

COUPLING PDES ON XD-YD DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. This note summarizes numerical experiments investigating stabilization techniques for coupled multiscale problems using Lagrange multipliers. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

Key words. elliptic problems, high dimensionality gap, essential coupling conditions, Lagrange multipliers

AMS subject classifications. n.a.

1. Babuška problem. Given bounded $\Omega \subset \mathbb{R}^2$, let $\Gamma_D, \Gamma_L \subset \partial\Omega$ be such that $|\Gamma_i| \neq 0, i \in \{D, L\}$, $\bigcup_i \Gamma_i = \partial\Omega$. We then consider the Poisson problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \Gamma_D, \\ u &= g && \text{on } \Gamma_L, \end{aligned}$$

which upon introducing the Lagrange multiplier $p \in Q = (H_{00}^{1/2}(\Gamma_L))'$ leads to a variational problem: Find $u \in V = H_{0,\Gamma_D}^1(\Omega)$, $p \in Q$ such that

$$(1.1) \quad \begin{aligned} \int_{\Omega} \nabla u \cdot \nabla v &+ \int_{\Gamma_L} p v = \int_{\Omega} f v && \forall v \in V, \\ \int_{\Gamma_L} q u &= \int_{\Gamma} g q && \forall q \in Q. \end{aligned}$$

In the following experiments $\Omega = (0, 1)^2$, $\Gamma_L = \{(x, y) \in \partial\Omega; x = 0\}$ while $\Gamma_D = \partial\Omega \setminus \Gamma_L$. Letting Ω_H, Γ_h denote respectively the discretizations of Ω and Γ_L we shall consider two different geometrical settings, cf. Figure 1.1, (i) either Γ_h is the trace mesh (on Γ_L) of Ω , i.e. there is a one-to-one mapping between vertices/cells of the two meshes or (ii) Γ_h is not the trace mesh of Ω and Γ_h is finer relative to the trace mesh. We shall subsequently refer to the two cases as *matching* and *non-matching*.

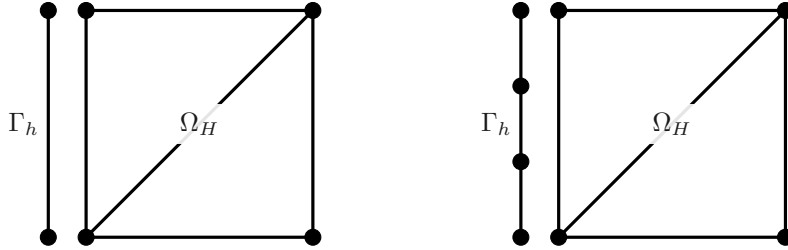


FIGURE 1.1. *Matching (left) and non-matching (right) setting considered for the Babuška problem.*

1.1. Discretization by P1-P1 elements. We consider finite element discretization of (1.1) in terms continuous linear Lagrange elements (P1) for both spaces V and Q .

1.1.1. Matching case. We remark that Dirichlet boundary conditions $p = 0$ on $\partial\Gamma_L$ need to be enforced on the linear system in order to obtain a non-singular system matrix. Then the discretization is inf-sup stable as can be seen in Table 1.1 by observing that the MinRes iterations using the Riesz map preconditioner remain bounded in the discretization parameter. Note that the Q block of (the diagonal)

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h	$\ u - u_H\ _1$	$\ p - p_h\ _0$	n	κ	$\ u - u_H\ _1$	$\ p - p_h\ _0$	n	κ
8.84E-02	6.91E+00(-)	6.37E+00(-)	29	5.131	6.90E+00(-)	5.94E+00(-)	31	4.836
4.42E-02	3.54E+00(0.97)	1.61E+00(1.98)	26	5.176	3.54E+00(0.96)	1.76E+00(1.75)	29	4.846
2.21E-02	1.78E+00(0.99)	5.45E-01(1.56)	26	5.191	1.78E+00(0.99)	5.95E-01(1.57)	28	4.851
1.10E-02	8.93E-01(1.00)	3.11E-01(0.81)	25	5.195	8.93E-01(1.00)	3.24E-01(0.88)	26	4.853
5.52E-03	4.46E-01(1.00)	2.12E-01(0.55)	24	5.196	4.46E-01(1.00)	2.16E-01(0.59)	26	4.853
2.76E-03	2.23E-01(1.00)	1.49E-01(0.51)	24	5.196	2.23E-01(1.00)	1.50E-01(0.52)	24	4.854
1.38E-03	1.12E-01(1.00)	1.06E-01(0.50)	22	-	1.12E-01(1.00)	1.06E-01(0.51)	22	-

