Projection Stabilization of Lagrange Multipliers for the Imposition of Constraints on Interfaces and Boundaries

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Projection stabilization applied to general Lagrange multiplier finite element methods is introduced and analyzed in an abstract framework. We then consider some applications of the stabilized methods: (i) the weak imposition of boundary conditions, (ii) multiphysics coupling on unfitted meshes, (iii) a new interpretation of the classical residual stabilized Lagrange multiplier method introduced in Barbosa and Hughes, Comput Methods Appl Mech Eng 85 (1991), 109–128. 30: 567–592, 2014

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I. INTRODUCTION

The use of Lagrange multipliers to impose constraints in the finite element method is a well-known and powerful technique. To obtain a stable method, the finite element spaces for the primal variable and the multiplier must be carefully matched so as to satisfy an inf-sup condition uniformly in the mesh parameter (see Babuska [1], Brezzi [2], Pitkäranta [3, 4]). If an unstable pair is used, stability can be recovered using a stabilized method [5, 6].

In many cases such as when imposing incompressibility for flow problems, there are several choices available, both to design inf-sup stable velocity-pressure pairs (see for instance [7]) and to design stabilized methods for pairs that do not satisfy the inf-sup condition. A class of method that has been particularly successful recently are projection stabilization methods. Loosely speaking such methods ensure stability by adding a term that penalizes the difference between the pressure solution and its projection onto some inf-sup stable space [8–11].

Recently, there has been renewed interest in Lagrange multiplier method in the context of imposing constraints on embedded boundaries and multiscale or multiphysics coupling problems

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[12–16]. Also, here care must be taken to chose pairs of finite element spaces that satisfy the appropriate inf-sup condition, in order to avoid spurious oscillations or locking.

In some of these cases, although the choice of stable space is known, it may be inconvenient. Either the spaces may be very complicated to design or use from an implementation point of view, or the multiplier space simply is too small to give sufficient control of the constraint. Here, the state of the art method for stabilization is the residual-based formulation introduced by Barbosa and Hughes [5]. This method has been shown to be closely related to Nitsche's method, in cases where the Lagrange multiplier can be eliminated locally [17]. It can also be applied for interface coupling proplems, with a large flexibility in the choice of multiplier space, see for instance [18].

It appears that the idea of projection stabilization, that has been very successful for Stokes' problem, has not yet been exploited to its full potential in the context of other type of problems featuring Lagrange multipliers. However, it appears that such an approach can give certain advantages.

- For domain decomposition with nonmatching meshes, it allows for the use of a Lagrange multiplier that is defined on a third mesh which can be chosen arbitrarily (typically structured). In this case, the stabilization operator only acts on the multiplier space, see [19]. This reduces the problem of interpolating between two fully unstructured meshes to that of interpolating from two unstructured meshes to one structured mesh.
- Another example is fictitious domain methods where the multiplier can be chosen piecewise constant per element and distributed in the interface zone if projection stabilization is used [20]. This choice is advantageous from the point of view of implementation, but normally prohibited as the inf-sup condition fails [21].
- Compared to Nitsche type methods or the Barbosa–Hughes stabilized method the projection stabilized multiplier method does not use the trace of the stress tensor explicitly. This is particularly advantageous in the nonlinear case, as the nonlinearity then appears only in the bulk and not in the interface terms.

Stabilized Lagrange methods seem to be attracting increasing attention, in particular for the imposition of embedded Dirichlet boundary conditions [13, 20, 22–24]. It is interesting to note that the extension to extended finite element (XFEM) type interface coupling methods is practically always straightforward.

The focus of the present article is on the generality of this type of method. We prove a well-posedness result for discrete solutions and a best approximation result in an abstract framework. Then, we show how to apply the ideas to the analysis and design of stabilized Lagrange multiplicator methods first in the simple case of the weak imposition of boundary conditions and then sketching an unfitted finite element method for multiphysics coupling.

As a last example of the applicability of our framework, we give a new interpretation of the nonsymmetric version of the method of Barbosa and Hughes [5], for the imposition of boundary conditions. In these methods, the stabilization acts on the difference between the multiplier and the gradient of the primal variable. Using a recent stability result for the penalty-free, nonsymmetric Nitsche's method [25], we show that the nonsymmetric version of such stabilized Lagrange multiplier methods are in fact closely related to projection stabilization methods by the inf-sup stability of the Lagrange multiplier space consisting of normal gradients of the primal variable on the boundary trace mesh.

As a model problem, the reader may consider the Poisson problem set on an open connected domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, with polygonal (or polyhedral) boundary. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The weak formulation of this problem, using Lagrange multipliers to impose the boundary constraints, takes the following form: find $(u, \lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ such that

$$\begin{gathered}\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \lambda v ds + \int_{\partial\Omega} \mu u ds = \int_\Omega f v dx \\ \forall (v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).\end{gathered} \tag{1.2}$$

We will frequently use the notation $a \lesssim b$ for $a \leq Cb$ where C is a constant independent of the mesh-size, but not necessarily of the local mesh geometry. We also assume quasiuniformity and shape regularity for all meshes.

II. ABSTRACT SETTING

We will here give an abstract framework for this type of method to give some understanding of the underlying idea. Our aim is to make the simplest possible framework. Let

$$a(\cdot, \cdot) : V \times V \to \mathbb{R}$$

and

$$b(\cdot, \cdot) : L \times V \to \mathbb{R}$$

be two bilinear forms representing the partial differential operator on weak form and the constraint, respectively. The abstract formulation then writes: find $(u, \lambda) \in V \times L$ such that

$$a(u, v) + b(\lambda, v) + b(\mu, u) = (f, v) \tag{2.1}$$

for all $(v, \mu) \in V \times L$. We assume that the spaces V and L are chosen such that the problem is well posed. First, we assume that the bilinear forms satisfy the following continuities

$$\begin{aligned} a(u, v) &\lesssim ||u||_V ||v||_V, \quad \forall u, v \in V \\ b(\lambda, v) &\lesssim ||\lambda||_L ||v||_V, \quad \forall \lambda \in L \text{ and } \forall u \in V \end{aligned}$$

and secondly that the form $a(u, v)$ is coercive on the kernel of $b(\lambda, v)$, that is,

$$||v||_V^2 \lesssim a(v, v), \text{ for all } v \text{ such that } b(\mu, v) = 0, \quad \forall \mu \in L.$$

Finally, we assume that the Babuska–Brezzi condition is satisfied so that $\forall \lambda \in L$ there holds

$$||\lambda||_L \lesssim \sup_{v \in V} \frac{b(\lambda, v)}{||v||_V}.$$

Example 1. In the case of the Poisson problem (1.1) above the bilinear forms are given by the weak formulation (1.2) as

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v dx, \tag{2.2}$$

and

$$b(\lambda, v) := \int_{\partial\Omega} \lambda v ds. \tag{2.3}$$

The spaces are given by $V := H^1(\Omega)$ and $L := H^{-\frac{1}{2}}(\partial\Omega)$.

Now, consider the discretization of the problem (2.1) in $V_h \subset V, L_h \subset L$. We assume that these spaces satisfy the discrete version of the inf-sup condition uniformly so that $\forall \lambda_h \in L_h$ there holds

$$||\lambda_h||_L \lesssim \sup_{v_h \in V_h} \frac{b(\lambda_h, v_h)}{||v_h||_V}. \tag{2.4}$$

It is known [26, 27] that the discrete inf-sup condition is equivalent to the existence of an interpolant $\pi_F : V \to V_h$ such that for any $v \in V$ there holds

$$b(v - \pi_F v, \mu_h) = 0 \quad \forall \mu_h \in L_h, \quad \text{and} \quad ||\pi_F v||_V \lesssim ||v||_V. \tag{2.5}$$

We introduce norms defined on functions in the discrete spaces $||\cdot||_{L_h}$ and $||\cdot||_{V_h}$ and assume that the bilinear forms also satisfy the following continuities,

$$a(u_h, v_h) \leq ||u_h||_{V_h} ||v_h||_{V_h}, \quad \forall u_h, v_h \in V_h$$
$$b(\lambda_h, v_h) \leq ||\lambda_h||_{L_h} ||v_h||_{V_h}, \quad \forall \lambda_h \in L_h \text{ and } \forall u_h \in V_h.$$

We will also assume that $||v_h||_V \lesssim ||v_h||_{V_h}$ for all $v_h \in V_h$.

Example 2 For $V_h \subset H^1(\Omega)$ and $L_h \subset H^{-\frac{1}{2}}(\partial\Omega) \cap L^2(\partial\Omega))$, we may take

$$||\mu_h||_{L_h} := ||h^{\frac{1}{2}} \mu_h||_{L^2(\partial\Omega)}$$

and

$$||v_h||_{V_h} := ||\nabla v_h||_{L^2(\Omega)} + ||h^{-\frac{1}{2}} v_h||_{L^2(\partial\Omega)}.$$

It follows immediately by the Cauchy–Schwarz inequality that the following continuities hold

$$a(u_h, v_h) \lesssim ||u_h||_{V_h} ||v_h||_{V_h} \quad \forall u_h, v_h \in V_h \tag{2.6}$$

and

$$b(\lambda_h, v_h) \lesssim ||\lambda_h||_{L_h} ||v_h||_{V_h} \quad \forall \lambda_h \in L_h, \forall v_h \in V_h. \tag{2.7}$$

This leads to the following formulation: find $\{u_h, \lambda_h\} \in V_h \times L_h$ such that

$$a(u_h, v_h) + b(\lambda_h, v_h) - b(u_h, \mu_h) = (f, v_h), \quad \forall \{v_h, \mu_h\} \in V_h \times L_h. \tag{2.8}$$

Then, we know that the discrete problem is well posed and we may prove optimal convergence provided the spaces have optimal approximation properties. We will denote the kernel of $b(\cdot, \cdot)$ by

$$K_h := \{v_h \in V_h : b(\mu_h, v_h) = 0, \forall \mu_h \in L_h\}.$$

Consider now the case where we do not want to use the space L_h because it leads to inconvenient interpolation problems. We want to work with the possibly completely unrelated, richer, space $\Lambda_h \subset L$, for which no stability is known to hold, but which is convenient from the point of view of implementation. We also assume that there exists a projection $\pi_L : \Lambda_h \to L_h$ so that the following continuity holds for all $v \in V$,

$$b(\lambda_h - \pi_L \lambda_h, v) \lesssim ||\lambda_h - \pi_L \lambda_h||_{L_h} ||v||_V. \tag{2.9}$$

This is a technical assumption that only constrains the choice of π_L used in the analysis and not in practice, as we shall see later. When the Fortin interpolant is used for the analysis as we do here this assumption is convenient as otherwise one must work in the norm $||\cdot||_L$ when designing the stabilization term. Under (2.9), one may use the discrete norm directly. An alternative route for the analysis is to use a discrete inf-sup condition in the discrete norm and associated analysis.

Instead of (2.4), we then have the following stability property.

