Infinitesimal groupoids and quotient spaces

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Declaration of originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

This dissertation develops a theory of infinitesimal groupoids in algebraic geometry, a natural generalisation of algebraic foliations to the singular or positive characteristic case. Three applications are presented thereafter. The first one is a criterion to establish when the quotient of a scheme by a smooth connected equivalence relation is a scheme. The second one establishes finite generation of the invariant sections of a smooth groupoid in some special cases. The third one concerns a strategy to approximate formal solutions of differential equations by solutions on étale neighbourhoods.

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Chapter 0

Introduction

0.1 Motivation

Cutting and glueing topological spaces are among the first surgeries that young mathematicians learn to perform in universities. The archetypal example is to glue the unit square along opposite edges in order to obtain a torus.

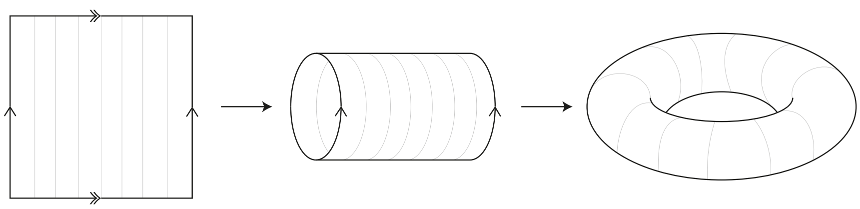


Figure 1: Building a torus

Slightly more formally, one defines an equivalence relation on the unit square by declaring two points to be equivalent if they are either the same point or if they are on opposite edges of the unit square at the same distance from the vertices. Then, the *quotient space* of the equivalence relation is the torus. Conversely, performing a longitudinal cut on the torus followed by a latitudinal cut will give back the unit square. As far as topological spaces are concerned, cutting and glueing are well understood surgical procedures. Indeed, it can be shown that there is a bijective correspondence constructed as follows:

$$\left\{ \begin{array}{c} R \to X \times X \mid R \text{ is an} \\ \text{equivalence relation on } X \end{array} \right\} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \left\{ \begin{array}{c} f : X \to Y \mid f \text{ is a} \\ \text{surjective function} \end{array} \right\} \tag{0.1.0.1}$$

$$\begin{aligned} (R \to X \times X) &\xrightarrow{\alpha} (f : X \to X/R) \\ (X \times_Y X \to X \times X) &\xleftarrow{\beta} (f : X \to Y)\,, \end{aligned}$$

where $\alpha(R) = f : X \to X/R$ maps an element of X to its equivalence class in the quotient space X/R.

What if the space X is endowed with geometric structures such as the structure of a smooth manifold or the structure of an algebraic scheme? This question is much more subtle and quotient spaces do not always exist. In other words, whilst the quotient exists as a topological space, the differential or algebraic structure need not descend to the quotient (vid. Example 0.3.1 (Hironaka's threefold)). Nonetheless, the question is so fundamental that even partial answers describing sufficient conditions for quotient spaces to exist are of great importance.

The next two subsections in this section illustrate two applications of the construction of quotient spaces. The first one is to construct moduli spaces of objects and the second one is to run the minimal model program.

Subsequently, Section 0.2 (Groupoids and quotient spaces) motivates the definition of groupoids and categorical quotients. In Section 0.3 (Quotient spaces in algebraic geometry), the discussion is restricted once and for all to algebraic geometry. Therein, it is observed that constructing categorical quotients of algebraic schemes is a hopeless task to accomplish in full generality. Nonetheless, plenty of theorems have been proved about the topic. These are discussed in this section together with some open problems. Section 0.4 (Infinitesimal groupoids) describes the main idea of this dissertation: to study groupoids and quotient spaces infinitesimally. Section 0.5 (Main results) lists the main results of this dissertation, describes the strategy to prove them and discusses how they fit in with the existing literature. Section 0.6 (A guide for the reader) contains a description of the structure and content of this dissertation. Finally, the content of Section 0.7 (Notation and conventions) is self-explanatory. The tone of this chapter is informal and all relevant concepts hereafter presented will be properly defined in the main body of the dissertation.

Moduli spaces

An important aim of several human endeavours is the classification of objects. Mathematics is no exception and, in particular, the aim of many branches of geometry is to accomplish the classification of spaces. The classification of conic sections is one of the earliest examples of classifications in geometry. It was accomplished by Apollonius in the third century BC and it can be summarised by stating that, up to Euclidean transformation, a conic is determined by its eccentricity, a non-negative real number. Therefore, the parameter space of conic sections is the set of non-negative real numbers. Due to the influence of Bernhard Riemann, parameter spaces are nowadays known as *moduli spaces*. The construction of quotients is an essential part of the construction of moduli spaces. In general, one starts by over-parametrising the spaces of interest and then constructs the quotient by a group action to dispose of those objects that were counted multiple times. In order to better understand this technique, consider this example due to Michael Artin ([Alp23, §0.2, page 17]).

Example 0.1.1 (Moduli space of triangles)**.** This example shows how to construct the moduli space of triangles up to similarity. Recall that two triangles are similar if and only if their angles have the same amplitude. In order to formally describe a triangle, it is convenient to label its vertices with letters A, B and C. Let α, β and γ be the amplitude of the angles corresponding to the vertices A, B and C and recall that $\alpha+\beta+\gamma = 180°$. A labelled triangle can be uniquely described, up to similarity, by an ordered pair of non-zero angles (α, β) such that $\alpha + \beta < 180°$. Therefore the moduli space M of *labelled* triangles can be represented as in Figure 2. However, triangles have been over-counted. In-

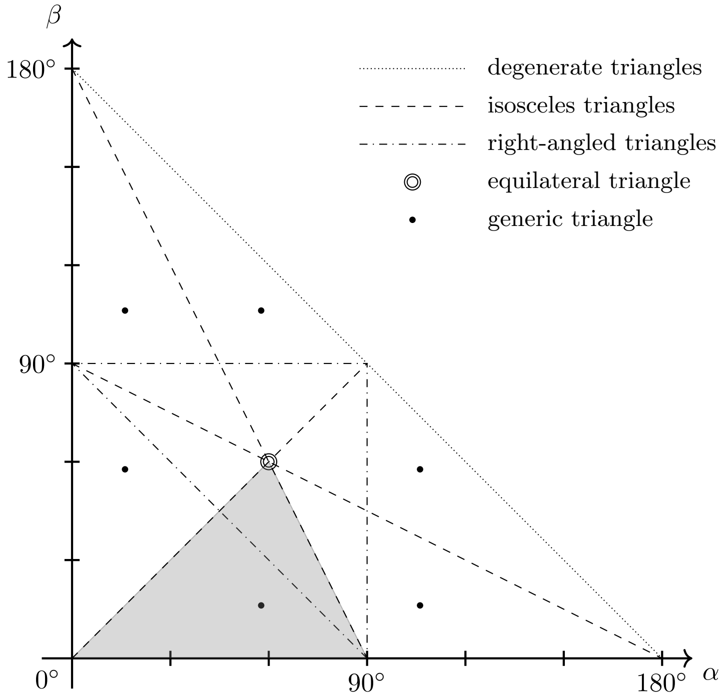


Figure 2: Moduli space of triangles

deed, the same triangle can have several labellings and, in order to construct the moduli space, all such triangles have to be identified. Note that S_3, the symmetric group on three letters, acts naturally on the moduli space of labelled triangles by permuting the labels of the vertices. For instance, a generic triangle (or more precisely a scalene triangle) can be labelled in six different ways by permuting the labels of its three vertices. The marked points in Figure 2 form the orbit of a generic triangle under this permutation. The group action induces an equivalence relation on M where two labelled triangles are equivalent if and only if the unlabelled triangles are similar. Therefore, the moduli space of triangles should be the quotient space

$$T := M/S_3. \tag{0.1.1.1}$$

By (0.1.0.1), this exists as a topological space. Since M has the structure of a smooth manifold, so it is expected of the space T. In fact, T is a smooth manifold with boundary and it can be identified with a fundamental domain of the action. This is the shaded area of Figure 2. Note it is a triangle.

Similarly, when trying to construct moduli spaces of algebraic schemes, it is expected that the moduli space will be an algebraic scheme itself. Therefore, it is necessary to show existence of quotient spaces in the category of algebraic schemes. This technique has already been successfully employed in the construction of moduli spaces of curves of genus greater than two ([DM69]) and, more generally, of varieties of general type.

Birational geometry

The minimal model program is an algorithm whose aim is to simplify the geometric properties of a given variety by contracting the areas where the variety is positively curved. It is a program whose roots are entrenched in the Italian school of algebraic geometry which flourished between the nineteenth and the twentieth century under the direction of Guido Castelnuovo, Federigo Enriques and Francesco Severi. The school developed such program in dimension two in order to complete the classification of algebraic surfaces. In higher dimensions, this is still an active area of research. A step of the minimal model program involves the construction of a quotient by an equivalence relation. This is defined by selecting a negative extremal ray of the cone of curves and declaring two points to be equivalent if they are connected by curves in the selected numerical equivalence class. This approach is evident in the proof of the abundance conjecture for varieties of dimension three ([Kol92, §9, page 103]). Depending on the *rank* of the equivalence relation, this quotient is either a fibre contraction, a divisorial contraction or a flipping contraction.

Example 0.1.2 (Quotient of blowing up)**.** Let X be the blowing up of $\mathbb{P}^2_k$ in a point. Define an equivalence relation R on X by declaring two points to be equivalent if they are either the same point or if they both belong to the exceptional divisor. Then, it should hold that

$$\mathbb{P}^2_k = X/R. \tag{0.1.2.1}$$

In order to run the minimal model program, it is necessary to show the existence of these quotients. The cases of divisorial and flipping contractions are treated as part of the cone theorem ([KM98, part (3) of Theorem 3.7, page 76]). However, the case of a fibre contraction and the determination of the rank of the equivalence relation still are outstanding problems related to the abundance conjecture.

0.2 Groupoids and quotient spaces

Having agreed that showing existence of quotient spaces is a task worthy of consideration, it is necessary to ask what a quotient space really is. Actually, it is convenient to develop a theory of quotient spaces for objects which are slightly more general than equivalence relations: groupoids. This is motivated in the first subsection. The second subsection defines the notion of categorical quotient.

Groupoids

In order to motivate the definition of a groupoid, the definition of equivalence relation is rephrased in the following way: it is an injective function

$$j := t \times s : R \to X \times X \tag{0.2.0.1}$$

admitting the existence of (unique) functions

$$e : X \to R \qquad x \to (x, x) \tag{0.2.0.2}$$
$$i : R \to R \qquad (x, y) \to (y, x) \tag{0.2.0.3}$$
$$c : R \times_{(s,t)} R \to R \qquad (x, y, y, z) \to (x, z). \tag{0.2.0.4}$$

These correspond to the properties of being reflexive, symmetric and transitive.

Equivalence relations may not always be general enough when constructing quotients. In Example 0.1.1 (Moduli space of triangles), constructing the moduli space involved constructing the quotient of a space by a group action. In details, the action of a group G on a space X is a function

$$\begin{aligned} \sigma : G \times X &\to X \\ (g, x) &\to g \cdot x \end{aligned} \tag{0.2.0.5}$$

satisfying some compatibility conditions with the properties of a group. Let e_G be the identity element of G and let $\mathrm{pr}_2 : G \times X \to X$ be the projection to the second factor. It is straightforward to verify that the image of the function

$$j := \sigma \times \mathrm{pr}_2 : G \times X \to X \times X \tag{0.2.0.6}$$

is an equivalence relation which relates x and y if and only if there exists a $g \in G$ such that $x = g \cdot y$. Similarly to the case of equivalence relations, a group action admits the existence of functions

$$e : X \to G \times X \qquad x \to (e_G, x) \tag{0.2.0.7}$$
$$i : G \times X \to G \times X \qquad (g, x) \to (g^{-1}, g \cdot x) \tag{0.2.0.8}$$
$$c : (G \times X) \times_X (G \times X) \to G \times X \qquad (g, x, h, y) \to (gh, y). \tag{0.2.0.9}$$

However, a group action is not in general an equivalence relation since j need not be injective. In fact, it is an equivalence relation if and only if every point of X has trivial stabiliser.

It is desirable to find a simultaneous generalisation of group actions and equivalence relations. A first attempt may involve defining such generalisation to be a function as in (0.2.0.1) which is not necessarily injective, but whose image is an equivalence relation. This is known as a *pre-equivalence relation*. However, a group action has much more structure which, in this generality, is lost.

A *groupoid* is a function as in (0.2.0.1) endowed with functions

$$e : X \to R \tag{0.2.0.10}$$
$$i : R \to R \tag{0.2.0.11}$$
$$c : R \times_{(s,t)} R \to R \tag{0.2.0.12}$$

satisfying natural compatibility conditions. The functions s, t, e, i and c are called the *source*, the *target*, the *identity*, the *inverse* and the *composition* function respectively. As in the case of group actions, the image $j(R)$ is an equivalence relation on X and the functions e, i and c should be thought as a choice of lift of the property of being reflexive, symmetric and transitive. By construction, a groupoid generalises both the notion of group action and the notion of equivalence relation. Furthermore, since a group action on a point is precisely a group, a groupoid also generalises the notion of a group.

A great advantage of groupoids over group actions is that they can be pulled back: if $U \subseteq X$ is a subspace of the space X, the groupoid R can be restricted to U. This is valid for group actions if and only if the subspace U is G-invariant.

Categorical quotients

Let R be a groupoid on a space X. A function $f : X \to Y$ to another space Y is R-invariant if the two functions

$$s \circ f : R \to X \tag{0.2.0.13}$$
$$t \circ f : R \to X \tag{0.2.0.14}$$

are equal. If R is induced by a group action, this means that $f(x) = f(g \cdot x)$ for all $x \in X$ and for all $g \in G$. This means precisely that f is G-invariant. If R is an equivalence relation, this means that f maps equivalence classes to points.

The *categorical quotient* of X by R is an R-invariant function $q : X \to Q$ satisfying the following universal property: for all R-invariant functions $f : X \to Y$, there exists a unique function $g : Q \to Y$ such that the following diagram

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "f"] & \\ Q \arrow[r, "g"] & Y. \end{tikzcd} \tag{0.2.0.15}$$

is commutative. By general category theory, if it exists, it is unique. It should be thought as the R-invariant function whose fibres are as small as possible.

Equivalently, the categorical quotient of X by R is the categorical colimit of the diagram

$$R \underset{t}{\overset{s}{\rightrightarrows}} X. \tag{0.2.0.16}$$

The term *quotient space* or simply *quotient* will indicate the categorical quotient, unless otherwise specified.

0.3 Quotient spaces in algebraic geometry

This dissertation lives in the realms of algebraic geometry, hence it is concerned with the construction of quotient spaces in algebraic geometry. As a result, the spaces hereafter considered will be algebraic varieties or, more generally, schemes. These are spaces which can be described as the zero locus of systems of polynomial equations.

The first subsection describes the inherent difficulties of the construction of quotient spaces in algebraic geometry. The second subsection presents an overview of what is known about them. The third subsection discusses open problems.

Why are quotients hard?

As already anticipated, quotients do not always exist in algebraic geometry. This fact admits a purely categorical explanation. Consider the contravariant Spec functor from the category of commutative rings to the category of schemes.

$$\begin{aligned} \{A \mid A \text{ is a commutative ring}\} &\longrightarrow \{X \mid X \text{ is a scheme}\} \\ A &\longrightarrow \operatorname{Spec} A. \end{aligned} \tag{0.3.0.1}$$

Recall that categorical quotients are a particular instance of finite categorical colimits and that the category of commutative rings admits all finite limits. Therefore, quotients exist in the category of affine schemes. This is constructed as follows: if R is an affine groupoid on an affine scheme $\operatorname{Spec} A$, the affine quotient is the spectrum of the subring of R-invariant sections of A, denoted by A^R. Recall that there is an adjunction between Spec and the global section functor $\Gamma(\_)$ given by

$$\operatorname{Hom}\left(A, \Gamma\left(X, \mathscr{O}_X\right)\right) = \operatorname{Hom}\left(X, \operatorname{Spec} A\right). \tag{0.3.0.2}$$

The existence of this adjunction directly implies that the Spec functor maps colimits of rings to limits of schemes. On the other hand, it does not map limits of rings to colimits of schemes. In particular, the categorical quotient in the category of affine schemes need not be the categorical quotient in the category of schemes. This phenomenon is observed in Example 6.2.6 (Non-closed equivalence relations). As a result, it is not possible to construct quotients locally and then glue, as in the case of fibre products. In fact, it may not be possible to construct quotients at all.

Example 0.3.1 (Hironaka's threefold)**.** There exists a proper smooth scheme X of dimension three over a field and a proper and free action of $\mathbb{Z}/2\mathbb{Z}$ on X (or equivalently an involution of X without any fixed points) such that the categorical quotient of X by $\mathbb{Z}/2\mathbb{Z}$ does not exist.

Another source of issues is the fact that, if A is a finitely generated algebra over a field k and R is an affine groupoid on $\operatorname{Spec} A$, the ring of R-invariant sections need not be finitely generated over k. The first example of this phenomenon was discovered by Masayoshi Nagata. The interested reader will find such example and several others in the survey [Fre01].

What is known about quotients?

The first satisfactory theory to construct quotients by group actions in algebraic geometry was developed by David Mumford in [MFK94], firstly published in 1965. A key step of the theory is to show that, if G is a reductive group acting on an affine scheme $X = \operatorname{Spec} A$ over a field k, where A is a finitely generated algebra over k, the morphism induced by the inclusion of the G-invariant sections

$$\operatorname{Spec} A \to \operatorname{Spec} A^G \tag{0.3.1.1}$$

is the categorical quotient *in the category of schemes* and A^G is a finitely generated algebra over k ([MFK94, Theorem 1.1, page 27]). This is a complete answer for group actions of reductive groups on affine schemes. Furthermore, an easy argument shows that, if a reductive group acts on a scheme X and every point of X admits an affine G-invariant open neighbourhood, the categorical quotient of X by G exists. For instance, this happens when a finite group acts on a quasi-projective variety. Of course, this excludes several cases of interest and this is the reason for introducing linearisations of ample invertible sheaves.

In 1997, using a different approach, Seán Keel and Shigefumi Mori showed a remarkable result, now known as the Keel–Mori theorem ([KM97, Theorem 1.1, page 193]). Some preliminary definitions are needed for the statement. An *algebraic stack* is (the datum of) a smooth groupoid R on a scheme X, i.e. a groupoid whose source morphism s is smooth. It is *separated* if $j = t \times s$ is finite. An *algebraic space* is (the datum of) an étale groupoid R on a scheme X, i.e. a groupoid whose source morphism s is étale. It is *separated* if $j = t \times s$ is a closed immersion. Then, the Keel–Mori theorem states that the quotient of a separated algebraic stack exists as a separated algebraic space. This can be thought as a partial quotient. Example 0.3.1 (Hironaka's threefold) shows that one cannot expect in general to obtain a scheme as opposed to an algebraic space. The Keel–Mori theorem is often employed in the construction of moduli spaces of schemes, where the typical strategy is the following:

1. Over-parametrise schemes of the moduli problem by endowing them with extra data. This amounts to constructing an algebraic stack.
2. Showing that the quotient space of the algebraic stack by the extra data is an algebraic space by use of the Keel–Mori theorem.
3. Employing properties of the moduli problem to show that such algebraic space is in fact a projective scheme.

This approach, and its application, are described in details in [Alp23].

On the other hand, if one is interested in tackling problems related to birational geometry, rather than moduli spaces, Artin's theorem on contractions ([Art70, Theorem 3.1, page 99]) may be of assistance: let T be a reduced closed subscheme of a scheme X satisfying suitable negativity hypotheses and let R be an equivalence relation which only relates all points in T. Then the quotient exists as an algebraic space.

Open problems

Note that in Example 0.3.1 (Hironaka's threefold), the group is disconnected. Therefore, it seems reasonable to ask the following vague question.

Question 0.3.2 (Scheme quotient)**.** *Let $j : R \to X \times X$ be a groupoid on a scheme X and suppose that the source morphism s is smooth with geometrically connected fibres and j is a finite morphism. Does the quotient space exist as a scheme?*

Answers to this question may be useful in the construction of moduli spaces: it may be easier to show that a scheme, as opposed to an algebraic space, is a projective scheme. Note that any algebraic space which is not a scheme is not a counterexample to the question. Indeed, an algebraic space with geometrically connected fibres is automatically trivial. It may be worth to point out that the strategy used in the proof of the Keel–Mori theorem cannot be easily adapted to answer the question. The strategy consists in restricting to a transversal of the action to reduce the construction of quotients by smooth groupoids to the construction of quotients by finite flat groupoids. But a finite flat groupoid is disconnected, unless trivial, and in this case, one cannot in general hope for the quotient space to be a scheme.

A similar related question concerns finite generation of the ring of R-invariant sections. There are examples of actions by non-reductive groups with non-finitely generated ring of invariant sections. In these cases, non-finite generation occurs where the group has non-trivial stabiliser.

Question 0.3.3 (Finite generation)**.** *Let R be an equivalence relation on an affine scheme X and suppose that the source morphism s is smooth with geometrically connected fibres. Is the ring of R-invariant sections finitely generated?*

It may also be possible to weaken the smoothness assumption by requiring the singularities to be at most log-canonical, a type of singularity occurring in the minimal model program. Paragraph 6.3.6 (Finite generation and singularities) elaborates on this idea and attempts to provide compelling evidence.

0.4 Infinitesimal groupoids

The slogan of this dissertation is the following:

Completing a groupoid along the identity may simplify problems regarding the existence of quotients spaces.

The first subsection describes the advantages of completing along the identity. The second subsection describes the technical difficulties encountered when doing so. The third subsection explores the relation with algebraic foliations.

Completing along the identity

Part of the data of a groupoid is the identity morphism $e : X \to R$. This is a section of the target morphism $t : R \to X$, therefore it is an immersion of schemes. Let $\mathfrak{R}$ be the tubular neighbourhood of X in R. Algebraically, this corresponds to completing the scheme R along the morphism e and obtain $\mathfrak{R}$ as a formal scheme. Since R is a groupoid in the category of schemes, it induces a groupoid structure on $\mathfrak{R}$ in the category of formal schemes. This is an example of *infinitesimal groupoid* or, when j is a monomorphism, an *infinitesimal equivalence relation.* The notion of $\mathfrak{R}$-invariant morphism still makes sense. For historical reasons, the $\mathfrak{R}$-invariant sections of X are called *first integrals.*

Example 0.4.1 (De Rham space)**.** When $R = X \times X$, $\mathfrak{R}$ is called the *de Rham space* of X. This is denoted by $\mathfrak{D}_X$. Its topological space is X and its structure sheaf is the sheaf of jets of X. In particular, $\mathfrak{D}_X$ contains the datum of the sheaf of differentials, since this is precisely the sheaf of jets of order one. The first integrals of $\mathfrak{D}_X$ are the locally constant functions (Example 5.2.9 (Locally constant functions)).

Example 0.4.2 (Lie algebras)**.** When $X = \operatorname{Spec} k$ is a point, a groupoid R on X is precisely a group scheme over k. Then $\mathfrak{R}$ should be thought as a higher order Lie algebra of R. Its first order data is the actual Lie algebra of R. More generally when X is a scheme and R is induced by the action of a group G, $\mathfrak{R}$ should be thought as the infinitesimal action of the Lie algebra of G on X.

Whilst more technically involved, replacing R with $\mathfrak{R}$ may have several advantages.

1. *Invariance is local.* Being R-invariant is not a local property on X. For instance, if R is induced by a free action of a finite group, every section of X is locally R-invariant. However, if the action is not trivial, not every global section of X is R-invariant. On the other hand, being an $\mathfrak{R}$-invariant morphism is a local property on X. In particular, there exists a sheaf of rings $\mathscr{O}_X^{\mathfrak{R}}$ on X of first integrals (Paragraph 5.1.6 (Sheaf of first integrals)).

2. *Restriction commutes with localisation.* The restriction of R to the local scheme $X_x := \operatorname{Spec} \mathscr{O}_{X,x}$ is an unpleasant object. On the other hand, the restriction of $\mathfrak{R}$ to X_x is simply the complete localisation of $\mathfrak{R}$ in x (Lemma 4.4.8 (Restriction to local schemes)).

3. *Frobenius' theorem.* If $\mathfrak{R}$ is a *regular* infinitesimal equivalence relation on a complex manifold X, for any $x \in X$, the restriction of $\mathfrak{R}$ to an analytic neighbourhood of x is induced by a trivial fibration. This is the content of Frobenius' theorem. Whilst Luna's étale slice theorem is a local structure theorem for algebraic stacks, Frobenius' theorem can be thought as a local structure theorem for infinitesimal stacks.

4. *Cohomology.* There shall exist a cohomology theory for the category of $\mathfrak{R}$-equivariant modules. This shall be a generalisation of algebraic de Rham cohomology.

As observed in (3), a special role is played by those infinitesimal groupoids induced by groupoids whose source morphism is smooth. There are called *regular infinitesimal groupoids* and should be thought as the infinitesimal counterpart of algebraic stacks. For instance, every action of a group on a scheme over a field of characteristic zero induces a regular infinitesimal groupoid. They are better behaved than arbitrary groupoids and most results in this dissertation only concern regular infinitesimal groupoids.

Regular infinitesimal groupoids shall help in tackling Question 0.3.2 (Scheme quotient) when R is an equivalence relation on X. Indeed, let $\mathfrak{R}$ be the regular infinitesimal equivalence relation induced by an equivalence relation R whose source morphism is smooth with geometrically connected fibres. Then the categorical quotients of $\mathfrak{R}$ and R are the same (Proposition 5.2.1 (First integrals and invariance)). Furthermore, locally analytically, the quotient of X by $\mathfrak{R}$ exists by Frobenius' theorem and one may hope of using the fact that $\mathfrak{R}$ is induced by an actual groupoid to lift the analytic quotient to a Zariski local quotient. This process is described in §6.4 (Local quotient spaces). Note that this strategy may only work when the source morphism of R has geometrically connected fibres, else the quotient of X by R would differ from the quotient by $\mathfrak{R}$. In particular, there is no blatant contradiction with the existence of algebraic spaces which are not schemes.

Formal schemes

Formal schemes are the algebraic analogue of tubular neighbourhoods of embeddings of complex varieties. Therefore, they seem to be the right object to use when constructing the tubular neighbourhood of the identity morphism of a groupoid. However, formal schemes are not schemes. Hence, a disadvantage of replacing R with $\mathfrak{R}$ is that one is forced to leave the category of schemes. Even worse, it seems necessary to work with non-Noetherian formal schemes. Consider the following example.

Example 0.4.3 (Restriction to completion)**.** Let $\mathfrak{R}$ be the de Rham space on the affine line $X := \operatorname{Spec} k[t]$ over a field k. Let $X_0 := \operatorname{Spec} k[t]_{(t)}$ be the spectrum of the localisation of X in the origin. It is desirable to be able to restrict $\mathfrak{R}$ to X_0. This should be equal to the completion along the diagonal morphism

$$\operatorname{Spec} k[t]_{(t)} \to \operatorname{Spec} \left(k[x]_{(x)} \otimes_k k[y]_{(y)}\right). \tag{0.4.3.1}$$

It can be shown that the completion is Noetherian (vid. Lemma 4.4.8 (Restriction to local schemes)). However, since the localisation morphism $k[t] \to k[t]_{(t)}$ is not of finite type, it is not clear whether the ring $k[x]_{(x)} \otimes_k k[y]_{(y)}$ is Noetherian. More generally, it is desirable to restrict to Hensel localisations and to completions. In the latter case, one is definitely forced to consider non-Noetherian rings. In fact, as pointed out in [Nic23], the ring

$$k[\![x]\!] \otimes_k k[\![y]\!] \tag{0.4.3.2}$$

is not Noetherian.

Given the desired level of generality, the standard theory of formal schemes, as developed in [Gro60, §10, page 180], is not sufficient. In fact, a complete satisfactory theory for general formal schemes was developed in [McQ02]. Unfortunately, the article contains a mistake in page 4, where it is erroneously claimed that the category of complete and separated topological groups endowed with a linear topology is Abelian. This was pointed out and partly corrected in [Yas09]. However, the theory therein developed is much more complicated as several technical difficulties are encountered.

The approach taken in this thesis is hybrid: formal schemes are defined as in [McQ02] in order to apply the theory of locally ringed spaces. On the other hand, it is shown that a formal scheme induces a formal scheme in the sense of [Yas09] in order to employ the fact that a suitable Abelian category of sheaves exists. This is needed to show properties about closed immersions of formal schemes in Section 3.3 (Immersions of formal schemes). The author is conscious that such approach may result unpleasant, however it is confined to a handful of sections in Chapter 3 (Formal schemes) and, when a more comprehensive theory about formal schemes will be available, it can be readily substituted.

Foliations

At the beginning of the author's doctoral studies, Micheal McQuillan peremptorily stated to him that the correct way to think about foliations is as infinitesimal equivalence relations. Recall that a foliation is a subsheaf of the tangent sheaf which is closed under the Lie bracket. Intuitively, a foliation is the first order information of an infinitesimal equivalence relation.

Example 0.4.4 (Foliation of the de Rham space)**.** Let $\mathfrak{D}_X$ be the de Rham space of a scheme X. Recall that its structure sheaf is the sheaf of jets and the associated foliation only remembers the first order information. This consists of the first order jets, i.e. the sheaf of differentials. Therefore, the induced foliation is the tangent sheaf of X.

Foliations may seem to contain substantially less information than infinitesimal equivalence relations, however, when X is a complex manifold, Frobenius' theorem gives a bijective correspondence between smooth foliations and regular infinitesimal equivalence relations. Under this correspondence, the existence of the Lie bracket is equivalent to the existence of the composition morphism c of the infinitesimal equivalence relation, i.e. the transitivity property. In other words, there is no incentive to keep track of the higher order data.

The insightful reader may contest the need to introduce infinitesimal groupoids. This is done for two reasons:

1. To study foliations with singularities.
2. To study foliations in positive and mixed characteristic.

It is rare to encounter a smooth algebraic foliation defined on a projective variety. For this reason, studying singular foliations is particularly important. An interesting definition of foliation singularity was suggested by Michael McQuillan. In order to motivate it, consider the action of a smooth group G on a scheme X. The induced infinitesimal groupoid is always regular. However, the induced foliation is singular in the points of X with non-trivial *stabiliser*.

Example 0.4.5 (Hyperbolic foliation)**.** Let k be a field, let $B = k[x, y]$ and let $X := \mathbb{A}^2_k = \operatorname{Spec} B$. Suppose that $G := \mathbb{G}_m = k^\times$ acts on X with weights $(1, -1)$. This induces a regular infinitesimal groupoid $\mathfrak{G}$ which in turn induces a foliation $\mathscr{F}_G$ on X given by the subsheaf of $\mathscr{T}_{X/k}$ generated by the vector field $x\,\partial/\partial x - y\,\partial/\partial y$. This is represented in Figure 3.

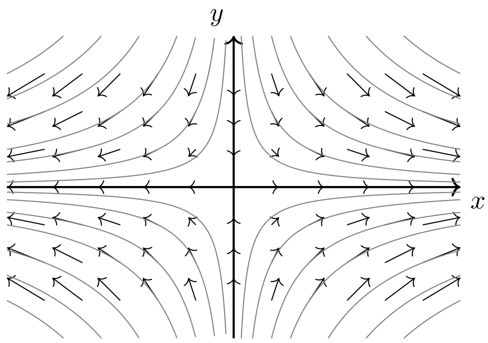


Figure 3: Hyperbolic foliation

The grey lines are the orbits of the action. Note that these are tangent to the induced vector field. Furthermore, the vector field has a singularity at the origin, the only point with non-trivial stabiliser.

This suggests that a foliation could be defined as a regular infinitesimal groupoid $j : \mathfrak{R} \to X \times X$ where j is generically a monomorphism. Its singularities would be the points where j is not a monomorphism.

Positive characteristic is another area where the groupoid approach could be beneficial. Let X be a scheme over a field of positive characteristic p. In this setting, the classical Frobenius' theorem is hopelessly false. Essentially, this is due to the fact that, for any section f of X, the derivative applied to the p^{th} power $d(f^p) = 0$. Hence the p^{th} power of any section is invariant with respect to any foliation. With the exception of trivial cases, this will never yield a smooth fibration. Working with an infinitesimal equivalence relation gives meaning to the expression $d(f^p) \neq 0$. Indeed, whilst its first order derivative is zero, the p^{th} order derivative is not necessarily zero (vid. Example 5.3.4 (Factorisations of Frobenius morphism)).

