NOTES ON ROBUST SOLVERS FOR SYSTEMS OF COUPLED MULTISCALE PDES

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Abstract. We summarize numerical experiments related to the task of developing mesh and parameter indepent solvers for the $3d$-$1d$ coupled diffusion problems. These are personal notes written to keep track of the developments on this topic and are to be kept confidential.

Key words. elliptic problems, high dimensionality gap, essential coupling conditions, Lagrange multipliers

AMS subject classifications. n.a.

1. Introduction.

Let $\langle \cdot, \cdot \rangle_\Lambda$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. This note concerns robust solvers for the problem: Find $u \in H^1_0(\Omega)$, $u_\odot \in H^1_0(\Lambda)$, $\lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$(u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \lambda_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, V)_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1_0(\Lambda)\,, \tag{1.1a}$$

$$\langle \overline{\mathcal{T}}_\Lambda u - u_\odot, \mu_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} = 0 \quad \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda)\,, \tag{1.1b}$$

where $\overline{\mathcal{T}}_\Lambda$ denotes the composition of operators $\overline{(\cdot)} \circ \mathcal{T}_\Gamma$. In particular, we aim at devising algorithms that are robust with respect to the averaging radius in $\overline{\mathcal{T}}_\Lambda$.

The (continuous) problem (1.1) is well posed, in particular the Lagrange multiplier $\mu_\odot$ exists in a suitable fractional Sobolev space on Λ. Following operator preconditioning framework [4] preconditioners for the problem rely on fractional Sobolev norms. At the moment the stability of the discrete problem can be shown using the Lagrange multiplier defined in the space of piecewise constant functions over the collection of tetrahedra (of the $3d$ mesh) that are intersected by Λ. However, it is not clear how fractional Sobolev norms on such a space should be constructed (in contrast to the case where the multiplier is defined on Λ). We shall therefore establish the preconditioners for the *discrete* problem using different norms than those used in the analysis of the *continuous* one.

As a stepping stone to (1.1) we shall consider a related $2d$-$1d$ problem

$$\begin{aligned} -\Delta u^i &= f && \text{in } \Omega^i, i \in \{+,-\}\,,\\ [\![u]\!] &= 0 && \text{on } \Gamma,\\ -\Delta \hat{u} + \hat{u} - [\![\nabla u \cdot n]\!] &= \hat{f} && \text{on } \Gamma,\\ \epsilon u - \hat{u} &= h && \text{on } \Gamma. \end{aligned} \tag{1.2}$$

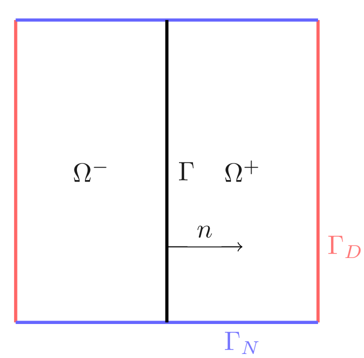

FIGURE 1.1. *Schematic domain of coupled multiscale problem.*

Here (1.2) shall be equipped with Dirichlet boundary conditions (for u^i) on Γ_D, while Neumann conditions are used on Γ_N (for both the unknowns). Note that the role of constant parameter ϵ is similar to that of averaging radius in (1.1).

Let $\Omega = \Omega^+ \cup \Omega^-$. Introducing a Lagrange multiplier $p = \epsilon^{-1}[\![\nabla u \cdot n]\!] \in Q = H^{-1/2}(\Gamma)$ the weak form

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ϵ	h			
	2^{-2}	2^{-3}	2^{-4}	2^{-5}
10^6	7.73	7.82	7.86	7.87
10^4	7.73	7.82	7.86	7.87
10^2	7.73	7.82	7.86	7.87
1	7.39	7.79	8.01	8.13
10^{-2}	2.62	2.63	2.63	2.65
10^{-4}	2.62	2.62	2.62	2.62
10^{-6}	2.62	2.62	2.62	2.62

ϵ	h					
	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
10^6	21	22	22	21	21	19
10^4	21	22	22	21	21	19
10^2	21	22	22	21	21	19
1	24	28	26	26	27	25
10^{-2}	10	10	10	12	13	14
10^{-4}	6	6	6	6	7	7
10^{-6}	4	4	4	4	4	4