Lemma 2.1 *For all $\lambda_h \in \Lambda_h$, there holds*

$$||\lambda_h||_L \lesssim \sup_{v_h \in V_h} \frac{b(\lambda_h, v_h)}{||v_h||_V} + ||\lambda_h - \pi_L \lambda_h||_{L_h},$$

where $\pi_L : \Lambda_h \to L_h$ denotes an interpolation operator from Λ_h to L_h such that (2.9) holds.

Proof. By the continuous inf-sup condition there holds for all $\lambda_h \in \Lambda_h$,

$$||\lambda_h||_L \lesssim \sup_{v \in V} \frac{b(\lambda_h, v)}{||v||_V}.$$

As $\pi_L \lambda_h \in L_h$, the condition (2.4) holds and hence by (2.9)

$$||\lambda_h||_L \lesssim \sup_{v \in V} \frac{b(\lambda_h - \pi_L \lambda_h, v) + b(\pi_L \lambda_h, \pi_F v)}{||v||_V} \lesssim ||\lambda_h - \pi_L \lambda_h||_{L_h} + \frac{b(\pi_L \lambda_h, \pi_F v)}{||\pi_F v||_V}.$$

We may then add and subtract λ_h in the last term in the right-hand side to obtain using (2.9)

$$\frac{b(\pi_L \lambda_h, \pi_F v)}{||\pi_F v||_V} = \frac{b(\lambda_h, \pi_F v) + b(\pi_L \lambda_h - \lambda_h, \pi_F v)}{||\pi_F v||_V} \lesssim ||\lambda_h - \pi_L \lambda_h||_{L_h} + \sup_{v_h \in V_h} \frac{b(\lambda_h, v_h)}{||v_h||_V}.$$

This means that, provided that we can control the distance $||\lambda_h - \pi_L \lambda_h||_{L_h}$ from the approximation in the space Λ_h to the space L_h, which satisfies the Babuska-Brezzi condition, we will have stability using the space Λ_h. The simplest way of obtaining this is to add a symmetric operator $s(\lambda_h, \mu_h)$, designed so that

$$||\lambda_h - \pi_L \lambda_h||^2_{L_h} \lesssim s(\lambda_h, \lambda_h) \tag{2.10}$$

to the formulation (2.8). Since the effect of $s(\cdot, \cdot)$ is to reduce the effective dimension of the space Λ_h it can be thought of as a coarsening operator.

This leads to the stabilized formulation:

$$a(u_h, v_h) + b(\lambda_h, v_h) + b(\mu_h, u_h) - s(\lambda_h, \mu_h) = (f, v_h)$$
$$\text{for all } (v_h, \mu_h) \in V_h \times \Lambda_h. \tag{2.11}$$

The signs in (2.11) have been chosen so as to preserve symmetry, note, however, that the problem is indefinite due to the saddle point structure. For the operator $s(\cdot,\cdot)$, the following design criteria are advantageous:

- minimal dependence of the stable subspace L_h
- the smallest possible stencil
- optimal weak consistency.

Often $s(\cdot,\cdot)$ may be chosen as the jump of the function or of function derivatives over element faces in the multiplier space and we will explore this possibility further below.

When we work with the multiplier space Λ_h, it is no longer sufficient to assume that $a(u_h, v_h)$ is coercive on the kernel K_h of $b(\lambda_h, v_h)$, for $\lambda_h \in \Lambda_h$. Indeed, the stabilization term could upset the coercivity. To ensure that the constraint remains strong enough compared to the penalty term we assume that for all $u_h \in V_h$ there exists $\xi_h(u_h) \in \Lambda_h$ such that

$$\begin{aligned}\alpha_\xi \|u_h\|_{V_h}^2 &\le a(u_h, u_h) + b(\xi_h(u_h), u_h)\\ s(\xi_h(u_h), \xi_h(u_h))^{\frac{1}{2}} &\le c_s \|u_h\|_{V_h},\end{aligned} \tag{2.12}$$

where c_s can be made small by choosing the stabilization parameter small. $\xi_h(u_h)$ is related to the constraint that one wishes to impose. For the case of weak boundary conditions typically $\xi_h(u_h)$ is the projection of the trace of u_h onto the Lagrange multiplier space as we shall see later. We first state and prove the obtained coercivity result in a lemma and then conclude this section by our main theorem, showing a best approximation property for the formulation (2.11). ■

Lemma 2.2 *For all* $\{u_h, \lambda_h\} \in V_h \times L_h$ *there holds*

$$\|u_h\|_{V_h}^2 + s(\lambda_h, \lambda_h) \lesssim a(u_h, u_h) + b(\lambda_h, u_h) - b(\lambda_h - \xi_h(u_h), u_h) + s(\lambda_h, \lambda_h - \xi_h(u_h)). \tag{2.13}$$

Proof. Starting from the right-hand side of (2.13) we have using (2.12) and an arithmetic-geometric inequality

$$\begin{aligned}&a(u_h, u_h) + b(\lambda_h, u_h) - b(\lambda_h - \xi_h(u_h), u_h) + s(\lambda_h, \lambda_h - \xi_h(u_h))\\ &\quad\ge \alpha_\xi \|u_h\|_{V_h}^2 + \frac{1}{2} s(\lambda_h, \lambda_h) - \frac{1}{2} s(\xi_h(u_h), \xi_h(u_h)).\end{aligned}$$

Using now the second inequality of (2.12) we may conclude, assuming c_s small enough. ■

Remark. If $\xi_h(u_h)$ may be chosen such that $s(\xi_h(u_h), v_h) = 0$, $\forall v_h \in \Lambda_h$ then (2.13) holds without constraints on c_s.

Existence and uniqueness of a solution $\{u_h, \lambda_h\}$ of the system (2.11) as well as the best approximation result are consequences of the Lemmas 2.2 and 2.1 as we now show.

Lemma 2.3 *Assume that the coercivity condition (2.12) holds for* $V_h \times \Lambda_h$ *and that there exists a space* L_h *such that the condition (2.4) holds for the pair* $V_h \times L_h$*. Then, the system (2.11) admits a unique solution* $\{u_h, \lambda_h\}$.

Proof. As the left-hand side of (2.11) represents a square linear system, it is sufficient to show that $f = 0$ in (2.11) implies $\{u_h, \lambda_h\} \equiv \{0, 0\}$. By Lemma 2.2 and Eq. (2.11), we have

$$||u_h||^2_{V_h} + s(\lambda_h, \lambda_h) = 0 \tag{2.14}$$

by which we conclude that $u_h = 0$. Once again by (2.11) we then have $b(\lambda_h, v_h) = 0$ for all $v_h \in V_h$, which we may use together with (2.14) and (2.10) in (2.4) to conclude that $\lambda_h = 0$. ∎

Theorem 2.4 *Assume that the coercivity condition (2.12) holds for $V_h \times \Lambda_h$ and that there exists a space L_h such that the condition (2.4) holds for the pair $V_h \times L_h$. Then, the solution $\{u_h, \lambda_h\}$ of the system (2.11) satisfies the following best approximation property*

$$||u - u_h||_V + ||\lambda - \lambda_h||_L \lesssim \inf_{y_h \in V_h} ||u - y_h||_V + \inf_{v_h \in \Lambda_h} \left(||\lambda - v_h||_L + s(v_h, v_h)^{\frac{1}{2}}\right).$$

Proof. Assume that u_h and λ_h exist. Now by the triangular inequality

$$||u - u_h||_V \leq ||u - \pi_F u||_V + ||\pi_F u - u_h||_{V_h},$$

where π_F is the Fortin interpolant associated to the spaces $V_h \times L_h$. Set $\eta_h = u_h - \pi_F u$ and $\zeta_h = \lambda_h - v_h$. By Lemma 2.2, we have

$$\begin{aligned} &||\eta_h||^2_{V_h} + s(\zeta_h, \zeta_h) \\ &\lesssim a(\eta_h, \eta_h) + b(\zeta_h, \eta_h) - b(\zeta_h - \xi_h(\eta_h), \eta_h) + s(\zeta_h, \zeta_h - \xi_h(\eta_h)). \end{aligned} \tag{2.15}$$

Subtracting (2.11) from (2.1) with $v = v_h$, $\mu = \mu_h$, gives the Galerkin orthogonality

$$a(u - u_h, v_h) + b(\lambda - \lambda_h, v_h) + b(\mu_h, u - u_h) + s(\lambda_h, \mu_h) = 0. \tag{2.16}$$

Taking $v_h = \eta_h$ and $\mu_h = -(\zeta_h - \xi_h(\eta_h))$ in (2.16) and adding the left-hand side of (2.16) to the right-hand side of (2.15) yields

$$\begin{aligned} ||\eta_h||^2_{V_h} + s(\zeta_h, \zeta_h) &\lesssim a(u - \pi_F u, \eta_h) + b(\lambda - v_h, \eta_h) \\ &-b(\zeta_h - \xi_h(\eta_h), u - \pi_F u) + s(v_h, \zeta_h - \xi_h(\eta_h)). \end{aligned} \tag{2.17}$$

Since $b(\mu_h, u - \pi_F u) = 0$ for all $\mu_h \in L_h$ there holds

$$\begin{aligned} ||\eta_h||^2_{V_h} + s(\zeta_h, \zeta_h) &\lesssim a(u - \pi_F u, \eta_h) + b(\lambda - v_h, \eta_h) + b(\xi_h(\eta_h) - \pi_L \xi_h(\eta_h), u - \pi_F u) \\ &-b(\zeta_h - \pi_L \zeta_h, u - \pi_F u) + s(v_h, \zeta_h - \xi_h(\eta_h)). \end{aligned}$$

Using the continuity (2.9), we have

$$\begin{aligned} &b(\xi_h(\eta_h) - \pi_L \xi_h(\eta_h), u - \pi_F u) - b(\zeta_h - \pi_L \zeta_h, u - \pi_F u) \\ &\lesssim \left(||\xi_h(\eta_h) - \pi_L \xi_h(\eta_h)||_{L_h} + ||\zeta_h - \pi_L \zeta_h||_{L_h}\right) ||u - \pi_F u||_V \end{aligned}$$

and together with the continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ and the bound $||\eta_h||_V \lesssim ||\eta_h||_{V_h}$ this leads to

$$||\eta_h||^2_{V_h} + s(\zeta_h,\zeta_h) \lesssim (||u-\pi_F u||_V + ||\lambda - \nu_h||_L)\,||\eta_h||_{V_h}$$
$$+||u-\pi_F u||_V \left(||\xi_h(\eta_h) - \pi_L \xi_h(\eta_h)||_{L_h} + ||\zeta_h - \pi_L\zeta_h||_{L_h}\right)$$
$$+s(\nu_h,\nu_h)^{\frac{1}{2}}\left(s(\zeta_h,\zeta_h)^{\frac{1}{2}} + s(\xi_h(\eta_h),\xi_h(\eta_h))^{\frac{1}{2}}\right). \tag{2.18}$$

