COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

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1. Introduction. We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators form a 3D domain, Ω, to 1D manifold, Λ, and vice versa. This is a non trivial objective because the standard trace operator form a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω.

2. Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex open set. Let $\Omega_\ominus$ be a generalized cylinder embedded into Ω and let $\Omega_\oplus = \Omega \setminus \overline{\Omega}_\ominus$ be the complementary set of the cylinder. We also introduce the set Λ, a 1D manifold that represents the centerline of $\Omega_\ominus$. We define the arc-length coordinate along Λ, denoted by $s \in (0, S)$. We denote with $\mathcal{D}(s)$ and $\partial\mathcal{D}(s)$ a cross section of $\Omega_\ominus$ and its boundary, respectively, and we assume that $\min|\mathcal{D}(s)| > 0$, where $|\mathcal{D}(s)|$ is the measure of $\mathcal{D}(s)$. We also assume that $\Omega_\ominus$ crosses Ω from side to side and we call Γ the lateral (cylindrical) surface of $\Omega_\ominus$, while the upper and lower side faces of $\Omega_\ominus$ belong to $\partial\Omega$. We refer to Figure 2.1 for an illustration of the notation.

We consider the problem arising from *Dirichlet-Neumann* conditions. It consists to find $u_\oplus, u_\ominus$ s.t.:

$$
\begin{aligned}
&(2.1a) && -\Delta u_\oplus + u_\oplus = f && \text{in } \Omega_\oplus,\\
&(2.1b) && -\Delta u_\ominus + u_\ominus = g && \text{in } \Omega_\ominus,\\
&(2.1c) && -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus && \text{on } \Gamma,\\
&(2.1d) && u_\ominus = u_\oplus && \text{on } \Gamma,\\
&(2.1e) && u_\oplus = 0 && \text{on } \partial\Omega\,.
\end{aligned}
$$

The objective of this work is to derive and analyze a simplified version of problem (2.1), where the domain $\Omega_\ominus$ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely $\mathcal{D}$. This new problem setting will be called the *reduced* problem. Form the mathematical standpoint it is more challenging than (2.1), because it involves the coupling of 3D/1D elliptic problems.

For the model reduction process, we decompose integrals as follows, for any sufficiently regular function w,

$$
\int_{\Omega_\ominus} w d\omega = \int_\Lambda \int_{\mathcal{D}} w d\sigma ds = \int_\Lambda |\mathcal{D}| \overline{\overline{w}} ds\,, \quad \int_\Gamma w d\sigma = \int_\Lambda \int_{\partial\mathcal{D}} w d\gamma ds = \int_\Lambda |\partial\mathcal{D}| \overline{w} ds\,,
$$

where $\overline{\overline{w}}$, $\overline{w}$ denote the following mean values respectively,

$$
\overline{\overline{w}} = |\mathcal{D}|^{-1} \int_{\mathcal{D}} w d\sigma\,, \quad \overline{w} = |\partial\mathcal{D}|^{-1} \int_{\partial\mathcal{D}} w d\gamma\,.
$$

We apply the model reduction approach at the level of the variational formulation. We start from the variational formulation of problem (2.1), that is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ u_\ominus \in H^1_{\partial\Omega_\ominus\setminus\Gamma}(\Omega_\ominus),\ \lambda \in H^{-\frac{1}{2}}(\partial\Omega_\ominus)$

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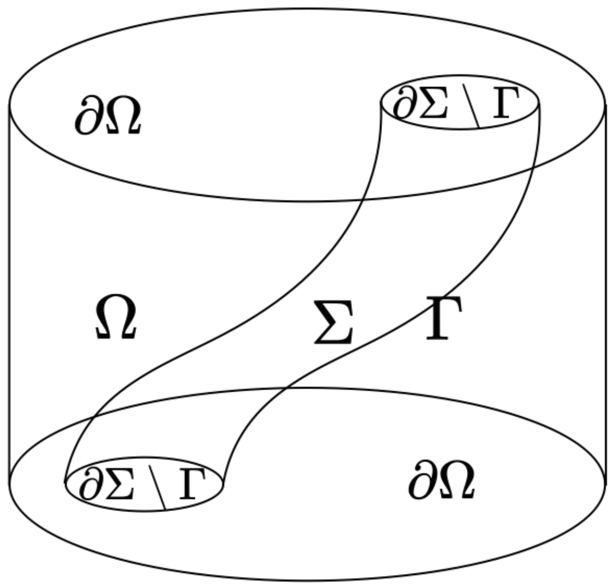


FIGURE 2.1. *Geometrical setting of the problem*

s.t.

$$(u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Omega_\ominus)} + \langle v_\oplus - v_\ominus, \lambda\rangle_{H^{-\frac{1}{2}}(\Gamma)} = (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Omega_\ominus)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Omega_\ominus\setminus\Gamma}(\Omega_\ominus) \tag{2.2a}$$

$$\langle u_\oplus - u_\ominus, \mu\rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,, \tag{2.2b}$$

where $\langle v, \mu\rangle_{H^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. In this case, the additional variable λ is equivalent to $\lambda = \nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

Using a model reduction approach based on averaging, we end up with two different formulations of a reduced problem for the unknown u defined on the entire 3D domain Ω, coupled with the unknown $u_\odot$, defined on the 1D manifold Λ and a Lagrange multiplier defined either on Γ (problem 1) or on Λ (problem 2). The scope of this work is to compare them, with the aim to determine which is the most suitable to set up a computational model for 3D-1D PDEs coupled with Dirichlet-Neumann constraint.

2.1. Topological model reduction.

Model reduction of the problem on $\Omega_\ominus$. We apply the averaging technique to equation (2.1b). In particular, we consider an arbitrary portion $\mathcal{P}$ of the cylinder $\Omega_\ominus$, with lateral surface $\Gamma_\mathcal{P}$ and bounded by two perpendicular sections to Λ, namely $\mathcal{D}(s_1)$, $\mathcal{D}(s_2)$ with $s_1 < s_2$. We have,

$$\int_\mathcal{P} -\Delta u_\ominus + u_\ominus d\omega = -\int_{\partial\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus\, d\sigma + \int_\mathcal{P} u_\ominus d\omega = \int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma - \int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma + \int_\mathcal{P} u_\ominus d\omega$$

By the fundamental theorem of integral calculus combined with the Reynolds transport Theorem, being ν the normal deformation of the boundary along $(0, S)$, we have,

$$\int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma = -\int_{s_1}^{s_2} d_s \int_{\mathcal{D}(s)} \partial_s u_\ominus d\sigma ds = -\int_{s_1}^{s_2} d^2_{ss} \int_{\mathcal{D}(s)} u_\ominus\, d\sigma\, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus\, d\gamma \right) ds\,,$$

and assuming that $\mathcal{D}(s)$ can not change shape, we have

$$
\begin{aligned}
-\int_{s_1}^{s_2} d_{ss}^2 \int_{\mathcal{D}(s)} u_\ominus \, d\sigma \, ds + \int_{s_1}^{s_2} d_s \left(\int_{\partial\mathcal{D}(s)} \nu u_\ominus \, d\gamma \right) ds &= -\int_{s_1}^{s_2} \left[d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) - d_s(\nu |\partial\mathcal{D}(s)| \overline{u}_\ominus) \right] ds \\
&= -\int_{s_1}^{s_2} \left[d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) - d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) \right] ds.
\end{aligned}
$$

Moreover, we have

$$
\int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma = \int_{\Gamma_\mathcal{P}} \lambda \, d\sigma = \int_{s_1}^{s_2} \int_{\partial\mathcal{D}(s)} \lambda d\gamma \, ds = \int_{s_1}^{s_2} |\partial\mathcal{D}| \overline{\lambda} \, ds \, .
$$

From the combination of all the above terms with the right hand side, we obtain that the solution $u_\ominus$ of (2.1b) satisfies,

$$
\int_{s_1}^{s_2} \left[-d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) + d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) + |\mathcal{D}(s)| \overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)| \overline{\lambda} \right] ds = \int_{s_1}^{s_2} |\mathcal{D}(s)| \overline{\overline{g}} \, ds \, .
$$

Since the choice of the points s_1, s_2 is arbitrary, we conclude that the following equation holds true,

$$
-d_{ss}^2 (|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) + d_s \left(d_s(|\mathcal{D}(s)|) \overline{u}_\ominus \right) + |\mathcal{D}(s)| \overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)| \overline{\lambda} = |\mathcal{D}(s)| \overline{\overline{g}} \quad \text{on } \Lambda \, , \tag{2.3}
$$

which is complemented by the following conditions at the boundary of Λ,

$$
|\mathcal{D}(s)| d_s \overline{\overline{u}}_\ominus = 0, \quad d_s |\mathcal{D}(s)| = 0, \quad \text{on} \quad s = 0, S. \tag{2.4}
$$

Then, we consider variational formulation of the averaged equation (2.3). After multiplication by a test function $v_\odot \in H^1(\Lambda)$, integration on Λ and suitable application of integration by parts, we obtain,

$$
\begin{aligned}
\int_\Lambda d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) d_s v_\odot \, ds - d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) v_\odot |_{s=0}^{s=S} - \int_\Lambda (d_s |\mathcal{D}(s)|) \overline{u}_\ominus d_s v_\odot \, ds + (d_s |\mathcal{D}(s)|) \overline{u}_\ominus v_\odot |_{s=0}^{s=S} \\
+ \int_\Lambda |\mathcal{D}(s)| \overline{\overline{u}}_\ominus v_\odot - \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda} v_\odot \, ds = \int_\Lambda |\mathcal{D}(s)| \overline{\overline{g}} V \, ds \, .
\end{aligned}
$$

Using boundary conditions, the identity $d_s(|\mathcal{D}(s)| \overline{\overline{u}}_\ominus) = |\mathcal{D}(s)| d_s \overline{\overline{u}}_\ominus + d_s(|\mathcal{D}(s)|) \overline{\overline{u}}_\ominus$ and reminding that $d_s |\mathcal{D}(s)|) / |\partial\mathcal{D}(s)| = \nu$, we obtain,

$$
(d_s \overline{\overline{u}}_\ominus, d_s v_\odot)_{\Lambda,|\mathcal{D}|} + (\nu(\overline{\overline{u}}_\ominus - \overline{u}_\ominus), d_s v_\odot)_{\Lambda,|\partial\mathcal{D}|} + (\overline{\overline{u}}_\ominus, v_\odot)_{\Lambda,|\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}}, V)_{\Lambda,|\mathcal{D}|} \, . \tag{2.5}
$$

where we have introduced the following weighted inner product notation,

$$
(u_\odot, v_\odot)_{\Lambda,w} = \int_0^S w(s) u_\odot(s) v_\odot(s) ds \, .
$$

Let us now formulate the modelling assumption that allows us to reduce equation (2.5) to a solvable one-dimensional (1D) model. More precisely, we assume that:

A1 the function $u_\ominus$ has a *uniform profile* on each cross section $\mathcal{D}(s)$, namely $u_\ominus(r, s, t) = u_\odot(s)$.

