Mathematica code for the simulations of the paper: "Scaling limits of planar maps under the Smith embedding"

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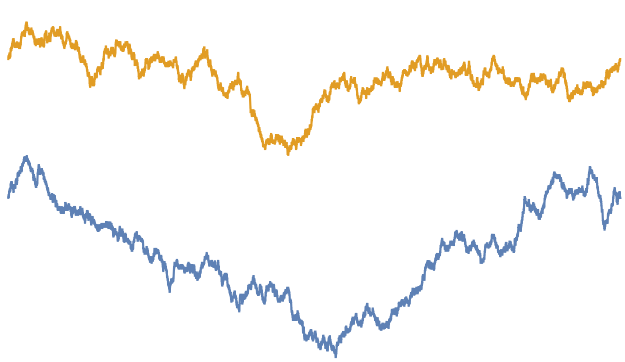
Abstract

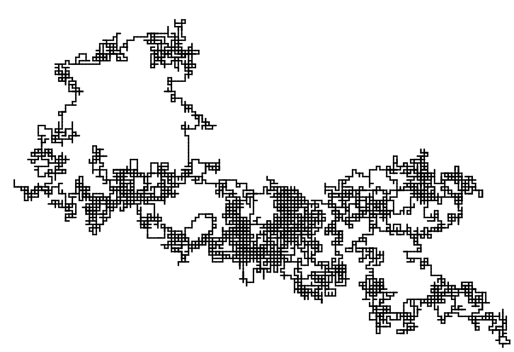
In this document, we provide the Mathematica code used to produce the simulations of the Smith embedding of a planar map contained in the paper "Scaling limits of planar maps under the Smith embedding".

1 Code and outputs

The Mathematica code below generates a random walk (X, Y) in $\mathbb{Z}^2$ conditioned to end where it started after n steps. The code starts by computing (A, B), which is the 45-degree-rotated version of (X, Y) whose coordinates are independent one-dimensional walks conditioned to end at zero. It then displays two figures, one showing the trace of the walk in $\mathbb{Z}^2$ and the other plotting the X and Y coordinates separately as functions of time.

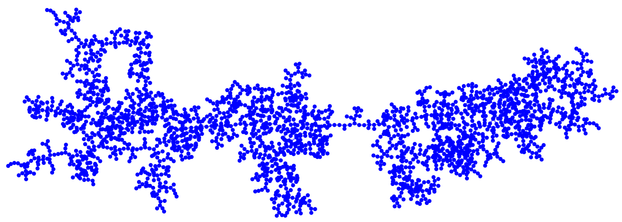
```
n = 2000;
Z = Table[0, {j, 1, n + 1}];
A = Z;
B = Z;

For[j = 1, j < n, ++j,
  A[[j + 1]] = A[[j]] + If[2 RandomReal[] > 1 + A[[j]]/(n - j), 1, -1];
  B[[j + 1]] = B[[j]] + If[2 RandomReal[] > 1 + B[[j]]/(n - j), 1, -1]
];

X = n/2 + (A + B)/2;
Y = n/2 + (A - B)/2;

{
  ListPlot[{X, n + Sqrt[n] - Y}, PlotJoined -> True, Axes -> False],
  Graphics[
    Table[Line[{{X[[j]], Y[[j]]}, {X[[j + 1]], Y[[j + 1]]}}], {j, 1, n}]
  ]
}
```

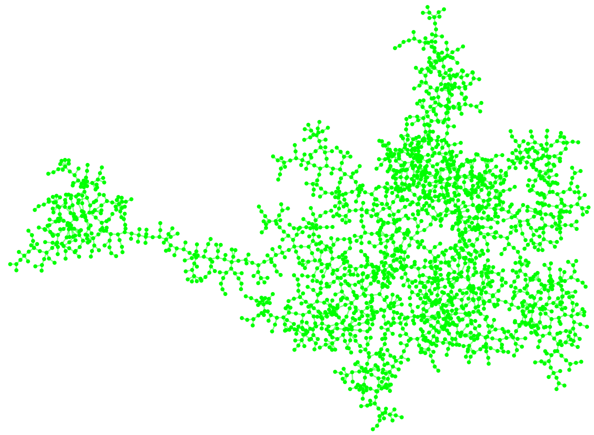




The graph of X determines a tree obtained by identifying points connected by a chord under the graph, with the two endpoints of $[0, n]$ identified so that chords can wrap around. An embedding of such a tree is shown below in blue (with a similar tree for Y shown in green).