TABLE 1.1

Problem (1.3) discretized with P1 elements. Condition number (left) and iterations counts (right) using preconditioner (1.5).

of (1.2) reads: Find $u \in V = H^1_{0,\Gamma_D}(\Omega)$, $\hat{v} \in \hat{V} = H^1(\Gamma)$, $p \in Q$ such that

$$
\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x && && + \epsilon \int_\Gamma p v \,\mathrm{d}s &&= \int_\Omega f v \,\mathrm{d}x \quad \forall v \in V,\\
& && + \int_\Gamma \nabla \hat{u} \cdot \nabla \hat{v} + \hat{u}\hat{v} \,\mathrm{d}x && - \int_\Gamma p \hat{v} \,\mathrm{d}s &&= \int_\Gamma \hat{f} \hat{v} \,\mathrm{d}s \quad \forall \hat{v} \in \hat{V},\\
&\epsilon \int_\Gamma u q \,\mathrm{d}s && - \int_\Gamma \hat{u} q \,\mathrm{d}s && &&= \int_\Gamma h q \,\mathrm{d}s \quad \forall q \in Q.
\end{aligned}
\tag{1.3}
$$

In the following it will be convenient to work with the operator form of (1.3). In particular the left-hand-side defines an operator

$$
\mathcal{A} = \begin{pmatrix} -\Delta_\Omega & & -\epsilon T' \\ & -\Delta_\Gamma + I & -I \\ -\epsilon T & -I & \end{pmatrix} = \begin{pmatrix} A_2 & & B_2' \\ & A_1 & -B_1' \\ B_2 & -B_1 & \end{pmatrix}.
\tag{1.4}
$$

In [3] it is shown that $\mathcal{A}$ is an isomorphism from W to its dual space where

$$
W = H^1(\Omega)_{0,\Gamma_D} \times H^1(\Gamma) \times (\epsilon H^{-1/2}(\Gamma) \cap H^{-1}(\Gamma)).
$$

It is further shown that discretization in terms of continuous linear Lagrange elements (P1) for all the unknowns is inf-sup stable and in turn the diagonal operator

$$
\mathcal{B} = \operatorname{diag}(-\Delta_\Omega, -\Delta_\Gamma + I, \epsilon^2(-\Delta_\Gamma)^{-1/2} + (-\Delta_\Gamma + I)^{-1})^{-1}
\tag{1.5}
$$

is a robust preconditioner for $\mathcal{A}$. For the sake of completness (and verifying the implementation) Table 1.1 shows the number of MinRes iterations and condition numbers[1] of the preconditioned problem for different levels of refinement and values of the parameter ϵ. The stability is evident.

Preconditioner (1.5) was derived under the assumption that the finite element mesh Γ_h of Γ consists of facets of the finite element mesh Ω_h of Ω, see also Figure 1.2 (i). This assumption is too limiting; for practical purposes the meshes should be independent and a more common case is that the trace mesh of Ω_h on Γ is coarser than Γ_h. In the following we shall pursue formulations (and corresponding solvers) of (1.3) which allow for different meshes and resolutions. Our ultimate goal is a stable formulation in which the multiplier is not defined on Γ_h but rather on cells of Ω_h intersected by the curve, cf. Figure 1.2 (iv) and (v).

The rest of the report is structured as follows. Considering (1.4) we first treat the problems related to domains Ω, i.e. $\mathcal{A}_2 = [A_2, B_2'; B_2, 0]$, and Γ, i.e. $\mathcal{A}_1 = [A_1, B_1'; B_1, 0]$, individually[2]. More precisely, in §2 a set of preconditioners for different formulations of the Babuška problem is presented culminating in a

[1] Unless specified otherwise all the experiments are done on Ω a unit square triangulated in $2N^2$ isosceles triangles. MinRes iterations are always started from a random initial vector and terminate one the relative preconditioned residual norm is reduced by factor 10^8. Condition numbers of the preconditioned problem are computed by direct solvers, i.e. from the full spectrum.

[2] This is done for clarity of exposition, to better fix the ideas and debugging.