Using the upper bound $||\zeta_h - \pi_L\zeta_h||^2_{L_h} \lesssim s(\zeta_h,\zeta_h)$ of (2.10) and (2.10) combined with the second relation of (2.12) to obtain

$$||\xi_h(\eta_h) - \pi_L\xi_h(\eta_h)||_{L_h} \lesssim s(\xi_h(\eta_h),\xi_h(\eta_h))^{\frac{1}{2}} \lesssim ||\eta_h||_{V_h}$$

we observe that

$$||\eta_h||^2_{V_h} + s(\zeta_h,\zeta_h) \lesssim \left(||u-\pi_F u||_V + ||\lambda-\nu_h||_L + s(\nu_h,\nu_h)^{\frac{1}{2}}\right)$$
$$\times\left(||\eta_h||^2_{V_h} + s(\zeta_h,\zeta_h)\right)^{\frac{1}{2}}. \tag{2.19}$$

This gives the following upper bound for $||\eta_h||_{V_h}$

$$||\eta_h||_{V_h} + s(\zeta_h,\zeta_h)^{\frac{1}{2}} \lesssim ||u-\pi_F u||_V + \inf_{\nu_h\in\Lambda_h}\left(||\lambda-\nu_h||_L + s(\nu_h,\nu_h)^{\frac{1}{2}}\right).$$

By the stability of π_F we have, for $v_h \in V_h$

$$||u-\pi_F u||_V \le ||u-v_h||_V + ||v_h - \pi_F u||_V$$
$$= ||u-v_h||_V + ||\pi_F(v_h-u)||_V \lesssim ||u-v_h||_V. \tag{2.20}$$

We conclude that

$$||u-u_h||_V + s(\zeta_h,\zeta_h)^{\frac{1}{2}} \lesssim \inf_{v_h\in V_h}||u-v_h||_V + \inf_{\nu_h\in\Lambda_h}\left(||\lambda-\nu_h||_L + s(\nu_h,\nu_h)^{\frac{1}{2}}\right).$$

For the bound on $\lambda - \lambda_h$, we use the triangle inequality to write

$$||\lambda-\lambda_h||_L \le ||\lambda-\nu_h||_L + ||\zeta_h||_L$$

followed by the result of Lemma 2.1:

$$||\zeta_h||_L \lesssim \sup_{v_h\in V_h}\frac{b(\zeta_h,v_h)}{||v_h||_V} + ||\zeta_h - \pi_L\zeta_h||_{L_h} \lesssim \sup_{v_h\in V_h}\frac{b(\zeta_h,v_h)}{||v_h||_V} + s(\zeta_h,\zeta_h)^{\frac{1}{2}}.$$

As we already have the desired bound for the stabilization term, we only need to consider the first term of the right-hand side. By the Galerkin orthogonality (2.16), with $\mu_h = 0$ and the continuities of the bilinear forms we have

$$b(\zeta_h,v_h) = b(\lambda-\nu_h,v_h) + a(u-u_h,v_h) \lesssim (||\lambda-\nu_h||_L + ||u-u_h||_V)\,||v_h||_V.$$

We deduce the upper bound on $||\zeta_h||_L$,

$$||\zeta_h||_L \lesssim ||\lambda-\nu_h||_L + ||u-u_h||_V + s(\zeta_h,\zeta_h)^{\frac{1}{2}}. \tag{2.21}$$

This concludes the best approximation result. ∎

III. STABILIZATION USING JUMP PENALTY OPERATORS

The design of the stabilization operator $s(\cdot,\cdot)$ is important, indeed if the construction of the projection π_L requires a too detailed understanding of the inf-sup stable space L_h the advantages of the stabilized method may be lost. Typically, this is the case if π_L is chosen to be the L^2-projection. Fortunately, there are some operators that can handle a relatively large set of spaces $\{L_h, \Lambda_h\}$. The two natural choices are local projection stabilization or interior penalty stabilization. Herein, we will only discuss the second choice. For examples of local projection stabilization methods that can be used in this context we refer to [24, 25, 28] where such methods have been proposed in a different context. The extension to the present case is straightforward. Below, we will focus on the construction relevant for weak imposition of boundary conditions. The extension to domain decomposition is straightforward. We assume that $b(\cdot,\cdot)$ is defined by (2.2).

We consider only one side Γ of the polygonal boundary $\partial\Omega$. Denote the trace mesh of V_h by Γ_V. Let the space L_h be defined on a trace mesh Γ_L,

$$L_h := \left\{ l_h \in L^2(\Gamma_L) : l_h|_K \in P_k(K), \forall K \in \Gamma_L \right\},$$

and Λ_h on a trace mesh Γ_Λ,

$$\Lambda_h := \left\{ \lambda_h \in L^2(\Gamma_\Lambda) : \lambda_h|_K \in P_k(K), \forall K \in \Gamma_\Lambda \right\}.$$

The mesh function on the trace meshes will be denoted $h_{\Gamma,X}$, with $X = V, L$, or Λ. We assume that there are positive constants c_1, c_2, and c_3 such that

$$h_{\Gamma,\Lambda}(x) \leq c_1 h_{\Gamma,V}(x) \leq c_2 h_{\Gamma,L}(x) \leq c_3 h_{\Gamma,\Lambda}(x), \quad \text{for all } x \in \Gamma.$$

We first note that using these spaces it is straightforward to design π_L so that (2.9) holds, the only requirement is orthogonality against constants on the elements of Γ_L. Indeed, if π_L is chosen as the L^2-projection on L_h it follows from the definition of $||\cdot||_{L_h}$ that it can be replaced by any interpolant in L_h using the stability of the L^2-projection and the quasiuniformity constraint on the mesh parameter

$$||\lambda_h - \pi_L\lambda_h||^2_{L_h} \leq \sum_{K\in\Gamma_L} c_2 h_{\Gamma,L}|_K ||\lambda_h - \pi_L\lambda_h||^2_{L^2(K)} = \inf_{v_h\in L_h} \sum_{K\in\Gamma_L} c_2 h_{\Gamma,L}|_K ||\lambda_h - v_h||^2_{L^2(K)}.$$

It follows that v_h may be chosen as any interpolation of λ_h in L_h. For imposition of boundary conditions and more generally for domain decomposition methods, the classical condition for inf-sup stability is that $h_{\Gamma,L} > C h_{\Gamma,V}$ for some constant $C > 1$, (see [1]). Using the projection stabilization, this condition may be relaxed for the space Λ_h, as the stabilization controls the unstable modes. The relative difference in mesh size should be accounted for in the stabilization parameter to tune the constant of the inf-sup condition. Numerical evidence, however, indicate that this dependence is relatively weak. Assume for simplicity that $h_{\Gamma,\Lambda} < h_{\Gamma,L}$. Let the interpolation operator $\tilde{\pi}_L : \Lambda_h \to L_h$ denote the quasi interpolation operator such that for $u_h \in \Lambda_h$

$$\tilde{\pi}_L u_h(x_i) := N_x^{-1} \sum_{\{K\in\Gamma_\Lambda : x_i\in K\}} u_h(x_i)|_K, \quad \text{for all nodes } x_i \text{ of } \Gamma_L,$$

where N_x denotes the cardinality of the set $\{K \in \Gamma_\Lambda : x \in K\}$. Now, consider any element in the trace mesh $K_\Gamma \in \Gamma_L$ and map it to a reference element $\hat{K}_{\Gamma_L}$. Also, map the subset Γ'_Λ for which $\Gamma'_\Lambda := \left\{K' : K' \cap K_\Gamma \neq \emptyset\right\}$ and denote the interior faces of $\hat{\Gamma}'_\Lambda$ by $\mathcal{F}'$. It then follows that

$$||\hat{\lambda}_h - \tilde{\pi}_L\hat{\lambda}_h||_{L_h,\hat{K}_\Gamma} \leq \sum_{\hat{F}\in\mathcal{F}'}\sum_{i=0}^{k}\int_{\hat{F}} [\![\hat{\partial}_n^i\hat{\lambda}_h]\!]^2 \mathrm{d}\hat{s},$$

where $[[x]]$ denotes the jump of the quantity x over an element face, with $[[x]] = 0$ for faces on the boundary and ∂_n^i denotes the normal derivative of order i, with n the outward pointing normal from the element K' and with ∂_n^0, defined to be the identity.

This upper bound on the reference element follows by the observation that if the jump of $\hat{\lambda}_h$ and all its normal derivatives are zero, then $\hat{\lambda}_h$ is a polynomial over all of $\hat{K}_\Gamma$, but as $\tilde{\pi}_L$ interpolates this polynomial $\left(\hat{\lambda}_h - \tilde{\pi}_L\hat{\lambda}_h\right)|_{\hat{K}_\Gamma} \equiv 0$. To show that $\hat{\lambda}_h$ is a polynomial over all of $\hat{K}_\Gamma$, it is enough to consider one face $\hat{F}$ and the associated elements such that $\hat{F} = \hat{K}_1 \cap \hat{K}_2$. We choose the coordinate system so that $\hat{F} \subset \left\{\left(\hat{x},\hat{y}\right) : \hat{y} = 0\right\}$. We let $\hat{p}_i\left(\hat{x},\hat{y}\right) = \hat{\lambda}_h|_{\hat{K}_i}$ and define the polynomial $\delta p_F\left(\hat{x},\hat{y}\right) = \hat{p}_1 - \hat{p}_2$ on $\hat{K}_1 \cup \hat{K}_2$. We must then show that

$$\left.\begin{array}{r} \delta p_F\left(\hat{x},\hat{y}\right)|_{y=0} = 0, \forall \hat{x} \in \hat{F} \\ \partial_{\hat{y}}^i \delta p_F\left(\hat{x},\hat{y}\right)|_{\hat{y}=0} = 0, i = 1,...,k, \forall \hat{x} \in \hat{F} \end{array}\right\} \to \delta p_F\left(\hat{x},\hat{y}\right) \equiv 0.$$

This is straightforward by noting that a polynomial of order k has $(k+1)(k+2)/2$ degrees of freedom and that $\delta p_F\left(\hat{x},\hat{y}\right)|_{y=0} = 0$ implies $k+1$ independent equations and that for each $i \in \{1,...,k\}$ the relation $\partial_{\hat{y}}^i \delta p_F\left(\hat{x},\hat{y}\right)|_{y=0} = 0$ gives $k - i + 1$ independent equations. Summing up the independent equations, we get

$$k + 1 + \sum_{i=1}^{k}(k - i + 1) = \frac{(k+1)(k+2)}{2}$$

and we conclude that $\delta p_F\left(\hat{x},\hat{y}\right) \equiv 0$. It follows that $\hat{\lambda}_h$ is defined by one polynomial over $\hat{K}_1 \cup \hat{K}_2$. The result on $\hat{K}_\Gamma$ is obtained by repeating the argument for all faces $\hat{F} \in \mathcal{F}'$.

