COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

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AMS subject classifications. n.a.

1. Introduction. We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators form a 3D domain, Ω, to 1D manifold, Λ, and vice versa. This is a non trivial objective because the standard trace operator form a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω.

2. Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex open set. Let Σ be a generalized cylinder embedded into Ω and be $\Omega_\oplus = \Omega \setminus \overline{\Sigma}$ be the complementary set of the cylinder. We also introduce the set Λ, a 1D manifold that is the centerline of Σ. We define the arc-length coordinate along Λ, denoted by $s \in (0, S)$. We denote with $\mathcal{D}(s)$ and $\partial\mathcal{D}(s)$ a cross section of Σ and its boundary, respectively. In what follows, we assume for simplicity of notation that Σ has a constant cross section, but this is not a restriction of the approach. We also assume that Σ crosses Ω from side to side and we call Γ the lateral (cylindrical) surface of Σ, while the upper and lower side faces of Σ belong to $\partial\Omega$. We refer to Figure 2.1 for an illustration of the notation.

We consider the problem arising from *Dirichlet-Neumann* conditions. It consists to find $u_\oplus, u_\ominus$ s.t.:

$$
\begin{aligned}
&(2.1a) && -\Delta u_\oplus + u_\oplus = f && \text{in } \Omega_\oplus,\\
&(2.1b) && -\Delta u_\ominus + u_\ominus = g && \text{in } \Sigma,\\
&(2.1c) && -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus && \text{on } \Gamma,\\
&(2.1d) && u_\ominus = u_\oplus && \text{on } \Gamma,\\
&(2.1e) && u_\oplus = 0 && \text{on } \partial\Omega\,.
\end{aligned}
$$

The objective of this work is to derive and alalyze a simplified version of problem (2.1), where the domain Σ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely $\mathcal{D}$. This new problem setting will be called the *reduced* problem. Form the mathematical standpoint it is more challenging than (2.1), because it involves the coupling of 3D/1D elliptic problems. For the model reduction process, we decompose integrals as follows, for any sufficiently regular function w,

$$\int_\Sigma w d\omega = \int_\Lambda \int_{\mathcal{D}} w d\sigma ds = \int_\Lambda |\mathcal{D}| \overline{\overline{w}} ds\,, \quad \int_\Gamma w d\sigma = \int_\Lambda \int_{\partial\mathcal{D}} w d\gamma ds = \int_\Lambda |\partial\mathcal{D}| \overline{w} ds\,,$$

where $\overline{\overline{w}}$, $\overline{w}$ denote the following mean values respectively,

$$\overline{\overline{w}} = |\mathcal{D}|^{-1} \int_{\mathcal{D}} w d\sigma\,, \quad \overline{w} = |\partial\mathcal{D}|^{-1} \int_{\partial\mathcal{D}} w d\gamma\,.$$

We apply the model reduction approach at the level of the variational formulation. We start from the variational formulation of problem (2.1), that is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ u_\ominus \in H^1_{\partial\Sigma\setminus\Gamma}(\Sigma),\ \lambda \in H^{-\frac{1}{2}}(\partial\Sigma)$ s.t.

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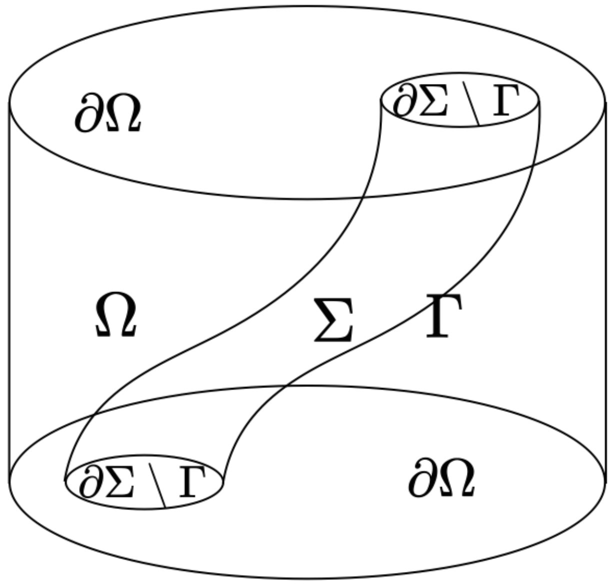

FIGURE 2.1. *Geometrical setting of the problem*

$$
\begin{aligned}
&(u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Sigma)} + \langle v_\ominus - v_\oplus, \lambda\rangle_{H^{-\frac{1}{2}}(\Gamma)} \\
&\qquad = (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Sigma)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Sigma\setminus\Gamma}(\Sigma)
\end{aligned} \tag{2.2a}
$$

$$
\langle u_\ominus - u_\oplus, \mu\rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,, \tag{2.2b}
$$

where $\langle v, \mu\rangle_{H^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. In this case, the additional variable λ is equivalent to $\lambda = -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

Using the averaging tools for the model reduction approach, we end up with a reduced problem for the unknown u defined on the entire 3D domain Ω, coupled with the unknown $u_\odot$, defined on the 1D manifold Λ and a Lagrange multiplier L defined on Λ. In the reduced problem the multiplier assumes the following interpretation,

$$
L = -\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}} \lambda\,.
$$

We consider two alternative formulations. The scope of this work is to compare them, with the aim to determine which is the most suitable as a computational model based on 3D-1D coupled PDEs.

2.1. Problem 1. The idea is to couple a 3D PDE with a 1D one, using a Lagrange multiplier space defined on a 2D surface that surrounds the 1D manifold. Let $\langle\cdot,\cdot\rangle_\Gamma$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. The problem consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ L \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$
\begin{aligned}
&(u, v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\
&\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda)
\end{aligned} \tag{2.3a}
$$

$$
\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{2.3b}
$$

Here, $\Pi_1 : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ and $\Pi_2 : H^1_0(\Lambda) \to H^{\frac{1}{2}}_{00}(\Gamma)$ and we remark that Σ may be considered as a virtual surface not necessarily of the same size as the underlying physical structure that is modeled. The Π_1 and Π_2 operators may be defined in terms of the averaging operators above, but may also be realized in terms of e.g. Green functions (I don't know if this is a good idea). Furthermore, Γ may be discretized in terms of facets neighboring Λ and may as such not be represented as a separate structure in the implementation.

2.2. Problem 2. Another form of the reduced problem uses Lagrange multipliers defined directly on the 1D manifold. In this case, $\langle\cdot,\cdot\rangle_\Lambda$ denotes the duality pairing between $H^{\frac{1}{2}}_{00}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. The problem requires to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ L \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}
&(u,v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)} \\
&\qquad = (f,v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1_0(\Lambda) \qquad (2.4a)
\end{aligned}$$

$$|\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \qquad (2.4b)$$

We notice that all the integrals of the reduced problem are well defined because $u,v \in H^1_0(\Omega)$, $u,v|_\Gamma \in H^{\frac{1}{2}}_{00}(\Gamma)$ and thus $\overline{u},\overline{v} \in H^{\frac{1}{2}}_{00}(\Lambda)$. More precisely, $\Pi_1 : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Lambda$ such that $\Pi_1 u = \overline{(u|_\Gamma)}$ is the combination between the trace on Γ and the average on $\partial\mathcal{D}$. The operator $\Pi_2 : H^1_0(\Lambda) \to H^{\frac{1}{2}}_{00}(\Lambda)$ is the injection form $H^1_0(\Lambda)$ and $H^{\frac{1}{2}}_{00}(\Lambda)$. It is apparent that problems (2.2) and (4.1) share the same mathematical structure. For this reason, the well-posedness of (4.1) can be studied in the framework of the classical theory of saddle point problems.

3. Saddle-point problem analysis. Let us consider the general saddle point problem of the form: find $u \in X$, $p \in Q$ s.t.

$$\begin{cases} a(u,v) + b(v,p) = f(v) & \forall v \in X \\ b(u,q) = g(q) & \forall q \in Q \end{cases} \qquad (3.1)$$

which embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem. We denote with A and B the operators associated to the bilinear forms a and b, namely $A : X \longrightarrow X'$ with $\langle Au, v\rangle_{X',X} = a(u,v)$ and $\langle Bv, q\rangle_{X',Q} = b(v,q)$.

THEOREM 3.1 (theorem 2.34 Ern-Guermond). *Problem (3.1) is well posed iff*

$$\begin{cases} \exists \alpha > 0 : \inf_{u \in ker(B)} \sup_{v \in ker(B)} \frac{a(u,v)}{\|u\|_X \|v\|_X} \geq \alpha \\ \forall v \in ker(B),\ (\forall u \in ker(B),\ a(u,v) = 0) \implies v = 0. \end{cases} \qquad (3.2)$$

and

$$\exists \beta > 0 : \inf_{q \in Q} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \|q\|_Q} \geq \beta. \qquad (3.3)$$

Notice that if a is coercive on $ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1. It consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ L \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned}
&(u,v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\
&\qquad = (f,v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1_0(\Lambda) \qquad (3.4a)
\end{aligned}$$

$$\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,, \qquad (3.4b)$$

Here, $\Pi_1 : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ is the trace operator while Π_2 is the uniform extension from $H^1_0(\Lambda)$ to $H^{\frac{1}{2}}_{00}(\Gamma)$. We notice that the trace operator is surjective from $H^1_0(\Omega)$ to $H^{\frac{1}{2}}_{00}(\Gamma)$. We apply theorem 3.1 in the following spaces $X = H^1_0(\Omega) \times H^1_0(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$, where X is equipped with the norm $|||[u, u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + |\mathcal{D}|\|u_\odot\|^2_{H^1(\Lambda)}$. We prove that:

- a coercive $\implies$ (3.2) is fulfilled

- We have to prove that $\forall M \in H^{-\frac{1}{2}}(\Gamma)$, $\exists \beta > 0$:

$$\sup_{v\in H_0^1(\Omega),\, v_\odot \in H_0^1(\Lambda)} \frac{\langle \Pi_1 v - \Pi_2 v_\odot, M\rangle_\Gamma}{|||[v, v_\odot]|||} \geq \beta \sup_{q\in H^{\frac{1}{2}}(\Gamma)} \frac{\langle q, M\rangle}{\|q\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

We choose $v_\odot \in H_0^1(\Lambda)$ such that $\Pi_2 v_\odot = 0$. Therefore,

$$\sup_{v\in H_0^1(\Omega),\, v_\odot \in H_0^1(\Lambda)} \frac{\langle \Pi_1 v - \Pi_2 v_\odot, M\rangle_\Gamma}{|||[v, v_\odot]|||} \geq \sup_{v\in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M\rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

The trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. Indeed, $\forall \xi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we can find v solution of

$$\begin{aligned} -\Delta v &= 0 \quad \text{in } \Omega \\ v &= 0 \;\text{ on } \partial\Omega \\ v &= \xi \;\text{ on } \Gamma. \end{aligned}$$

We denote with E the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|E\| \in \mathbb{R}$ such that $v = E(\xi)$ and $\|v\|_{H^1(\Omega)} \leq \|E\|\|\xi\|_{H_{00}^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$\sup_{v\in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M\rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi\in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle \xi, M\rangle_\Gamma}{\|E\|\|\xi\|_{H_{00}^{\frac{1}{2}}(\Gamma)}} = \|E\|^{-1}\|M\|_{H^{-\frac{1}{2}}(\Gamma)}, \tag{3.5}$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H_{00}^{\frac{1}{2}}(\Gamma))^*$.