TABLE 1.1

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (1.1) (κ) discretized with P1-P1 elements. (left) Matching case. (right) Non-matching case.

preconditioner represents the action of the inverse of $-\Delta_{00}^{1/2}$. In particular, the fractional operator is based on the eigenvalue problem

$$\begin{aligned} -\Delta p &= \lambda p & \text{in } \Gamma_L, \\ p &= 0 & \text{on } \partial\Gamma_L. \end{aligned}$$

1.1.2. Non-matching case. Here $\dim Q_h$ is (considerably) larger than the trace space of V_H and stabilization of [2] is needed to obtain inf-sup stability. More precisely, the discretization of (1.1) reads: Find $u \in V_H$, $q \in Q_h$ such that

$$\begin{aligned} (1.2) \quad & \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma_L} p v = \int_{\Omega} f v \quad \forall v \in V_H, \\ & \int_{\Gamma_L} q u - \gamma \sum_{K \in \Gamma_h} \int_K h_K^3 \nabla p \cdot \nabla q = \int_{\Gamma} g q \quad \forall q \in Q_h. \end{aligned}$$

Here γ is a stabilization parameter and we use $\gamma = 1$ in the following. As before, the trial and test functions of the space Q_h are constructed such that they satisfy the boundary conditions $p = 0$ on $\partial\Gamma_L$.

MinRes iteration counts with the Riesz map preconditioner are shown in Table 1.1. Here, the Q block of the preconditioner is based on the inverse of the bilinear form

$$\langle -\Delta_{00}^{1/2} p, q \rangle + \sum_{K \in \Gamma_h} \int_K h_K^3 \nabla p \cdot \nabla q, \quad p, q \in Q_h.$$

Concerning approximation properties of the discretization, note that the error for the multiplier is reported in the L^2 norm and not the natural $H^{-1/2}$ norm. Nevertheless, the rate seems sub-optimal. This is likely due to the fact that the multiplier in the manufactured test case does not satisfy the Dirichlet boundary conditions employed in the discrete problem.

1.2. Discretization by P1-P0 elements. We consider finite element discretization of (1.1) in terms P1 elements for the space V while the multiplier space uses piecewise constants elements (P0). In both the matching and non-matching case the discretization is unstable and inf-sup stability is obtained by using stabilization [1]. That is, the discretization of (1.1) reads: Find $u \in V_H$, $q \in Q_h$ such that

$$\begin{aligned} (1.3) \quad & \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma_L} p v = \int_{\Omega} f v \quad \forall v \in V_H, \\ & \int_{\Gamma_L} q u - \gamma \sum_{\mathcal{V}^0} \{ \{ h_K \} \}^2 [p][q] = \int_{\Gamma} g q \quad \forall q \in Q_h, \end{aligned}$$

where \mathcal{V}^0 is the set of all interior vertices of Γ_h .

Table 1.2 shows stability and convergence properties of the approximation. Note that the error in the multiplier converges linearly, cf. $\frac{1}{2}$ for the P1-P1 discretization in Table 1.1.

h	$\ u - u_H\ _1$	$\ p - p_h\ _0$	n	κ	$\ u - u_H\ _1$	$\ p - p_h\ _0$	n	κ
8.84E-02	6.88E+00(-)	2.83E+00(-)	26	4.749	6.91E+00(-)	1.17E+01(-)	35	6.940
4.42E-02	3.54E+00(0.96)	2.12E+00(0.42)	28	4.817	3.54E+00(0.96)	2.33E+00(2.33)	35	6.973
2.21E-02	1.78E+00(0.99)	7.42E-01(1.51)	28	4.840	1.78E+00(0.99)	4.91E-01(2.24)	30	6.987
1.10E-02	8.93E-01(1.00)	2.17E-01(1.78)	25	4.850	8.93E-01(1.00)	1.46E-01(1.75)	28	6.994
5.52E-03	4.46E-01(1.00)	8.90E-02(1.28)	23	4.853	4.46E-01(1.00)	5.90E-02(1.31)	27	6.996
2.76E-03	2.23E-01(1.00)	4.28E-02(1.06)	23	4.853	2.23E-01(1.00)	2.77E-02(1.09)	26	6.995
1.38E-03	1.12E-01(1.00)	2.12E-02(1.01)	23	-	1.12E-01(1.00)	1.36E-02(1.02)	25	-

