COUPLING PDES ON XD-YD DOMAINS WITH LAGRANGE MULTIPLIERS

MIROSLAV KUCHTA, FEDERICA LAURINO, KENT-ANDRE MARDAL, PAOLO ZUNINO,*

Abstract. This note summarizes numerical experiments investigating stabilization techniques for coupled multiscale problems using Lagrange multipliers. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

Key words. elliptic problems, high dimensionality gap, essential coupling conditions, Lagrange multipliers

AMS subject classifications. n.a.

1. Babuška problem. Given bounded $\Omega \subset \mathbb{R}^2$, let Γ_D, $\Gamma_L \subset \partial\Omega$ be such that $|\Gamma_i| \neq 0$, $i \in \{D, L\}$, $\bigcup_i \Gamma_i = \partial\Omega$. We then consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma_D,\\ u &= g \quad \text{on } \Gamma_L, \end{aligned}$$

which upon introducing the Lagrange multiplier $p \in Q = (H^{1/2}_{00}(\Gamma_L))'$ leads to a variational problem: Find $u \in V = H^1_{0,\Gamma_D}(\Omega)$, $p \in Q$ such that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \quad &+ \int_{\Gamma_L} pv = \int_\Omega fv \quad \forall v \in V,\\ \int_{\Gamma_L} qu \quad & \quad = \int_\Gamma gq \quad \forall q \in Q. \end{aligned} \tag{1.1}$$

In the following experiments $\Omega = (0,1)^2$, $\Gamma_L = \{(x,y) \in \partial\Omega; x = 0\}$ while $\Gamma_D = \partial\Omega \setminus \Gamma_L$. Letting Ω_H, Γ_h denote respectively the discretizations of Ω and Γ_L we shall consider two different geometrical settings, cf. Figure 1.1, (i) either Γ_h is the trace mesh (on Γ_L) of Ω, i.e. there is a one-to-one mapping between vertices/cells of the two meshes or (ii) Γ_h is not the trace mesh of Ω and Γ_h is finer relative to the trace mesh. We shall subsequently refer to the two cases as *matching* and *non-matching*.

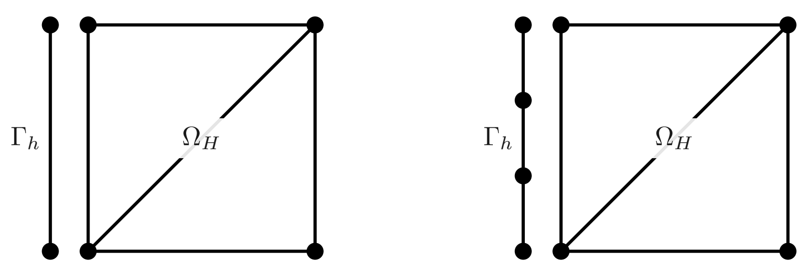

FIGURE 1.1. *Matching (left) and non-mathching (right) setting considered for the Babuška problem.*

1.1. Discretization by P1-P1 elements. We consider finite element discretization of (1.1) in terms continuous linear Lagrange elements (P1) for both spaces V and Q.

1.1.1. Matching case. We remark that Dirichlet boundary conditions $p = 0$ on $\partial\Gamma_L$ need to be enforced on the linear system in order to obtain a non-singular system matrix. Then the discretization is inf-sup stable as can be seen in Table 1.1 by observing that the MinRes iterations using the Riesz map preconditioner remain bounded in the discretization parameter. Note that the Q block of (the diagonal)

*Authors are listed in alphabetical order

h	$\|u-u_H\|_1$	$\|p-p_h\|_0$	n	κ	$\|u-u_H\|_1$	$\|p-p_h\|_0$	n	κ
8.84E-02	6.91E+00(–)	6.37E+00(–)	29	5.131	6.90E+00(–)	5.94E+00(–)	31	4.836
4.42E-02	3.54E+00(0.97)	1.61E+00(1.98)	26	5.176	3.54E+00(0.96)	1.76E+00(1.75)	29	4.846
2.21E-02	1.78E+00(0.99)	5.45E-01(1.56)	26	5.191	1.78E+00(0.99)	5.95E-01(1.57)	28	4.851
1.10E-02	8.93E-01(1.00)	3.11E-01(0.81)	25	5.195	8.93E-01(1.00)	3.24E-01(0.88)	26	4.853
5.52E-03	4.46E-01(1.00)	2.12E-01(0.55)	24	5.196	4.46E-01(1.00)	2.16E-01(0.59)	26	4.853
2.76E-03	2.23E-01(1.00)	1.49E-01(0.51)	24	5.196	2.23E-01(1.00)	1.50E-01(0.52)	24	4.854
1.38E-03	1.12E-01(1.00)	1.06E-01(0.50)	22	–	1.12E-01(1.00)	1.06E-01(0.51)	22	–