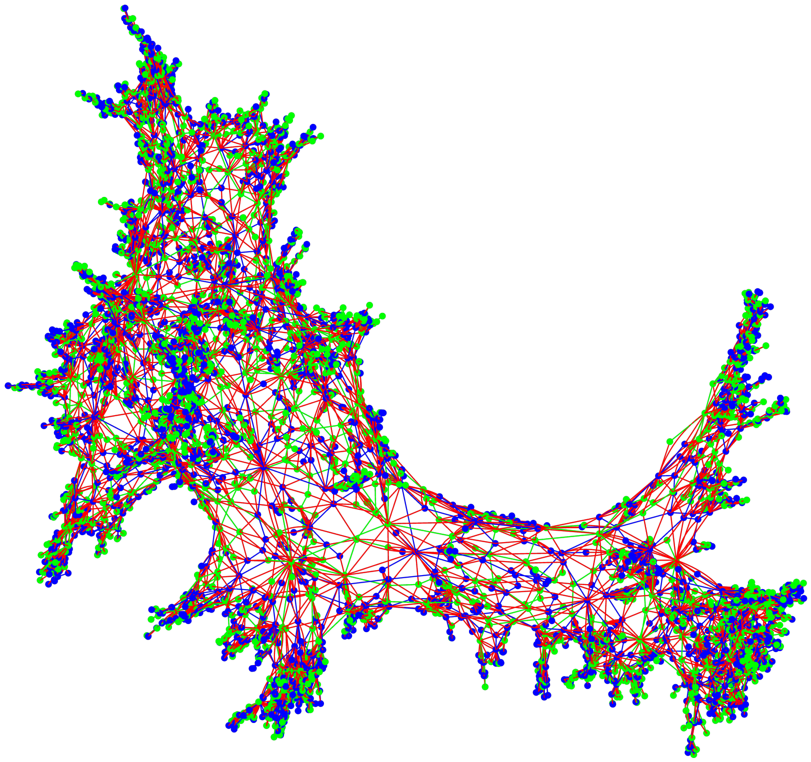
```
vertnumX = Z + 1;
vertnumY = Z + 1;
last = Z + 1;
minxloc = 1;
minyloc = 1;

For[j = 1, j < n + 1, ++j,
  If[X[[j]] < X[[minxloc]], minxloc = j];
  If[Y[[j]] < Y[[minyloc]], minyloc = j]
];

count = 1;

For[j = minxloc, j < n + 1, ++j,
  If[X[[j + 1]] > X[[j]],
    vertnumX[[j + 1]] = ++count;
    last[[X[[j + 1]]]] = count,
    vertnumX[[j + 1]] = last[[X[[j + 1]]]]
  ]
];

vertnumX[[1]] = vertnumX[[n + 1]];

For[j = 1, j < minxloc - 1, ++j,
  If[X[[j + 1]] > X[[j]],
    vertnumX[[j + 1]] = ++count;
    last[[X[[j + 1]]]] = count,
    vertnumX[[j + 1]] = last[[X[[j + 1]]]]
  ]
];

vertnumY[[minyloc]] = ++count;
last = Z + count;

For[j = minyloc, j < n + 1, ++j,
  If[Y[[j + 1]] > Y[[j]],
    vertnumY[[j + 1]] = ++count;
    last[[Y[[j + 1]]]] = count,
    vertnumY[[j + 1]] = last[[Y[[j + 1]]]]
  ]
];

vertnumY[[1]] = vertnumY[[n + 1]];

For[j = 1, j < minyloc - 1, ++j,
  If[Y[[j + 1]] > Y[[j]],
    vertnumY[[j + 1]] = ++count;
    last[[Y[[j + 1]]]] = count,
    vertnumY[[j + 1]] = last[[Y[[j + 1]]]]
  ]
];

{
  GraphPlot[
    SimpleGraph[Table[Style[vertnumX[[j]] <-> vertnumX[[j + 1]], Blue], {j, 1, n}]],
    VertexStyle -> Blue,
    GraphLayout -> {"SpringEmbedding"}
  ],
  GraphPlot[
    SimpleGraph[Table[Style[vertnumY[[j]] <-> vertnumY[[j + 1]], Green], {j, 1, n}]],
    VertexStyle -> Green,
    GraphLayout -> {"SpringEmbedding"}
  ]
}
```