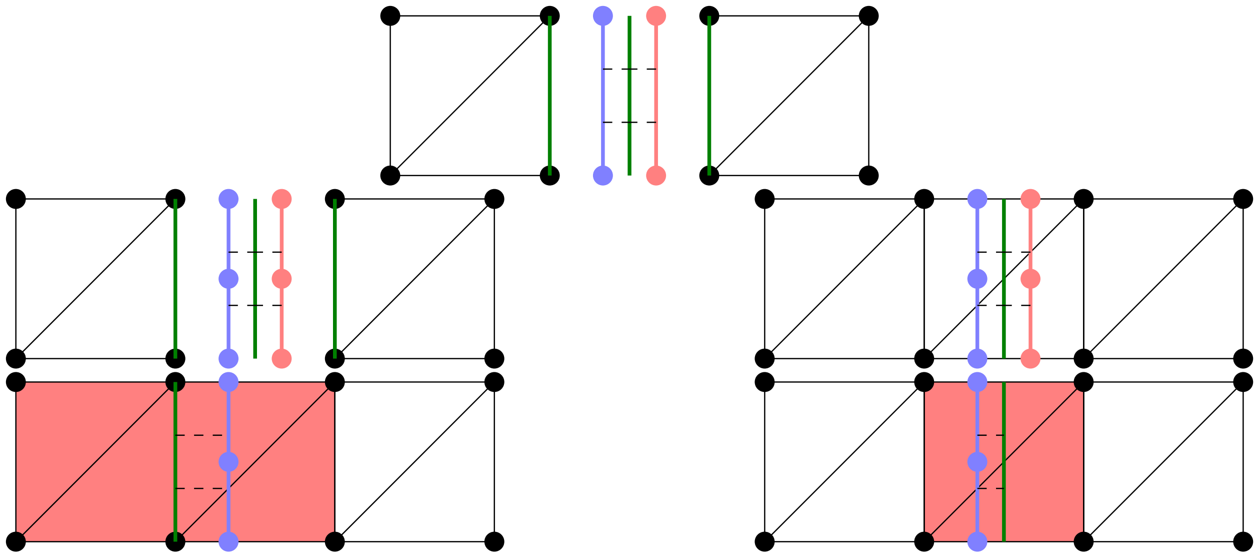

FIGURE 1.2. *Schematic geometric setting and discretization of multiscale problems. Position of interface Γ is denoted by ▬ . Meshes underlying discrete approximation spaces $\hat{V}_h$ and Q_h of $\hat{V}$ and Q are denoted respectively by ▬ and ▬ . Geometric correspondence is signified by dashed line. Cases: (i) trace mesh of V_h is used for $\hat{V}_h$ and Q_h, (ii) Γ does not intersect Ω_h cell interiors, (iii) Γ does intersect cell interiors, (iv) as (ii) with Q_h defined on intersected cells and (v) as (iii) with Q_h defined on intersected cells.*

formulation having the discrete Lagrange multiplier on the cut cells. In §3 we deal with instabities in $\mathcal{A}_1$ which arise due to choices of discretization of the Lagrange multiplier space in $\mathcal{A}_2$. Finally in §**??** the insights from individual problems are combined to yield robust solvers for the coupled problem.

2. Preconditioning $\mathcal{A}_2$. With Figure 1.1 in mind we wish to solve the $2d$-$1d$ coupled problem: Find $u \in V = H^1_{0,\Gamma_D}(\Omega)$, $p \in Q = H^{-1/2}(\Gamma)$ such that

$$
\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Gamma pv = \int_\Omega fv \,\mathrm{d}x \quad \forall v \in V,\\
&\int_\Gamma qu \,\mathrm{d}s + \qquad\quad = \int_\Gamma gq \,\mathrm{d}s \quad \forall q \in Q.
\end{aligned}
, \quad \text{or equivalently} \quad \mathcal{A}_2 \begin{pmatrix} u \\ p \end{pmatrix} = L. \tag{2.1}
$$

It can be seen that $\mathcal{A}_2$ is an isomorphism from $W = V \times Q$ to $W^{'}$ and in turn by [4] operator

$$\mathcal{B}_2 = \operatorname{diag}(-\Delta_\Omega, (-\Delta_\Gamma + I)^{-1/2})^{-1}$$

is a cannonical preconditioner for $\mathcal{A}_2$. Considering first the case (i) from Figure 1.2, i.e. trace mesh of Ω_h is Γ_h and using (stable) P1-P1 discretization Table 2.1 shows the both the condition number and the number of MinRes iterations of $\mathcal{B}_2\mathcal{A}_2$ are bounded.

