omega_i$$
$$u_1 - u_2 = 0, \quad (\sigma_1(u_1) - \sigma_2(u_2)) \cdot n_\Gamma \quad \text{on } \Gamma.$$

Using the LM method, the coupling term in the weak formulation is $b(\lambda, v) = \int_\Gamma \lambda(v_1 - v_2)$.

Discretization

- background mesh $\mathcal{T}_h$ of Ω (non conforming with Γ), from which we extract two non conforming triangulation $\mathcal{T}_1$ and $\mathcal{T}_2$ for Ω_1 and Ω_2
- for LM mesh we choose the mesh $\mathcal{G}$ constituted by all the volume elements of $\mathcal{T}_h$ which has a nonnull intersection with Γ.

using

- P^1-elements for the solutions in Ω_1 and Ω_2 and
- P^0-elements for the LM

Stabilization

The choice for the stabilization term is the following

$$s(\lambda_h, \mu_h) = \sum_{K \in \mathcal{G}} \int_{Internalfaces} h[\![\lambda_h]\!][\![\mu_h]\!] ds.$$

The proof of the inf-sup condition for the stabilized problem relies on the construction of a space L_h defined on a coarser mesh such that the inf-sup condition holds for the original problem and the construction of an interpolation operator π_L from the LM space and L_h. The coarser mesh of elements intersecting Γ is built assembling the elements of $\mathcal{G}$ in macro patches $(F_j)_j$ and π_L is defined as the projection onto P^0 functions on $(F_j)_j$. The following relation must hold $\|\lambda_h - \pi_L \lambda_h\|_{L_h} \leq C s(\lambda_h, \lambda_h)$.

It is not clear to me how this example can be extended to the case of P^1-continuous LM, in the sense that I would not know how to define an interpolation operator from P^1 functions defined on the elements of $\mathcal{G}$ to these macro patches $(F_j)_j$. An idea could be (as they also do for the analysis) to build around each F_j a shape regular macro element w_j, basically attaching to F_j other elements of the background mesh $\mathcal{T}_h$ and project on these new macro elements. However, it is not clear to me how the satbilization term could be then deduced from the projection operation, that's why I read the following second paper in which they use P^1 continuous LM but in the 'fitted' case.

Interior-penalty-stabilized LM methods (Burman, Hansbo, 2010)
Two domains Ω_1 and Ω_2 separated by an interface Γ.The problem is the following

$$\begin{aligned} -\Delta u_i = f \quad \text{on } \Omega_i \\ u_i = 0 \quad \text{on } \partial\Omega \\ u_1 - u_2 = 0 \quad \text{on } \Gamma \\ \nabla u_1 \cdot \mathbf{n}_1 + \nabla u_2 \cdot \mathbf{n}_2 = 0 \quad \text{on } \Gamma \end{aligned}$$

They use the LM method to impose the second constraint on Γ. The coupling term in the weak formulation is $b(v, \lambda) = \int_\Gamma \lambda [\![v]\!]$, with $[\![v]\!] = (v_1 - v_2)_\Gamma$.

Discretization

- conforming mesh $\mathcal{T}_1$ of Ω_1 and corresponding trace mesh $\partial\mathcal{T}_1$ on Γ with mesh size h_1
- conforming mesh $\mathcal{T}_2$ of Ω_2 and corresponding trace mesh $\partial\mathcal{T}_2$ on Γ with mesh size h_2($\mathcal{T}_1$ and $\mathcal{T}_2$ not matching)
- multiplier mesh $\mathcal{G}$ defined on Γ with mesh size h_Γ(different from the trace meshes)

with $c_1 h_1 \leq h_\Gamma \leq c_2 h_1$, $c_1 h_2 \leq h_\Gamma \leq c_2 h_2$ and using

- P^1-elements for solution u_1 in Ω_1
- P^1-elements for solution u_2 in Ω_2
- P^0 or continuous P^1-elements for LM

Stabilization

- In case of P^0 LM,

$$j(\lambda, \mu) = \sum_{K \in \mathcal{G}} \int_{\partial K} \gamma h_\Gamma^2 [\lambda][\mu] ds$$

- In case of P^1 LM,

$$j(\lambda, \mu) = \sum_{K \in \mathcal{G}} \int_{\partial K} \gamma h_\Gamma^4 [\nabla\lambda][\nabla\mu] ds$$

  or

$$j(\lambda, \mu) = \sum_{K \in \mathcal{G}} \int_{K} \gamma h_\Gamma^3 \nabla\lambda \nabla\mu dx$$

The proof of the stability of the stabilized global biliear form relies on the following choice for the discrete norm $|||(v,\mu)|||^2 = \|\nabla v\|^2_{L^2(\Omega_1)} + \|\nabla v\|^2_{L^2(\Omega_2)} + \|h^{\frac{1}{2}}\mu\|^2_{L^2(\Gamma)} + j(\mu,\mu)$; on the construction of projection operators π_1 and π_2 from the multiplier space $\mathcal{G}$ to the trace spaces $\partial\mathcal{T}_1$ and $\partial\mathcal{T}_2$: in the case of P^1 elements they are just the nodal interpolation operator whereas in the P^0 case is a sort of average of the nodal values. These operators satisfy $\|h^{\frac{1}{2}}(\lambda^h - \pi_i\lambda^h)\|^2_{L^2(\Gamma)} \leq Cj(\lambda^h,\lambda^h)$; on the fact that the choice of $j(\lambda,\mu)$ are made s.t. j defines a norm on the space of functions $\lambda^h - \pi_i\lambda^h$ defined on the set of elements of $\mathcal{G}$ intersecting the generic element K of the trace mesh $\partial\mathcal{T}_i$.

Kent thinks that our case should be simpler because we don't have the jump of the solution as coupling term in the weak formulation; we have instead just the L-2 product of the multiplier and the trace of the solution. So 'his feeling' if I understood well is that we can avoid the construction of these projection operators to control the coupling term. Instead, we should be able to control it directly in the analysis using a stabilization of (grad, grad) type. What I don't understand is how to choose then the coefficient and the power of h that multiplies the stabilization term..