# CHERN CHARACTER OF COHERENT SHEAVES ON HYPERSURFACE SINGULARITIES

FEDERICO BONGIORNO

## Introduction

Any respectable theorem in Birational Geometry typically satisfies the following two properties:

(1) The theorem is about varieties which are not necessarily smooth.
(2) The theorem deduces an algebraic property starting from purely numerical assumptions.

Nakai's criterion, Mori's bend and break, existence of flips or the abundance conjecture are some instances of this phenomenom. An important tool in proving some of the above theorems is the Riemann-Roch theorem for (not necessarily smooth) algebraic schemes ([3, §18.3]) which relates algebraic properties and numerical properties of sheaves. However, this theorem is not as powerful as it could be: the Chern character is there defined only for sheaves and not for general complexes. This makes the reasonable statement

$$\mathrm{ch}(\mathbf{L}f^*\mathscr{F}) = f^*\mathrm{ch}(\mathscr{F})$$

meaningless. Indeed, $\mathbf{L}f^*\mathscr{F}$ is, in general, an infinite complex of sheaves.

The original motivation to attempt to develop this machinery is to prove termination of flips. The image of the Chern Character in an appropriate cohomology theory is a finite dimensional $\mathbb{Q}$-vector space. Showing its dimension decreases after a birational contraction (blow-up or flip) implies termination of flips. In the case of the standard flip, this result is already likely to hold ([5, Proposition 11.23]).

The aim of this note is to propose an alternative definition of Chern Character suited to all complexes. It does not seem straightforward to prove it is well-defined and, in fact, it may be wrong altogether.

The first section gives the definition of Chern Character and some examples suggesting consistency with intuition. All the material in this section is conjectural. The second section proves a theorem about the structure of the resolution of a sheaf on a hypersurface singularity, which may be useful to prove the Chern Character is well-defined. This section is mathematically rigorous.

Throughout the document, $X$ will be a quasi-projective scheme of finite type over a ring $S$ of dimension $d$. Recall that, in this case, $X$ has enough locally frees.

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## 1. The Chern Character

In [3, Chapter 17], a definition of Chern Character is given for coherent locally free sheaves $\mathscr{E}$ on $X$. It is denoted by $\mathrm{ch}(\mathscr{E})$ and it belongs to $A^\star(X)_\mathbb{Q}$, the Chow cohomology of $X$ with rational coefficients. This is a graded algebra over $\mathbb{Q}$, not necessarily finitely generated, whose graded components are zero outside of grading 0 to $d$. The Chern Character is alternating additive with respect to exact sequences and multiplicative with respect to tensor product.

Suppose now that $\mathscr{F}$ is a coherent sheaf on $X$ affording a finite coherent locally free resolution

$$0 \to \mathscr{E}_n \to ... \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

Exploiting the additivity property of the Chern Character, one can define

$$\mathrm{ch}(\mathscr{F}) = \sum_{k=0}^{n} (-1)^k \mathrm{ch}(\mathscr{E}_k).$$

This definition does not depend on the resolution.

The aim of this note is to generalise the above definition to the case where $\mathscr{F}$ does not have a finite locally free resolution. For instance, in the case when $X$ is singular.

**Definition 1.1.** Let $\mathscr{F}$ be a coherent sheaf on a scheme $X$ and let $\mathscr{E}_. \to \mathscr{F}$ be a locally free resolution. Let $P_{\mathscr{E}_.}(z)$ be the analytic continuation of the power series

$$\sum_{k \in \mathbb{N}} \mathrm{ch}(\mathscr{E}_k) z^k.$$

Then the Chern Character of $\mathscr{F}$ is defined to be

$$\mathrm{ch}(\mathscr{F}) = P_{\mathscr{E}_.}(-1).$$

*Remark* 1.2. (a) The power series is valued in a ring. Summing and multiplying are always allowed, however, when dividing, care has to be taken to ensure that the divisor is invertible, i.e. its grade zero component is non-zero.
(b) In [2, Infinite Free Resolutions], Corollary 4.1.5 suggests that the radius of convergence of this series is always positive. It is not clear whether an analytic continuation exists and whether it is defined at $z = -1$. It is expected that the power series is a rational function, i.e. $P_{\mathscr{E}_.}(z) \in A^\star(X)_\mathbb{Q}(z)$.
(c) This definition generalises the case where the locally free resolution is finite and, furthermore, it applies to any complex.
(d) If all resolutions are allowed, the Chern Character above defined depends on the resolution chosen (see Example 1.6). However, if the resolution satisfies some minimality condition to be yet defined, it is expected that the Chern Character will be independent of the resolution. By analogy, note that, in the non-singular case, the minimality condition states that the resolution need to be eventually zero.

