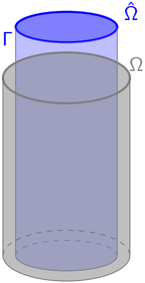
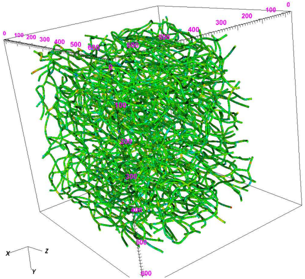
Simple multiscale 3d-1d model of interstitial flow

Vasculature resolved as a three-dimensional structure

$$
\begin{aligned}
-\nabla \cdot (\kappa \nabla u) &= f && \text{in } \Omega, \\
-\nabla \cdot (\hat{\kappa} \nabla \hat{u}) &= \hat{f} && \text{in } \hat{\Omega}, \\
u - \hat{u} &= g && \text{on } \Gamma, \\
\kappa \nabla u \cdot n - \hat{\kappa} \nabla \hat{u} \cdot n &= h && \text{on } \Gamma.
\end{aligned}
$$





Impractical for real geometries

Multiscale 3d-1d model by dimensional reduction

Current models lead to “dense” operators

$$(\Pi_R u)(y) = (2\pi R)^{-1} \int_{\mathcal{C}_R(y)} u \circ C_R \mathrm{d}l,\ R \ll \mathrm{diam}(\Omega)$$

- D'Angelo, Quarteroni: assymetric continuous problem
  $\mathcal{A} : V \mapsto \hat{V}'$

$$\mathcal{A} = \begin{pmatrix} -\kappa\Delta_\Omega & -T' \\ -\beta\Pi_R & -R^2\hat{\kappa}\Delta_\Lambda \end{pmatrix}$$

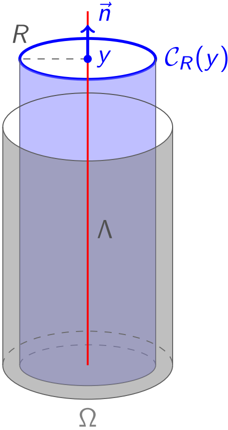
  with $V = H^1_\alpha(\Omega) \times H^1(\Gamma)$, $\hat{V} = H^1_{-\alpha}(\Omega) \times H^1(\Gamma)$

- Cerrroni, Laurino, Zunnino: symmetric problem
  $\mathcal{A} : V \mapsto V'$

$$\mathcal{A} = \begin{pmatrix} -\kappa\Delta_\Omega + \Pi_R{}'\Pi_R & -\beta\Pi_R{}' \\ -\Pi_R & -R^2\hat{\kappa}\Delta_\Lambda \end{pmatrix}$$

  with $V = H^1(\Omega) \times H^1(\Gamma)$

Can we use common black-box preconditioners?



Lagrange multiplier $3d$-$1d$ formulation

Offers possibly more flexible coupling

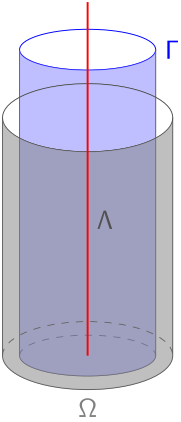
Consider $\mathcal{A} : H^1(\Omega) \times H^1(\Lambda) \times Q$

$$\mathcal{A} = \begin{pmatrix} -\kappa\Delta_\Omega & & \Pi' \\ & -R^2\hat{\kappa}\Delta_\Lambda & -\hat{\Pi}' \\ \Pi & -\hat{\Pi} & \end{pmatrix}$$

Two options for the Lagrange multiplier space

- Q defined on Λ
- Q defined in *virtual* coupling surface Γ

We wish to construct block-diagonal preconditioners



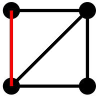
Formulation with line multiplier-conforming, P1

Riesz map preconditioner based on existence analysis

$$\mathcal{B} = \begin{pmatrix} -\Delta_\Omega & & \\ & -R^2\Delta_\Lambda & \\ & & (-\Delta_\Gamma)^{-1/2} \end{pmatrix}^{-1}$$

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}
$\mathrm{cond}(\mathcal{B}\mathcal{A})$	40.5	40.6	40.6	40.6

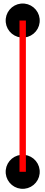
V_h



Schur complement of $\mathcal{A}$ is R dependent, $\Pi = \Pi_R$, $\hat{\Pi} = I$

$$R^2(\Pi)(-\Delta_\Omega)^{-1}(\Pi)' + (\hat{\Pi})(-\Delta_\Lambda)^{-1}(\hat{\Pi})'$$

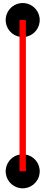
$\hat{V}_h$



R-robust preconditioner

$$\mathcal{B} = \begin{pmatrix} -\Delta_\Omega & & \\ & -R^2\Delta_\Lambda & \\ & & R^2(-\Delta_\Gamma)^{-1/2} + (-\Delta_\Gamma)^{-1} \end{pmatrix}^{-1}$$

Q_h

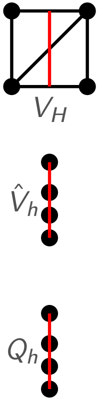


α	h: 2^{-2}	2^{-3}	2^{-4}	2^{-5}
10^{-1}	4.5162	4.5201	4.5608	4.6189
10^{-2}	4.5314	4.5366	4.5574	4.6137
10^{-3}	4.5311	4.5312	4.5315	4.5328

Nonconforming line multiplier, P1-P1-P0 elements

Stabilized FEM formulation

$$\mathcal{A} = \begin{pmatrix} -\kappa\Delta_\Omega & & \Pi' \\ & -R^2\hat{\kappa}\Delta_\Lambda & -\hat{\Pi}' \\ \Pi & -\hat{\Pi} & -h^3\Delta_{\Gamma,h} \end{pmatrix} \quad \langle p, -\Delta_{\Gamma,h} q\rangle = \sum \{\{h\}\}^{-1} [\![p]\!][\![q]\!]$$



Conforming-case preconditioner with stabilization term

α	h			
	2^{-2}	2^{-3}	2^{-4}	2^{-5}
10^{-1}	4.5316	4.5319	4.5289	4.5506
10^{-2}	4.5412	4.5399	4.5364	4.5337
10^{-3}	4.5413	4.5310	4.5364	4.5439

Formulation with surface multiplier-conforming, P1

Coupling via

- $\Pi = T$ (standard $3d$-$2d$ trace)
- $\hat{\Pi}$ extension operator $\Lambda \to \Gamma$

Riesz map preconditioner based on existence analysis

$$\mathcal{B} = \begin{pmatrix} -\Delta_\Omega & & \\ & -R^2\Delta_\Lambda & \\ & & (-\Delta_\Gamma)^{-1/2} \end{pmatrix}^{-1}$$

At the moment only h-robust

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}
cond$(\mathcal{B}\mathcal{A})$	27.4	27.5	27.6	27.6

Fractional preconditioner computationally expensive