In order to arrive at the formulation where the discrete multiplier space Q_h is setup on the Γ-intersected cells of Ω_h let us first consider an intermediate formulation with a piecewise constant (P0) multiplier defined on Γ_h, cf. Figure 1.2 (ii) and (iii). It is well known, e.g. [5, Ch 11.3], that if trace mesh of Ω_h is Γ_h then P1-P0 discretization is unstable. However the pair is stable if Γ_h is coarser than the trace mesh or a stabilization is employed. In the application we have in mind the mesh of Ω is in general coarser than that of Γ and thus stabilized formulation are of interest. To highlight the difference in resolution between the meshes we shall use different subscripts, i.e. Ω_H and Γ_h. We remark that stabilization for piecewise linear Lagrange multipliers (on Γ) is discussed in [2].

Let $V_H = V_H(\Omega_H)$, $Q_h = Q_h(\Gamma_h)$ be the finite element approximations of V and Q in terms of P1 and P0 elements respectively. Following [1] the stabilized formulation of (2.1) reads: Find $u \in V_H$ and $p \in Q_h$

such that

$$\begin{aligned} \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \quad & \int_\Gamma pv\,\mathrm{d}s = \int_\Omega fv\,\mathrm{d}x \quad \forall v\in V_H,\\ \int_\Gamma qu\,\mathrm{d}s + \quad -\sum_{F\in\mathcal{F}} & \int_F h^2[\![p]\!][\![q]\!]\mathrm{d}s = \int_\Gamma gq\,\mathrm{d}s \quad \forall q\in Q_h, \end{aligned} \quad \text{, or equivalently} \quad \mathcal{A}_{2,\Gamma_h}\begin{pmatrix}u\\p\end{pmatrix} = L.$$

Here $\mathcal{F}$ is the union of internal and Γ_N-intersecting facets of Γ_h. A possible preconditioner for $\mathcal{A}_{2,\Gamma_h}$, which is based on the mapping properties of the continuous problem is a Riesz map with respect to the inner product in induced by $\mathcal{B}^{-1}_{2,\Gamma_h}$

$$\langle\mathcal{B}^{-1}_{2,\Gamma_h}\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\rangle = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \langle(-\Delta_\Gamma+I)^{-1/2}p,q\rangle + \sum_{F\in\mathcal{F}}\int_F h^2[\![p]\!][\![q]\!]\mathrm{d}s.$$

Robustness of the preconditioner can be seen in Table 2.1 in both the case where Γ_h does not and does intersect the cell interior of Ω_h (cases (iii) and (iv) in Figure 1.2). A potential difficulty with $\mathcal{B}_{2,\Gamma_h}$ is, however, its generalization of the call of Q_h defined on intersected cells (interiors). In particular, the fractional norm can be an problematic. To avoid the issue, we therefore consider an alternative preconditioner based on [1, §4.A]

$$\langle\tilde{\mathcal{B}}^{-1}_{2,\Gamma_h}\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\rangle = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Gamma h^{-1}uv\,\mathrm{d}s + \int_\Gamma hpq\,\mathrm{d}s + \sum_{F\in\mathcal{F}}\int_F h^2[\![p]\!][\![q]\!]\mathrm{d}s.$$

Note that the fractional norm on the multiplier has been replaced by the h-weighted L^2 norm. Furthermore, there is an additional control of the trace of u on the curve by h^{-1}-weighted L^2 norm. We remark that [1] proves the inf-sup condition for $\mathcal{A}_{2,\Gamma_h}$ using the norms on V_H and Q_h as follows

$$\|u\|^2_{V_H} = \||\nabla u|\|^2_0 + \int_\Gamma h^{-1}uv\,\mathrm{d}s \quad\text{and}\quad \|p\|_{Q_h} = \|h^{1/2}p\|_0.$$

Robustness of the preconditioner is shown in Table 2.1; in the cut case (iii) the preconditioner is more efficinet than $\mathcal{B}_{2,\Gamma_h}$. We remark that the error convergence of $\|u-u_H\|_1$ is reduced in this case due to the fact that the kink of the solution cannot be captured within the element.