TABLE 1.2

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (1.1) (κ) discretized with $P1$ - $P0$ elements. (left) Matching case. (right) Non-matching case.

2. Coupled multiscale problem. As a stepping stone towards the coupled $3d$ - $1d$ - $1d$ problem we consider first a coupled multiscale problem where the dimensionality gap is 1. Given two bounded domains $\Omega^i \subset \mathbb{R}^2$, $i \in \{-, +\}$, let Γ be the intersection of the domain boundaries, cf. Figure 2.1. We are then interested in solving

$$\begin{aligned}
 (2.1) \quad & -\Delta u^i = f && \text{in } \Omega^i, i \in \{+, -\}, \\
 & u^i = g^i && \text{on } \Gamma_D^i = \partial\Omega^i \setminus \Gamma, \\
 & \llbracket u \rrbracket = 0 && \text{on } \Gamma, \\
 & -\Delta \hat{u} - \llbracket \nabla u \cdot n \rrbracket = \hat{f} && \text{on } \Gamma, \\
 & \hat{u} = \hat{g} && \text{on } \partial\Gamma, \\
 & u - \hat{u} = h && \text{on } \Gamma.
 \end{aligned}$$

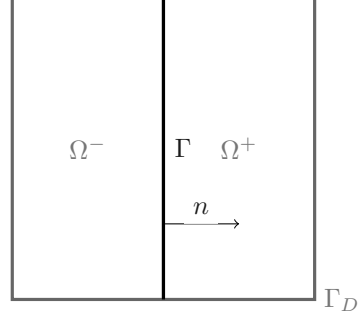


FIGURE 2.1. Schematic domain of coupled multiscale problem.

Let $\Omega = \Omega^+ \cup \Omega^-$ and $\Gamma_D = \Gamma_D^+ \cup \Gamma_D^-$. Introducing a Lagrange multiplier $p = \llbracket \nabla u \cdot n \rrbracket \in Q = (H_{00}^{-1/2}(\Gamma))'$ the weak form of (2.1) reads: Find $u \in V = H_{0, \Gamma_D}^1(\Omega)$, $\hat{v} \in \hat{V} = H_{0, \Gamma}^1(\Gamma)$, $p \in Q$ such that

$$\begin{aligned}
 (2.2) \quad & \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma} p v = \int_{\Omega} f v \quad \forall v \in V, \\
 & + \int_{\Gamma} \nabla \hat{u} \cdot \nabla \hat{v} - \int_{\Gamma} p \hat{v} = \int_{\Gamma} \hat{f} \hat{v} \quad \forall \hat{v} \in \hat{V}, \\
 & \int_{\Gamma} u q - \int_{\Gamma} \hat{u} q = \int_{\Gamma} h q \quad \forall q \in Q.
 \end{aligned}$$

In the following $\Omega = (0, 1)$, $\Gamma = \{(x, y) \in \Omega; x = \frac{1}{2}\}$ and based on the properties of the finite element meshes

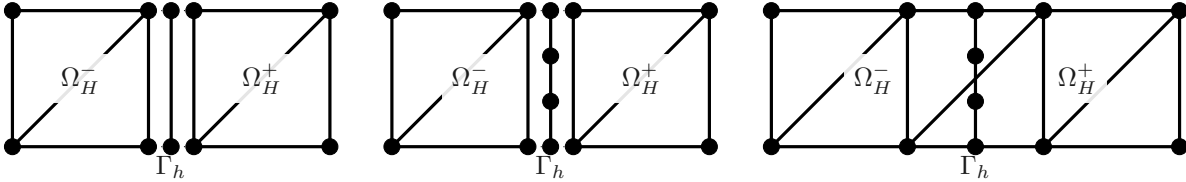


FIGURE 2.2. Different mesh settings considered for the coupled problem (2.2): conforming matching, conforming non-matching, non-conforming (from left to right).