3.2. Problem 2. This problem requires to find $u \in H_0^1(\Omega),\ u_\odot \in H_0^1(\Lambda),\ L \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned} &(u, v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v_\odot - \Pi_2 v, L\rangle_\Lambda \\ &\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda) \end{aligned} \tag{3.6a}$$

$$|\partial\mathcal{D}|\langle \Pi_1 u_\odot - \Pi_2 u, M\rangle_\Lambda = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{3.6b}$$

Here, $\Pi_1 : H_0^1(\Lambda) \to H_{00}^{\frac{1}{2}}(\Lambda)$ is the immersion operator and $\Pi_2 : H_0^1(\Omega) \to H_{00}^{\frac{1}{2}}(\Lambda)$ is defined as the composition of the trace operator $T_\Gamma : H_0^1(\Omega) \to H_{00}^{\frac{1}{2}}(\Gamma)$ and the average operator $\bar{()} : H_{00}^{\frac{1}{2}}(\Gamma) \to H_{00}^{\frac{1}{2}}(\Lambda)$, namely $\Pi_2 = \bar{()} \circ T_\Gamma$. First of all we prove that if $u \in H_0^1(\Omega)$, than $\Pi_2 u \in H_{00}^{\frac{1}{2}}(\Lambda)$. In particular, from standard trace theory, we have that $T_\Gamma u \in H_{00}^{\frac{1}{2}}(\Gamma)$, therefore we have to prove that if $u \in H_{00}^{\frac{1}{2}}(\Gamma)$ then $\overline{u} \in H_{00}^{\frac{1}{2}}(\Lambda)$.

Lemma 3.2. *When Γ is a cylinder, if $u \in H_{00}^{\frac{1}{2}}(\Gamma)$, then $\overline{u} \in H_{00}^{\frac{1}{2}}(\Lambda)$.*

Proof. Let us denote as ϕ_{ij} and ρ_{ij}, for $i = 1, 2, \ldots,\ j = 0, 1, \ldots$, the eigenfunctions and the eigenvalues of the laplacian on Γ, and with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the laplacian on Λ. In particular,

$$\begin{aligned} \phi_{ij}(s, \theta) &= sin(i\pi s)\,(cos(j\theta) + sin(j\theta))\,, \\ \rho_{ij} &= i\pi^2 + \frac{j^2}{R^2}, \\ \phi_i(s) &= sin(i\pi s), \\ \rho_i &= i\pi^2. \end{aligned}$$

It is easy to verify that

$$\int_0^{2\pi} \phi_{ij}(s,\theta) = 0 \quad \forall j > 0, \forall i \tag{3.7}$$

$$\int_0^{2\pi} \phi_{ij}(s,\theta) = 2\pi R\, \sin(i\pi s) \quad \text{if } j = 0, \forall i. \tag{3.8}$$

(3.9)

Moreover we recall that $\phi_{i,j}(s,\theta)$ and $\phi_i(s)$ are orthogonal basis of $L^2(\Gamma)$ and $L^2(\Lambda)$ respectively. Therefore,

$$\begin{aligned} \overline{u}(s) = \frac{1}{2\pi R}\int_0^{2\pi} u(s,\theta) R\, d\theta = \frac{1}{2\pi R}\int_0^{2\pi} \sum_{i,j} a_{i,j}\phi_{i,j}(s,\theta) R\, d\theta \\ = \frac{1}{2\pi R}\sum_{i,j} a_{i,j}\int_0^{2\pi} \phi_{i,j}(s,\theta) R\, d\theta = \sum_i a_{i,0}\phi_i(s). \end{aligned}$$

From [4, Lemma 4.11] we have

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} (1+\rho_{ij})^{\frac{1}{2}}\, |a_{ij}|^2, \text{ with } a_{ij} = \int_0^1\int_0^{2\pi} u(s,\theta)\phi_{ij}\, R d\theta ds. \tag{3.10}$$

and

$$\|\overline{u}\|^2_{H^{\frac{1}{2}}(\Lambda)} = \sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}}\, |\overline{a}_i|^2, \text{ with } \overline{a}_i = \int_0^1 \overline{u}(s)\phi_i(s) ds.$$

Therefore, we have

$$\begin{aligned} \|\overline{u}\|^2_{H^{\frac{1}{2}}(\Lambda)} &= \sum_{i=1}^{\infty}\left(1+i^2\pi^2\right)^{\frac{1}{2}}\left(\int_0^1 \overline{u}(s) sin(i\pi s)\, ds\right)^2 \\ &= \sum_{i=1}^{\infty}\left(1+i^2\pi^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty} a_{j,0}\int_0^1 \sin(j\pi s)\sin(i\pi s)\, ds\right)^2 \\ &= \sum_{i=1}^{\infty}\frac{1}{4}\left(1+i^2\pi^2\right)^{\frac{1}{2}} a_{i,0}^2 \\ &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+i^2\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}} |a_{i,j}|^2 = \|u\|^2_{H^{\frac{1}{2}}(\Gamma)}, \end{aligned}$$

where we have used the fact that

$$\int_0^1 \sin(i\pi s)\sin(j\pi s)\, ds = 0 \quad \text{if } i \neq j$$
$$\int_0^1 \sin(i\pi s)\sin(j\pi s)\, ds = \frac{1}{2} \quad \text{if } i = j.$$

c.v.d. ∎

Lemma 3.3. *If Σ is a straight cylinder, if $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ is constant on each cross section, namely $u(s,\theta) = u(s)$, then*

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = 2\pi R \|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)}.$$

Proof. From (3.10),

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}|a_{ij}|^2 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+i\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}\left(\int_0^1\int_0^{2\pi}u(s,\theta)\phi_{ij}\,Rd\theta ds\right)^2$$

$$= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+i\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}\left(\int_0^1 u(s)\int_0^{2\pi}\phi_{ij}\,Rd\theta ds\right)^2,$$

and using (3.7) and (3.8), we obtain

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\left(1+i\pi^2\right)^{\frac{1}{2}}\left(\int_0^1 u(s)sin(i\pi s)2\pi Rds\right)^2$$

$$= 4\pi^2R^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}|a_i|^2 = 4\pi^2R^2\|u\|^2_{H^{\frac{1}{2}}(\Lambda)}. \qquad \square$$

Then, we apply Theorem 3.1 with the following spaces $X = H_0^1(\Omega)\times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped again with the norm $|||[u,u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + |\mathcal{D}|\|u_\odot\|^2_{H^1(\Lambda)}$, Q equipped with the norm

$$\|M\|_{H^{-\frac{1}{2}}} := \sup_{q\in H^{\frac{1}{2}}(\Lambda)}\frac{\langle q,M\rangle}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}$$

Then, the following properties hold true:

- the form $a([u,u_\odot],[v,v_\odot]) = (u,v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot,v_\odot)_{H^1(\Lambda)}$ is coercive $\implies$ (3.2) is fulfilled. Indeed, we have,

$$a([u,u_\odot],[u,u_\odot]) = (u,u)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot,u_\odot)_{H^1(\Lambda)} = |||[u,u_\odot]|||^2.$$

- We have that $\forall M\in H^{-\frac{1}{2}}(\Lambda)$, $\exists\beta>0$:

$$\sup_{v\in H_0^1(\Omega),\,v_\odot\in H_0^1(\Lambda)}\frac{\langle\Pi_1 v_\odot-\Pi_2 v,M\rangle_\Lambda}{\sqrt{\|v\|^2_{H^1(\Omega)}+|\mathcal{D}|\|v_\odot\|^2_{H^1(\Lambda)}}} \geq \beta\sup_{q\in H^{\frac{1}{2}}(\Lambda)}\frac{\langle q,M\rangle_\Lambda}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}.$$

  We choose $v_\odot = 0$ and we obtain

$$\sup_{v\in H_0^1(\Omega),\,v_\odot\in H_0^1(\Lambda)}\frac{\langle\Pi_1 v_\odot-\Pi_2 v,M\rangle_\Lambda}{\sqrt{\|v\|^2_{H^1(\Omega)}+|\mathcal{D}|\|v_\odot\|^2_{H^1(\Lambda)}}} \geq \sup_{v\in H_0^1(\Omega)}\frac{\langle\Pi_2 v,M\rangle_\Lambda}{\|v\|_{H^1(\Omega)}}.$$

  For any $q\in H_{00}^{\frac{1}{2}}(\Lambda)$, we consider its uniform extension to Γ (we denote it again with q) and then we consider the harmonic extension $v = E(q)\in H_0^1(\Omega)$. It follows that $\Pi_2 v = q$. Therefore,

$$\sup_{v\in H_0^1(\Omega)}\langle\Pi_2 v,M\rangle_\Lambda \geq \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)}\langle q,M\rangle_\Lambda.$$

  Moreover, using Lemma 3.3 we obtain

$$\|v\|_{H_0^1(\Omega)} \leq \|E\|\|q\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}|\|E\|\|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)}.$$

  Therefore,

$$\sup_{v\in H_0^1(\Omega)}\frac{\langle\Pi_2 v,M\rangle_\Lambda}{\|v\|_{H^1(\Omega)}} \geq \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)}\frac{\langle q,M\rangle_\Lambda}{\|v\|_{H^1(\Omega)}} \geq |\partial\mathcal{D}|^{-1}\|E\|^{-1}\sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)}\frac{\langle q,M\rangle_\Lambda}{\|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)}}$$

$$= |\partial\mathcal{D}|^{-1}\|E\|^{-1}\|M\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

Remark 3.1. *The results of* (3.2) *and* (3.3) *can be generalized to the case of a different geometry of* Γ*, for example a parallelepiped.*

4. Finite element approximation (Different meshes for solution and Lagrange multiplier).

Let us introduce an admissibile triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_h^\Lambda$ of Λ. We denote by $X_h^0(\Omega) \subset H_0^1(\Omega)$ the conforming finite element space of continuous piecewise linear functions defined on Ω and by $X_h(\Lambda) \subset H_0^1(\Lambda)$ the space of continuous piecewise linear functions defined on Λ. Moreover, Q_H denotes a suitable trial space for the lagrange multiplier L_H, defined on a different triangulation of Γ with mesh size H. In particular, $Q_H \subset H^{-\frac{1}{2}}(\Gamma)$ in the case of Problem 1 and $Q_H \subset H^{-\frac{1}{2}}(\Lambda)$ in the case of Problem 2.

