COUPLING PDES ON 3D-1D DOMAINS WITH LAGRANGE MULTIPLIERS

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Abstract. These are personal notes written to keep track of the developments on this topic, to be kept confidential.

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AMS subject classifications. n.a.

1. Introduction. We address the geometrical configuration of the problem for a 3D coupled problem formulation based on from Dirichlet-Neumann interface conditions. Then, we apply a model reduction technique that transforms the problem into 3D-1D coupled PDEs. We develop and analyze a robust definition of the coupling operators form a 3D domain, Ω, to 1D manifold, Λ, and vice versa. This is a non trivial objective because the standard trace operator form a domain Ω to a subset Λ is not well posed if Λ is a manifold of co-dimension two of Ω.

2. Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex open set. Let Σ be a generalized cylinder embedded into Ω and be $\Omega_\oplus = \Omega \setminus \overline{\Sigma}$ be the complementary set of the cylinder. We also introduce the set Λ, a 1D manifold that is the centerline of Σ. We define the arc-length coordinate along Λ, denoted by $s \in (0, S)$. We denote with $\mathcal{D}(s)$ and $\partial\mathcal{D}(s)$ a cross section of Σ and its boundary, respectively. In what follows, we assume for simplicity of notation that Σ has a constant cross section, but this is not a restriction of the approach. We also assume that Σ crosses Ω from side to side and we call Γ the lateral (cylindrical) surface of Σ, while the upper and lower side faces of Σ belong to $\partial\Omega$. We refer to Figure 2.1 for an illustration of the notation.

We consider the problem arising from *Dirichlet-Neumann* conditions. It consists to find $u_\oplus, u_\ominus$ s.t.:

$$-\Delta u_\oplus + u_\oplus = f \qquad \text{in } \Omega_\oplus, \tag{2.1a}$$

$$-\Delta u_\ominus + u_\ominus = g \qquad \text{in } \Sigma, \tag{2.1b}$$

$$-\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus \qquad \text{on } \Gamma, \tag{2.1c}$$

$$u_\ominus = u_\oplus \qquad \text{on } \Gamma, \tag{2.1d}$$

$$u_\oplus = 0 \qquad \text{on } \partial\Omega\,. \tag{2.1e}$$

The objective of this work is to derive and alalyze a simplified version of problem (2.1), where the domain Σ shrinks to its centerline Λ and the corresponding partial differential equation is averaged on the cylinder cross section, namely $\mathcal{D}$. This new problem setting will be called the *reduced* problem. Form the mathematical standpoint it is more challenging than (2.1), because it involves the coupling of 3D/1D elliptic problems. For the model reduction process, we decompose integrals as follows, for any sufficiently regular function w,

$$\int_\Sigma w d\omega = \int_\Lambda \int_{\mathcal{D}} w d\sigma ds = \int_\Lambda |\mathcal{D}| \overline{\overline{w}} ds\,, \quad \int_\Gamma w d\sigma = \int_\Lambda \int_{\partial\mathcal{D}} w d\gamma ds = \int_\Lambda |\partial\mathcal{D}| \overline{w} ds\,,$$

where $\overline{\overline{w}}$, $\overline{w}$ denote the following mean values respectively,

$$\overline{\overline{w}} = |\mathcal{D}|^{-1} \int_{\mathcal{D}} w d\sigma\,, \quad \overline{w} = |\partial\mathcal{D}|^{-1} \int_{\partial\mathcal{D}} w d\gamma\,.$$

We apply the model reduction approach at the level of the variational formulation. We start from the variational formulation of problem (2.1), that is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ u_\ominus \in H^1_{\partial\Sigma\setminus\Gamma}(\Sigma),\ \lambda \in H^{-\frac{1}{2}}(\partial\Sigma)$ s.t.

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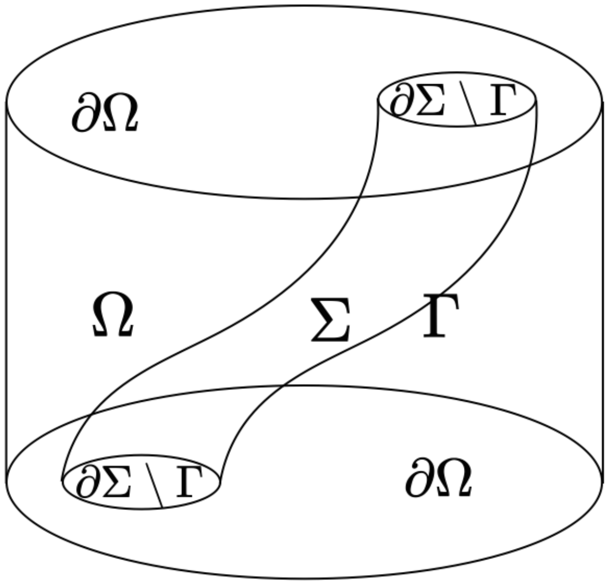