Ω_H , Γ_h we shall distinguish three different settings, cf. Figure 2.2. Here, a conforming case implies that the trace mesh of Ω_H on Γ consists only of the existing facets of the mesh. In the conforming setting we consider

h	$\ u - u_H\ _1$	$\ \hat{u} - \hat{u}_h\ _1$	$\ p - p_h\ _0$	n	κ	$\ u - u_H\ _1$	$\ \hat{u} - \hat{u}_h\ _1$	$\ p - p_h\ _0$	n	κ
8.8E-2	8.8E-1(–)	2.0E0(–)	8.9E-1(–)	20	8.0	8.6E-1(–)	6.7E-1(–)	1.7E-1(–)	23	4.5
4.4E-2	4.4E-1(1.01)	1.0E0(0.99)	2.1E-1(2.10)	22	8.4	4.4E-1(0.99)	3.4E-1(1.00)	4.5E-2(1.89)	25	4.5
2.2E-2	2.2E-1(1.00)	5.0E-1(1.00)	5.1E-2(2.03)	22	8.5	2.2E-1(1.00)	1.7E-1(1.00)	1.2E-2(1.95)	25	4.6
1.1E-2	1.1E-1(1.00)	2.5E-1(1.00)	1.3E-2(2.01)	21	8.6	1.1E-1(1.00)	8.4E-2(1.00)	3.0E-3(1.98)	24	4.7
5.5E-3	5.5E-2(1.00)	1.3E-1(1.00)	3.2E-3(2.00)	21	8.7	5.5E-2(1.00)	4.2E-2(1.00)	7.5E-4(1.99)	24	4.7
2.8E-3	2.7E-2(1.00)	6.3E-2(1.00)	7.9E-4(2.00)	21	8.7	2.7E-2(1.00)	2.1E-2(1.00)	1.9E-4(1.99)	22	4.7
1.4E-3	1.4E-2(1.00)	3.1E-2(1.00)	2.0E-4(2.00)	21	–	1.4E-2(1.00)	1.0E-2(1.00)	4.7E-5(2.00)	22	–

TABLE 2.1

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) (κ) discretized with P1-P1-P1 elements. Conforming (left) matching and (right) non-matching case.

h	$\ u - u_h\ _1$	$\ \hat{u} - \hat{u}_h\ _1$	$\ p - p_h\ _0$	n	κ	$\ u - u_h\ _1$	$\ \hat{u} - \hat{u}_h\ _1$	$\ p - p_h\ _0$	n	κ
8.8E-2	9.3E-1(–)	2.0E0(–)	2.9E0(–)	21	4.5	8.6E-1(–)	6.7E-1(–)	9.2E-1(–)	29	7.8
4.4E-2	4.4E-1(1.06)	1.0E0(0.99)	1.1E0(1.43)	28	4.8	4.4E-1(0.99)	3.4E-1(1.00)	3.4E-1(1.42)	31	8.1
2.2E-2	2.2E-1(1.02)	5.0E-1(1.00)	4.1E-1(1.39)	28	5.0	2.2E-1(1.00)	1.7E-1(1.00)	1.4E-1(1.34)	32	8.3
1.1E-2	1.1E-1(1.00)	2.5E-1(1.00)	1.7E-1(1.28)	27	5.1	1.1E-1(1.00)	8.4E-2(1.00)	5.6E-2(1.26)	30	8.4
5.5E-3	5.5E-2(1.00)	1.3E-1(1.00)	7.5E-2(1.19)	27	5.2	5.5E-2(1.00)	4.2E-2(1.00)	2.5E-2(1.18)	30	8.5
2.8E-3	2.7E-2(1.00)	6.3E-2(1.00)	3.5E-2(1.11)	27	5.2	2.7E-2(1.00)	2.1E-2(1.00)	1.1E-2(1.11)	29	8.5
1.4E-3	1.4E-2(1.00)	3.1E-2(1.00)	1.7E-2(1.07)	27	–	1.4E-2(1.00)	1.0E-2(1.00)	5.5E-3(1.06)	29	–

TABLE 2.2

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) (κ) discretized with P1-P1-P0 elements. Conforming (left) matching and (right) non-matching case.

matching and non-matching cases as defined above. We remark that the approximation of the space \hat{V} shall always be constructed on the multiplier mesh Γ_h .