Therefore, observing that $u_\odot = \overline{u}_\ominus = \overline{\overline{u}}_\ominus$, problem (2.5) consists to find $u_\odot \in H^1(\Lambda)$ such that

$$
(d_s u_\odot, d_s v_\odot)_{\Lambda,|\mathcal{D}|} + (u_\odot, v_\odot)_{\Lambda,|\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}}, v_\odot)_{\Lambda,|\mathcal{D}|} \quad \forall v_\odot \in H^1(\Lambda) \, . \tag{2.6}
$$

Topological model reduction of the problem on $\Omega_\oplus$. We focus here on the subproblem of (2.1a) related to $\Omega_\oplus$. We multiply both sides of (2.1a) by a test function $v \in H_0^1(\Omega)$ and integrate on $\Omega_\oplus$. Integrating by parts and using boundary and interface conditions, we obtain

$$
\begin{aligned}
\int_{\Omega_\oplus} f v \, d\omega &= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\omega - \int_{\partial\Omega_\oplus} \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v \, d\sigma + \int_{\Omega_\oplus} u_\oplus v \\
&= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\Omega - \int_\Gamma \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v + \int_{\Omega_\oplus} u_\oplus v \\
&= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v \, d\Omega + \int_\Gamma \lambda v + \int_{\Omega_\oplus} u_\oplus v.
\end{aligned}
$$

Then, we make the following modelling assumptions:

A2 we identify the domain $\Omega_\oplus$ with the entire Ω, and we correspondingly omit the subscript $\oplus$ to the functions defined on $\Omega_\oplus$, namely

$$\int_{\Omega_\oplus} u_\oplus \, d\omega \simeq \int_\Omega u \, d\omega \, .$$

Therefore, we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\lambda, v)_\Gamma = (f, v)_\Omega$$

and combining with (2.6) we obtain the first formulation of the reduced problem.

Problem 1 (3D-2D-1D). Let $\langle \cdot, \cdot \rangle_\Gamma$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. The problem consists to find $u \in H^1_0(\Omega)$, $\lambda \in H^{-\frac{1}{2}}(\Gamma)$, $u_\odot \in H^1_0(\Lambda)$, such that

$$\begin{aligned} &(u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda), |\mathcal{D}|} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \lambda \rangle_\Gamma \\ &\qquad = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda), |\mathcal{D}|} \quad \forall v \in H^1_0(\Omega), \ v_\odot \in H^1(\Lambda) \end{aligned} \tag{2.7a}$$

$$\langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, \mu \rangle_\Gamma = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma) \, . \tag{2.7b}$$

Here, $\mathcal{T}_\Gamma : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ denotes the trace operator from Ω to Γ and $\mathcal{E}_\Gamma : H^1_0(\Lambda) \to H^1_0(\Gamma)$ denotes the uniform extension from Λ to Γ and we exploited the fact that

$$\left(\overline{\lambda}, v_\odot\right)_{\Lambda, |\partial\mathcal{D}|} = \int_\Lambda |\partial\mathcal{D}| \left(\frac{1}{|\partial\mathcal{D}|} \int_{\partial\mathcal{D}} \lambda \, d\gamma \right) v_\odot \, ds = (\lambda, \mathcal{E}_\Gamma v_\odot)_\Gamma.$$

Now, we apply a topological model reduction of the interface conditions, namely we go from a 3D-2D-1D formulation involving sub-problems on Ω and Λ and coupling operators defined on Γ to a 3D-1D-1D formulation where the coupling terms are set on Λ. To this purpose, let us write the Lagrange multiplier and the test functions on every cross section $\partial\mathcal{D}(s)$ as their average plus some fluctuation,

$$\lambda = \overline{\lambda} + \tilde{\lambda}, \qquad v = \overline{v} + \tilde{v}, \quad \text{on } \partial\mathcal{D}(s) \, ,$$

where $\overline{\tilde{\lambda}} = \overline{\tilde{v}} = 0$. Therefore, the coupling term on Γ can be decomposed as,

$$\int_\Gamma \lambda v \, d\sigma = \int_\Lambda \int_{\partial\mathcal{D}(s)} (\overline{\lambda} + \tilde{\lambda})(\overline{v} + \tilde{v}) d\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda} \overline{v} \, ds + \int_\Lambda \int_{\partial\mathcal{D}(s)} \tilde{\lambda} \tilde{v} d\gamma ds \, .$$

Then, we make the following modelling assumptions:

A3 we assume that the product of fluctuations is small, namely

$$\int_{\partial\mathcal{D}(s)} \tilde{\lambda} \tilde{v} d\gamma \simeq 0$$

and the term $(\mathcal{T}_\Gamma v, \lambda)_\Gamma$ becomes $\left(\overline{\mathcal{T}}_\Lambda v, \overline{\lambda}\right)_{\Lambda, |\partial\mathcal{D}|}$, where $\overline{\mathcal{T}}_\Lambda$ denotes the composition of operators $\mathcal{T}_\Gamma \circ \overline{(\cdot)}$. Combined with (2.6), this leads to the second formulation of the reduced problem.

Problem 2 (3D-1D-1D). Let $\langle \cdot, \cdot \rangle_\Lambda$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. The problem requires to find $u \in H^1_0(\Omega)$, $u_\odot \in H^1_0(\Lambda)$, $\lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned} &(u, v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda), |\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \lambda_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} \\ &\qquad = (f, v)_{L^2(\Omega)} + (\overline{\overline{g}}, V)_{L^2(\Lambda), |\mathcal{D}|} \quad \forall v \in H^1_0(\Omega), \ v_\odot \in H^1_0(\Lambda) \, , \end{aligned} \tag{2.8a}$$

$$\langle \overline{\mathcal{T}}_\Lambda u - u_\odot, \mu_\odot \rangle_{H^{-\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} = 0 \quad \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda) \, , \tag{2.8b}$$

where $\overline{\mathcal{T}}_\Lambda$ denotes the composition of operators $\mathcal{T}_\Gamma \circ \overline{(\cdot)}$. We notice that all the integrals of the reduced problem are well defined because $\overline{\mathcal{T}}_\Lambda : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Lambda)$ as shown in the following.

From the theory of interpolation of Sobolev spaces illustrated in [3], we can define the fractional norms $\|\cdot\|_{H^{\frac{1}{2}}(\Gamma)}$ and $\|\cdot\|_{H^{\frac{1}{2}}(\Lambda)}$ as functions of the eigenvalues and eigenfunctions of the Laplacian on Γ and Λ respectively. In particular, from [3, Lemma 4.11], if we denote as ϕ_{ij} and ρ_{ij}, for $i = 1, 2, \dots,\ j = 0, 1, \dots,$ the eigenfunctions and the eigenvalues of the Laplacian on Γ with homogeneous Dirichlet condition at the boundary, for any function $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ we have

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} = \left(\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (1+\rho_{ij})^{\frac{1}{2}} |a_{ij}|^2 \right)^{\frac{1}{2}}, \text{ with } a_{ij} = (u, \phi_{ij})_\Gamma . \tag{2.9}$$

Similarly, denoting with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the Laplacian on Λ with homogenous Dirichlet boundary conditions, if $u \in H^{\frac{1}{2}}(\Lambda)$ we have

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)} = \left(\sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}} |a_i|^2 \right)^{\frac{1}{2}}, \text{ with } a_i = (u, \phi_i)_\Lambda . \tag{2.10}$$

In the same way, we can define the equivalent weighted norm in $H^{\frac{1}{2}}_{00}(\Lambda)$

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} = \left(\sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}} |a_i|^2 \right)^{\frac{1}{2}}, \text{ with } a_i = (u, \phi_i)_{\Lambda,|\partial\mathcal{D}|} . \tag{2.11}$$

Lemma 2.1. *Let Γ be a tensor product domain, $\Gamma = (0, X) \times (0, Y)$. For any regular $u(x, y)$ in Γ, let $\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x, y)\,dy$. Then, for any $u \in H^{\frac{1}{2}}_{00}(\Gamma)$, then $\overline{u}(x) \in H^{\frac{1}{2}}_{00}((0, X))$. Moreover, if $u(x, y) \in H^{\frac{1}{2}}_{00}(\Gamma)$ is constant with respect to y, namely $u(x, y) = u(x)$, then*

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = Y\|u\|_{H^{\frac{1}{2}}_{00}(0,X)}.$$

Proof. In order to apply (2.9) and (2.10), let us consider the eigenvalue problems for the Laplace operator on Γ with homogeneous Dirichlet conditions at $x = 0, X$ and periodic boundary conditions at $y = 0, Y$. Let us also consider the Laplace eigeinproblem on $(0, X)$ with homogeneous Dirichlet conditions. Let us denote as $\phi_{ij}(x, y)$ and ρ_{ij}, for $i = 1, 2, \dots,\ j = 0, 1, \dots,$ the eigenfunctions and the eigenvalues of the Laplacian on Γ, and with $\phi_i(x)$ and ρ_i the eigenfunctions and the eigenvalues of the laplacian on $(0, X)$. In particular,

$$\begin{aligned} \phi_{ij}(x,y) = \sin\left(\frac{i\pi x}{X}\right)\left(cos\left(\frac{j2\pi y}{Y}\right) + sin\left(\frac{j2\pi y}{Y}\right)\right), \\ \rho_{ij} = \left(\frac{i\pi}{X}\right)^2 + \left(\frac{j2\pi}{Y}\right)^2, \\ \phi_i(x) = \sin\left(\frac{i\pi x}{X}\right), \\ \rho_i = \left(\frac{i\pi}{X}\right)^2. \end{aligned}$$

It is easy to verify that

$$\int_0^Y \phi_{ij}(x,y) = 0 \quad \forall j > 0, \forall i \tag{2.12}$$

$$\int_0^Y \phi_{ij}(x,y) = Y\, \sin\left(\frac{i\pi x}{X}\right) \quad \text{if } j = 0, \forall i. \tag{2.13}$$

Moreover we recall that $\phi_{i,j}(x,y)$ and $\phi_i(x)$ form an orthogonal basis of $L^2(\Gamma)$ and $L^2(0,X)$ respectively. Therefore,

$$\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x,y)\,dy = \frac{1}{Y}\int_0^Y \sum_{i,j} a_{i,j}\phi_{i,j}(x,y)\,dy$$
$$= \frac{1}{Y}\sum_{i,j} a_{i,j}\int_0^Y \phi_{i,j}(x,y)\,dy = \sum_i a_{i,0}\phi_i(x).$$