After scaling back to physical space and summing over all elements in Γ_L we obtain, if $\mathcal{F}_\Lambda$ denotes the set of interior faces in Γ_Λ,

$$||\lambda_h - \tilde{\pi}_L\lambda_h||_{L_h}^2 \leq \sum_{F\in\mathcal{F}_\Lambda}\sum_{i=0}^{k}\int_F h^{s_0+2i}[\![\partial_n^i\lambda_h]\!]^2 ds.$$

The order s_0 depends on L_h and follows from the scaling argument, in our example where the L_h -norm is the $h^{\frac{1}{2}}$-weighted L^2-norm over Γ we have $s_0 = 2$. We conclude that the interior penalty stabilization operator may be written

$$s\left(\lambda_h, \mu_h\right) := \sum_{F\in\mathcal{F}_\Lambda}\sum_{i=0}^{k}\int_F h^{s_0+2i}\left[\!\left[\partial_n^i\lambda_h\right]\!\right]\left[\!\left[\partial_n^i\mu_h\right]\!\right] ds.$$

It may be inconvenient to compute all normal derivatives up to polynomial order and an equivalent local projection approach may be used instead as suggested in the references given above. Observe that above we have assumed that Λ_h and L_h have the same polynomial everywhere in the domain. If this is not the case the analysis has to be modified accordingly.

IV. PENALTY STABILIZATION OF LAGRANGE MULTIPLIER FORMULATIONS: APPLICATIONS

As an example of the above theory, we recall a stabilized method introduced as a fictitious domain method in [19] and using the results of [21] for the underlying stable spaces. Here, we will first present the method in the simple case of weak imposition of boundary condition and then propose an extension to unfitted finite element methods. Both cases are considered in two space dimensions, but the extension to three dimensions is straightforward.

A. Weak Imposition of Boundary Conditions

In this section, we will consider the problem (1.1), that we recall here for the readers convenience.

Let Ω be a bounded domain in $\mathbb{R}^2$, with polygonal boundary $\partial\Omega$. The Poisson equation that we propose as a model problem is given by

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

where $\partial\Omega$ denotes the boundary of the domain Ω, $f \in L^2(\Omega)$, and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Under these assumptions (4.1) has a unique solution $u \in H^1(\Omega)$ satisfying $||u||_{H^1(\Omega)} \lesssim ||f||_{L^2(\Omega)}$. As already suggested, we define $V := H^1(\Omega)$ and $L := H^{-\frac{1}{2}}(\partial\Omega)$.

The usual L^2-scalar product on the domain Ω will be denoted by $(\cdot,\cdot)_\Omega$ or on the boundary $\langle\cdot,\cdot\rangle_{\partial\Omega}$. We also introduce the discrete norms

$$||\lambda||^2_{\frac{1}{2},h,\partial\Omega} = \langle h^{-1}\lambda,\lambda\rangle_{\partial\Omega}, \quad ||\lambda||^2_{-\frac{1}{2},h,\partial\Omega} = \langle h\lambda,\lambda\rangle_{\partial\Omega}$$

and

$$||u||^2_{1,h} := ||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{\frac{1}{2},h,\partial\Omega}.$$

Recall that $\forall\lambda,\mu \in L^2(\partial\Omega)$ there holds

$$\langle\lambda,\mu\rangle_{\partial\Omega} \le ||\lambda||_{-\frac{1}{2},h,\partial\Omega}||\mu||_{\frac{1}{2},h,\partial\Omega}. \tag{4.2}$$

The weak formulation of the problem is given by (1.2) with $a(\cdot,\cdot)$ defined by (2.2) and $b(\cdot,\cdot)$ by (2.3).

Finite Element Formulation. We introduce a triangulation $\mathcal{T}_h$, fitted to the boundary of Ω. The set of faces of triangles that form the boundary $\partial\Omega$ of Ω is denoted $\mathcal{F}$.

We will use the following notation for mesh related quantities. Let h_K be the diameter of K and $h = \max_{K\in\mathcal{T}_h} h_K$. We introduce the finite element spaces

$$V_h := \left\{v \in H^1(\Omega) : v|_K \in P_1(\Omega), \forall K \in \mathcal{T}_h\right\}$$

and

$$\Lambda_h := \left\{\mu \in L^2(\partial\Omega) : \mu|_F \in P_0(F), \forall F \in \mathcal{F}\right\}.$$

It is known that this choice of spaces does not satisfy (2.4).

The standard finite element formulation reads: find $u_h \in V_h$ and $\lambda_h \in \Lambda_h$ such that

$$a(u_h, v_h) + b(\lambda_h, v_h) + b(\mu_h, u_h) = (f, v_h) + b(\mu_h, g) \quad \text{for all } (v_h, \mu_h) \in V_h \times \Lambda_h. \tag{4.3}$$

Assume that L_h denotes a coarsened version of $\Lambda_h, L_h \subset \Lambda_h$ such that the inf-sup condition is uniformly satisfied for the pair $V_h \times L_h$, we know that this is always possible if L_h is chosen coarse enough, that is, if H denotes the mesh size of L_h thereholds $H > ch$, for some $c > 1$ and we assume that there exists a positive constant c_H such that $c_H H \leq h$. We let π_L denote the L^2-projection on the space L_h . As proposed in the previous section, we may stabilize the formulation (4.3) by adding the penalty term

$$s(\lambda_h, \mu_h) = \langle h(\lambda_h - \pi_L \lambda_h), \mu_h - \pi_L \mu_h \rangle_{\partial\Omega}.$$

Clearly, the space Λ_h is more convenient to work in as it does not require any special meshing of the boundary. If we now let $\mathcal{X} := \{x_i\}$ be the set of all the mesh nodes in $\partial\Omega$ excluding corner nodes. Then, there holds, by the arguments of Section III

$$||\lambda_h - \pi_L \lambda_h||^2_{-\frac{1}{2},h,\partial\Omega} \leq c \sum_{x_j \in \mathcal{X}} h^2 [[\lambda_h]]\,|^2_{x_j}.$$

This prompts the stabilization operator

$$s(\lambda_h, \mu_h) := \sum_{x_j \in \mathcal{X}} h^2 [[\lambda_h]]\,|_{x_j} [[\mu_h]]\,|_{x_j}.$$

Observe that penalizing the jump of λ_h over a corner node leads to an inconsistent method even for smooth u, as λ will jump across the corner due to the jump in the boundary normal. The stabilized method reads: find $u_h \in V_h$ and $\lambda_h \in \Lambda_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(\lambda_h, v_h) + b(\mu_h, u_h) - \gamma s(\lambda_h, \mu_h) \\ = (f, v_h) + b(\mu_h, g) \quad \text{for all } (v_h, \mu_h) \in V_h \times \Lambda_h. \end{aligned} \tag{4.4}$$

We will outline the analysis of the penalty stabilized Lagrange multiplier method using the abstract framework derived in Section II.

Satisfaction of the Assumptions of the Abstract Analysis. We may now use the abstract analysis of Theorem 2.4 combined with Lemma 2.1 to prove a best approximation result. We will use the discrete norms

$$||u_h||_{V_h} = ||u_h||_{1,h}, ||\lambda_h||_{L_h} := ||\lambda_h||_{-\frac{1}{2},h,\partial\Omega}.$$

By assumption L_h satisfies the inf-sup condition (2.4), for π_L defined as the L^2-projection on the piecewise constants (2.9) holds and hence we have the stabilized inf-sup condition (II.1). It is easy to see that the continuities (2.6) and (2.7) hold. The condition (2.12) also holds by taking $\xi_h(u_h) := \delta h^{-1} \pi_L u_h$, where $\delta \in \mathbb{R}^+$. The satisfaction of (2.12) now follows from the construction of $s(\cdot,\cdot)$, the quasiuniformity between H and h, the stability of the L^2-projection and the definition of the L_h and V_h norms,

$$c_0 s(\xi_h(u_h), \xi_h(u_h))^{\frac{1}{2}} \leq ||h^{-1} \pi_L u_h||_{L_h} \leq ||u_h||_{\frac{1}{2},h,\partial\Omega} \leq ||u_h||_{V_h}. \tag{4.5}$$

The second relation of (2.12) is satisfied using the approximation property of the projection π_L

$$||u_h - \pi_L u_h||_{\frac{1}{2},h,\partial\Omega} \leq c_0 ||\nabla u_h \times n_{\partial\Omega}||_{-\frac{1}{2},h,\partial\Omega}$$

and a discrete trace inequality $||\nabla u_h \times n_{\partial\Omega}||_{-\frac{1}{2},h,\partial\Omega} \leq c_t ||\nabla u_h||_{L^2(\Omega)}$, leading to

$$||u_h||^2_{\frac{1}{2},h,\partial\Omega} \leq 2||u_h - \pi_L u_h||^2_{\frac{1}{2},h,\partial\Omega} + 2||\pi_L u_h||^2_{\frac{1}{2},h,\partial\Omega} \leq 2||\pi_L u_h||^2_{\frac{1}{2},h,\partial\Omega} + 2c_0^2 c_t^2 ||\nabla u_h||^2_{L^2(\Omega)}. \tag{4.6}$$

It follows that

$$a(u_h, u_h) + b(\xi_h(u_h), u_h) \geq \left(1 - c_0^2 c_t^2 \delta\right) ||\nabla u_h||^2_{L^2(\Omega)} + \frac{\delta}{2} ||u_h||^2_{\frac{1}{2},h,\partial\Omega}$$

and hence for $\delta < c_0^{-2} c_t^{-2}$ the coercivity assumption is satisfied.

We conclude that the assumptions of Theorem 2.4 are satisfied and that the formulation (4.4) is wellposed and satisfies a best approximation result.

B. Unfitted Finite Element Methods and Multimodel Coupling

Here, we will consider the coupling of two models of elasticity over a smooth interface that is not fitted to the computational mesh. This type of method can be useful for problems where the interface itself is an unknown and repeated computations have to be performed with different interface positions, for instance for transient problems where an interface moves through the mesh or for inverse identification where the interface will move during iterations.