**Example 1.3.** Let $X$ be cut out by $y^2 - x^2$ inside $\mathbb{P}^2_S$ and let $P$ be its singular point. Then its structure sheaf $\mathscr{O}_P$ can be resolved by

$$... \to \mathscr{O}_X(-3)^{\oplus 2} \xrightarrow{\begin{pmatrix} y & x \\ -x & -y \end{pmatrix}} \mathscr{O}_X(-2)^{\oplus 2} \xrightarrow{\begin{pmatrix} y & x \\ -x & -y \end{pmatrix}} ...$$

$$... \to \mathscr{O}_X(-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathscr{O}_X \to 0$$

where $\mathscr{O}_X(-1)$ is the pullback of the tautological invertible sheaf on $\mathbb{P}^2_S$. Using Definition 1.1,

$$\mathrm{ch}(\mathscr{O}_P) = 1 + \frac{2e^{-H}z}{1 - e^{-H}z} = \frac{1}{2}H$$

where $H$ is the first Chern class of $\mathscr{O}_X(1)$. Note that its rank is computed to be zero.

**Example 1.4.** More generally, suppose $X$ is the union of $d$ lines crossing in one point, $y^d - x^d$ inside $\mathbb{P}^2_S$ and let $P$ be its singular point. Then its structure sheaf $\mathscr{O}_P$ can be resolved by

$$... \to \mathscr{O}_X(-d-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} y^{d-1} & x^{d-1} \\ -x & -y \end{pmatrix}} \mathscr{O}_X(-2) \oplus \mathscr{O}_X(-d) \xrightarrow{\begin{pmatrix} y & x^{d-1} \\ -x & -y^{d-1} \end{pmatrix}} ...$$

$$... \to \mathscr{O}_X(-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathscr{O}_X \to 0$$

where $\mathscr{O}_X(-1)$ is the pullback of the tautological invertible sheaf on $\mathbb{P}^2_S$. Using the same notation as in the previous example,

$$\begin{aligned}
\mathrm{ch}(\mathscr{O}_P) &= 1 + \lim_{z\to -1} \frac{2e^{-H}z + e^{-2H}z^2 + e^{-dH}z^2}{1 - e^{-dH}z^2} \\
&= 1 + \lim_{z\to -1} \left( \frac{1}{1-z^2} - \frac{dz^2}{(1-z^2)^2} H \right) \left( (2z + 2z^2) - (2z + (d+2)z^2)H \right) \\
&= 1 + \lim_{z\to -1} \frac{2z + 2z^2}{1 - z^2} + \lim_{z\to -1} \frac{(2-d)z^4 + (2-2d)z^3 - (d+2)z^2 - 2z}{(1-z^2)^2} H \\
&= 1 + \lim_{z\to -1} \frac{2z}{1-z} + \lim_{z\to -1} \frac{(2-d)z^2 - 2z}{(1-z)^2} H = \left(1 - \frac{d}{4}\right) H.
\end{aligned}$$

Note that the Chern Character is able to distinguish how many lines pass through the singular point.

**Example 1.5.** Let $X$ be the cone over a conic in $\mathbb{P}^3_k$ given by the equation $y^2 - xz$. Consider the line $L$ cut by the equations $y = z = 0$ and its ideal sheaf $\mathscr{I}_L$ ([4, Example 6.5.2 §2.6]). The rank and first Chern class of $\mathscr{I}(L)$ are expected to be 1 and $-\frac{1}{2}H$ respectively, where $H$ is the first Chern class of the pullback of $\mathscr{O}_{\mathbb{P}^3}(1)$. A resolution of $\mathscr{I}_X(L)$ is given by

$$... \to \mathscr{O}_X(-3)^{\oplus 2} \xrightarrow{\begin{pmatrix} x & y \\ -y & -z \end{pmatrix}} \mathscr{O}_X(-2)^{\oplus 2} \xrightarrow{\begin{pmatrix} z & y \\ -y & -x \end{pmatrix}} \mathscr{O}_X(-1)^{\oplus 2} \to 0.$$

It follows that $\mathrm{ch}(\mathscr{I}_L) = 1 - \frac{1}{2}H$.