TABLE 1.1

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (1.1) (κ) discretized with P1-P1 elements. (left) Matching case. (right) Non-matching case.

preconditioner represents the action of the inverse of $-\Delta_{00}^{1/2}$. In particular, the fractional operator is based on the eigenvalue problem

$$\begin{aligned} -\Delta p &= \lambda p \qquad \text{in } \Gamma_L, \\ p &= 0 \qquad \text{on } \partial\Gamma_L. \end{aligned}$$

1.1.2. Non-matching case. Here $\dim Q_h$ is (considerably) larger than the trace space of V_H and stabilization of [2] is needed to obtain inf-sup stability. More precisely, the discretization of (1.1) reads: Find $u \in V_H$, $q \in Q_h$ such that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \qquad\qquad &+ \int_{\Gamma_L} pv = \int_\Omega fv \quad \forall v \in V_H, \\ \int_{\Gamma_L} qu \qquad -\gamma \sum_{K\in\Gamma_h} &\int_K h_K^3 \nabla p \cdot \nabla q = \int_\Gamma gq \quad \forall q \in Q_h. \end{aligned} \tag{1.2}$$

Here γ is a stabilization parameter and we use $\gamma = 1$ in the following. As before, the trial and test functions of the space Q_h are constructed such that they satisfy the boundary conditions $p = 0$ on $\partial\Gamma_L$.

MinRes iteration counts with the Riesz map preconditioner are shown in Table 1.1. Here, the Q block of the preconditioner is based on the inverse of the bilinear form

$$\langle -\Delta_{00}^{1/2} p, q\rangle + \sum_{K\in\Gamma_h} \int_K h_K^3 \nabla p \cdot \nabla q, \quad p, q \in Q_h.$$

Concerning approximation properties of the discretization, note that the error for the multiplier is reported in the L^2 norm and not the natural $H^{-1/2}$ norm. Nevertheless, the rate seems sub-optimal. This is likely due to the fact that the multiplier in the manufactured test case does not satisfy the Dirichlet boundary conditions employed in the discrete problem.

1.2. Discretization by P1-P0 elements. We consider finite element discretization of (1.1) in terms P1 elements for the space V while the multiplier space uses piecewise constants elements (P0). In both the matching and non-matching case the discretization is unstable and inf-sup stability is obtained by using stabilization [1]. That is, the disretization of (1.1) reads: Find $u \in V_H$, $q \in Q_h$ such that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \qquad\qquad &+ \int_{\Gamma_L} pv = \int_\Omega fv \quad \forall v \in V_H, \\ \int_{\Gamma_L} qu \qquad -\gamma \sum_{\mathcal{V}^0} &\{\{h_K\}\}^2 [\![p]\!][\![q]\!] = \int_\Gamma gq \quad \forall q \in Q_h, \end{aligned} \tag{1.3}$$

where $\mathcal{V}^0$ is the set of all interior vertices of Γ_h.

Table 1.2 shows stability and convergence properties of the approximation. Note that the error in the multiplier converges linearly, cf. $\frac{1}{2}$ for the P1-P1 discretization in Table 1.1.

h	$\|u-u_H\|_1$	$\|p-p_h\|_0$	n	κ	$\|u-u_H\|_1$	$\|p-p_h\|_0$	n	κ
8.84E-02	6.88E+00(–)	2.83E+00(–)	26	4.749	6.91E+00(–)	1.17E+01(–)	35	6.940
4.42E-02	3.54E+00(0.96)	2.12E+00(0.42)	28	4.817	3.54E+00(0.96)	2.33E+00(2.33)	35	6.973
2.21E-02	1.78E+00(0.99)	7.42E-01(1.51)	28	4.840	1.78E+00(0.99)	4.91E-01(2.24)	30	6.987
1.10E-02	8.93E-01(1.00)	2.17E-01(1.78)	25	4.850	8.93E-01(1.00)	1.46E-01(1.75)	28	6.994
5.52E-03	4.46E-01(1.00)	8.90E-02(1.28)	23	4.853	4.46E-01(1.00)	5.90E-02(1.31)	27	6.996
2.76E-03	2.23E-01(1.00)	4.28E-02(1.06)	23	4.853	2.23E-01(1.00)	2.77E-02(1.09)	26	6.995
1.38E-03	1.12E-01(1.00)	2.12E-02(1.01)	23	–	1.12E-01(1.00)	1.36E-02(1.02)	25	–