We consider a geometrical setting where a polygonal Ω is decomposed in two subdomains, Ω_1 and Ω_2 and a separating interface Γ. In each subdomain Ω_i we consider the following partial differential equation:

$$\nabla \cdot \sigma_i(u_i) = f, \quad \text{in } \Omega_i$$

where $u_i \in V_i := \left[H^1(\Omega_i)\right]^2$ denotes a displacement field, $\sigma_i(u_i) \in [H(\text{div}; \Omega_i)]^2$ the stress tensor and $f \in L^2(\Omega)$ the applied force. Across the interface we assume that the following matching conditions hold

$$u_1 - u_2 = 0, (\sigma_1(u_1) - \sigma_2(u_2)) \cdot n_\Gamma = 0.$$

For simplicity, we assume that $u = 0$ on the outer boundary $\partial\Omega$. Let

$$V := \left\{(v_1, v_2) \in V_1 \times V_2 : v_i|_{\partial\Omega_i \cap \partial\Omega} = 0\right\}$$

and L be the dual space to the space of traces of V on Γ. We propose the following norm on V:

$$||u||_V := \sum_{i=1}^{2} ||\nabla u||_{\Omega_i} + ||u_1 - u_2||_{\frac{1}{2},\Gamma}. \tag{4.7}$$

We assume that the following coercivity and continuity properties hold for the continuous problem. There exists positive constants α_0, α_1, M such that

$$\alpha_0 \sum_i ||\nabla u_i||^2_{\Omega_i} \leq \sum_{i=1}^{2} \left((\sigma_i (u_i), \nabla u_i)_{\Omega_i} + ||u_i||^2_{\partial\Omega_i \cap \partial\Omega} \right) + ||u_1 - u_2||^2_\Gamma, \quad \forall (u_1, u_2) \in V_1 \times V_2, \tag{4.8}$$

$$\alpha_1 ||u||^2_V \leq \sum_{i=1}^{2} (\sigma_i (u_i), \nabla u_i)_{\Omega_i}, \quad \forall u \in \{v \in V : \langle \nu, v_1 - v_2 \rangle_\Gamma = 0, \forall \nu \in L\}, \tag{4.9}$$

$$\left| \sum_{i=1}^{2} (\sigma_i (u_i), \nabla v_i)_{\Omega_i} \right| \leq M ||u||_V ||v||_V, \quad \forall u, v \in V. \tag{4.10}$$

Note that (4.8) typically implies a Korn's inequality and that (4.9) is a consequence of (4.8), the boundary and interface conditions and the Poincaré inequality. We propose a weak formulation using Lagrange multipliers that takes the form, find $(u, \lambda) \in V \times L$ such that

$$a (u, v) + b (\lambda, v) + b (\mu, u) = (f, v) \quad \text{for all } (v, \mu) \in V \times L, \tag{4.11}$$

where

$$a (u, v) := \sum_{i=1}^{2} (\sigma_i (u_i), \nabla v_i)_{\Omega_i}, \quad b (\lambda, v) = \langle \lambda, (v_1 - v_2) \rangle_\Gamma. \tag{4.12}$$

Note that the continuity $b (\lambda, v) \leq M_b ||\lambda||_L ||v||_V$ holds. If in addition to (4.8), (4.9), and the above continuities we assume that $\sigma_i (u_i)$ are linear, this formulation is wellposed by the Babuska–Brezzi Theorem (see [1, 2]). Observe that there are some differences in the functional analytical framework depending on whether or not Γ intersects the Dirichlet boundary. These differences are irrelevant for the present discussion and will be neglected.

Finite Element Formulation. Consider the mesh family $\{\mathcal{T}_h\}_h$ where we let $\mathcal{T}_h := \{K\}$ be a triangulation of Ω that is constructed without fitting the element nodes or sides to the interface Γ. For any $\mathcal{T}_h$, we now extract two subtriangulations, $\mathcal{T}_i := \left\{K \in \mathcal{T}_h : K \cap \Omega_i \neq \emptyset\right\}$, $i = 1,2$. We define two finite element spaces, one for Ω_1 and one for Ω_2 by

$$V_{ih} := \left\{v \in V_i : v|_K \in [P_1 (K)]^2, \forall K \in \mathcal{T}_i \text{ and } v|_{\partial\Omega \cap \mathcal{T}_i} = 0\right\}.$$

Let $\tilde{G}_h := \left\{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\right\}$. We assume that the mesh is fine enough so that, for all $K \in \tilde{G}_h$, $\Gamma \cap K$ can be approximated by a line segment, i.e. that Γ intersects the boundary of K in two points and that there exists $c > 0$ so that meas $(\Gamma \cap K) < ch$ for all elements and all meshes.

Observe that the finite element functions extend to all of the mesh domain $\mathcal{T}_i$ which can lead to conditioning problems if there are elements in $\tilde{G}_h$ with very small intersection with the physical domain. On the set $\tilde{G}_h$, we define the following multiplier space

$$\Lambda_h := \left\{\lambda_h \in \left[L^2 \left(\tilde{G}_h\right)\right]^2 : \lambda_h|_K \in [P_0 (K)]^2, \forall K \in \tilde{G}_h\right\}.$$

The Lagrange multiplier is defined on the same elements as the primal variables and hence has been extended in space, the advantage of this is that the stabilization of the multiplier can be designed on the standard volume elements (here in $\mathcal{R}^2$) and we do not need to consider a trace mesh of Γ.

The finite element method once again is on the generic form find $u_h := \{u_{h1}, u_{h2}\} \in V_{1h} \times V_{2h} =: V_h$ and $\lambda_h \in \Lambda_h$ such that

$$a(u_h, v_h) + b(\lambda_h, v_h) + b(\mu_h, u_h) - s(\lambda_h, \mu_h) = (f, v_h) \quad \text{for all } (v_h, \mu_h) \in V_h \times \Lambda_h, \tag{4.13}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (4.12) and $s(\cdot,\cdot)$ will be detailed below. We know that if we instead looked for λ_h in a space L_h defined on macro elements with diameter H such that $H > c_h h$, with c_h sufficiently large the inf-sup condition would be satisfied. We also assume that there exists $c_H > 0$ so that $c_H H \leq h$. We assume that the space L_h is constructed by assembling elements in $\tilde{G}_h$ into macro patches F_j such that for every $jH \leq \text{meas}\left(F_j \cap \Gamma\right) \leq H + h$. By the constraints on the mesh with respect to the interface, we may conclude that the cardinality of the set $\left\{K : K \cap F_j \neq \emptyset\right\}$ is upper bounded uniformly in j and h by some $M_F \in \mathbb{N}^+$. To each boundary patch F_j, we associate a shape regular macro patch $\omega_j^i \subset \Omega_i$ consisting $F_j \cap \Omega_i$ and a sufficient number of interior elements $K \subset \mathcal{T}_{ih} \cap \Omega_i$ so that $\text{meas}\left(\omega_j^i \cap \Omega_i\right) = O\left(H^2\right)$. It follows by construction that $\overline{\omega}_j^1 \cap \overline{\omega}_j^2 = F_j$ and we assume that for fixed i, the interiors of the patches ω_j^i are disjoint. The rationale for the patches ω_j^i is that for all $u_j \in H^1\left(\omega_j^i\right)$ the following trace inequality holds

$$H^{-\frac{1}{2}}||u_j - \pi_L u_j||_{\Gamma\cap\omega_j^i} \leq c_P||\nabla u_j||_{\omega_j^i} \tag{4.14}$$

where π_L denotes the projection onto piecewise constant functions on F_j and c_P is independent of the mesh interface intersection. This inequality is proven by mapping to a reference patch $\hat{\omega}$, there applying a trace inequality followed by a Poincaré type inequality (see Corollary B.65 of [27]) and then mapping back to the physical patch ω_j^i, using the shape regularity of the patch for uniformity. For completeness, we sketch a proof of the construction of the Fortin interpolant in appendix. Observe that using the stable pair $V_h \times L_h$ and taking $s(\cdot,\cdot) = 0$ then leads to a best approximation for the inf-sup stable unfitted finite element method using Theorem 2.4.

As before, we get the abstract stabilization operator

$$s(\lambda_h, \mu_h) := \langle h(\lambda_h - \pi_L\lambda_h), \mu_h - \pi_L\mu_h\rangle_\Gamma. \tag{4.15}$$

In practice, as we do not want to be concerned with the construction of L_h, we apply the ideas of Section III and instead work with the operator

$$s(\lambda_h, \mu_h) := \sum_{K\in\tilde{G}_h} \int_{\partial K\setminus\partial\tilde{G}_h} h\,[[\lambda_h]]\cdot[[\mu_h]]\,ds, \tag{4.16}$$

where $[[x]]$ denotes the jump of the quantity x over the interior faces of the elements in the set $\tilde{G}_h$.

Remark. Note that although the operator of (4.15) is defined on Γ, the operator (4.16) is defined on the interior faces of elements in G_h. This convenient trick introduced in [19], allows us to use the volume mesh structure for stabilization and we never need to worry about the actual intersections of Γ with element boundaries. Uniformity of the stabilization relies on the mesh regularity.

Satisfaction of the Assumptions of the Abstract Analysis. For the method to be robust with respect to the mesh-interface intersection, the constants in the bounds in the above abstract analysis must all be independent of the cut. This holds for the approximation using the inf-sup stable space $V_h \times L_h$, thanks to the robustness of the Fortin interpolant and the properties of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. Therefore, we only show that the inequalities (2.12) also can be made independent of the cut, under the above assumptions. Similarly, as in the case of weak boundary condition we introduce the following norms on the discrete spaces

$$||\lambda_h||^2_{\frac{1}{2},h,\Gamma} = \langle h^{-1}\lambda_h, \lambda_h\rangle_\Gamma, ||\lambda_h||^2_{-\frac{1}{2},h,\Gamma} = \langle h\lambda_h, \lambda_h\rangle_\Gamma,$$

$$||u_h||^2_{1,h} := \sum_{i=1}^{2} ||\nabla u_{ih}||^2_{L^2(\Omega_i)} + ||u_{1h} - u_{2h}||^2_{\frac{1}{2},h,\Gamma}.$$

To prove that the hypothesis of Theorem 2.4 are satisfied, we chose the norms $||\cdot||_{V_h}$ and $||\cdot||_{L_h}$ as follows,

$$||u||_{V_h} := ||u||_{1,h}, ||\lambda||_{L_h} := ||\lambda||_{-\frac{1}{2},h,\Gamma}.$$

To satisfy the coercivity condition of (2.12), we take $\xi_h(u_h)|_{F_i} := \delta H^{-1}\pi_L(u_1 - u_2)$. We recall that π_L is defined by the projection on the space L_h with mesh size H,

$$\pi_L(u_{1h} - u_{2h})|_{F_i} := |F_i \cap \Gamma|^{-1} \int_{F_i\cap\Gamma} (u_{1h} - u_{2h})\, ds.$$

By this choice, using the orthogonality of π_L we have

$$b(\xi_h(u_h), u_h) = \delta \sum_i \langle H^{-1}\pi_L(u_{1h} - u_{2h}), \pi_L(u_{1h} - u_{2h})\rangle_{F_i\cap\Gamma}$$

$$= \delta \sum_j ||H^{-\frac{1}{2}}(\pi_L - I)(u_{1h} - u_{2h}), ||^2_{F_j\cap\Gamma} + \delta \sum_j ||H^{-\frac{1}{2}}(u_{1h} - u_{2h}), ||^2_{F_j\cap\Gamma}$$

$$\geq -2\delta \sum_{i=1}^{2}\sum_j ||H^{-\frac{1}{2}}(\pi_L - I)u_{ih}||^2_{\omega^i_j\cap\Gamma} + \delta \sum_j ||H^{-\frac{1}{2}}(u_{1h} - u_{2h})||^2_{F_j\cap\Gamma}.$$

Applying (4.14) in the right-hand side of the last inequality, it then follows that

$$a(u_h,u_h) + b(\xi_h(u_h),u_h) \geq \sum_{i=1}^{2} (\sigma_i(u_{ih}), \nabla u_{ih})_{\Omega_i} - 2\delta c_P^2 \sum_{i=1}^{2} ||\nabla u_{ih}||^2_{\Omega_i} + \delta c_H ||u_{1h} - u_{2h}||^2_{\frac{1}{2},h,\Gamma}.$$

We then apply (4.8) in the right-hand side, recalling that $u_{ih} \in V_i, i = 1,2$ to obtain

$$a(u_h,u_h) + b(\xi_h(u_h),u_h) \geq \left(\alpha_0 - 2\delta c_P^2\right) \sum_{i=1}^{2} ||\nabla u_{ih}||^2_{\Omega_i} + (\delta c_H - h)\, ||u_{1h} - u_{2h}||^2_{\frac{1}{2},h,\Gamma}$$

and we conclude by choosing $\delta = \frac{\alpha_0}{4c_P^2}$ and taking $h < \delta c_H$.