4.1. Problem 1.

It consists to find $u_h \in X_h(\Omega),\ u_{\odot h} \in X_h(\Lambda),\ L_H \in Q_H(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned}
&(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + \langle \Pi_1 v_h - \Pi_2 v_{\odot h}, L_H\rangle_\Gamma \\
&\qquad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h} \in X_h(\Lambda) \qquad \text{(4.1a)}
\end{aligned}$$

$$\langle \Pi_1 u_h - \Pi_2 u_{\odot h}, M_H\rangle_\Gamma = 0 \quad \forall M_H \in Q_H(\Gamma)\,, \tag{4.1b}$$

Theorem 4.1. $\exists \gamma_1 > 0$ *s.t.*

$$\inf_{M_H \in Q_H(\Gamma)} \sup_{\substack{v_h \in X_h(\Omega),\\ v_{\odot h} \in X_h(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_H\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||\,\|M_H\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \gamma_1. \tag{4.2}$$

Proof. Let $M_H \in Q_H(\Gamma)$. As in the continuos case, let us choose $v_{\odot h} = 0$, therefore

$$\sup_{\substack{v_h \in X_h(\Omega),\\ v_{\odot h} \in X_h(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_H\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_1 v_h, M_H\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Following [6, Theorem 11.5], it can be shown under the following assumptions

- the mesh size h of the trial space $X_h(\Omega)$ is sufficiently small compared to the mesh size H of $Q_H(\Gamma)$, i.e. $h \leq c_0 H$ with $c_0 < 1$, and
- a global inverse inequality for the trial space $Q_H(\Gamma)$ holds,

that exists a positive constant c_S

$$c_S \|M_H\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \qquad \forall M_H \in Q_H(\Gamma), \tag{4.3}$$

being $W_h(\Gamma)$ the trace space of the functions in $X_h(\Omega)$, namely the space of the restrictions of the functions in $X_h(\Omega)$ to Γ. Using the boundedness of the extension operator E from $H_0^{\frac{1}{2}}(\Gamma)$ to $H_0^1(\Omega)$ introduced in the previous section, we have

$$\sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|E\| \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|E w_h\|_{H^1(\Omega)}}.$$

Let $R_h : H_0^1(\Omega) \to X_h(\Omega)$ be a quasi interpolation operator satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq C_R \|v\|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Therefore, we obtain

$$\|E\| \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|E w_h\|_{H^1(\Omega)}} \leq \|E\| C_R \sup_{w_h \in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|R_h E w_h\|_{H^1(\Omega)}}$$

and using (4.3), we have

$$c_S\|M_H\|_{H^{-\frac{1}{2}}(\Gamma)} \le \sup_{w_h\in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \le \|E\|C_R \sup_{w_h\in W_h(\Gamma)} \frac{\langle w_h, M_H\rangle_\Gamma}{\|Ew_h\|_{H^1(\Gamma)}}$$
$$= \|E\|C_R \sup_{w_h\in W_h(\Gamma)} \frac{\langle \Pi_1\ R_h E w_h, M_H\rangle_\Gamma}{\|R_hEw_h\|_{H^1(\Omega)}} \le \|E\|C_R \sup_{v_h\in X_h(\Omega)} \frac{\langle \Pi_1 v_h, M_H\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}. \tag{4.4}$$

Therefore the inf-sup condition (5.5) holds with $\gamma_1 = c_S\|E\|^{-1}C_R^{-1}$. ∎

4.2. Problem 2. This problem requires to find $u_h \in X_h(\Omega),\ u_{\odot h} \in X_h(\Lambda),\ L_H \in Q_H(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}
&(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v_{\odot h} - \Pi_2 v_h, L_H\rangle_\Lambda \\
&\qquad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\overline{g}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h} \in X_h(\Lambda) \\
&|\partial\mathcal{D}|\langle \Pi_1 u_{\odot h} - \Pi_2 u_h, M_H\rangle_\Lambda = 0 \quad \forall M_H \in Q_H(\Lambda)\,.
\end{aligned}$$

THEOREM 4.2. $\exists \gamma_2 > 0$ *s.t.*

$$\inf_{M_H\in Q_H(\Lambda)} \sup_{\substack{vh\in X_h(\Omega),\\ v_{\odot h}\in X_h(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_H\rangle_\Lambda}{|||[v_h, v_{\odot h}]|||\,\|M_H\|_{H^{-\frac{1}{2}}(\Lambda)}} \ge \gamma_2. \tag{4.6}$$

Proof. Let M_H be arbitrarly chosen in $Q_H(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that (4.6) reduces to prove

$$\gamma_2\|M_H\|_{H^{\frac{1}{2}}(\Lambda)} \le \sup_{v_h\in X_h(\Omega)} \frac{\langle \Pi_2 v_h, M_H\rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}} \qquad \forall M_H \in Q_H(\Lambda).$$

Assume that

- the mesh size h of the trial space $X_h(\Omega)$ is sufficiently small compared to the mesh size H of $Q_H(\Lambda)$, i.e. $h \le c_1 H$ with $c_1 < 1$,
- a global inverse inequality for the trial space $Q_H(\Lambda)$ holds and
- the space $W_h(\Lambda)$, defined as the space of the restrictions on Γ of the functions in $X_h(\Omega)$ averaged on the cross section, has the approximation property, namely if Q_h^σ denotes the projection from $H^\sigma(\Lambda)$ to $W_h(\Lambda)$, we have

$$\|w - Q_h^\sigma w\|_{H^\sigma(\Lambda)} \le c_A h^{s-\sigma}|w|_{H^s(\Lambda)} \qquad \forall w \in H^s(\Lambda).$$

(Actually $W_h(\Lambda)$ concides with the space of piecewise linear continuous polinomials on Λ).

Under the previous assumptions, an inequality similar to (4.3) holds also for $W_h(\Lambda)$. In particular, following the same proof in Steinbach, we can prove

$$c_{S_2}\|M_H\|_{H^{-\frac{1}{2}}(\Lambda)} \le \sup_{w_h\in W_h(\Lambda)} \frac{\langle w_h, M_H\rangle_\Lambda}{\|w_h\|_{H^{\frac{1}{2}}(\Lambda)}} \qquad \forall M_H \in Q_H(\Lambda). \tag{4.7}$$

If we denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ, using Lemma 3.3, we easily have for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}|\|w\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Consequently, using again the extension operator E from $H_0^{\frac{1}{2}}(\Omega)$ to $H_0^1(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X_h(\Omega)$, we obtain

$$
\begin{aligned}
(4.8)\quad c_{S_2}\|M_H\|_{H^{-\frac{1}{2}}(\Lambda)} &\le \sup_{w_h\in W_h(\Lambda)} \frac{\langle w_h, M_H\rangle_\Lambda}{\|w_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\
&\le |\partial\mathcal{D}| \sup_{w_h\in W_h(\Lambda)} \frac{\langle w_h, M_H\rangle_\Lambda}{\|\mathcal{U}_E w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \le |\partial\mathcal{D}|\|E\| \sup_{w_h\in W_h(\Lambda)} \frac{\langle w_h, M_H\rangle_\Lambda}{\|E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\le |\partial\mathcal{D}|\|E\|C_R \sup_{w_h\in W_h(\Lambda)} \frac{\langle w_h, M_H\rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&= |\partial\mathcal{D}|\|E\|C_R \sup_{w_h\in W_h(\Lambda)} \frac{\langle \Pi_1 R_h E\mathcal{U}_E w_h, M_H\rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\le |\partial\mathcal{D}|\|E\|C_R \sup_{v_h\in X_h(\Omega)} \frac{\langle \Pi_2 v_h, M_H\rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned}
$$

Therefore, (4.6) holds with $\gamma_2 = c_{S_2}|\partial\mathcal{D}|^{-1}\|E\|^{-1}C_R^{-1}$ ∎

5. Finite element approximation (Same mesh for solution and Lagrange multiplier). Let us introduce a shape-regular triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_h^\Lambda$ of Λ. We denote by $X^0_{h,k}(\Omega) \subset H^1_0(\Omega)$ the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satysfing homogeneous Dirichlet conditions on the boundary and by $X^0_{h,k}(\Lambda) \subset H^1_0(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ, satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. Moreover, Q_h denotes a suitable trial space for the lagrange multiplier L_h. In particular, $Q_h \subset H^{-\frac{1}{2}}(\Gamma)$ in the case of Problem 1 and $Q_h \subset H^{-\frac{1}{2}}(\Lambda)$ in the case of Problem 2.