When studying singularities of varieties in positive characteristic, merely studying the sheaf of differentials is not sufficient for most purposes. Indeed, the methods of Frobenius splittings and the notions of F-purity and F-regularity have been in use for a long time in order to study singularities in positive characteristic. Some results of Karen Smith suggest that the study of Frobenius splittings is equivalent to the study of higher order jets ([Smi95]). By extension, when studying the singularities of a foliation in positive characteristic, it would seem natural to consider the whole infinitesimal equivalence relation rather that its first order data.

Finally, the author would like to convince the reader that studying foliations in positive and mixed characteristic is worthwhile. The motivation comes from a conjecture by Torsten Ekedahl, Nicholas Shepherd-Barron and Richard Taylor in the style of Shigefumi Mori's bend and break lemma. The conjecture describes an equivalent condition for a foliation to be induced by an equivalence relation, rather than an infinitesimal equivalence relation ([ESBT99, Conjecture F, page 1]). In this case, the foliation is said to be *algebraically integrable*.

Conjecture 0.4.6 (Conjecture F)**.** *A foliation on a variety over a field of characteristic zero is algebraically integrable if and only if its reduction modulo p is p-closed for all primes $p \gg 0$.*

Being p-closed is an arithmetic condition which is automatically satisfied in the case of an infinitesimal equivalence relation (vid. Example 4.5.12 (Exponential foliation)). Therefore, in good circumstances, an infinitesimal equivalence relation defined on a scheme in mixed characteristic is expected to be induced by an equivalence relation. Part of the interest aroused by this conjecture is to do with its opaque relation with the abundance conjecture.

0.5 Main results

At this point, it may be worth to mention the following:

The main objective of this dissertation is to develop
a theory of infinitesimal groupoids.

As byproduct, some applications to quotient spaces are presented.

Theorem 6.2.5 (Existence of quotients)**.** *Let $j : R \to X \times_L X$ be a smooth equivalence relation with geometrically connected fibres on a normal integral scheme X of finite type over a Noetherian scheme L, where j is a closed immersion. Then, there exists an infinitesimal, effective and universal categorical quotient*

$$q : X(\mathrm{AlgSmooth}) \to Q \tag{6.2.5.1}$$

of X by R, where Q is separated and of finite type over L.

$X(\mathrm{AlgSmooth})$ is the open subset of *algebraically smooth* points. A point $x \in X$ belongs to this set if, locally around x, the morphism induced by the inclusion of the first integrals is smooth of relative dimension equal to the rank of R. Therefore, the theorem states that it suffices to check smoothness locally to show existence of a quotient space. Intuitively, the open set of algebraically smooth points should be the smooth locus of the induced foliation. Therefore, when R is a smooth equivalence relation, every point should be algebraically smooth and, in this case, Question 0.3.2 (Scheme quotient) would admit a positive answer. Recall that a morphism is smooth if and only if it is formally smooth and locally of finite presentation. Being of finite presentation follows if the ring of invariant sections is finitely generated. In this direction, the following encouraging statement is proved.

Corollary 6.3.4 (Finite generation of invariant sections)**.** *Let X be an integral normal affine scheme of finite type over a field k with generic point η. Let R be a smooth groupoid with geometrically connected fibres on X over k and let G_η be the generic stabiliser of R. Assume that*

$$\dim X - \operatorname{rel\,dim}_\eta R + \dim G_\eta \leq 2. \tag{6.3.4.1}$$

Then, the R-invariant sections are finitely generated over k.

The corollary is obtained from Theorem 6.3.3 (Finite generation of first integrals), a result about finite generation of the rings of first integrals of a regular infinitesimal groupoid.

A strategy to prove formal smoothness is outlined in Section 6.4 (Local quotient spaces). This involves employing Frobenius' theorem to show existence of a quotient space locally analytically and then lifting formal first integrals to étale first integrals using Artin's approximation theorems ([Art69, Theorem 1.12, page 26]). This is equivalent to lifting formal analytic solutions of differential equations to algebraic solutions and is related to the theory of Artin approximation with constraints (vid. [PR19]).

Strategy of proof

Both applications make use of infinitesimal groupoids in an essential way.

Theorem 6.2.5 (Existence of quotients)

The basic idea is to glue rings of $\mathfrak{R}$-invariant sections over suitable open subsets, where $\mathfrak{R}$ is the regular infinitesimal groupoid induced by R. The assumptions imply that the quotient of X by R is equal to the quotient of X by $\mathfrak{R}$ (Proposition 5.2.1 (First integrals and invariance)), hence the two quotients can be used interchangeably. Example 4.5.11 (Radial foliation) and its sequel Example 6.2.1 (Glueing first integrals) show how to use this method to obtain the projective line as a quotient space of the affine plane by the action of $\mathbb{G}_m$ with weights $(1, 1)$. The first step is to show that constructing quotients can be carried out locally around a point (Lemma 6.2.2 (Localisation of construction of quotients)). This is standard. The proof then splits into two parts.

The first part is the content of Proposition 6.2.4 (Quotient around algebraically smooth points). Suppose that x is an algebraically smooth point and let $x \in$

$U \to Y$ be the morphism induced by the inclusion of the ring of first integrals in a neighbourhood of x. By assumption, $U \to Y$ is a smooth morphism of relative dimension equal to the rank of R. There is no harm in replacing U with X. The proposition shows three statements: R and $X \times_Y X$ induce the same infinitesimal equivalence relation on X, $X \to Y$ may be assumed surjective and the fraction field of Y is the same as the field of rational first integrals. This is the definition of *infinitesimal quotient.* Firstly, using Lemma 5.1.11 (Localisation and first integrals), it is shown that $X \to Y$ may be assumed surjective. The assumptions imply that there is a morphism $R \to X \times_Y X$ and the induced infinitesimal equivalence relations are regular of the same rank. Hence, there cannot be enough space for them to be different (Lemma 5.4.8 (Smooth groupoids of equal rank)). The third statement is not dealt with here.

The second part is the content of Proposition 6.1.7 (Infinitesimal quotients are effective). It shows that the three properties listed above are enough to imply that $R = X \times_Y X$. Given that their induced infinitesimal equivalence relations are equal, Proposition 5.2.1 (First integrals and invariance) would imply the result if the fibres of $X \to Y$ were all geometrically connected. This is not true in general and one must use the fact that j is a closed immersion (vid. Example 6.2.6 (Non-closed equivalence relations)). In the proof, it is firstly shown that a general fibre of $X \to Y$ is geometrically irreducible. This is a consequence of the fact that the field of rational first integrals is algebraically closed in the field of fractions (Proposition 5.3.3 (First integrals are algebraically closed)). Now a density argument, coupled with the necessary assumption that j is a closed immersion, implies that $R = X \times_Y X$.

Combining the reduction to the local case and the two parts implies that a global categorical quotient exists.

Corollary 6.3.4 (Finite generation of invariant sections)

The basic strategy is to use a theorem of Oscar Zariski and Masayoshi Nagata ([Nag56b]). The theorem states that, under suitable hypotheses, when B is a finitely generated subring of a field F and M is a subfield of F of transcendence degree less than or equal to two, the intersection $B \cap M$ is finitely generated. As before, the R-invariant sections can be confused with the first integrals of $\mathfrak{R}$. Therefore F will be the field of fractions of X, B will be the ring corresponding to X and M will be the field of rational first integrals. The proof splits into two parts.

In the first step, it has to be shown that a section of X is a first integral if and only if it is generically a first integral. This is equivalent to stating that $B \cap M$ is the ring of first integrals. This is not true in general (vid. Example 6.3.5 (Finite generation of blowing up)). However, the result is true under regularity assumptions (Lemma 5.1.12 (First integrals at generic point)).

In the second step, a relation between the rank of the groupoid and the transcendence degree of M is established. More precisely, it is shown that

$$\operatorname{tr}\deg M \leq \dim X - \operatorname{rel}\dim R + \dim G, \tag{0.5.0.1}$$

where G is the generic group stabiliser. This is the content of Proposition 5.4.9 (Rank inequality). Whilst the formula is intuitive and easy to show in characteristic zero, the general proof relies crucially on a lesser-known result of Alexandre Grothendieck (Lemma 1.2.3 (Grothendieck–Sharp)).

The concatenation of the two steps, together with the result of Oscar Zariski and Masayoshi Nagata, proves the corollary.

Literature overview

Recently, Michael McQuillan published an article where he develops a theory of foliations as infinitesimal groupoids ([McQ22]). The definitions employed in this dissertation are very similar to the ones in *op. cit.* with one notable exception: in the latter theory, a foliation is required to be a regular infinitesimal groupoid such that j is generically a monomorphism. In here, this is not required in order to be able to restrict an infinitesimal groupoid by an arbitrary morphism.

The applications of this dissertation are an extension of the main results of [Bon21] by the same author. The work in *op. cit.* is based on the theory of foliations and it is only valid in characteristic zero. Furthermore, [Bon21, Theorem 5.0.1, page 39] (i.e. the corresponding result of Theorem 6.2.5 (Existence of quotients)) uses a stronger definition of algebraically smooth point (therein called stable point), where it is further required that the induced morphism from the inclusion of the first integrals has geometrically connected fibres. This is not needed anymore.

The idea of constructing quotient spaces by foliations in algebraic geometry first appeared in [GM89]. Therein, Xavier Gómez-Mont shows that, in the terminology of this thesis, for an equivalence relation on a projective integral variety X over the complex numbers, the generic point of X is algebraically smooth for the induced foliation ([GM89, Theorem 3, page 452]). His argument uses Hilbert schemes.

The Keel–Mori theorem ([KM97]) also implies the main result of [GM89], generalising it to arbitrary schemes. Indeed, a separated algebraic space contains an open dense subset which is a scheme ([Sta23, Proposition 06NH]). Hence, the quotient constructed is a scheme in the generic point. In the terminology of this dissertation, this implies that the generic point of X is algebraically smooth. However, since the methods employed in the proof of the Keel–Mori theorem are different from the ones presented in this dissertation, this is not mentioned in Theorem 6.2.5 (Existence of quotients).

Finally, it is worth mentioning that Jean-Benoît Bost showed Conjecture 0.4.6 (Conjecture F) under an addition technical constraint ([Bos01, Theorem 2.1, page 168]).

0.6 A guide for the reader

The content of this dissertation is implicitly split into two parts.

The first part consists of the first three chapters. The ultimate aim of this part is to develop a theory of formal schemes in enough generality to support a theory of infinitesimal groupoids. This part should be skipped on a first reading.

The second part consists of the last three chapters. This is where the core ideas are developed and the main results are proved. The reader is advised to start with this part, on assuming all the necessary properties of formal schemes.

The reader is assumed to have a solid background in algebraic geometry, however, no prior knowledge of formal schemes is assumed.

This guide gives a short summary of each chapter and supplies the reader with a map to navigate in between the sections.

Chapter 1 (Preliminaries)

This chapter contains several results which will be used to develop the theory of formal schemes. Section 1.1 (Thickenings of rings) introduces the starting point of the theory of topological rings and formal schemes: thickenings of rings. Section 1.2 (Commutative algebra) contains complementary results from commutative algebra. Finally, Section 1.3 (Pro-categories) describes the process of adjoining all filtered limits to a category. This will give an alternative way to study topological modules over a topological ring.

Chapter 2 (Topological algebra)

This chapter consists of the affine theory of formal schemes. The affine local models of formal schemes are admissible rings. These are defined in Section 2.1 (Admissible Rings). Section 2.2 (Pro-modules) defines an alternative category of modules over admissible rings. It is shown that it is an Abelian category. Section 2.3 (Pro-ideals) defines the affine analogue of closed immersions of formal schemes. These generalise closed immersions of schemes. Section 2.4 (Thickenings of admissible rings) defines thickenings of formal schemes and what it means for a thickening to be regular. Thickenings are the correct morphisms to consider when describing the universal property that a tubular neighbourhood should satisfy. Finally, Section 2.5 (Affine infinitesimal neighbourhoods) shows that tubular neighbourhoods indeed satisfy such universal property.

Chapter 3 (Formal schemes)

This chapter globalises the affine theory developed in the previous chapter. Section 3.1 (Formal schemes) defines formal schemes as locally topologically ringed spaces which are locally spectra of admissible rings. Section 3.2 (Quasi-coherent pro-sheaves) defines an Abelian category of sheaves on a formal scheme. Having an Abelian category is important to show that the notion of closed immersion of admissible rings globalises well to the setting of formal schemes. This is done in Section 3.3 (Immersions of formal schemes). Section 3.4 (Thickenings of formal schemes) globalises thickenings and regular thickenings to the setting of formal schemes. Section 3.5 (Infinitesimal neighbourhoods) globalises the tubular neighbourhood construction previously developed in the affine case.

Chapter 4 (Groupoids)

This chapter defines the main object of this thesis: infinitesimal groupoids. The standard definition of groupoid is revised in Section 4.1 (Groupoids of sets) and Section 4.2 (Groupoids of schemes). The former treats the case of sets and the latter treats the case of schemes. Section 4.3 (Infinitesimal groupoids) finally introduces infinitesimal groupoids. These are groupoids whose identity morphism is a thickening. Section 4.4 (Regular infinitesimal groupoids) discusses properties of those infinitesimal groupoids whose identity morphism is a regular thickening. These are the infinitesimal analogue of smooth groupoids. Section 4.5 (Foliations) describes the relation with foliations.

Chapter 5 (First integrals)

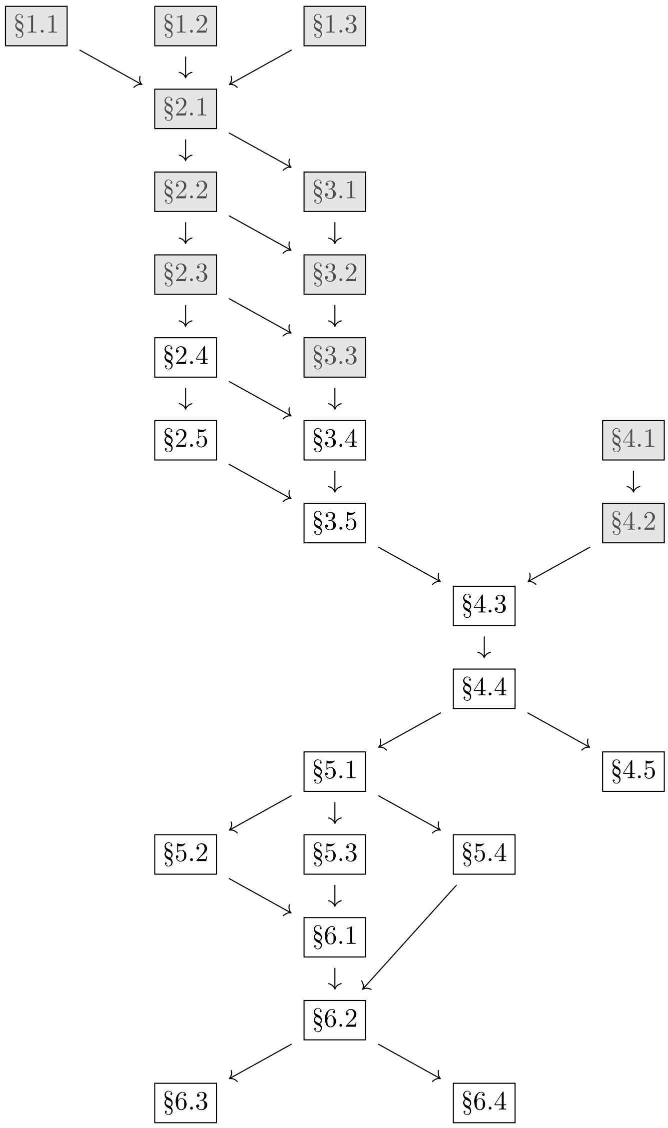
This chapter studies the first integrals of an infinitesimal groupoid. Section 5.1 (First integrals) defines what first integrals are and proves they form a sheaf. Section 5.2 (First integrals and invariance) proves that, under suitable hypotheses, being an invariant section of a groupoid is equivalent to being a first integral of the induced infinitesimal groupoid. Section 5.3 (Properties of first integrals) shows some important algebraic properties of first integrals. Section 5.4 (Rank of infinitesimal groupoids) defines the rank of an infinitesimal groupoid and proves a bound on the transcendence degree of its rational first integrals.

Chapter 6 (Quotient spaces)

This chapter presents some applications of the theory so far developed. Section 6.1 (Properties of quotient spaces) defines what an infinitesimal quotient is and proves it is a categorical quotient. Section 6.2 (Construction of quotient spaces) proves Theorem 6.2.5 (Existence of quotients). Section 6.3 (Finite generation) proves Theorem 6.3.3 (Finite generation of first integrals) and Corollary 6.3.4 (Finite generation of invariant sections). Finally, section 6.4 (Local quotient spaces) describes a strategy to lift formal first integrals to algebraic integrals. This is potentially useful in the study of local quotients.

Interdependence of sections

The following diagram shows the interdependence of sections. The sections enclosed in a gray box mainly summarise material in the literature. The sections enclosed in a white box consist, for the most part, of new material. A summary of the content of each section with appropriate references can be found at the beginning of each section.



0.7 Notation and conventions

Logic

When writing $x := y$ or $y =: x$, it is understood that the variable x is being defined as equal to y.

The statement P implies Q is written as $P \to Q$. Given two statements P and Q, the *forward implication* is $P \to Q$ and the *backward implication* is $P \leftarrow Q$.

Category theory

Let x be an object in a category. Its identity morphism is denoted by $\mathbb{1}_X$, or simply by $\mathbb{1}$, when X is clear from the context.

A *filtered set* Λ is a set together with a pre-order such that any pair of elements has an upper bound. In the literature, this is also known as a directed set. A *filtered system* of objects *along* morphisms over Λ is a collection of objects $\{x_\lambda\}_{\lambda\in\Lambda}$ indexed by Λ and endowed with morphisms $x_{\lambda'} \to x_\lambda$ for all $\lambda' \geq \lambda$. In the literature, this is also known as an inverse system. A *cofiltered system* of objects *along* morphisms is the dual notion: a collection of objects $\{x_\lambda\}_{\lambda\in\Lambda}$ indexed by Λ and endowed with morphisms $x_\lambda \to x_{\lambda'}$ for all $\lambda' \geq \lambda$. In the literature, this is also known as a directed system. Given a filtered system and a cofiltered system, their respective *filtered limit* and *filtered colimit* are denoted by

$$\lim_{\lambda\in\Lambda} x_\lambda \quad \text{and} \quad \operatorname*{colim}_{\lambda\in\Lambda} x_\lambda \tag{0.7.0.1}$$

respectively, when they exist.

Let $x \to z$ and $y \to z$ be morphisms in a category with fibre products. The projections $x \times_z y \to x$ and $x \times_z y \to y$ are denoted by pr_1 and pr_2 respectively, unless otherwise specified. Let w be another object and let $f : w \to x$ and $g : w \to y$ be two morphisms commuting over z. There is a unique induced morphism to the fibre product denoted by

$$f \times g : w \to x \times_z y. \tag{0.7.0.2}$$

Given two morphisms $f : x \to z$ and $g : y \to z$ of objects in a category, where g is a monomorphism admitting the existence of a (unique) morphism $h : x \to y$ such that $f = g \circ h$, the fibre product satisfies $x \times_z y = x$.

Lastly, the word formal is used with two distinct meanings. Typically, either in conjunction with the word *scheme*, to mean a formal scheme, or in conjunction with the word *property*, to mean a property which can be deduced by unravelling the definitions or by performing basic diagram chasing. It shall be clear from the context which meaning is intended.

Commutative algebra

All rings are commutative with unity and all morphisms of rings are unital.

Let A be a ring and let I be an ideal of A. Then $\operatorname{rad}_A I$ and $\operatorname{nilrad} A$ denote the *radical ideal* of I and the *nilradical ideal* of A respectively.

Let $\varphi : A \to B$ be a morphism of rings and let I be an ideal of A. The *ideal extension* of I by the morphism φ is denoted by $\varphi(I) \cdot B$, or by $I \cdot B$ when φ is understood. If J is another ideal of A, then

$$IJ \cdot B = (I \cdot B)\,(J \cdot B)\,. \tag{0.7.0.3}$$

When φ is surjective, the correspondence between ideals of A containing the kernel ideal of f and ideals of B is used without mention.

When φ is the localisation of a multiplicatively closed subset S, the correspondence between prime ideals of A not containing S and prime ideals of B is used without mention.

Given a prime idea $\mathfrak{p}$ of A, $\dim_{\mathfrak{p}} A$ is the dimension of the local ring $A_{\mathfrak{p}}$.

Scheme theory

The anti-equivalence of categories between rings and affine schemes is constantly used without mention.

Let X be a scheme. A *local section* of X over an open subset U, or simply a section, is an element of $\Gamma(U, \mathscr{O}_X)$. This is equivalent to the datum of a morphism $U \to \mathbb{A}^1_{\mathbb{Z}}$. When $U = X$, it is simply called a *global section*.

A morphism of schemes $f : X \to Y$ is given by a morphism of topological spaces and a morphism of sheaves of rings denoted by

$$f^{\#} : \mathscr{O}_Y \to f_* \mathscr{O}_X\,. \tag{0.7.0.4}$$

When $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine schemes, by a slight abuse of notation, the induced morphism of rings $f^{\#} : A \to B$ is also denoted by $f^{\#}$. Ditto for formal schemes.

Rings are typically denoted by letters in the top half of the alphabet, whereas schemes are typically denoted by letters in the bottom half of the alphabet. A formal scheme is denoted by a letter in fraktur font. For instance, X is a usual scheme and $\mathfrak{X}$ is a formal scheme.

Chapter 1

Preliminaries

1.1 Thickenings of rings

Formal schemes are colimit of schemes along thickenings. This section takes the first step of defining thickenings of rings. Some basic properties of thickenings of rings are shown, such as base change and localisation. Contrarily to popular belief, a thickening is not necessarily a morphism of rings whose kernel ideal is nilpotent. This is however true for Noetherian rings and it is explained at the end of the section.

Definition 1.1.1 (Thickenings of rings)**.** Let $\varphi : A \to B$ be a morphism of rings. Then φ is a *thickening* if it is surjective and the induced map of spectra

$$\operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A \tag{1.1.1.1}$$

is a homeomorphism of topological spaces.

Lemma 1.1.2 (Radical and pre-image)**.** *Let $\varphi : A \to B$ be a morphism of rings and let J be an ideal of B. Then*

$$\operatorname{rad}_A \varphi^{-1}(J) = \varphi^{-1}(\operatorname{rad}_B J). \tag{1.1.2.1}$$

Proof. This is a straightforward computation.

$$\begin{aligned} \operatorname{rad}_A \varphi^{-1}(J) &= \{x \in A \mid x^{n+1} \in \varphi^{-1}(J) \text{ for some } n \in \mathbb{N}\} \\ &= \{x \in A \mid \varphi(x^{n+1}) \in J \text{ for some } n \in \mathbb{N}\} \\ &= \{x \in A \mid \varphi(x)^{n+1} \in J \text{ for some } n \in \mathbb{N}\} \\ &= \{x \in A \mid \varphi(x) \in \operatorname{rad}_B J\} \\ &= \varphi^{-1}(\operatorname{rad}_B J). \end{aligned} \tag{1.1.2.2}$$

Therefore the claim holds. □

It is not necessarily true that the kernel ideal of a thickening is nilpotent. However the next lemma shows that every element in the ideal is nilpotent. The former notion can be thought as being globally nilpotent, whereas the latter as being locally nilpotent.

Lemma 1.1.3 (Characterisation of thickenings)**.** *Let $\varphi : A \to B$ be a surjective morphism of rings. Then φ is a thickening if and only if*

$$\ker \varphi \subseteq \operatorname{nilrad} A, \tag{1.1.3.1}$$

where $\operatorname{nilrad} A = \operatorname{rad}_A 0$ *is the nilradical ideal of A.*

Proof. By definition, the surjective morphism φ is a thickening if and only if $\operatorname{Spec} \varphi$ induces a bijection between the prime ideals of B and the prime ideals of A. Now, since φ is surjective, $\operatorname{Spec} \varphi$ induces a bijection between the prime ideals of B and the prime ideals of A containing $\ker \varphi$. It follows that φ is a thickening if and only if all the prime ideals of A contain $\ker \varphi$. Since the intersection of all the prime ideals of A is $\operatorname{nilrad} A$ ([Eis95, Corollary 2.12, page 71]), the result follows. ☐

Lemma 1.1.4 (Base change of thickenings of rings)**.** *The base change of a thickening of rings is a thickening of rings.*

Proof. Let $\varphi : A \to B$ be a thickening of rings and let $\psi : A \to A'$ be morphism of rings. Let

$$\varphi' : A' \to A' \otimes_A B \tag{1.1.4.1}$$

denote the base change. It is shown that φ' is a thickening of rings. Since φ is surjective and the tensor product functor is right exact, φ' is surjective, hence, by Lemma 1.1.3 (Characterisation of thickenings), it suffices to show that

$$\ker \varphi' \subseteq \operatorname{nilrad} A'. \tag{1.1.4.2}$$

By assumption, φ is a thickening, hence

$$\ker \varphi \subseteq \operatorname{nilrad} A. \tag{1.1.4.3}$$

Since φ' is the base change of ϕ and the tensor product functor is right exact,

$$\ker \varphi' = \psi \left(\ker \varphi\right) \cdot A'. \tag{1.1.4.4}$$

Furthermore, by Lemma 1.1.2 (Radical and pre-image),

$$\begin{aligned} \psi^{-1}(\operatorname{nilrad} A') &= \psi^{-1}(\operatorname{rad}_{A'} 0) \\ &= \operatorname{rad}_A \left(\psi^{-1}(0)\right) \\ &\supseteq \operatorname{rad}_A 0 = \operatorname{nilrad} A. \end{aligned} \tag{1.1.4.5}$$

It follows that

$$\begin{aligned} \ker \varphi' &= \psi \left(\ker \varphi\right) \cdot A' \\ &\subseteq \psi \left(\operatorname{nilrad} A\right) \cdot A' \\ &\subseteq \psi \left(\psi^{-1} \left(\operatorname{nilrad} A'\right)\right) \cdot A' \\ &\subseteq \operatorname{nilrad} A'. \end{aligned} \tag{1.1.4.6}$$

This proves the lemma. ☐

Lemma 1.1.5 (Localisation of thickenings of rings)**.** *The localisation of a thickening of rings is a thickening of rings.*

Proof. Let $\varphi : A \to B$ be a thickening of rings and let S be a multiplicatively closed subset of A. By Lemma 1.1.4 (Base change of thickenings of rings), the localised morphism

$$S^{-1}\varphi : S^{-1}A = S^{-1}A \otimes_A A \to S^{-1}A \otimes_A B = S^{-1}B. \tag{1.1.5.1}$$

is a thickening of rings. □

Lemma 1.1.6 (Thickenings from reduced rings)**.** *Let $\varphi : A \to B$ be a thickening of rings. Suppose that A is reduced, then φ is an isomorphism.*

Proof. By Lemma 1.1.3 (Characterisation of thickenings), $\ker\varphi$ is contained in the nilradical of A. Since A is reduced, its nilradical is the zero ideal. Therefore $\ker\varphi$ is the zero ideal and φ is an isomorphism. □

Definition 1.1.7 (Order of thickenings of rings)**.** Let $\varphi : A \to B$ be a thickening of rings. Consider the set

$$S = \{n \in \mathbb{N} \,|\, (\ker\varphi)^{n+1} = 0\}. \tag{1.1.7.1}$$

If S is non-empty, then φ is a *thickening of finite order* and its *order* is $\min S$.

The next lemma shows that, if the kernel ideal of a thickening is finitely generated, it is globally nilpotent.

Lemma 1.1.8 (Powers of finitely generated ideals)**.** *Let A be a ring and let I and K be ideals of A. Suppose that K is a finitely generated ideal and that*

$$K \subseteq \operatorname{rad}_A I. \tag{1.1.8.1}$$

Then, there exists an $n \in \mathbb{N}$ such that

$$K^{n+1} \subseteq I. \tag{1.1.8.2}$$

Proof. Pick a finite number of generators $x_1, ..., x_r$ of K. By assumption, for all $i \le r$, there exists $n_i \in \mathbb{N}$ such that $x_i^{n_i+1} \in I$. Set $n := \sum_{i=1}^r n_i$. Now, any element of K^{n+1} can be written as a sum of monomials of type

$$a\, x_1^{k_1} ... x_r^{k_r}, \tag{1.1.8.3}$$

where $a \in A$ and $k_i \in \mathbb{N}$ such that $\sum_{i=1}^r k_i = n+1$. Therefore, it is enough to show that all such monomials are in I. By construction, there must exist an $i \le r$ such that $k_i > n_i$. Indeed, if it were not the case,

$$n + 1 = \sum_{i=1}^r k_i \le \sum_{i=1}^r n_i = n. \tag{1.1.8.4}$$

Therefore, $ax_1^{k_1}...x_r^{k_r} \in I$ and $K^{n+1} \subseteq I$. □

Lemma 1.1.9 (Thickenings of Noetherian rings)**.** *Let $\varphi : A \to B$ be a thickening of rings. Suppose that A is a Noetherian ring. Then φ is a thickening of finite order.*

Proof. Since A is Noetherian, $\ker \varphi$ is finitely generated. By Lemma 1.1.3 (Characterisation of thickenings), $K \subseteq \operatorname{nilrad} A$. Therefore, Lemma 1.1.8 (Powers of finitely generated ideals) implies that there exists an $n \in \mathbb{N}$ such that $K^{n+1} \subseteq 0$. This shows that φ is a thickening of finite order. $\square$

Example 1.1.10 (Thickenings of a point)**.** Let k be a field and let $n \in \mathbb{N}$. Then, the k-morphism

$$\begin{aligned} \frac{k[x]}{(x^{n+1})} &\to k \\ x &\to 0 \end{aligned} \tag{1.1.10.1}$$

is a thickening of rings of finite order n. Similarly, the k-morphism

$$\begin{aligned} \frac{k[x_1, x_2, x_3, ...]}{(x_1, x_2^2, x_3^3, ...)} &\to k \\ x_i &\to 0 \end{aligned} \tag{1.1.10.2}$$

is a thickening of rings, however it is not of finite order.

1.2 Commutative algebra

This section is a collection of results in commutative algebra which could not be referenced directly.

The first two lemmata show that filtered limits and filtered colimits of local rings remain local. This is used to prove that a formal scheme is a locally ringed space.

Lemma 1.2.1 (Filtered limit of local rings)**.** *Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a filtered system of local rings along morphisms of local rings. Then*

$$A := \lim_{\lambda \in \Lambda} A_\lambda \tag{1.2.1.1}$$

is a local ring and the canonical morphisms $A \to A_\lambda$ are morphisms of local rings for all $\lambda \in \Lambda$.

Proof. Let $\mathfrak{m}_\lambda$ be the maximal ideal of A_λ and define

$$\mathfrak{m} := \lim_{\lambda \in \Lambda} \mathfrak{m}_\lambda. \tag{1.2.1.2}$$

This filtered system exists since the morphisms of $\{A_\lambda\}_{\lambda \in \Lambda}$ are morphisms of local rings. It is shown that this is the unique maximal ideal of A. It is readily checked that this is an ideal. In order to show it is the unique maximal ideal, it suffices to show that any element of $A \setminus \mathfrak{m}$ is a unit. Let $x \in A \setminus \mathfrak{m}$. By definition, this is given by a compatible sequence $\{x_\lambda\}_{\lambda \in \Lambda}$ such that $x_\lambda \in A_\lambda \setminus \mathfrak{m}_\lambda$ for all $\lambda \in \Lambda$. Since A_λ is a local ring, there exists a unique $y_\lambda \in A_\lambda$ such that $x_\lambda y_\lambda = 1$. By uniqueness, the sequence $\{y_\lambda\}_{\lambda \in \Lambda}$ is a well-defined element of A and, by construction, it is the inverse of x. This shows that A is a local ring and that, by definition of $\mathfrak{m}$, the morphism $A \to A_\lambda$ is a morphism of local rings. □

Lemma 1.2.2 (Filtered colimit of local rings). *Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a cofiltered system of local rings along morphisms of local rings. Then*

$$A := \operatorname*{colim}_{\lambda \in \Lambda} A_\lambda \tag{1.2.2.1}$$

is a local ring and the canonical morphisms $A_\lambda \to A$ are morphisms of local rings for all $\lambda \in \Lambda$.