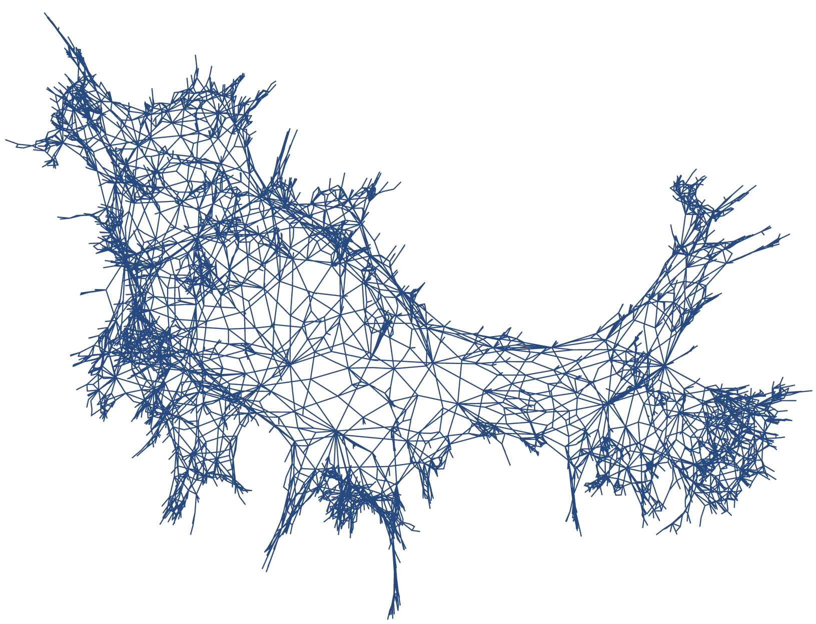
We can then draw in the red edges between the blue and green trees to construct and illustrate a graph object as follows.

```
g = Table[0 -> 0, {j, 1, 3 n/2}];
count = 1;

For[j = 1, j <= n, ++j,
  g[[count++]] = Style[vertnumX[[j]] <-> vertnumY[[j]], Red]
];

For[j = 1, j <= n, ++j,
  If[X[[j]] < X[[j + 1]],
    g[[count++]] = Style[vertnumX[[j]] <-> vertnumX[[j + 1]], Blue]
  ]
];

For[j = 1, j <= n, ++j,
  If[Y[[j]] < Y[[j + 1]],
    g[[count++]] = Style[vertnumY[[j]] <-> vertnumY[[j + 1]], Green]
  ]
];

GraphPlot3D[
  g,
  VertexStyle -> Table[i -> If[i < vertnumY[[minyloc]], Blue, Green], {i, 1, n/2 + 2}],
  GraphLayout -> {"SpringElectricalEmbedding"}
]
```



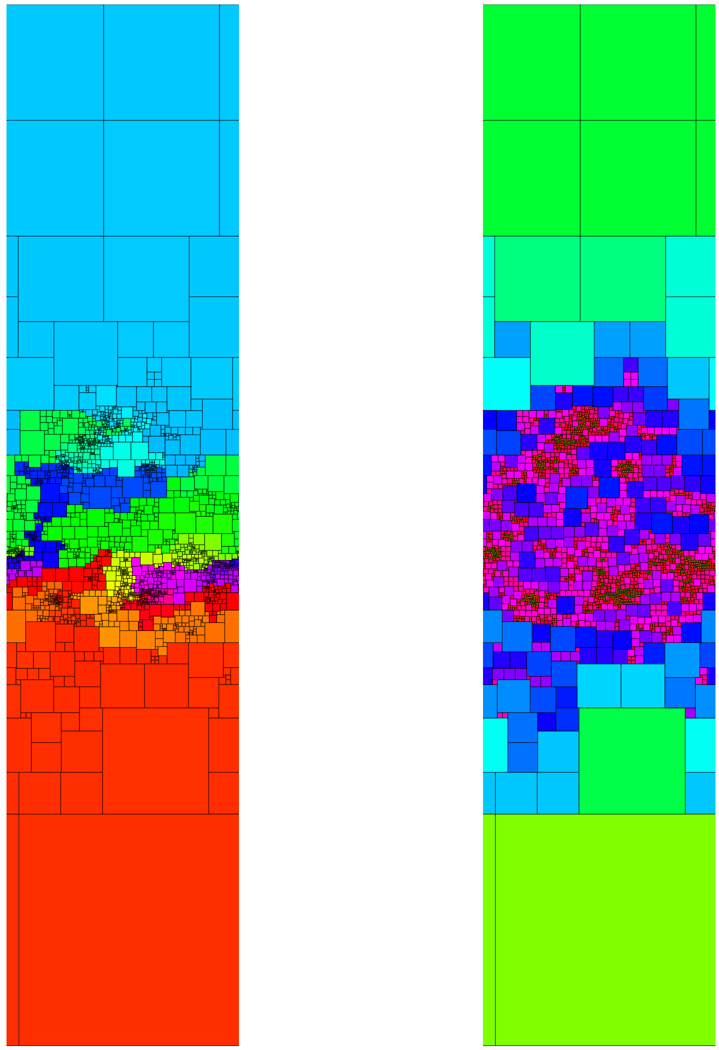
Alternatively, one can represent the graph using a sparse array M, which is more efficient when the number of vertices is large. The code below displays only the red edges, so the map is a quadrangulation.

```
M = SparseArray[Table[0, {j, 1, n/2 + 2}, {k, 1, n/2 + 2}]];
For[j = 1, j < n + 1, j++,
  M[[vertnumX[[j]], vertnumY[[j]]]] += 1
];
x = GraphPlot3D[M,
  GraphLayout -> {"SpringElectricalEmbedding"},
  VertexShapeFunction -> None
]
```