**Example 1.6.** Let $X$ be a scheme and $\mathscr{E}$ a coherent locally free sheaf. Then

$$... \to \mathscr{E} \xrightarrow{\mathbb{1}} \mathscr{E} \xrightarrow{0} \mathscr{E} \xrightarrow{\mathbb{1}} \mathscr{E} \to 0$$

is a resolution of the zero sheaf. Using Definition 1.1, the Chern Character of the zero sheaf is computed to be

$$\mathrm{ch}(0) = \lim_{z\to -1} \sum_{k\in\mathbb{N}} \mathrm{ch}(\mathscr{E}) z^k = \frac{\mathrm{ch}(\mathscr{E})}{1-z} = \frac{1}{2}\mathrm{ch}(\mathscr{E}).$$

This is absurd. However, note that the resolution is the trivial complex and it cannot be regarded to be *minimal*.

In the next proposition, the tensor product is assumed derived.

**Proposition 1.7.** *Let $\mathscr{F}$ and $\mathscr{G}$ be two coherent sheaves on a scheme $X$. Then*

$$\mathrm{ch}(\mathscr{F} \otimes \mathscr{G}) = \mathrm{ch}(\mathscr{F}) \cdot \mathrm{ch}(\mathscr{G}).$$

*Proof.* Let $\mathscr{E}.$ and $\mathscr{H}.$ be a locally free resolutions of $\mathscr{F}$ and $\mathscr{G}$ respectively. Then $\mathscr{E}. \otimes \mathscr{H}.$ (tensor product of complexes) resolves $\mathscr{F} \otimes \mathscr{G}$. In a small enough radius of convergence, it holds true that

$$P_{\mathscr{E}.\otimes\mathscr{H}.}(z) = P_{\mathscr{E}.}(z) \cdot P_{\mathscr{H}.}(z).$$

Indeed by the Cauchy product rule

$$\begin{aligned}
P_{\mathscr{E}.\otimes\mathscr{H}.}(z) &= \sum_{k\in\mathbb{N}} \mathrm{ch}(\mathscr{E}. \otimes \mathscr{H}.)_k z^k \\
&= \sum_{k\in\mathbb{N}} \left( \sum_{i+j=k} \mathrm{ch}(\mathscr{E}_i \otimes \mathscr{H}_j) \right) z^k \\
&= \sum_{k\in\mathbb{N}} \left( \sum_{i+j=k} \mathrm{ch}(\mathscr{E}_i) z^i \cdot \mathrm{ch}(\mathscr{H}_j) z^j \right) \\
&= \left( \sum_{i\in\mathbb{N}} \mathrm{ch}(\mathscr{E}_i) z^i \cdot \sum_{j\in\mathbb{N}} \mathrm{ch}(\mathscr{H}_j) z^j \right) \\
&= \sum_{i\in\mathbb{N}} \mathrm{ch}(\mathscr{E}_i) z^i \cdot \sum_{j\in\mathbb{N}} \mathrm{ch}(\mathscr{H}_j) z^j \\
&= P_{\mathscr{E}.}(z) \cdot P_{\mathscr{H}.}(z)
\end{aligned}$$

By uniquenes of the analytic continuation,

$$\mathrm{ch}(\mathscr{F} \otimes \mathscr{G}) = P_{\mathscr{E}.\otimes\mathscr{H}.}(-1) = P_{\mathscr{E}.}(-1) \cdot P_{\mathscr{H}.}(-1) = \mathrm{ch}(\mathscr{F}) \cdot \mathrm{ch}(\mathscr{G}).$$

□

*Remark* 1.8. In a similar way, it is possible to deduce similar statements about the behaviour of the Chern Character with respect to exact sequences and pullbacks. However, in proving such statements, caution has to be taken. Firstly, the power series are valued in an algebra, hence, one would need to check that the relevant results about their radii of convergence are still applicable. Secondly, one would need to check that applying the required constructions, such as tensor product, direct sum or pullback, preserves *minimality* of the complexes.