From (2.10) we have

$$\begin{aligned}
\|\overline{u}\|^2_{H^{\frac{1}{2}}(0,X)} &= \sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}\,a_i^2 \\
&= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X \overline{u}(x)\sin\left(\frac{i\pi x}{X}\right)dx\right)^2 \\
&= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}a_{j,0}\int_0^X \sin\left(\frac{j\pi x}{X}\right)\sin\left(\frac{i\pi x}{X}\right)dx\right)^2 \\
&= \sum_{i=1}^{\infty}\frac{X^2}{4}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}} a_{i,0}^2 \\
&\leq \frac{X^2}{4}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}|a_{i,j}|^2 = \frac{X^2}{4}\|u\|^2_{H^{\frac{1}{2}}(\Gamma)},
\end{aligned}$$

where we have used the fact that

$$\int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)dx = 0 \quad \text{if } i \neq j$$
$$\int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)dx = \frac{X}{2} \quad \text{if } i = j$$

and we have applied (2.9) in the last equality. Moreover, in the case in which u is constant with respect to y, we have

$$\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}|a_{ij}|^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X\int_0^Y u(x,y)\phi_{ij}(x,y)\,dy\,dx\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X u(x)\int_0^Y \phi_{ij}(x,y)\,dydx\right)^2,
\end{aligned}$$

and using (2.12) and (2.13), we obtain

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X Y u(x)\,\sin\left(\frac{i\pi x}{X}\right)dx\right)^2$$
$$= Y^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}|a_i|^2 = Y^2\|u\|^2_{H^{\frac{1}{2}}(0,X)}. \quad \square$$

COROLLARY 2.2. *Let Γ be the lateral surface of a cylinder of radius R. Let $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ and let $\overline{u}(s) = \frac{1}{2\pi R}\int_0^{2\pi} u(s,\theta) R\, d\theta$. Then $\overline{u} \in H^{\frac{1}{2}}_{00}(0,S)$ and there exists a constant C_Γ such that*

$$\|\overline{u}\|_{H^{\frac{1}{2}}(0,S)} \leq C_\Gamma \|u\|_{H^{\frac{1}{2}}(\Gamma)}.$$

COROLLARY 2.3. *Let Γ be the lateral surface of a generalized cylinder, being $\partial\mathcal{D}(s)$ the boundary of the cross section and Λ its centerline. Let $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ and let $\overline{u}(s) = \frac{1}{|\partial\mathcal{D}(s)|}\int_{\partial\mathcal{D}} u\, d\gamma$. Then $\overline{u} \in H^{\frac{1}{2}}_{00}(\Lambda)$ and there exists a constant C_Γ such that*

$$\|\overline{u}\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \leq C_\Gamma \|u\|_{H^{\frac{1}{2}}(\Gamma)}.$$

The well-posedness of (2.7) and (2.8) can be studied in the framework of the classical theory of saddle point problems as shown in the following.

3. Saddle-point problem analysis. Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bounded bilinear forms. Let us consider the general saddle point problem of the form: find $u \in X$, $\lambda \in Q$ s.t.

$$\begin{cases} a(u,v) + b(v,\lambda) = c(v) & \forall v \in X \\ b(u,\mu) = d(\mu) & \forall \mu \in Q. \end{cases} \tag{3.1}$$

We denote with A and B the operators associated to the bilinear forms a and b, namely $A : X \longrightarrow X'$ with $\langle Au, v\rangle_{X',X} = a(u,v)$ and $B : X \longrightarrow Q'$ with $\langle Bv, \mu\rangle_{Q',Q} = b(v,\mu)$. Problem (3.1) embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem.

THEOREM 3.1. *[4, Theorem 2.34] Problem (3.1) is well posed iff*

$$\begin{cases} \exists \alpha > 0 : \inf_{u \in ker(B)} \sup_{v \in ker(B)} \frac{a(u,v)}{\|u\|_X \|v\|_X} \geq \alpha \\ \forall v \in ker(B),\ (\forall u \in ker(B),\ a(u,v) = 0) \implies v = 0. \end{cases} \tag{3.2}$$

and

$$\exists \beta > 0 : \inf_{\mu \in Q} \sup_{v \in X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Q} \geq \beta. \tag{3.3}$$

Notice that if a is coercive on $ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1. It consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ \lambda \in H^{-\frac{1}{2}}(\Gamma)$, solutions of (3.1), where

$$a([u,u_\odot],[v,v_\odot]) = (u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|}$$

$$b([v,v_\odot],\mu) = \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma$$

$$c([v,v_\odot]) = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|}$$

$$d(\mu) = 0$$

We prove that the hypothesis of 3.1 are fullfilled choosing $X = H^1_0(\Omega) \times H^1_0(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$, where X is equipped with the norm $|||[u,u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + \|u_\odot\|^2_{H^1(\Lambda),|\mathcal{D}|}$ and Q equipped with the norm

$$\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} := \sup_{q \in H^{\frac{1}{2}}_{00}(\Gamma)} \frac{\langle q, \mu\rangle_\Gamma}{\|q\|_{H^{\frac{1}{2}}(\Gamma)}}$$

.

LEMMA 3.2. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The bilinear form $a(\cdot\ ,\ \cdot)$ is clearly bounded since

$$a([u,u_\odot],[v,v_\odot]) \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + \|u_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|} \leq 2|||[u,u_\odot]|||\,|||[v,v_\odot]|||.$$

Concerning the bilinear form $b(\cdot\ ,\ \cdot)$ we have

$$\begin{aligned}
b([v,v_\odot],\mu) &= \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma \leq \|\mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot\|_{H^{\frac{1}{2}}(\Gamma)}\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\
&\leq \left(\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^{\frac{1}{2}}(\Gamma)}\right)\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \left(C_T\|v\|_{H^1(\Omega)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^1(\Gamma)}\right)\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\
&\leq \left(C_T\|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\right)\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \\
&\leq \left(C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\right)|||[v,v_\odot]|||\,\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \square
\end{aligned}$$

LEMMA 3.3. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive .*

Proof. Indeed, we have, $\square$

$$a([u,u_\odot],[u,u_\odot]) = (u,u)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot,u_\odot)_{H^1(\Lambda)} = |||[u,u_\odot]|||^2.$$

LEMMA 3.4. *The inf-sup inequality (3.3) is fulfilled, namely $\exists\beta_1 > 0$ such that $\forall\mu \in H^{-\frac{1}{2}}(\Gamma)$:*

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v,v_\odot]|||} \geq \beta_1 \sup_{q\in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle q,\mu\rangle_\Gamma}{\|q\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

Proof. We choose $v_\odot \in H_0^1(\Lambda)$ such that $\mathcal{E}_\Gamma v_\odot = 0$. Therefore,

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v,v_\odot]|||} \geq \sup_{v\in H_0^1(\Omega)} \frac{\langle \mathcal{T}_\Gamma v,\mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

We notice that the trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. Indeed, $\forall\xi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we can find v solution of

$$\begin{aligned}
-\Delta v &= 0 \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial\Omega \\
v &= \xi \quad \text{on } \Gamma.
\end{aligned}$$

We denote with $\mathcal{E}_\Omega$ the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|\mathcal{E}_\Omega\| \in \mathbb{R}$ such that $v = \mathcal{E}_\Omega(\xi)$ and $\|v\|_{H^1(\Omega)} \leq \|\mathcal{E}_\Omega\|\|\xi\|_{H^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$\sup_{v\in H_0^1(\Omega)} \frac{\langle \mathcal{T}_\Gamma v,\mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi\in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle \xi,\mu\rangle_\Gamma}{\|\mathcal{E}_\Omega\|\|\xi\|_{H^{\frac{1}{2}}(\Gamma)}} = \|\mathcal{E}_\Omega\|^{-1}\|\mu\|_{H^{-\frac{1}{2}}(\Gamma)}, \tag{3.4}$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H_{00}^{\frac{1}{2}}(\Gamma))^*$. $\square$

3.2. Problem 2. This problem requires to find $u \in H_0^1(\Omega),\ u_\odot \in H_0^1(\Lambda),\ \lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, solution of (3.1) with

$$a([u,u_\odot],[v,v_\odot]) = (u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|}$$

$$b([v,v_\odot],\mu_\odot) = \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}$$

$$c([v,v_\odot]) = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|}$$

$$d(\mu_\odot) = 0$$

We prove that the hypotesis of Theorem 3.1 are fulfilled with the following spaces $X = H_0^1(\Omega)\times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped again with the norm $|||[\cdot,\cdot]|||$ and Q equipped with the norm $\|\cdot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}$, defined as

$$\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} := \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q,\mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}}$$

Then, we have the following lemmas.

Lemma 3.5. *The bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ are bounded.*

Proof. The boundedness of $a(\cdot\ ,\ \cdot)$ can be proved as in Lemma 3.2. Concerning $b(\cdot\ ,\ \cdot)$, we have

$$\begin{aligned}
b([v,v_\odot],\mu_\odot) &= \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \le \|\overline{\mathcal{T}}_\Lambda v - v_\odot\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(\|\overline{\mathcal{T}}_\Lambda v\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} + \|v_\odot\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(C_\Gamma\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} + \|v_\odot\|_{H^1(\Lambda),|\partial\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(C_\Gamma C_T\|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|}\right)\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}\\
&\le \left(C_\Gamma C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}\right)|||[v,v_\odot]|||\|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}. \qquad \square
\end{aligned}$$

Lemma 3.6. *The bilinear form $a(\cdot\ ,\ \cdot)$ is coercive.*

Lemma 3.7. *The inf-sup inequality (3.3) holds, namely $\exists\beta_2>0$ such that $\forall\mu_\odot\in H^{-\frac{1}{2}}(\Lambda)$, :*

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v,v_\odot]|||} \ge \beta_2\|\mu_\odot\|_{H^{\frac{1}{2}}(\Lambda)}.$$

We choose $v_\odot = 0$ and we obtain

$$\sup_{\substack{v\in H_0^1(\Omega),\\ v_\odot\in H_0^1(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v,v_\odot]|||} \ge \sup_{v\in H_0^1(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}}.$$

For any $q\in H_{00}^{\frac{1}{2}}(\Lambda)$, we consider its uniform extension to Γ named as $\mathcal{E}_\Gamma q$ and then we consider the harmonic extension $v = \mathcal{E}_\Omega\mathcal{E}_\Gamma q \in H_0^1(\Omega)$. It follows that $\overline{\mathcal{T}}_\Lambda v = q$. Therefore,

$$\sup_{v\in H_0^1(\Omega)} \langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \ge \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \langle q,\mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}\,.$$

Moreover, using Lemma 2.1 we obtain

$$\|v\|_{H_0^1(\Omega)} \leq \|\mathcal{E}_\Omega\| \|\mathcal{E}_\Gamma q\|_{H^{\frac{1}{2}}(\Gamma)} = \|\mathcal{E}_\Omega\| \|q\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.$$

Therefore,

$$\sup_{v\in H_0^1(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} \geq \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} \geq \|\mathcal{E}_\Omega\|^{-1} \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)} \frac{\langle q, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}}$$
$$= \|\mathcal{E}_\Omega\|^{-1} \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.$$

4. Finite element approximation. In this section we consider the discretization of Problem 1 and 2 by means of the finite element method. We address two main challenges; first we aim to identify a suitable approximation space for the Lagrange multiplier and to analyze the stability of the discrete saddle point problem; second we aim to derive a stable discretization method that uses indepent and conforming computational meshes for Ω, Γ and Λ. Let us introduce a shape-regular triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_h^\Lambda$ of Λ. We analyze two different cases: the one in which the 3D mesh is conforming to the interface Γ, namely the set of the intersections of the 3D elements of $\mathcal{T}_h^\Omega$ with Γ is constituted by facets of such elements, and the non conforming case, namely the interface Γ cuts the mesh arbitrarly. The discrete equivalent of (3.1) reads as finding $u_h \in X_h \subset X$, $\lambda_h \in Q_h \subset Q$ s.t.

$$\begin{cases} a(u_h, v_h) + b(v_h, \lambda_h) = c(v_h) & \forall v_h \in X_h \\ b(u_h, \mu_h) = d(\mu_h) & \forall \mu_h \in Q_h. \end{cases} \tag{4.1}$$

Let $B_h : Q_h' \longrightarrow Q_h$ be the operator induced by b such that $\langle B_h v_h, \mu_h\rangle_{Q_h',Q_h} = b(v_h, \mu_h)$. The well posedness of such problem is governed by the classical inf-sup theory in Banach spaces. The main result is reported below.