5.1. Problem 1. It consists to find $u_h \in X^0_{h,k}(\Omega)$, $u_{\odot h} \in X^0_{h,k}(\Lambda)$, $L_h \in Q_h(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$
\begin{aligned}
(5.1a)\quad & (u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + \langle \Pi_1 v_h - \Pi_2 v_{\odot h}, L_h\rangle_\Gamma \\
&\qquad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X^0_{h,k}(\Omega),\ v_{\odot h} \in X^0_{h,k}(\Lambda) \\
(5.1b)\quad & \langle \Pi_1 u_h - \Pi_2 u_{\odot h}, M_h\rangle_\Gamma = 0 \quad \forall M_h \in Q_h(\Gamma)\,,
\end{aligned}
$$

Let us denote with $W^0_{h,k}(\Gamma) \subset H^{\frac{1}{2}}_{00}(\Gamma)$ the trace space of functions running in $X^0_{h,k}(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$. We choose $Q_h(\Gamma) = W^0_{h,k}(\Gamma)$, therefore we impose homogeneous Dirichlet boundary condition on $\partial\Omega$ also for the Lagrange multiplier. For this choice of $Q_h(\Gamma)$ we can prove the well-posedness of the discrete problem, as shown in the following.

Lemma 5.1. *Let $P_h : H^{\frac{1}{2}}_{00}(\Gamma) \longrightarrow W^0_{h,k}(\Gamma)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Gamma)$ by*

$$
(P_h v, \psi)_\Gamma = (v, \psi)_\Gamma \qquad \forall \psi \in W^0_{h,k}(\Gamma).
$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Gamma)$, namely

$$
\|P_h v\|_{H^{\frac{1}{2}}_{00}(\Gamma)} \le C\|v\|_{H^{\frac{1}{2}}_{00}(\Gamma)}, \tag{5.2}
$$

where C is a positive constant independent of h.

Proof. We prove that P_h is continuous on $L^2(\Gamma)$ and on $H^1_0(\Gamma)$ following [5, Section 1.6.3]. Then, the inequality (5.2) can be dirived by Hilbertian interpolation. For the L^2-continuity, we exploit the fact that, from the definition of P_h,

$$
(v - P_h v, P_h v)_\Gamma = 0.
$$

Therefore, by Pythagoras identity,

$$\|v\|^2_{L^2(\Gamma)} = \|v - P_h v\|^2_{L^2(\Gamma)} + \|P_h v\|^2_{L^2(\Gamma)} \geq \|P_h v\|_{L^2(\Gamma)}.$$

Let us now consider $v \in H^1_0(\Gamma)$.The Scott-Zhang interpolation operator SZ_h from $H^1_0(\Gamma)$ to $W^0_{h,k}(\Gamma)$ satisfies the following inequalities,

$$\|SZ_h v\|_{H^1(\Gamma)} \leq C_1 \|v\|_{H^1(\Gamma)} \tag{5.3}$$

$$\|v - SZ_h v\|_{L^2(\Gamma)} \leq C_2 h \|v\|_{H^1(\Gamma)}. \tag{5.4}$$

Therefore,

$$\begin{aligned}\|\nabla P_h v\|_{L^2(\Gamma)} &\leq \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + \|\nabla SZ_h v\|_{L^2(\Gamma)} \\ &\leq \ \text{(using (5.3))}\ \|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)}\end{aligned}$$

and by using the inverse inequality we obtain

$$\begin{aligned}\|\nabla(P_h v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} &\leq \frac{C_3}{h}\|P_h v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &= \frac{C_3}{h}\|P_h(v - SZ_h v)\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &\leq \ \text{(Stability of } P_h \text{ in } L^2\text{)}\ \frac{C_3}{h}\|v - SZ_h v\|_{L^2(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &\leq \ \text{(using (5.4))}\ \frac{C_3}{h}C_2 h\|v\|_{H^1(\Gamma)} + C_1\|v\|_{H^1(\Gamma)} \\ &\leq (C_2C_3 + C_1)\|v\|_{H^1(\Gamma)},\end{aligned}$$

from which we obtain the continuity in $H^1_0(\Gamma)$. ∎

LEMMA 5.2. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \geq \gamma \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. Let M_h be in $W^0_{h,k}(\Gamma)$. From the continuous case, in particular from (3.5), we have

$$\|E\|^{-1}\|M_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H^1_0(\Omega)} \frac{\langle \Pi_1 v, M_h\rangle}{\|v\|_{H^1(\Omega)}} = \sup_{v \in H^1_0(\Omega)} \frac{\langle T_\Gamma v, M_h\rangle}{\|v\|_{H^1(\Omega)}}$$

and by the trace inequality $\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \leq K_1\|v\|_{H^1(\Omega)}$ (see [1, 7.56]), we obtain

$$\sup_{v \in H^1_0(\Omega)} \frac{\langle T_\Gamma v, M_h\rangle}{\|v\|_{H^1(\Omega)}} \leq K_1 \sup_{v \in H^1_0(\Omega)} \frac{\langle T_\Gamma v, M_h\rangle}{\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}}.$$

By the definition of P_h and (5.2)

$$\begin{aligned}K_1 \sup_{v \in H^1_0(\Omega)} \frac{\langle T_\Gamma v, M_h\rangle}{\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} &= K_1 \sup_{v \in H^1_0(\Omega)} \frac{\langle P_h(T_\Gamma v), M_h\rangle}{\|T_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)}} \\ &\leq K_1 C \sup_{v \in H^1_0(\Omega)} \frac{\langle P_h(T_\Gamma) v, M_h\rangle}{\|P_h(T_\Gamma v)\|_{H^{\frac{1}{2}}(\Gamma)}} \\ &= K_1 C \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}}.\end{aligned}$$ ∎

THEOREM 5.3 (Discrete inf-sup). $\exists \gamma_1 > 0$ *s.t.*

$$\inf_{M_h \in W^0_{h,k}(\Gamma)} \sup_{\substack{v_h \in X^0_{h,k}(\Omega),\\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]||| \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \gamma_1. \tag{5.5}$$

Proof. Let $M_h \in W^0_{h,k}$. As in the continuos case, we choose $v_{\odot h} = 0$ and we have

$$\sup_{\substack{v_h \in X^0_{h,k}(\Omega),\\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_h\rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle \Pi_1 v_h, M_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Therefore, we want to prove that there exists γ_1 such that

$$\sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle \Pi_1 v_h, M_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} = \sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle T_\Gamma v_h, M_h\rangle}{\|v_h\|_{H^1(\Omega)}} \geq \gamma_1 \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall M_h \in W^0_{h,k}.$$

Using Lemma 5.2 and the boundedness of the armonic extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ introduced in the previous section, we have

$$\gamma \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|E\| \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle_\Gamma}{\|E q_h\|_{H^1(\Omega)}}.$$

Let $R_h : H^1_0(\Omega) \to X^0_{h,k}(\Omega)$ be a quasi interpolation operator satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq C_R \|v\|_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega).$$

Therefore, we obtain

$$\|E\| \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle_\Gamma}{\|E q_h\|_{H^1(\Omega)}} \leq \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle_\Gamma}{\|R_h E q_h\|_{H^1(\Omega)}}$$

and we have

$$\begin{aligned} \gamma \|M_h\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle q_h, M_h\rangle_\Gamma}{\|R_h E q_h\|_{H^1(\Gamma)}} \\ &= \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Gamma)} \frac{\langle \Pi_1 \ R_h E q_h, M_h\rangle_\Gamma}{\|R_h E q_h\|_{H^1(\Omega)}} \leq \|E\| C_R \sup_{v_h \in X_{h,k}(\Omega)} \frac{\langle \Pi_1 v_h, M_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}. \end{aligned} \tag{5.6}$$

Therefore the inf-sup condition (5.5) holds with $\gamma_1 = \gamma \|E\|^{-1} C_R^{-1}$. ∎

REMARK 5.1. *We notice that to prove the result in Lemma 5.2 (and then the discrete inf-sup condition) basically we need a projection operator $P_h : H^{\frac{1}{2}}_{00} \longrightarrow W^0_{h,k}(\Gamma)$ orthogonal in the multiplier space $Q_h(\Gamma)$, namely such that $\langle P_h v, M_h\rangle = \langle v, M_h\rangle$, $\forall M_h \in Q_h(\Gamma)$, and continuous in $H^{\frac{1}{2}}(\Gamma)$. Therefore, in principle different choices than $Q_h(\Lambda) = W^0_{h,k}(\Gamma)$ could be considered if we can build an operator P_h satisfying these properties. In [2] such operator P_h is built for a particular choice of $Q_h(\Gamma)$ but it is not clear how they prove the H^1-stability inequality (and consequently the $H^{\frac{1}{2}}$-stability) with a constant independent of the mesh size h...*

5.2. Problem 2. This problem requires to find $u_h \in X^0_{h,k}(\Omega)$, $u_{\odot h} \in X^0_{h,k}(\Lambda)$, $L_h \in Q_h(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned} &(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot h}, v_{\odot h})_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v_{\odot h} - \Pi_2 v_h, L_h\rangle_\Lambda \\ &\qquad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h} \in X_h(\Lambda) \\ &|\partial\mathcal{D}|\langle \Pi_1 u_{\odot h} - \Pi_2 u_h, M_h\rangle_\Lambda = 0 \quad \forall M_h \in Q_h(\Lambda)\,. \end{aligned}$$

We introduce the space $W^0_{h,k}(\Lambda) \subset H^{\frac{1}{2}}_{00}(\Lambda)$, which is the averaged trace space of functions running in $H^1_0(\Omega)$. It coincides with the space of continuous piecewise polynomials of degree k defined on Λ and satisfying homogeneous Dirichlet boundary condition. (Add assumptions..) We choose $Q_h(\Lambda) = W^0_{h,k}(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda \cap \partial\Omega$ also for the Lagrange multiplier. With this choice for $Q_h(\Lambda)$, we can prove the well-posedness of the discrete problem. In particular, following the same steps as for Problem 1, we can prove the following results.

LEMMA 5.4. *Let $P_h : H^{\frac{1}{2}}_{00}(\Lambda) \longrightarrow W^0_{h,k}(\Lambda)$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Lambda)$ by*

$$(P_h v, \psi)_\Lambda = (v, \psi)_\Lambda \qquad \forall \psi \in W^0_{h,k}(\Lambda).$$

Then, P_h is continuous on $H^{\frac{1}{2}}_{00}(\Lambda)$, namely

$$\|P_h v\|_{H^{\frac{1}{2}}_{00}(\Lambda)} \leq C \|v\|_{H^{\frac{1}{2}}_{00}(\Lambda)},$$

where C is a positive constant independent of h.