TABLE 1.2

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (1.1) (κ) discretized with P1-P0 elements. (left) Matching case. (right) Non-matching case.

2. Coupled multiscale problem. As a stepping stone towards the coupled $3d$-$1d$-$1d$ problem we consider first a coupled multiscale problem where the dimensionality gap is 1. Given two bounded domains $\Omega^i \subset \mathbb{R}^2$, $i \in \{-,+\}$, let Γ be the intersection of the domain boundaries, cf. Figure 2.1. We are then interested in solving

$$
\begin{aligned}
-\Delta u^i &= f \qquad \text{in } \Omega^i, i \in \{+,-\}, \\
u^i &= g^i \qquad \text{on } \Gamma^i_D = \partial\Omega^i \setminus \Gamma, \\
[\![u]\!] &= 0 \qquad \text{on } \Gamma, \\
-\Delta \hat{u} - [\![\nabla u \cdot n]\!] &= \hat{f} \qquad \text{on } \Gamma, \\
\hat{u} &= \hat{g} \qquad \text{on } \partial\Gamma, \\
u - \hat{u} &= h \qquad \text{on } \Gamma.
\end{aligned} \tag{2.1}
$$

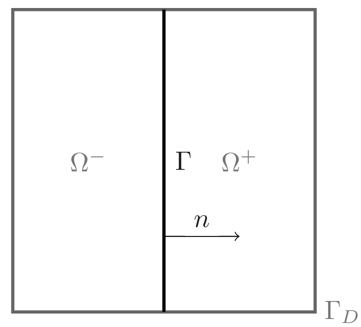

FIGURE 2.1. *Schematic domain of coupled multiscale problem.*

Let $\Omega = \Omega^+ \cup \Omega^-$ and $\Gamma_D = \Gamma^+_D \cup \Gamma^-_N$. Introducing a Lagrange multiplier $p = [\![\nabla u \cdot n]\!] \in Q = (H^{-1/2}_{00}(\Gamma))'$ the weak form of (2.1) reads: Find $u \in V = H^1_{0,\Gamma_D}(\Omega)$, $\hat{v} \in \hat{V} = H^1_{0,\partial\Gamma}(\Gamma)$, $p \in Q$ such that

$$
\begin{aligned}
\int_\Omega \nabla u \cdot \nabla v \qquad\qquad\qquad + \int_\Gamma pv &= \int_\Omega fv \quad \forall v \in V, \\
+\int_\Gamma \nabla \hat{u} \cdot \nabla \hat{v} - \int_\Gamma p\hat{v} &= \int_\Gamma \hat{f}\hat{v} \quad \forall \hat{v} \in \hat{V}, \\
\int_\Gamma uq \qquad - \int_\Gamma \hat{u}q \qquad\qquad &= \int_\Gamma hq \quad \forall q \in Q.
\end{aligned} \tag{2.2}
$$

In the following $\Omega = (0,1)$, $\Gamma = \left\{(x,y) \in \Omega; x = \frac{1}{2}\right\}$ and based on the properties of the finite element meshes

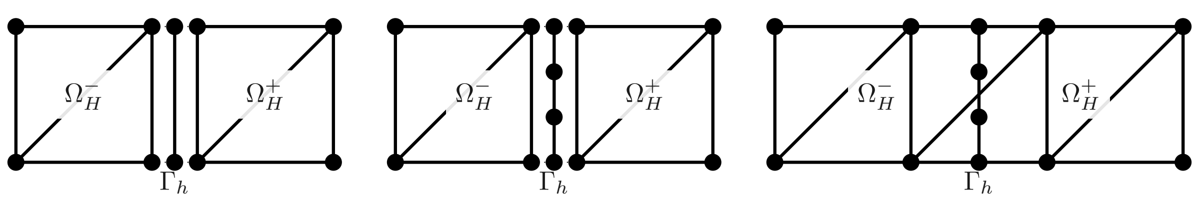

FIGURE 2.2. *Different mesh settings considered for the coupled problem (2.2): conforming matching, conforming non-matching, non-conforming (from left to right).*