Corollary 4.1. *[4, Theorem 2.42]Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be continuous bilinear forms. Problem (4.1) is well-posed if and only if*

$$\exists \alpha_h > 0 : \inf_{u_h\in ker(B_h)} \sup_{v_h\in ker(B_h)} \frac{a(u_h, v_h)}{\|u_h\|_X \|v_h\|_X} \geq \alpha_h \tag{4.2}$$

and

$$\exists \beta_h > 0 : \inf_{\mu_h\in Q_h} \sup_{v_h\in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X \|\mu_h\|_Q} \geq \beta_h. \tag{4.3}$$

This corollary is the discrete counterpart of Theorem 3.1 where at the discrete level condition (4.2) implies both of (3.2). Conversely, (4.3) does not follow from the conformity of the finte element spaces and it must be analysed independetly of (3.3). Let us notice that for both problem 1 and problem 2 the bilinear form $a(\cdot,\cdot)$ is coercive as stated in Lemmas (3.3) and (3.6). Consequently, (4.2) is automatically satisfied, being α_h the coercivity constant.

4.1. $\mathcal{T}_h^\Omega$ conforming to Γ. We first analyze the case in which the 3D mesh is conforming to the interface Γ. With this aim, we define conformity conditions between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$ with Γ. More precisely we require that the intersection of $\mathcal{T}_h^\Omega$ and Γ is made of entire faces of elemets $K \in \mathcal{T}_h^\Omega$. Furthermore, we also set a restriction between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$. We assume that Λ is a piecewise linear manifold. We want that the intersection of Γ with any orthogonal plane to Λ that crosses Λ at the internal nodes of $\mathcal{T}_h^\Lambda$, consists of entire edges of $\mathcal{T}_h^\Omega$. Namely the intersection of Γ with orthogonal planes to Λ is conformal with $\mathcal{T}_h^\Lambda$.

4.1.1. Problem 1. We denote by $X_{h,0}^k(\Omega) \subset H_0^1(\Omega)$, with $k > 0$, the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satisfying homogeneous Dirichlet conditions on the boundary and by $X^k_{h,0}(\Lambda) \subset H^1_0(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ, satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. Problem 1 consists to find $u_h \in X^k_{h,0}(\Omega)$, $u_{\odot h} \in X^k_{h,0}(\Lambda)$, $\lambda_h \in Q_h \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned} (4.4a)\qquad & (u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Lambda v_{\odot h}, \lambda_h\rangle_\Gamma \\ & \qquad = (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X^k_{h,0}(\Omega),\ v_{\odot h} \in X^k_{h,0}(\Lambda) \\ (4.4b)\qquad & \langle \mathcal{T}_\Gamma u_h - \mathcal{E}_\Lambda u_{\odot h}, \mu_h\rangle_\Gamma = 0 \quad \forall \mu_h \in Q_h\,, \end{aligned}$$

The space Q_h must be suitably chosen such that (4.3) holds. Let Q_h be the trace space of functions running in $X^k_{h,0}(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$.As a result, $Q_h = X^k_{h,0}(\Gamma)$. Therefore we impose homogeneous Dirichlet boundary condition on $\partial\Omega$ also for the Lagrange multiplier. For this choice of Q_h we can prove the well-posedness of the discrete problem, as shown in the following.

LEMMA 4.2. *Let $P_h : H^{\frac{1}{2}}_{00}(\Gamma) \longrightarrow Q_h$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Gamma)$ by*

$$(P_h v, \psi_h)_\Gamma = (v, \psi_h)_\Gamma \qquad \forall \psi_h \in Q_h.$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Gamma)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}(\Gamma)} \le C\|v\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{4.5}$$

where C is a positive constant independent of h.

Proof. We prove that P_h is continuous on $L^2(\Gamma)$ and on $H^1_0(\Gamma)$ following [4, Section 1.6.3]. Then, the inequality (4.5) can be dirived by Hilbertian interpolation. For the L^2-continuity, we exploit the fact that, from the definition of P_h,

$$(v - P_h v, P_h v)_\Gamma = 0.$$

Therefore, by Pythagoras identity,

$$\|v\|^2_{L^2(\Gamma)} = \|v - P_h v\|^2_{L^2(\Gamma)} + \|P_h v\|^2_{L^2(\Gamma)} \ge \|P_h v\|^2_{L^2(\Gamma)}.$$

Let us now consider $v \in H^1_0(\Gamma)$.The Scott-Zhang interpolation operator SZ_h from $H^1_0(\Gamma)$ to Q_h satisfies the following inequalities,

$$\|SZ_h v\|_{H^1(\Gamma)} \le C_1\|v\|_{H^1(\Gamma)} \tag{4.6}$$

$$\|v - SZ_h v\|_{L^2(\Gamma)} \le C_2 h\|v\|_{H^1(\Gamma)}. \tag{4.7}$$

Therefore, using (4.6),

$$\begin{aligned} \|\nabla P_h v\|_{L^2(\Gamma)} &\le \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + \|\nabla SZ_h v\|_{L^2(\Gamma)} \\ &\le \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \end{aligned}$$

and by using the inverse inequality we obtain

$$\begin{aligned} \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} &\le \frac{C_3}{h}\|P_h v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &= \frac{C_3}{h}\|P_h(v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &\le \ \text{(Stability of } P_h \text{ in } L^2)\ \frac{C_3}{h}\|v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &\le \ \text{(using (4.7))}\ \frac{C_3}{h}C_2 h\|v\|_{H^1(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &\le (C_2C_3 + C_1)\|v\|_{H^1(\Gamma)}, \end{aligned}$$ ∎

from which we obtain the continuity in $H^1_0(\Gamma)$.

LEMMA 4.3. *There exists a constant $\gamma > 0$ such that for any $\mu_h \in Q_h$*

$$\sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \geq \gamma \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. Let μ_h be in Q_h. From the continuous case, in particular from (3.4), we have

$$\|\mathcal{E}_\Omega\|^{-1} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|v\|_{H^1(\Omega)}}$$

and by the trace inequality $\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_T \|v\|_{H^1(\Omega)}$ (see [1, 7.56]), we obtain

$$\sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|v\|_{H^1(\Omega)}} \leq C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

By the definition of P_h and (4.5)

$$\begin{aligned} C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} &= C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle P_h(\mathcal{T}_\Gamma v), \mu_h \rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} \\ &\leq C_T C \sup_{v \in H^1_0(\Omega)} \frac{\langle P_h(\mathcal{T}_\Gamma v), \mu_h \rangle}{\|P_h(\mathcal{T}_\Gamma v)\|_{H^{\frac{1}{2}}(\Gamma)}} \\ &= C_T C \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}}. \end{aligned}$$

∎

THEOREM 4.4 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists \beta_{h,1} > 0$ s.t.*

$$\inf_{\mu_h \in Q_h} \sup_{\substack{v_h \in X^k_{h,0}(\Omega), \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Gamma v_{\odot h}, \mu_h \rangle_\Gamma}{|||[v_h, v_{\odot h}]||| \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \beta_{h,1}. \tag{4.8}$$

Proof. Let $\mu_h \in Q_h$. As in the continuos case, we choose $v_{\odot h} = 0$ and we have

$$\sup_{\substack{v_h \in X^k_{h,0}(\Omega), \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Gamma v_{\odot h}, \mu_h \rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h, \mu_h \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Therefore, we want to prove that there exists $\beta_{h,1}$ such that

$$\sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h, \mu_h \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,1} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall \mu_h \in Q_h.$$

Using Lemma 4.3 and the boundedness of the armonic extension operator $\mathcal{E}_\Omega$ from $H^{\frac{1}{2}}_{00}(\Gamma)$ to $H^1_0(\Omega)$ introduced in the previous section, we have

$$\gamma \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}_\Omega\| \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle_\Gamma}{\|\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}}.$$

Let $R_h : H^1_0(\Omega) \to X^k_{h,0}(\Omega)$ be a quasi interpolation operator (such as the Scott-Zhang operator) satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq C_R \|v\|_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega).$$

Therefore, we obtain

$$\|\mathcal{E}_\Omega\| \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}} \le \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}}$$

and we have

$$\begin{aligned}(4.9)\quad \gamma\|\mu_h\|_{H^{-\frac12}(\Gamma)} &\le \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|q_h\|_{H^{\frac12}(\Gamma)}} \le \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}_\Omega q_h\|_{H^1(\Gamma)}}\\ &= \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle \mathcal{T}_\Gamma R_h\mathcal{E}_\Omega q_h,\mu_h\rangle_\Gamma}{\|R_h\mathcal{E}_\Omega q_h\|_{H^1(\Omega)}} \le \|\mathcal{E}_\Omega\| C_R \sup_{v_h\in X_{h,k}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h,\mu_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.\end{aligned}$$

Therefore the inf-sup condition (4.8) holds with $\beta_{h,1} = \gamma\|\mathcal{E}_\Omega\|^{-1}C_R^{-1}$. We notice that in (4.9) we exploit the fact that the operator $\mathcal{T}_\Gamma R_h \mathcal{E}_\Omega$ coincides with the identity on the space Q_h, thanks to the conformity of $\mathcal{T}_h^\Omega$ to the interface Γ. ∎

4.1.2. Problem 2. This problem requires to find $u_h \in X^k_{h,0}(\Omega)$, $u_{\odot h} \in X^k_{h,0}(\Lambda)$, $\lambda_{\odot h} \in Q_h \subset H^{-\frac12}(\Lambda)$, such that

$$\begin{aligned}(4.10a)\qquad &(u_h,v_h)_{H^1(\Omega)} + (u_{\odot h},v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v_h - v_{\odot h},\lambda_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}\\ &\quad = (f,v_h)_{L^2(\Omega)} + (\overline{\overline{g}},v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h}\in X_h(\Lambda)\\ (4.10b)\qquad &\langle \overline{\mathcal{T}}_\Lambda u_h - u_{\odot h},\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|} = 0 \quad \forall \mu_{\odot h}\in Q_h\,.\end{aligned}$$

We choose $Q_h = X^k_{h,0}(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda\cap\partial\Omega$ also for the Lagrange multiplier. With this choice for Q_h, we can prove the well-posedness of the discrete problem. In particular, following the same steps as for Problem 1, we can prove the following results.