FIGURE 2.1. *Geometrical setting of the problem*

$$(u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Sigma)} + \langle v_\ominus - v_\oplus, \lambda\rangle_{H^{-\frac{1}{2}}(\Gamma)} \tag{2.2a}$$
$$= (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Sigma)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Sigma\setminus\Gamma}(\Sigma)$$
$$\langle u_\ominus - u_\oplus, \mu\rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,, \tag{2.2b}$$

where $\langle v, \mu\rangle_{H^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. In this case, the additional variable λ is equivalent to $\lambda = -\nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

Using the averaging tools for the model reduction approach, we end up with a reduced problem for the unknown u defined on the entire 3D domain Ω, coupled with the unknown $u_\odot$, defined on the 1D manifold Λ and a Lagrange multiplier L defined on Λ. In the reduced problem the multiplier assumes the following interpretation,

$$L = -\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}} \lambda\,.$$

We consider two alternative formulations. The scope of this work is to compare them, with the aim to determine which is the most suitable as a computational model based on 3D-1D coupled PDEs.

2.1. Problem 1. The idea is to couple a 3D PDE with a 1D one, using a Lagrange multiplier space defined on a 2D surface that surrounds the 1D manifold. Let $\langle\cdot,\cdot\rangle_\Gamma$ denote the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. The problem consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ L \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$(u, v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \tag{2.3a}$$
$$= (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda)$$
$$\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{2.3b}$$

Here, $\Pi_1 : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ and $\Pi_2 : H^1_0(\Lambda) \to H^{\frac{1}{2}}_{00}(\Gamma)$ and we remark that Σ may be considered as a virtual surface not necessarily of the same size as the underlying physical structure that is modeled. The Π_1 and Π_2 operators may be defined in terms of the averaging operators above, but may also be realized in terms of e.g. Green functions (I don't know if this is a good idea). Furthermore, Γ may be discretized in terms of facets neighboring Λ and may as such not be represented as a separate structure in the implementation.

2.2. Problem 2. Another form of the reduced problem uses Lagrange multipliers defined directly on the 1D manifold. In this case, $\langle\cdot,\cdot\rangle_\Lambda$ denotes the duality pairing between $H^{\frac{1}{2}}_{00}(\Lambda)$ and $H^{-\frac{1}{2}}(\Lambda)$. The problem requires to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ L \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned}
&(u,v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)} \\
&\qquad = (f,v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1_0(\Lambda)
\end{aligned} \tag{2.4a}$$

$$|\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{2.4b}$$

We notice that all the integrals of the reduced problem are well defined because $u,v \in H^1_0(\Omega)$, $u,v|_\Gamma \in H^{\frac{1}{2}}_{00}(\Gamma)$ and thus $\overline{u},\overline{v} \in H^{\frac{1}{2}}_{00}(\Lambda)$. More precisely, $\Pi_1 : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Lambda$ such that $\Pi_1 u = \overline{(u|_\Gamma)}$ is the combination between the trace on Γ and the average on $\partial\mathcal{D}$. The operator $\Pi_2 : H^1_0(\Lambda) \to H^{\frac{1}{2}}_{00}(\Lambda)$ is the injection form $H^1_0(\Lambda)$ and $H^{\frac{1}{2}}_{00}(\Lambda)$. It is apparent that problems (2.2) and (4.1) share the same mathematical structure. For this reason, the well-posedness of (4.1) can be studied in the framework of the classical theory of saddle point problems.

3. Saddle-point problem analysis. Let us consider the general saddle point problem of the form: find $u \in X$, $p \in Q$ s.t.

$$\begin{cases} a(u,v) + b(v,p) = f(v) & \forall v \in X \\ b(u,q) = g(q) & \forall q \in M \end{cases} \tag{3.1}$$

which embraces problems 1 and 2 described before. For the analysis of such problems we apply the following general abstract theorem. We denote with A and B the operators associated to the bilinear forms a and b, namely $A : X \longrightarrow X'$ with $\langle Au, v\rangle_{X',X} = a(u,v)$ and $\langle Bv, q\rangle_{X',Q} = b(v,q)$.

Theorem 3.1 (theorem 2.34 Ern-Guermond). *Problem (3.1) is well posed iff*

$$\begin{cases} \exists \alpha > 0 :\ \inf_{u \in ker(B)} \sup_{v \in ker(B)} \frac{a(u,v)}{\|u\|_X \|v\|_X} \geq \alpha \\ \forall v \in ker(B),\ (\forall u \in ker(B),\ a(u,v) = 0) \implies v = 0. \end{cases} \tag{3.2}$$

and

$$\exists \beta > 0 :\ \inf_{q \in Q} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \|q\|_Q} \geq \beta. \tag{3.3}$$

Notice that if a is coercive on $ker(B)$, (3.2) is clearly fulfilled.