Proof. Let $\mathfrak{m}_\lambda$ be the maximal ideal of A_λ and define

$$\mathfrak{m} := \operatorname*{colim}_{\lambda \in \Lambda} \mathfrak{m}_\lambda. \tag{1.2.2.2}$$

This cofiltered system exists since the morphisms of $\{A_\lambda\}_{\lambda \in \Lambda}$ are morphisms of local rings. It is shown that this is the unique maximal ideal of A. It is readily checked that this is an ideal. In order to show it is the unique maximal ideal, it suffices to show that any element of $A \setminus \mathfrak{m}$ is a unit. Let $x \in A \setminus \mathfrak{m}$. By definition, this is can be represented by $x_\lambda \in A_\lambda$ for some $\lambda \in \Lambda$, where $x_\lambda \in A_\lambda \setminus \mathfrak{m}_\lambda$. This means that there exists a unique $y_\lambda \in A_\lambda$ such that $x_\lambda y_\lambda = 1$. Let y be the image of y_λ in A, then $xy = 1$. This shows that A is a local ring and that, by definition of $\mathfrak{m}$, the morphism $A_\lambda \to A$ is a morphism of local rings. □

The final lemma of this section is a slightly more refined version of a special case of a result of Alexandre Grothendieck, in the Noetherian case, and Rodney Sharp, in the non-Noetherian case.

Lemma 1.2.3 (Grothendieck–Sharp). *Let $M \subseteq F$ be a finitely generated field extension of transcendence degree $d \in \mathbb{N}$. Let $\mathfrak{m}$ be the kernel ideal of the multiplication morphism*

$$\begin{aligned} F \otimes_M F &\to F \\ a \otimes b &\to ab. \end{aligned} \tag{1.2.3.1}$$

Then

$$\dim_{\mathfrak{m}} (F \otimes_M F) = d. \tag{1.2.3.2}$$

The original result implies that the dimension of $F \otimes_M F$ is d. This lemma proves that the dimension of $F \otimes_M F$ *at its diagonal point* is d. This is needed in the proof of Proposition 5.4.9 (Rank inequality). Its proof is essentially the same as the one in [Gro67, Remarque 4.2.1.4, page 350], however, it has been here rewritten for the sake of completeness. It consists of a reduction to the case where $M \subseteq F$ is a purely transcendental field extension.

Proof of Lemma 1.2.3 (Grothendieck–Sharp). Since $M \subseteq F$ is a field extension of transcendence degree d, there exists a factorisation

$$M \subseteq M(t_1, ..., t_d) =: F' \subseteq F, \tag{1.2.3.3}$$

where $M \subseteq F'$ is a purely transcendental field extension of degree d and $F' \subseteq F$ is an algebraic field extension. Let $\mathfrak{m}'$ be the kernel ideal of the multiplication morphism

$$F \otimes_M F' \to F. \tag{1.2.3.4}$$

It is verified that

$$\dim_{\mathfrak{m}'} (F \otimes_M F') = d. \tag{1.2.3.5}$$

Note that F' is a localisation of the ring $M[x_1, ..., x_d]$ via the morphism mapping x_i to t_i. Since localisation commutes with tensor product, $F \otimes_M F'$ is a localisation of the ring $F \otimes_M M[x_1, ..., x_d] = F[x_1, ..., x_d]$. Let $\mathfrak{n}$ be the pre-image of the maximal ideal $\mathfrak{m}$ under the morphism

$$F[x_1, ..., x_d] \to F \otimes_M F'. \tag{1.2.3.6}$$

Since $x_i - t_i$ is mapped to $\mathfrak{m}$, $x_i - t_i \in \mathfrak{n}$ for all $i \leq d$. This implies that $\mathfrak{n}$ is a maximal ideal with residue field F. Since $F \otimes_M F'$ is a localisation of $F[x_1, ..., x_d]$, $\mathfrak{m}'$ and $\mathfrak{n}$ have the same height in $F \otimes_M F'$ and $F[x_1, ..., x_d]$ respectively. But now, since $\mathfrak{m}'$ and $\mathfrak{n}$ are both maximal ideals,

$$\begin{aligned} \dim_{\mathfrak{m}'} (F \otimes_M F') &= \operatorname{height} \mathfrak{m}' \\ &= \operatorname{height} \mathfrak{n} \\ &= \dim_{\mathfrak{n}} (F[t_1, ..., t_d]) = d, \end{aligned} \tag{1.2.3.7}$$

where the last equality follows from [Sta23, Lemma 00OP].

To complete the proof, it has to be shown that

$$\dim_{\mathfrak{m}} (F \otimes_M F) = \dim_{\mathfrak{m}'} (F \otimes_M F'). \tag{1.2.3.8}$$

Consider the commutative diagram

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & F \otimes_M F' \arrow[r] \arrow[d] & F \arrow[d] \\ F \arrow[r] & F \otimes_M F \arrow[r] & F \otimes_{F'} F, \end{tikzcd} \tag{1.2.3.9}$$

where $F \otimes_M F' \to F$ is the morphism in (1.2.3.4) with kernel ideal $\mathfrak{m}'$. Note that every square of the diagram is Cartesian. Since $F' \subseteq F$ is an algebraic field extension, the left-most vertical arrow in the diagram is a flat and integral morphism of rings. The base change of a flat morphism is flat and the base change of an integral morphism is integral (part (1) of [Sta23, Lemma 02JK]). Therefore, both the centre and right-most vertical morphisms are flat and integral morphisms of rings. Note that $F \otimes_{F'} F$ is a ring which is integral over a field.

Therefore, [Sta23, Lemma 00GT] implies that every prime ideal of $F \otimes_{F'} F$ is minimal, hence

$$\dim\,(F \otimes_{F'} F) = 0. \tag{1.2.3.10}$$

Finally, since all field extensions are finitely generated, [Sta23, Lemma 045I] implies that both $F \otimes_M F'$ and $F \otimes_M F$ are Noetherian rings. Applying [Sta23, Lemma 00ON] to the flat morphism of Noetherian local rings

$$(F \otimes_M F')_{\mathfrak{m}'} \to (F \otimes_M F)_{\mathfrak{m}} \tag{1.2.3.11}$$

yields

$$\begin{aligned} \dim_{\mathfrak{m}}\,(F \otimes_M F) &= \dim_{\mathfrak{m}'}\,(F \otimes_M F') + \dim\,(F \otimes_{F'} F) \\ &= \dim_{\mathfrak{m}'}\,(F \otimes_M F') + 0 &(1.2.3.12) \\ &= \dim_{\mathfrak{m}'}\,(F \otimes_M F')\,. &(1.2.3.13) \end{aligned}$$

This shows that $\dim_{\mathfrak{m}}\,(F \otimes_M F) = d$. □

1.3 Pro-categories

This section is a selection of results from [AM86, Appendix, page 147]. Informally, the pro-completion of a category is the process of adjoining all filtered limits. The pro-completion of the category of modules will be used as a replacement for the category of topological modules, since the former is Abelian, whereas the latter is not.

Definition 1.3.1 (Pro-categories)**.** Given a category $\mathcal{C}$, the *pro-completion* of $\mathcal{C}$, or the *pro-category* of $\mathcal{C}$, denoted by $\operatorname{Pro}(\mathcal{C})$, is the category whose objects are formal filtered limits

$$\lim_{\lambda \in \Lambda} x_\lambda, \tag{1.3.1.1}$$

where Λ is a filtered set and $x_\lambda \in \mathcal{C}$, and whose morphisms between $x := \lim\limits_{\lambda \in \Lambda} x_\lambda$ and $y := \lim\limits_{\mu \in M} y_\mu$ in $\operatorname{Pro}(\mathcal{C})$ are given by

$$\operatorname{Hom}_{\operatorname{Pro}(\mathcal{C})}\,(x, y) = \lim_{\mu \in M} \left(\operatorname*{colim}_{\lambda \in \Lambda} \operatorname{Hom}_{\mathcal{C}}\,(x_\lambda, y_\mu) \right). \tag{1.3.1.2}$$

Concretely a morphism is given by specifying for all $\mu \in M$ compatible morphisms

$$x_{\lambda_\mu} \to y_\mu \tag{1.3.1.3}$$

for some $\lambda_\mu \in \Lambda$ depending on μ. This is a *representation* of the morphism.

1.3.2 (Pro-completions of categories)**.** There is a natural fully faithful inclusion of categories

$$\mathcal{C} \to \operatorname{Pro}(\mathcal{C}) \tag{1.3.2.1}$$

which maps an object of $\mathcal{C}$ to the trivial filtered limit over the singleton set. If $\mathcal{C}$ is a category admitting all filtered limits, then there is a functor

$$\mathrm{Pro}\,(\mathcal{C}) \to \mathcal{C} \tag{1.3.2.2}$$

mapping a filtered system to its limit. However the two functors need not induce an equivalence of categories.

Lemma 1.3.3 (Pro-completion of Abelian categories)**.** *The pro-completion of an Abelian category is an Abelian category.*

Proof. This is the content of [AM86, Proposition 4.5]. □

The remaining part of this section studies epimorphisms and monomorphisms in pro-categories. This will be relevant since a closed immersion of formal schemes will be defined as an epimorphism in the pro-category of modules.

Lemma 1.3.4 (Epimorphisms in pro-categories)**.** *Let $\mathcal{C}$ be a category and let*

$$f : x := \lim_{\lambda \in \Lambda} x_\lambda \to \lim_{\mu \in M} y_\mu =: y \tag{1.3.4.1}$$

be a morphism in $\mathrm{Pro}\,(\mathcal{C})$*. Then f is an epimorphism if and only if it can be represented by*

$$\{f_\mu : x_{\lambda\mu} \to y_\mu\}_{\mu \in M}, \tag{1.3.4.2}$$

where f_μ is an epimorphism for all $\mu \in M$.

Proof. ($\rightarrow$). This is [AM86, Proposition 4.6] for epimorphisms.

($\leftarrow$). Suppose there is a commutative diagram

$$x \xrightarrow{f} y \underset{h}{\overset{g}{\rightrightarrows}} z := \lim_{\nu \in N} z_\nu. \tag{1.3.4.3}$$

After selecting $\mu \in M$ large enough, pick representations of g and h with a common domain

$$g_\nu, h_\nu : y_\mu \rightrightarrows z_\nu, \tag{1.3.4.4}$$

for all $\nu \in N$. By assumption, $f_\mu : x_\lambda \to y_\mu$ is an epimorphism for some $\lambda \in \Lambda$. Therefore $g_\nu = h_\nu$ for all $\nu \in N$. This shows that $g = h$. □

1.3.5 (Filtered subsystems)**.** Let $\mathcal{C}$ be a category and let $\Lambda' \subseteq \Lambda$ be filtered sets. Suppose $\{x_\lambda\}_{\lambda \in \Lambda}$ is a filtered system in $\mathcal{C}$. Then, there is a natural morphism in the pro-category of $\mathcal{C}$

$$x := \lim_{\lambda \in \Lambda} x_\lambda \to \lim_{\lambda' \in \Lambda'} x_{\lambda'} =: x', \tag{1.3.5.1}$$

obtained as a limit of identity morphisms

$$\{\mathbb{1}_{x_\lambda} : x_\lambda \to x_{\lambda'}\}_{\lambda' \in \Lambda'}, \tag{1.3.5.2}$$

where λ is chosen to be equal to the given λ'. It follows readily from the construction and Lemma 1.3.4 (Epimorphisms in pro-categories) that this is an epimorphism in $\mathrm{Pro}\,(\mathcal{C})$.

If $\mathcal{C}$ is an Abelian category, the same result above can be proved for monomorphisms.

Lemma 1.3.6 (Monomorphisms in Abelian pro-categories)**.** *Let $\mathcal{C}$ be an Abelian category and let*

$$f : x := \lim_{\lambda\in\Lambda} x_\lambda \to \lim_{\mu\in M} y_\mu =: y \tag{1.3.6.1}$$

be a morphism in $\mathrm{Pro}\,(\mathcal{C})$*. Then f is a monomorphism if and only if it can be represented by*

$$\{f_\mu : x_{\lambda\mu} \to y_\mu\}_{\mu\in M}, \tag{1.3.6.2}$$

where f_μ is a monomorphism for all $\mu \in M$ and the natural morphism of Paragraph 1.3.5 (Filtered subsystems)

$$\lim_{\lambda\in\Lambda} x_\lambda \to \lim_{\mu\in N} x_{\lambda\mu} \tag{1.3.6.3}$$

is an isomorphism.

Proof. ($\to$). By [AM86, Proposition 4.6] for monomorphisms, f can be represented by monomorphisms. Furthermore, by assumption, the composition

$$\lim_{\lambda\in\Lambda} x_\lambda \to \lim_{\mu\in M} x_{\lambda\mu} \to \lim_{\mu\in M} y_\mu. \tag{1.3.6.4}$$

is a monomorphism. Therefore, the morphism in (1.3.6.3), which is the first morphism in (1.3.6.4), is a monomorphism. On the other hand, by Paragraph 1.3.5 (Filtered subsystems), (1.3.6.3) is an epimorphism. Since $\mathrm{Pro}\,(\mathcal{C})$ is an Abelian category (Lemma 1.3.3 (Pro-completion of Abelian categories)), (1.3.6.3) is an isomorphism.

($\leftarrow$). Let

$$x' := \lim_{\mu\in M} x_{\lambda\mu} \tag{1.3.6.5}$$

and suppose there is a commutative diagram

$$\lim_{\nu\in N} z_\nu =: z \overset{g}{\underset{h}{\rightrightarrows}} x \xrightarrow{f} y. \tag{1.3.6.6}$$

Pick representations of g and h

$$g_\mu, h_\mu : z_\nu \rightrightarrows x_{\lambda\mu}. \tag{1.3.6.7}$$

for all λ_μ. Then, since f_μ is a monomorphism, $g_\mu = h_\mu$. This shows that $g : z \to x'$ is equal to $h : z \to x'$. By assumption x is isomorphic to x', hence $g = h$. ☐

Chapter 2

Topological algebra

2.1 Admissible Rings

The aim of this section is to develop the theory of admissible rings, the affine local model of formal schemes. These are rings which can be obtained as filtered limits of rings along thickenings. Surprisingly, they can be described purely topologically, and often, it is easier and more intuitive to work in this setting. This section starts with the definition of topological rings and the property of being Cauchy complete. It then covers, in decreasing order of generality, rings with linear topologies, admissible rings and adic rings. The standard constructions of tensor product, localisation and quotient by an ideal are also described. Finally, the formal spectrum Spf of a ring is defined. The material in this section follows closely [McQ02, §1, page 3].

Definition 2.1.1 (Topological rings)**.** A *topological ring* A is a ring endowed with a topology such that the operations

$$+ : A \times A \to A \text{ and} \tag{2.1.1.1}$$

$$\cdot : A \times A \to A \tag{2.1.1.2}$$

are continuous, where $A \times A$ has the product topology. A morphism of topological rings $A \to B$ is a continuous morphism of rings. The set of topological rings with their morphisms forms a category. The category of rings embeds fully faithfully in the category of topological rings by endowing a ring with the discrete topology.

Lemma 2.1.2 (Open ideal is closed)**.** *Let A be a topological ring and let $I \subseteq A$ be an open ideal. Then I is closed.*

Proof. It is shown that $A \setminus I$ is open. Let $a \in A \setminus I$ be an arbitrary element. It suffices to show that there exists an open neighbourhood of a contained in

$A \setminus I$. Note that

$$\begin{aligned} +a : A &\to A \\ x &\to a + x \end{aligned} \tag{2.1.2.1}$$

is a homeomorphism with inverse $-a$. Therefore, since I is open, $(a + I)$ is an open subset of A containing a. Since $a \notin I$, $(a + I)$ is disjoint from I. $\square$

2.1.3 (Separation and completeness)**.** Any topological ring A is a uniform space. Therefore, the notions of Cauchy sequences, separatedness and completeness may be defined. Let Λ be a filtered set and let $\{(a_\lambda)\}_{\lambda \in \Lambda}$ be a net. It is said to be a *Cauchy net* if, for any open neighbourhood U of $0 \in A$, there exists a $\lambda_0 \in \Lambda$, such that for all λ and λ' greater than λ_0,

$$a_\lambda - a_{\lambda'} \in U. \tag{2.1.3.1}$$

A net converges to $a \in A$ if for any open neighbourhood U of a, there exists a $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $a_\lambda \in U$. The ring A is *separated* if every Cauchy net admits at most one limit. The ring A is *complete* if every Cauchy net admits at least one limit. The ring A is separated and complete if every Cauchy net admits a unique limit.

2.1.4 (Linear topologies)**.** Let A be a topological ring, then the topology near $0 \in A$ determines the topology of A. In more details, let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a fundamental system of open neighbourhoods of 0. This means that a subset $V \subseteq A$ containing 0 is a neighbourhood of 0 if and only if there exists a $\lambda \in \Lambda$ such that $0 \in U_\lambda \subseteq V$. Then, an arbitrary subset $V \subseteq A$ is open if and only if, for all $a \in V$, there exists a $\lambda \in \Lambda$ such that $a \in (U_\lambda + a) \subseteq V$. This follows from the fact that the morphism $+a : A \to A$ is a homeomorphism. In particular, if $I \subseteq A$ is an ideal of A, it is open if and only if there exists a $\lambda \in \Lambda$ such that $0 \in U_\lambda \subseteq I$. Indeed, if $U_\lambda \subseteq I$, then $(U_\lambda + a) \subseteq (I + a) = I$ for all $a \in I$. If a topological ring admits a fundamental system of open neighbourhoods of 0 consisting of ideals $\{I_\lambda\}_{\lambda \in \Lambda}$, then its topology is *linear*.

The next lemma gives a criterion for a set of ideals to define a linear topology on a ring.

Lemma 2.1.5 (Generated linear topologies)**.** *Let A be a ring and let $\mathcal{C}$ be a collection of ideals of A. Consider the following properties.*

(a) If $I \in \mathcal{C}$ and $I \subseteq J$, then $J \in \mathcal{C}$.

(b) If I and J are in $\mathcal{C}$, then $I \cap J \in \mathcal{C}$.

Then the following statements hold.

1. *Suppose that A is endowed with a linear topology. Then the collection of open ideals satisfies (a) and (b).*
2. *Suppose the $\mathcal{C}$ is a collection of ideals of A satisfying (a) and (b). Then there exists a unique linear topology on A such that $\mathcal{C}$ is the set of open ideals of A.*

Proof. This is the result of [Mur05, Proposition 4, page 2]. Note that the author in *op. cit* requires a further property to be satisfied by a collection of ideals for them to define a linear topology on A. However, this is automatically satisfied when A is commutative. $\square$

Rings endowed with linear topologies are important since they can always be completed.

2.1.6 (Separated completion of topological rings)**.** If A is a topological ring endowed with a linear topology, there exists a topological ring $\hat{A}$ endowed with a morphism $A \to \hat{A}$ with the following universal property: $\hat{A}$ is separated and complete and, for any separated and complete ring B endowed with a morphism $A \to B$, there exists a unique morphism $\hat{A} \to B$ such that the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \hat{A} \arrow[d] \\ & B \end{tikzcd} \tag{2.1.6.1}$$

is commutative. The topological ring $\hat{A}$ is the *separated completion* of A. It is constructed as follows. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a fundamental system of open neighbourhoods of 0 consisting of ideals, where Λ is a filtered set. Define

$$\hat{A} := \lim_{\lambda\in\Lambda} A/I_\lambda, \tag{2.1.6.2}$$

where the limit is in the category of topological rings and A/I_λ has the quotient topology for all $\lambda \in \Lambda$. Since I_λ is open and closed (Lemma 2.1.2 (Open ideal is closed)), that is the discrete topology. The kernel of the completion morphism is

$$\bigcap_{\lambda\in\Lambda} I_\lambda. \tag{2.1.6.3}$$

The topological ring A is separated (resp. complete) if and only if $A \to \hat{A}$ is injective (resp. surjective).

Next, a series of four examples is presented in order to motivate the definition of admissible rings. The first example is the well-known ring of formal power series in two variables. This can be thought as an analytic neighbourhood of the origin in the plane.

Example 2.1.7 (Formal power series)**.** Let k be a field and consider the ring $k[x, y]$ endowed with the linear topology induced by the collection of ideals $\{(x, y)^{n+1}\}_{n\in\mathbb{N}}$ (vid. Lemma 2.1.5 (Generated linear topologies)). This topology is called the (x, y)-adic topology. Then the separated completion

$$k[\![x, y]\!] := \lim_{n\in\mathbb{N}} \left(\frac{k[x, y]}{(x, y)^{n+1}} \right) \tag{2.1.7.1}$$

is the ring of formal power series in two variables. An element $f \in k[\![x, y]\!]$ is a power series

$$f(x, y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij}\, x^i y^j, \tag{2.1.7.2}$$

where $a_{ij} \in k$.

The next example describes adic rings, the simplest instance of admissible rings. Theses will be defined later (vid. Definition 2.1.18 (Adic rings)).

Example 2.1.8 (Filtered limit of discrete rings I)**.** The *I-adic completion* is a key example of a topological ring. Let A be a ring and $I \subseteq A$ an ideal. For each $n \in \mathbb{N}$, endow A/I^{n+1} with the discrete topology. Let

$$\hat{A} := \lim_{n \in \mathbb{N}} A/I^{n+1}, \tag{2.1.8.1}$$

be the filtered limit, where $\hat{A}$ has the limit topology. Equivalently, this is obtained by endowing A with the linear topology induced by the set of ideals $\{I^{n+1}\}_{n \in \mathbb{N}}$ and then completing. Then $\hat{A}$ is a separated and complete topological ring. Let $\pi_n : \hat{A} \to \hat{A}/I^{n+1}$ denote the canonical projection. Let V be a subset of $\hat{A}$ containing 0. Then, by definition of limit topology, V is a neighbourhood of 0 if and only if there exists an $n \in \mathbb{N}$ such that $I^{n+1} \subseteq V$. This implies that I is satisfies following property: for all $n \in \mathbb{N}$, I^{n+1} is an open ideal of $\hat{A}$ and, for all open neighbourhoods V of 0, there exists an $n \in \mathbb{N}$ such that $I^{n+1} \subseteq V$. In particular, $\{I^{n+1}\}_{n \in \mathbb{N}}$ is a fundamental system of open neighbourhoods of 0 for the topology of $\hat{A}$. Furthermore, since a prime ideal contains I if and only if it contains I^{n+1}, the set of open prime ideals of $\hat{A}$ is the same as the set of prime ideals of $\hat{A}/I^{n+1} = A/I^{n+1}$, for any $n \in \mathbb{N}$.

The next example describes those topological rings which are filtered limits of rings along thickenings of finite order. This is the standard definition of admissible rings in the literature.

Example 2.1.9 (Filtered limit of discrete rings II)**.** Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a filtered system of discrete rings, where for all $\lambda' \geq \lambda \in \Lambda$, the morphism $A_{\lambda'} \to A_\lambda$ is a thickening of finite order. Let

$$\hat{A} := \lim_{\lambda \in \Lambda} A_\lambda \tag{2.1.9.1}$$

be the filtered limit. Then $\hat{A}$ is a separated and complete topological ring. Let $\pi_\lambda : \hat{A} \to A_\lambda$ denote the canonical projection and let I_λ be its kernel ideal. Similarly to Example 2.1.8 (Filtered limit of discrete rings I), $\{I_\lambda\}_{\lambda \in \Lambda}$ is a fundamental system of open neighbourhoods for the topology of $\hat{A}$. However, unlike the previous example, it is not necessarily true that I_λ^{n+1} is an open ideal, for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Fix $\lambda \in \Lambda$ and let V be a subset of A containing 0. By definition, V is a neighbourhood of 0 if and only if there exists $\lambda' \in \Lambda$ such that $I_{\lambda'} \subseteq V$. Pick $\lambda' \geq \lambda$ sufficiently large and consider the surjective morphisms of rings

$$\hat{A} \xrightarrow{\pi_{\lambda'}} A_{\lambda'} \to A_\lambda. \tag{2.1.9.2}$$

By construction, $I_{\lambda'} \subseteq I_\lambda$. Furthermore, since $A_{\lambda'} \to A_\lambda$ is a thickening of finite order with kernel ideal $I_\lambda/I_{\lambda'}$, there exists an $n \in \mathbb{N}$ such that $(I_\lambda/I_{\lambda'})^{n+1} = 0$. This means precisely that

$$I_\lambda^{n+1} \subseteq I_{\lambda'} \subseteq V. \tag{2.1.9.3}$$

This shows that a subset $V \subseteq \hat{A}$ is a neighbourhood of 0 if and only if there exists an $n \in \mathbb{N}$ such that $I_\lambda^{n+1} \subseteq V$. This property may be described by saying that I_λ *globally tends to zero.*

The final example describes admissible rings in full generality.

Example 2.1.10 (Filtered limit of discrete rings III)**.** Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an filtered system of discrete rings, where for all $\lambda' \geq \lambda \in \Lambda$, the morphism $A_{\lambda'} \to A_\lambda$ is a thickening. Let

$$\hat{A} := \lim_{\lambda\in\Lambda} A_\lambda \tag{2.1.10.1}$$

be the filtered limit. Then $\hat{A}$ is a separated and complete topological ring. Let $\pi_\lambda : \hat{A} \to A_\lambda$ denote the canonical projection and let I_λ be its kernel ideal. Similarly to Example 2.1.9 (Filtered limit of discrete rings II), $\{I_\lambda\}_{\lambda\in\Lambda}$ is a fundamental system of open neighbourhoods for the topology of $\hat{A}$. However, unlike the previous example, it is not necessarily true that I_λ globally tends to zero for all $\lambda \in \Lambda$. On the other hand, something slightly weaker still holds. Fix $\lambda \in \Lambda$ and let V be a subset of A containing 0. Then V is a neighbourhood of 0 if and only if there exists $\lambda' \in \Lambda$ such that $I_{\lambda'} \subseteq V$. Pick $\lambda' \geq \lambda$ sufficiently large and consider the surjective morphisms of rings

$$\hat{A} \xrightarrow{\pi_{\lambda'}} A_{\lambda'} \to A_\lambda. \tag{2.1.10.2}$$

By construction, $I_{\lambda'} \subseteq I_\lambda$. Furthermore, since $A_{\lambda'} \to A_\lambda$ is a thickening with kernel ideal $I_\lambda/I_{\lambda'}$, by Lemma 1.1.3 (Characterisation of thickenings),

$$I_\lambda/I_{\lambda'} \subseteq \operatorname{nilrad} A_{\lambda'}. \tag{2.1.10.3}$$

This means precisely that, for all $x \in I_\lambda \subseteq \hat{A}$, there exists an $n \in \mathbb{N}$ such that

$$x^{n+1} \in I_{\lambda'} \subseteq V. \tag{2.1.10.4}$$

This property may be described by saying that I_λ *locally tends to zero.*

Finally, here is the definition of admissible rings used in this dissertation, which first appeared in [McQ02, Definition 1.2, page 3]. The reason for adopting this definition is to deal with non-Noetherian rings. In [Sta23], these are referred as *weakly admissible rings* (vid. [Sta23, Definition 0AMV]).

Definition 2.1.11 (Admissible rings)**.** Let A be a topological ring endowed with a linear topology. An ideal $I \subseteq A$ *locally tends to zero* if, for all $x \in I$ and for all open neighbourhood U of $0 \in A$, there exists an $n \in \mathbb{N}$ such that $x^{n+1} \in U$. An ideal $I \subseteq A$ is an *ideal of definition* if I is open and it locally tends to 0. The topological ring A is *pre-admissible* if the set of ideals of definition forms a fundamental system of neighbourhoods of 0. It is *admissible* if it is pre-admissible, complete and separated. Given a pre-admissible ring, its separated completion is admissible.

Pre-admissible rings can be characterised as those rings endowed with a linear topology whose topological nilradical is open. Here is the definition of topological nilradical of a ring.

Definition 2.1.12 (Topological radicals)**.** Let A be a topological ring endowed with a linear topology. The *topological radical* of an ideal I of A, denoted by $\operatorname{top\,rad}_A I$ consists of those elements $x \in A$ such that, for all open subsets U containing I, there exists an $n \in \mathbb{N}$ such that $x^{n+1} \in U$. Since the topology is linear, it follows that this is an ideal of A. If I is an open ideal, it is clear that $\operatorname{top\,rad}_A I = \operatorname{rad}_A I$. In particular, if A is a discrete ring, the topological radical and the classical radical of any ideal coincide. The *topological nilradical* of a topological ring A, denoted by $\operatorname{top\,nilrad} A$, is the topological radical of 0 in A. By definition, this is the largest ideal of A which locally tends to zero.

Lemma 2.1.13 (Topological nilradicals and morphisms)**.** *Let $\varphi : A \to B$ be a morphism of topological rings endowed with linear topologies and let N_A and N_B be the topological nilradicals of A and B respectively. Then $N_A \subseteq \varphi^{-1}(N_B)$.*

Proof. Let $x \in N_A$ and let U be an open neighbourhood of $0 \in B$. Since φ is continuous, $\varphi^{-1}(U)$ is an open neighbourhood of $0 \in A$. By definition, there exists an $n \in \mathbb{N}$ such that $x^{n+1} \in \varphi^{-1}(U)$. But then

$$\varphi(x)^{n+1} = \varphi\left(x^{n+1}\right) \in \varphi\left(\varphi^{-1}\left(U\right)\right) \subseteq U. \qquad (2.1.13.1)$$

This shows that $\varphi(x) \in N_B$. □

Lemma 2.1.14 (Characterisations of pre-admissible rings)**.** *Let A be a topological ring endowed with a linear topology and let N be its topological nilradical. Then the following are equivalent:*

1. *A is pre-admissible.*
2. *N is open.*
3. *N is an ideal of definition.*
4. *There exists an ideal of definition.*

Proof. (1) → (2). Let I be an ideal of definition. By definition of locally tending to zero, I is contained in N. Since the topology is linear and N is an ideal, N is open.

(2) → (3). By definition of topological nilradical, N locally tends to zero. By assumption, it is open, hence it is an ideal of definition.

(3) → (4). Obvious.

(4) → (1). Suppose there exists an ideal of definition I. It has to be checked that the set of ideals of definition forms a fundamental system of neighbourhoods of 0. To this end, let V be a neighbourhood of 0. Since the topology of A is linear, there exists an open ideal J such that $J \subseteq V$. But then $I \cap J$ is a subset of I, therefore it locally tends to zero. Since it is the intersection of two open ideals, it is an open ideal. Therefore it is an ideal of definition and $I \cap J \subseteq V$. □

The next lemma shows that the topological nilradical of a pre-admissible ring is its largest ideal of definition.

Lemma 2.1.15 (Largest ideal of definition)**.** *Let A be a pre-admissible ring and let N be its topological nilradical. Then, for any ideal of definition I of A,*

$$I \subseteq \operatorname{rad}_A I = N. \tag{2.1.15.1}$$

In particular, N is the largest ideal of definition of A.

Proof. It is firstly shown that $N \subseteq \operatorname{rad}_A I$. Suppose that $x \in N$, then for any open neighbourhood U of 0, there exists an $n \in \mathbb{N}$ such that $x^{n+1} \in U$. Pick $U = I$, then $x^{n+1} \in I$ so that $x \in \operatorname{rad}_A I$.

Conversely, it is shown that $\operatorname{rad}_A I \subseteq N$. Suppose that $x^{n+1} \in I$ for some $n \in \mathbb{N}$. Since I locally tends to zero, for any open neighbourhood U of 0, there exists an $m \in \mathbb{N}$ such that $x^{(n+1)(m+1)} \in U$. This shows that $x \in N$.

Therefore N is an ideal of definition of A ((1) $\to$ (3) implication of Lemma 2.1.14 (Characterisations of pre-admissible rings)) which contains every ideal of definition. □

Next, as promised at the beginning of this section, it is shown that admissible rings are precisely filtered limits of rings along thickenings.