LEMMA 4.5. *Let $P_h : H^{\frac12}_{00}(\Lambda) \longrightarrow Q_h$ be the orthogonal projection operator defined for any $v\in H^{\frac12}_{00}(\Lambda)$ by*

$$(P_h v,\psi)_{\Lambda,|\partial\mathcal{D}|} = (v,\psi)_{\Lambda,|\partial\mathcal{D}|} \qquad \forall\psi\in Q_h.$$

Then, P_h is continuous on $H^{\frac12}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac12}(\Lambda),|\partial\mathcal{D}|} \le C\|v\|_{H^{\frac12}(\Lambda),|\partial\mathcal{D}|},$$

where C is a positive constant independent of h.

LEMMA 4.6. *There exist a constant $\gamma>0$ such that*

$$\sup_{q_h\in Q_h} \frac{\langle q_h,\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac12}(\Lambda),|\partial\mathcal{D}|}} \ge \gamma\|\mu_{\odot h}\|_{H^{-\frac12}(\Lambda)} \qquad \forall \mu_{\odot h}\in Q_h.$$

The proofs are equivalent ot the ones of Lemmas 4.5 and 4.3.

THEOREM 4.7 (Discrete inf-sup). *The inequality (4.3) holds, namely $\exists\beta_{h,2}>0$ s.t.*

$$(4.11)\qquad \inf_{\mu_h\in Q_h} \sup_{\substack{v_h\in X^k_{h,0}(\Omega),\\ v_{\odot h}\in X^k_{h,0}(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h - v_{\odot h},\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h,v_{\odot h}]|||\,\|\mu_{\odot h}\|_{H^{-\frac12}(\Lambda)}} \ge \beta_{h,2}.$$

Proof. Let $\mu_{\odot h}$ be arbitrarly chosen in Q_h. Again, we choose $v_{\odot h}=0$, so that the proof reduces to show that there exists $\beta_{h,2}$ such that

$$\sup_{v_h\in X^k_{h,0}(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h,\mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \ge \beta_{h,2}\|\mu_{\odot h}\|_{H^{-\frac12}(\Lambda)} \qquad \forall\mu_{\odot h}\in Q_h.$$

From Lemma 2.1 and its corollaries, for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{E}_\Gamma w\|_{H^{\frac{1}{2}}(\Gamma)} = \|w\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.$$

Consequently, from Lemma 4.6, using again the extension operator $\mathcal{E}_\Omega$ from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^k_{h,0}(\Omega)$, we obtain

$$
\begin{aligned}
(4.12)\quad \gamma\|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h\in Q_h} \frac{\langle q_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \\
&= \sup_{q_h\in Q_h} \frac{\langle q_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{E}_\Gamma q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|\mathcal{E}_\Omega\| \sup_{q_h\in Q_h} \frac{\langle q_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{E}_\Omega\mathcal{E}_\Gamma q_h\|_{H^1(\Omega)}} \\
&\leq \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle q_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|R_h\mathcal{E}_\Omega\mathcal{E}_\Gamma q_h\|_{H^1(\Omega)}} \\
&= \|\mathcal{E}_\Omega\| C_R \sup_{q_h\in Q_h} \frac{\langle \overline{\mathcal{T}}_\Lambda R_h\mathcal{E}_\Omega\mathcal{E}_\Gamma q_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|R_h\mathcal{E}_\Omega\mathcal{E}_\Gamma w_h\|_{H^1(\Omega)}} \\
&\leq \|\mathcal{E}_\Omega\| C_R \sup_{v_h\in X_h(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h, \mu_{\odot h}\rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned}
$$

Also in this case to prove the disrete inf-sup condition we exploit the conformity of the meshes on Ω, Γ and Λ and the fact that the operator $\overline{\mathcal{T}}_\Lambda R_h\mathcal{E}_\Omega\mathcal{E}_\Gamma$ coincides with the identity if applied to functions in Q_h. ∎

4.2. $\mathcal{T}_h^\Omega$ non conforming to Γ. We analyze now the case in which the elements of the 3D mesh $\mathcal{T}_h^\Omega$ cut the interface Γ. It is easy to understand that the formulation of Problem 2 is more suitable. Therefore we focus on the analysis of Problem 2.

4.2.1. Problem 2. We consider for the solutions u_h and $u_{\odot h}$ the spaces $X^1_{h,0}(\Omega)$ and $X^1_{h,0}(\Lambda)$, see the previous subsection for the definition. Notice that in this case we suppose that the mesh sizes of the 3D mesh $\mathcal{T}_h^\Omega$ and the 1D mesh $\mathcal{T}_{h'}^\Lambda$ are different, in particular we suppose the 1D mesh is finer. Concerning the multiplier space, let $\mathcal{G}_h = \{K \in \mathcal{T}_h^\Omega : K \cap \Lambda \neq \emptyset\}$, namely the set of the 3D elements which intersect Λ, and let us define $Q_h = \{\lambda_{\odot h} : \lambda_{\odot h} \in P^0(K)\,\forall K \in \mathcal{G}_h\}$. We notice that we are extending the multiplier to the 3D elements: equivalenty we could have defined λ_h as a piecewise constant function on $K \cap \Lambda$ for any $K \in \mathcal{G}_h$, but we use this trick so that we do not need to consider the 1D mesh given by the intersection of the elements of $\mathcal{G}_h$ with Λ. With this choice the problem is not inf-sup stable, therefore the idea is to add a stabilization term $s(\lambda_{\odot h}, \mu_{\odot h})$ to (4.10a) following the approach introduced in [2]. In particular, we build a new multiplier space L_h for which the discrete inf-sup condition is fulfilled and we build a projection operator $\pi_L : Q_h \to L_h$ such that for any $[v, v_\odot] \in X$

$$b([v, v_\odot], \lambda_{\odot h} - \pi_L\lambda_{\odot h}) \lesssim |||[v, v_\odot]||| \|\lambda_{\odot h} - \pi_L\lambda_{\odot h}\|_{L_h}, \tag{4.13}$$

where for X and $|||[\cdot,\cdot]|||$ we use the definitions of section 3.2 and $\|\cdot\|_{L_h}$ denotes a suitable discrete norm on L_h. Then, the following lemma holds

Lemma 4.8. *[2, Lemma2.1] Under the previous assumption, $\forall\lambda_{\odot h} \in Q_h$,*

$$\|\lambda_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)} \lesssim \sup_{\substack{v_h\in X^1_{h,0}(\Omega)\\ v_{\odot h}\in X^1_{h,0}(\Lambda)}} \frac{b([v_h, v_{\odot h}], \lambda_{\odot h})}{|||[v_h, v_{\odot h}]|||} + \|\lambda_{\odot h} - \pi_L\lambda_{\odot h}\|_{L_h}$$

Based on this projection operator, we build the stabilization term $s(\lambda_{\odot h}, \mu_{\odot h})$ satisfying

$$\|\lambda_{\odot h} - \pi_L\lambda_{\odot h}\|_{L_h} \lesssim s(\lambda_{\odot h}, \lambda_{\odot h})$$

and prove that $\forall[u_h, u_{\odot h}]$, there exists $\xi_h([u_h, u_{\odot h}]) \in Q_h$ s.t.

$$a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \geq \alpha_\xi |||[u_h, u_{\odot h}]|||_{X^1_{h,0}(\Omega)\times X^1_{h,0}(\Lambda)}, \tag{4.14}$$

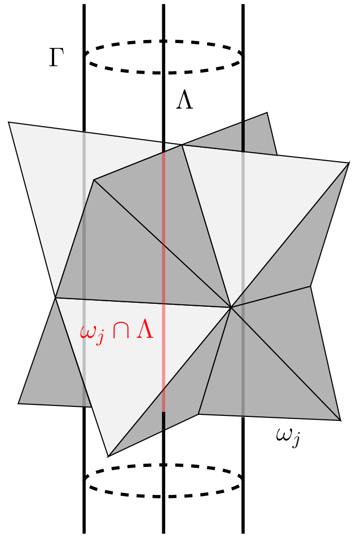


FIGURE 4.1. *Extended patches ω_j.*

$$(s(\xi_h,\xi_h))^{\frac{1}{2}} \leq c_s |||[u_h,u_{\odot h}]|||_{X^1_{h,0}(\Omega)\times X^1_{h,0}(\Lambda)}, \tag{4.15}$$

being $|||[\cdot\,,\,\cdot]|||_{X^1_{h,0}(\Omega)\times X^1_{h,0}(\Lambda)}$ a suitable discrete norm. Then, the stabilized problem

$$\begin{aligned} &a([u_h,u_{\odot h}],[v_h,v_{\odot h}]) + b([v_h,v_{\odot h}],\lambda_{\odot h}) + \\ &\qquad b([u_h,u_{\odot h}],\mu_{\odot h}) - s(\lambda_{\odot h},\mu_{\odot h}) = c(v_h) + d(\mu_{\odot h}) \\ &\qquad\qquad \forall [v_h,v_{\odot h}] \in X^1_{h,0}(\Omega)\times X^1_{h,0}(\Lambda),\ \forall \mu_{\odot_h} \in Q_h \end{aligned} \tag{4.16}$$

is well posed thanks to the following lemma.

LEMMA 4.9. *[2, Lemma2.3] Under the previous assumptions, system (4.16) admits a unique solution* $\{[u_h,u_{\odot h}],\lambda_{\odot h}\}$.