3.1. Problem 1. It consists to find $u \in H^1_0(\Omega),\ u_\odot \in H^1_0(\Lambda),\ L \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned}
&(u,v)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot, v_\odot)_{H^1(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\
&\qquad = (f,v)_{L^2(\Omega)} + |\mathcal{D}(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda)
\end{aligned} \tag{3.4a}$$

$$|\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,, \tag{3.4b}$$

Here, $\Pi_1 : H^1_0(\Omega) \to H^{\frac{1}{2}}_{00}(\Gamma)$ is the trace operator while Π_2 is the uniform extension from $H^1_0(\Lambda)$ to $H^{\frac{1}{2}}_{00}(\Gamma)$. We notice that the trace operator is surjective from $H^1_0(\Omega)$ to $H^{\frac{1}{2}}_{00}(\Gamma)$. We apply theorem 3.1 in the following spaces $X = H^1_0(\Omega) \times H^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$ and we prove that:

- a coercive $\implies$ (3.2) is fulfilled

- We have to prove that $\forall M \in H^{-\frac{1}{2}}(\Gamma)$, $\exists \beta > 0$:

$$\sup_{v\in H_0^1(\Omega),\, v_\odot \in H_0^1(\Lambda)} \frac{\langle \Pi_1 v - \Pi_2 v_\odot, M\rangle_\Gamma}{\sqrt{\|v\|^2_{H^1(\Omega)} + \|v_\odot\|^2_{H^1(\Lambda)}}} \geq \beta \sup_{q \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle q, M\rangle}{\|q\|_{H_{00}^{\frac{1}{2}}(\Gamma)}}.$$

We choose $v_\odot \in H_0^1(\Lambda)$ such that $\Pi_2 v_\odot = 0$. Therefore,

$$\sup_{v\in H_0^1(\Omega),\, v_\odot \in H_0^1(\Lambda)} \frac{\langle \Pi_1 v - \Pi_2 v_\odot, M\rangle_\Gamma}{\sqrt{\|v\|^2_{H^1(\Omega)} + \|v_\odot\|^2_{H^1(\Lambda)}}} \geq \sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M\rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.$$

The trace operator is surjective from $H_0^1(\Omega)$ to $H_{00}^{\frac{1}{2}}(\Gamma)$. Consequently, $\forall \xi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we find v solution of

$$\begin{aligned} -\Delta v &= 0 \quad \text{in } \Omega \\ v &= 0 \ \text{ on } \partial\Omega \\ v &= \xi \ \text{ on } \Gamma. \end{aligned}$$

We denote with E the harmonic extension operator defined above. The boundedness/stability of this operator ensures that there exists $\|E\| \in \mathbb{R}$ such that $v = E\xi$ and $\|v\|_{H^1(\Omega)} \leq \|E\|\|\xi\|_{H_{00}^{\frac{1}{2}}(\Gamma)}$. Substituting in the previous inequalities we obtain

$$\sup_{v \in H_0^1(\Omega)} \frac{\langle \Pi_1 v, M\rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \geq \sup_{\xi \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle \xi, M\rangle_\Gamma}{\|E\|\|\xi\|_{H_{00}^{\frac{1}{2}}(\Gamma)}} = \|E\|^{-1}\|M\|_{H^{-\frac{1}{2}}(\Gamma)},$$

where in the last inequality we exploited the fact that $H^{-\frac{1}{2}}(\Gamma) = (H_{00}^{\frac{1}{2}}(\Gamma))^*$.

3.2. Problem 2. This problem requires to find $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $L \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$\begin{aligned} &(u, v)_{H^1(\Omega)} + |\mathcal{D}|(U, V)_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 V - \Pi_2 v, L\rangle_\Lambda \\ &\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ V \in H^1(\Lambda) \end{aligned} \tag{3.5a}$$

$$|\partial\mathcal{D}|\langle \Pi_1 U - \Pi_2 u, M\rangle_\Lambda = 0 \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{3.5b}$$

Here, $\Pi_1 : H_0^1(\Lambda) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$ is the immersion operator and $\Pi_2 : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$ is defined as the composition of the trace operator $T_\Gamma : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$ and the average operator $\bar{()} : H_{00}^{\frac{1}{2}}(\Gamma) \rightarrow H_{00}^{\frac{1}{2}}(\Lambda)$, namely $\Pi_2 = \bar{()} \circ T_\Gamma$. First of all we prove that if $v \in H_0^1(\Omega)$, than $\Pi_2 v \in H_{00}^{\frac{1}{2}}(\Lambda)$. In particular, from standard trace theory, we have that $T_\Gamma v \in H_{00}^{\frac{1}{2}}(\Gamma)$, therefore we have to prove that if $v \in H^{\frac{1}{2}}(\Gamma)$ then $\overline{v} \in H^{\frac{1}{2}}(\Lambda)$.