Ω_H, Γ_h we shall distinguish three different settings, cf. Figure 2.2. Here, a conforming case implies that the trace mesh of Ω_H on Γ consists only of the existing facets of the mesh. In the conforming setting we consider

h	$\|u-u_H\|_1$	$\|\hat{u}-\hat{u}_h\|_1$	$\|p-p_h\|_0$	n	κ	$\|u-u_H\|_1$	$\|\hat{u}-\hat{u}_h\|_1$	$\|p-p_h\|_0$	n	κ
8.8E-2	8.8E-1(–)	2.0E0(–)	8.9E-1(–)	20	8.0	8.6E-1(–)	6.7E-1(–)	1.7E-1(–)	23	4.5
4.4E-2	4.4E-1(1.01)	1.0E0(0.99)	2.1E-1(2.10)	22	8.4	4.4E-1(0.99)	3.4E-1(1.00)	4.5E-2(1.89)	25	4.5
2.2E-2	2.2E-1(1.00)	5.0E-1(1.00)	5.1E-2(2.03)	22	8.5	2.2E-1(1.00)	1.7E-1(1.00)	1.2E-2(1.95)	25	4.6
1.1E-2	1.1E-1(1.00)	2.5E-1(1.00)	1.3E-2(2.01)	21	8.6	1.1E-1(1.00)	8.4E-2(1.00)	3.0E-3(1.98)	24	4.7
5.5E-3	5.5E-2(1.00)	1.3E-1(1.00)	3.2E-3(2.00)	21	8.7	5.5E-2(1.00)	4.2E-2(1.00)	7.5E-4(1.99)	24	4.7
2.8E-3	2.7E-2(1.00)	6.3E-2(1.00)	7.9E-4(2.00)	21	8.7	2.7E-2(1.00)	2.1E-2(1.00)	1.9E-4(1.99)	22	4.7
1.4E-3	1.4E-2(1.00)	3.1E-2(1.00)	2.0E-4(2.00)	21	–	1.4E-2(1.00)	1.0E-2(1.00)	4.7E-5(2.00)	22	–

TABLE 2.1

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) (κ) discretized with P1-P1-P1 elements. Conforming (left) matching and (right) non-matching case.

h	$\|u-u_h\|_1$	$\|\hat{u}-\hat{u}_h\|_1$	$\|p-p_h\|_0$	n	κ	$\|u-u_h\|_1$	$\|\hat{u}-\hat{u}_h\|_1$	$\|p-p_h\|_0$	n	κ
8.8E-2	9.3E-1(–)	2.0E0(–)	2.9E0(–)	21	4.5	8.6E-1(–)	6.7E-1(–)	9.2E-1(–)	29	7.8
4.4E-2	4.4E-1(1.06)	1.0E0(0.99)	1.1E0(1.43)	28	4.8	4.4E-1(0.99)	3.4E-1(1.00)	3.4E-1(1.42)	31	8.1
2.2E-2	2.2E-1(1.02)	5.0E-1(1.00)	4.1E-1(1.39)	28	5.0	2.2E-1(1.00)	1.7E-1(1.00)	1.4E-1(1.34)	32	8.3
1.1E-2	1.1E-1(1.00)	2.5E-1(1.00)	1.7E-1(1.28)	27	5.1	1.1E-1(1.00)	8.4E-2(1.00)	5.6E-2(1.26)	30	8.4
5.5E-3	5.5E-2(1.00)	1.3E-1(1.00)	7.5E-2(1.19)	27	5.2	5.5E-2(1.00)	4.2E-2(1.00)	2.5E-2(1.18)	30	8.5
2.8E-3	2.7E-2(1.00)	6.3E-2(1.00)	3.5E-2(1.11)	27	5.2	2.7E-2(1.00)	2.1E-2(1.00)	1.1E-2(1.11)	29	8.5
1.4E-3	1.4E-2(1.00)	3.1E-2(1.00)	1.7E-2(1.07)	27	–	1.4E-2(1.00)	1.0E-2(1.00)	5.5E-3(1.06)	29	–

TABLE 2.2

Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) (κ) discretized with P1-P1-P0 elements. Conforming (left) matching and (right) non-matching case.

matching and non-matching cases as defined above. We remark that the approximation of the space $\hat{V}$ shall always be constructed on the multiplier mesh Γ_h.