Lemma 2.1.16 (Characterisation of admissible rings)**.** *A topological ring A is admissible if and only if it is a filtered limit of discrete rings along thickenings.*

Proof. *($\to$)* Suppose that A is admissible and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of ideals of definition. By definition, they form a fundamental system of open neighbourhoods of 0. Let

$$A \to \hat{A} = \lim_{\lambda\in\Lambda} A/I_\lambda \tag{2.1.16.1}$$

be the separated completion of A. Since A is already separated and complete, this is an isomorphism and it suffices to show that $\hat{A}$ is a filtered limit of discrete rings along thickenings. Note that, since I_λ is open and closed (Lemma 2.1.2 (Open ideal is closed)), A/I_λ is a discrete ring. Furthermore, if $\lambda' \geq \lambda$, the induced morphism

$$\pi := \pi_{\lambda',\lambda} : A/I_{\lambda'} \to A/I_\lambda \tag{2.1.16.2}$$

is a thickening. Indeed, since I_λ locally tends to zero and $I_{\lambda'}$ is an open neighbourhood of 0, for all $x \in I_\lambda$ there exists an $n \in \mathbb{N}$ such that $x^{n+1} \in I_{\lambda'}$. This means precisely that $I_\lambda \subseteq \operatorname{rad}_A I_{\lambda'}$ which implies that $\ker \pi \subseteq \operatorname{nilrad} A/I_{\lambda'}$. By Lemma 1.1.3 (Characterisation of thickenings), π is a thickening. Therefore $\hat{A}$ is a filtered limit of discrete rings along thickenings.

($\leftarrow$) This is precisely the content of Example 2.1.10 (Filtered limit of discrete rings III). □

In fact, it is also true that morphisms of admissible rings are precisely morphisms of filtered limits. This gives an equivalence of categories.

Lemma 2.1.17 (Admissible rings and pro-category of rings)**.** *The category of admissible rings is equivalent to the full subcategory of the pro-category of rings consisting of filtered limits along thickenings.*

Proof. Let $\mathcal{D}$ be the category of admissible rings and let $\mathcal{E}$ be the full subcategory of the pro-category of rings consisting of filtered limits along thickenings. Given an admissible ring A, the forward implication of Lemma 2.1.16 (Characterisation of admissible rings) guarantees that A can be written as a limit of discrete rings $\{A_\lambda\}_{\lambda\in\Lambda}$ along thickenings. Define a functor

$$\begin{aligned} F : \mathcal{D} &\to \mathcal{E} \\ A &\to \lim_{\lambda\in\Lambda} A_\lambda. \end{aligned} \tag{2.1.17.1}$$

If $f : A \to B := \lim_{\mu\in M} B_\mu$ is a morphism of admissible rings, by continuity of f, for any $\mu \in M$, there exists a $\lambda \in \Lambda$ such that f descends to $f_\mu : A_\lambda \to B_\mu$. This gives an element

$$F(f) \in \lim_{\mu\in M} \left(\operatorname*{colim}_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{C}}\,(A_\lambda, B_\mu) \right). \tag{2.1.17.2}$$

Conversely, using the backward implication of Lemma 2.1.16 (Characterisation of admissible rings), define a functor

$$\begin{aligned} G : \mathcal{E} &\to \mathcal{D} \\ \lim_{\lambda\in\Lambda} A_\lambda &\to \lim_{\lambda\in\Lambda} A_\lambda, \end{aligned} \tag{2.1.17.3}$$

where the limit on the right hand side is in the category of topological rings. If g is a morphism in $\mathcal{E}$ represented by $\{g_\mu : A_\lambda \to B_\mu\}_{\mu\in M}$, the universal property of limits gives a morphism

$$G(g) : \lim_{\lambda\in\Lambda} A_\lambda \to \lim_{\mu\in M} B_\mu \tag{2.1.17.4}$$

of admissible rings. It is readily verified that F and G are inverse to each other on objects as well as on morphisms. This gives an equivalence of categories. □

Next, adic rings are defined. These are filtered limits of rings along special types of thickenings: those induced by the powers of a fixed finitely generated ideal. This definition of adic ring is stronger than the one in the literature. The reason for adopting it is that being adic becomes a local property of a ring. In the terminology of [Sta23], this is called an adic* ring, whereas an adic ring is an adic* ring whose adic ideal of definition is not necessarily finitely generated.

Definition 2.1.18 (Adic rings)**.** Let A be a topological ring endowed with a linear topology. An ideal $I \subseteq A$ is an *adic ideal of definition* if I is finitely generated and $\{I^{n+1}\}_{n\in\mathbb{N}}$ is a fundamental system of open neighbourhoods of 0. Note that an adic ideal of definition is an ideal of definition. The topological ring A is *pre-adic* if there exists an adic ideal of definition. It is *adic* if it is pre-adic, complete and separated. Given a pre-adic ring, its separated completion is adic.

Next, taut morphisms are introduced. When thinking of admissible rings as formal schemes, these are morphisms satisfying the following property: the preimage of a usual scheme is a usual scheme. A taut morphism between adic rings is also known as an adic morphism.

Definition 2.1.19 (Taut morphisms)**.** Let $\varphi : A \to B$ be a morphism of preadmissible rings and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of ideals of definition of A. Then φ is a *taut morphism* if the closure of the extension ideals

$$\{\overline{\varphi(I_\lambda)\cdot B}\}_{\lambda\in\Lambda} \tag{2.1.19.1}$$

is a collection of ideals of definition of B forming a a fundamental system of neighbourhoods of 0.

Adic rings arise as completions of Noetherian rings with respect to an ideal.

2.1.20 (Completion of Noetherian rings)**.** Let A be a Noetherian ring and let I be an ideal of A. Let $\hat{A}$ denote the I-adic completion of A. Then $\hat{A}$ is an adic Noetherian ring and $I \cdot \hat{A}$ is an adic ideal of definition ([Sta23, Lemma 05GH]). Furthermore, the completion morphism $A \to \hat{A}$ is flat (part (1) of [Sta23, Lemma 00MB]) and taut ([Sta23, Lemma 0GX2]).

The tensor product of admissible rings is not, in general, complete. Completing it gives rise to a tensor product functor in the category of admissible rings.

2.1.21 (Tensor product of admissible rings)**.** Let A, B and C be admissible rings and let $\{I_\lambda\}_{\lambda\in\Lambda}$ and $\{J_\mu\}_{\mu\in M}$ be the sets of ideals of definition of A and B respectively. Let $f : C \to A$ and $g : C \to B$ be morphisms of admissible rings. The *completed tensor product* is defined as

$$A\hat{\otimes}_C B := \lim_{(\lambda,\mu)\in\Lambda\times M} \left(A/I_\lambda \otimes_C B/J_\mu\right), \tag{2.1.21.1}$$

where the topological rings in the filtered system are endowed with the discrete topology. Applying Lemma 1.1.4 (Base change of thickenings of rings) repeatedly yields that $A\hat{\otimes}_C B$ is a filtered limit of discrete rings along thickenings. By Lemma 2.1.16 (Characterisation of admissible rings), $A\hat{\otimes}_C B$ is an admissible ring. It is readily verified that it is the colimit of the solid diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, dashed, "g'"] \\ B \arrow[r, dashed, "f'"] & A\hat{\otimes}_C B \end{tikzcd} \tag{2.1.21.2}$$

in the category of admissible rings, where f' and g' are defined via the universal property of filtered limits. Note that, if A and B are adic rings, the completed tensor product $A\hat{\otimes}_C B$ is an adic ring (part (4) of [Sta23, Lemma 0GB4]).

The localisation of an admissible ring in a multiplicatively closed subset is not, in general, complete. The completion of the localisation is known as *complete localisation* and satisfies the usual universal property in the category of admissible rings.

2.1.22 (Complete localisation of admissible rings)**.** Let A be an admissible ring and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of ideals of definition of A. Let $S \subseteq A$ be a multiplicatively closed subset. The *completed localisation* is defined as

$$\{S^{-1}\}A := \lim_{\lambda\in\Lambda} \left(\frac{S^{-1}A}{S^{-1}I_\lambda}\right), \tag{2.1.22.1}$$

where the topological rings in the filtered system are endowed with the discrete topology. Applying Lemma 1.1.5 (Localisation of thickenings of rings) yields that $\{S^{-1}\}A$ is a filtered limit of discrete rings along thickenings. By Lemma 2.1.16 (Characterisation of admissible rings), $\{S^{-1}\}A$ is an admissible ring. It is readily verified that it satisfies the following universal property: if $A \to B$ is a morphism of admissible rings which maps every element of S to a unit in B, then there exists a unique morphism of admissible rings $\{S^{-1}\}A \to B$ factorising

$$A \to \{S^{-1}\}A \to B. \tag{2.1.22.2}$$

When S is the complement of an open prime ideal $\mathfrak{p}$, the complete localisation is denoted by $A_{\{\mathfrak{p}\}}$ and is the filtered limit of local rings along morphisms of local rings. This is a local ring (Lemma 1.2.1 (Filtered limit of local rings)). When S consists of $\{1, f, f^2, ...\}$ for some $f \in A$, the complete localisation is denoted by $A_{\{f\}}$.

2.1.23 (Complete localisation of adic Noetherian rings)**.** If A is an adic Noetherian ring with adic ideal of definition I, then $\{S^{-1}\}A$ is the $S^{-1}I$-adic completion of $S^{-1}A$. The latter is a Noetherian ring ([Sta23, Lemma 00FN]). Therefore, Paragraph 2.1.20 (Completion of Noetherian rings) implies that $\{S^{-1}\}A$ is an adic Noetherian ring flat and taut over $S^{-1}A$, hence flat and taut over A. Furthermore, it is shown that, if I is a prime ideal of A and $I \cap S = \varnothing$, then $S^{-1}A \to \{S^{-1}\}A$ is injective. To see this, it has to be shown that the infinite intersection

$$H := \bigcap_{n\in\mathbb{N}} \left(S^{-1}I\right)^{n+1} = 0 \subseteq S^{-1}A. \tag{2.1.23.1}$$

Since, for all $n \in \mathbb{N}$, $H \subseteq \left(S^{-1}I\right)^{n+1}$, the pre-image of H in A is contained in the pre-image of $\left(S^{-1}I\right)^{n+1}$ in A. By the correspondence of prime ideals under localisation, the pre-image of $S^{-1}I$ in A is I. It follows that, for all $n \in \mathbb{N}$, the pre-image of $\left(S^{-1}I\right)^{n+1}$ is I^{n+1}. But now, the pre-image of H is contained in

$$\bigcap_{n\in\mathbb{N}} I^{n+1} = 0, \tag{2.1.23.2}$$

since I is an adic ideal of definition of A. Using again the correspondence of ideals under localisation, it follows that $H = 0$.

The quotient of an admissible ring is not, in general, complete. Therefore, it is necessary to identify those ideals which give rise to complete quotient rings. For adic Noetherian local rings, there is a satisfactory answer.

2.1.24 (Quotients of admissible rings)**.** Let A be an admissible ring and let K be an ideal. Endow A/K with the quotient topology induced by the morphism

$A \to A/K$, then A/K is a pre-admissible ring. Indeed, by definition of quotient topology, the quotient morphism is taut and the result follows from [Sta23, Lemma 0GX7]. Furthermore, A/K is separated (resp. discrete) if and only if K is a closed (resp. open and closed) ideal. Indeed, the point set $\{0\} \subseteq A/K$ is closed (resp. open and closed) if and only if its pre-image K in A is closed (resp. open and closed). But then $\{0\}$ is closed (resp. open and closed) if and only if A/K is separated (resp. discrete).

2.1.25 (Quotients of adic Noetherian local rings)**.** Let A be an adic Noetherian local ring and let I be an adic ideal of definition of A. Let K be an ideal. Then, the topological quotient A/K is already separated and complete. To see this, note that, by definition of quotient topology, A/K is a pre-adic Noetherian local ring and $\bar{I} := (I, K)/K$ is an adic ideal of definition. Now, $\bar{I}$ is contained in the maximal ideal of A/K, hence, by Krull's intersection theorem ([Sta23, Lemma 00IP]),

$$\bigcap_{n \in \mathbb{N}} \bar{I} = 0. \tag{2.1.25.1}$$

This shows that A/K is separated, so K is a closed ideal of A. Since K is a closed subspace of a complete topological space, it is complete. Now, completion with respect to the I-adic topology is an exact functor in Noetherian rings ([Sta23, Lemma 00MA]). Hence, applying it to the exact sequence

$$0 \to K \to A \to A/K \to 0 \tag{2.1.25.2}$$

and noting that K and A are already complete with respect to the I-adic topology yields that A/K is complete. In particular, if $C \to A$ and $C \to B$ are morphisms of topological rings, where A is an adic Noetherian local ring and $C \to B$ is a surjective morphism of discrete rings, the tensor products

$$A \otimes_C B = A \hat{\otimes}_C B \tag{2.1.25.3}$$

are equal.

The formal spectrum of an admissible ring is the corresponding notion of spectrum of a ring. It only keeps track of open prime ideals.

2.1.26 (Formal spectrum of admissible ring)**.** Let A be an admissible ring and let $\{I_\lambda\}_{\lambda \in \Lambda}$ be the set of ideals of definition of A. The *formal spectrum* of A, denoted by $\operatorname{Spf} A$, is the topological space

$$\operatorname{Spf} A = \operatorname*{colim}_{\lambda \in \Lambda} \left(\operatorname{Spec} A/I_\lambda\right). \tag{2.1.26.1}$$

Since A is a filtered limit of discrete rings along thickenings (Lemma 2.1.16 (Characterisation of admissible rings)), for all $\lambda' \geq \lambda$,

$$\operatorname{Spec} A/I_{\lambda'} \to \operatorname{Spec} A/I_\lambda \tag{2.1.26.2}$$

is a homeomorphism. Therefore, for any $\lambda \in \Lambda$,

$$\operatorname{Spf} A = \operatorname{Spec} A/I_\lambda. \tag{2.1.26.3}$$

For any $f \in A$, define

$$D(f) = \operatorname*{colim}_{\lambda \in \Lambda} \left(\operatorname{Spec} A_f / I_{\lambda f} \right). \tag{2.1.26.4}$$

As before, this is a colimit of homeomorphic subspaces. Since $\operatorname{Spec} A_f / I_{\lambda f}$ is an open subset of $\operatorname{Spec} A/I$, $D(f)$ is an open subset of $\operatorname{Spf} A$. By construction,

$$D(f) = \operatorname{Spf} A_{\{f\}}. \tag{2.1.26.5}$$

It is straightforward to verify that $\operatorname{Spf} A$ is canonically homeomorphic to the subspace of $\operatorname{Spec} A$ consisting of open prime ideals of A. Indeed, a prime ideal $\mathfrak{p}$ of A is open if and only if it contains I_λ for some $\lambda \in \Lambda$. Since $\mathfrak{p}$ is prime, this is true if and only if it contains $I_{\lambda'}$ for all $\lambda' \in \Lambda$.

2.1.27 (Formal spectrum of morphisms)**.** Let $\varphi : A \to B$ be a morphism of admissible rings and let $\{I_\lambda\}_{\lambda \in \Lambda}$ and $\{J_\mu\}_{\mu \in M}$ be the sets of ideals of definition of A and B respectively. Then φ induces a morphism of topological spaces

$$\operatorname{Spf} \varphi : \operatorname{Spf} B = \operatorname*{colim}_{\mu \in M} (\operatorname{Spec} B/J_\mu) \to \operatorname*{colim}_{\lambda \in \Lambda} (\operatorname{Spec} A/I_\lambda) = \operatorname{Spf} A. \tag{2.1.27.1}$$

Indeed, since φ is continuous, for all $\mu \in M$, there exists a $\lambda \in \Lambda$ such that the factorisation

$$\operatorname{Spec} B/J_\mu \to \operatorname{Spec} A/I_\lambda \to \operatorname{Spf} A. \tag{2.1.27.2}$$

exists. By the universal property of colimits, there is a unique morphism $\operatorname{Spf} B \to \operatorname{Spf} A$. Alternatively, $\operatorname{Spf} \varphi$ is the map of topological spaces which maps an open prime ideal $\mathfrak{q}$ of B to the open prime ideal $\varphi^{-1}(\mathfrak{q})$ of A.

An advantage of considering only open prime ideals is that the inclusion of the origin into its analytic neighbourhood in the plane is a homeomorphism.

Example 2.1.28 (Formal spectrum of power series)**.** Consider the topological ring $k[\![x, y]\!]$ of Example 2.1.7 (Formal power series). Then, the topological space of $\operatorname{Spf} k[\![x, y]\!]$ consists of one single point. This corresponds to the unique open prime ideal (x, y). In contrast, $\operatorname{Spec} k[\![x, y]\!]$ is a larger topological space For instance, it contains the point corresponding to the prime ideal (0).

2.2 Pro-modules

This section should be titled *Topological modules*. Unfortunately, the category of topological modules is not Abelian. Since the category of admissible rings is a full subcategory of the pro-category of rings, it is plausible to replace the category of topological modules by the category of pro-modules. Firstly, topological modules and pro-modules are defined. Then, it is shown that the category of pro-modules is Abelian and the relation with the category of topological modules is explored. This section employs the ideas and results of [Yas09, §1.2, page 2422].

Definition 2.2.1 (Topological modules)**.** Let A be an admissible ring and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of ideals of definition of A. A topological module M over A is an A-module endowed with a topology such that the operations

$$+ : M \times M \to M \tag{2.2.1.1}$$
$$\cdot : A \times M \to M \tag{2.2.1.2}$$

are continuous, where $M \times M$ and $A \times M$ have the product topology. The topological module M is A-linear if there exists a fundamental system of open neighbourhoods of $0 \in M$ for the topology of M consisting of submodules $\{P_\gamma\}_{\gamma\in\Gamma}$. A morphism of topological modules over A is a morphism of A-modules which is continuous as a map of topological spaces. The set of topological modules with their morphisms forms a category.

2.2.2 (Completion of topological modules)**.** The notions of separated and completeness extend to topological modules. Indeed, in Paragraph 2.1.6 (Separated completion of topological rings), only the group structure of the ring was used. Let A be an admissible ring and let M be an A-linear topological module over A. Then the separated completion of M is

$$\hat{M} := \lim_{\gamma\in\Gamma} M/P_\gamma, \tag{2.2.2.1}$$

where $\{P_\gamma\}_{\gamma\in\Gamma}$ is a fundamental system of open submodules of $0 \in M$. It is readily verified that $\hat{M}$ is a complete and separated A-linear topological module which satisfies the following universal property: for any separated and complete A-linear topological module N endowed with an A-morphism $M \to N$, there exists a unique A-morphism $\hat{M} \to N$ such that the following diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & \hat{M} \arrow[d] \\ & N \end{tikzcd} \tag{2.2.2.2}$$

is commutative. The module $\hat{M}$ is the *separated completion* of M. The module M is *separated and complete* if the morphism $M \to \hat{M}$ is an isomorphism.

If a pro-module were defined as an object of the pro-completion of all modules over a given ring, the topological data of the ring would be lost. For this reason, only *discrete modules* ([Yas09, Definition 1.7, page 2422]) will be considered.

Definition 2.2.3 (Discrete modules)**.** Let A be an admissible ring. A *discrete module* M over A is a topological module over A endowed with the discrete topology.

2.2.4 (Discrete modules)**.** Let A be an admissible ring and let M be a usual module over A. Then M is a discrete module over A if and only if the scalar multiplication

$$A \times M \to M \tag{2.2.4.1}$$

is continuous. This holds if and only if, for all $m \in M$, there exists an ideal of definition $I \subseteq A$ such that $I \subseteq \mathrm{Ann}_A(m)$. Therefore, the notion of discrete module is equivalent to the one given in [Yas09, Definition 1.7, page 2422].

The category of discrete modules is still Abelian.

Lemma 2.2.5 (Abelian category of discrete modules)**.** *Let A be an admissible ring, then the category of discrete modules is Abelian.*

Proof. Note that the category of discrete modules over A is a full subcategory of the category of modules over A. Hence, in order to check it is Abelian, it suffices to check that it has the zero object and that it is closed under direct sums, kernels and cokernels ([Rot09, Proposition 5.92, page 310]). This is readily verified. $\square$

Now, the much anticipated definition (vid. [Yas09, Definition 1.8, page 2423]).

Definition 2.2.6 (Pro-modules)**.** Let A be an admissible ring. The category of *pro-modules* over A is the pro-completion of the category of discrete A-modules. Elements of this category are *pro-modules* and a morphism of pro-modules is a morphism in this category.

Example 2.2.7 (Pro-Abelian groups)**.** Consider the ring of integers $\mathbb{Z}$ with the discrete topology. An Abelian group is a module over $\mathbb{Z}$. In fact, since (0) is an ideal of definition of $\mathbb{Z}$, an abelian group is precisely a discrete module over $\mathbb{Z}$. Therefore, the category of pro-Abelian groups and the category of pro-modules over $\mathbb{Z}$ are equivalent.

Lemma 2.2.8 (Category of pro-modules is Abelian)**.** *Let A be an admissible ring. The category of pro-modules over A is Abelian.*

Proof. Since the category of discrete modules is Abelian (Lemma 2.2.5 (Abelian category of discrete modules)) and the pro-completion of an Abelian category is Abelian (Lemma 1.3.3 (Pro-completion of Abelian categories)), the result follows. $\square$

Not only the category of pro-modules is Abelian. It also has a tensor product.

2.2.9 (Tensor product of pro-modules)**.** Given pro-modules

$$M = \varprojlim_{\gamma\in\Gamma} M_\gamma \text{ and } N = \varprojlim_{\delta\in\Delta} N_\delta \tag{2.2.9.1}$$

over an admissible ring A, the tensor product is defined as

$$M\hat{\otimes}_A N := \varprojlim_{(\gamma,\delta)\in\Gamma\times\Delta} M_\gamma \otimes_A N_\delta. \tag{2.2.9.2}$$

This is well-defined since $M_\gamma \otimes_A N_\delta$ is a discrete module over A. This can be shown to satisfy the universal property of tensor products in the category of pro-modules over A (vid. [Yas09, Definition 1.9, page 2423]).

2.2.10 (Localisation of pro-modules)**.** Given a pro-module $M = \varprojlim_{\gamma\in\Gamma} M_\gamma$ over an admissible ring A and a multiplicatively closed subset S of A, the localisation is defined as

$$\{S^{-1}\}M = M\hat{\otimes}_A\{S^{-1}\}A, \tag{2.2.10.1}$$

where $\{S^{-1}\}A$ is the complete localisation of A in S (vid. Paragraph 2.1.22 (Complete localisation of admissible rings)).

After this abstract definition, it is important to understand how it relates with the category of topological modules. As in the case of admissible rings, there are two functors, however, these are not inverse to each other. Essentially, the difference is due to the fact that the morphisms in a filtered system of modules need not be surjective.

2.2.11 (Topological modules and pro-modules)**.** Let A be an admissible ring and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of ideals of definition of A. Let $\mathcal{D}$ be the category of topological A-linear modules and let $\mathcal{E}$ be the category of pro-modules. Define a functor

$$\begin{aligned} F : \mathcal{D} &\to \mathcal{E} \\ M &\to \lim_{\gamma\in\Gamma} M/P_\gamma, \end{aligned} \tag{2.2.11.1}$$

where $\{P_\gamma\}_{\gamma\in\Gamma}$ is a fundamental system of open submodules of $0 \in M$. For all $\gamma \in \Gamma$, M/P_γ is a discrete A-module, hence the functor F is well-defined. Conversely, define a functor

$$\begin{aligned} G : \mathcal{E} &\to \mathcal{D} \\ \lim_{\gamma\in\Gamma} M_\gamma &\to \lim_{\gamma\in\Gamma} M_\gamma, \end{aligned} \tag{2.2.11.2}$$

where the limit on the right hand side is in the category of topological modules. By definition of limit topology, the submodules

$$\left\{ \ker\left(\lim_{\gamma\in\Gamma} M_\gamma \to M_\gamma \right) \right\}_{\gamma\in\Gamma} \tag{2.2.11.3}$$

form a fundamental system of open submodules of $0 \in M$, hence the functor is well-defined. By construction, for any $M \in \mathcal{D}$, there exists a morphism

$$M \to G(F(M)) = \hat{M}, \tag{2.2.11.4}$$

which is precisely the separated completion morphism of Paragraph 2.2.2 (Completion of topological modules). Therefore $G \circ F$ is the identity when restricted to separated and complete topological modules. On the other hand, there exist pro-modules $M \in \mathcal{E}$ such that

$$F(G(M)) \neq M. \tag{2.2.11.5}$$

Therefore the category of topological modules is not equivalent to the category of pro-modules over A (vid. Example 2.3.9 (Completion is weakly surjective)).

2.3 Pro-ideals

Given that the category of pro-modules is Abelian, monomorphisms and epimorphisms are well-behaved. Therefore, the corresponding notions of pro-ideal and weakly surjective morphism may be defined. These will be used to define closed immersion of formal schemes in the corresponding section of the next chapter. This section is the affine analogue of [Yas09, §3.2, page 2439].

Definition 2.3.1 (Pro-ideals)**.** Let A be an admissible ring. A *pro-ideal* of A is a monomorphism $0 \to K \to A$ in the Abelian category of pro-modules over A.

2.3.2 (Characterisation of pro-ideals)**.** Let A be an admissible ring and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of ideals of definition of A. Furthermore, let $A_\lambda := A/I_\lambda$. Then a pro-module $K \to A$ is a pro-ideal of A if and only if there exist ideals $K_\lambda \subseteq A_\lambda$ such that K is isomorphic to $\varprojlim_{\lambda\in\Lambda} K_\lambda$, where K_λ is regarded as a discrete module over A via $A \to A_\lambda$. This follows from Lemma 1.3.6 (Monomorphisms in Abelian pro-categories) together with the fact that the category of pro-modules over A is Abelian (Lemma 2.2.8 (Category of pro-modules is Abelian)).

The next paragraph explores the relation between ideals and pro-ideals. It turns out that pro-ideals generalise closed ideals.

2.3.3 (Ideals and pro-ideals)**.** Notation as in Paragraph 2.3.2 (Characterisation of pro-ideals), let K be a usual ideal of A. Then K induces a pro-ideal $F(K)$ of A via the construction in Paragraph 2.2.11 (Topological modules and pro-modules) given by

$$\varprojlim_{\lambda\in\Lambda} \frac{K}{I_\lambda \cap K}. \tag{2.3.3.1}$$

Then, since filtered limits are left exact, $G(F(K))$ is an ideal of A. The morphism $K \to G(F(K))$ is the separated completion, which, for ideals corresponds to the closure. In particular, if K is a closed ideal, $K = G(F(K))$.

Next, the dual notion of pro-ideal is defined. These morphisms are called *weakly surjective* and they are the analogue of surjective morphisms of rings. Indeed, when the topological rings are discrete, this is the usual definition of surjective.

Definition 2.3.4 (Weakly surjective morphisms of admissible rings)**.** Let $\varphi : A \to B$ be a morphism of admissible rings. Then φ is *weakly surjective* if it is an epimorphism in the category of pro-modules over A, where B is regarded as a pro-module over A via φ.

Lemma 2.3.5 (Weakly surjective morphisms of admissible rings)**.** *Let $\varphi : A \to B$ be a morphism of admissible rings. Then φ is weakly surjective if and only if the composition*

$$A \xrightarrow{\varphi} B \to B/J \tag{2.3.5.1}$$

is surjective, for all open ideals $J \subseteq B$. In particular, a surjective morphism of admissible rings is weakly surjective.

Proof. By the results of Lemma 1.3.4 (Epimorphisms in pro-categories) and Lemma 2.1.17 (Admissible rings and pro-category of rings), φ is weakly surjective if and only if, for all ideals of definition $J \subseteq B$, there exists an ideal of definition $I \subseteq A$ such that the dashed morphism in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A/I \arrow[r, dashed] & B/J \end{tikzcd} \tag{2.3.5.2}$$

exists and is surjective. Since the vertical morphisms are both surjective, the dashed morphism is surjective if and only if the composition in (2.3.5.1) is surjective. It then clear that a surjective morphism of admissible rings is weakly surjective. □

2.3.6 (Weakly surjective morphisms to discrete rings)**.** Note that, if $\varphi : A \to B$ is a morphism of admissible rings where B is a discrete ring, then φ is weakly surjective if and only if it is surjective. In particular, its associated pro-ideal is a topological ideal of A. On the other hand, if B is not discrete, $\varphi : A \to B$ could be weakly surjective but not surjective. Indeed, the filtered limit of surjective morphisms need not be surjective.

Unlike the case of schemes, a pro-ideal need not give rise to a morphism of admissible rings.

2.3.7 (Pro-ideals and weakly surjective morphisms)**.** Let $\varphi : A \to B$ be a weakly surjective morphism of admissible rings. Then $\ker \varphi$ is a pro-ideal of A and the induced morphism in the Abelian category of pro-modules

$$A/\ker \varphi \to B \tag{2.3.7.1}$$

is an isomorphism. This follows from the axioms of an Abelian category. On the other hand, not every pro-ideal induces a morphism of admissible rings. Indeed, the quotient of an admissible ring by a pro-ideal need not be an admissible ring.

The next lemma is a formal consequence of the definitions and the fact that the category of admissible rings embeds fully faithfully in the category of pro-rings.

Lemma 2.3.8 (Weakly surjective morphisms are epimorphisms)**.** *A weakly surjective morphism $\varphi : A \to B$ is an epimorphism in the category of admissible rings.*

Proof. Let

$$A \xrightarrow{\varphi} B \underset{\psi'}{\overset{\psi}{\rightrightarrows}} C \tag{2.3.8.1}$$

be a commutative diagram with morphisms ψ and ψ' to another admissible ring C. It has to be shown that $\psi = \psi'$. Note that B and C are pro-modules over A via the construction in 2.2.11 (Topological modules and pro-modules). Hence, since φ is an epimorphism in the category of pro-modules over A, ψ and ψ' are equal in the category of pro-modules over A. On the other hand, by definition, the pro-category of rings over A embeds faithfully into the category of pro-modules over A. Therefore ψ and ψ' are equal in the pro-category rings over A. Finally, by Lemma 2.1.17 (Admissible rings and pro-category of rings), the category of admissible rings over A embeds fully faithfully into the category of pro-modules over A and the result follows. □

The next example shows a weakly surjective morphism which is not surjective.

Example 2.3.9 (Completion is weakly surjective)**.** Let $A = k[x]$ with the discrete topology. Then the (x)-adic completion

$$k[x] \to k[\![x]\!] \tag{2.3.9.1}$$

is weakly surjective. To see this, note that the morphism can be represented by epimorphisms

$$k[x] \to \frac{k[x]}{(x^{n+1})} \tag{2.3.9.2}$$

as n varies in $\mathbb{N}$. By Lemma 2.3.5 (Weakly surjective morphisms of admissible rings), the completion is weakly surjective. Its kernel is

$$\lim_{n \in \mathbb{N}} \left(x^{n+1}\right). \tag{2.3.9.3}$$

Note that, if the filtered limit is computed in the category of topological modules over A, it is readily observed to be zero. On the other hand, it is not zero in the category of pro-modules over A. Indeed, using the definition of morphisms in the pro-completion, it follows that the identity morphism is different from the zero morphism. This shows that, even for usual discrete rings, the category of pro-modules contains strictly more objects than the category of modules.

2.4 Thickenings of admissible rings

In this section, the notion of thickening is generalised to morphisms of admissible rings. This is needed because the tubular neighbourhood of an immersion should be a thickening of admissible rings. After proving a characterisation, the standard operations of base change and localisation are discussed. Subsequently, the notion of regular thickening is proposed. This is an infinitesimal analogue of a regular immersion. Finally, some properties of regular thickenings are shown. These include invariance under flat base change and sufficient conditions to prove that an element of a regular thickening is zero.

Definition 2.4.1 (Thickenings of admissible rings)**.** Let $\varphi : A \to B$ be a morphism of admissible rings. Then φ is a *thickening* if it is a weakly surjective morphism of admissible rings and the induced map of formal spectra

$$\operatorname{Spf} \varphi : \operatorname{Spf} B \to \operatorname{Spf} A \tag{2.4.1.1}$$

is a homeomorphism.

2.4.2 (Thickenings of discrete rings)**.** If $\varphi : A \to B$ is a morphism of discrete rings, then φ is a thickening of admissible rings if and only if it is a thickening of rings. This follows from Paragraph 2.3.6 (Weakly surjective morphisms to discrete rings) and Paragraph 2.1.27 (Formal spectrum of morphisms).