LEMMA 5.5. *There exist a constant $\gamma > 0$ such that*

$$\sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, M_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \geq \gamma \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)} \qquad \forall M_h \in W^0_{h,k}(\Lambda).$$

THEOREM 5.6 (Discrete inf-sup). *$\exists \gamma_2 > 0$ s.t.*

$$\inf_{M_h \in Q_h(\Lambda)} \sup_{\substack{v_h \in X^0_{h,k}(\Omega),\\ v_{\odot h} \in X^0_{h,k}(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot h}, M_h \rangle_\Lambda}{|||[v_h, v_{\odot h}]||| \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \gamma_2. \tag{5.8}$$

Proof. Let M_h be arbitrarily chosen in $W^0_{h,k}(\Lambda)$. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to show that there exists γ_2 s.t.

$$\gamma_2 \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)} \leq \sup_{v_h \in X^0_{h,k}(\Omega)} \frac{\langle \Pi_2 v_h, M_h \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}} \qquad \forall M_h \in W^0_{h,k}(\Lambda).$$

Let us denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ. Using Lemma 3.3, we easily have for any $w \in H^{\frac{1}{2}}(\Lambda)$,

$$\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}| \|w\|_{H^{\frac{1}{2}}(\Lambda)}.$$

Consequently, from Lemma 5.5, using again the extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H^1_0(\Omega)$ and the quasi interpolation operator R_h from $H^1_0(\Omega)$ to $X^0_{h,k}(\Omega)$, we obtain

$$\begin{aligned}
\gamma \|M_h\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|q_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\
&= |\partial\mathcal{D}| \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|\mathcal{U}_E q_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq |\partial\mathcal{D}| \|E\| \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|E\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}| \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle q_h, M_h \rangle_\Lambda}{\|R_h E\mathcal{U}_E q_h\|_{H^1(\Omega)}} \\
&= |\partial\mathcal{D}| \|E\| C_R \sup_{q_h \in W^0_{h,k}(\Lambda)} \frac{\langle \Pi_1 R_h E\mathcal{U}_E q_h, M_h \rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}| \|E\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, M_h \rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}. \qquad \square
\end{aligned} \tag{5.9}$$

6. A benchmark problem with analytical solution. Let $\Omega = [0,1]^3$, $\Lambda = \{x = \frac{1}{2}\} \times \{y = \frac{1}{2}\} \times [0,1]$ and $\Sigma = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}] \times [0,1]$. Finally we let $\partial\mathcal{D}$ be the cross section of the virtual interface $\Gamma = \partial\Sigma$. As a benchmark for the two formulations we consider the following coupled problems

$$-\Delta u = f \quad \text{in } \Omega \tag{6.1a}$$
$$-d_{zz}^2 u_\odot = g \quad \text{on } \Lambda \tag{6.1b}$$
$$u = h \quad \text{on } \partial\Omega, \tag{6.1c}$$

where for formulation (2.3) the mix-dimensional coupling constraint reads

$$\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1 \quad \text{on } \Gamma, \tag{6.2}$$

while for (2.4) we set

$$\overline{u} - u_\odot = q_2 \quad \text{on } \Lambda. \tag{6.3}$$

In (6.1)-(6.3) the right-hand side data shall be defined as

$$f = 8\pi^2 \sin(2\pi x)\sin(2\pi y), \qquad g = \pi^2 \sin(\pi z)\,, \quad h = \sin(2\pi x)\sin(2\pi y),$$
$$q_1 = \sin(2\pi x)\sin(2\pi y) - \sin(\pi z)\,, \quad q_2 = -\sin(\pi z)\,.$$

The exact solution of (6.1), regardless of the coupling constraint, is given by

$$u = \sin(2\pi x)\sin(2\pi y) \tag{6.4}$$
$$u_\odot = \sin(\pi z)\,. \tag{6.5}$$

Let us notice that $u_\odot$ satisfies homogeneous Dirichlet conditions at the boundary of Λ. Moreover, the solution (6.4)-(6.5) satisfies on Γ the relation

$$\lambda = \nabla u \cdot \mathbf{n}_\oplus = d_z u_\odot n_{\oplus,z} = 0, \tag{6.6}$$

with $n_{\oplus,z}$ the $z-$component of the normal unit vector to Γ.

We prove that (6.1) is solution of (2.4) in the simplified case in which the starting 3D-3D problem is

$$-\Delta u_\oplus = f \quad \text{in } \Omega_\oplus, \tag{6.7a}$$
$$-\Delta u_\ominus = g \quad \text{in } \Sigma, \tag{6.7b}$$
$$-\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus \quad \text{on } \Gamma, \tag{6.7c}$$
$$u_\ominus - u_\oplus = q_i \quad \text{on } \Gamma, \tag{6.7d}$$
$$u_\oplus = h \quad \text{on } \partial\Omega. \tag{6.7e}$$

instead of (2.1). Therefore the reduced problem in (2.3) and (2.4) become respectively

$$(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, L\rangle_\Gamma$$
$$= (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda) \tag{6.8a}$$
$$\langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, M\rangle_\Gamma = \langle q_1, M\rangle_\Gamma \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{6.8b}$$

and

$$(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \overline{v} - v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)}$$
$$= (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda) \tag{6.9a}$$
$$|\partial\mathcal{D}|\langle \overline{u} - u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \overline{q_2}, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{6.9b}$$

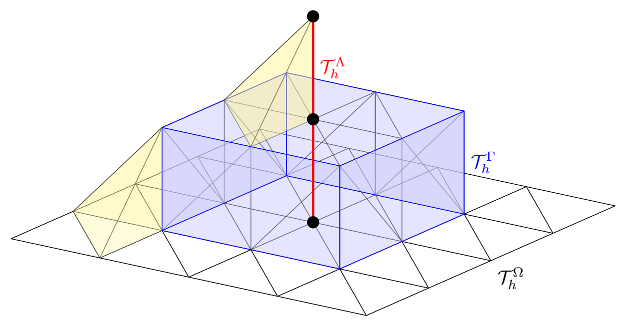

FIGURE 6.1. Λ *and* Γ *conforming discretization of* Ω *used for* (6.8). *For* (6.9) *only conformity to* Λ *is needed.*

Let us prove that (6.4)-(6.5) is solution of (6.9). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (6.9a) we have

$$
\begin{aligned}
&-(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d^2_{ss}u_\odot, v_\odot)_{L^2(\Lambda)} + |\mathcal{D}|\langle \overline{v} - v_\odot, L\rangle_\Lambda \\
&\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda).
\end{aligned}
$$

Since $L = \overline{\lambda} = 0$ and (6.4) satisfies (6.1a) and (6.5) satisfies (6.1b), we have that

$$
\begin{aligned}
-(\Delta u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} \\
-|\partial\mathcal{D}|(d^2_{ss}u_\odot, v_\odot)_{L^2(\Lambda)} &= |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)},
\end{aligned}
$$

Thus (6.4)-(6.5) satisfy (6.9a). The fact that the solution satisfy (6.9b) follows from (6.3).

We can prove in a similar way that (6.4)-(6.5), with $L = \lambda = 0$ satisfy (6.8). Note in particular that q_1 is such that $\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1$ on Γ.

6.1. Numerical experiments. Using the benchmark problem (6.1) we now investigate convergence properties of the two formulations. To this end we consider a *uniform* tessilation of $\mathcal{T}_h^\Omega$ of Ω consisting of tetrahedra with diameter h. Further, the discretization shall be geometrically *conforming* to both Λ and Γ such that the tessilations $\mathcal{T}_h^\Gamma$, $\mathcal{T}_h^\Lambda$ are made up of facets and edges of $\mathcal{T}_h^\Omega$ respectively, cf. Figure 6.1 for illustration.

Formulations (6.8) and (6.9) shall be discretized using contiuous Lagrange element of order 1 (P_1) for all the spaces involved. We recall that since are are investigating the conforming case, the tripplet P_1-P_1-P_1 satisfies the discrete inf-sup condition. The resulting linear systems are solved using the minimal residual method (MinRes) with stopping criterion requiring the relative preconditioned residual norm to be less than 10^{-12}. As the preconditioner we use the (approximate) Riesz mapping with respect to the inner products of the spaces in which the two formulations were proved to be well posed. In particular, the preconditioner for the Lagrange multiplier relies on (the inverse of) the fractional Laplacian $-\Delta^{-1/2}$ on Γ for (6.8) and Λ for (6.9). We remark that the size of the linear systems on the finest meshes considered here prevents the use direct solvers. Therefore iterative solvers were necessary.