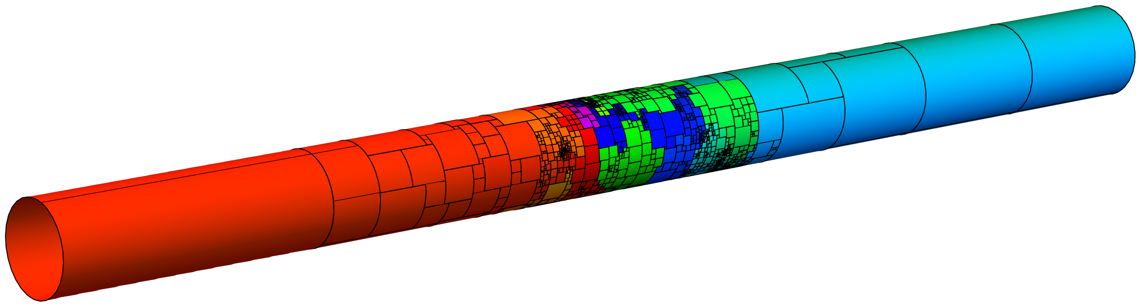
Once M is computed as above we can use some linear algebra to compute a function w that is discrete harmonic except at the points a and b corresponding to the root and dual root of the two trees. We can then display the corresponding Smith embedding by positioning the squares one by one. The functions `sq` and `ssq` give the two different color schemes, one by size, one by order w.r.t. the space-filling path that snakes between the tree and dual tree.

```
M = M + Transpose[M];
deg = Table[Sum[M[[i, j]], {i, 1, n/2 + 2}], {j, 1, n/2 + 2}];
L = Table[If[i == j, -deg[[i]], M[[i, j]]], {i, 1, n/2 + 2}, {j, 1, n/2 + 2}];
a = vertnumX[[minxloc]];
b = vertnumY[[minyloc]];

For[j = 1, j <= n/2 + 2, ++j,
  L[[a, j]] = 0;
  L[[b, j]] = 0;
];
L[[a, a]] = 1;
L[[b, b]] = 1;

v = Table[0, {j, 1, n/2 + 2}];
v[[a]] = 1;
w = LinearSolve[N[L], N[v]];

horiz = Table[0, {j, 1, n + 1}];
vertgap = Table[w[[vertnumY[[j]]]] - w[[vertnumX[[j]]]], {j, 1, n + 1}];
horiz[[1]] = 0;

For[j = 1, j <= n, ++j,
  horiz[[j + 1]] = horiz[[j]] + vertgap[[j]];
];

horizgap = Abs[horiz[[n + 1]]];
g = Table[0, {j, 1, n}];
count = 1;

sq[bot_, top_, left_, hue_] := {Hue[hue], EdgeForm[Thin], Rectangle[{left, bot}, {left + (
    top - bot), top}]};
ssq[bot_, top_, left_, hue_] := {Hue[-Log[Abs[top - bot] + .0000001]/6], EdgeForm[Thin],
    Rectangle[{left, bot}, {left + (top - bot), top}]};

g1 = Table[sq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n], {j, 1, n}];
g2 = Table[ssq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n], {j, 1, n}];

{
  Graphics[{g1, Translate[g1, {horizgap, 0}], Translate[g1, {2 horizgap, 0}]}, PlotRange ->
    {{0, horizgap}, {0, 1}}],
  Graphics[{g2, Translate[g2, {horizgap, 0}], Translate[g2, {2 horizgap, 0}]}, PlotRange ->
    {{0, horizgap}, {0, 1}}]
}
```