For discrete rings, the nilradical of a ring detects if a surjective morphism is a thickening. For admissible rings, the same is true up to replacing the nilradical with the topological nilradical.

Lemma 2.4.3 (Thickenings and topological nilradicals)**.** *Let $\varphi : A \to B$ be a weakly surjective morphism of admissible rings and let N_A and N_B be the topological nilradicals of A and B respectively. Then the following statements are equivalent:*

1. *φ is a thickening.*
2. *For any ideal of definition $J \subseteq B$ and for some (or any) ideal of definition I of A such that $I \subseteq \varphi^{-1}(J)$, the induced morphism*
$$\varphi_{I,J} : A/I \to B/J \tag{2.4.3.1}$$
is a thickening of rings.
3. *$\varphi^{-1}(N_B) \subseteq N_A$.*
4. *$\varphi^{-1}(N_B) = N_A$.*

Proof. (1) $\to$ (2). Let $J \subseteq B$ be an ideal of definition and let I be any ideal of definition of A such that $I \subseteq \varphi^{-1}(J)$. Since φ is a weakly surjective morphism of rings, Lemma 2.3.5 (Weakly surjective morphisms of admissible rings) implies that the induced morphism $\varphi_{I,J}$ is surjective. Furthermore, by Paragraph 2.1.26 (Formal spectrum of admissible ring), there is a factorisation

$$\operatorname{Spf}\varphi : \operatorname{Spf} B = \operatorname{Spec} B/J \xrightarrow{\operatorname{Spec}\varphi_{I,J}} \operatorname{Spec} A/I = \operatorname{Spf} A. \tag{2.4.3.2}$$

Since φ is a thickening, $\operatorname{Spf}\varphi$ is a homeomorphism, therefore $\operatorname{Spec}\varphi_{I,J}$ is a homeomorphism. This shows that $\varphi_{I,J}$ is a thickening of rings.

(2) $\to$ (3). Since $\varphi_{I,J}$ is a thickening of rings, Lemma 1.1.3 (Characterisation of thickenings) implies that

$$\varphi_{I,J}^{-1}(0) \subseteq \operatorname{nilrad} A/I. \tag{2.4.3.3}$$

Therefore

$$\varphi^{-1}(J) \subseteq \operatorname{rad}_A I. \tag{2.4.3.4}$$

By Lemma 2.1.15 (Largest ideal of definition), $N_A = \operatorname{rad}_A I$ for any ideal of definition I of A. Since B is admissible, N_B is an ideal of definition of B ((1) $\to$ (3) implication of Lemma 2.1.14 (Characterisations of pre-admissible rings)). Therefore, the assumption implies that $\varphi^{-1}(N_B) \subseteq N_A$.

(3) → (4). Since φ is continuous, Lemma 2.1.13 (Topological nilradicals and morphisms) implies that $\varphi^{-1}(N_B) \supseteq N_A$. But then the assumption implies that $\varphi^{-1}(N_B) = N_A$.

(4) → (1). Since B is admissible, N_B is an ideal of definition of B. By Lemma 2.3.5 (Weakly surjective morphisms of admissible rings), there is a surjective morphism of rings

$$A \to B/N_B. \tag{2.4.3.5}$$

By assumption, there is an isomorphism of discrete rings

$$A/N_A \xrightarrow{\sim} B/N_B. \tag{2.4.3.6}$$

Since N_A is an ideal of definition of A, replacing I with N_A and J with N_B in (2.4.3.2) gives that $\operatorname{Spf}\varphi$ is a homeomorphism. Therefore φ is a thickening of admissible rings. □

Lemma 2.4.4 (Base change of thickenings of admissible rings)**.** *The base change of a thickening of admissible rings is a thickening of admissible rings.*

Proof. Let $\varphi : A \to B$ be a thickening of admissible rings and let $\psi : A \to A'$ be morphism of admissible rings. Let

$$\varphi' : A' \to A'\hat{\otimes}_A B =: B' \tag{2.4.4.1}$$

denote the base change. It is shown that φ' is a thickening of admissible rings. Let J' be an ideal of definition of B'. By construction, there exist ideals of definition I' and J of A' and B respectively such that J' is the ideal generated by the extension ideals of J and I' in B'. Pick an ideal of definition $I \subseteq \varphi^{-1}(J) \cap \psi^{-1}(I')$ of A. By construction, there is a co-Cartesian diagram

$$\begin{tikzcd} A/I \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\psi}"] & B/J \arrow[d, "\overline{\psi}'"] \\ A'/I' \arrow[r, "\overline{\varphi}'"] & B'/J'. \end{tikzcd} \tag{2.4.4.2}$$

By the (2) → (1) implication of Lemma 2.4.3 (Thickenings and topological nilradicals), it suffices to show that $\overline{\varphi}'$ is a thickening of rings. Since the base change of a thickening of rings is a thickening of rings (Lemma 1.1.4 (Base change of thickenings of rings)), it suffices to show that $\overline{\varphi}$ is a thickening of rings. This is true by assumption and the (1) → (2) implication of Lemma 2.4.3 (Thickenings and topological nilradicals). □

The next lemma proves that the kernel of a thickening is an adic ideal of definition.

Lemma 2.4.5 (Kernel of thickenings)**.** *Let $\varphi : A \to B$ be a thickening of adic rings where A is an adic ring and B is a discrete ring. Let $K := \ker\varphi$ be the the kernel of φ. If K is a finitely generated ideal, then it is an adic ideal of definition of A.*

Proof. Since B is discrete, K is open and closed in A. Therefore,

$$I^{n+1} \subseteq K \tag{2.4.5.1}$$

for some $n \in \mathbb{N}$. This implies that

$$I^{(n+1)(m+1)} = \left(I^{n+1}\right)^{m+1} \subseteq K^{m+1} \tag{2.4.5.2}$$

for all $m \in \mathbb{N}$. Hence K^{m+1} is an open ideal. On the other hand, since φ is a thickening, by the implication (1) $\to$ (3) of Lemma 2.4.3 (Thickenings and topological nilradicals),

$$K = \varphi^{-1}(0) \subseteq \varphi^{-1}(N_B) \subseteq N_A, \tag{2.4.5.3}$$

where N_A and N_B are the topological nilradicals of A and B respectively. Since I is an ideal of definition, Lemma 2.1.15 (Largest ideal of definition) implies that $\operatorname{rad}_A I = N_A$, so that

$$K \subseteq \operatorname{rad}_A I. \tag{2.4.5.4}$$

Since K is finitely generated, Lemma 1.1.8 (Powers of finitely generated ideals) implies that

$$K^{m+1} \subseteq I \tag{2.4.5.5}$$

for some $m \in \mathbb{N}$. Therefore, K is an adic ideal of definition of A. □

The next lemma states that, in some cases, being a local ring descends along thickenings.

Lemma 2.4.6 (Descent of local rings along thickenings)**.** *Let $\varphi : A \to B$ be a thickening of adic rings where A is an adic ring and B is a discrete ring. Let $K := \ker \varphi$ be the the kernel of φ and suppose it is finitely generated. Then, if B is a local ring with maximal ideal $\mathfrak{n}$, A is a local ring with maximal ideal $\varphi^{-1}(\mathfrak{n})$.*

For schemes, the result is obvious. Indeed, the two rings have the same spectra, hence they both have a unique maximal ideal. For formal schemes, it follows that both rings have a unique *open* maximal ideal, hence the difficulty lies in showing that there are no other maximal ideals.

Proof of Lemma 2.4.6 (Descent of local rings along thickenings). Since K is an adic ideal of definition (Lemma 2.4.5 (Kernel of thickenings)), there is an isomorphism

$$A = \lim_{n \in \mathbb{N}} A/K^{n+1}. \tag{2.4.6.1}$$

For all $n \in \mathbb{N}$, the morphism

$$\varphi_n : A/K^{n+1} \to B \tag{2.4.6.2}$$

is a thickening of rings ((1) → (2) of Lemma 2.4.3 (Thickenings and topological nilradicals)). In particular, it induces a homeomorphism of topological spaces, hence $\varphi_n^{-1}(\mathfrak{n})$ is the maximal ideal of A/K^{n+1}. By construction, $A/K^{m+1} \to A/K^{n+1}$ is a morphism of local rings for all $m \geq n$. But now (2.4.6.1) is a filtered limit of local rings along morphisms of local rings, hence A is a local ring with maximal ideal $\varphi^{-1}(\mathfrak{n})$ (Lemma 1.2.1 (Filtered limit of local rings)). □

The tubular neighbourhood of a regular immersion should be, in some sense, regular too. The next definition attempts to capture its properties.

Definition 2.4.7 (Regular thickenings of adic rings)**.** Let $\varphi : A \to B$ be a thickening of adic rings where A is an adic ring and B is a discrete ring. Let $K := \ker \varphi$ be the associated topological ideal of φ. Then φ is a *regular thickening* if K satisfies the following properties:

1. K is finitely generated.
2. K/K^2 is a projective B-module of finite rank.
3. For all $n \in \mathbb{N}$, the natural morphism

$$\operatorname{Sym}_B^n K/K^2 \xrightarrow{\sim} K^n/K^{n+1} \tag{2.4.7.1}$$

is an isomorphism.

In particular, K^n/K^{n+1} is a projective B-module for all $n \in \mathbb{N}$.

2.4.8 (Quasi-regular sequences)**.** There is an alternative characterisation of Definition 2.4.7 (Regular thickenings of adic rings). In the same notation, the ideal K is said to be *locally generated by a regular sequence* if for all open prime ideals $\mathfrak{p}$ of A, or equivalently all prime ideals of B, there exists a $g \in A \setminus \mathfrak{p}$ such that $K_g \subseteq A_g$ is generated by a quasi-regular sequence. By [Sta23, Lemma 063H], K is locally generated by a quasi-regular sequence if and only if K satisfies properties (1), (2) and (3) of the definition.

A flat base change of a regular immersion is a regular immersion. A similar statement is proved here.

Lemma 2.4.9 (Flat base change of regular thickenings)**.** *Let $\varphi : A \to B$ be a regular thickening of adic rings where A is an adic ring and B is a discrete ring. Let $A \to A'$ be a taut morphism of admissible rings which is flat as a morphism of rings and let*

$$\varphi' : A' \to A'\hat{\otimes}_A B =: B' \tag{2.4.9.1}$$

denote the base change. Then φ' is a regular thickening.

Proof. By Lemma 2.4.5 (Kernel of thickenings), $K := \ker \varphi$ is an adic ideal of definition of A. Since $A \to A'$ is a taut morphism to a separated and complete ring, [Sta23, Lemma 0APU] implies that $K' := K \cdot A'$ is an adic ideal of definition for the topology of A'.

By assumption, $A \to A'$ is flat as a morphism of rings, hence

$$K \otimes_A A' = K \cdot A' = K', \tag{2.4.9.2}$$

where the tensor product is uncompleted. By the backward implication of Paragraph 2.4.8 (Quasi-regular sequences), K is locally generated by a quasi-regular sequence. It is shown that K' is locally generated by a quasi-regular sequence. Let $\mathfrak{p}'$ be an open prime ideal of A' and let $\mathfrak{p}$ denote its pre-image in A. Since K is locally generated by a quasi-regular sequence, there exists a $g \in A \setminus \mathfrak{p}$ such that $K_g \subseteq A_g$ is generated by a quasi-regular sequence. Let g' be the image of g in A'. Then $A_g \to A_{g'}$ is flat and $K'_{g'} = K_g \otimes_{A_g} A_{g'}$ is generated by a quasi-regular sequence ([Sta23, Lemma 065L]). Since $g \notin \mathfrak{p}$, $g' \notin \mathfrak{p}'$. This shows that K' is locally generated by a quasi-regular sequence.

Finally, since the tensor product is a right exact functor

$$B' = A'/K', \tag{2.4.9.3}$$

where the quotient is taken in the category of rings. Since K' is an open and closed ideal, B' is a discrete ring which is also the quotient in the category of admissible rings (Paragraph 2.1.24 (Quotients of admissible rings)). Since φ is a thickening of admissible rings, the base change φ' is a thickening of admissible rings (Lemma 1.1.4 (Base change of thickenings of rings)). But now A' is an adic ring with adic ideal of definition K' and B' is a discrete ring. Furthermore, K' is locally generated by a quasi-regular sequence, hence the forward implication of Paragraph 2.4.8 (Quasi-regular sequences) shows that φ' is a regular thickening. □

The next lemma is concerned with the property of being zero. Suppose that a function is defined in a tubular neighbourhood of a variety and the function is zero along the variety and along the fibres of the tubular neighbourhood. Under regularity assumptions, the function should be zero.

Lemma 2.4.10 (Trickle down)**.** *Let $\varphi : A \to B$ be a regular thickening of adic rings where A is an adic ring and B is a discrete integral domain. Let $K := \ker \varphi$. Suppose that $x \in K$ and $ax = 0 \in A$ for some $a \in A \setminus K$. Then $x = 0$.*

Proof. Since φ is a regular thickening, K is finitely generated and Lemma 2.4.5 (Kernel of thickenings) implies that K is an adic ideal of definition of A. Hence, it suffices to show that $x \in K^{n+1}$ for all $n \in \mathbb{N}$. This is shown by induction on n.

(Base case). By assumption, $x \in K$. Hence the case $n = 0$ holds.

(Inductive step). Suppose $x \in K^n$. By assumption, there exists an $a \notin K$ such that $ax \in K^{n+1}$. Therefore,

$$ax = 0 \in K^n/K^{n+1}. \tag{2.4.10.1}$$

Since K^n/K^{n+1} is a projective B-module and B is an integral domain, either $a = 0 \in B$ or $x \in K^{n+1}$. But $a \notin K$, hence $a \neq 0 \in B$. Therefore $x \in K^{n+1}$. $\square$

The final lemma shows that, over a field of positive characteristic, the Frobenius morphism in injective on the adic ideal of definition. This should be thought as a mild regularity property.

Lemma 2.4.11 (Regular thickenings and Frobenius)**.** *Let $\varphi : A \to B$ be a regular thickening of adic rings where A is an adic ring and B is a discrete integral domain of characteristic $p > 0$. Let $K := \ker \varphi$. Suppose that there exists $x \in K$ such that $x^p = 0 \in A$. Then $x = 0 \in A$.*

Proof. Since φ is a regular thickening, K is finitely generated and Lemma 2.4.5 (Kernel of thickenings) implies that K is an adic ideal of definition of A. Hence, it suffices to show that $x \in K^{n+1}$ for all $n \in \mathbb{N}$. This is shown by induction on n.

(Base case). By assumption, $x \in K$. Hence the case $n = 0$ holds.

(Inductive step). Suppose $x \in K^n$ and consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n_B K/K^2 & \xrightarrow{\mathrm{Fr}} & \mathrm{Sym}^{np}_B K/K^2 \\ \downarrow & & \downarrow \\ K^n/K^{n+1} & \xrightarrow{\mathrm{Fr}} & K^{np}/K^{np+1}, \end{array}$$

where the vertical arrows are isomorphisms by the regularity assumption, and the horizontal arrows are elevation to the p^{th} power. These latter morphisms are morphisms of B-modules since the characteristic of B is p. Note that the top horizontal arrow maps x to the p^{th} symmetric power $x \otimes x \otimes ... \otimes x$. Since being zero is a local property on B, after localising by appropriate elements, K/K^2 may be assumed a free B-module of constant rank r ([Sta23, Lemma 01C9]). Let $\{u_i\}_{i \leq r}$ be a free basis over B. This induces a basis on $\mathrm{Sym}^n_B K/K^2$ denoted by $\{u_\upsilon\}$, where $\upsilon = (i_1, i_2, .., i_n)$ is an index parametrising unordered n-tuples of numbers between 1 and r. Let Υ be such indexing set. Now, suppose that

$$x = \sum_{\upsilon \in \Upsilon} b_\upsilon u_\upsilon \in K^n/K^{n+1} \tag{2.4.11.1}$$

for unique $b_\upsilon \in B$. Since Fr is a morphism of B-modules,

$$\mathrm{Fr}(x) = \sum_{\upsilon \in \Upsilon} b_\upsilon^p u_\upsilon^{\otimes p} \in K^{np}/K^{np+1}. \tag{2.4.11.2}$$

By the previous part, $\{u_\upsilon^{\otimes p}\}_{\upsilon \in \Upsilon}$ is a linearly independent set over B. By assumption, $\mathrm{Fr}(x) = 0$, hence $b_\upsilon^p = 0$. Since B is an integral domain, $b_\upsilon = 0$ for all $\upsilon \in \Upsilon$. This shows that $x = 0 \in K^n/K^{n+1}$, thus completing the induction. $\square$

2.5 Affine infinitesimal neighbourhoods

This section defines the affine tubular neighbourhood of an immersion. Hereafter, it will be called an infinitesimal neighbourhood in order to conceal its analytic origins. The infinitesimal neighbourhood of an immersion is defined as the thickening containing all possible thickenings factoring through the given immersion. Firstly, a definition through a universal property is given. Then, a candidate thickening is constructed and finally, it is shown that it satisfies the universal property. Some applications to the case of discrete rings follow.

Definition 2.5.1 (Affine infinitesimal neighbourhoods)**.** Let $\varphi : A \to B$ be a weakly surjective morphism of admissible rings. The *affine infinitesimal neighbourhood* $\hat{\varphi} : \hat{A}_\varphi \to B$ is a thickening of admissible rings satisfying the following universal property: for any solid commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\varphi"] \arrow[d, "\psi"'] & \hat{A}_\varphi \arrow[r, "\hat{\varphi}"'] \arrow[dl, dashed] & B \arrow[d, "\psi'"] \\ C \arrow[rr, "\varphi'"] & & D, \end{tikzcd} \tag{2.5.1.1}$$

where $\varphi' : C \to D$ is a thickening of admissible rings, the dashed morphism making the diagram commute exists and is unique. By general category theory, if the affine infinitesimal neighbourhood exists, it is unique.

Construction 2.5.2 (Affine infinitesimal neighbourhoods)**.** Let $\varphi : A \to B$ be a weakly surjective morphism of admissible rings. Let $\overline{A}_\varphi$ be the same ring A endowed with a coarser topology described hereafter. Let $\mathcal{C}$ be the collection of open ideals of A which further satisfy the following property:

$$\varphi^{-1}(N_B) \subseteq \operatorname{rad}_A I, \tag{2.5.2.1}$$

where N_B is the topological nilradical of B. Intuitively, this forces the equality $N_{\overline{A}_\varphi} = \varphi^{-1}(N_B)$ to be true, where $N_{\overline{A}_\varphi}$ is the topological nilradical of $\overline{A}_\varphi$.

It has to be shown that the collection $\mathcal{C}$ generates a linear topology on A. To this end, Lemma 2.1.5 (Generated linear topologies) is employed. By its part (2), it suffices to show that properties (a) and (b) are satisfied.

Suppose that $I \in \mathcal{C}$ and $I \subseteq J$. By part (1) of Lemma 2.1.5 (Generated linear topologies) J is open and it follows that

$$N_B \subseteq \operatorname{rad}_A I \subseteq \operatorname{rad}_A J \tag{2.5.2.2}$$

This shows that $J \in \mathcal{C}$, hence (a) is satisfied.

Now suppose that I and J are in $\mathcal{C}$. Certainly $I \cap J$ is open and furthermore

$$N_B \subseteq \operatorname{rad}_A I \cap \operatorname{rad}_A J = \operatorname{rad}_A (I \cap J)\,. \tag{2.5.2.3}$$

This shows that $I \cap J \in \mathcal{C}$, hence (b) is satisfied. Therefore $\overline{A}_\varphi$ is topological ring endowed with a linear topology.

Since A and $\overline{A}_\varphi$ are isomorphic as rings, there are morphisms of rings

$$\varphi : A \to \overline{A}_\varphi \xrightarrow{\overline{\varphi}} B \tag{2.5.2.4}$$

By construction, the topology of $\overline{A}_\varphi$ is coarser that the topology of A, hence the morphism $A \to \overline{A}_\varphi$ is a morphism of topological rings. Furthermore $\overline{\varphi}$ is still continuous with the newly defined coarser topology. To see this, let J be an ideal of definition of B. It has to be shown that $\overline{\varphi}^{-1}(J)$ is open in $\overline{A}_\varphi$. Certainly $\varphi^{-1}(J)$ is open in A. By Lemma 2.1.15 (Largest ideal of definition), $\operatorname{rad}_B J = N_B$. Therefore, using Lemma 1.1.2 (Radical and pre-image),

$$\begin{aligned} \varphi^{-1}(N_B) &= \varphi^{-1}(\operatorname{rad}_B J) \\ &= \operatorname{rad}_A \varphi^{-1}(J). \end{aligned} \tag{2.5.2.5}$$

By definition, $\overline{\varphi}^{-1}(J)$ is open in $\overline{A}_\varphi$.

Next, it is shown that $\overline{A}_\varphi$ is pre-admissible. By the (2) $\to$ (1) implication of Lemma 2.1.14 (Characterisations of pre-admissible rings), it suffices to show that $N_{\overline{A}_\varphi}$ is open in $\overline{A}_\varphi$. To this end, it suffices to show that

$$\varphi^{-1}(N_B) \subseteq N_{\overline{A}_\varphi} \subseteq A. \tag{2.5.2.6}$$

Indeed, since B is admissible, N_B is open in B, therefore $\varphi^{-1}(N_B)$ is open in A. Therefore (2.5.2.6) implies that $N_{\overline{A}_\varphi}$ is open in A, and hence that it is open in $\overline{A}_\varphi$. Suppose that $x \in \varphi^{-1}(N_B)$ and let I be an open ideal of $\overline{A}_\varphi$. By construction, $\varphi^{-1}(N_B) \subseteq \operatorname{rad}_A I$, hence $x \in \operatorname{rad}_A I$. Therefore $x^{n+1} \in I$ for some $n \in \mathbb{N}$. This shows that $x \in N_{\overline{A}_\varphi}$, hence $\overline{A}_\varphi$ is pre-admissible.

Let $\hat{A}_\varphi$ denote the separated completion of $\overline{A}_\varphi$. Since $\overline{A}_\varphi$ is pre-admissible, $\hat{A}_\varphi$ is an admissible ring. By the universal property of separated completion, there are morphisms of topological rings

$$\varphi : A \to \overline{A}_\varphi \to \hat{A}_\varphi \xrightarrow{\hat{\varphi}} B. \tag{2.5.2.7}$$

Note that $A \to \hat{A}_\varphi$ is, by construction, a weakly surjective morphism of admissible rings. When the morphism φ is clear from the context, $\overline{A}_\varphi$ and $\hat{A}_\varphi$ are simply denoted by $\overline{A}$ and $\hat{A}$ respectively.

Lemma 2.5.3 (Affine infinitesimal neighbourhoods)**.** *Notation as in Construction 2.5.2 (Affine infinitesimal neighbourhoods),*

$$\varphi : A \to \hat{A} \xrightarrow{\hat{\varphi}} B \tag{2.5.3.1}$$

satisfies the universal property of Definition 2.5.1 (Affine infinitesimal neighbourhoods). Furthermore, $A \to \hat{A}$ is weakly surjective.

Proof. Firstly, it is verified that $\hat{\varphi}$ is a thickening. Since φ is a weakly surjective morphism of admissible rings, $\hat{\varphi}$ is a weakly surjective morphism of admissible rings. Let $N_{\overline{A}}$ be the topological nilradical of $\overline{A}$. According to the (3) $\to$ (1)

implication of Lemma 2.4.3 (Thickenings and topological nilradicals), it suffices to show that

$$\varphi^{-1}(N_B) \subseteq N_{\overline{A}}. \tag{2.5.3.2}$$

This is shown in (2.5.2.6). Hence $\hat{\varphi}$ is a thickening of admissible rings.

Next, the universal property of Diagram (2.5.1.1) is verified. Since C is separated and complete, by the universal property of separated completion, it suffices to show that there exists a unique morphism $\overline{A} \to C$ making the diagram commute. This morphism has to be set-theoretically identical to the morphism $\psi : A \to C$. This implies that, if it exists, it is unique and that, in order to show existence, it suffices to show that ψ is continuous with respect to the coarser topology. Let N_B, N_C and N_D be the topological nilradicals of B, C and D respectively. Let K be an ideal of definition of C. It has to be shown that $\psi^{-1}(K)$ is an open ideal of $\overline{A}$. Since it is already open in A, it suffices to show that

$$\varphi^{-1}(N_B) \subseteq \operatorname{rad}_A \psi^{-1}(K). \tag{2.5.3.3}$$

Since φ' is a thickening, by the (1) $\to$ (4) implication of Lemma 2.4.3 (Thickenings and topological nilradicals),

$$N_C = \left(\varphi'\right)^{-1}(N_D). \tag{2.5.3.4}$$

By Lemma 2.1.15 (Largest ideal of definition),

$$\operatorname{rad}_C K = N_C. \tag{2.5.3.5}$$

By Lemma 2.1.13 (Topological nilradicals and morphisms),

$$N_B \subseteq \left(\psi'\right)^{-1}(N_D) \tag{2.5.3.6}$$

Putting everything together and using Lemma 1.1.2 (Radical and pre-image) yields

$$\begin{aligned}
\varphi^{-1}(N_B) &\subseteq \varphi^{-1}\left(\left(\psi'\right)^{-1}(N_D)\right) \\
&= \psi^{-1}\left(\left(\varphi'\right)^{-1}(N_D)\right) \\
&= \psi^{-1}(N_C) \\
&= \psi^{-1}(\operatorname{rad}_C K) \\
&= \operatorname{rad}_A \psi^{-1}(K).
\end{aligned} \tag{2.5.3.7}$$

This shows (2.5.3.3).

Finally, for any open ideal I of $\hat{A}$,

$$A \to \overline{A}/I \xrightarrow{\sim} \hat{A}/I \tag{2.5.3.8}$$

is, by construction, surjective. Therefore $A \to \hat{A}$ is weakly surjective. $\square$

The next lemma applies the construction to discrete rings and proves that the infinitesimal neighbourhood of a regular immersion is regular.

Lemma 2.5.4 (Infinitesimal neighbourhoods of rings)**.** *Let $\varphi : A \to B$ be a surjective morphism of discrete rings and let $\hat{\varphi} : \hat{A} \to B$ be the affine infinitesimal neighbourhood. Let K and $\hat{K}$ be the kernel ideals of φ and $\hat{\varphi}$ respectively.*

1. *If K is finitely generated, the infinitesimal neighbourhood $\hat{A}$ is an adic ring and $\hat{K}$ is an adic ideal of definition.*
2. *If K is finitely generated and A/K is Noetherian, then $\hat{A}$ is an adic Noetherian ring and $\hat{K}$ is an adic ideal of definition.*
3. *If A is Noetherian and K is a quasi-regular ideal, $\hat{\varphi}$ is a regular thickening.*

Proof. 1. By Construction 2.5.2 (Affine infinitesimal neighbourhoods), it has to be shown that an ideal I is open in $\overline{A}$ if and only if $K^{n+1} \subseteq I$ for some $n \in \mathbb{N}$. If I is open then, by definition, $K \subseteq \operatorname{rad}_A I$. Therefore, since K is finitely generated, Lemma 1.1.8 (Powers of finitely generated ideals) implies that $K^{n+1} \subseteq I$. Conversely, if $K^{n+1} \subseteq I$, I is open since K^{n+1} is open in the discrete ring A. This implies that $\hat{A}$ is the K-adic completion of A. By [Sta23, Lemma 05GG], $\hat{A}$ is an adic ring and $\hat{K} = K \cdot A$ is an adic ideal of definition.

2. By part (1), it suffices to show that $\hat{A}$ is Noetherian. Since A/K is Noetherian, the result follows from ([Sta23, Lemma 05GH]).
3. Since A is Noetherian, part (3) of [Sta23, Lemma 00MA] implies that $\hat{K} = K \otimes_A \hat{A}$, where the tensor product is uncompleted. Furthermore, the K-adic completion $A \to \hat{A}$ is flat and taut (Paragraph 2.1.20 (Completion of Noetherian rings)). Now [Sta23, Lemma 065L] implies that $\hat{K} = K \otimes_A \hat{A}$ is locally generated by a quasi-regular sequence and the forward implication of Paragraph 2.4.8 (Quasi-regular sequences) shows that $\hat{\varphi}$ is a regular thickening. ☐

The final example compares adic completions and infinitesimal neighbourhoods.

Example 2.5.5 (Completion of rings)**.** Let A be a discrete local ring with maximal ideal $\mathfrak{m}$. Let $\hat{A}$ denote the infinitesimal neighbourhood of the surjection $\varphi : A \to A/\mathfrak{m} =: \kappa$.

If $\mathfrak{m}$ is finitely generated, by part (1) of Lemma 2.5.4 (Infinitesimal neighbourhoods of rings), $\hat{A}$ is the $\mathfrak{m}$-adic completion of A.

If furthermore A is a regular local ring, A is Noetherian and $\mathfrak{m}$ is generated by a quasi-regular sequence. Therefore, by part (3) of Lemma 2.5.4 (Infinitesimal neighbourhoods of rings), $\hat{\varphi} : \hat{A} \to \kappa$ is a regular thickening.

On the other hand, if $\mathfrak{m}$ is not finitely generated, $\hat{A}$ need not be an adic ring. It is the completion of A with respect to all ideals $I \subseteq A$ such that

$$\mathfrak{m} = \operatorname{rad}_A I. \tag{2.5.5.1}$$

Chapter 3

Formal schemes

3.1 Formal schemes

This section is a globalisation of the concepts developed in §2.1 (Admissible Rings). A locally topologically ringed space is defined and it is shown that the formal spectrum of an admissible ring is an instance of such space. A formal scheme is then defined as a space which is locally the formal spectrum of an admissible ring, or equivalently, a filtered colimit of affine schemes along thickenings. The theory then proceeds as in the case of schemes. The material in this section is based on [Sta23, Section 0AHY], [Gro60, §10, page 180] and [McQ02, §4, page 8]. Note that the definition of admissible ring used in [Gro60] is stronger, however, given the equivalence of categories of Lemma 2.1.17 (Admissible rings and pro-category of rings), the results of *op. cit.* apply to this more general setting without alteration.

3.1.1 (Topologically ringed spaces)**.** Let $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ be a ringed space where $\mathscr{O}_{\mathfrak{X}}$ is a sheaf valued in the category of topological rings. This is a *topologically ringed space.* A morphism of topologically ringed spaces

$$\left(f, f^{\#}\right) : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \tag{3.1.1.1}$$

is a morphism of ringed spaces such that the morphism of sheaves is a morphism of sheaves of topological rings.

3.1.2 (Locally topologically ringed spaces)**.** Notation as in Paragraph 3.1.1 (Topologically ringed spaces), let $x \in \mathfrak{X}$. The *stalk* in x is the colimit

$$\mathscr{O}_{\mathfrak{X},x} := \operatorname*{colim}_{U \ni x} \mathscr{O}_{\mathfrak{X}}(U) \tag{3.1.2.1}$$

in the category of rings *without topology*, over all open subsets $U \subseteq \mathfrak{X}$ containing x. Then, $\mathfrak{X}$ is a *locally topologically ringed space* if $\mathscr{O}_{\mathfrak{X},x}$ is a local ring for all $x \in \mathfrak{X}$. A morphism of locally topologically ringed spaces is a morphism of topologically ringed spaces such that the induced morphism of rings without topology

$$\mathscr{O}_{\mathfrak{Y},f(x)} \to \mathscr{O}_{\mathfrak{X},x} \tag{3.1.2.2}$$

is a morphism of local rings. The set of locally topologically ringed spaces with their morphisms forms a category. A locally topologically ringed space induces a locally ringed space by forgetting the topological structure. Furthermore, by definition, the category of locally topologically ringed spaces embeds faithfully in the category of locally ringed spaces. Hence, there is no harm in confusing a locally topologically ringed space with the locally ringed space it induces. This is essentially the approach in [Sta23, Section 0AHY] and [Gro60].