For the second inequality of (2.12) observe that by the fact that an interface segment F_j can only be cut by a uniformly upper bounded number of elements, the mesh condition, $c_h h \leq H \leq c_H^{-1} h$, and that the $\xi_h(u_h)$ are constant over each macro patch F_j we have

$$\sum_{K \in \tilde{G}_h} \int_{\partial K \setminus \partial \tilde{G}_h} h[[\xi_h(u_h)]]^2 dx \lesssim \sum_{K \in \tilde{G}_h} h^2 |\xi_h(u_h)|_K|^2$$
$$\lesssim M_F \sum_j h^2 |\xi_h(u_h)|_{F_j}|^2 \lesssim ||\xi_h(u_h)||_{L_h}^2.$$

Then, using the stability of the L^2-projection and the mesh conditions linking h and H, we conclude

$$||\xi_h(u_h)||_{L_h} = \delta ||H^{-1}\pi_L(u_{1h} - u_{2h})||_{-\frac{1}{2},h,\Gamma} \lesssim ||\pi_L(u_{1h} - u_{2h})||_{\frac{1}{2},h,\Gamma} \lesssim ||u_h||_{V_h}.$$

We conclude that the results of Theorem 2.4 hold in this case as well.

Remark. By using suitable extensions of the solution following [20] and [29] optimal convergence may be obtained for smooth solutions. The conditioning of the system, however, depends on how the interface cuts the mesh and must be handled either following the ideas introduced in [25] or by preconditioning.

C. Nitsche's Method and Stabilized Lagrange Multiplier Methods: A Different Approach

The close relation between the residual-based stabilized methods for Lagrange multipliers as introduced by Barbosa and Hughes and Nitsche's method was discussed by Stenberg in [17]. The idea of that paper was that if the Lagrange multiplier can be eliminated locally by solving the constraint equation, Nitsche's method is recovered. Other authors have recently discussed the need of penalty for Nitsche's method and its close relation to Lagrange multiplier methods, see for instance [30–32].

Herein, we will show the connection between the nonsymmetric variant of Nitsche's method, the projection stabilized methods discussed above and the residual-based stabilization of the Lagrange multiplier. Let us first recall the nonsymmetric version of the method of Barbosa and Hughes: find $\{u_h, \lambda_h\} \in V_h \times \Lambda_h$ such that

$$\begin{aligned} &A_{BH}[(u_h, \lambda_h), (v_h, \mu_h)] \\ &:= a(u_h, v_h) + b(\lambda_h, v_h) - b(u_h, \mu_h) + \gamma \langle h(\lambda_h + \nabla u_h \cdot n), \mu_h + \nabla v_h \cdot n \rangle_{\partial\Omega} \\ &= (f, v_h), \forall \{v_h, \mu_h\} \in V_h \times \Lambda_h, \end{aligned} \tag{4.17}$$

with $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (2.2) and (2.3), corresponding to the weak imposition of boundary conditions. Recalling that formally the Lagrange multiplier is given by the diffusive flux $\lambda = -\nabla u \cdot n$, we immediately conclude that the method is consistent. Stability is then typically proven by testing with $v_h = u_h$ and $\mu_h = \lambda_h$ using the positivity of the form to obtain control of $||h^{\frac{1}{2}}\lambda_h||_{\partial\Omega}$ by absorbing all the other terms in the stabilization using the H^1-seminorm of u_h over the domain. Control of u_h on the boundary is then obtained in a second step by choosing μ_h suitably.

We will now consider the stabilization used in (4.17) as a penalty on the distance to a stable subspace. This would mean using the space N_h of normal derivatives of V_h on the trace mesh as multiplier space, together with V_h for the primal variable. Since in that case V_h and N_h no longer can be chosen independently this method may be written: find $u_h \in V_h$ such that

$$A_{Nit}(u_h, v_h) := a(u_h, v_h) + b(-\nabla u_h \cdot n, v_h) - b(u_h, -\nabla v_h \cdot n)$$
$$= (f, v_h), \forall v_h \in V_h. \tag{4.18}$$

We have eliminated the Lagrange multiplier in the formulation using its equivalence with the diffusive flux. Writing out this variational formulation leads to

$$\int_\Omega \nabla u_h \cdot \nabla v_h dx - \int_{\partial\Omega} \nabla u_h \cdot n v_h ds + \int_{\partial\Omega} \nabla v_h \cdot n u_h ds = \int_\Omega f v dx,$$

which we identify as the nonsymmetric version of Nitsche's method, without penalty. For the argument to make sense, we now need a stability result for this method. The question of the inf-sup stability of the nonsymmetric version of Nitsche's method, without penalty, was recently treated in [30], where the following stability result was proven.

Lemma 4.1 *Let V_h be the standard space of piecewise polynomial continuous finite element functions. Assume that the each face of the polygonal Ω is meshed with a sufficient number of elements (depending only on the shape regularity), then for some $\zeta \geq c_0 > 0$, with c_0 independent of h, but not of the mesh geometry, there holds*

$$||u_h||_{1,h} \lesssim \sup_{v_h \in V_h} \frac{A_{Nit}(u_h, v_h)}{||v_h||_{1,h}},$$

where

$$||u_h||_{1,h}^2 := ||\nabla u_h||_{L^2(\Omega)}^2 + \zeta ||h^{-\frac{1}{2}} u_h||_{L^2(\partial\Omega)}^2, \quad \zeta > 0.$$

It follows that we have the required stability and we may prove stability of the residual-based stabilization using the techniques discussed above.

Remark. The above lemma can be rewritten as $\exists w_h \in V_h$ such that

$$c_w ||u_h||_{1,h}^2 \leq A_{\mathrm{Nit}}(u_h, w_h) \tag{4.19}$$

with $w_h := u_h + \zeta \varphi_\partial$, $c_w > 0$ and

$$||\varphi_\partial||_{1,h} \leq c_\partial ||u_h||_{\frac{1}{2},h,\partial\Omega}. \tag{4.20}$$

The function φ_∂ ensures the control of the boundary contribution.

We now give an alternative proof of the equivalent of Lemma 4.1 for the formulation (4.17) using the framework of penalty on the distance to the stable subspace. The result holds for multiplier spaces satisfying the following compatibility assumption.

Assumption [A1]. *The following continuity holds for the spaces V_h and Λ_h. For every $v_h \in V_h$, there exists $z_h(v_h) \in \Lambda_h$ such that*

$$b\left(u_h, \nabla v_h \cdot n + z_h\left(v_h\right)\right) \leq C_z ||\nabla u_h||_{L^2(\Omega)} ||\nabla v_h||_{L^2(\Omega)}, ||h^{\frac{1}{2}} z_h\left(v_h\right)||_{L^2(\partial\Omega)} \leq c_z ||\nabla v_h||_{L^2(\Omega)}. \tag{4.21}$$

Theorem 4.2 *Let $V_h \times \Lambda_h$ satisfy assumption* **[A1]** *. Then, for all $\{u_h, \lambda_h\} \in V_h \times \Lambda_h$ there holds*

$$||u_h||_{1,h} + ||\lambda_h||_{L_h} \lesssim \sup_{\{w_h, \nu_h\} \in V_h \times \Lambda_h} \frac{A_{BH}\left[\left(u_h, \lambda_h\right), \left(w_h, \nu_h\right)\right]}{||w_h||_{1,h} + ||\nu_h||_{L_h}}. \tag{4.22}$$

Proof. First, take $v_h = u_h$ and $\mu_h = \lambda_h$ to obtain

$$\begin{aligned} &||\nabla u_h||^2_{L^2(\Omega)} + \gamma ||h^{\frac{1}{2}}\left(\lambda_h + \nabla u_h\right)||^2_{L^2(\partial\Omega)} \\ &= a\left(u_h, u_h\right) + b\left(\lambda_h, u_h\right) - b\left(\lambda_h, u_h\right) + \gamma \langle h\left(\lambda_h + \nabla u_h \cdot n\right), \lambda_h + \nabla u_h \cdot n\rangle_{\partial\Omega} \end{aligned} \tag{4.23}$$

We add and subtract $\nabla u_h \cdot n$ and $\nabla v_h \cdot n$ in the $b(\cdot, \cdot)$ forms of the formulation

$$\begin{aligned} &a\left(u_h, v_h\right) + b\left(\lambda_h, v_h\right) - b\left(u_h, \mu_h\right) + \gamma \langle h\left(\lambda_h + \nabla u_h \cdot n\right), \mu_h + \nabla v_h \cdot n\rangle_{\partial\Omega} \\ &= a\left(u_h, v_h\right) + b\left(-\nabla u_h \cdot n, v_h\right) + b\left(\nabla u_h \cdot n + \lambda_h, v_h\right) \\ &-b\left(u_h, -\nabla v_h \cdot n\right) - b\left(u_h, \nabla v_h \cdot n + \mu_h\right) + \gamma \langle h\left(\lambda + \nabla u_h \cdot n\right), \mu_h + \nabla v_h \cdot n\rangle_{\partial\Omega} \\ &= A_{\mathrm{Nit}}\left(u_h, v_h\right) \\ &+b\left(\nabla u_h \cdot n + \lambda_h, v_h\right) - b\left(u_h, \nabla v_h \cdot n + \mu_h\right) + \gamma \langle h\left(\lambda_h + \nabla u_h \cdot n\right), \mu_h + \nabla v_h \cdot n\rangle_{\partial\Omega}. \end{aligned} \tag{4.24}$$

We will first show that by taking $v_h = w_h$ [of (4.19)–(4.20)] and $\mu_h := \lambda_h + z_h(\zeta\varphi_\partial)$ we have

$$||u_h||^2_{1,h} + \gamma h ||\lambda_h + \nabla u_h \cdot n||^2_{L^2(\partial\Omega)} \lesssim A_{BH}\left[\left(u_h, \lambda_h\right), \left(w_h, \lambda_h + z_h\left(\zeta\varphi_\partial\right)\right)\right].$$

First, note that by the construction of w_h and μ_h and the form (4.24) we have

$$\begin{aligned} A_{BH}\left[\left(u_h, \lambda_h\right), \left(w_h, \mu_h\right)\right] &= A_{\mathrm{Nit}}\left(u_h, w_h\right) + \gamma ||h^{\frac{1}{2}}\left(\lambda_h + \nabla u_h \cdot n\right)||^2_{L^2(\partial\Omega)} \\ &+ \zeta b\left(\nabla u_h \cdot n + \lambda_h, \varphi_\partial\right) - \zeta b\left(u_h, \nabla\varphi_\partial \cdot n + z_h\left(\varphi_\partial\right)\right) \\ &+ \gamma\zeta \langle h\left(\nabla u_h \cdot n + \lambda_h\right), \nabla\varphi_\partial \cdot n + z_h\left(\varphi_\partial\right)\rangle_{\partial\Omega}. \end{aligned} \tag{4.25}$$