Finally, we extend $\mathcal{A}_{2,\Gamma_h}$ to the formulation with Lagrange mutliplier on the intersected elements. To this end, let $\mathcal{S}_H$ denote elements of Ω_H intersected by Γ_h. The space Q_H now consists of piecewise constants on $\mathcal{S}_H$. Following [1, §4.B] we consider problem: Find $u\in V_H$, $p\in Q_H$ such that

$$\begin{aligned} \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \quad & \int_\Gamma pv\,\mathrm{d}s = \int_\Omega fv\,\mathrm{d}x \quad \forall v\in V_H,\\ \int_\Gamma qu\,\mathrm{d}s + \quad -\sum_{F\in\partial\mathcal{S}} & \int_F H[\![p]\!][\![q]\!]\mathrm{d}s = \int_\Gamma gq\,\mathrm{d}s \quad \forall q\in Q_H, \end{aligned} \quad \text{, or equivalently} \quad \mathcal{A}_{2,\Omega_H}\begin{pmatrix}u\\p\end{pmatrix} = L.$$

Here $\partial\mathcal{S}$ denotes interior and Neumann boundary intersecting facets of $\mathcal{S}$. We remark that the norms for the inf-sup condition (shown in [1] of $\mathcal{A}_{2,\Omega_H}$ are identical to $\mathcal{A}_{2,\Gamma_g}$.

Note that the stabilization is newly applied on edges (points previously in $\mathcal{A}_{2,\Gamma_h}$) and relative to $\mathcal{A}_{2,\Gamma_h}$ we componsate for the increased dimensionality of the facets by decreasing the exponent of the H-weight. In a similar manner the preconditioner $\tilde{\mathcal{B}}_{2,\Gamma_h}$ can be generalized yielding $\tilde{\mathcal{B}}_{2,\Omega_H}$

$$\langle\tilde{\mathcal{B}}^{-1}_{2,\Omega_H}\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\rangle = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Gamma h^{-1}uv\,\mathrm{d}s + \int_\Gamma hpq\,\mathrm{d}s + \sum_{F\in\partial\mathcal{S}}\int_F H[\![p]\!][\![q]\!]\mathrm{d}s.$$

In Table 2.1 we observe that MinRes iterations with $\tilde{\mathcal{B}}_{2,\Omega_H}\mathcal{A}_{2,\Omega_H}$ are bounded. The decreasing increments of the condition numbers suggest that κ is bounded as well.

At this point we have at our disposal an efficient solver for $\mathcal{A}_2$, that is a $2d$-$1d$ subcomponent of the coupled problem (1.3). We now turn attention to the $1d$-$1d$ component; the idea being that the solver for the coupled problem in some sense consists of solvers for $\mathcal{A}_2$ and $\mathcal{A}_1$.

$\frac{H}{H_0}$	$\mathcal{B}_2$			
	$\|u-u_H\|_1$	$\|\lambda-\lambda_H\|_0$	#	
1	8.62E-1(–)	1.04E0(–)	21	7.8
2^{-1}	4.4E-1(0.99)	3.8E-1(1.46)	24	7.9
2^{-2}	2.2E-1(1.00)	1.4E-1(1.49)	22	7.9
2^{-3}	1.1E-1(1.00)	4.8E-2(1.50)	22	–
2^{-4}	5.5E-2(1.00)	1.7E-2(1.50)	21	–
2^{-5}	2.7E-2(1.00)	6.0E-3(1.50)	21	–

$\frac{H}{H_0}$	$\mathcal{B}_{2,\Gamma_h}$				$\tilde{\mathcal{B}}_{2,\Gamma_h}$	
	$\|u-u_H\|_1$	$\|\lambda-\lambda_h\|_0$	#	κ	#	κ
1	8.6E-1(–)	5.4E-1(–)	29	6.5	25	16.9
2^{-1}	4.3E-1(0.98)	2.3E-1(1.20)	29	6.6	31	17.7
2^{-2}	2.2E-1(1.00)	1.0E-1(1.19)	28	6.6	34	18.1
2^{-3}	1.1E-1(1.00)	4.7E-2(1.13)	28	–	37	–
2^{-4}	5.5E-2(1.00)	2.2E-2(1.08)	28	–	37	–
2^{-5}	2.7E-2(1.00)	1.1E-2(1.04)	27	–	37	–