3.1.3 (Affine formal schemes)**.** Let A be an admissible ring and let $\operatorname{Spf} A$ be its formal spectrum. Define an affine formal scheme $(\operatorname{Spf} A, \mathscr{O}_{\operatorname{Spf} A})$, or simply $\operatorname{Spf} A$, associated to A in the following way: for any $f \in A$, define a pre-sheaf on the basis of distinguished open subsets

$$\mathscr{O}_{\operatorname{Spf} A}(D(f)) = A_{\{f\}}. \tag{3.1.3.1}$$

This is a pre-sheaf on a basis valued in the category of admissible rings. The restriction morphisms of this pre-sheaf are well-defined by the universal property of complete localisation. This is in fact a sheaf on the given basis, thus it induces a unique sheaf on $\operatorname{Spf} A$. This is the *structure sheaf* of $\operatorname{Spf} A$ and it is also denoted by $\mathscr{O}_{\operatorname{Spf} A}$. This is a sheaf valued in the category of topological (i.e. not admissible) rings. Indeed, if U is not a distinguished open subset, $\mathscr{O}_{\operatorname{Spf} A}(U)$ need not be an admissible ring. By construction,

$$\Gamma\left(\operatorname{Spf} A, \mathscr{O}_{\operatorname{Spf} A}\right) = A. \tag{3.1.3.2}$$

Furthermore, if $x \in \operatorname{Spf} A$ corresponds to a prime ideal $\mathfrak{p}$ of A/I_λ, for some $\lambda \in \Lambda$, then

$$\mathscr{O}_{\operatorname{Spf} A,x} = \operatorname*{colim}_{f \notin \mathfrak{p}} A_{\{f\}} = \operatorname*{colim}_{f \notin \mathfrak{p}} \left(A_{\{f\}}\right)_{\mathfrak{p}}, \tag{3.1.3.3}$$

where the second equality follows from the universal property of usual localisation. This is not the complete localisation in $\mathfrak{p}$, however, by Lemma 1.2.2 (Filtered colimit of local rings), it is a local ring. Therefore $\operatorname{Spf} A$ is a locally topologically ringed space. An *affine formal scheme* is a locally topologically ringed space isomorphic to $\operatorname{Spf} A$ for some A. An affine formal scheme is *adic* (resp. *Noetherian*) if it is isomorphic to $\operatorname{Spf} A$ for some adic (resp. Noetherian) ring A.

3.1.4 (Anti-equivalence of categories)**.** Recall that a morphism $\varphi : A \to B$ of admissible rings induces a map of topological spaces

$$f := \operatorname{Spf} \varphi : \operatorname{Spf} B \to \operatorname{Spf} A. \tag{3.1.4.1}$$

Furthermore, φ induces a morphism of sheaves of topological rings

$$f^{\#} : \mathscr{O}_{\operatorname{Spf} A} \to f_* \mathscr{O}_{\operatorname{Spf} B} \tag{3.1.4.2}$$

which is a morphism of local rings in the stalks. Thus

$$\left(f, f^{\#}\right) : \operatorname{Spf} B \to \operatorname{Spf} A \tag{3.1.4.3}$$

is a morphism of locally topologically ringed spaces. This gives a map

$$\left\{ \begin{array}{c} \varphi : A \to B \,|\, \varphi \text{ is a morphism} \\ \text{of admissible rings.} \end{array} \right\} \xrightarrow{\mathrm{Spf}} \left\{ \begin{array}{c} f : \mathfrak{X} \to \mathfrak{Y} \,|\, f \text{ is a morphism} \\ \text{of affine formal schemes.} \end{array} \right\}. \tag{3.1.4.4}$$

This is a bijection whose inverse is given by the functor of global sections ([Gro60, Proposition 10.2.2, page 182] or [McQ02, Fact 4.4, page 9]).

Definition 3.1.5 (Formal schemes)**.** A *formal scheme* $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$, or simply $\mathfrak{X}$, is a locally topologically ringed space such that every point has an open neighbourhood isomorphic to an affine formal scheme. A formal scheme is *adic* (resp. *locally Noetherian*) if every point has an open neighbourhood isomorphic to an adic (resp. Noetherian) affine formal scheme.

The next paragraph justifies the choice of definition of an adic ring.

3.1.6 (Local properties of formal schemes)**.** Let $\mathfrak{X}$ be a formal scheme. Then being adic and locally Noetherian are local properties. More precisely, if there exists an affine open cover of $\mathfrak{X}$ by affine formal schemes which are adic (resp. Noetherian), then every affine open formal subscheme is adic (resp. Noetherian). This is proved in [Sta23, Lemma 0AKX] where the property of being adic is called adic*.

Most formal schemes occurring naturally are actually global filtered colimits of schemes along thickenings.

3.1.7 (Colimit of schemes along thickenings)**.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a cofiltered system of schemes along thickenings. Then, it is shown that

$$\mathfrak{X} := \operatorname*{colim}_{\lambda \in \Lambda} X_\lambda \tag{3.1.7.1}$$

is naturally a formal scheme. If X_{λ_0} is affine for some $\lambda_0 \in \Lambda$, then X_λ is affine for all $\lambda \in \Lambda$. This is a consequence of the fact that, if $X_\lambda \to X_{\lambda'}$ is a thickening of schemes, X_λ is affine if and only if $X_{\lambda'}$ is affine. The forward implication is given by [Sta23, Lemma 06AD] and the backward implication is given by the fact that closed immersions are affine morphisms ([Sta23, Lemma 01IN]). Define the topological space of $\mathfrak{X}$ to be the topological space of X_λ for any $\lambda \in \Lambda$. Furthermore, define $\mathscr{O}_{\mathfrak{X}}$ on the basis of open sets U where X_λ is affine by

$$\mathscr{O}_{\mathfrak{X}}(U) = \lim_{\lambda \in \Lambda} \mathscr{O}_{X_\lambda}(U). \tag{3.1.7.2}$$

On this basis, $\mathscr{O}_{\mathfrak{X}}$ is a filtered limit of discrete rings along thickenings. By Lemma 2.1.16 (Characterisation of admissible rings), it is an admissible ring. Therefore $\mathfrak{X}$ is a formal scheme.

Formal schemes, and morphisms of formal schemes, can be glued along open immersions.

3.1.8 (Glueing formal schemes)**.** Consider the setup of [Sta23, Section 01JA]. Therein, there is a collection of locally ringed spaces together with isomorphisms between open subsets satisfying appropriate compatibility conditions. This is referred as *glueing data*. Suppose that the glueing data consists of formal schemes $\{U_i\}_{i\in I}$, for an indexing set I. Then, [Sta23, Lemma 01JB] implies that there exists a locally ringed space $\mathfrak{X}$ obtained by glueing the formal schemes U_i along open subsets. Since $\mathfrak{X}$ is, by construction, locally isomorphic to a formal scheme, it is a formal scheme. Let $U_{ij} := U_i \cap U_j \subseteq \mathfrak{X}$. The same lemma shows that, if $\mathfrak{Y}$ is another formal scheme, then

$$\mathrm{Hom}\,(\mathfrak{X},\mathfrak{Y}) = \left\{(f_i)_{i\in I} \,|\, f_i \to \mathfrak{Y} \text{ and } f_i|_{U_{ij}} = f_j|_{U_{ij}}\right\} \tag{3.1.8.1}$$

$$\mathrm{Hom}\,(\mathfrak{Y},\mathfrak{X}) = \left\{ \begin{array}{c} (g_i)_{i\in I} \,|\, g_i : V_i \to U_i \text{ and } V_i \subseteq \mathfrak{Y} \\ \text{satisfying compatibility conditions} \end{array} \right\}, \tag{3.1.8.2}$$

where morphisms are morphisms of locally ringed spaces. Since, by assumption, f_i and g_i are morphisms of topologically ringed spaces, the morphisms in (3.1.8.1) and (3.1.8.2) are morphisms of locally topologically ringed spaces. This follows from the fact that the structure sheaves are sheaves together with the fact that the category of topological rings admits all limits and limits commute with the forgetful functor to the category of rings ([Sta23, Lemma 0B23]).

The next two statements follow from the fact that formal schemes can be glued along open immersions.

3.1.9 (Morphisms to affine formal schemes)**.** Let $\mathfrak{X}$ be a formal scheme and let A be an admissible ring. Then, the map obtained by taking global sections

$$\left\{ \begin{array}{c} f : \mathfrak{X} \to \mathrm{Spf}\, A \,|\, f \text{ is a} \\ \text{morphism of formal schemes} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \varphi : A \to \Gamma\,(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \,|\, \varphi \text{ is a} \\ \text{morphism of topological rings} \end{array} \right\}. \tag{3.1.9.1}$$

is a bijection. Note that $\Gamma\,(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ need not be an admissible ring. Using Paragraph 3.1.4 (Anti-equivalence of categories), the proof of the analogous result for schemes applies here unaltered ([Gro60, Proposition 10.4.6, page 186]).

Lemma 3.1.10 (Fibre products of formal schemes)**.** *Let $\mathfrak{X} \to \mathfrak{Z}$ and $\mathfrak{Y} \to \mathfrak{Z}$ be morphisms of formal schemes. Then, their fibre product*

$$\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \tag{3.1.10.1}$$

exists. If $\mathfrak{X}$ and $\mathfrak{Y}$ are adic formal schemes, then so is $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$.

Proof. As in the case of schemes, with assistance from Paragraph 3.1.9 (Morphisms to affine formal schemes), existence of fibre products is readily reduced to the case where all formal schemes are affine. This is the result of [Gro60, Proposition 10.7.3, page 193]. But then the results follow from Paragraph 2.1.21 (Tensor product of admissible rings). ☐

3.2 Quasi-coherent pro-sheaves

This section is a globalisation of the concepts developed in §2.2 (Pro-modules). Affine locally, the theory of pro-modules is satisfactory. However, it does not globalise well. This is explained in the first paragraph. The remaining part of the section presents a workaround to define a sheaf of pro-modules. This section follows closely [Yas09, §2 - §3.1, page 2427-2435].

3.2.1 (Quasi-separated quasi-compact basis)**.** Pro-categories admit finite limits but do not admit arbitrary limits. As a result, it is not immediately clear what it means for a pre-sheaf of pro-Abelian groups to be a sheaf. Recall that a topological space is quasi-separated if the intersection of any two quasi-compact open subsets is quasi-compact. In [Yas09], the author defines pre-sheaves of pro-Abelian groups on a topological space X to be pre-sheaves of pro-Abelian groups on the basis of open subsets which are quasi-separated and quasi-compact. This has the following advantage: if U is a quasi-separated and quasi-compact open subset and $\{U_i\}_{i\in I}$ is a finite covering of U by quasi-compact and quasi-separated open subsets, $U_{ij} := U_i \cap U_j$ can be covered by finitely many quasi-separated and quasi-compact open subsets U_{ijk} as k varies in a finite indexing set K. Therefore, since all products are finite, it makes sense to ask whether the sequence

$$0 \to \mathscr{F}(U) \to \prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_{ijk}) \tag{3.2.1.1}$$

is exact in the category of pro-Abelian groups.

Definition 3.2.2 (Pro-sheaves)**.** Let $\mathfrak{X}$ be a formal scheme. A *pro-sheaf* on $\mathfrak{X}$ is a pre-sheaf $\mathscr{F}$ of pro-Abelian groups defined on quasi-separated and quasi-compact open subsets such that, for all finite collection of open subsets, the sequence of (3.2.1.1) is exact. Given pro-sheaves $\mathscr{F}$ and $\mathscr{G}$, a *morphism of pro-sheaves* f is a morphism of sheaves of pro-Abelian groups defined on quasi-separated and quasi-compact open subsets. A pro-sheaf is a *pro-sheaf of pro-modules* if, for every affine open subset $U \subseteq \mathfrak{X}$, $\mathscr{F}(U)$ is a pro-module over the admissible ring $\mathscr{O}_{\mathfrak{X}}(U)$ in a functorial way. Given pro-sheaves of pro-modules $\mathscr{F}$ and $\mathscr{G}$, a *morphism of pro-sheaves of pro-modules* f is a morphism of pro-sheaves such that for every affine open subset $U \subseteq \mathfrak{X}$,

$$f(U) : \mathscr{F}(U) \to \mathscr{G}(U) \tag{3.2.2.1}$$

is a morphism of pro-modules over $\mathscr{O}_{\mathfrak{X}}(U)$.

In [Yas09], the author defines formal schemes in an alternative way and, in order to apply the results of *op. cit.*, it has to be checked that the notion of formal scheme therein employed is equivalent to the one used in this dissertation.

3.2.3 (Locally admissibly pro-ringed spaces)**.** In [Yas09], the author takes a different approach to formal schemes. In the terminology of *op. cit.* an *admissibly pro-ringed space* $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a topological space $\mathfrak{X}$ and a pro-sheaf of pro-Abelian groups $\mathscr{O}_{\mathfrak{X}}$ such that there exists a basis of open subsets U for which $\mathscr{O}_{\mathfrak{X}}(U)$ is an *admissible pro-ring.* In the terminology of this dissertation, the category of admissible pro-rings is the sub-category of the pro-completion of rings consisting of filtered limits along thickenings. By Lemma 2.1.17 (Admissible rings and pro-category of rings), this is equivalent to the category of admissible rings. Furthermore, the *reduced stalk* in $x \in \mathfrak{X}$ is defined to be

$$\mathscr{O}_{\mathfrak{X},x}^{\mathrm{red}} := \operatorname*{colim}_{U \ni x} \frac{\mathscr{O}_{\mathfrak{X}}(U)}{N_U}, \tag{3.2.3.1}$$

where the colimit is taken over open subsets U such that $\mathscr{O}_{\mathfrak{X}}(U)$ is an admissible ring and N_U is its topological nilradical. If this is a local ring, then this is a *locally admissibly pro-ringed space* (vid. [Yas09, Definition 2.8, page 2429]). The author defines an affine formal scheme to be an affine locally admissibly pro-ringed space (vid. [Yas09, Definition 2.15, page 2431]) and then proceeds to show that such category is dual to the category of admissible rings ([Yas09, Corollary 2.14, page 2431]). By Paragraph 3.1.3 (Affine formal schemes), both categories of affine formal schemes are dual to the category of admissible rings. Therefore, the two definitions of formal schemes are equivalent. As a result, a formal scheme $\mathfrak{X}$ will be a locally topologically ringed space and, at the same time, all the results of [Yas09] may be applied.

As in the case of schemes, a pro-module over an admissible ring A induces a pro-sheaf of pro-modules on $\operatorname{Spf} A$.

3.2.4 (Pro-sheaves induced by pro-modules)**.** Let A be an admissible ring and let M be a pro-module over A. The *pro-sheaf of pro-modules associated to M* on $\operatorname{Spf} A$, denoted by $M^{\triangle}$, is the pro-sheaf of pro-Abelian groups associated to the pre-sheaf on a basis

$$\operatorname{Spf} A_f \to M_{\{f\}}, \tag{3.2.4.1}$$

for all $f \in A$. By construction, there is a basis of affine open subsets of $\operatorname{Spf} A$ over which $M^{\triangle}$ is a pro-module over A. This shows that $M^{\triangle}$ is a pro-sheaf of pro-modules. In particular, $\mathscr{O}_{\mathfrak{X}} = A^{\triangle}$ is a pro-sheaf of pro-modules.

A quasi-coherent pro-sheaf is defined to be a pro-sheaf which is locally induced by a pro-module. This definition is motivated by the well-known characterisation of quasi-coherent sheaves on schemes and is already present in [McQ02, Intermission/Definition 5.2, page 11]. In [Yas09, Definition 3.2, 2434], the author uses the term *semicoherent promodule*.

Definition 3.2.5 (Quasi-coherent pro-sheaves)**.** Let $\mathfrak{X}$ be a formal scheme. A *quasi-coherent pro-sheaf* $\mathscr{F}$ is pro-sheaf of pro-modules on $\mathfrak{X}$ with the following property: for every $x \in \mathfrak{X}$, there exists an affine open neighbourhood $\operatorname{Spf} A \ni x$ such that

$$\mathscr{F}|_{\operatorname{Spf} A} = M^{\triangle}, \tag{3.2.5.1}$$

where $M = \Gamma(\operatorname{Spf} A, \mathscr{F})$. A *morphism of quasi-coherent pro-sheaves* is a morphism of pro-sheaves of pro-modules. The set of quasi-coherent pro-sheaves with their morphisms forms a category.

A very important feature of quasi-coherent pro-sheaves is that they are induced by a pro-module on *every* affine open subset.

3.2.6 (Quasi-coherent pro-sheaves over affine opens)**.** Let $\mathfrak{X}$ be a formal scheme and let $\mathscr{F}$ be a pro-sheaf over $\mathfrak{X}$. Any affine open formal subscheme U of $\mathfrak{X}$ is quasi-separated and quasi-compact. Therefore, if $\mathscr{F}$ is quasi-coherent, for every affine open subset U,

$$\mathscr{F}(U) = \Gamma\left(U, \mathscr{F}\right)^{\triangle}. \tag{3.2.6.1}$$

This is [Yas09, Proposition 3.4, page 2434]. As a result, the functor $\triangle$ induces an equivalence of categories between the category of pro-modules over A and the category of quasi-coherent pro-sheaves over X. The inverse functor is the global section functor. This is the result of [Yas09, Corollary 3.5, page 2435]. In particular a morphism of quasi-coherent pro-sheaves is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism) over every affine open subset.

This allows to show that the category of quasi-coherent pro-sheaves is Abelian.

3.2.7 (Abelian category of quasi-coherent pro-sheaves)**.** Let $\mathfrak{X}$ be a formal scheme. The category of quasi-coherent pro-sheaves over X is Abelian. Using Paragraph 3.2.6 (Quasi-coherent pro-sheaves over affine opens), the question is readily reduced to the affine case where the claim holds by Paragraph 2.2.8 (Category of pro-modules is Abelian). This is [Yas09, Corollary 3.6, page 2435].

3.3 Immersions of formal schemes

This section is a globalisation of the concepts developed in §2.3 (Pro-ideals). The aim of this section is to define closed immersions of formal schemes and prove some basic properties. Closed immersions are not defined as closed immersions of locally ringed spaces as these are not stable under composition. They are defined using the corresponding algebraic notion of weakly surjective morphism. Using the fact that the category of quasi-coherent pro-sheaves is Abelian, it is shown that most properties of closed immersions of schemes are valid in this more general setting. The material in this section is based on [Yas09, §3.5, page 2439].

3.3.1 (Sheaves of pro-ideals)**.** Let $\mathfrak{X}$ be a formal scheme. A sheaf of pro-ideals $\mathscr{I}$ is a quasi-coherent pro-sheaf together with a monomorphism

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathfrak{X}} \tag{3.3.1.1}$$

in the Abelian category of quasi-coherent pro-sheaves.

Definition 3.3.2 (Immersions)**.** A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ between formal schemes is an *open immersion* if it induces an open immersion of locally ringed spaces ([Sta23, Definition 01HE]). It is a *closed immersion* if f is a homeomorphism onto a closed subset and the associated morphism

$$f^{\#} : \mathscr{O}_{\mathfrak{Y}} \to f_*\mathscr{O}_{\mathfrak{X}} \tag{3.3.2.1}$$

is an epimorphism in the category of quasi-coherent pro-sheaves. The pro-ideal associated to f is the kernel of $f^{\#}$. It is an *immersion* if f factors in

$$f : \mathfrak{X} \xrightarrow{g} \mathfrak{Z} \xrightarrow{h} \mathfrak{Y}, \tag{3.3.2.2}$$

where g is a closed immersion and h is an open immersion.

Lemma 3.3.3 (Composition of immersions)**.** *The composition of two open immersions is an open immersion. The composition of two closed immersions is a closed immersion. The composition of two immersions is an immersion.*

Proof. The composition of two open immersions is clearly an open immersion.

The composition of closed immersions is a closed immersion since the composition of epimorphisms is an epimorphism and the composition of homeomorphisms onto closed subsets is a homeomorphism onto a closed subset.

Finally, the composition of two immersions is an immersion. Indeed, as in the proof of [Sta23, Lemma 02V0], the claim reduces to the proof that the composition of two closed immersions is a closed immersion. □

Lemma 3.3.4 (Base change of immersions)**.** *The base change of an open immersion is an open immersion. The base change of a closed immersion is a closed immersion. The base change of an immersion is an immersion.*

Proof. It is clear that the base change of an open immersion is an open immersion.

The case of closed immersion is proved in [Yas09, Proposition 3.27, page 2440].

The base change of an immersion is an immersion by the previous two claims and Lemma 3.3.3 (Composition of immersions). □

The next paragraph shows that closed immersions of affine formal schemes are precisely weakly surjective morphisms of admissible rings.

3.3.5 (Closed immersions of affine formal schemes)**.** Suppose that $f : \mathfrak{X} := \operatorname{Spf} B \to \operatorname{Spf} A =: \mathfrak{Y}$ is a closed immersion of affine formal schemes. Since the functor $\triangle$ induces an equivalence of categories (Paragraph 3.2.6 (Quasi-coherent pro-sheaves over affine opens)), $f^{\#} : \mathscr{O}_{\mathfrak{Y}} \to f_*\mathscr{O}_{\mathfrak{X}}$ is an epimorphism of quasi-coherent pro-sheaves if and only if $f^{\#} : A \to B$ is an epimorphism in the category of pro-modules over A. This means precisely that $f^{\#} : A \to B$ is weakly surjective.

Lemma 3.3.6 (Closed immersions are local on the target)**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes. Suppose that there is an affine open cover $\{V_i\}_{i\in I}$ of $\mathfrak{Y}$ such that the base change*

$$f_i : U_i := V_i \times_{\mathfrak{Y}} \mathfrak{X} \to V_i \tag{3.3.6.1}$$

is a closed immersion for all $i \in I$. Then f is a closed immersion.

Proof. It is readily checked that f is a homeomorphism onto a closed subset. Therefore, showing that f is a closed immersion amounts to showing that the morphism

$$f^{\#} : \mathscr{O}_{\mathfrak{Y}} \to f_* \mathscr{O}_{\mathfrak{X}} \tag{3.3.6.2}$$

is an epimorphism in the category of quasi-coherent pro-sheaves. By Paragraph 3.2.6 (Quasi-coherent pro-sheaves over affine opens), it suffices to check that (3.3.6.2) is an epimorphism over an affine open cover of $\mathfrak{Y}$. Let $\{V_i\}_{i \in I}$ be such open cover. Since $U_i \subseteq \mathfrak{X}$ is an open immersion of formal schemes, there is a morphism

$$\mathscr{O}_{\mathfrak{Y}}(V_i) = \mathscr{O}_{V_i}(V_i) \to (f_i)_* \mathscr{O}_{U_i}(V_i) = f_* \mathscr{O}_{\mathfrak{X}}(V_i) = \mathscr{O}_{\mathfrak{X}}(U_i). \tag{3.3.6.3}$$

Since $U_i \to V_i$ is a closed immersion, the morphism in (3.3.6.3) is an epimorphism. $\square$

The next lemma proves that the pre-image of an affine open subset under a closed immersion is an affine open subset.

Lemma 3.3.7 (Closed immersions are affine)**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a closed immersion. Then f is an affine morphism.*

The proof consists of three steps. The statement is firstly shown in the case where the domain is a scheme and the codomain is a formal scheme. In this case, the proof is the same as for schemes. It does not work immediately for arbitrary formal schemes since, in the notation of the proof below, B is not, a priori, an admissible ring. Thus, it does not make sense to show that $\mathfrak{X} = \operatorname{Spf} B$. In the second step, it is shown that the topological space of $\mathfrak{X}$ is quasi-separated and quasi-compact. This implies that $\mathfrak{X}$ is a filtered colimit of schemes. This is used in the third step to reduce the proof to the case where $\mathfrak{X} = X$ is a usual scheme.

Proof of Lemma 3.3.7 (Closed immersions are affine). Assume $\mathfrak{Y} = \operatorname{Spf} A$ is an affine formal scheme. It has to be shown that $\mathfrak{X}$ is affine.

Firstly, suppose that $\mathfrak{X} = X$ is a usual scheme. Since $\mathfrak{Y}$ is affine, Paragraph 3.1.9 (Morphisms to affine formal schemes) implies that $X \to \mathfrak{Y}$ corresponds uniquely to a morphism of topological rings

$$A \to \Gamma(X, \mathscr{O}_X) =: B, \tag{3.3.7.1}$$

where, since X is quasi-compact, B is a discrete ring. Since $\mathscr{O}_{\mathfrak{Y}} \to f_* \mathscr{O}_X$ is an epimorphism, Paragraph 3.2.6 (Quasi-coherent pro-sheaves over affine opens) implies that (3.3.7.1) is an epimorphism in the category of pro-modules over A. Since B is discrete, it is surjective. It is shown that $X = \operatorname{Spec} B$. Since the functor $\triangle$ is an equivalence of categories (Paragraph 3.2.6 (Quasi-coherent pro-sheaves over affine opens)), by construction,

$$f_* \mathscr{O}_X = B^{\triangle}. \tag{3.3.7.2}$$

as pro-sheaves over Y. Since X is a scheme and B is a discrete ring, there is no harm in assuming these are sheaves over the topological space of $\mathfrak{Y}$. Localising (3.3.7.2) in the stalk over points in $\mathfrak{Y}$ gives a bijective correspondence between points of X and points of $\operatorname{Spec} B$. Indeed, if $y \in Y$ corresponds to the open prime ideal $\mathfrak{p} \subseteq A$,

$$\begin{aligned} y \in X &\leftrightarrow (f_* \mathscr{O}_X)_y \neq 0 \\ &\leftrightarrow (f_* \mathscr{O}_{\operatorname{Spec} B})_y \neq 0 \\ &\leftrightarrow \ker (A \to B) \subseteq \mathfrak{p}. \end{aligned} \tag{3.3.7.3}$$

But then there is a homeomorphism between X and $\operatorname{Spec} B$ and (3.3.7.2) gives an isomorphism of structure sheaves. Therefore $X = \operatorname{Spec} B$.

Now suppose that $\mathfrak{X}$ is an arbitrary formal scheme. It is shown that it is quasi-separated and quasi-compact. By definition, $\mathfrak{X}$ is a closed subspace of a quasi-compact space $\operatorname{Spf} A$, hence it is quasi-compact. It is shown that $\mathfrak{X}$ is quasi-separated. Let V_1 and V_2 be two quasi-compact open subsets of $\mathfrak{X}$. It has to be shown that $V_1 \cap V_2$ is quasi-compact. Since $\mathfrak{Y}$ is, by assumption, quasi-separated, it suffices to show that, for any quasi-compact open subset V of $\mathfrak{X}$, there exists a quasi-compact open subset U of $\mathfrak{Y}$ such that $U \cap \mathfrak{X} = V$. By definition of subspace topology, there exists an open subset U' such that $U' \cap \mathfrak{X} = V$. Since $\mathfrak{Y}$ is a formal scheme, it has a basis of quasi-separated quasi-compact open subsets ([Yas09, Proposition 2.18, page 2432]). Hence, $U' = \bigcup_{i \in I} U_i$ for some quasi-compact open subsets U_i, for $i \in I$. Since V is quasi-compact, there are finitely many open subsets U_i such that

$$\bigcup_{i \in J} (U_i \cap \mathfrak{X}) = V, \tag{3.3.7.4}$$

where J is a finite subset of I. Set U to be the union of the open subsets U_i for all $i \in J$. Since U is a finite union of quasi-compact open subsets, it is quasi-compact and, by construction, $U \cap \mathfrak{X} = V$. It follows that $\mathfrak{X}$ is quasi-separated.

Therefore [Yas09, Proposition 3.32, page 2442] may be applied to show that

$$\mathfrak{X} = \operatorname*{colim}_{\lambda \in \Lambda} X_\lambda \tag{3.3.7.5}$$

is a colimit of schemes along thickenings. Pick $\lambda \in \Lambda$. It is shown that $X_\lambda \to \mathfrak{X}$ is a closed immersion of formal schemes. By Lemma 3.3.6 (Closed immersions are local on the target), it suffices to work affine locally on $\mathfrak{X}$. In this case,

$$\mathfrak{X} = \operatorname{Spf} B := \operatorname{Spf} \left(\lim_{\lambda \in \Lambda} B_\lambda \right) \tag{3.3.7.6}$$

is a colimit of affine schemes along thickenings, hence the morphism $B \to B_\lambda$ is surjective. Therefore $X_\lambda \to \mathfrak{X}$ is a closed immersion of formal schemes.

Finally $X_\lambda \to \mathfrak{X} \to \mathfrak{Y}$ is a closed immersion of formal schemes (Lemma 3.3.3 (Composition of immersions)). By the previous paragraph, X_λ is an affine scheme. Now Paragraph 3.1.7 (Colimit of schemes along thickenings) implies that $\mathfrak{X}$ is an affine formal scheme. □

A closed immersion of formal schemes between usual schemes is precisely a closed immersion of schemes.

3.3.8 (Closed immersions of schemes)**.** Suppose that $f : \mathfrak{X} := X \to Y =: \mathfrak{Y}$ is a closed immersion of formal schemes where X and Y are schemes. Then it is in fact a closed immersion of schemes. Indeed, by Lemma 3.3.6 (Closed immersions are local on the target) and Lemma 3.3.7 (Closed immersions are affine), $X := \operatorname{Spec} B$ and $Y := \operatorname{Spec} A$ may be assumed affine schemes. By Paragraph 3.3.5 (Closed immersions of affine formal schemes), this corresponds to a weakly surjective morphism $A \to B$. By Paragraph 2.3.6 (Weakly surjective morphisms to discrete rings), $A \to B$ is surjective. Hence f is a closed immersion.

Now that the hard work has been completed, several of the usual statements for schemes generalise to formal schemes.

Lemma 3.3.9 (Diagonal is an immersion)**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes. Then the diagonal*

$$\Delta_f : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \tag{3.3.9.1}$$

is an immersion.

Proof. The formal scheme $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is covered by open formal subschemes of the form $U \times_V U$, where $U = \operatorname{Spf} B \subseteq \mathfrak{X}$ and $V = \operatorname{Spf} A \subseteq \mathfrak{Y}$ are affine open formal subschemes such that $f(U) \subseteq V$. Then, $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is covered by affine open formal subschemes of the form $\operatorname{Spf}\left(B \hat{\otimes}_A B\right)$. Therefore, by Lemma 3.3.6 (Closed immersions are local on the target), it suffices to check that

$$\operatorname{Spf} B \to \operatorname{Spf}\left(B \hat{\otimes}_A B\right) \tag{3.3.9.2}$$

is a closed immersion. By Paragraph 3.3.5 (Closed immersions of affine formal schemes), this is equivalent to showing that

$$B \hat{\otimes}_A B \to B \tag{3.3.9.3}$$

is a weakly surjective morphism of admissible rings. Since the composition

$$B \otimes_A B \to B \hat{\otimes}_A B \to B \tag{3.3.9.4}$$

is surjective, $B \hat{\otimes}_A B \to B$ is surjective, therefore weakly surjective. □

Lemma 3.3.10 (Factorisation of immersion)**.** *Let*

$$f : \mathfrak{X} \xrightarrow{g} \mathfrak{Z} \xrightarrow{h} \mathfrak{Y} \tag{3.3.10.1}$$

be morphisms of formal schemes. If f is an immersion, then g is an immersion.

Proof. For schemes, this is part (1) of [Sta23, Lemma 07RK]. Note however that the proof also works for formal schemes. Indeed, the proof relies on the fact that the composition of two immersions is an immersion (Lemma 3.3.3 (Composition of immersions)), the fact that the base change of an immersion is an immersion (Lemma 3.3.4 (Base change of immersions)) and on [Sta23, Lemma 01KR]. The latter also holds for formal schemes, since it uses general category theory and the fact that the diagonal morphism is an immersion (Lemma 3.3.9 (Diagonal is an immersion)). □

Lemma 3.3.11 (Immersions are monomorphisms)**.** *An immersion of formal schemes is a monomorphism in the category of formal schemes.*

Proof. Since the composition of monomorphisms is a monomorphism in any category, it suffices to show that open immersions are monomorphisms and closed immersions are monomorphisms.

Open immersions are monomorphisms in the (larger) category of locally ringed spaces ([Sta23, Lemma 01HI]).