Lemma 3.2. *When Γ is a cylinder, if $u \in H_{00}^{\frac{1}{2}}(\Gamma)$, then $\overline{u} \in H_{00}^{\frac{1}{2}}(\Lambda)$.*

Proof. Let us denote as ϕ_{ij} and ρ_{ij}, for $i = 1, 2, \ldots, j = 0, 1, \ldots,$ the eigenfunctions and the eigenvalues of the laplacian on Γ, and with ϕ_i and ρ_i the eigenfunctions and the eigenvalues of the laplacian on Λ. In particular,

$$\begin{aligned} \phi_{ij}(s,\theta) &= sin(i\pi s)\,(cos(j\theta) + sin(j\theta))\,, \\ \rho_{ij} &= i\pi^2 + \frac{j^2}{R^2}, \\ \phi_i(s) &= sin(i\pi s), \\ \rho_i &= i\pi^2. \end{aligned}$$

It is easy to verify that

$$\int_0^{2\pi} \phi_{ij}(s,\theta) = 0 \quad \forall j > 0, \forall i \tag{3.6}$$

$$\int_0^{2\pi} \phi_{ij}(s,\theta) = 2\pi R \sin(i\pi s) \quad \text{if } j = 0, \forall i. \tag{3.7}$$

(3.8)

Moreover we recall that $\phi_{i,j}(s,\theta)$ and $\phi_i(s)$ are orthogonal basis of $L^2(\Gamma)$ and $L^2(\Lambda)$ respectively. Therefore,

$$\begin{aligned}\overline{u}(s) = \frac{1}{2\pi R}\int_0^{2\pi} u(s,\theta) R\,d\theta = \frac{1}{2\pi R}\int_0^{2\pi} \sum_{i,j} a_{i,j}\phi_{i,j}(s,\theta) R\,d\theta \\ = \frac{1}{2\pi R}\sum_{i,j} a_{i,j}\int_0^{2\pi} \phi_{i,j}(s,\theta) R\,d\theta = \sum_i a_{i,0}\phi_i(s).\end{aligned}$$

From [**?**, Lemma 4.11] we have

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} (1+\rho_{ij})^{\frac{1}{2}}\,|a_{ij}|^2, \text{ with } a_{ij} = \int_0^1\int_0^{2\pi} u(s,\theta)\phi_{ij}\, Rd\theta ds. \tag{3.9}$$

and

$$\|\overline{u}\|^2_{H^{\frac{1}{2}}(\Lambda)} = \sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}}\,|\overline{a}_i|^2, \text{ with } \overline{a}_i = \int_0^1 \overline{u}(s)\phi_i(s)ds.$$

Therefore, we have

$$\begin{aligned}\|\overline{u}\|^2_{H^{\frac{1}{2}}(\Lambda)} &= \sum_{i=1}^{\infty}\left(1+i^2\pi^2\right)^{\frac{1}{2}}\left(\int_0^1 \overline{u}(s) sin(i\pi s)\,ds\right)^2 \\ &= \sum_{i=1}^{\infty}\left(1+i^2\pi^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty} a_{j,0}\int_0^1 \sin(j\pi s)\sin(i\pi s)\,ds\right)^2 \\ &= \sum_{i=1}^{\infty}\frac{1}{4}\left(1+i^2\pi^2\right)^{\frac{1}{2}} a_{i,0}^2 \\ &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+i^2\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}|a_{i,j}|^2 = \|u\|^2_{H^{\frac{1}{2}}(\Gamma)},\end{aligned}$$

where we have used the fact that

$$\int_0^1 \sin(i\pi s)\sin(j\pi s)\,ds = 0 \quad \text{if } i \neq j$$
$$\int_0^1 \sin(i\pi s)\sin(j\pi s)\,ds = \frac{1}{2} \quad \text{if } i = j.$$

c.v.d. □

Lemma 3.3. *If Σ is a straight cylinder, if $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ is constant on each cross section, namely $u(s,\theta) = u(s)$, then*

$$\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = 2\pi R\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)}.$$

Proof. From (3.9),

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}|a_{ij}|^2 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+i\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}\left(\int_0^1\int_0^{2\pi}u(s,\theta)\phi_{ij}\,Rd\theta ds\right)^2$$

$$= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+i\pi^2+\frac{j^2}{R^2}\right)^{\frac{1}{2}}\left(\int_0^1 u(s)\int_0^{2\pi}\phi_{ij}\,Rd\theta ds\right)^2,$$

and using (3.6) and (3.7), we obtain

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \sum_{i=1}^{\infty}\left(1+i\pi^2\right)^{\frac{1}{2}}\left(\int_0^1 u(s)sin(i\pi s)2\pi R ds\right)^2$$

$$= 4\pi^2R^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}|a_i|^2 = 4\pi^2R^2\|u\|^2_{H^{\frac{1}{2}}(\Lambda)}. \quad \square$$