Considering uniform refinements of the initial mesh, Table 6.1 lists the errors of formulations (6.8) and (6.9) on the benchmark problem. It can be seen the error in u and $u_\odot$ in H^1 norm converges linearly (as can be expected due to P_1 element discretization). Moreover, the error of the Lagrange multiplier approximation in $H^{-1/2}$ norm decreases quadratically. In the light of P_1 discretization this rate appears superconvergent. We speculate that the result is due to the fact that the exact solution is particularly simple, $L = 0$. In case of the results for (6.8) the rate can also be due to the fact that the error is interpolated into the same finite element space as the approximation L_h. We remark that for u and $u_\odot$ the error is interpolated into FE space of piecewise quadratic *discontinous* functions. For (6.9) we evaluate the fractional norm and interpolate the error using piecewise continuous cubic functions. Evaluating the fractional norm in higher order spaces for formulation with the multiplier space on Γ is prohibitively costly.

h	$\|u-u_h\|_{1,\Omega}$	$\|u_\odot-u_{\odot,h}\|_{1,\Lambda}$	$\|L-L_h\|_{-1/2,\Gamma}$
4.3E-1	3.4E0(–)	5.3E-1(–)	2.9E0(–)
2.2E-1	1.7E0(0.99)	2.6E-1(1.06)	6.1E-1(2.25)
1.1E-1	8.7E-1(0.99)	1.3E-1(1.02)	1.4E-1(2.13)
5.4E-2	4.4E-1(1.00)	6.3E-2(1.00)	3.4E-2(2.03)
2.7E-2	2.2E-1(1.00)	3.1E-2(1.00)	8.6E-3(2.00)

h	$\|u-u_h\|_{1,\Omega}$	$\|u_\odot-u_{\odot,h}\|_{1,\Lambda}$	$\|L-L_h\|_{-1/2,\Lambda}$
4.3E-1	3.1E0(–)	5.4E-1(–)	4.4E-2(–)
2.2E-1	1.7E0(0.87)	2.6E-1(1.06)	1.1E-2(2.01)
1.1E-1	8.6E-1(0.96)	1.3E-1(1.02)	2.7E-3(2.01)
5.4E-2	4.4E-1(0.99)	6.3E-2(1.00)	6.7E-4(2.01)
2.7E-2	2.2E-1(1.00)	3.1E-2(1.00)	1.7E-4(2.01)
1.4E-2	1.1E-1(1.00)	1.6E-2(1.00)	4.1E-5(2.01)

Table 6.1
Error convergence of (6.8) and (6.9) on a benchmark problem (6.1). Continuous linear Lagrange elements are used.

h	$\dim V_h$	$\dim V_{\odot,h}$	$\dim Q_h^\Gamma$	#	$T\,[s]$	$\dim Q_h^\Lambda$	#	$T\,[s]$
4.33E-01	125	5	40	27	0.03	5	9	0.01
2.17E-01	729	9	144	55	0.10	9	19	0.02
1.08E-01	4913	17	544	62	0.25	17	36	0.14
5.41E-02	35937	33	2112	64	1.97	33	42	1.08
2.71E-02	274625	65	8320	64	18.01	65	36	8.24
1.35E-02	–	–	–	–	–	129	31	61.37

Table 6.2
Cost comparison of the two formulations. Number of MinRes iterations is denoted by #. Time till convergence of the iterative solver (excluding the setup) is shown as T.

In Table 6.1 one can observe that the two formulations yield practically identical approximations of u and $u_\odot$. However, the solution cost of the two approaches differs. In Table 6.2 we summarize the size of the linear systems solved at each level of refinement and the time for the iterative solver to converge. Let us first note that the proposed preconditioners seem robust with respect to discretization parameter as the iteration counts are clearly bounded. We then see that the solution time for (6.8) is about 2 times longer compared to (6.9). This is in addition to the higher setup costs of the preconditioner which in our implementation involve solving an eigenvalue problem for the fractional Laplacian. Therefore it is advantageous to keep the multiplier space as small as possible. We remark that the missing results for (6.8) in Table 6.2 and 6.1 are due to the memory limitations which we encounter when solving the eigenvalue problem for the Laplacian, which for finest mesh involves cca 32 thousand eigenvalues.

Some plots of the solutions

7. Unfitted P1-P1-P0 case.

7.1. Problem 1. All the observations and the proofs are an application of [3] to our case. Let $\mathcal{T}_h^\Omega$ and $\mathcal{T}_{h'}^\Lambda$ denote a shape-regular triangulation of Ω and an admissible partition of Λ respectively. Let us also introduce the set $\mathcal{G}_h$ of elements of $\mathcal{T}_h^\Omega$ intersecting the interface Γ, namely $\mathcal{G}_h = \left\{K \in \mathcal{T}_h^\Omega : K \cap \Gamma \neq \emptyset\right\}$. For the discrete solutions u_h and $u_{\odot h}$ of the problem in Ω and on Λ we choose the conforming spaces $X_{h,1}^0(\Omega) \subset H_0^1(\Omega)$ and $X_{h,1}^0(\Lambda) \subset H_0^1(\Lambda)$ respectively. We extend the Lagrange multiplier λ_h to the volume elements of $\mathcal{G}_h$ and we define $Q_h = \left\{\lambda_h : \lambda_{h|K} \in P^0(K)\, \forall K \in \mathcal{G}_h\right\}$. With this choice of the LM space, the problem is not inf-sup stable. We add a stabilization term $-s(\lambda_h, \mu_h)$ and we have

$$a([u_h, u_{\odot h}], [v_h, v_{\odot h}]) + b([v_h, v_{\odot h}], \lambda_h) + b([u_h, u_{\odot h}], \mu_h) - s(\lambda_h, \mu_h) = (f, [v_h, v_{\odot h}]) \qquad \forall [v_h, v_{\odot h}] \in X_h(\Omega) \times X_h(\Lambda),\ \forall \mu_h \in Q_h.$$

The stabilization $s(\lambda_h, \mu_h)$ is connected to a projector π_L between Q_h and a new space L_h for the LM for which we can prove the inf-sup stability.
Therefore we have to build this new space L_h, prove that the inf-sup condition is fulfilled, build a projection operator $\pi_L : Q_h \to L_h$, build $s(\lambda_h, \mu_h)$ and prove that $\forall [u_h, u_{\odot h}]$, there exists $\xi_h([u_h, u_{\odot h}]) \in Q_h$ s.t.

$$a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \geq \alpha_\xi |||[u_h, u_{\odot h}]|||_{X_h(\Omega)\times X_h(\Lambda)}, \tag{7.1}$$

$$(s(\xi_h, \xi_h))^{\frac{1}{2}} \leq c_s |||[u_h, u_{\odot h}]|||_{X_h(\Omega)\times X_h(\Lambda)}, \tag{7.2}$$

being $|||\cdot|||_{X_h(\Omega)\times X_h(\Lambda)}$ a suitable discrete norm.
The space L_h is built as the space of P^0 functions defined on macro patches $\left\{F_j\right\}_j$ of elements of $\mathcal{G}_h$. These

patches are such that $diam(F_j) \leq H$ and $H \leq |F_j \cap \Gamma| \leq H + h$. Moreover, there exist constants c_h and c_H such that $c_h h \leq H \leq c_H^{-1} h$. To each patch F_j we associate two shape regular macro elements ω_j and $\tilde{\omega}_j$: ω_j is built adding to $F_j \cap \Omega_\oplus$ a sufficient number of elements of $\mathcal{T}_h^\Omega$ contained in $\Omega_\oplus$, whereas $\tilde{\omega}_j$ is obtained adding to $F_j \cap \Sigma$ a sufficient number of elements of $\mathcal{T}_h^\Omega$ contained in Σ. Thanks to the shape regularity of these macro elements, we have that the following trace and Poincar inequalites hold. For every function $v \in H^1(\omega_j)$,

$$\|Tv\|_{\Gamma \cap \omega_j} \lesssim H^{-\frac{1}{2}} \|v\|_{L^2(\omega_j)} \tag{7.3}$$

$$\|v - \pi_L v\|_{L^2(\omega_j)} \leq c_P H \|\nabla v\|_{L^2(\omega_j)}, \tag{7.4}$$

being π_L the projection onto piecewise constant functions on F_j. The same also in $\tilde{\omega_j}$ for $v \in H^1(\tilde{\omega_j})$. This choice leads to the following stabilization

$$s(\lambda_h, \mu_h) = \sum_{K \in \mathcal{G}_h} \int_{\partial K \setminus \partial \mathcal{G}_h} h [\![\lambda_h]\!] [\![\mu_h]\!],$$

being $[\![\lambda_h]\!]$ the jump of λ_h across the internal faces of $\mathcal{G}_h$.

Is L_h inf-sup stable with constants independent of the cuts?. We have to prove that $\forall l_h \in L_h$, $\exists \beta > 0$ s.t.

$$\sup_{\substack{v_h \in X^0_{h,1}(\Omega), \\ v_{\odot h} \in X^0_{h',1}(\Lambda)}} \frac{b([v_h, v_{\odot h}], l_h)}{|\!|\!|[v_h, v_{\odot h}]|\!|\!|} \geq \beta \|l_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

As in the continuous case, we can choose $v_{\odot h} = 0$ and we prove that

$$\sup_{v_h \in X^0_{h,1}(\Omega), v_{\odot h} \in X^0_{h,1}(\Lambda)} \frac{b([v_h, v_{\odot h}], l_h)}{|\!|\!|[v_h, v_{\odot h}]|\!|\!|} \geq \sup_{v_h \in X^0_{h,1}(\Omega)} \frac{b(v_h, l_h)}{\|v_h\|_{H^1(\Omega)}} \geq \beta \|l_h\|_{H^{-\frac{1}{2}}(\Gamma)},$$

where $b(v_h, l_h) = (Tv_h, l_h)_\Gamma$ with a little abuse of notation. Proving the last inequality it is equivalent to find the Fortin operator $\pi_F : H^1_0(\Omega) \to X^0_{h,1}(\Omega)$, such that

$$b(v - \pi_F v, l_h) = 0, \quad \forall v \in H^1_0(\Omega),\, l_h \in L_h$$

and

$$\|\pi_F v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.$$

We define

$$\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{F_j \cap \Gamma} T(v - I_h v)}{\int_{F_j \cap \Gamma} T\varphi_j}$$

and $\varphi_j \in X^0_{h,1}(\Omega)$ s.t. $\text{supp}(\varphi_j) \subset \bar{\omega}_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$\int_{F_j \cap \Gamma} T\varphi_j = O(H) \text{ and } \|\nabla \varphi\|_{L^2(\omega_j)} = O(1).$$

This construction is always possible provided H is sufficiently larger that h (usually $H = 3h$). Then $b(v - \pi_F v, l_h) = 0 \, \forall l_h$ follows by construction. Indeed:

$$
\begin{aligned}
b(v - \pi_F v, l_h) = \int_\Gamma T(v - \pi_F v) l_h &= \sum_j \int_{F_j \cap \Gamma} \left[T(v - I_h v) - \sum_i \alpha_i T \varphi_i \right] l_h \\
&= (\mathrm{supp}\varphi \subset \omega_j) \sum_j \int F_j \cap \Gamma \left[T(v - I_h v) - \alpha_j T \varphi_j \right] l_h \\
&= \sum_j \int_{F_j \cap \Gamma} T(v - I_h v) l_h - \frac{\int_{F_j \cap \Gamma} T(v - I_h v)}{\int_{F_j \cap \Gamma} T \varphi_j} \int_{F_j \cap \Gamma} T \varphi_j l_h \\
&= (\text{using } l_h \text{ constant on } F_j \cap \Gamma) \, 0.
\end{aligned}
$$