Then, note that by the continuity of $b(\cdot, \cdot)$ and by using the Cauchy–Schwarz inequality in the penalty term we have

$$\begin{aligned} \zeta b\left(\nabla u_h \cdot n + \lambda_h, \varphi_\partial\right) &\leq c_b ||h^{\frac{1}{2}}\left(\lambda_h + \nabla u_h \cdot n\right)||_{L^2(\partial\Omega)} \zeta c_\partial ||u_h||_{\frac{1}{2},h,\partial\Omega} \\ &\leq \frac{1}{4}\gamma ||h^{\frac{1}{2}}\left(\lambda_h + \nabla u_h \cdot n\right)||^2_{L^2(\partial\Omega)} + \gamma^{-1}\zeta^2 c_b^2 c_\partial^2 ||u_h||^2_{\frac{1}{2},h,\partial\Omega}, \end{aligned} \tag{4.26}$$

$$\begin{aligned} \zeta b\left(u_h, \nabla\varphi_\partial \cdot n + z_h\left(\varphi_\partial\right)\right) &\leq \zeta C_z ||\nabla u_h||_{L^2(\Omega)} ||\varphi_\partial||_{1,h} \\ &\leq \frac{1}{2} c_w ||\nabla u_h||^2_{L^2(\Omega)} + C_z^2 c_w^{-1} c_\partial^2 \zeta^2 ||u_h||^2_{\frac{1}{2},h,\partial\Omega} \end{aligned} \tag{4.27}$$

and

$$\gamma\zeta\left(h\left(\nabla u_h\cdot n+\lambda_h\right),\nabla\varphi_\partial\cdot n+z_h\left(\varphi_\partial\right)\right)\lesssim\frac{1}{4}\gamma||h^{\frac{1}{2}}\left(\lambda_h+\nabla u_h\cdot n\right)||^2_{L^2(\partial\Omega)}$$
$$+2\left(c_t^2c_\partial^2+c_z^2\right)\zeta^2\gamma||u_h||^2_{\frac{1}{2},h,\partial\Omega}. \tag{4.28}$$

If we choose ζ small enough, it follows from (4.25), (4.19)–(4.20) and the bounds (4.26)–(4.28) that

$$A_{BH}\left[\left(u_h,\lambda_h\right),\left(w_h,\mu_h\right)\right]\geq\frac{1}{2}c_w||u_h||^2_{1,h}+\frac{1}{2}\gamma||h^{\frac{1}{2}}\left(\lambda_h+\nabla u_h\cdot n\right)||^2_{L^2(\partial\Omega)}.$$

Since $||h^{\frac{1}{2}}\lambda_h||_{L^2(\partial\Omega)}\leq||h^{\frac{1}{2}}\left(\lambda_h+\nabla u_h\cdot n\right)||_{L^2(\partial\Omega)}+c_t||\nabla u_h||_{L^2(\Omega)}$ we deduce that

$$||u_h||^2_{1,h}+||h^{\frac{1}{2}}\lambda_h||^2_{L^2(\partial\Omega)}\lesssim A_{BH}\left[\left(u_h,\lambda_h\right),\left(w_h,\mu_h\right)\right].$$

It only remains to show that

$$||u_h+\zeta\varphi_\partial||_{1,h}+||h^{\frac{1}{2}}\left(\lambda_h+z_h\left(\zeta\varphi_\partial\right)\right)||_{L^2(\partial\Omega)}\lesssim||u_h||_{1,h}+||h^{\frac{1}{2}}\lambda_h||_{L^2(\partial\Omega)}.$$

This is immediate by the triangle inequality and the stability

$$||\varphi_\partial||_{1,h}+||h^{\frac{1}{2}}z_h\left(\zeta\varphi_\partial\right)||_{L^2(\partial\Omega)}\lesssim||u_h||_{\frac{1}{2},h,\partial\Omega}.$$

∎

Remark. The condition (4.21) is easily satisfied for any reasonable space Λ_h. For spaces including discontinuous functions on boundary elements F_j take $z_h\left(v_h\right)|_{F_j}:=-\text{meas}\left(F_j\right)^{-1}\int_{F_j}\nabla v_h\cdot nds$. If the spaces Λ_h consists of continuous functions decompose the boundary in macro patches F_j consisting of a sufficient number of elements for the construction of functions $z_h\left(v_h\right)\in H_0^1\left(F_j\right)$ such that $\int_{F_j}z_h\left(v_h\right)\mathrm{d}s=-\int_{F_j}\nabla v_h\cdot nds$. Then, on each subdomain F_j there holds (with π_L denoting the L^2-projection on constant functions on F_j)

$$\langle u_h,\nabla v_h\cdot n+z_h\left(v_h\right)\rangle_{F_j}=\langle u_h-\pi_Lu_h,\nabla v_h\cdot n+z_h\left(v_h\right)\rangle_{F_j}.$$

It also follows that whenever the choice $z_h\left(v_h\right)=-\nabla v_h\cdot n$ is possible, the right-hand sides of (4.27) and (4.28) are zero and, therefore, the stability is obtained independently of the stability parameter γ. It is then straightforward to show, using the above inf-sup argument, that the solution u_h of (4.17) converges to that of (4.18) in the limit $\gamma\to\infty$. This is consistent with the argument of [17], as the local elimination of the Lagrange multiplier in (4.17) yields the nonsymmetric version of Nitsche's method with a penalty that vanishes in the limit $\gamma\to\infty$.

Remark. It follows from Theorem 4.2 that the nonsymmetric version of the stabilization of Barbosa and Hughes, can be interpreted as a penalty on the distance to the stable subspace, consisting of the normal derivatives of the primal finite element space in the setting of the nonsymmetric Nitsche method. Loosely speaking, we can consider the nonsymmetric Nitsche method as a special member of the set of inf-sup stable Lagrange multiplier methods. An associated stabilization based on penalty on the distance to a stable subspace is the Barbosa–Hughes method. In case the Lagrange multiplier can be eliminated locally the two methods are equivalent and the stabilized method is robust for large values of the penalty parameter.

V. NUMERICAL EXAMPLE

The aim of this section is to compare the performance of the different methods in the simple case of weak imposition of boundary conditions. All computations were carried out using Freefem++ [33].

We consider the Poisson problem in the unit square, $\Omega := (0, 1) \times (0, 1)$. The source term and boundary terms are chosen so that

$$u(x, y) = \frac{1}{2\pi^2} \cos(\pi x) \cos(\pi y) + 0.25x(1 - x)y(1 - y).$$

We compute the solution using the nonsymmetric residual stabilized method, a projection stabilization method, and penalty free Nitsche methods. Below, the forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are given by (2.2) and (2.3). We impose Dirichlet boundary conditions on the boundaries $y = 0$ and $y = 1$ (denoted $\partial\Omega_D$ below). On the other two boundaries, we impose Neumann conditions. In all cases the primal variable u_h is approximated using continuous finite elements, of first or second polynomial order,

$$V_h^k := \{v_h \in C^0(\overline{\Omega}) : v_h|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\}, \quad k = 1, 2.$$

Let $G_h := \{F\}$ denote a trace mesh on $\partial\Omega_D$, coinciding with the trace mesh of $\mathcal{T}_h$ and $G_{\tilde{h}} := \{F\}$ a trace mesh on $\partial\Omega_D$, such that the local mesh size in $G_{\tilde{h}}$ is half that of $\mathcal{T}_h, h = 2\tilde{h}$. Define the Lagrange multiplier spaces by

$$\Lambda_h^1 := \left\{v_h \in L^2(\partial\Omega_D) : v_h|_F \in \mathbb{P}_0, \forall F \in G_{\tilde{h}}\right\},$$
$$\Lambda_h^2 := \left\{v_h \in L^2(\partial\Omega_D) : v_h|_F \in \mathbb{P}_2, \forall F \in G_h\right\}.$$

These spaces are chosen so that the pair $V_h^k \times \Lambda_h^k, k = 1, 2$ are unstable. The stabilizing spaces were then both chosen as

$$L_h^k := \{v_h \in C^0(\partial\overline{\Omega}_D) : v_h|_F \in \mathbb{P}_1, \forall F \in G_h\}, \quad k = 1, 2.$$

Here, $C^0(\partial\overline{\Omega}_D)$ stands for functions continuous on each separate connected component of $\partial\Omega_D$. It is straightforward to verify that the spaces $V_h^k \times L_h^k$ are stable for our problem. In all figures below square markers refer to methods using $k = 1$ and circles to methods using $k = 2$. Empty markers indicate convergence of the error in the H^1-norm and filled markers in the L^2-norm. We have also plotted for reference the slopes corresponding to $O(h^\alpha)$ convergence with $\alpha = 1$ in dotted line, $\alpha = 2$ in a dashed line, and $\alpha = 3$ in dash dotted line. These reference plots are the same for all methods so that the relative performance can be assessed. In all cases, the stabilization parameter has been set to $\gamma = 1$. This parameter appeared to give a resonable result for all methods. We observed that increasing the parameter can improve the accuracy in the multiplier at the expense of the primal variable and vice versa.

We consider the formulation (2.11) with the stabilization given by

$$s(\lambda_h, \mu_h) = \langle \gamma h(\lambda_h - \pi_L \lambda_h), \mu_h - \pi_L \mu_h \rangle_{\partial\Omega_D}$$

and the finite element spaces proposed above. In Fig. 1, left plot, we give the convergence plots for $k = 1$ and $k = 2$. Then, we consider the method (4.17) and give the same convergence curves in the right plot of (5). For comparison, we also present the results obtained using the inf-sup stable

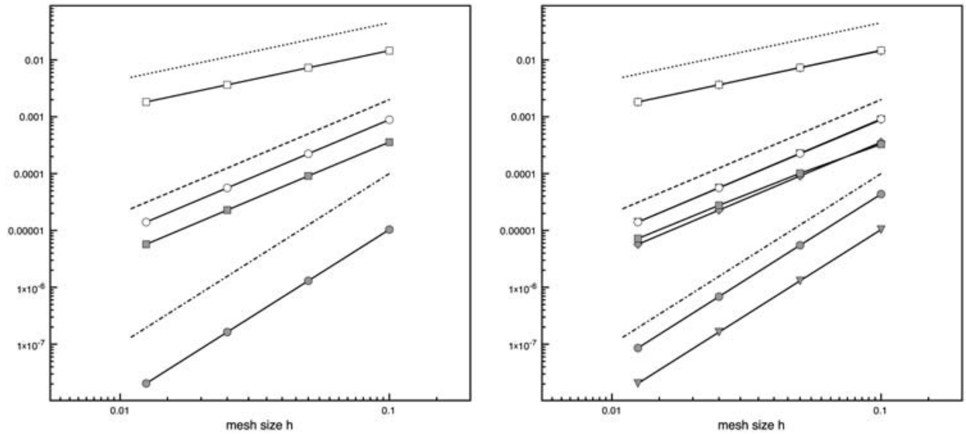

FIG. 1. Left plot: convergence of the projection stabilized methods, ($k = 1$ square marker, $k = 2$ round marker, markers for L^2-error filled). Right plot: convergence of the method (4.17) ($k = 1$ square marker, $k = 2$ round marker, markers for L^2-error filled) and the stable Lagrange multiplier method ($k = 1$ diamond marker, $k = 2$ triangular marker, markers for L^2-error filled).

finite element pairs $V_h^k \times L_h^k$. Finally, we consider the penalty free version of Nitsche's method, both the nonsymmetric version given by Eq. (4.18) and its symmetric equivalent that may be written

$$\int_\Omega \nabla u_h \cdot \nabla v_h dx - \int_{\partial\Omega} \nabla u_h \cdot n v_h ds - \int_{\partial\Omega} \nabla v_h \cdot n u_h ds = \int_\Omega f v dx - \int_{\partial\Omega} \nabla v_h \cdot n g ds.$$

Observe that the stability properties of this latter method are unknown, but for the computations considered herein the method remained stable and optimally convergent. We report the convergence of the Nitsche type methods in the left plot of Fig. 2. The symmetric version is distinguished by thick lines. A consequence of the close relation between the Barbosa–Hughes method and Nitsche's method is that in both the symmetric and the nonsymmetric case, the unpenalized Nitsche methods are recovered in the limit as the stabilization goes to infinity. This is illustrated in the right plot of (6) where we show the variation L^2-error of the difference between the solution obtained by the Barbosa–Hughes stabilization and Nitsche's method on a 20×20 mesh as the penalty parameter goes to infinity. In both the nonsymmetric and the symmetric case the penalty free Nitsche type methods are recovered. Observe the strong increase in the error for the symmetric case at approximately 2.1 where the matrix becomes singular. For higher values of the penalty parameter no instabilities were observed.