Then, we apply Theorem 3.1 with the following spaces $X = H_0^1(\Omega)\times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Lambda)$. Let us consider X equipped with the norm $|||[u,u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + |\mathcal{D}|\|u_\odot\|^2_{H^1(\Lambda)}$, Q equipped with the norm

$$\|M\|_{H^{-\frac{1}{2}}} := \sup_{q\in H^{\frac{1}{2}}(\Lambda)}\frac{\langle q, M\rangle}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}$$

Then, the following properties hold true:

- the form $a([u,U],[v,V]) = (u,v)_{H^1(\Omega)} + |\mathcal{D}|(U,V)_{H^1(\Lambda)}$ is coercive $\Longrightarrow$ (3.2) is fulfilled. Indeed, we have,

$$a([u,u_\odot],[u,u_\odot]) = (u,u)_{H^1(\Omega)} + |\mathcal{D}|(u_\odot,u_\odot)_{H^1(\Lambda)} = |||[u,u_\odot]|||^2.$$

- We have that $\forall M \in H^{-\frac{1}{2}}(\Lambda)$, $\exists\beta > 0$:

$$\sup_{v\in H_0^1(\Omega),\,V\in H_0^1(\Lambda)}\frac{\langle \Pi_1 V - \Pi_2 v, M\rangle_\Lambda}{\sqrt{\|v\|^2_{H^1(\Omega)}+\|V\|^2_{H^1(\Lambda)}}} \geq \beta \sup_{q\in H^{\frac{1}{2}}(\Lambda)}\frac{\langle q, M\rangle_\Lambda}{\|q\|_{H^{\frac{1}{2}}(\Lambda)}}.$$

  We choose $V = 0$ and we obtain

$$\sup_{v\in H_0^1(\Omega),\,V\in H_0^1(\Lambda)}\frac{\langle \Pi_1 V - \Pi_2 v, M\rangle_\Lambda}{\sqrt{\|v\|^2_{H^1(\Omega)}+\|V\|^2_{H^1(\Lambda)}}} \geq \sup_{v\in H_0^1(\Omega)}\frac{\langle \Pi_2 v, M\rangle_\Lambda}{\|v\|_{H^1(\Omega)}}.$$

  For any $q \in H_{00}^{\frac{1}{2}}(\Lambda)$, we consider its uniform extension to Γ and then we consider the harmonic extension $v = E(q) \in H_0^1(\Omega)$. It follows that $\Pi_2 v = q$. Therefore,

$$\sup_{v\in H_0^1(\Omega)}\langle \Pi_2 v, M\rangle_\Lambda \geq \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)}\langle q, M\rangle_\Lambda.$$

  Moreover, using Lemma 3.3 we obtain

$$\|v\|_{H_0^1(\Omega)} \leq \|E\|\|q\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}|\|E\|\|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)}.$$

  Therefore,

$$\sup_{v\in H_0^1(\Omega)}\frac{\langle \Pi_2 v, M\rangle_\Lambda}{\|v\|_{H^1(\Omega)}} \geq \sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)}\frac{\langle q, M\rangle_\Lambda}{\|v\|_{H^1(\Omega)}} \geq |\partial\mathcal{D}|^{-1}\|E\|^{-1}\sup_{q\in H_{00}^{\frac{1}{2}}(\Lambda)}\frac{\langle q, M\rangle_\Lambda}{\|q\|_{H_{00}^{\frac{1}{2}}(\Lambda)}}$$

$$= |\partial\mathcal{D}|^{-1}\|E\|^{-1}\|M\|_{H^{-\frac{1}{2}}(\Lambda)}.$$

REMARK 3.1. *The results of* (3.2) *and* (3.3) *can be generalized to the case of a different geometry of* Γ, *for example a parallelepiped.*

4. Finite element approximation. Let us introduce an admissibile triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_{\tilde{h}}^\Lambda$ of Λ. We denote with $X_h(\Omega) \subset H_0^1(\Omega)$ the conforming finite element space of continuous piecewise linear functions defined on Ω and by $X_{\tilde{h}}(\Lambda) \subset H_0^1(\Lambda)$ the space of continuous piecewise linear functions defined on Λ. Moreover, Q_H denotes a suitable trial space for the lagrange multiplier L_H: $Q_H \subset H^{-\frac{1}{2}}(\Gamma)$ in the case of Problem 1 and $Q_H \subset H^{-\frac{1}{2}}(\Lambda)$ in the case of Problem 2.