$\frac{H}{H_0}$	$\mathcal{B}_{2,\Gamma_h}$				$\tilde{\mathcal{B}}_{2,\Gamma_h}$	
	$\|u-u_H\|_1$	$\|\lambda-\lambda_h\|_0$	#	κ	#	κ
1	1.3E0(–)	1.7E0(–)	31	6.7	21	2.5
2^{-1}	8.2E-1(0.68)	9.2E-1(0.92)	32	6.6	21	2.5
2^{-2}	5.4E-1(0.63)	4.6E-1(1.00)	31	6.6	21	2.5
2^{-3}	3.6E-1(0.58)	2.3E-1(1.01)	30	–	20	–
2^{-4}	2.5E-1(0.54)	1.1E-1(1.01)	28	–	20	–
2^{-5}	1.7E-1(0.52)	5.7E-2(1.01)	28	–	18	–

$\frac{H}{H_0}$	$\mathcal{B}_{2,\Omega_H}$			
	$\|u-u_h\|_1$	$\|\lambda-\lambda_h\|_0$	#	κ
1	1.7E0(–)	2.0E0(–)	22	4.5
2^{-1}	1.0E0(0.71)	1.2E0(0.71)	27	5.1
2^{-2}	5.9E-1(0.78)	7.7E-1(0.69)	29	5.4
2^{-3}	3.3E-1(0.85)	4.9E-1(0.64)	29	–
2^{-4}	1.7E-1(0.91)	3.3E-1(0.59)	28	–
2^{-5}	9.1E-2(0.95)	2.2E-1(0.55)	26	–

$\frac{H}{H_0}$	$\mathcal{B}_{2,\Omega_H}$			
	$\|u-u_h\|_1$	$\|\lambda-\lambda_h\|_0$	#	κ
1	2.0E0(–)	1.3E0(–)	24	5.8
2^{-1}	1.4E0(0.53)	7.0E-1(0.93)	29	7.1
2^{-2}	9.1E-1(0.62)	3.5E-1(1.02)	34	8.0
2^{-3}	5.7E-1(0.68)	1.6E-1(1.08)	36	–
2^{-4}	3.5E-1(0.71)	7.9E-2(1.05)	36	–
2^{-5}	2.2E-1(0.69)	4.3E-2(0.89)	34	–

TABLE 2.1

Iteration counts (#) and condition numbers (κ) of different formulations of (2.1), $H_0 = 2^{-4}$ and $h = H/3$. The cases correspond to setups (i)-(v) from Figure 1.2. First row (i) with $\mathcal{A}_2$. Second row (ii) and (iii) with $\mathcal{A}_{2,\Gamma_h}$. Third row (iv) and (v) with $\mathcal{A}_{2,\Omega_H}$.

3. Preconditioning $\mathcal{A}_1$.

Let $V = H^1(\Gamma)$ We consider the minimization problem

$$\min_{u\in V} \quad \|u\|_1^2 - \int_\Gamma f u \,\mathrm{d}x \quad \text{subject to} \quad u = g \text{ in } \Gamma.$$

Denoting p the Lagrange multiplier for the constraint the extremal points of the related Lagrangian are found as a solution to

$$\begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{A}_1 \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} -\Delta + I & I \\ I & \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}. \tag{3.1}$$

Setting $Q = H^{-1}(\Gamma)$, the operator $\mathcal{A}_1 : W \to W'$ can be seen to be an isomorphism with $W = V \times Q$. In particular the inf-sup condition holds with constant 1 (by definition of the H^{-1} norm).

While it can be easily seen that P1-P1 discretization is inf-sup stable for the problem the pair which we are interested in is P1-P0 as piece-wise constant elements were used in the previous section. We recall that stable P1-P1 discretization for $\mathcal{A}_2$ (using different meshes for Ω, Γ is also possible, cf. [2], however, it is not clear how to extend this result to $3d$-$1d$ problem (1.1).