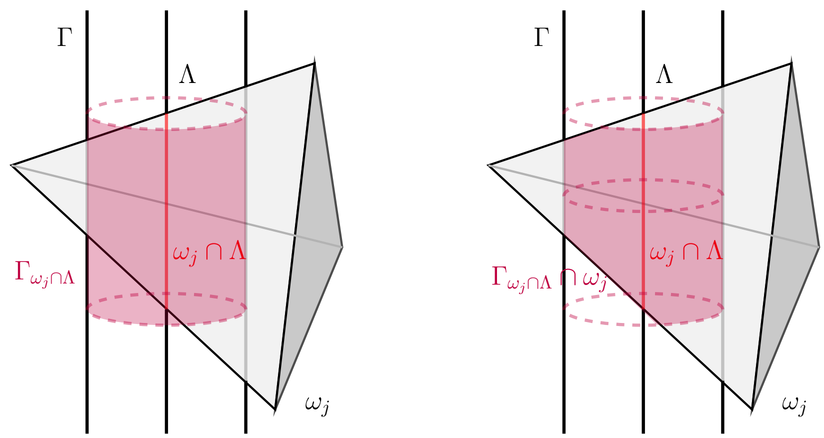
We recall that in the case of Problem 2,

$$b([u_h,u_{\odot h}],\lambda_{\odot h}) = \left(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \lambda_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}.$$

The construction of the inf-sup stable space L_h is based on macro elements of diameter H, where H is sufficiently larger than h. In particular, we assume that there exists positive constants c_h and c_H such that $c_h h \leq H \leq c_H^{-1} h$, with c_h is sufficiently large. The space is constructed assembling the 3D elements of $\mathcal{G}_h$ into macro patches ω_j such that $H \leq |\omega_j \cap \Lambda| \leq cH$. Namely, $\omega_j = \cup_{i=0}^{M_j} K_i$, where $K_i \in \mathcal{G}_h$ and M_j is uniformly bounded in j by some $M \in \mathbb{N}$ and $H = \min_j |\omega_j \cap \Lambda|$. We assume that the interiors of the patches ω_j are disjoint. We define

$$L_h = \left\{ l_{\odot h} : \, l_{\odot h} \in P^0(\omega_j)\, \forall j \right\}.$$

Moreover, we associate to each patch ω_j a shape-regular extended patch, still denoted by ω_j for notational simplicity, which is built adding to ω_j a sufficient number of elements of $\mathcal{T}_h^\Omega$ and we assume that the interiors of the new extended patches ω_j are still disjoint (see Figure 4.1). Here we are using the classical definition of shape-regularity, see for example [4], namely there exist a constant $C > 0$ such that for any ω_j, $\frac{\tilde\rho_j}{\bar\rho_j} \leq C$, being $\tilde\rho_j$ the diameter of ω_j and $\bar\rho_j$ the diameter of the largest ball that can be inscribed in ω_j. The extended patches ω_j are built such that they fulfill the conditions $\mathrm{meas}(\omega_j) = \mathcal{O}(H^3)$ and $\mathrm{diam}(\Gamma_{\omega_j\cap\Lambda} \cap \omega_j) = O(H)$, where $\Gamma_{\omega_j\cap\Lambda}$ is the portion of Γ with centerline $\omega_j \cap \Lambda$. See Figure 4.2 for a representation in the simple



(a) $\Gamma_{\omega_j\cap\Lambda}$ (b) $\Gamma_{\omega_j\cap\Lambda} \cap \omega_j$

FIGURE 4.2. *$\Gamma_{\omega_j\cap\Lambda}$, the portion of Γ generated by $\omega_j \cap \Lambda$ in (a) and the intersection between $\Gamma_{\omega_j\cap\Lambda}$ and ω_j in (b). Here for simplicity ω_j is represented as a single tetrahedron but actually it is a collection of tetrahedra as shown in Figure 4.1.*

case in which ω_j is composed just by one tetrahedron. The latter assumption is required to ensure that the intersection of $\Gamma_{\omega_j\cap\Lambda}$ and ω_j is not too small and it will be needed later on to prove the inf-sup stability of the space L_h in Lemma 4.10. We equipe the space L_h with the discrete norm

$$\|l_\odot\|_{L_h} = \|l_\odot\|_{-\frac{1}{2},h,\Lambda} = \|h^{\frac{1}{2}}\ l_\odot\|_{L^2(\Lambda)}.$$

As shown in [2, Section III], we can always choose π_L as the L_2 orthogonal projection operator from Λ_h to L_h in order to satisfy (4.13) and then in practice replace it with any interpolation $\tilde{\pi}_L$ of Λ_h in L_h. In particular, we define $\forall \lambda_h \in \Lambda_h$,

$$\tilde{\pi}_L \lambda_{\odot_{|\omega_j}} = N_j^{-1} \sum_{i:\ K_i\in\mathcal{G}_h,\ K_i\cap\omega_j\neq\emptyset} \lambda_{\odot|K_i} \qquad \text{for all } \omega_j,$$

being N_j the cardinality of the set $\{i :\ \ K_i \in \mathcal{G}_h,\ K_i \cap \omega_j \neq \emptyset\}$. These choice leads to the following stabilization

$$s(\lambda_{\odot h}, \mu_{\odot h}) = \sum_{K\in\mathcal{G}_h} \int_{\partial K\setminus\partial\mathcal{G}_h} h[\![\lambda_{\odot h}]\!][\![\mu_{\odot h}]\!],$$

being $[\![\lambda_{\odot h}]\!]$ the jump of $\lambda_{\odot h}$ across the internal faces of $\mathcal{G}_h$.

Thanks to the shape regularity of these extended patches, we have that the following discrete trace and Poincarè-type inequalites hold. More precisely, for any function $v \in H^1(\omega_j)$,

$$\|\mathcal{T}_\Gamma v\|_{L^2(\Gamma\cap\omega_j)} \lesssim H^{-\frac{1}{2}} \|v\|_{L^2(\omega_j)} \tag{4.17}$$

$$\|v - \mathcal{E}_{\omega_j}\pi_L \overline{\mathcal{T}}_\Lambda v\|_{L^2(\omega_j)} \leq c_P H \|\nabla v\|_{L^2(\omega_j)}, \tag{4.18}$$

where for any $w \in L^1(\Lambda)$, $\mathcal{E}_{\omega_j}\pi_L w \in L_h$ and with a little abuse of notation we denote as π_L the operator

$$\pi_L w_{|\omega_j\cap\Lambda} = \frac{1}{|\Gamma_{\omega_j\cap\Lambda}|} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| w \quad \forall j,$$

whereas the operator $\mathcal{E}_{\omega_j}$ simply extends the constant $\pi_L \overline{\mathcal{T}}_\Lambda v_{|\omega_j\cap\Lambda}$ to ω_j. Moreover $\forall u_h \in X^1_{h,0}(\Omega)$ we have the following average inequality

$$\sum_j \|\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \leq \sum_j \|\mathcal{T}_\Gamma u_h\|^2_{L^2(\omega_j\cap\Gamma)}.$$

Indeed, by the definition of $\overline{\mathcal{T}}_\Lambda$ and Jensen inequality, we have

$$
\begin{aligned}
\sum_j \|\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda u_h^2 \\
&= \int_\Lambda |\partial\mathcal{D}| \left(\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}} \mathcal{T}_\Gamma u_h\right)^2 \\
\text{(Jensen)} &\leq \int_\Lambda\int_{\partial\mathcal{D}} (\mathcal{T}_\Gamma u_h)^2 = \int_\Gamma (\mathcal{T}_\Gamma u_h)^2 \\
(\omega_j\cap\Gamma \text{ partition of } \Gamma) &= \sum_j \int_{\omega_j\cap\Gamma} (\mathcal{T}_\Gamma u_h)^2 = \sum_j \|\mathcal{T}_\Gamma u_h\|^2_{L^2(\omega_j\cap\Gamma)}.
\end{aligned}
$$

LEMMA 4.10. *The space L_h is inf-sup stable, namely $\forall l_{\odot h} \in L_h$, $\exists \beta > 0$ s.t.*

$$
\sup_{\substack{v_h\in X^1_{h,0}(\Omega),\\ v_{\odot h}\in X^1_{h',0}(\Lambda)}} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]|||} \geq \beta \|l_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}.
$$

and the constant is independent of the cuts.

Proof. As in the continuous case, we can choose $v_{\odot h} = 0$ and we prove that

$$
\sup_{v_h\in X^1_{h,0}(\Omega)} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta \|l_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda)}.
$$

Proving the last inequality it is equivalent to find the Fortin operator $\pi_F : H^1_0(\Omega) \to X^1_{h,0}(\Omega)$, such that

$$
\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|} = 0, \quad \forall v \in H^1_0(\Omega),\ l_{\odot h} \in L_h
$$

and

$$
\|\pi_F v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.
$$

We define

$$
\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j}
$$

where $I_h : H^1(\Omega) \to X^1_{h,0}$ denotes an $H^1(\Omega)$-stable interpolant and $\varphi_j \in X^1_{h,0}(\Omega)$ is such that $\mathrm{supp}(\varphi_j) \subset \omega_j$, $\mathrm{supp}(\mathcal{T}_\Gamma\varphi_j) \subset \Gamma_{\omega_j\cap\Lambda} \cap \omega_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$
\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j = O(H) \text{ and } \|\nabla\varphi_j\|_{L^2(\omega_j)} = O(1). \tag{4.19}
$$

We notice that $\mathrm{supp}(\mathcal{T}_\Gamma\varphi_j) \subset \Gamma_{\omega_j\cap\Lambda} \cap \omega_j$ ensures that $\overline{\mathcal{T}}_\Lambda\varphi_j \subset \omega_j \cap \Lambda$. Therefore, since for construction the interiors of $\omega_j \cap \Lambda$ are disjoint and $\varphi_j = 0$ on $\partial\omega_j$, the functions $\overline{\mathcal{T}}_\Lambda\varphi_j\, \forall j$ have all disjoint supports. This construction is always possible since $\mathrm{meas}(\omega_j) = \mathcal{O}(H^3)$ and $\mathrm{diam}(\Gamma_{\omega_j\cap\Lambda} \cap \omega_j) = \mathcal{O}(H)$, provided H is sufficiently larger that h. Indeed, this guarantees that the functions φ_j and their traces $\mathcal{T}_\Gamma\varphi_j$ have a sufficiently large support so that they can be built in order to satisfy (4.19). Then we have

$$\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|} = \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \sum_i \alpha_i \overline{\mathcal{T}}_\Lambda \varphi_i\right] l_{\odot h}$$

$$(\mathrm{supp}(\overline{\mathcal{T}}_\Lambda \varphi_i) \subset \omega_i \cap \Lambda\, \forall i) = \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \alpha_j \overline{\mathcal{T}}_\Lambda \varphi_j\right] l_{\odot h}$$

$$= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v) l_{\odot h} - \frac{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j l_{\odot h}$$

$$(\text{using } l_h \text{ constant on } \omega_j \cap \Lambda) = 0.$$

Concerning the continuity of π_F, we exploit the assumptions that the interiors of ω_j are disjoint and $\mathrm{supp}(\varphi_j) \subset \omega_j$ and we have

$$\|\nabla \pi_F v\|_{L^2(\Omega)} \le \|\nabla I_h v\|_{L^2(\Omega)} + \left(\sum_j \alpha_j^2 \|\nabla\varphi_j\|^2_{L^2(\omega_j)}\right)^{\frac{1}{2}}$$