## 2. Hypersurface Singularities

**Definition 2.1.** $X$ is said to enjoy *hypersurface singularities* if there exists a closed immersion $\iota : X \hookrightarrow M$ of $S$-schemes into a regular scheme $M$ together with an invertible sheaf $\tilde{\mathscr{L}}$ on $M$ and a section $f \in H^0(M, \tilde{\mathscr{L}})$ such that the sequence

$$0 \to \tilde{\mathscr{L}}^\vee \xrightarrow{\cdot f} \mathscr{O}_M \to \iota_*\mathscr{O}_X \to 0$$

is exact.

*Remark* 2.2. Requiring the map $\cdot f$ to be injective is equivalent to requiring $f$ be a non-zero divisor. In this case, Krull's Principal Ideal Theorem guarantees that for all $P \in X$, $\dim \mathscr{O}_{M,Q} = \dim \mathscr{O}_{X,P} + 1$ where $Q = \iota(P)$.

The next theorem is a global version of what is proven in [1, Lemma 0.1, Proposition 5.1 & Theorem 6.1]. It says that every coherent sheaf on a hypersurface singularity has a resolution which is eventually biperiodic.

**Theorem 2.3.** *Let $\mathscr{F}$ be a coherent sheaf on $X$, a scheme enjoying hypersurface singularities. Fix an embedding in a regular ambient scheme $M$ and let $\mathscr{L}$ be the normal invertible sheaf on $X$. Then there exists a locally free resolution $\mathscr{E}_\cdot \to \mathscr{F}$ such that for some $k$, $0 \le k \le d$, and for all $i \ge 0$,*

$$\mathscr{E}_{k+2i} = \mathscr{E}_k \otimes \mathscr{L}^{\vee \otimes i}$$
$$\mathscr{E}_{k+2i+1} = \mathscr{E}_{k+1} \otimes \mathscr{L}^{\vee \otimes i}.$$

In the first step of the proof, there is a reduction to the case where $\mathscr{F}$ is a maximal Cohen-Macaulay sheaf. Its definition is recalled.

**Definition 2.4.** A sheaf $\mathscr{F}$ on a scheme $X$ is *maximal Cohen-Macaulay* if for all $P \in X$, the $\mathscr{O}_{X,P}$-module $\mathscr{F}_P$ satisfies

$$\mathrm{depth}_{\mathscr{O}_{X,P}} \mathscr{F}_P = \dim \mathscr{O}_{X,P}.$$

*Remark* 2.5. If $X$ is regular, then, by the Auslander-Buchsbaum-Serre theorem, being maximal Cohen-Macaulay is equivalent to being locally free.

*Proof of Theorem 2.3.* It is enough to show the statement when $\mathscr{F}$ is a maximal Cohen-Macaulay sheaf on $X$. Indeed, the next Lemma shows that high enough syzygies are maximal Cohen-Macaulay sheaves.

**Lemma 2.6.** *Let $\mathscr{E}_\cdot \to \mathscr{F}$ be a resolution. Then* $\mathrm{im}\partial_k$ *is a maximal Cohen-Macaulay sheaf for all $k \ge d$.*

Therefore, suppose $\mathscr{F}$ is a maximal Cohen-Macaulay sheaf. Now $X$ is a hypersurface singularity, hence there exists a closed immersion $\iota : X \hookrightarrow M$, an invertible sheaf $\tilde{\mathscr{L}}$ and a global section $f \in H^0(M, \tilde{\mathscr{L}})$ satisfying Definition 2.1. The next Lemma shows that the resolution of $\iota_*\mathscr{F}$ on $M$ is simple.

**Lemma 2.7.** *The sheaf $\iota_*\mathscr{F}$ on $M$ can be resolved by a two term complex*

$$0 \to \tilde{\mathscr{E}}_1 \to \tilde{\mathscr{E}}_0 \to \iota_*\mathscr{F} \to 0.$$

Assuming the two lemmas, the resolution in now constructed. There exists a sequence of $\mathscr{O}_M$-morphisms

$$\ldots \to \tilde{\mathscr{L}}^{\vee \otimes 2} \xrightarrow{f} \tilde{\mathscr{L}}^\vee \xrightarrow{f} \mathscr{O}_M \to 0.$$