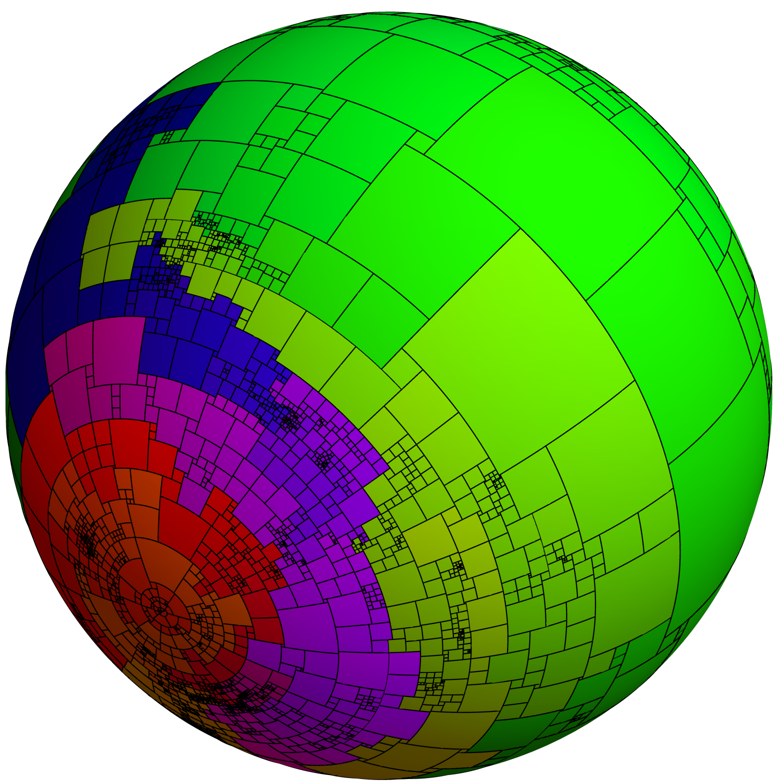
Replacing the square-embedding function `sq` with a cylinder-embedding function `rsvq` gives an embedding in the cylinder. Here we only display squares of size about some `cutoff` so that computational time is not wasted on drawing squares that may be too small to see.

```
cutoff = .0001;
rvsq[bot_, top_, left_, hue_] := RevolutionPlot3D[
{1, \[Theta]},
{\[Theta], (2 Pi/horizgap) bot, (2 Pi/horizgap) top + .0000001},
{p, (2 Pi/horizgap) left, (2 Pi/horizgap) (left + (top - bot) + .0000001)},
Mesh -> None,
PlotStyle -> Hue[hue],
BoundaryStyle -> {None, Black}
];

count = 0;
r = Table[0, {j, 1, n}];

For[j = 1, j <= n, ++j,
If[Abs[w[[vertnumX[[j]]]] - w[[vertnumY[[j]]]]] > .0001,
r[[++count]] = rvsq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n]
];
];

Show[Table[r[[j]], {j, 1, count}], PlotRange -> All, Boxed -> False, Axes -> False]
```



A similar function **spsq** (using a Mercator projection) gives an embedding in the sphere instead of cylinder.

```
cutoff = .0001;

spsq[bot_, top_, left_, hue_] := SphericalPlot3D[
  1,
  {p, 2 ArcTan[Exp[(2 Pi/horizgap) (bot - 1/2)]], 2 ArcTan[Exp[(2 Pi/horizgap) (top - 1/2)]
    + .0000001]},
  {\[Theta], (2 Pi/horizgap) left, (2 Pi/horizgap) (left + (top - bot)) + .0000001},
  Mesh -> None,
  PlotStyle -> Hue[hue],
  BoundaryStyle -> {None, Black}
];

count = 0;
For[j = 1, j <= n, ++j,
  If[Abs[w[[vertnumX[[j]]]] - w[[vertnumY[[j]]]]] > .0001,
    r[[++count]] = spsq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n]
  ];
];

Show[Table[r[[j]], {j, 1, count}], PlotRange -> All, Boxed -> False, Axes -> False]
```



We remark that one can also type the following after any of the 3D figures to export an animation showing a rotating version of the image.

```
ResourceFunction["ExportRotatingGIF"]["C:\\filename.gif", %, ImageSize -> 1200]
```