$$(\text{stability of } I_h) \lesssim \|\nabla v\|_{L^2(\Omega)} + \left(\sum_j \alpha_j^2 \|\nabla\varphi_j\|^2_{L^2(\omega_j)}\right)^{\frac{1}{2}}$$

and for the second term

$$\sum_j \alpha_j^2 \|\nabla\varphi_j\|^2_{L^2(\omega_j)} \le$$

$$\left(\text{using } \|\nabla\varphi_j\|_{L^2(\omega_j)} = O(1)\right) \lesssim \sum_j \frac{\left(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\right)^2}{\left(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j\right)^2}$$

$$\left(\text{since } \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda \varphi_j = O(H)\right) \lesssim \frac{1}{H^2} \sum_j \left(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\right)^2$$

$$(\text{Jensen}) \lesssim \frac{1}{H^2} \sum_j |\omega_j \cap \Lambda| \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|^2 (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)^2$$

$$(\text{being } |\omega_j \cap \Lambda| \le cH) \lesssim \frac{1}{H} \sum_j \|\overline{\mathcal{T}}_\Lambda (v - I_h v)\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|}$$

$$(\text{average inequality}) \lesssim \frac{1}{H} \sum_j \|\mathcal{T}_\Gamma (v - I_h v)\|^2_{L^2(\omega_j\cap\Gamma)}$$

$$(\text{trace inequality}) \lesssim \frac{1}{H^2} \sum_j \|v - I_h v\|^2_{L^2(\omega_j)} \lesssim \frac{1}{H^2} \|v - I_h v\|^2_{L^2(\Omega)}$$

$$(\text{approximation properties of } I_h) \lesssim \|\nabla v\|^2_{L^2(\Omega)}$$

and the continuity of π_F follows. ◻

We choose the following discrete norm

$$|||[u_h, u_{\odot h}]|||^2_{X_h(\Omega)\times X_{h'}(\Lambda)} = \|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|},$$

where $\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|} = \|h^{-\frac{1}{2}}(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\|^2_{L^2(\Lambda),|\partial\mathcal{D}|}$. Then, we have the following lemma.

Lemma 4.11. *The inequalities* (4.14) *and* (4.15) *hold.*

Proof. Concerning the coercivity property (4.14), we have to show that $\forall [u_h, u_{\odot h}]$, there esists $\xi_h \in Q_h$ s.t.

$$(u_h, u_h)_{H^1(\Omega)} + (u_{\odot h}, u_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \xi_h)_{\Lambda,|\partial\mathcal{D}|} \\ \geq \alpha_\xi (\|u_h\|^2_{H^1(\Omega)} + \|u_{\odot h}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}.$$

We choose

$$\xi_{h|\omega_j\cap\Lambda} = \delta \frac{1}{H} \pi_L (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})_{|\omega_j\cap\Lambda}$$

and we recall that

$$\pi_L (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})_{|\omega_j\cap\Lambda} = \frac{1}{|\Gamma_{\omega_j\cap\Lambda}|} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}).$$

Actually, $\mathcal{E}_{\omega_j}\xi_h \in L_h \subset Q_h$. Then,

$$\begin{aligned}
\big(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \xi_h\big)_{\Lambda,|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}) \xi_h \\
\text{(definition of } \xi_h) &= \delta \frac{1}{H} \sum_j \pi_L (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}) \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}) \\
&= \delta \frac{1}{H} \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| (\pi_L (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}))^2 \\
\text{(orthogonality of } \pi_L) &= -\delta \frac{1}{H} \|(\pi_L - \mathcal{I})(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h})\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} + \delta \frac{1}{H} \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\geq -2\delta \frac{1}{H} \sum_j \|(\pi_L - \mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} - 2\delta \frac{1}{H} \sum_j \|(\pi_L - \mathcal{I}) u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\quad + \delta \frac{1}{H} \sum_j \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|}.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
\sum_j \|(\pi_L - \mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| (\pi_L \overline{\mathcal{T}}_\Lambda u_h - \overline{\mathcal{T}}_\Lambda u_h)^2 \\
\text{(Average inequality)} &\leq \sum_j \int_{\omega_j\cap\Gamma} (\mathcal{E}_\Gamma \pi_L \overline{\mathcal{T}}_\Lambda u_h - \mathcal{T}_\Gamma u_h)^2 \\
\text{(trace inequality)} &\leq \sum_j \frac{1}{H} \int_{\omega_j} (\mathcal{E}_{\omega_j} \pi_L \overline{\mathcal{T}}_\Lambda u_h - u_h)^2 \\
\text{(Poincare, see [4, Corollary B.65])} &\leq \sum_j H c_P^2 \|\nabla u_h\|^2_{L^2(\omega_j)}.
\end{aligned}$$

For the second term we have

$$
\begin{aligned}
\sum_j \|(\pi_L - \mathcal{I})u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L u_{\odot h} - u_{\odot h})^2 \\
\text{(Poincare, [4, Corollary B.65])} &\lesssim \sum_j H^2 c_P^2 \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\nabla u_{\odot h})^2 \\
\text{(since } H \text{ is fixed, we can find a constant s.t. } H|\partial\mathcal{D}| \lesssim |\mathcal{D}|) &\lesssim \sum_j H c_P^2 \int_{\omega_j\cap\Lambda} |\mathcal{D}|(\nabla u_{\odot h})^2 \\
&\lesssim \sum_j H c_P^2 \|\nabla u_{\odot h}\|^2_{L^2(\omega_j\cap\Lambda),|\mathcal{D}|}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&a([u_h, u_{\odot h}],[u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \geq \\
&\qquad (1-2\delta c_P^2)\|\nabla u_h\|^2_{L^2(\Omega)} + (1-2\delta c_P^2)\|\nabla u_{\odot h}\|^2_{L^2(\Lambda),|\mathcal{D}|} + \delta c_H \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}
\end{aligned}
$$

and choosing $\delta = \frac{1}{4c_P^2}$ we obtain the coercivity inequality.
Concerning the stability inequality (4.15), the proof is analogous to the one in [2]. ∎

REMARK 4.1. *We notice that if we choose $Q_h = X^1_{h',0}(\Lambda)$, the constant in the inf-sup inequality (4.3) depends on the mesh size h'. Indeed,*

$$
\begin{aligned}
\sup_{\substack{v_h\in X^1_{h,0}(\Omega),\\ v_{\odot h}\in X^1_{h',0}(\Lambda)}} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, l_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]|||} &\geq \sup_{v_{\odot h}\in X^1_{h',0}(\Lambda)} \frac{(-v_{\odot h}, l_{\odot h})_{\Lambda,|\partial\mathcal{D}|}}{\|v_{\odot h}\|_{H^1(\Lambda)}} \geq \frac{\|l_{\odot h}\|^2_{L^2(\Lambda)}}{\|v_{\odot h}\|_{H^1(\Lambda)}} \\
(\textit{inverse inequality}) &\geq \frac{{h'}^2}{c_I}\|l_{\odot h}\|_{L^2(\Lambda)} \geq \frac{{h'}^2}{c_I}\|l_{\odot h}\|_{H^{-\frac{1}{2}},(\Lambda)}
\end{aligned} \tag{4.20}
$$

being c_I the constant in the inverse inequality

$$\|l_{\odot h}\|_{H^1(\Lambda)} \leq \frac{c_I}{{h'}^2}\|l_{\odot h}\|_{L^2(\Lambda)}.$$

5. A benchmark problem with analytical solution. Let $\Omega = [0,1]^3$, $\Lambda = \{x = \frac{1}{2}\} \times \{y = \frac{1}{2}\} \times [0,1]$ and $\Sigma = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}] \times [0,1]$. Finally we let $\partial\mathcal{D}$ be the cross section of the virtual interface $\Gamma = \partial\Sigma$. As a benchmark for the two formulations we consider the following coupled problems

$$-\Delta u = f \quad \text{in } \Omega \tag{5.1a}$$

$$-d^2_{zz} u_\odot = g \quad \text{on } \Lambda \tag{5.1b}$$

$$u = h \quad \text{on } \partial\Omega, \tag{5.1c}$$

where for formulation (2.7) the mix-dimensional coupling constraint reads

$$\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1 \quad \text{on } \Gamma, \tag{5.2}$$

while for (2.8) we set

$$\overline{u} - u_\odot = q_2 \quad \text{on } \Lambda. \tag{5.3}$$

In (5.1)-(5.3) the right-hand side data shall be defined as

$$
\begin{aligned}
f &= 8\pi^2 \sin(2\pi x)\sin(2\pi y), \qquad g = \pi^2 \sin(\pi z), \qquad h = \sin(2\pi x)\sin(2\pi y), \\
q_1 &= \sin(2\pi x)\sin(2\pi y) - \sin(\pi z), \qquad q_2 = -\sin(\pi z).
\end{aligned}
$$

The exact solution of (5.1), regardless of the coupling constraint, is given by

$$u = \sin(2\pi x)\sin(2\pi y) \tag{5.4}$$

$$u_\odot = \sin(\pi z)\,. \tag{5.5}$$

Let us notice that $u_\odot$ satisfies homogeneous Dirichlet conditions at the boundary of Λ. Moreover, the solution (5.4)-(5.5) satisfies on Γ the relation

$$\lambda = \nabla u \cdot \mathbf{n}_\oplus = d_z u_\odot n_{\oplus,z} = 0, \tag{5.6}$$

with $n_{\oplus,z}$ the $z-$component of the normal unit vector to Γ.

We prove that (5.1) is solution of (2.8) in the simplified case in which the starting 3D-3D problem is

$$-\Delta u_\oplus = f \qquad \text{in } \Omega_\oplus, \tag{5.7a}$$

$$-\Delta u_\ominus = g \qquad \text{in } \Sigma, \tag{5.7b}$$

$$-\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus \qquad \text{on } \Gamma, \tag{5.7c}$$

$$u_\ominus - u_\oplus = q_i \qquad \text{on } \Gamma, \tag{5.7d}$$

$$u_\oplus = h \qquad \text{on } \partial\Omega. \tag{5.7e}$$

instead of (2.1). Therefore the reduced problem in (2.7) and (2.8) become respectively

$$\begin{aligned}(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, L\rangle_\Gamma \\ = (f,v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\end{aligned} \tag{5.8a}$$

$$\langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, M\rangle_\Gamma = \langle q_1, M\rangle_\Gamma \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{5.8b}$$

and

$$\begin{aligned}(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \overline{v} - v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)} \\ = (f,v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\end{aligned} \tag{5.9a}$$

$$|\partial\mathcal{D}|\langle \overline{u} - u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \overline{q_2}, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{5.9b}$$

Let us prove that (5.4)-(5.5) is solution of (5.9). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (5.9a) we have

$$\begin{aligned}-(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d_{ss}^2 u_\odot, v_\odot)_{L^2(\Lambda)} + |\mathcal{D}|\langle \overline{v} - v_\odot, L\rangle_\Lambda \\ = (f,v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda).\end{aligned}$$

Since $L = \overline{\lambda} = 0$ and (5.4) satisfies (5.1a) and (5.5) satisfies (5.1b), we have that

$$\begin{aligned}-(\Delta u, v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} \\ -|\partial\mathcal{D}|(d_{ss}^2 u_\odot, v_\odot)_{L^2(\Lambda)} = |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)},\end{aligned}$$

Thus (5.4)-(5.5) satisfy (5.9a). The fact that the solution satisfy (5.9b) follows from (5.3).