To illustrate that P1-P0 is not inf-sup stable for $\mathcal{A}_1$ consider a uniform mesh Γ_h and for simplicity assume homogeneous Dirichlet boundary conditions on V. Let mid_K be the midpoint of $K \in \Gamma_h$. Then

$$\int_\Gamma pv = \sum_{K\in\Gamma_h} p(\text{mid}_K) v(\text{mid}_K)$$

and it can be seen that a “checker-board” function p renders the integral zero, that is, the inf-sup condition does not hold.

Let C be the constant from the inf-sup condition of $\mathcal{A}_2$. Since $\mathcal{A}_1$ is only a sub-problem of (1.4) the inf-sup condition for the operator of the coupled problem can be obtained as

$$\sup_{(v,\hat{v})\in V\times\hat{V}} \frac{\int_\Gamma (v-\hat{v})p}{\|(v,\hat{v})\|_{V\times\hat{V}}} \geq \sup_{v\in V} \frac{\int_\Gamma vp}{\|v\|_V} \geq C\|p\|_Q.$$

ϵ	h				
	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
10^6	54	57	59	59	96
10^4	121	116	113	107	109
10^2	100	86	69	121	172
1	177	244	287	311	329
2^{-1}	231	303	372	415	500
2^{-2}	350	453	500	500	500
10^{-1}	500	500	500	500	500

ϵ	h				
	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
10^6	51	54	55	57	58
10^4	51	54	55	57	58
10^2	51	54	56	57	58
1	36	42	47	49	50
10^{-2}	32	38	44	47	49
10^{-4}	32	38	44	47	49
10^{-6}	32	38	44	47	49

TABLE 3.1

Iteration counts for the Stokes problem. (left) Formulation without stabilization using preconditioner $\mathcal{B}$. (right) Stabilized formulation with preconditioner $\mathcal{B}_s$. Maximum number of iterations allowed is 500.

If we apply similar argument to the discrete inf-sup condition it seems that the unstable discretization of $\mathcal{A}_1$ does not present issue. We shall now give an example which demonstrates that for *robust* preconditioning the inf-sup condition should be established in the (intersection) space in which *both* subproblems are inf-sup stable.

EXAMPLE 3.1 (Stokes problem). *Consider the problem of finding u_1, u_2, p satisfying*

$$\begin{aligned} -\Delta u_1 - \epsilon\nabla p &= f_1 \ \textit{in}\ \Omega, \\ -\Delta u_2 + \nabla p &= f_2 \ \textit{in}\ \Omega, \\ \nabla\cdot(\epsilon u_1 - u_2) &= g \ \textit{in}\ \Omega. \end{aligned}$$

We shall assume that the $u_1 = 0$, $u_2 = 0$ on $\Gamma_D \subset \partial\Omega$ and $|\Gamma_D| \neq |\partial\Omega|$. Let now $V_1 = H^1_{0,\Gamma_D}(\Omega)$, $V_2 = H^1_{0,\Gamma_D}(\Omega)$ and $Q = L^2(\Omega)$. Then the operator defined by the weak form of the problem

$$\mathcal{A} = \begin{pmatrix} -\Delta & & -\epsilon\nabla \\ & -\Delta & \nabla \\ \epsilon\nabla\cdot & -\nabla\cdot & \end{pmatrix} : V_1 \times V_2 \times Q \to (V_1 \times V_2 \times Q)'$$

is an isomorphism. However, with these spaces the Riesz map preconditioner is not parameter robust. Instead $Q = \epsilon L^2(\Omega) \cap L^2(\Omega)$ shall be used.

To illustrate the issue with unstable discretization let V_1 be discretized by P2 elements while V_2 uses P1 elements. As P1 elements are used for Q the first Stokes problem is stable while the latter one is not. We shall first consider

$$\mathcal{B} = \textit{diag}(-\Delta, -\Delta, (\epsilon^2+1)I)^{-1}$$

as a preconditioner for $\mathcal{A}$.