4.1. Problem 1. It consists to find $u_h \in X_h(\Omega),\ u_{\odot_{\tilde{h}}} \in X_{\tilde{h}}(\Lambda),\ L_H \in Q_H(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{aligned} &(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot_{\tilde{h}}}, v_{\odot_{\tilde{h}}})_{H^1(\Lambda)} + \langle \Pi_1 v_h - \Pi_2 v_{\odot_{\tilde{h}}}, L_H\rangle_\Gamma \\ &\qquad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}(\overline{g}, v_{\odot_{\tilde{h}}})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega),\ v_{\odot_{\tilde{h}}} \in X_{\tilde{h}}(\Lambda) \end{aligned} \tag{4.1a}$$

$$\langle \Pi_1 u_h - \Pi_2 u_{\odot_{\tilde{h}}}, M_H\rangle_\Gamma = 0 \quad \forall M_H \in Q_H(\Gamma)\,, \tag{4.1b}$$

THEOREM 4.1. $\exists \gamma_1 > 0$ *s.t.*

$$\inf_{L_H \in Q_H(\Gamma)} \sup_{\substack{v_h \in X_h(\Omega),\\ v_{\odot_{\tilde{h}}} \in X_{\tilde{h}}(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot_{\tilde{h}}}, L_H\rangle_\Gamma}{|||[v_h, v_{\odot_{\tilde{h}}}]||| \|L_H\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \gamma_1. \tag{4.2}$$

Proof. Let $L_H \in Q_H(\Gamma)$. As in the continuos case, let us choose $v_{\odot_{\tilde{h}}} = 0$, therefore

$$\sup_{\substack{v_h \in X_h(\Omega),\\ v_{\odot_{\tilde{h}}} \in X_{\tilde{h}}(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot_{\tilde{h}}}, L_H\rangle_\Gamma}{|||[v_h, v_{\odot_{\tilde{h}}}]|||} \geq \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_1 v_h, L_H\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Following Steinbach, Theorem 11.5, it can be shown under the following assumptions

- the mesh size h of the trial space $X_h(\Omega)$ is sufficiently small compared to the mesh size H of $Q_H(\Gamma)$, i.e. $h \leq c_0 H$ with $c_0 < 1$, and
- a global inverse inequality for the trial space $Q_H(\Gamma)$ holds,

that exists a positive constant c_S

$$c_S \|L_H\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \qquad \forall L_H \in Q_H(\Gamma), \tag{4.3}$$

being $X_h(\Gamma)$ the trace space of the functions in $X_h(\Omega)$, namely the space of the restrictions of the functions in $X_h(\Omega)$ to Γ. Using the boundedness of the extension operator E from $H^{\frac{1}{2}}(\Gamma)$ to $H_0^1(\Omega)$ introduced in the previous section, we have

$$\sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|E\| \sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|E w_h\|_{H^1(\Omega)}}.$$

Let $R_h : H^1(\Omega) \to X_h(\Omega)$ be a quasi interpolation operator satisfying

$$\|R_h v\|_{H^1(\Omega)} \leq c_R \|v\|_{H^1(\Omega)} \qquad \forall v \in H^1(\Omega).$$

Therefore, we obtain

$$\|E\| \sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|E w_h\|_{H^1(\Omega)}} \leq \|E\| c_R \sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|R_h E w_h\|_{H^1(\Omega)}}$$

and using (4.3), we have

$$\begin{aligned} c_S \|L_H\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|E\| c_R \sup_{w_h \in X_h(\Gamma)} \frac{\langle w_h, L_H\rangle_\Gamma}{\|E w_h\|_{H^1(\Gamma)}} \\ &= \|E\| c_R \sup_{w_h \in X_h(\Gamma)} \frac{\langle \Pi_1\ R_h E w_h, L_H\rangle_\Gamma}{\|R_h E w_h\|_{H^1(\Omega)}} \leq \|E\| c_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_1 v_h, L_H\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}. \end{aligned} \tag{4.4}$$

Therefore the inf-sup condition (4.2) holds with $\gamma_1 = c_S \|E\|^{-1} c_R^{-1}$. $\square$

4.2. Problem 2. This problem requires to find $u_h \in X_h(\Omega),\ u_{\odot_{\tilde h}} \in X_{\tilde h}(\Lambda),\ L_H \in Q_H(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$
\begin{aligned}
&(u_h, v_h)_{H^1(\Omega)} + |\mathcal{D}|(u_{\odot_{\tilde h}}, v_{\odot_{\tilde h}})_{H^1(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v_{\odot_{\tilde h}} - \Pi_2 v_h, L_H\rangle_\Lambda \\
&\qquad = (f, v_h)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_{\odot_{\tilde h}})_{L^2(\Lambda)} \quad \forall v_h \in X_h(\Omega),\ v_{\odot_{\tilde h}} \in X_{\tilde h}(\Lambda) \\
&|\partial\mathcal{D}|\langle \Pi_1 u_{\odot_{\tilde h}} - \Pi_2 u_h, M_H\rangle_\Lambda = 0 \quad \forall M_H \in Q_H(\Lambda)\,.
\end{aligned}
$$

THEOREM 4.2. $\exists \gamma_2 > 0$ *s.t.*

$$
\inf_{L_H \in Q_H(\Lambda)} \sup_{\substack{vh \in X_h(\Omega), \\ v_{\odot_{\tilde h}} \in X_{\tilde h}(\Lambda)}} \frac{\langle \Pi_1 v_h - \Pi_2 v_{\odot_{\tilde h}}, L_H\rangle_\Lambda}{|||[v_h, v_{\odot_{\tilde h}}]||| \|L_H\|_{H^{-\frac{1}{2}}(\Lambda)}} \geq \gamma_2. \tag{4.6}
$$