2.1. P1-P1-P1 discretization with Γ conformity. Let us first consider a conforming setting and an approximation of all the spaces involved in (2.2) in terms of P1 elements. As with the Babuška problem (1.1) the discrete multiplier space is such that $q = 0$ on $\partial\Gamma$ for all $q \in Q_h$. In fact, the boundary conditions are necessary to obtain an invertible linear system.

Table 2.1 summarizes the results for both matching and non-matching tessilations. In the latter case, a stabilization term [2] is added similar to (1.2) (we set $\gamma = 1$). In both cases the results show evidence of the inf-sup stability of the discretization. We note that the L^2 error of the multiplier converges quadratically; in contrast to the test problem in (1.1) the solution here *does* satisfy $p = 0$ on $\partial\Gamma$.

2.2. P1-P1-P0 discretization with Γ conformity. We shall next have the multiplier space be spanned by piecewise constant functions. Following (1.1) a stabilization [1] is employed in both the matching and non-matching cases. Table 2.2 confirms that the discretization is inf-sup stable.

2.3. Non-conforming setting. Finally a non-conforming setting is considered with P1-P1-P1 and P1-P1-P0 discretizations. In both cases stabilization terms are employed to obtain inf-sup stability, cf. Table 2.3. Note that compared to the conforming case the multiplier convergence with P1 elements is only linear. The order from the conforming case is preserved with P0 elements. Note also that with both elements u_h converges with order $\frac{1}{2}$; the approximation struggles to capture the kink of the solution which now occurs within the finite element cells.

REFERENCES

- [1] E. BURMAN, *Projection stabilization of Lagrange multipliers for the imposition of constraints on interfaces and boundaries*, Numerical Methods for Partial Differential Equations, 30 (2014), pp. 567–592.
- [2] E. BURMAN AND P. HANSBO, *Interior-penalty-stabilized Lagrange multiplier methods for the finite-element solution of elliptic interface problems*, IMA journal of numerical analysis, 30 (2009), pp. 870–885.

h	$\ u - u_H\ _1$	$\ \hat{u} - \hat{u}_h\ _1$	$\ p - p_h\ _0$	n	κ	$\ u - u_H\ _1$	$\ \hat{u} - \hat{u}_h\ _1$	$\ p - p_h\ _0$	n	κ
8.3E-2	1.3E0(–)	7.0E-1(–)	1.2E0(–)	28	5.2	1.3E0(–)	7.0E-1(–)	1.7E0(–)	31	8.6
4.3E-2	8.1E-1(0.68)	3.5E-1(1.04)	6.4E-1(0.95)	29	5.6	8.1E-1(0.68)	3.5E-1(1.04)	7.6E-1(1.18)	34	9.3
2.2E-2	5.3E-1(0.62)	1.8E-1(1.02)	3.3E-1(0.99)	29	5.8	5.3E-1(0.62)	1.8E-1(1.02)	3.6E-1(1.10)	34	9.7
1.1E-2	3.6E-1(0.57)	8.8E-2(1.01)	1.6E-1(1.00)	27	6.0	3.6E-1(0.57)	8.8E-2(1.01)	1.8E-1(1.05)	34	9.9
5.5E-3	2.5E-1(0.54)	4.4E-2(1.01)	8.3E-2(1.00)	27	6.0	2.5E-1(0.54)	4.4E-2(1.01)	8.7E-2(1.03)	33	10.0
2.8E-3	1.7E-1(0.52)	2.2E-2(1.00)	4.1E-2(1.00)	26	6.1	1.7E-1(0.52)	2.2E-2(1.00)	4.3E-2(1.01)	32	10.0
1.4E-3	1.2E-1(0.51)	1.1E-2(1.00)	2.1E-2(1.00)	26	–	1.2E-1(0.51)	1.1E-2(1.00)	2.1E-2(1.01)	31	10.0

TABLE 2.3

Non-conforming setting. Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) (κ) discretized with $P1$ - $P1$ - $P1$ (left) and $P1$ - $P1$ - $P0$ (right) elements.