We can prove in a similar way that (5.4)-(5.5), with $L = \lambda = 0$ satisfy (5.8). Note in particular that q_1 is such that $\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1$ on Γ.

5.1. Numerical experiments. Using the benchmark problem (5.1) we now investigate convergence properties of the two formulations. To this end we consider a *uniform* tessilation of $\mathcal{T}_h^\Omega$ of Ω consisting of tetrahedra with diameter h. Further, the discretization shall be geometrically *conforming* to both Λ and Γ such that the tessilations $\mathcal{T}_h^\Gamma$, $\mathcal{T}_h^\Lambda$ are made up of facets and edges of $\mathcal{T}_h^\Omega$ respectively, cf. Figure 5.1 for illustration.

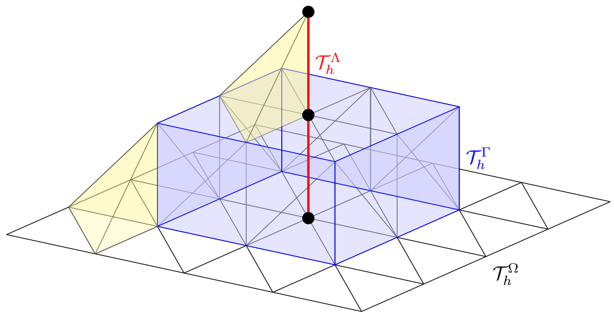


FIGURE 5.1. *Λ and Γ conforming discretization of Ω used for (5.8). For (5.9) only conformity to Λ is needed.*

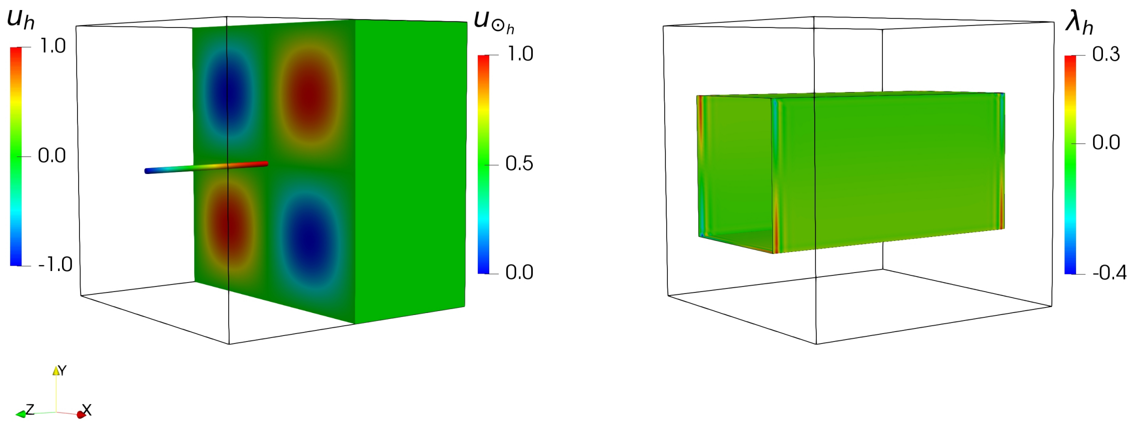


FIGURE 5.2. *Numerical solution of problem (5.8): functions u_h and $u_{\odot,h}$ on the left and the Lagrance multiplier λ_h on the right.*

Formulations (5.8) and (5.9) shall be discretized using contiuous Lagrange element of order 1 (P_1) for all the spaces involved. We recall that since are are investigating the conforming case, the tripplet P_1-P_1-P_1 satisfies the discrete inf-sup condition. The resulting linear systems are solved using the minimal residual method (MinRes) with stopping criterion requiring the relative preconditioned residual norm to be less than 10^{-12}. As the preconditioner we use the (approximate) Riesz mapping with respect to the inner products of the spaces in which the two formulations were proved to be well posed. In particular, the preconditioner for the Lagrange multiplier relies on (the inverse of) the fractional Laplacian $-\Delta^{-1/2}$ on Γ for (5.8) and Λ for (5.9). We remark that the size of the linear systems on the finest meshes considered here prevents the use direct solvers. Therefore iterative solvers were necessary. We plot the numerical solution of problem (5.8) and (5.9) in Figure 5.2 and 5.3, respectively.

Considering uniform refinements of the initial mesh, Table 5.1 lists the errors of formulations (5.8) and (5.9) on the benchmark problem. It can be seen the error in u and $u_\odot$ in H^1 norm converges linearly (as can be expected due to P_1 element discretization). Moreover, the error of the Lagrange multiplier approximation in $H^{-1/2}$ norm decreases quadratically. In the light of P_1 discretization this rate appears superconvergent. We speculate that the result is due to the fact that the exact solution is particularly simple, $L = 0$. In case of the results for (5.8) the rate can also be due to the fact that the error is interpolated into the same finite element space as the approximation L_h. We remark that for u and $u_\odot$ the error is interpolated into FE space of piecewise quadratic *discontinous* functions. For (5.9) we evaluate the fractional norm and interpolate the error using piecewise continuous cubic functions. Evaluating the fractional norm in higher order spaces for formulation with the multiplier space on Γ is prohibitively costly.

In Table 5.1 one can observe that the two formulations yield practically identical approximations of u and $u_\odot$. However, the solution cost of the two approaches differs. In Table 5.2 we summarize the size of the linear systems solved at each level of refinement and the time for the iterative solver to converge. Let us first

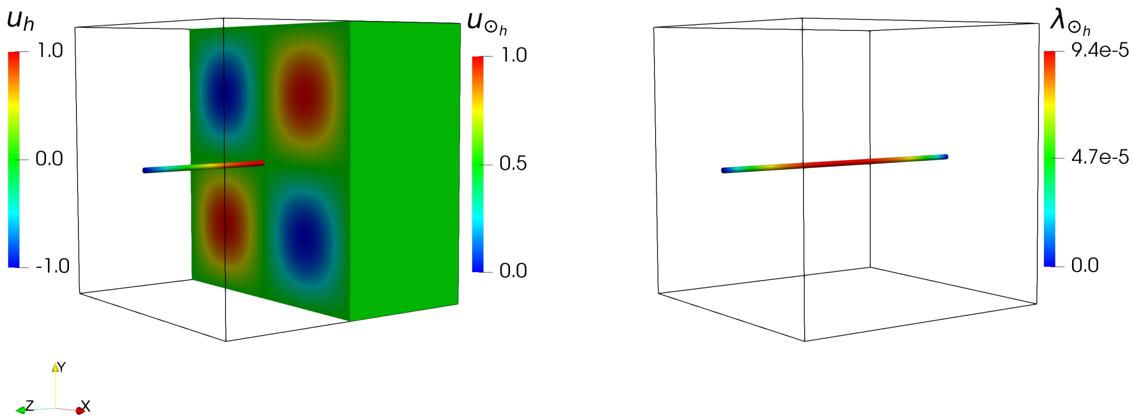


FIGURE 5.3. *Numerical solution of problem* (5.9)*: functions* u_h *and* $u_{\odot,h}$ *on the left and the Lagrance multiplier* $\lambda_{\odot_h}$ *on the right.*

h	$\lVert u - u_h \rVert_{1,\Omega}$	$\lVert u_\odot - u_{\odot,h} \rVert_{1,\Lambda}$	$\lVert L - L_h \rVert_{-1/2,\Gamma}$
4.3E-1	3.4E0(–)	5.3E-1(–)	2.9E0(–)
2.2E-1	1.7E0(0.99)	2.6E-1(1.06)	6.1E-1(2.25)
1.1E-1	8.7E-1(0.99)	1.3E-1(1.02)	1.4E-1(2.13)
5.4E-2	4.4E-1(1.00)	6.3E-2(1.00)	3.4E-2(2.03)
2.7E-2	2.2E-1(1.00)	3.1E-2(1.00)	8.6E-3(2.00)

h	$\lVert u - u_h \rVert_{1,\Omega}$	$\lVert u_\odot - u_{\odot,h} \rVert_{1,\Lambda}$	$\lVert L - L_h \rVert_{-1/2,\Lambda}$
4.3E-1	3.1E0(–)	5.4E-1(–)	4.4E-2(–)
2.2E-1	1.7E0(0.87)	2.6E-1(1.06)	1.1E-2(2.01)
1.1E-1	8.6E-1(0.96)	1.3E-1(1.02)	2.7E-3(2.01)
5.4E-2	4.4E-1(0.99)	6.3E-2(1.00)	6.7E-4(2.01)
2.7E-2	2.2E-1(1.00)	3.1E-2(1.00)	1.7E-4(2.01)
1.4E-2	1.1E-1(1.00)	1.6E-2(1.00)	4.1E-5(2.01)

TABLE 5.1

Error convergence of (5.8) *and* (5.9) *on a benchmark problem* (5.1). *Continuous linear Lagrange elements are used.*

note that the proposed preconditioners seem robust with respect to discretization parameter as the iteration counts are clearly bounded. We then see that the solution time for (5.8) is about 2 times longer compared to (5.9). This is in addition to the higher setup costs of the preconditioner which in our implementation involve solving an eigenvalue problem for the fractional Laplacian. Therefore it is advantageous to keep the multiplier space as small as possible. We remark that the missing results for (5.8) in Table 5.2 and 5.1 are due to the memory limitations which we encounter when solving the eigenvalue problem for the Laplacian, which for finest mesh involves cca 32 thousand eigenvalues.

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h	$\dim V_h$	$\dim V_{\odot,h}$	$\dim Q_h^{\Gamma}$	#	$T\,[s]$	$\dim Q_h^{\Lambda}$	#	$T\,[s]$
4.33E-01	125	5	40	27	0.03	5	9	0.01
2.17E-01	729	9	144	55	0.10	9	19	0.02
1.08E-01	4913	17	544	62	0.25	17	36	0.14
5.41E-02	35937	33	2112	64	1.97	33	42	1.08
2.71E-02	274625	65	8320	64	18.01	65	36	8.24
1.35E-02	–	–	–	–	–	129	31	61.37

TABLE 5.2

Cost comparison of the two formulations. Number of MinRes iterations is denoted by #. Time till convergence of the iterative solver (excluding the setup) is shown as T.