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a closed immersion of formal schemes and let

$$\mathfrak{T} \underset{h}{\overset{g}{\rightrightarrows}} \mathfrak{X} \xrightarrow{f} \mathfrak{Y} \tag{3.3.11.1}$$

be a commutative diagram. By definition, f is injective on points, hence g and h are set-theoretically the same morphism. Therefore, to show that $g = h$, it suffices to prove that there exists an affine open cover $\{V_j = \operatorname{Spf} B_j\}_{j \in J}$ of $\mathfrak{X}$ and, for all $j \in J$, an affine open cover $\{U_{ij} = \operatorname{Spf} C_{ij}\}_{i \in I}$ of $g^{-1}(V_j) = h^{-1}(V_j)$ such that the induced morphisms

$$B_j \underset{h_{ij}^{\#}}{\overset{g_{ij}^{\#}}{\rightrightarrows}} C_{ij} \tag{3.3.11.2}$$

are equal for all $i \in I$ and $j \in J$. This follows from the description of morphisms between formal schemes in Paragraph 3.1.8 (Glueing formal schemes). Now, let $\{W_j =: \operatorname{Spf} A_j\}_{j \in J}$ be an affine open cover of $\mathfrak{Y}$. Set $V_j := f^{-1}(W_j)$ for $j \in J$. Since closed immersions are affine (Lemma 3.3.7 (Closed immersions are affine)), this is an affine open cover of $\mathfrak{X}$. But now, for all $i \in I$ and $j \in J$, there are morphisms of admissible rings

$$A_j \xrightarrow{f_j^{\#}} B_j \underset{h_{ij}^{\#}}{\overset{g_{ij}^{\#}}{\rightrightarrows}} C_{ij}, \tag{3.3.11.3}$$

where $f_j^{\#}$ is a weakly surjective morphism of admissible rings (Paragraph 3.3.5 (Closed immersions of affine formal schemes)) for all $j \in J$. By Lemma 2.3.8 (Weakly surjective morphisms are epimorphisms), $f_j^{\#}$ is an epimorphism of admissible rings, hence $g_{ij}^{\#} = h_{ij}^{\#}$ and the lemma follows. □

3.4 Thickenings of formal schemes

This section is a globalisation of the concepts developed in §2.4 (Thickenings of admissible rings). After the natural definition of thickenings, the standard properties are proved. Since thickenings are closed immersions, these mostly follow from the previous section. Finally, the notion of regular thickening is globalised.

Definition 3.4.1 (Thickenings of formal schemes)**.** A *thickening* $f : \mathfrak{X} \to \mathfrak{Y}$ of formal schemes is a closed immersion inducing a homeomorphism of topological spaces.

3.4.2 (Thickenings are surjective closed immersion)**.** Equivalently, a thickening of formal schemes can be defined as a surjective closed immersion. Indeed, a surjective closed immersion is a bijective closed morphism, hence a homeomorphism.

Lemma 3.4.3 (Composition of thickenings)**.** *The composition of two thickenings is a thickening.*

Proof. Being surjective is preserved by composition. Being a closed immersion is preserved by composition (Lemma 3.3.3 (Composition of immersions)). □

Lemma 3.4.4 (Base change of thickenings)**.** *The base change of a thickening is a thickening.*

Proof. Being surjective is preserved by base change. Being a closed immersion is preserved by base change (Lemma 3.3.4 (Base change of immersions)). □

3.4.5 (Thickenings of affine formal schemes)**.** It follows immediately from Paragraph 3.3.5 (Closed immersions of affine formal schemes) and the definition of thickening of admissible rings, that a thickening of affine formal schemes is precisely a thickening of admissible rings.

3.4.6 (Thickenings are local on the target)**.** Let

$$f : \mathfrak{X} \to \mathfrak{Y} \tag{3.4.6.1}$$

be a morphism of formal schemes. Suppose that there are affine open subsets V_i of $\mathfrak{Y}$ such that the base change

$$f_i : U_i := V_i \times_{\mathfrak{Y}} \mathfrak{X} \to V_i \tag{3.4.6.2}$$

is a thickening for all $i \in I$. Then f is a thickening. Indeed, it is clear that f has to be surjective. By Lemma 3.3.6 (Closed immersions are local on the target), f is a closed immersion.

3.4.7 (Thickenings are affine morphisms)**.** Since a thickening is a closed immersion and a closed immersion is an affine morphism (Lemma 3.3.7 (Closed immersions are affine)), a thickening is an affine morphism.

3.4.8 (Thickenings of schemes)**.** It follows immediately from Paragraph 3.3.8 (Closed immersions of schemes) and the definition of thickenings of schemes, that a thickening of formal schemes between schemes is precisely a thickening of schemes.

The next lemma is frequently used to show that an infinitesimal groupoid on an affine scheme is an affine formal scheme.

Lemma 3.4.9 (Image of thickening is affine)**.** *Let X be a scheme and let $f : X \to \mathfrak{Y}$ be a thickening to a formal scheme $\mathfrak{Y}$. Suppose that X is affine, then $\mathfrak{Y}$ is affine.*

The proof of this lemma is a reduction to the case where $\mathfrak{Y}$ is a usual scheme. The assumptions imply that $\mathfrak{Y}$ is a filtered colimit of schemes and it suffices to show that f factors through a scheme in the cofiltered system defining $\mathfrak{Y}$. This is shown using the definition of morphism in the pro-category of rings and the fact that $\mathfrak{Y}$ is quasi-compact.

Proof of Lemma 3.4.9 (Image of thickening is affine). Note that, since f is a homeomorphism and X is quasi-separated and quasi-compact, so is $\mathfrak{Y}$. Therefore [Yas09, Proposition 3.32, page 2442] applies to show that

$$\mathfrak{Y} = \operatorname*{colim}_{\mu \in M} Y_\mu. \tag{3.4.9.1}$$

is a filtered colimit of schemes.

It is shown that there exists a $\mu \in M$ such that f factors through

$$f_\mu : X \to Y_\mu. \tag{3.4.9.2}$$

To this end, cover $\mathfrak{Y}$ by finitely many affine open subsets $V := \operatorname{Spf} A$ indexed by a finite set I, where

$$A := \lim_{\mu \in M} A_\mu. \tag{3.4.9.3}$$

Let $U := f^{-1}(V) \subseteq X$. Since thickenings are affine morphisms (Paragraph 3.4.7 (Thickenings are affine morphisms)), $U = \operatorname{Spec} B$ for some discrete ring B. This induces a morphism of admissible rings $A \to B$. Since the category of admissible rings embeds fully faithfully in the pro-category of rings (Lemma 2.1.17 (Admissible rings and pro-category of rings)), by definition of morphism in the latter category, there exists a μ_i and a factorisation

$$A \to A_{\mu_i} \to B. \tag{3.4.9.4}$$

Let μ be an upper bound in M of the finite set $\{\mu_i\}_{i \in I}$. Then the morphism $X \to \mathfrak{Y}$ factors through Y_μ over an affine open cover. By the description of morphisms between formal schemes in Paragraph 3.1.8 (Glueing formal schemes), there is a factorisation $f_\mu : X \to Y_\mu$.

Next, it is shown that f_μ is a thickening of schemes. By definition of formal spectrum of a morphism, f_μ induces a homeomorphism of topological spaces. By Lemma 3.3.10 (Factorisation of immersion), f_μ is an immersion. Since f_μ is surjective, it has to be a closed immersion.

Finally, [Sta23, Lemma 06AD] implies that Y_μ is an affine scheme. By Paragraph 3.1.7 (Colimit of schemes along thickenings), $\mathfrak{Y}$ is an affine formal scheme. □

This is the global counterpart to the notion of a regular thickening of admissible rings.

Definition 3.4.10 (Regular thickenings of formal schemes)**.** Let $f : X \to \mathfrak{Y}$ be a thickening of formal schemes where X is a scheme and $\mathfrak{Y}$ is an adic formal scheme. Let $x \in X$. Then f is a *regular thickening* in $x \in X$, if there exists an affine open neighbourhood $V = \operatorname{Spf} A \ni f(x)$ such that the induced thickening of admissible rings

$$A \to B, \tag{3.4.10.1}$$

where $\operatorname{Spec} B = \mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spf} A$, is a regular thickening of adic rings. If f is regular thickening in all $x \in X$, then f is a *regular thickening.*

Finally, it is shown that, if a thickening is regular over an affine open cover, it is regular over any affine open subset.

3.4.11 (Being a regular thickening is a local property)**.** Let $f : X \to \mathfrak{Y}$ be a regular thickening of affine formal schemes where X is a scheme and $\mathfrak{Y}$ is an adic formal scheme. This corresponds to a thickening of admissible rings $A \to B$ where A is an adic ring and B is a discrete ring (Paragraph 3.4.5 (Thickenings of affine formal schemes)). Then it is a regular thickening of adic rings. This follows from the fact that being finitely generated is an affine local property on A, being projective is an affine local property on B and being an isomorphism is an affine local property on B.

3.5 Infinitesimal neighbourhoods

This section is a globalisation of the concepts developed in §3.5 (Infinitesimal neighbourhoods). One expects that the universal property of affine infinitesimal neighbourhoods can be globalised since it is a categorical limit. This is done in steps and it relies on the fact that formal schemes are locally ringed spaces in order to glue formal schemes as well as morphisms of formal schemes.

Definition 3.5.1 (Infinitesimal neighbourhoods of immersions)**.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an immersion of formal schemes. The *infinitesimal neighbourhood*

$$\hat{f} : \mathfrak{X} \to \hat{\mathfrak{Y}}_{\mathfrak{X}} \tag{3.5.1.1}$$

is a thickening of formal schemes satisfying the following universal property: for any solid commutative diagram

$$\begin{tikzcd} \mathfrak{W} \arrow[rr, "f'"] \arrow[d, "g"] & & \mathfrak{Z} \arrow[d, "g'"] \arrow[dl, dashed] \\ \mathfrak{X} \arrow[r, "\hat{f}"] \arrow[rr, bend right, "f"'] & \hat{\mathfrak{Y}}_{\mathfrak{X}} \arrow[r] & \mathfrak{Y} \end{tikzcd} \tag{3.5.1.2}$$

where $f' : \mathfrak{W} \to \mathfrak{Z}$ is a thickening of formal schemes, the dashed morphism making the diagram commute exists and is unique. By general category theory, if the infinitesimal neighbourhood exists, it is unique.

Firstly, it is shown that the immersion of formal schemes may be assumed closed.

Lemma 3.5.2 (Infinitesimal neighbourhoods and open immersions)**.** *Let*

$$f : \mathfrak{X} \xrightarrow{f'} \mathfrak{Y}' \xrightarrow{\iota} \mathfrak{Y} \tag{3.5.2.1}$$

be immersions of formal schemes where ι is an open immersion. Then the infinitesimal neighbourhood of f exists if and only if the infinitesimal neighbourhood of f' exists. Furthermore, whenever they do exists, they are naturally isomorphic.

Proof. It is shown that the dashed morphisms and the dotted morphisms making the diagram

$$\begin{tikzcd}
\mathfrak{W} \arrow[rr, "f'"] \arrow[d] & & \mathfrak{Z} \arrow[dl, dashed] \arrow[d, dotted] \\
\mathfrak{X} \arrow[r, "\hat{f}"] \arrow[rr, bend right, "f"'] & \mathfrak{Y}' \arrow[r, "\iota"] & \mathfrak{Y}
\end{tikzcd} \tag{3.5.2.2}$$

commute are in bijective correspondence, where f' is a thickening. If this holds, then the lemma is a formal categorical consequence.

Clearly, given a dashed morphism, post-composition with ι gives a dotted morphism. Since ι is an open immersion, it is a monomorphism (Lemma 3.3.11 (Immersions are monomorphisms)), therefore this correspondence is injective. It is shown it is surjective. Suppose a dotted morphism is given. By commutativity of the diagram and the fact that f' is a homeomorphism, the image of $\mathfrak{Z}$ under the dotted morphism is contained in the image of $\mathfrak{Y}'$ under ι. Since ι is an open immersion, the dotted morphism factors through the dashed morphism, proving the claim. $\square$

Next, it is shown that infinitesimal neighbourhoods commute with fibre products, when they exist. This is not surprising as they are both categorical limits.

Lemma 3.5.3 (Infinitesimal neighbourhoods and fibre products)**.** *Let*

$$\begin{tikzcd}
\mathfrak{X}' \arrow[r, "f'"] \arrow[d] & \mathfrak{Y}' \arrow[d] \\
\mathfrak{X} \arrow[r, "f"] & \mathfrak{Y}
\end{tikzcd} \tag{3.5.3.1}$$

be a Cartesian diagram of formal schemes where f is an immersion. Suppose that the infinitesimal neighbourhood $\hat{\mathfrak{Y}}$ of f exists, then f' is an immersion and its formal neighbourhood $\hat{\mathfrak{Y}}'$ exists. This is given by the fibre product

$$\hat{\mathfrak{Y}}' = \hat{\mathfrak{Y}} \times_{\mathfrak{Y}} \mathfrak{Y}'. \tag{3.5.3.2}$$

Proof. Firstly, f' is an immersion by Lemma 3.3.4 (Base change of immersions). Now the lemma becomes completely formal. Consider the solid commutative diagram of formal schemes

$$
\begin{tikzcd}
\mathfrak{W} \arrow[rr, "f''"] \arrow[d] & & \mathfrak{Z} \arrow[d] \arrow[dl, dashed] \\
\mathfrak{X}' \arrow[r] \arrow[d] & \hat{\mathfrak{Y}}' \arrow[r] \arrow[d] & \mathfrak{Y}' \arrow[d] \\
\mathfrak{X} \arrow[r] & \hat{\mathfrak{Y}} \arrow[r] & \mathfrak{Y},
\end{tikzcd}
\tag{3.5.3.3}
$$

where the two bottom squares are Cartesian and f'' is a thickening of formal schemes. Suppose that $\hat{\mathfrak{Y}}$ is the infinitesimal neighbourhood of f, it is shown that the fibre product $\hat{\mathfrak{Y}}'$ is the infinitesimal neighbourhood of f'. To this end, it suffices to show that the dashed morphism exists and is unique. By the universal property of the infinitesimal neighbourhood $\hat{\mathfrak{Y}}$, there exists a unique morphism $\mathfrak{Z} \to \hat{\mathfrak{Y}}$. By the universal property of the fibre product $\hat{\mathfrak{Y}}'$, there exists a unique dashed morphism $\mathfrak{Z} \to \hat{\mathfrak{Y}}'$. □

Next, it is shown that the affine infinitesimal neighbourhood of an immersion of affine formal schemes is the infinitesimal neighbourhood in the category of formal schemes.

Lemma 3.5.4 (Affine infinitesimal neighbourhoods in formal schemes)**.** *Let $f : \mathfrak{X} = \operatorname{Spf} B \to \operatorname{Spf} A = \mathfrak{Y}$ be a closed immersion of affine formal schemes. Let $\hat{A}$ be the affine infinitesimal neighbourhood of the weakly surjective morphism*

$$A \to \hat{A} \to B \tag{3.5.4.1}$$

from Construction 2.5.2 (Affine infinitesimal neighbourhoods). Let $\hat{\mathfrak{Y}} = \operatorname{Spf} \hat{A}$. Then the factorisation

$$f : \mathfrak{X} \to \hat{\mathfrak{Y}} \to \mathfrak{Y}. \tag{3.5.4.2}$$

is the infinitesimal neighbourhood of f in the category of formal schemes.

Proof. It has to be shown that the dashed morphism in Diagram (3.5.1.2) exists and is unique. Suppose firstly that $\mathfrak{Z}$ is an affine formal scheme. Since f' is a thickening, it is an affine morphism (Paragraph 3.4.7 (Thickenings are affine morphisms)), hence $\mathfrak{W}$ is also an affine formal scheme. Therefore all the schemes are affine and the question readily reduces to Construction 2.5.2 (Affine infinitesimal neighbourhoods). By Lemma 2.5.3 (Affine infinitesimal neighbourhoods), the dashed morphism exists and is unique.

Next cover $\mathfrak{Z}$ by affine open formal subschemes $\mathfrak{Z}_i$, for $i \in I$. By the previous paragraph there exists a unique morphism $\mathfrak{Z}_i \to \hat{\mathfrak{Y}}$ for all $i \in I$. Since the morphisms are unique, they have to agree on every intersection. Indeed, every point in the intersection has an affine open neighbourhood which admits a unique morphism to $\hat{\mathfrak{Y}}$. By the description of morphisms between formal schemes in Paragraph 3.1.8 (Glueing formal schemes), the morphisms glue to give a unique morphism $\mathfrak{Z} \to \hat{\mathfrak{Y}}$. □

After the preliminary results, the main construction follows from the affine case.

Construction 3.5.5 (Infinitesimal neighbourhood of immersions)**.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an immersion of formal schemes. The infinitesimal neighbourhood

$$f : \mathfrak{X} \xrightarrow{\hat{f}} \hat{\mathfrak{Y}} \to \mathfrak{Y} \tag{3.5.5.1}$$

is constructed and it is shown that $\hat{\mathfrak{Y}} \to \mathfrak{Y}$ is an immersion of formal schemes.

Firstly, by Lemma 3.5.2 (Infinitesimal neighbourhoods and open immersions), f may be assumed a closed immersion. Indeed, the resulting infinitesimal neighbourhood would be naturally isomorphic.

Now cover $\mathfrak{Y}$ by affine open formal subschemes $\mathfrak{Y}_i$, for $i \in I$. Let

$$f_i : \mathfrak{X}_i \to \mathfrak{Y}_i \tag{3.5.5.2}$$

be the base change of f by the open immersion $\mathfrak{Y}_i \to \mathfrak{Y}$. By Lemma 3.3.4 (Base change of immersions), f_i is a closed immersion. Since $\mathfrak{Y}_i$ is affine, so is $\mathfrak{X}_i$ (Lemma 3.3.7 (Closed immersions are affine)). In this case, Lemma 3.5.4 (Affine infinitesimal neighbourhoods in formal schemes) implies that the infinitesimal neighbourhood $\hat{\mathfrak{Y}}_i$ of f_i exists.

Now let $\mathfrak{Y}_{ijk}$ denote the triple intersection of $\mathfrak{Y}_i$, $\mathfrak{Y}_j$ and $\mathfrak{Y}_k$ for $i, j, k \in I$. If the indices are repeated, the double intersection $\mathfrak{Y}_{ijj}$ is simply written as $\mathfrak{Y}_{ij}$. Let $f_{ijk} : \mathfrak{X}_{ijk} \to \mathfrak{Y}_{ijk}$ be the base change of f_i by the open immersion $\mathfrak{Y}_{ijk} \to \mathfrak{Y}_i$. By Lemma 3.5.3 (Infinitesimal neighbourhoods and fibre products), the infinitesimal neighbourhood $\hat{\mathfrak{Y}}_{ijk}$ of f_{ijk} exists and is an open formal subscheme of $\mathfrak{Y}_i$ (Lemma 3.3.4 (Base change of immersions)). Now, the collection of affine formal schemes $\{\mathfrak{Y}_i\}_{i \in I}$ is glued with glueing morphisms $\mathfrak{Y}_{ij} \to \mathfrak{Y}_i$, for any $i, j \in I$. Using the universal property of infinitesimal neighbourhoods, it is clear that $\hat{\mathfrak{Y}}_{ij}$ and $\hat{\mathfrak{Y}}_{ji}$ are uniquely isomorphic, both being infinitesimal neighbourhoods of $f_{ij} = f_{ji}$. By the same argument,

$$\hat{\mathfrak{Y}}_{ijk} = \hat{\mathfrak{Y}}_{kij} = \hat{\mathfrak{Y}}_{jki} \tag{3.5.5.3}$$

By Paragraph 3.1.8 (Glueing formal schemes), the affine formal schemes can be glued to obtain a formal scheme $\hat{\mathfrak{Y}}$ and a factorisation

$$f : \mathfrak{X} \xrightarrow{\hat{f}} \hat{\mathfrak{Y}} \to \mathfrak{Y}. \tag{3.5.5.4}$$

By Lemma 3.3.6 (Closed immersions are local on the target), whether $\hat{\mathfrak{Y}} \to \mathfrak{Y}$ is a closed immersion of formal schemes may be checked locally. This follows from Lemma 2.5.3 (Affine infinitesimal neighbourhoods).

Finally, it is verified that $\hat{\mathfrak{Y}}$ satisfies the universal property of infinitesimal neighbourhoods. For any $i, j \in I$, let $\mathfrak{Z}_{ij} \to \mathfrak{Y}_{ij}$ be the the base change of $\mathfrak{Z} \to \mathfrak{Y}$ by the open immersion $\mathfrak{Y}_{ij} \to \mathfrak{Y}$. By the universal property of $\hat{\mathfrak{Y}}_{ij}$, there exists a unique compatible morphism $\mathfrak{Z}_{ij} \to \hat{\mathfrak{Y}}_{ij}$. Therefore, the morphisms $\mathfrak{Z}_{ij} \to \hat{\mathfrak{Y}}$, for $i, j \in I$, glue to give a unique morphism $\mathfrak{Z} \to \hat{\mathfrak{Y}}$.

This is the global counterpart to Lemma 2.5.4 (Infinitesimal neighbourhoods of rings).

Lemma 3.5.6 (Infinitesimal neighbourhoods of schemes)**.** *Let $f : X \to Y$ be an immersion of schemes and let*

$$f : X \xrightarrow{\hat{f}} \mathfrak{Y} \to Y \tag{3.5.6.1}$$

be the infinitesimal neighbourhood of f.

1. *If f is locally of finite presentation, then $\mathfrak{Y}$ is an adic formal scheme.*
2. *If f is locally of finite presentation and X is locally Noetherian, $\mathfrak{Y}$ is an adic locally Noetherian formal scheme.*
3. *If Y is locally Noetherian and f is a quasi-regular immersion, $\hat{f}$ is a regular thickening.*

Proof. By Lemma 3.5.2 (Infinitesimal neighbourhoods and open immersions), f may be assumed a closed immersion. Let $\mathscr{K}$ be the sheaf of ideals of the closed immersion f. Since closed immersions are affine and all statements are local, f may be assumed a closed immersion of affine schemes. Therefore, let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and $\mathscr{K} = K^{\sim}$ for some discrete rings A and B and an ideal K of A such that $B = A/K$.

1. The assumptions imply that K is finitely generated. By part (1) of Lemma 2.5.4 (Infinitesimal neighbourhoods of rings), $\hat{A}$ is an adic ring. Therefore, $\mathfrak{Y}$ is an adic formal scheme.
2. The assumptions imply that K is finitely generated and B is Noetherian. By part (2) of Lemma 2.5.4 (Infinitesimal neighbourhoods of rings), $\hat{A}$ is an adic Noetherian ring. Therefore, $\mathfrak{Y}$ is an adic locally Noetherian formal scheme.
3. The assumptions imply that A is Noetherian and that K is locally generated by a quasi-regular sequence. By part (3) of Lemma 2.5.4 (Infinitesimal neighbourhoods of rings), $\hat{A} \to B$ is a regular thickening. Since being a regular thickening is a property local on the target (Paragraph 3.4.11 (Being a regular thickening is a local property)), $\hat{f}$ is a regular thickening. □

The final example shows that any morphism of schemes induces a morphism of infinitesimal neighbourhoods.

Example 3.5.7 (Completion of morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes. If $x \in X$ and $y := f(x) \in Y$, there is an induced morphism of local schemes

$$f_x : X_x := \operatorname{Spec} \mathscr{O}_{X,x} \to \operatorname{Spec} \mathscr{O}_{Y,f(x)} =: Y_y \tag{3.5.7.1}$$

Let $\mathfrak{X}_x$ and $\mathfrak{Y}_y$ denote the infinitesimal neighbourhoods of the closed immersions $\operatorname{Spec}\kappa(x) \to X_x$ and $\operatorname{Spec}\kappa(y) \to Y_y$ respectively. By construction, there is a solid commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\kappa(x) \arrow[r] \arrow[d] & \mathfrak{X}_x \arrow[r] \arrow[d, dashed] & X_x \arrow[r] \arrow[d, "f_x"] & X \arrow[d, "f"] \\
\operatorname{Spec}\kappa(y) \arrow[r] & \mathfrak{Y}_y \arrow[r] & Y_y \arrow[r] & Y.
\end{tikzcd} \tag{3.5.7.2}$$

By the universal property of the infinitesimal neighbourhood $\operatorname{Spec}\kappa(y) \to \mathfrak{Y}_y$, the dashed morphism exists and is unique. This is shown in Figure 4.

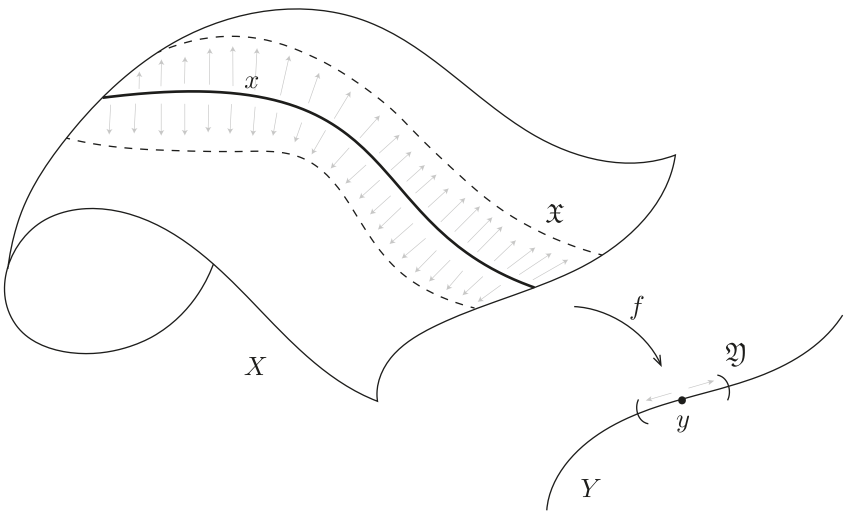


Figure 4: Infinitesimal neighbourhoods

Chapter 4

Groupoids

4.1 Groupoids of sets

This section introduces groupoids of sets. After the terse well-known definition, it is explained how it generalises both equivalence relations and group actions. Then, for any morphism, the restriction and pullback of a groupoid are defined. The former can be thought as restricting an equivalence relation or restricting a group action to a subset. The latter can be thought as a base change of the groupoid. The material in this section is based on [KM97, §2, page 196].

Definition 4.1.1 (Groupoids of sets)**.** Let X be a set. A *groupoid structure* on X is a small category $\mathcal{R}$ whose set of objects is X and where every morphism in $\mathcal{R}$ is an isomorphism.

4.1.2 (Groupoids of sets)**.** The definition of groupoid structure is abstract and requires to be unravelled. Let X be a set and let $\mathcal{R}$ be a groupoid structure on X. Define R to be the set of all morphisms in $\mathcal{R}$. In symbols,

$$R = \bigsqcup_{x,y \in X} \mathrm{Hom}_{\mathcal{R}}(x, y). \tag{4.1.2.1}$$

Then the groupoid structure is characterised by five maps:

1. The *source* $s : R \to X$ mapping a morphism $f : x \to y$ to its domain x.
2. The *target* $t : R \to X$ mapping a morphism $f : x \to y$ to its codomain y.
3. The *identity* $e : X \to R$ mapping an object x to the identity morphism $\mathbb{1}_x : x \to x$.
4. The *inverse* $i : R \to R$ mapping a morphism $f : x \to y$ to its inverse $f^{-1} : y \to x$.
5. The *composition* $c : R \times_{(s,t)} R \to R$ mapping a pair of morphisms $(g : y \to z, f : x \to y)$ to their composition $g \circ f : x \to z$.

The source and target functions induce a map $j := t \times s : R \to X \times X$. The set R is the *groupoid set* associated to $\mathcal{R}$. In the following, it will be understood that it comes equipped with all of the above maps, even if these are not explicitly stated.

Lemma 4.1.3 (Properties of groupoids of sets)**.** *Let $\mathcal{R}$ be a groupoid structure on a set X. Then it follows that*

1. $s \circ e = t \circ e = \mathbb{1}_X$.
2. $s \circ i = t$ *and* $t \circ i = s$.
3. $i \circ i = \mathbb{1}_R$.
4. $i \circ e = e$.

Proof. The statements all follow from the fact that $\mathcal{R}$ is a category.

1. Given $x \in X$, the domain and codomain of its identity morphism are both equal to x.
2. The domain of the inverse of a morphism is the codomain of the morphism.
3. The inverse of the inverse of a morphism is the morphism itself.
4. The inverse of the identity morphism is the identity morphism. □

The next paragraph shows that the groupoid set associated to a groupoid is a generalisation of an equivalence relation.

4.1.4 (Groupoids and equivalence relations of sets)**.** Let $\mathcal{R}$ be a groupoid structure on a set X. The image of the map $j : R \to X \times X$ is an equivalence relation. To see this, note that reflexivity follows from the existence of the map $e : X \to R$. Indeed, the composition $j \circ e$ is the diagonal map $\Delta_X : X \to X \times X$. Symmetry is a consequence of the existence of i. Transitivity follows from the existence of c. It follows that, if j is a monomorphism (or equivalently injective), R is an equivalence relation on X. Conversely, if R is an equivalence relation on X, it is a groupoid on X such that j is a monomorphism. To see this, define a pre-order category $\mathcal{R}$ whose set of objects is X and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{R}}(x, y) = \begin{cases} \{*\} & \text{if } x \sim_R y \\ \varnothing & \text{otherwise,} \end{cases} \tag{4.1.4.1}$$

for any x and y in X. The composition law exists by transitivity of the relation and it is associative since the set of morphisms between any two objects is either a singleton or empty. Furthermore, the identity morphism exists by reflexivity of the relation. Finally, every morphism is an isomorphism by symmetry of the relation. Given x and y in X, writing $x \sim_R y$ will be the shorthand for writing that $(x, y) \in X \times X$ is in the image of j.

Groupoids also generalise group actions on sets.

Example 4.1.5 (Group actions on sets)**.** An important example of groupoid is that of a group G acting on a set X. Let $G \times X \to X$ be a group action and let e_G denote the identity element of G. Define a category $\mathcal{R}$ whose set of objects is X and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{R}}(x, y) = \{g \in G \,|\, g \cdot x = y\}\,, \tag{4.1.5.1}$$

for any x and y in X. Using the properties of group actions, one can verify that this is a groupoid. More concretely, there is a bijection $G \times X \xrightarrow{\sim} R$ given by mapping (g, x) to $g \in \mathrm{Hom}_{\mathcal{R}}(x, g \cdot x) \subseteq R$. This gives the following maps:

1. $s(g, x) = x$.
2. $t(g, x) = g \cdot x$.
3. $e(x) = (e_G, x)$.
4. $i(g, x) = (g^{-1}, g \cdot x)$.
5. $c(g, x, h, y) = (gh, y)$, where $x = h \cdot y$.

The next paragraph motivates the definition of *group stabiliser* of a groupoid by first looking at group actions.

4.1.6 (Stabilisers of group actions on sets)**.** Let G be a group acting on a set X and let $x \in X$ be an element. The stabiliser G_x of G in x can be defined by the Cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \{x\} \arrow[d, "\Delta_x"] \\ G \times X \arrow[r, "j"] & X \times X, \end{tikzcd} \tag{4.1.6.1}$$

where

$$\Delta_x : \{x\} \to X \xrightarrow{\Delta_X} X \times X \tag{4.1.6.2}$$

is the diagonal inclusion. It is well-known that it is a group.

4.1.7 (Stabilisers of groupoids of sets)**.** Let $\mathcal{R}$ be a groupoid structure on a set X and let $x \in X$ be an element. The *group stabiliser* of $\mathcal{R}$ in x is the fibre product

$$G_x := R \times_{(X \times X)} \{x\} \tag{4.1.7.1}$$

By definition, the set G_x consists of those morphisms in R whose source and target is x. Therefore, it is the set of automorphisms of x, hence it is a group.

Example 4.1.8 (Fundamental groupoid)**.** Let X be a topological space. The *fundamental groupoid* $\pi_1(X)$ is a groupoid structure on X defined as follows: for any points x and y in X, define the set of paths modulo homotopy equivalence

$$\mathrm{Hom}_{\pi_1(X)}\,(x, y) = \frac{\{\gamma : [0, 1] \to X \,|\, \gamma(0) = x \text{ and } \gamma(1) = y\}}{\text{homotopy equivalence}}. \tag{4.1.8.1}$$

Then $\pi_1(X)$ is a groupoid. The source and target functions return the start and end point of a given path respectively. The identity function returns the constant path at any given element $x \in X$. The inverse function returns the path with the opposite orientation. The composition function returns the composition of paths whenever the start point of the second path is the end point of the first one. Finally, the group stabiliser at any $x \in X$ is the fundamental group with base point x.

The set of groupoids naturally forms a category.

Definition 4.1.9 (Morphisms of groupoids of sets)**.** Let X and Y be sets and let $\mathcal{R}$ and $\mathcal{S}$ be groupoid structures on X and Y respectively. A *morphism of groupoids* is a functor $F : \mathcal{R} \to \mathcal{S}$. The set of all groupoids with their morphisms forms a category.