This sequence can be tensored with the complex

$$0 \to \tilde{\mathscr{E}}_1 \xrightarrow{\partial} \tilde{\mathscr{E}}_0 \to 0$$

in order to obtain commutative squares of morphisms

$$\tilde{a}^{bc}$$

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathscr{E}}_1 \arrow[r, "\partial"] & \tilde{\mathscr{E}}_0 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & 0 & & 0 \\
0 \arrow[r] & \tilde{\mathscr{E}}_1 \arrow[r, "\partial"] \arrow[u] & \tilde{\mathscr{E}}_0 \arrow[r] \arrow[u] & 0 & \mathscr{O}_M \arrow[u] \\
0 \arrow[r] & \tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^\vee \arrow[r, "\partial\otimes\mathbb{1}"] \arrow[u, "\mathbb{1}\otimes f"] & \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^\vee \arrow[r] \arrow[u, "\mathbb{1}\otimes f"'] & 0 & \tilde{\mathscr{L}}^\vee \arrow[u, "f"'] \\
0 \arrow[r] & \tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^{\vee\otimes 2} \arrow[r, "\partial\otimes\mathbb{1}"] \arrow[u, "\mathbb{1}\otimes f"] & \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^{\vee\otimes 2} \arrow[r] \arrow[u, "\mathbb{1}\otimes f"'] & 0 & \tilde{\mathscr{L}}^{\vee\otimes 2} \arrow[u, "f"'] \\
 & \cdots \arrow[u] & \cdots \arrow[u] & & \cdots \arrow[u]
\end{tikzcd}$$

The crucial step in constructing the resolution is to show that there exists a morphism $\sigma$ making the following diagram commute

$$\begin{tikzcd}
\tilde{\mathscr{E}}_1 \arrow[r, "\partial"] & \tilde{\mathscr{E}}_0 \\
\tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^\vee \arrow[r, "\partial\otimes\mathbb{1}"'] \arrow[u, "\mathbb{1}\otimes f"] & \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^\vee . \arrow[u, "\mathbb{1}\otimes f"'] \arrow[ul, "\sigma"']
\end{tikzcd}$$

This can be thought of as a homotopy between $f$ and the zero morphism. To show its existence, the argument goes as follows. Since $\iota_*\mathscr{F}$ is supported on the zero locus of $f$,

$$\operatorname{im}(\mathbb{1} \otimes f) \subseteq \operatorname{im}(\partial) \subseteq \tilde{\mathscr{E}}_0$$

hence there is a factorisation

$$\begin{tikzcd}
\tilde{\mathscr{E}}_1 \arrow[r, "\partial"] & \operatorname{im}(\partial) \\
\tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^\vee \arrow[r, "\partial\otimes\mathbb{1}"'] \arrow[u, "\mathbb{1}\otimes f"] & \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^\vee . \arrow[u, "\mathbb{1}\otimes f"']
\end{tikzcd}$$

Furthermore, since $\partial$ is injective, it is an isomorphism between $\tilde{\mathscr{E}}_1$ and $\operatorname{im}(\partial)$. Define $\sigma$ to be $\partial^{-1} \circ (\mathbb{1} \otimes f)$. It is straightforward to see that the diagram is commutative. Tensoring by $\tilde{\mathscr{L}}^\vee$ preserves exact sequences, hence the same argument may be repeated to construct the following sequence of $\mathscr{O}_M$-morphisms

$$\begin{tikzcd}
\tilde{\mathscr{E}}_1 \arrow[r, "\partial"] & \tilde{\mathscr{E}}_0 \\
\tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^\vee \arrow[r, "\partial\otimes\mathbb{1}"] & \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^\vee \arrow[ul, "\sigma"'] \\
\tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^{\vee\otimes 2} \arrow[r, "\partial\otimes\mathbb{1}"] & \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^{\vee\otimes 2} \arrow[ul, "\sigma"'] \\
\ldots & \ldots \arrow[ul, "\sigma"']
\end{tikzcd}$$

which, after unraveling, gives

$$\ldots \to \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^{\vee\otimes 2} \xrightarrow{\sigma} \tilde{\mathscr{E}}_1 \otimes \tilde{\mathscr{L}}^\vee \xrightarrow{\partial\otimes\mathbb{1}} \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}}^\vee \xrightarrow{\sigma} \tilde{\mathscr{E}}_1 \xrightarrow{\partial} \tilde{\mathscr{E}}_0 \to 0. \tag{2.1}$$

This is not a complex of sheaves on $M$, however when pulled back to $X$, it becomes a resolution of $\mathscr{F}$.