We make the following observations. The H^1-norm error is almost identical for all methods. For the L^2-norm error, all adjoint consistent methods have very similar error curves, whereas the lack of adjoint consistency is expressed only as a larger error constant and not in a loss of convergence order as expected from the analysis. In experiments not reported here, we imposed Dirichlet conditions all around the domain to see the effects on the corners in the nonsymmetric Nitsche method, but optimal convergence was still attained on the finest meshes. We also studied the error in the fluxes approximated by the multiplier and the results were similar to that of the L^2-norm error.

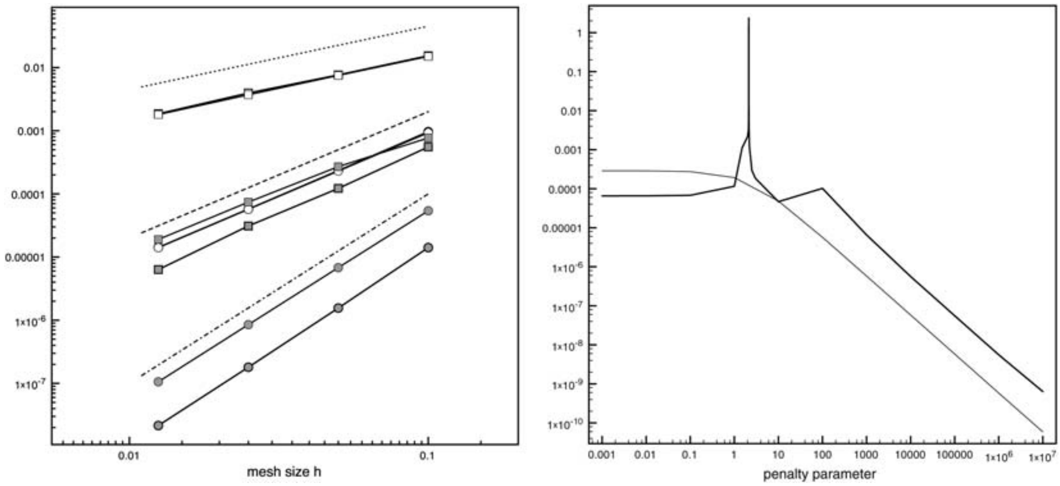

FIG. 2. Left plot: convergence of the penalty free Nitsche type methods, ($k = 1$ square marker, $k = 2$ round marker, markers for L^2-error filled, symmetric version plotted with thick line). Right plot: asymptotic behavior of the difference in the L^2-norm between the solution of the Barbosa–Hughes method and the corresponding Nitsche type method (symmetric version plotted with thick line).

VI. CONCLUSIONS

We have given an analysis of projection stabilized Lagrange multipliers in an abstract framework and shown some applications of this theory. We then showed how the residual-based stabilization method of Barbosa–Hughes can be interpreted as a method penalizing the distance to a stable subspace by relating it to the inf-sup stable penalty free Nitsche method. The methods were tested and compared numerically on a simple model problem. All these methods appear to have very similar properties. In particular, optimal convergence was observed in both the H^1- and the L^2-norms independently of adjoint consistency. Nevertheless adjoint consistent methods have smaller errors in L^2-norm for a fixed mesh size and similarly for the approximation of the fluxes. The observed difference was a moderate factor. One may conclude from this that it is reasonable that one may base the choice of method entirely on what is the easiest to implement for a given application. Two open problems are the question of stability of the penalty free symmetric Nitsche method and accuracy in the L^2-norm of the nonsymmetric Nitsche method. Both of which are observed.

APPENDIX CONSTRUCTION OF THE FORTIN INTERPOLANT

We will use the notation of Section IVB and prove that the Fortin interpolant $\pi_F v$ satisfying (2.5) exists and that the stability constant is independent of how the interface Γ cuts the mesh $\mathcal{T}_h$. Note that the stability of the Fortin interpolant writes

$$\sum_{i=1}^{2} ||\nabla \pi_F v||^2_{\Omega_i} + ||\pi_F v_1 - \pi_F v_2||^2_{\frac{1}{2},\Gamma} \lesssim \sum_{i=1}^{2} ||\nabla v||^2_{\Omega_i} + ||v_1 - v_2||^2_{\frac{1}{2},\Gamma}$$

where the hidden constant must be independent of the mesh-interface intersection. We introduce the extension operators $\mathbb{E}_i$ such that for all $v \in V_i, \mathbb{E}_i v \in H^1(\mathcal{T}_{ih}), \mathbb{E}_i v|_{\Omega_i} = v$, and $\|\mathbb{E}_i v\|_{H^1(\mathcal{T}_{ih})} \lesssim \|v\|_{H^1(\Omega_i)}$. Here, $\mathcal{T}_{ih}$ denotes the mesh-domain defined as $\cup_{K\in\mathcal{T}_{ih}} K$. Let $\mathcal{I}_h^i : H^1(\mathcal{T}_{ih}) \to V_{ih}$ denote an $H^1(\Omega)$-stable interpolant. For each j define the extended patch $\varpi_j^i := \omega_j^i \cup F_j$. Then, on each patch ϖ_j^i define a function $\varphi_j^i \in V_{ih}$ with supp$\varphi_j^i = \overline{\varpi}_j^i, \varphi|_{\partial\varpi_j^i\cap\Omega_i} = 0$ and

$$\int_{F_j} \varphi_j^i ds = O(H_i), \|\nabla\varphi_j^i\|_{\varpi_j^i} = O(1).$$

Define $\pi_F v_i := \mathcal{I}_h^i \mathbb{E}_i v_i + \sum_j \alpha_j^i \varphi_j^i$ where

$$\alpha_j^i := \frac{\int_{F_j} \left(v_i - \mathcal{I}_h^i \mathbb{E}_i v_i\right) ds}{\int_{F_j} \varphi_j^i ds}.$$

This construction is always possible, provided H is a given (fixed) factor larger than h, typically $H = 3h$ is sufficient.

Then, the orthogonality condition of (2.5) holds by construction. It remains to prove the H^1-stability. By the triangle inequality and the disjoint supports of the ϖ_j^i we have,

$$\|\nabla\pi_F v_i\|_{\Omega_i} \lesssim \|\nabla\mathcal{I}_h^i\mathbb{E}_i v_i\|_{\mathcal{T}_{ih}} + \left(\sum_j \left(\alpha_j^i\right)^2 \|\nabla\varphi_j^i\|_{\varpi_j^i}^2\right)^{\frac{1}{2}} = T_1 + T_2. \tag{6.1}$$

By the assumed stability of $\mathcal{I}_h^i$ and $\mathbb{E}_i$, we immediately have

$$T_1 \lesssim \|\nabla v_i\|_{\Omega_i}.$$

For T_2, we consider one term in the sum and get by the construction

$$\begin{aligned}|\alpha_i^i|\|\nabla\varphi_j^i\|_{\varpi_j^i} &\lesssim H^{-1}\int_{F_j}\left(v - \mathcal{I}_h^i v\right) ds \lesssim H^{-\frac{1}{2}}\|v - \mathcal{I}_h^i v\|_{F_j} \\ &\lesssim H^{-1}\|\mathbb{E}_i v - \mathcal{I}_h^i\mathbb{E}_i v\|_{\varpi_j^i} + \|\nabla\left(\mathbb{E}_i v - \mathcal{I}_h^i\mathbb{E}_i v\right)\|_{\varpi_j^i}.\end{aligned}$$

By the shape regularity of the ϖ_j^i, there is no dependence on the mesh domain intersection in the constants. Summing over j and using the fact that the ϖ_j^i are disjoint for fixed i we obtain that

$$T_2 \lesssim H^{-1}\|\mathbb{E}_i v_i - \mathcal{I}_h^i\mathbb{E}_i v_i\|_{\mathcal{T}_{ih}} + \|\nabla\left(\mathbb{E}_i v_i - \mathcal{I}_h^i\mathbb{E}_i v_i\right)\|_{\mathcal{T}_{ih}}$$

and the desired stability estimate follows by the approximation and stability properties of $\mathcal{I}_h^i$ and the stability of $\mathbb{E}_i$. It remains to prove that

$$\|\pi_F v_1 - \pi_F v_2\|_{\frac{1}{2},\Gamma}^2 \lesssim \sum_{i=1}^2 \|\nabla v_i\|_{\Omega_i}^2 + \|v_1 - v_2\|_{\frac{1}{2},\Gamma}^2.$$

This follows by adding and subtracting $v_1 - v_2$ in the left-hand side, and using a triangle inequality to obtain

$$||\pi_F v_1 - \pi_F v_2||^2_{\frac{1}{2},\Gamma} \lesssim ||v_1 - v_2||^2_{\frac{1}{2},\Gamma} + ||\pi_F v_1 - v_1||^2_{\frac{1}{2},\Gamma} + ||\pi_F v_2 - v_2||^2_{\frac{1}{2},\Gamma}.$$

We now proceed using a global trace inequality, the stability of the interpolant $\mathcal{I}_h^i$ and the above bound on the term T_2, to show that

$$||\pi_F v_i - v_i||^2_{\frac{1}{2},\Gamma} \lesssim ||\mathcal{I}_h^i v_i - v_i||^2_{H^1(\Omega_i)} + T_2 \lesssim ||\mathcal{I}_h^i \mathbb{E}_i v_i - \mathbb{E}_i v_i||^2_{H^1(\mathcal{T}_{ih})} + ||v||^2_V \lesssim ||v||^2_V.$$

Collecting the above bounds concludes the proof.

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