Table 3.1 shows that iterations are stable when $\epsilon \gg 1$ while they are unbounded for $\epsilon < 1$. A heuristic explanation for the observation can based on Schur complement reasoning. Indeed the Schur complement of $\mathcal{A}_h$ consists of two parts corresponding to the two Stokes problems

$$S = \epsilon^2\nabla_h\cdot(-\Delta_h)^{-1}\nabla_h + \nabla_h\cdot(-\Delta_h)^{-1}\nabla_h.$$

Owing the discrete inf-sup condition the spectrum of the first part is bounded by $\epsilon^2 I_h$ for all h. However, for the second component a similar bound does not hold (as the corresponding discretization is unstable) and this part can become dominant for small ϵ.

To obtain parameter robustness we consider a stabilized formulation of the Stokes problem in which $\gamma\Delta p = 0$ is added to the divergence constraint (with Dirichlet boundary conditions for p on $\partial\Omega\setminus\Gamma_D$) and γ depends on the square of the mesh size, see also [4, §7]. As a preconditioner for the stabilized operator we then consider

$$\mathcal{B}_s = \textit{diag}(-\Delta, -\Delta, (\epsilon^2+1)I - \beta\Delta)^{-1}.$$

Table 3.1 confirms robustness of the preconditioner. We remark that the stabilizing term has little effect on approximation properties of the formulation.

Example 3.1 motivates the need to find a stabilizing term for the P1-P0 discretization of $\mathcal{A}_1$. In addition, the stabilization should be consistent so as to not ruin the approximation properties of the formulation.

h	$\|u-u_h\|_1$	$\|p-p_h\|_0$	κ
1.56E-02	1.3715E-06(4.00)	1.0020E-02(1.00)	2.66592
7.81E-03	8.5716E-08(4.00)	5.0100E-03(1.00)	2.66596
3.91E-03	5.3572E-09(4.00)	2.5050E-03(1.00)	2.66599
1.95E-03	3.3482E-10(4.00)	1.2525E-03(1.00)	2.66599
9.77E-04	2.0926E-11(4.00)	6.2625E-04(1.00)	2.66599
4.88E-04	1.3079E-12(4.00)	3.1312E-04(1.00)	2.66575

TABLE 3.2
Conditioning and approximation properties of stabilized formulation of (3.1) *using P1-P0 elements.*

3.1. Stabilized $\mathcal{A}_1$. We construct the stabilizing term based on two observations. (i) In §2 the stabilization for $p \in Q_h \subset H^{-1/2}$ in takes the form the facet integral with integrand $h^2[\![p]\!][\![q]\!]$. (ii) For P0 element $-\Delta_h$ is spectrally equivalent to operator involving facet integrals with integrand $h^{-1}[\![p]\!][\![q]\!]$. In turn we assume that the stabilization for H^s norms takes the form $h^p[\![p]\!][\![q]\!]$ and by interpolation from (i) and (ii) $p = -2s + 1$. Thus the stabilizing term for H^{-1} shall involve the cube of the mesh size and we consider a discrete formulation of (3.1) in terms of the operator

$$\langle \mathcal{A}_{1,h} \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \rangle = \int_\Gamma \nabla u \cdot \nabla v + uv \,\mathrm{d}x + \int_\Gamma uq + vp \,\mathrm{d}x - \sum_{F \in \mathcal{F}} h^3 [\![p]\!][\![q]\!] \,\mathrm{d}s.$$

The preconditioner for $\mathcal{A}_{1,h}$ is then the Riesz mapping which includes the negative stabilized term

$$\langle \mathcal{B}_{1,h}^{-1} \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \rangle = \int_\Gamma \nabla u \cdot \nabla v + uv \,\mathrm{d}x + \langle (-\Delta + I)^{-1} p, q \rangle + \sum_{F \in \mathcal{F}} h^3 [\![p]\!][\![q]\!] \,\mathrm{d}s.$$

Table 3.2 shows error convergence of the proposed formulation together with the condition number of the preconditioner problem. We remark that the manufactured solution on the unit interval has $g = 0$ so that $p = f = \cos(\pi x)$ and $u = 0$ is the solution. It can be seen that the formulation is stable and yields converging solutions.

4. Preconditioning 2d coupled problem. 3

5. Preconditioning 3d coupled problem.

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