2.1. P1-P1-P1 discretization with Γ conformity. Let us first consider a conforming setting and an approximation of all the spaces involved in (2.2) in terms of P1 elements. As with the Babuška problem (1.1) the discrete multiplier space is such that $q = 0$ on $\partial\Gamma$ for all $q \in Q_h$. In fact, the boundary conditions are necessary to obtain an invertible linear system.

Table 2.1 summarizes the results for both matching and non-matching tessilations. In the latter case, a stabilization term [2] is added similar to (1.2) (we set $\gamma = 1$). In both cases the results show evidence of the inf-sup stability of the discretization. We note that the L^2 error of the multiplier converges quadratically; in contrast to the test problem in (1.1) the solution here *does* satisfy $p = 0$ on $\partial\Gamma$.

2.2. P1-P1-P0 discretization with Γ conformity. We shall next have the multiplier space be spanned by piecewise constant functions. Following (1.1) a stabilization [1] is employed in both the matching and non-matching cases. Table 2.2 confirms that the discretization is inf-sup stable.

2.3. Non-conforming setting. Finally a non-conforming setting is considered with P1-P1-P1 and P1-P1-P0 discretizations. In both cases stabilization terms are employed to obtain inf-sup stability, cf. Table 2.3. Note that compared to the conforming case the multiplier convergence with P1 elements is only linear. The order from the conforming case is preserved with P0 elements. Note also that with both elements u_h converges with order $\frac{1}{2}$; the approximation struggles to capture the kink of the solution which now occurs within the finite element cells.

3. Coupled multiscale problem in 3 mesh formulation. The discrete formulations thus far concerned only two meshes, Ω_H and Γ_h, and are in this sense similar to Problem 2 of the $3d$-$1d$ problem. Here we shall consider a $2d$ "analogue" of Problem 1. In particular the reduced domain and the coupling/Lagrange multiplier domain shall be different.

Let us recall that in Problem 1 (of the coupled $3d$-$1d$ problem) the coupling surface Γ isn't necessarily identical/related to the cylindrical surface whose centerline is the reduced domain Γ. We propose to take advantage of the fact that (to certain extent?) the surface may be chosen freely in the following construction which automatically satisfies the requirement of conformity of the triangulation of Ω to the multiplier surface.

Let Ω_H, Γ_h be the two *independent* discretizations of the bulk and the centerline. In particular, Γ_h is not assumed to consist only of the edges of the tetrahedra in Ω_H. However, we assume that the Γ_h is finer than Ω_H restricted to the line. Next, let $\mathcal{K}_H$ be the set of all tetrahedra in Ω_H intersected by elements of Γ_h. Finally, the coupling surface ω_H shall consist of those facets of elements in $\mathcal{K}_H$ which are *not* intersected by

h	$\|u-u_H\|_1$	$\|\hat{u}-\hat{u}_h\|_1$	$\|p-p_h\|_0$	n	κ	$\|u-u_H\|_1$	$\|\hat{u}-\hat{u}_h\|_1$	$\|p-p_h\|_0$	n	κ
8.3E-2	1.3E0(–)	7.0E-1(–)	1.2E0(–)	28	5.2	1.3E0(–)	7.0E-1(–)	1.7E0(–)	31	8.6
4.3E-2	8.1E-1(0.68)	3.5E-1(1.04)	6.4E-1(0.95)	29	5.6	8.1E-1(0.68)	3.5E-1(1.04)	7.6E-1(1.18)	34	9.3
2.2E-2	5.3E-1(0.62)	1.8E-1(1.02)	3.3E-1(0.99)	29	5.8	5.3E-1(0.62)	1.8E-1(1.02)	3.6E-1(1.10)	34	9.7
1.1E-2	3.6E-1(0.57)	8.8E-2(1.01)	1.6E-1(1.00)	27	6.0	3.6E-1(0.57)	8.8E-2(1.01)	1.8E-1(1.05)	34	9.9
5.5E-3	2.5E-1(0.54)	4.4E-2(1.01)	8.3E-2(1.00)	27	6.0	2.5E-1(0.54)	4.4E-2(1.01)	8.7E-2(1.03)	33	10.0
2.8E-3	1.7E-1(0.52)	2.2E-2(1.00)	4.1E-2(1.00)	26	6.1	1.7E-1(0.52)	2.2E-2(1.00)	4.3E-2(1.01)	32	10.0
1.4E-3	1.2E-1(0.51)	1.1E-2(1.00)	2.1E-2(1.00)	26	–	1.2E-1(0.51)	1.1E-2(1.00)	2.1E-2(1.01)	31	10.0