Concerning the continuity of π_F, we have

$$
\begin{aligned}
\|\nabla \pi_F v\|_{L^2(\Omega)} &\le \|\nabla I_h v\|_{L^2(\Omega)} + \sum_j |\alpha_j| \|\nabla \varphi_j\|_{L^2(\bar{\omega}_j)} \\
(\text{stability of } I_h) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \sum_j |\alpha_j| \|\nabla \varphi_j\|_{L^2(\bar{\omega}_j)} \\
(\text{using } \|\nabla \varphi\| = O(1)) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \sum_j \frac{\left| \int_{F_j \cap \Gamma} T(v - I_h v) \right|}{\left| \int_{F_j \cap \Gamma} T \varphi_j \right|} \\
\left(\text{since } \left| \int_{F_j \cap \Gamma} T \varphi_j \right| = O(H) \right) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \frac{1}{H} \sum_j \left| \int_{F_j \cap \Gamma} T(v - I_h v) \right| \\
(\text{Hölder}) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \frac{1}{H} \sum_j |F_j \cap \Gamma|^{\frac{1}{2}} \left(\int_{F_j \cap \Gamma} (T(v - I_h v))^2 \right)^{\frac{1}{2}} \\
(\text{being } |F_j \cap \Gamma| \le H + h) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \frac{1}{H^{\frac{1}{2}}} \sum_j \|v - I_h v\|_{F_j \cap \Gamma} \\
(\text{trace inequality}) &\lesssim \|\nabla v\|_{L^2(\Omega)} + \frac{1}{H} \sum_j \|v - I_h v\|_{L^2(\omega_j)} \lesssim \|\nabla v\|_{L^2(\Omega)} + \frac{1}{H} \|v - I_h v\|_{L^2(\Omega)} \\
&(\text{approximation properties of } I_h) \lesssim \|\nabla v\|_{L^2(\Omega)}.
\end{aligned}
$$

Satisfaction of the assumptions of the abstract analysis. We have to prove (7.1) and (7.2). We choose the following discrete norm

$$
|||[u_h, u_{\odot h}]|||^2_{X_h(\Omega) \times X_h(\Lambda)} = \|u_h\|^2_{H^1(\Omega)} + |D| \|u_{\odot h}\|^2_{H^1(\Lambda)} + \|u_h - u_{\odot h}\|^2_{-\frac{1}{2}, h, \Gamma},
$$

where $\|u_h - u_{\odot h}\|^2_{-\frac{1}{2}, h, \Gamma} = \|h^{\frac{1}{2}} (u_h - u_{\odot h})\|^2_{L^2(\Gamma)}$. Concerning the coercivity property (7.1), we have to show that $\forall [u_h, u_{\odot h}]$, there esists ξ_h s.t.

$$
(u_h, u_h)_{H^1(\Omega)} + |D| (u_{\odot h}, u_{\odot h})_{H^1(\Lambda)} + (T u_h - \mathcal{U}_E u_{\odot h}, \xi_h)_\Gamma \ge \alpha_\xi (\|u_h\|^2_{H^1(\Omega)} + |D| \|u_{\odot h}\|^2_{H^1(\Lambda)} + \|u_h - u_{\odot h}\|^2_{-\frac{1}{2}, h, \Gamma}).
$$

We choose

$$
\xi_{h|F_j \cap \Gamma} = \delta \frac{1}{H} \pi_L (u_h - u_{\odot h}) \qquad \text{with } \pi_L (u_h - u_{\odot h}) = \frac{1}{|F_j \cap \Gamma|} \int_{F_j \cap \Gamma} T u_h - \mathcal{U}_E u_{\odot h}.
$$

Actually, $\xi_h \in L_h \subset Q_h$. Then,

$$
\begin{aligned}
(Tu_h - \mathcal{U}_E u_{\odot h}, \xi_h)_\Gamma &= \sum_j \int_{F_j \cap \Gamma} (Tu_h - \mathcal{U}_E u_{\odot h})\xi_h \\
&= \delta \frac{1}{H} \sum_j \int_{F_j \cap \Gamma} (Tu_h - \mathcal{U}_E u_{\odot h}) \pi_L (u_h - u_{\odot h}) \\
&= \delta \frac{1}{H} \sum_j \int_{F_j \cap \Gamma} (\pi_L (u_h - u_{\odot h}))^2 \\
&= \delta \frac{1}{H} \sum_j \left(\|(\pi_L - \mathcal{I})(u_h - u_{\odot h})\|^2_{L^2(F_j\cap\Gamma)} + \|u_h - u_{\odot h}\|^2_{L^2(F_j \cap \Gamma)} \right) \\
&\geq -\delta \frac{1}{H} \sum_j \|(\pi_L - \mathcal{I}) u_h\|^2_{L^2(F_j\cap\Gamma)} - \delta \frac{1}{H} \sum_j \|(\pi_L - \mathcal{I}) u_{\odot h}\|^2_{L^2(F_j\cap\Gamma)} + \delta \frac{1}{H} \sum_j \|u_h - u_{\odot h}\|^2_{L^2(F_j\cap\Gamma)} \\
&\geq -\delta \sum_j c_P^2 \|\nabla u_h\|^2_{L^2(\omega_j)} - \delta \sum_j c_P^2 \|\nabla \mathcal{U}_E u_h\|^2_{L^2(\tilde{\omega}_j)} + \delta \frac{1}{H} \sum_j \|u_h - u_{\odot h}\|^2_{L^2(F_j\cap\Gamma)} \\
(\text{since } c_H H \leq h) &\leq -\delta c_P^2 \|\nabla u_h\|^2_{L^2(\Omega)} - \delta c_P^2 |\mathcal{D}| \|\nabla u_{\odot h}\|^2_{L^2(\Lambda)} + \delta c_H \|u_h - u_{\odot h}\|^2_{--\frac{1}{2},h,\Gamma}.
\end{aligned}
$$

With a little abuse of notation we are denoting with $\mathcal{U}_E$ both the uniform extension to Γ and $\tilde{\omega}_j$. Therefore, we obtain

$$
\begin{aligned}
&a([u_h, u_{\odot h}], [u_h, u_{\odot h}]) + b([u_h, u_{\odot h}], \xi_h([u_h, u_{\odot h}])) \geq \\
&\qquad (1 - \delta c_P^2) \|\nabla u_h\|^2_{L^2(\Omega)} + (1 - \delta c_P^2)|\mathcal{D}| \|\nabla u_{\odot h}\|^2_{L^2(\Lambda)} + \delta c_H \|u_h - u_{\odot h}\|^2_{-\frac{1}{2},h,\Gamma}
\end{aligned}
$$

and choosing $\delta = \frac{1}{2c_P^2}$ we obtain the coercivity inequality.
Concerning the stability inequality, the proof is analogous to the one in [3].

7.2. Problem 2. In the case of Problem 2 we consider $Q_h = \{\lambda_h : \lambda_h \in P^0(K) \forall K \in \mathcal{T}^\Lambda_{h'}\}$, namely the multiplier lives on the same mesh used for the 1D solution $u_{\odot h}$. Notice that in this case we suppose that the mesh sizes of the 3D mesh $\mathcal{T}^\Omega_h$ and the 1D mesh $\mathcal{T}^\Lambda_{h'}$ are different, in particular we suppose the 1D mesh is finer. With this choice the problem is not inf-sup stable, therefore we add a stabilization term as in the case of Problem 1. Again, we have to build a new space L_h, prove that the inf-sup condition is fulfilled, build a projection operator $\pi_L : Q_h \to L_h$, build $s(\lambda_h, \mu_h)$ and prove that $\forall [u_h, u_{\odot h}]$, there exists $\xi_h([u_h, u_{\odot h}]) \in Q_h$ s.t. (7.1) and (7.2) holds. We recall that in the case of problem 2,

$$
b([u_h, u_{\odot h}], \lambda_h) = |\partial \mathcal{D}| \left(\overline{Tu_h} - u_{\odot h}, \lambda_h \right)_\Lambda .
$$

Let us consider macro patches $\{F_j\}_j$ of elements of the 3D mesh $\mathcal{T}^\Omega_h$ intersecting the 1D manifold Λ. These patches are such that and $H \leq |F_j \cap \Lambda| \leq H + h$, where H is sufficiently larger than h. Moreover, there exist constants c_h and c_H such that $c_h h \leq H \leq c_H^{-1} h$. We define the space L_h as the space of functions which are P^0 on each intersection $F_j \cap \Lambda$. Moreover, we associate to each patch F_j a shape regular macro elements ω_j, which is built adding to F_j a sufficient number of elements of $\mathcal{T}^\Omega_h$.We make the following technical assumption: $\Gamma \subset \bigcup_j \omega_j$. Thanks to the shape regularity of these macro elements, again we have that the discrete trace and Poincar inequalites hold. Moreover $\forall u_h \in X^\Omega_h$ we have the following average inequality

$$
\sum_j |\partial \mathcal{D}| \|\overline{Tu_h}\|^2_{L^2(F_j \cap \Lambda)} \leq \sum_j \|Tu_h\|^2_{L^2(\omega_j \cap \Gamma)}.
$$

I think this inequality is valid but only globally. Indeed locally it is not guaranteed that the portion of Γ corresponding to $F_j \cap \Lambda$ is contained in $\omega_j \cap \Gamma$.
We define π_L as the projection onto piecewise constant functions on $F_j \cap \Lambda$. This choice leads to the following

stabilization

$$s(\lambda_h, \mu_h) = \sum_{K \in \mathcal{T}_{h'}^{\Lambda}} \int_{\partial K} h[\![\lambda_h]\!][\![\mu_h]\!],$$

being $[\![\lambda_h]\!]$ the jump of λ_h across the internal faces of $\mathcal{T}_{h'}^{\Lambda}$.