**Lemma 2.8.** *The pullback of the sequence of morphisms in 2.1 by $\iota$, given by*

$$\ldots \to \mathscr{E}_0 \otimes \mathscr{L}^{\vee\otimes 2} \xrightarrow{\sigma} \mathscr{E}_1 \otimes \mathscr{L}^\vee \xrightarrow{\partial\otimes\mathbb{1}} \mathscr{E}_0 \otimes \mathscr{L}^\vee \xrightarrow{\sigma} \mathscr{E}_1 \xrightarrow{\partial} \mathscr{E}_0 \to 0, \tag{2.2}$$

*where $\mathscr{E}_i = \iota^*\tilde{\mathscr{E}}_i$ and $\mathscr{L} = \iota^*\tilde{\mathscr{L}}$, is an $\mathscr{O}_X$-resolution of $\mathscr{F}$.*

This concludes the proof. □

*Remark* 2.9. Using Definition 1.1

$$P_{\mathscr{E}.}(z) = \frac{\mathrm{ch}(\mathscr{E}_0) + \mathrm{ch}(\mathscr{E}_1)z}{1 - \mathrm{ch}(\mathscr{L}^\vee)z^2}.$$

By inverting the polynomial on the bottom, $P_{\mathscr{E}.}(z)$ can be realised as an element of $A^\star(X)_{\mathbb{Q}}(z)$. Note that $P_{\mathscr{E}.}(z)$ is not a priori well-defined at $z = -1$. Indeed $1 - \mathrm{ch}(\mathscr{L}^\vee)$ is not invertible in $A^*(X)_{\mathbb{Q}}$. However, the numerator is also not invertible. Indeed, it is easy to see that $\mathrm{rk}(\mathscr{E}_0) = \mathrm{rk}(\mathscr{E}_1)$. This is a necessary but insufficient condition to show the existence of the limit.

*Proof of Lemma 2.6.* By definition, being maximal Cohen-Macaulay is a local property, hence, by localising at $P$, it may be assumed that the resolution is defined over the local ring $\mathscr{O}_{X,P}$. Let $x_1, x_2, \ldots, x_l$ be a system of parameters for $\mathscr{O}_{X,P}$. The aim is to show that $x_1, x_2, \ldots, x_l$ is an $\mathrm{im}\partial_{k,P}$-regular sequence. This is shown by induction on the regular sequence $x_1, x_2, \ldots, x_j$, the base case $j = 0$ being trivial. Suppose that $x_1, x_2, \ldots, x_{j-1}$ is an $\mathrm{im}\partial_{k,P}$-regular sequence. Denote

$$\mathscr{O}^j_{X,P} = \frac{\mathscr{O}_{X,P}}{(x_1, x_2, \ldots, x_j)}.$$

It is enough to show that $x_j$ is regular in $\mathscr{O}^{j-1}_{X,P} \otimes \mathrm{im}\partial_{k,P}$. There is a short exact sequence

$$0 \to \mathrm{im}\partial_{k,P} \to \mathscr{E}_{k-1,P} \to \mathrm{coker}\partial_{k,P} \to 0.$$

Tensoring by $\mathscr{O}^{j-1}_{X,P}$ yields an exact sequence

$$\mathrm{Tor}_1(\mathscr{O}^{j-1}_{X,P}, \mathrm{coker}\partial_{k,P}) \to \mathscr{O}^{j-1}_{X,P} \otimes \mathrm{im}\partial_{k,P} \to \mathscr{O}^{j-1}_{X,P} \otimes \mathscr{E}_{k-1,P}.$$

Now, $\mathrm{Tor}_1(\mathscr{O}^{j-1}_{X,P}, \mathrm{coker}\partial_{k,P}) = \mathrm{Tor}_k(\mathscr{O}^{j-1}_{X,P}, \mathscr{F}_P)$. But since the projective dimension $\mathrm{pd}_{\mathscr{O}_{X,P}} \mathscr{O}^{j-1}_{X,P} = j-1$ and $k \geq d \geq l \geq j$, it must follow that

$$\mathrm{Tor}_k(\mathscr{O}^{j-1}_{X,P}, \mathscr{F}_P) = 0.$$

This implies that there exists an injection

$$\kappa : \mathscr{O}^{j-1}_{X,P} \otimes \mathrm{im}\partial_{k,P} \hookrightarrow \mathscr{O}^{j-1}_{X,P} \otimes \mathscr{E}_{k-1,P}.$$