TABLE 2.3

Non-conforming setting. Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) (κ) discretized with P1-P1-P1 (left) and P1-P1-P0 (right) elements.

the curve. The process in the $2d$ setting of Figure 2.1 is shown in Figure 3.1. We remark that the coupling surface changes with discretization.

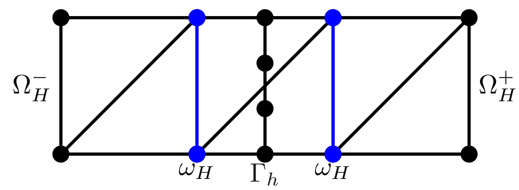

FIGURE 3.1. *Coupling surface ω_H consists of facets of elements of Ω_H not intersected by Γ_h.*

To investigate the idea we consider the multiscale problem (2.2). Note that its left hand side defines an operator

$$\begin{pmatrix} A & & T' \\ & \hat{A} & -I \\ T & -I & 0 \end{pmatrix},$$

where A, $\hat{A}$ are the Laplacians on Ω and Γ respectively and T, I are the trace and identity operators. Let now $V_H = H^1_0(\Omega)$, $\hat{V}_h = H^1_0(\Gamma_h)$, $Q = (H^{-1/2}_{00}(\omega_H))'$. We shall consider a modified problem defined by operator

$$\begin{pmatrix} A & & T' \\ & \hat{A} & -E'_\omega \\ T_\omega & -E_\omega & 0 \end{pmatrix}, \tag{3.1}$$

with $T_\omega : V \to Q'$ (trace) and $E_\omega : \hat{V} \to Q'$ (extension)

$$\langle T_\Omega u, q\rangle = \int_{\omega_H} uq, \qquad \langle E_\Omega \hat{u}, q\rangle = \int_{\omega_H} \hat{u}q.$$

In the following experiments we investigate stability and approximation properties of (3.1). To this end we consider a Riesz map preconditioner based on spaces V, $\hat{V}$ and Q and the manufactured test cased from §2. The spaces are all discretized with P1 elements and we have $H = 3h$. As before, we enforce boundary conditions on Q. The extension operator is defined as extension by constant from the nearest degree of freedom of $\hat{V}$.

Table 3.1 summarizes the preliminary results. With the exception of the multiplier convergence in all the variables can be seen. However, we recall that the true solution is here defined as $[\![\nabla u \cdot n]\!]$ on Γ while the discrete one lives on ω_H so there is at the very least some geometrical error. Observing the condition numbers we also see that the formulation is unstable.

3.1. TODO.

- Build (3.1) for just half of the domain, so ω_H is connected
- How is this problem related $2d$-$2d$-$2d$ problem with "thin" (middle) domain which becomes Γ?
- What are then the approximation properties?

h	$\|u - u_H\|_1$	$\|\hat{u} - \hat{u}_h\|_1$	$\|p - p_h\|_0$	n	κ
8.58E-2	1.49E0(–)	6.74E-1(–)	7.22E0(–)	31	13
4.35E-2	9.89E-1(0.60)	3.46E-1(0.98)	7.59E0(-0.07)	40	22
2.19E-2	6.66E-1(0.58)	1.74E-1(1.00)	7.37E0(0.04)	48	41
1.10E-2	4.57E-1(0.55)	8.75E-2(1.00)	7.25E0(0.02)	58	79
5.51E-3	3.17E-1(0.53)	4.37E-2(1.00)	7.14E0(0.02)	75	154
2.76E-3	2.22E-1(0.52)	2.18E-2(1.00)	7.09E0(0.01)	75	304
1.38E-3	1.56E-1(0.51)	1.09E-2(1.00)	7.06E0(0.01)	77	–

TABLE 3.1

Three mesh formulation. Approximation errors, number of preconditioned MinRes iterations (n) and condition number of the preconditioned problem (2.2) *(κ) discretized with P1-P1-P1 elements.*

- Stabilization

REFERENCES

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