Proof. Let L_H be arbitrarly chosen in $Q_H(\Lambda)$. Again, we choose $v_{\odot_{\tilde h}} = 0$, so that (4.6) reduces to prove

$$
\gamma_2 \|L_H\|_{H^{\frac{1}{2}}(\Lambda)} \leq \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, L_H\rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}} \qquad \forall L_H \in Q_H(\Lambda).
$$

Assume that

- the mesh size h of the trial space $X_h(\Omega)$ is sufficiently small compared to the mesh size H of $Q_H(\Lambda)$, i.e. $h \leq c_1 H$ with $c_1 < 1$,
- a global inverse inequality for the trial space $Q_H(\Lambda)$ holds and
- the space $X_h(\Lambda)$, defined as the space of the restrictions on Γ of the functions in $X_h(\Omega)$ averaged on the cross section, has the approximation property, namely if Q_h^σ denotes the projection from $H^\sigma(\Lambda)$ to $X_h(\Lambda)$, we have

$$
\|w - Q_h^\sigma w\|_{H^\sigma(\Lambda)} \leq c_A h^{s-\sigma} |w|_{H^s(\Lambda)} \qquad \forall w \in H^s(\Lambda).
$$

  (I think $X_h(\Lambda)$ concides with the space of piecewise linear continuous polinomials on Λ).

Under the previous assumptions, an inequality similar to (4.3) holds also for $X_h(\Lambda)$. In particular, following the same proof in Steinbach, we can prove

$$
c_{S_2} \|L_H\|_{H^{-\frac{1}{2}}(\Lambda)} \leq \sup_{w_h \in X_h(\Lambda)} \frac{\langle w_h, L_H\rangle_\Lambda}{\|w_h\|_{H^{\frac{1}{2}}(\Lambda)}} \qquad \forall L_H \in Q_H(\Lambda). \tag{4.7}
$$

If we denote with $\mathcal{U}_E$ the uniform extension operator from Λ to Γ, using Lemma 3.3, we easily have for any $w_h \in H^{\frac{1}{2}}(\Lambda)$,

$$
\|\mathcal{U}_E w\|_{H^{\frac{1}{2}}(\Gamma)} = |\partial\mathcal{D}| \|w\|_{H^{\frac{1}{2}}(\Lambda)}.
$$

Consequently, using again the extension operator E from $H^{\frac{1}{2}}(\Omega)$ to $H_0^1(\Omega)$ and the quasi interpolation operator R_h from $H^1(\Omega)$ to $X_h(\Omega)$, we obtain

$$
\begin{aligned}
c_{S_2} \|L_H\|_{H^{-\frac{1}{2}}(\Lambda)} &\leq \sup_{w_h \in X_h(\Lambda)} \frac{\langle w_h, L_H\rangle_\Lambda}{\|w_h\|_{H^{\frac{1}{2}}(\Lambda)}} \\
&\leq |\partial\mathcal{D}|^{-1} \sup_{w_h \in X_h(\Lambda)} \frac{\langle w_h, L_H\rangle_\Lambda}{\|\mathcal{U}_E w_h\|_{H^{\frac{1}{2}}(\Gamma)}} \leq |\partial\mathcal{D}|^{-1} \|E\| \sup_{w_h \in X_h(\Lambda)} \frac{\langle w_h, L_H\rangle_\Lambda}{\|E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}|^{-1} \|E\| C_R \sup_{w_h \in X_h(\Lambda)} \frac{\langle w_h, L_H\rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&= |\partial\mathcal{D}|^{-1} \|E\| C_R \sup_{w_h \in X_h(\Lambda)} \frac{\langle \Pi_1 R_h E\mathcal{U}_E w_h, L_H\rangle_\Lambda}{\|R_h E\mathcal{U}_E w_h\|_{H^1(\Omega)}} \\
&\leq |\partial\mathcal{D}|^{-1} \|E\| C_R \sup_{v_h \in X_h(\Omega)} \frac{\langle \Pi_2 v_h, L_H\rangle_\Lambda}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned} \tag{4.8}
$$

Therefore, (4.6) holds with $\gamma_2 = c_{S_2}|\partial\mathcal{D}|\|E\|^{-1}C_R^{-1}$ ∎

5. A benchmark problem with analytical solution. We consider the following 3D-1D coupled problem,

$$-\Delta u = f \quad \text{in } \Omega \tag{5.1a}$$

$$-d_{zz}^2 u_\odot = g \quad \text{on } \Lambda \tag{5.1b}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{5.1c}$$

$$u_\odot - \overline{u} = q \quad \text{on } \Lambda \tag{5.1d}$$

where $\Omega = [0,1]\times[0,1]\times[0,H]$, $\Lambda = \{x = 0.5\}\times\{y = 0.5\}\times[0,H]$ and

$$f = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$$
$$g = \frac{\pi^2}{H^2}\sin\left(\frac{\pi z}{H}\right)$$
$$q = \sin\left(\frac{\pi z}{H}\right).$$