If $x_j$ is not regular, then there exists an element $0 \neq m \in \mathscr{O}^{j-1}_{X,P} \otimes \mathrm{im}\partial_{k,P}$ such that $x_j \cdot m = 0$. But the above map is injective, hence $\kappa(m) \neq 0$. Furthermore $x_j \cdot \kappa(m) = 0$, hence $x_j$ is a zero divisor on $\mathscr{O}^{j-1}_{X,P} \otimes \mathscr{E}_{k-1,P}$. However

$$\mathscr{O}^{j-1}_{X,P} \otimes \mathscr{E}_{k-1,P} \cong {\mathscr{O}^{j-1}_{X,P}}^{\oplus r}$$

for some $r \in \mathbb{N}$, since $\mathscr{E}_{k-1,P}$ is a free $\mathscr{O}_{X,P}$-module. This contradicts the fact that $x_1, x_2, ..., x_l$ is a system of parameters for $\mathscr{O}_{X,P}$. □

*Proof of Lemma 2.7.* The aim is to show that the projective dimension

$$\mathrm{pd}_M \iota_* \mathscr{F} = 1.$$

It is well-known that for any sheaf $\mathscr{G}$,

$$\mathrm{pd}_M \mathscr{G} = \sup_{Q \in M} \mathrm{pd}_{\mathscr{O}_{M,Q}} \mathscr{G}_Q$$

so it is enough to show the statement locally, i.e. it is enough to show for all $Q \in M$ that

$$\mathrm{pd}_{\mathscr{O}_{M,Q}} (\iota_* \mathscr{F})_Q \leq 1.$$

Suppose first that there does not exist a $P \in X$ such that $\iota(P) = Q$. Then $(\iota_* \mathscr{F})_Q = 0$ and it holds trivially that

$$\mathrm{pd}_{\mathscr{O}_{M,Q}} (\iota_* \mathscr{F})_Q = 0 \leq 1.$$

Suppose instead that there exists a $P \in X$ such that $\iota(P) = Q$. Since $M$ is regular, by the Auslander-Buchsbaum-Serre theorem,

$$\mathrm{pd}_{\mathscr{O}_{M,Q}} (\iota_* \mathscr{F})_Q + \mathrm{depth}_{\mathscr{O}_{M,Q}} (\iota_* \mathscr{F})_Q = \dim \mathscr{O}_{M,Q}.$$

Furthermore, it is easy to see that $\mathrm{depth}_{\mathscr{O}_{M,Q}} (\iota_* \mathscr{F})_Q = \mathrm{depth}_{\mathscr{O}_{X,P}} \mathscr{F}_P$ and since $\mathscr{F}$ is a maximal Cohen-Macaulay sheaf, $\mathrm{depth}_{\mathscr{O}_{X,P}} \mathscr{F}_P = \dim \mathscr{O}_{X,P}$. Therefore

$$\mathrm{pd}_{\mathscr{O}_{M,Q}} (\iota_* \mathscr{F})_Q = \dim \mathscr{O}_{M,Q} - \dim \mathscr{O}_{X,P} = 1$$

since $X$ is a hypersurface in $M$ (Remark 2.2). □

*Proof of Lemma 2.8.* Firstly, it is clear that the zero$^{\mathrm{th}}$ homology is given by $\mathscr{F}$. Indeed, since pullback is right-exact, the zero$^{\mathrm{th}}$ homology is readily computed to be $\iota^* \iota_* \mathscr{F} \xrightarrow{\sim} \mathscr{F}$.

Next, since exactness is a local condition, after localising at $P \in X$, the resolution may be assumed to be defined over the local ring $\mathscr{O}_{X,P}$. Let $Q = \iota(P)$ where $\iota : \mathscr{O}_{M,Q} \twoheadrightarrow \mathscr{O}_{X,P}$. By construction, the composition of the morphisms over $\mathscr{O}_{M,Q}$ is $(\mathbb{1} \otimes f)_Q$. This becomes 0 after resticting to $\mathscr{O}_{X,P}$. Hence the sequence in (2.2) is indeed a complex over $\mathscr{O}_{X,P}$.