Is L_h inf-sup stable with constants independent of the cuts?. We have to prove that $\forall l_h \in L_h$, $\exists \beta > 0$ s.t.

$$\sup_{\substack{v_h \in X^0_{h,1}(\Omega),\\ v_{\odot h} \in X^0_{h',1}(\Lambda)}} \frac{|\partial\mathcal{D}| \left(\overline{Tv_h} - v_{\odot h}, l_h\right)_\Lambda}{|\!|\!|[v_h, v_{\odot h}]|\!|\!|} \geq \beta \|l_h\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

As in the continuous case, we can choose $v_{\odot h} = 0$ and we prove that

$$\sup_{v_h \in X^0_{h,1}(\Omega)} \frac{|\partial\mathcal{D}| \left(\overline{Tv_h}, l_h\right)_\Lambda}{\|v_h\|_{H^1(\Omega)}} \geq \beta \|l_h\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

Proving the last inequality it is equivalent to find the Fortin operator $\pi_F : H^1_0(\Omega) \to X^0_{h,1}(\Omega)$, such that

$$|\partial\mathcal{D}| \left(\overline{Tv} - \overline{T\pi_F v}, l_h\right)_\Lambda = 0, \quad \forall v \in H^1_0(\Omega),\, l_h \in L_h$$

and

$$\|\pi_F v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.$$

We define

$$\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{F_j \cap \Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})}{\int_{F_j \cap \Lambda} |\partial\mathcal{D}| \overline{T\varphi_j}}$$

and $\varphi_j \in X^0_{h,1}(\Omega)$ s.t. $\mathrm{supp}(\varphi_j) \subset \bar{\omega}_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$\int_{F_j \cap \Lambda} |\partial\mathcal{D}| \overline{T\varphi_j} = O(H) \text{ and } \|\nabla\varphi\|_{L^2(\omega_j)} = O(1).$$

This construction is always possible provided H is sufficiently larger that h. Then we have

$$\begin{aligned}
|\partial\mathcal{D}| \left(\overline{Tv} - \overline{T\pi_F v}, l_h\right)_\Lambda &= \sum_j \int_{F_j \cap \Lambda} |\partial\mathcal{D}| \left[\overline{Tv} - \overline{TI_h v} - \sum_i \alpha_i \overline{T\varphi_i}\right] l_h \\
&= (\mathrm{supp}\varphi \subset \omega_j) \sum_j \int_{F_j \cap \Lambda} |\partial\mathcal{D}| \left[\overline{Tv} - \overline{TI_h v} - \alpha_j \overline{T\varphi_j}\right] l_h \\
&= \sum_j \int_{F_j \cap \Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v}) l_h - \frac{\int_{F_j \cap \Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})}{\int_{F_j \cap \Lambda} |\partial\mathcal{D}| \overline{T\varphi_j}} \int_{F_j \cap \Lambda} |\partial\mathcal{D}| \overline{T\varphi_j} l_h \\
&= (\text{using } l_h \text{ constant on } F_j \cap \Lambda)\, 0.
\end{aligned}$$

Concerning the continuity of π_F, we have

$$\|\nabla \pi_F v\|_{L^2(\Omega)} \le \|\nabla I_h v\|_{L^2(\Omega)} + \left(\sum_j |\alpha_j|^2 \|\nabla \varphi_j\|^2_{L^2(\bar\omega_j)}\right)^{\frac{1}{2}}$$
$$(\text{stability of } I_h) \lesssim \|\nabla v\|_{L^2(\Omega)} + \left(\sum_j |\alpha_j|^2 \|\nabla \varphi_j\|^2_{L^2(\bar\omega_j)}\right)^{\frac{1}{2}}$$

and for the second term we have

$$\begin{aligned}
&\sum_j |\alpha_j|^2 \|\nabla \varphi_j\|^2_{L^2(\bar\omega_j)} \le \\
&(\text{using } \|\nabla \varphi_j\| = O(1)) \lesssim \sum_j \frac{\left(\left|\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})\right|\right)^2}{\left(\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j}\right)^2} \\
&\left(\text{since } \left|\int_{F_j\cap\Lambda} |\partial\mathcal{D}|\overline{T\varphi_j}\right| = O(H)\right) \lesssim \frac{1}{H^2}\sum_j \left(\left|\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tv} - \overline{TI_h v})\right|\right)^2 \\
&(\text{Jensen}) \lesssim \frac{1}{H^2}\sum_j |F_j\cap\Lambda| \int_{F_j\cap\Lambda} |\partial\mathcal{D}|^2(\overline{Tv} - \overline{TI_h v})^2 \\
&(\text{being } |F_j\cap\Lambda| \le H+h) \lesssim \frac{1}{H}\sum_j \|\overline{Tv} - \overline{TI_h v}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \\
&(\text{average inequality}) \lesssim \frac{1}{H}\sum_j \|T(v - I_h v)\|^2_{L^2(\omega_j\cap\Gamma)} \\
&(\text{trace inequality}) \lesssim \frac{1}{H^2}\sum_j \|v - I_h v\|^2_{L^2(\omega_j)} \lesssim \frac{1}{H^2}\|v - I_h v\|^2_{L^2(\Omega)} \\
&(\text{approximation properties of } I_h) \lesssim \|\nabla v\|^2_{L^2(\Omega)}
\end{aligned}$$

and the continuity of π_F follows.

Satisfaction of the assumptions of the abstract analysis. We have to prove (7.1) and (7.2). We choose the following discrete norm

$$|||[u_h, u_{\odot h}]|||^2_{X_h(\Omega)\times X_{h'}(\Lambda)} = \|u_h\|^2_{H^1(\Omega)} + |D|\|u_{\odot h}\|^2_{H^1(\Lambda)} + |\partial\mathcal{D}|\|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda},$$

where $|\partial\mathcal{D}|\|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda} = |\partial\mathcal{D}|\|h^{\frac{1}{2}}(\overline{Tu_h} - u_{\odot h})\|^2_{L^2(\Lambda)}$. Concerning the coercivity property (7.1), we have to show that $\forall [u_h, u_{\odot h}]$, there esists $\xi_h \in Q_h$ s.t.

$$\begin{aligned}
&(u_h, u_h)_{H^1(\Omega)} + |D|(u_{\odot h}, u_{\odot h})_{H^1(\Lambda)} + |\partial\mathcal{D}|(\overline{Tu_h} - u_{\odot h}, \xi_h)_\Lambda \\
&\qquad \ge \alpha_\xi(\|u_h\|^2_{H^1(\Omega)} + |D|\|u_{\odot h}\|^2_{H^1(\Lambda)} + |\partial\mathcal{D}|\|\overline{Tu_h} - u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda}.
\end{aligned}$$

We choose

$$\xi_{h|F_j\cap\Lambda} = \delta\frac{1}{H}\pi_L(\overline{Tu_h} - u_{\odot h}) \qquad \text{with } \pi_L(\overline{Tu_h} - u_{\odot h}) = \frac{1}{|\Gamma_{F_j\cap\Lambda}|}\int_{F_j\cap\Lambda} \|\partial\mathcal{D}\|(\overline{Tu_h} - u_{\odot h}),$$

being $\Gamma_{F_j\cap\Lambda}$ the portion of Γ with centerline $F_j\cap\Lambda$. Actually, $\xi_h \in L_h \subset Q_h$. Then,

$$
\begin{aligned}
|\partial\mathcal{D}|\left(\overline{Tu_h}-u_{\odot h},\xi_h\right)_\Lambda &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\overline{Tu_h}-u_{\odot h})\xi_h \\
&= \delta\frac{1}{H}\sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L(\overline{Tu_h}-u_{\odot h}))^2 \\
\text{(orthogonality of } \pi_L) &= \delta\frac{1}{H}\|(\pi_L-\mathcal{I})(\overline{Tu_h}-u_{\odot h})\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} + \delta\frac{1}{H}\|\overline{Tu_h}-u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\geq -\delta\frac{1}{H}\sum_j \|(\pi_L-\mathcal{I})\overline{Tu_h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} - \delta\frac{1}{H}\sum_j |\partial\mathcal{D}|\|(\pi_L-\mathcal{I})u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\quad + \delta\frac{1}{H}\sum_j |\partial\mathcal{D}|\|\overline{Tu_h}-u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|}.
\end{aligned}
$$

For the first term we have

$$
\begin{aligned}
\sum_j \|(\pi_L-\mathcal{I})\overline{Tu_h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L\overline{Tu_h}-\overline{Tu_h})^2 \\
\text{(Average inequality)} &\leq \sum_j \int_{\omega_j\cap\Gamma} (\pi_L\overline{Tu_h}-Tu_h)^2 \\
\text{(trace inequality)} &\leq \sum_j \frac{1}{H}\int_{\omega_j} (\pi_L\overline{Tu_h}-u_h)^2 \\
\text{(Poincare, see [5, Corollary B.65])} &\leq \sum_j Hc_P^2\|\nabla u_h\|^2_{L^2(\omega_j)}.
\end{aligned}
$$

For the second term we have

$$
\begin{aligned}
\sum_j \|(\pi_L-\mathcal{I})u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\partial\mathcal{D}|} &= \sum_j \int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\pi_L u_{\odot h}-u_{\odot h})^2 \\
\text{(Poincare, [5, Corollary B.65])} &\leq \sum_j H^2c_P^2\int_{F_j\cap\Lambda} |\partial\mathcal{D}|(\nabla u_{\odot h})^2 \\
\text{(since } H \text{ is fixed, we can find a constant s.t. } H|\partial\mathcal{D}| \lesssim |\mathcal{D}|) &\lesssim \sum_j Hc_P^2\int_{F_j\cap\Lambda} |\mathcal{D}|(\nabla u_{\odot h})^2 \\
&\lesssim \sum_j Hc_P^2\|\nabla u_{\odot h}\|^2_{L^2(F_j\cap\Lambda),|\mathcal{D}|}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&a([u_h,u_{\odot h}],[u_h,u_{\odot h}]) + b([u_h,u_{\odot h}],\xi_h([u_h,u_{\odot h}])) \geq \\
&\qquad (1-\delta c_P^2)\|\nabla u_h\|^2_{L^2(\Omega)} + (1-\delta c_P^2)|\mathcal{D}|\|\nabla u_{\odot h}\|^2_{L^2(\Lambda)} + \delta c_H|\partial\mathcal{D}|\|\overline{Tu_h}-u_{\odot h}\|^2_{-\frac{1}{2},h,\Lambda}
\end{aligned}
$$

and choosing $\delta = \frac{1}{2c_P^2}$ we obtain the coercivity inequality.
Concerning the stability inequality, the proof is analogous to the one in [3].

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