In this case the z coordinate coincides with the axial coordinate along Λ. We define $\Sigma = [0.25, 0.75]\times[0.25, 0.75]\times[0,H]$. The average of the 3D solution $\overline{u}$ in (5.1d) is computed on the cross section $\partial\mathcal{D}$ of the virtual interface $\Gamma = \partial\Sigma$. The exact solution of (5.1) is given by

$$u = \sin(2\pi x)\sin(2\pi y) \tag{5.2}$$

$$u_\odot = \sin\left(\frac{\pi z}{H}\right) \tag{5.3}$$

Let us notice that $u_\odot$ satisfies homogeneous Dirichlet conditions at the boundary of Λ. Moreover, the solution (5.2)-(5.3) satisfies on Γ the relation

$$\lambda = \nabla u \cdot \mathbf{n}_\oplus = d_z u_\odot n_{\oplus,z} = 0, \tag{5.4}$$

being $n_{\oplus,z}$ the $z-$component of the normal unit vector to Γ.

We prove that (5.2)-(5.3) is solution of (2.3) and (2.4) in the simplified case in which the starting 3D-3D problem is

$$-\Delta u_\oplus = f \qquad \text{in } \Omega_\oplus, \tag{5.5a}$$

$$-\Delta u_\ominus = g \qquad \text{in } \Sigma, \tag{5.5b}$$

$$-\nabla u_\ominus \cdot \boldsymbol{n}_\ominus = -\nabla u_\oplus \cdot \boldsymbol{n}_\ominus \qquad \text{on } \Gamma, \tag{5.5c}$$

$$u_\ominus - u_\oplus = q \qquad \text{on } \Gamma, \tag{5.5d}$$

$$u_\oplus = 0 \qquad \text{on } \partial\Omega\,. \tag{5.5e}$$

instead of (2.1). Therefore the reduced problems in the two different formulations (2.3) and (2.4) become respectively

$$\begin{aligned}(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda)\end{aligned} \tag{5.6a}$$

$$\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_\Gamma = \langle q, M\rangle_\Gamma \quad \forall M \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{5.6b}$$

and

$$\begin{aligned}(\nabla u, \nabla v)_{L^2(\Omega)} + |\mathcal{D}|(d_s u_\odot, d_s v_\odot)_{L^2(\Lambda)} + |\partial\mathcal{D}|\langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_{H^{-\frac{1}{2}}(\Lambda)} \\ = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, V)_{L^2(\Lambda)} \quad \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\end{aligned} \tag{5.7a}$$

$$|\partial\mathcal{D}|\langle \Pi_1 u - \Pi_2 u_\odot, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} = |\partial\mathcal{D}|\langle \overline{q}, M\rangle_{H^{-\frac{1}{2}}(\Lambda)} \quad \forall M \in H^{-\frac{1}{2}}(\Lambda)\,. \tag{5.7b}$$

Let us prove that (5.2)-(5.3) is solution of (5.6). Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (5.6a) we have

$$
\begin{aligned}
&-(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d^2_{ss} u_\odot, v_\odot)_{L^2(\Lambda)} + \langle \Pi_1 v - \Pi_2 v_\odot, L\rangle_\Gamma \\
&\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega),\ v_\odot \in H^1(\Lambda).
\end{aligned}
$$

Clearly, since (5.2) satisfies (5.1a) and (5.3) satisfies (5.1b), we have that

$$
\begin{aligned}
-(\Delta u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} \\
-|\mathcal{D}|(d^2_{ss} u_\odot, v_\odot)_{L^2(\Lambda)} &= |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)}
\end{aligned}
$$

and being $L = \lambda = 0$, we can conclude that (5.2)-(5.3) satisfy (5.6a). The fact that the solution satisfy (5.6b) follows from (5.1d). We can prove in the same way that (5.2)-(5.3) is solution of (5.7), exploiting the fact that in this case $L = \overline{\lambda} = 0$.

Remark 5.1. *Let us notice that the 3D solution (5.2) is such that $\overline{u} = 0$. Therefore in (5.1) it is like we are solving two separated problems, one in Ω and the other on Λ.*

Remark 5.2. *It would be interesting to make a comparison between the solution of the fully coupled 3D-3D problem (2.1) (also in the simplified case of type (5.5)) and the solution of the reduced problems (2.3) and (2.4). Therefore, we could set the values of the data of the problem such that the reduced formulation becomes non-trivial and fully coupled. Then, we will solve both the original and reduced problem to observe the differences in the solutions and the values of the Lagrange multiplier.*