Exactness is now proven. Pick isomorphisms $\tilde{\mathscr{E}}_{0,Q} \xrightarrow{\sim} \mathscr{O}^{\oplus r}_{M,Q}$ and $\tilde{\mathscr{E}}_{1,Q} \xrightarrow{\sim} \mathscr{O}^{\oplus s}_{M,Q}$. There is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{O}_{M,Q}^{\oplus s} \arrow[r, "\partial_Q"] \arrow[d, two heads, "\iota^{\oplus s}"'] & \mathscr{O}_{M,Q}^{\oplus r} \arrow[r, "\sigma_Q"] \arrow[d, two heads, "\iota^{\oplus r}"'] & \mathscr{O}_{M,Q}^{\oplus s} \arrow[r, "\partial_Q"] \arrow[d, two heads, "\iota^{\oplus s}"'] & \mathscr{O}_{M,Q}^{\oplus r} \arrow[r] \arrow[d, two heads, "\iota^{\oplus r}"'] & \cdots \\
\cdots \arrow[r] & \mathscr{O}_{X,P}^{\oplus s} \arrow[r, "\partial_P"'] & \mathscr{O}_{X,P}^{\oplus r} \arrow[r, "\sigma_P"'] & \mathscr{O}_{X,P}^{\oplus s} \arrow[r, "\partial_P"'] & \mathscr{O}_{X,P}^{\oplus r} \arrow[r] & \cdots
\end{tikzcd}$$

where $\sigma_Q \circ \partial_Q = f_Q \cdot \mathbb{1} = \partial_Q \circ \sigma_Q$. Suppose firstly that $u \in \ker\partial_P \subseteq \mathscr{O}_{X,P}^{\oplus s}$ and let $\tilde{u} \in \mathscr{O}_{M,Q}^{\oplus s}$ be a lift of $u$ under the surjection $\iota^{\oplus s}$. Since $u \in \ker\partial_P$, $\partial_Q\tilde{u} \in f_Q \cdot \mathscr{O}_{M,Q}^{\oplus r}$, say $\partial_Q\tilde{u} = f_Q \cdot \tilde{v}$ for some $\tilde{v} \in \mathscr{O}_{M,Q}^{\oplus r}$. But then, $\partial_Q\sigma_Q\tilde{v} = f_Q \cdot \tilde{v} = \partial_Q\tilde{u}$. This gives $\partial_Q(\sigma_Q\tilde{v} - \tilde{u}) = 0$. Since $\partial_Q$ is injective, $\sigma_Q\tilde{v} = \tilde{u}$. Letting $v = \iota^{\oplus r}\tilde{v}$ yields $\sigma_P v = u$. Lastly, suppose $u \in \ker\partial_P \subseteq \mathscr{O}_{X,P}^{\oplus r}$ and let $\tilde{u} \in \mathscr{O}_{M,Q}^{\oplus r}$ be a lift. By a similar argument, there exists a $\tilde{v} \in \mathscr{O}_{M,Q}^{\oplus s}$ such that $\sigma_Q\partial_Q\tilde{v} = \sigma_Q\tilde{u}$. Applying $\partial_Q$ to both sides gives $\partial_Q\sigma_Q\partial_Q\tilde{v} = \partial_Q\sigma_Q\tilde{u}$ which simplifies to $f_Q \cdot (\partial_Q\tilde{v} - \tilde{u}) = 0$. But $f_Q$ is a non-zero divisor, hence $\partial_Q\tilde{v} = \tilde{u}$. Letting again $v = \iota^{\oplus s}\tilde{v}$ yields $\sigma_P v = u$. □

*Remark* 2.10. The crux of the argument is the existence of the homotopy $\sigma$. This could be lifted using the fact that a maximal Cohen-Macaulay sheaf on a hypersurface $X$ can be resolved by a two term complex on the ambient space. When $X$ is not a hypersurface, the existence of $\sigma$ would follow from the vanishing of a first cohomology group, namely $H^1(M, \ker\partial \otimes \tilde{\mathscr{E}}_0 \otimes \tilde{\mathscr{L}})$, using the notation in the proof. Once this is achieved, the arguments in Theorem 2.3 and in [1, Theorem 7.1 & Theorem 7.2] may generalise to cover the case where $X$ is a local complete intersection.

## References

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