4.1.10 (Morphisms of groupoids of sets)**.** Let X and Y be sets and let $\mathcal{R}$ and $\mathcal{S}$ be groupoid structures on X and Y respectively. Let R and S be the groupoid sets associated to $\mathcal{R}$ and $\mathcal{S}$ respectively. Suppose that $F : \mathcal{R} \to \mathcal{S}$ is a morphism of groupoids, then F induces two functions $f : X \to Y$ and $F : R \to S$ such that $s \circ F = f \circ s$ and $t \circ F = f \circ t$. This follows immediately from the properties of functors.

Restriction of a groupoid is now defined. As a guiding principle, the restriction of (the groupoid induced by) a group action to an invariant open subset should be the restricted group action. Before the definition, a motivating example is presented.

Example 4.1.11 (Restriction of groupoids of sets)**.** Let $X = \mathbb{R}^2 \setminus \{0\}$ and define an equivalence relation R on X by

$$(x, y) \sim_R (x', y') \longleftrightarrow xy' = x'y. \tag{4.1.11.1}$$

Let Y be the circle of unit radius and let

$$g : Y \to X \tag{4.1.11.2}$$

be the inclusion. This is represented in Figure 5. The grey lines in the figure are the equivalence classes. The restriction of R by g, denoted by $R|_Y$, is the groupoid (in fact equivalence relation) on Y relating antipodal points. For instance, the points p and q are related.

Definition 4.1.12 (Restriction of groupoids of sets)**.** Let X be a set and let $\mathcal{R}$ be a groupoid structure on X. Let $g : Y \to X$ be a function. The *restriction* of $\mathcal{R}$ by g (or simply to Y when g is understood) is a groupoid $\mathcal{R}|_Y$ on Y endowed with a morphism of groupoids

$$G : \mathcal{R}|_Y \to \mathcal{R} \tag{4.1.12.1}$$

satisfying the following universal property: for any groupoid structure $\mathcal{S}$ on Y and any morphism of groupoids $H : \mathcal{S} \to \mathcal{R}$ with underlying map $g : Y \to X$, the solid commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, dashed] \arrow[dr, "H"'] & \mathcal{R}|_Y \arrow[d, "G"] \\ & \mathcal{R} \end{tikzcd} \tag{4.1.12.2}$$

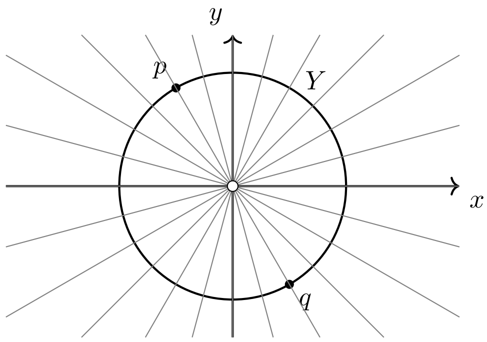


Figure 5: Restriction of a groupoid

admits a unique dashed morphism of groupoids over Y making the diagram commute. By general category theory, if the restriction exists, it is unique.

Construction 4.1.13 (Restriction of groupoids of sets)**.** Notation as in Definition 4.1.12 (Restriction of groupoids of sets), define $\mathcal{R}|_Y$ as follows: its set of objects is Y and, for any y_1 and y_2 in Y, the set of morphisms is defined to be

$$\mathrm{Hom}_{\mathcal{R}|_Y}(y_1, y_2) = \mathrm{Hom}_{\mathcal{R}}(g(y_1), g(y_2)). \tag{4.1.13.1}$$

Composition is defined via composition in $\mathcal{R}$. It is straightforward to see that this is a category and indeed a groupoid. Furthermore, there is a functor $G : \mathcal{R}|_Y \to \mathcal{R}$ which maps an object $y \in Y$ to $g(y) \in X$ and a morphism in $\mathrm{Hom}_{\mathcal{R}|_Y}(y_1, y_2)$ to a morphism in $\mathrm{Hom}_{\mathcal{R}}(g(y_1), g(y_2))$ via the equality in (4.1.13.1).

Lemma 4.1.14 (Universal property of restriction in sets)**.** *Notation as in Construction 4.1.13 (Restriction of groupoids of sets), the groupoid $\mathcal{R}|_Y$ is the restriction of $\mathcal{R}$ to Y.*

Proof. Define a functor $\mathcal{S} \to \mathcal{R}|_Y$ to be the identity on the underlying set Y and, for any y_1 and y_2 in Y,

$$\mathrm{Hom}_{\mathcal{S}}(y_1, y_2) \to \mathrm{Hom}_{\mathcal{R}}(g(y_1), g(y_2)) = \mathrm{Hom}_{\mathcal{R}|_Y}(y_1, y_2), \tag{4.1.14.1}$$

where the morphism is obtained by applying the functor $H : \mathcal{S} \to \mathcal{R}$ and the equality is obtained by (4.1.13.1). By construction, Diagram (4.1.12.2) is commutative. Finally, since the functor $\mathcal{R}|_Y \to \mathcal{R}$ is faithful and injective on objects, it is a monomorphism in the category of small categories. This implies that the functor $\mathcal{S} \to \mathcal{R}|_Y$ is unique. $\square$

4.1.15 (Restriction of groupoids of sets)**.** Notation as in Definition 4.1.12 (Restriction of groupoids of sets), let R and $R|_Y$ denote the groupoid sets of $\mathcal{R}$ and $\mathcal{R}|_Y$ respectively. It follows from the definition of $\mathcal{R}|_Y$, that there is a Cartesian diagram

$$\begin{tikzcd} R|_Y \arrow[r] \arrow[d] & R \arrow[d, "j"] \\ X \times X \arrow[r, "g \times g"] & Y \times Y. \end{tikzcd} \tag{4.1.15.1}$$

Next, the pullback of groupoids is discussed. To this end, invariant functions are introduced. These are functions which respect the equivalence classes determined by a groupoid.

Definition 4.1.16 (Invariant functions of sets)**.** Let X be a set and let $\mathcal{R}$ be a groupoid structure on X. A function $f : X \to Y$ is *$\mathcal{R}$-invariant* if the diagram

$$R \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{f} Y \tag{4.1.16.1}$$

is commutative, where R is the groupoid set associated to $\mathcal{R}$.

Given an invariant function, $\mathcal{R}$ can be thought as a groupoid on X over Y. The pullback of $\mathcal{R}$ by a function $g : Y' \to Y$ is then described by the base change. The next example describes the pullback in the familiar case of group actions.

Example 4.1.17 (Pullback of group action)**.** Let G be a group acting on a set X and let $f : X \to Y$ be a G-invariant function. Let $g : Y' \to Y$ be another function and let $g' : X' \to X$ be the base change of g by f. Then, there is a natural induced group action G on X' described as follows: for any $h \in G$ and $x' = (x, y') \in X \times_Y Y'$,

$$h \cdot x' = (h \cdot x, y') \in X \times_Y Y' = X'. \tag{4.1.17.1}$$

This is well-defined since f is G-invariant. The groupoid induced by this group action on X' is precisely the pullback by g of the groupoid induced by the action of G on X.

Definition 4.1.18 (Pullback of groupoids of sets)**.** Let X be a set and let $\mathcal{R}$ be a groupoid structure on X. Let $f : X \to Y$ be an $\mathcal{R}$-invariant function and let $g : Y' \to Y$ be a function. Let $f' : X' := X \times_Y Y' \to Y'$ and $g' : X' \to X$ denote the base change of f and g respectively. The *pullback* of $\mathcal{R}$ by g (or to X' when g is understood) is a groupoid $\mathcal{R}'$ on X' endowed with a morphism of groupoids

$$G : \mathcal{R}' \to \mathcal{R} \tag{4.1.18.1}$$

satisfying the following universal property: for any groupoid structure $\mathcal{S}$ on X' such that f' is $\mathcal{S}$-invariant, and any morphism of groupoids $H : \mathcal{S} \to \mathcal{R}$ with underlying map $g' : X' \to X$, the solid commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, dashed] \arrow[dr, "H"'] & \mathcal{R}' \arrow[d, "G"] \\ & \mathcal{R} \end{tikzcd} \tag{4.1.18.2}$$

admits a unique dashed morphism of groupoids over Y making the diagram commute. By general category theory, if the pullback exists, it is unique.

Construction 4.1.19 (Pullback of groupoids of sets)**.** Notation as in Definition 4.1.18 (Pullback of groupoids of sets), define $\mathcal{R}'$ as follows: its set of objects is X' and, for any x'_1 and x'_2 in X', the set of morphisms is defined to be

$$\mathrm{Hom}_{\mathcal{R}'}(x'_1, x'_2) = \begin{cases} \mathrm{Hom}_{\mathcal{R}}(g'(x'_1), g'(x'_2)) & \text{if } f'(x'_1) = f'(x'_2) \\ \varnothing & \text{otherwise.} \end{cases} \tag{4.1.19.1}$$

Composition is defined via composition in $\mathcal{R}$. It is straightforward to see that this is a category and indeed a groupoid. Furthermore, there is a functor $G : \mathcal{R}' \to \mathcal{R}$ which maps an object $x' \in X'$ to $g'(x') \in X$ and a morphism in $\mathrm{Hom}_{\mathcal{R}'}(x'_1, x'_2)$ to a morphism in $\mathrm{Hom}_{\mathcal{R}}(g'(x'_1), g'(x'_2))$ via the equality in (4.1.19.1).

Lemma 4.1.20 (Universal property of pullback in sets)**.** *Notation as in Construction 4.1.19 (Pullback of groupoids of sets), the groupoid $\mathcal{R}'$ is the pullback of $\mathcal{R}$ by g.*

Proof. Define a functor $\mathcal{S} \to \mathcal{R}'$ to be the identity on the underlying set X' and, for any x'_1 and x'_2 in X' satisfying $f'(x'_1) = f'(x'_2)$,

$$\mathrm{Hom}_{\mathcal{S}}(x'_1, x'_2) \to \mathrm{Hom}_{\mathcal{R}}(g'(x'_1), g'(x'_2)) = \mathrm{Hom}_{\mathcal{R}'}(x'_1, x'_2), \tag{4.1.20.1}$$

where the morphism is obtained by applying the functor $H : \mathcal{S} \to \mathcal{R}$ and the equality is obtained by (4.1.19.1). If $f'(x'_1) \neq f'(x'_2)$, then there is nothing to define. By construction, the Diagram (4.1.18.2) is commutative. Finally, since the functor $\mathcal{R}|_Y \to \mathcal{R}$ is faithful and injective on objects, it is a monomorphism in the category of small categories. This implies that the functor $\mathcal{S} \to \mathcal{R}'$ is unique. $\square$

4.1.21 (Pullback of groupoids of sets)**.** Notation as in Definition 4.1.18 (Pullback of groupoids of sets), let R and R' denote the groupoid sets of $\mathcal{R}$ and $\mathcal{R}'$ respectively. It follows from the definition of $\mathcal{R}'$ that there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "e'"] \arrow[d, "g'"] & R' \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] \arrow[d] & X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "e"] & R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.1.21.1}$$

where every square is Cartesian. The middle square is Cartesian either with s and s' or with t and t'.

The next paragraph states that, when g is a monomorphism, restriction and pullback are equal.

4.1.22 (Pullback and restriction of groupoids of sets)**.** Notation as in Definition 4.1.18 (Pullback of groupoids of sets), there is an inclusion functor $\mathcal{R}' \to \mathcal{R}|_{X'}$ which is the identity on objects of X' and the inclusion on morphisms. If g is a monomorphism, $\mathcal{R}' \to \mathcal{R}|_{X'}$ is an isomorphism. Indeed, if $g'(x'_1) \sim_R g'(x'_2)$, the condition $f'(x'_1) = f'(x'_2)$ always holds. To see this, note that, since f is $\mathcal{R}$-invariant,

$$\begin{aligned} g(f'(x'_1)) &= f(g'(x'_1)) \\ &= f(g'(x'_2)) \\ &= g(f'(x'_2)). \end{aligned} \tag{4.1.22.1}$$

But then, since g is a monomorphism,

$$f'(x'_1) = f'(x'_2). \tag{4.1.22.2}$$

4.2 Groupoids of schemes

This section upgrades the previous section to treat the case of schemes. It discusses the same notions of groupoids, restriction and pullback. Typically, the statements follow from the case of sets together with Yoneda lemma. One important difference is that all definitions are now relative to a base scheme. The material in this section is also based on [KM97, §2, page 196].

4.2.1 (Notation for set of morphisms)**.** Let L be a scheme and let X be a scheme over L. If T is another scheme over L, $X_L(T)$ will denote the set of T-valued points, that is $\mathrm{Hom}_L(T, X)$. Furthermore, if $f : X \to Y$ is a morphism over L, $f(T) : X_L(T) \to Y_L(T)$ will denote the map obtained by post-composing with f. Finally, if $g : S \to T$ is a morphism over L, $g(X) : X_L(T) \to X_L(S)$ will denote the map obtained by pre-composing with g.

Definition 4.2.2 (Groupoids of schemes)**.** Let L be a scheme and let X be a scheme over L. A *groupoid space* on X over L, or simply a *groupoid*, is a scheme R over L together with L-morphisms

1. $s : R \to X$.
2. $t : R \to X$.
3. $e : X \to R$.
4. $i : R \to R$.
5. $c : R \times_{(s,t)} R \to R$,

such that, for all schemes T over L, the induced maps

1. $s(T) : R_L(T) \to X_L(T)$.
2. $t(T) : R_L(T) \to X_L(T)$.
3. $e(T) : X_L(T) \to R_L(T)$.
4. $i(T) : R_L(T) \to R_L(T)$.
5. $c(T) : \left(R \times_{(s,t)} R\right)_L (T) = R_L(T) \times_{(s(T),t(T))} R_L(T) \to R_L(T)$

induce a groupoid structure $\mathcal{R}_T$ on $X_L(T)$ in a functorial way: given an L-morphism $g : S \to T$, the map

$$\mathcal{R}_T \to \mathcal{R}_S \tag{4.2.2.1}$$

obtained from $g(X) : X_L(T) \to X_L(S)$ and $g(R) : R_L(T) \to R_L(S)$ is a morphism of groupoids of sets.

Lemma 4.2.3 (Properties of groupoids of schemes)**.** *Let R be a groupoid on scheme X over a scheme L. Then, all parts of Lemma 4.1.3 (Properties of groupoids of sets) hold.*

Proof. The proofs of the statements employ the lemma for sets and then invoke the faithfulness of the Yoneda embedding to conclude. As an example, it is shown that $s \circ e = \mathbb{1}_X$. Let T be a scheme. By definition, $\mathcal{R}_T$ is a groupoid of sets. By part (1) of Lemma 4.1.3 (Properties of groupoids of sets),

$$(s \circ e)(T) : X_L(T) \xrightarrow{e(T)} R_L(T) \xrightarrow{s(T)} X_L(T) \tag{4.2.3.1}$$

is the identity function. But then, for all schemes T, $s \circ e$ and $\mathbb{1}_X$ induce the same function on $X_L(T)$. Since the Yoneda embedding is faithful ([Sta23, Lemma 001P]), $s \circ e = \mathbb{1}_X$. The proofs for the remaining statements are similar. □

Definition 4.2.4 (Equivalence relations on schemes)**.** Let L be a scheme and let X be a scheme over L. An *equivalence relation* on X over L is a morphism $j : R \to X \times_L X$ such that for all schemes T over L,

$$j(T) : R_L(T) \to (X \times_L X)_L\,(T) = X_L(T) \times X_L(T) \tag{4.2.4.1}$$

is an equivalence relation of sets.

Lemma 4.2.5 (Groupoids and equivalence relations of schemes)**.** *Let R be a groupoid on scheme X over a scheme L. Then R is an equivalence relation if and only if j is a monomorphism.*

Proof. Using Yoneda embedding, it is straightforward to reduce the statement to the case of sets. The latter case holds true by Paragraph 4.1.4 (Groupoids and equivalence relations of sets). □

Since the source and target morphisms in a groupoid are morphisms of schemes, they may satisfy further scheme-theoretic properties.

Definition 4.2.6 (Properties of groupoids of schemes)**.** Let R be a groupoid on a scheme X over a scheme L. Let P be a property in

$$\{\text{finitely presented, flat, smooth, étale}\}\,. \tag{4.2.6.1}$$

Then R is a groupoid with property P whenever both morphisms s and t have property P.

4.2.7 (Properties of groupoids of schemes)**.** Note that the properties in (4.2.6.1) are stable under base change and composition. Furthermore, it is enough to check property P on either s or t. Indeed, since i is an involution (part (3) of Lemma 4.2.3 (Properties of groupoids of schemes)), it has property P. By part (2) of Lemma 4.2.3 (Properties of groupoids of schemes), $s = i \circ t$ and $t = i \circ s$ and the result follows from the fact that P is stable under composition.

Since the identity morphism is a section of the source (or target) morphism, it satisfies additional properties.

Lemma 4.2.8 (Diagonal immersion of groupoids)**.** *Let R be a groupoid on a scheme X over a scheme L. Then $e : X \to R$ is an immersion.*

Proof. By part(1) of Lemma 4.2.3 (Properties of groupoids of schemes), there exists a factorisation

$$\Delta_{X/L} : X \xrightarrow{e} R \xrightarrow{j} X \times_L X. \tag{4.2.8.1}$$

Firstly recall that $\Delta_{X/L}$ is an immersion ([Sta23, Lemma 01KJ]). Since e is the first factor of an immersion, it is an immersion ([Sta23, Lemma 07RK]). ☐

Lemma 4.2.9 (Regular immersions of groupoids)**.** *Let R be a smooth groupoid on a scheme X over a scheme L. Then $e : X \to R$ is a regular immersion.*

Proof. By assumption, $s : R \to X$ is a smooth morphism. By part (1) of Lemma 4.2.3 (Properties of groupoids of schemes) e is a section of s. By [Sta23, Lemma 067R], e is a regular immersion. ☐

Example 4.2.10 (Groupoids induced by morphisms)**.** Let X be a scheme over a scheme L. A morphism $f : X \to Y$ of schemes over L induces a groupoid (in fact an equivalence relation) on X over L. Intuitively, two points x_1 and x_2 in X are defined to be equivalent if and only if $f(x_1) = f(x_2) \in Y$. More precisely, let

$$D_f := X \times_Y X \underset{t}{\overset{s}{\rightrightarrows}} X \tag{4.2.10.1}$$

with the following morphisms

$$e = \Delta_{X/Y} : X \to X \times_Y X, \tag{4.2.10.2}$$
$$i = s \times t : X \times_Y X \to X \times_Y X, \tag{4.2.10.3}$$
$$c = \mathrm{pr}_1 \times \mathrm{pr}_3 : X \times_Y X \times_Y X \to X \times_Y X. \tag{4.2.10.4}$$

It is readily verified that this is a groupoid. In fact, since $X \times_Y X \to X \times_L X$ is an immersion ([Sta23, Lemma 01KR]), hence a monomorphism ([Sta23, Lemma 01L7]), D_f is an equivalence relation. Two notable examples are when $f = \mathbb{1}_X : X \to X$ is the identity morphism and when $f : X \to L$ is the structural morphism. In the first instance, $D_f = X$ is the *trivial groupoid*. In the second instance, $D_f = X \times_L X$ is the *total groupoid*.

The next example shows the benefits of developing a theory of groupoids relative to a base scheme.

Example 4.2.11 (Group schemes)**.** Let $p : G \to L$ be a group scheme over a scheme L. Let $e_G : L \to G$, $i_G : G \to G$ and $c_G : G \times_L G \to G$ denote the identity, the inverse and the composition respectively. Then G is a groupoid on L over L with the following morphisms: $s := p =: t$, $e := e_G$, $i := i_G$ and $c := c_G$. If L is a field of characteristic zero, then p is smooth ([Sta23, Lemma 047N]), hence G is a smooth groupoid.

Example 4.2.12 (Group actions on schemes)**.** Let X be a scheme over a scheme L. Let G be a group scheme over L and let it act on X via $\sigma_G : G \times_L X \to X$. Let $e_G : L \to G$, $i_G : G \to G$ and $c_G : G \times_L G \to G$ denote the identity, the inverse and the composition respectively. Similarly to Example 4.1.5 (Group actions on sets), set $R_G = G \times_L X$ and define the following morphisms:

1. $s(g,x) = x$.
2. $t(g,x) = \sigma_G(g,x)$.
3. $e(x) = (e_G, x)$.
4. $i(g,x) = (i_G(g), \sigma_G(g,x))$.
5. $c(g,x,h,y) = (c_G(g,h), y)$, where $x = \sigma_G(h,y)$.

Using the properties of group actions, one can check that this is a groupoid. Note that G acts freely on X if and only if R_G is an equivalence relation. Furthermore, if G is flat (resp. smooth) over L, the induced groupoid R_G is flat (resp. smooth).

The next paragraph defines group stabilisers of a groupoid based on the corresponding notion for sets.

4.2.13 (Stabilisers of groupoids of schemes)**.** Let R be a groupoid on a scheme X over a scheme L. Let $x \in X$ be a point with residue field $\kappa(x)$. The *group stabiliser* of R in x is the fibre product

$$G_x := R \times_{(X \times_L X)} (\operatorname{Spec} \kappa(x)) \,. \tag{4.2.13.1}$$

By Paragraph 4.1.6 (Stabilisers of group actions on sets), $(G_x)_L(T)$ is a group for all schemes T over L. Applying Yoneda lemma yields that G_x is a group scheme over $\kappa(x)$.

The next paragraph states that the datum of a smooth groupoid is equivalent to the datum of an algebraic stack.

4.2.14 (Schemes, algebraic spaces and algebraic stacks)**.** Let R be a groupoid on a scheme X over a scheme L. By definition, R induces a functor X/R on the category of schemes over L and valued in the category of groupoids of sets

$$(X/R)(T) = \mathcal{R}_T. \tag{4.2.14.1}$$

If R is the trivial groupoid, then X/R is equivalent to the functor induced by X. If R is an étale equivalence relation, then X/R is, by definition, an algebraic space. If R is a smooth groupoid, then X/R is an algebraic stack ([Sta23, Theorem 04TK]).

Groupoids of schemes also form a category.

Definition 4.2.15 (Morphisms of groupoids of schemes)**.** Let X and Y be schemes over a scheme L. Let R and S be groupoids on X and Y respectively over L. A *morphism of groupoids* is a pair of morphisms $f : X \to Y$ and $F : R \to S$ such that, for all L-schemes T, the induced map between groupoids of sets

$$\mathcal{R}_T \to \mathcal{R}_S \tag{4.2.15.1}$$

obtained from $f(T) : X_L(T) \to Y_L(T)$ and $F(T) : R_L(T) \to S_L(T)$ is a morphism of groupoids of sets. The set of groupoids with their morphisms forms a category.

The notion of restriction is completely analogous to the case of sets.

Definition 4.2.16 (Restriction of groupoids of schemes)**.** Let R be a groupoid on a scheme X over a scheme L. Let $g : Y \to X$ be a morphism of schemes over L. The *restriction* of R by g (or simply to Y when g is understood) is a groupoid $R|_Y$ on Y endowed with a morphism of groupoids

$$G : R|_Y \to R \tag{4.2.16.1}$$

satisfying the following universal property: for any groupoid S on Y over L and any morphism of groupoids $H : S \to R$ with underlying map $g : Y \to X$, the solid commutative diagram

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[dr, "H"'] & R|_Y \arrow[d, "G"] \\ & R \end{tikzcd} \tag{4.2.16.2}$$

admits a unique dashed morphism of groupoids over Y making the diagram commute. By general category theory, if the restriction exists, it is unique.

Construction 4.2.17 (Restriction of groupoids of schemes)**.** Notation as in Definition 4.2.16 (Restriction of groupoids of schemes), define

$$R|_Y := R \times_{(X \times_L X)} (Y \times_L Y) \tag{4.2.17.1}$$

This is a groupoid on Y. Indeed, applying the functor $\mathrm{Hom}_L(T, \_)$ and using the characterisation of Paragraph 4.1.15 (Restriction of groupoids of sets) and Construction 4.1.13 (Restriction of groupoids of sets) yields that $(R|_Y)_L(T)$ is a groupoid of sets for all schemes T in a functorial way.

Note that, by definition, $R|_Y$ fits in a commutative diagram

$$\begin{tikzcd} R|_Y \arrow[r] \arrow[d] & R \times_{(t,f)} Y \arrow[r] \arrow[d] & Y \arrow[d, "f"'] \\ Y \times_{(f,s)} R \arrow[r] \arrow[d] & R \arrow[r, "t"] \arrow[d, "s"'] & X \\ Y \arrow[r, "f"] & X, & \end{tikzcd} \tag{4.2.17.2}$$

where every square is Cartesian.

Lemma 4.2.18 (Universal property of restriction in schemes)**.** *Notation as in Construction 4.2.17 (Restriction of groupoids of schemes), the groupoid $R|_Y$ is the restriction of R to Y.*

Proof. This follows directly from the definitions and Lemma 4.1.14 (Universal property of restriction in sets). □

Lemma 4.2.19 (Properties of restrictions of groupoids)**.** *Let X and Y be schemes over a scheme L. Let R be a groupoid on X over L and let $g : Y \to X$ be an L-morphism. Suppose that both R and f have property P listed in (4.2.6.1). Then $R|_Y$ also has property P.*

Proof. In Paragraph 4.2.7 (Properties of groupoids of schemes), it was observed that P is stable under base change and composition. Now observe Diagram (4.2.17.2). It it enough to show that the composition of morphisms in the top row has property P. By assumption and stability of P under base change, both morphisms in the top row have property P. The result follows. $\square$

The notions of invariance and pullback are also analogous to the case of sets.

Definition 4.2.20 (Invariant morphisms of schemes)**.** Let R be a groupoid on a scheme X over a scheme L. A morphism $f : X \to Y$ of schemes over L is R-invariant if the diagram

$$R \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{f} Y \tag{4.2.20.1}$$

is commutative.

Definition 4.2.21 (Pullback of groupoids of schemes)**.** Let R be a groupoid on a scheme X over a scheme L. Let $f : X \to Y$ be an R-invariant morphism of schemes over L and let $g : Y' \to Y$ be a morphism of schemes over L. Let $f' : X' := X \times_Y Y' \to Y'$ and $g' : X' \to X$ denote the base change of f and g respectively. The *pullback* of R by g (or to X' when g is understood) is a groupoid R' on X' endowed with a morphism of groupoids

$$G : R' \to R \tag{4.2.21.1}$$

satisfying the following universal property: for any groupoid S on X' such that f' is S-invariant, and any morphism of groupoids $H : S \to R$ with underlying map $g' : X' \to X$, the solid commutative diagram

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[dr, "H"'] & R' \arrow[d, "G"] \\ & R \end{tikzcd} \tag{4.2.21.2}$$

admits a unique dashed morphism of groupoids over Y making the diagram commute. By general category theory, if the pullback exists, it is unique.

Construction 4.2.22 (Pullback of groupoids of schemes)**.** Notation as in Definition 4.2.21 (Pullback of groupoids of schemes), define

$$R' := R \times_{Y'} Y \tag{4.2.22.1}$$

This is a groupoid on X'. Indeed, applying the functor $\mathrm{Hom}_L(T, \_)$ and using the characterisation of Paragraph 4.1.21 (Pullback of groupoids of sets) and Construction 4.1.19 (Pullback of groupoids of sets) yields that $R'_L(T)$ is a groupoid of sets for all schemes T in a functorial way. By construction, the Diagram (4.1.21.1) still applies.

Lemma 4.2.23 (Universal property of pullback in schemes)**.** *Notation as in Construction 4.2.22 (Pullback of groupoids of schemes), the groupoid R' is the pullback of R by g.*

Proof. This follows directly from the definitions and Lemma 4.1.20 (Universal property of pullback in sets). ☐

4.2.24 (Pullback and restriction of groupoids of schemes)**.** Notation as in Definition 4.2.21 (Pullback of groupoids of schemes), there is a morphism of groupoids $R' \to R|_{X'}$. This comes from the universal property of restriction of groupoids of schemes (Lemma 4.2.18 (Universal property of restriction in schemes)). If g is a monomorphism, $R' \to R|_{X'}$ is an isomorphism. Indeed, after reducing to the case of sets, the result follows from Paragraph 4.1.22 (Pullback and restriction of groupoids of sets).

4.3 Infinitesimal groupoids

This section is the Samarkand of this dissertation, where the theory of groupoids meets the theory of formal schemes. Firstly, formal groupoids are defined. These are simply groupoid objects in the category of formal schemes and the proofs of the previous section easily apply to this case. They generalise groupoids of schemes. The objects of real interest are infinitesimal groupoids. These are formal groupoids whose identity morphism is a thickening of formal schemes. The most important result of this section is that, given a groupoid of schemes, or more generally a formal groupoid, the infinitesimal neighbourhood of the identity morphism is an infinitesimal groupoid. Once this is shown, all the material of the previous section also applies to infinitesimal groupoids.

Definition 4.3.1 (Formal and infinitesimal groupoids)**.** Let $\mathfrak{X}$ be a formal scheme over a formal scheme $\mathfrak{L}$. A *formal groupoid* on $\mathfrak{X}$ over $\mathfrak{L}$ is a formal scheme $\mathfrak{R}$ together with five morphisms

1. $s : \mathfrak{R} \to \mathfrak{X}$.
2. $t : \mathfrak{R} \to \mathfrak{X}$.
3. $e : \mathfrak{X} \to \mathfrak{R}$.
4. $i : \mathfrak{R} \to \mathfrak{R}$.
5. $c : \mathfrak{R} \times_{(s,t)} \mathfrak{R} \to \mathfrak{R}$,

satisfying the functorial properties of Definition 4.2.2 (Groupoids of schemes). A formal groupoid $\mathfrak{R}$ is an *infinitesimal groupoid* if e is a thickening of formal schemes. A formal groupoid (resp. infinitesimal groupoid) $\mathfrak{R}$ is a *formal equivalence relation* (resp. *infinitesimal equivalence relation*) if $j = t \times s$ is a monomorphism of formal schemes. A morphism between two formal (or infinitesimal) groupoids $\mathfrak{R}$ on $\mathfrak{X}$ over $\mathfrak{L}$ and $\mathfrak{S}$ on $\mathfrak{Y}$ over $\mathfrak{L}$ is a pair of morphisms $\mathfrak{R} \to \mathfrak{S}$ and $\mathfrak{X} \to \mathfrak{Y}$ satisfying the functorial properties of Definition 4.2.15 (Morphisms of groupoids of schemes). The set of formal or infinitesimal groupoids and the morphisms in between them forms a category.

Lemma 4.3.2 (Properties of formal groupoids)**.** *Let $\mathfrak{R}$ be a formal groupoid on a formal scheme $\mathfrak{X}$ over a formal scheme $\mathfrak{L}$. Then, all parts of Lemma 4.2.3 (Properties of groupoids of schemes) hold.*

Proof. Note that the proof of the counterpart lemma for schemes (Lemma 4.2.3 (Properties of groupoids of schemes)) works in any category. □

A groupoid of schemes induces a functor from the category of formal schemes to the category of sets. If restricted to the category of schemes, this functor factors through the category of groupoids, however it is not immediately obvious that it factors through the category of groupoids when defined on the category of formal schemes. The next lemma shows that this is true using the fact that the category of schemes is, in some sense, dense in the category of formal schemes.

Lemma 4.3.3 (Groupoids are formal groupoids)**.** *Let R be a groupoid on a scheme X over a scheme L. Then it is a formal groupoid.*

Proof. The claim amounts to show that, if $\mathcal{R}_T$ is a groupoid of sets for all schemes T over L, then $\mathcal{R}_{\mathfrak{T}}$ is a groupoid of sets for all formal schemes $\mathfrak{T}$ over L. To this end, recall that the functor Hom preserves colimits. In details, if $\mathfrak{X}$ is a formal scheme and $\{\mathfrak{T}_\lambda\}_{\lambda\in\Lambda}$ is a cofiltered system of formal schemes, where Λ is a filtered set, then

$$\operatorname{Hom}\left(\operatorname*{colim}_{\lambda\in\Lambda}\mathfrak{T}_j,\mathfrak{X}\right)=\lim_{\lambda\in\Lambda}\operatorname{Hom}\left(\mathfrak{T}_j,\mathfrak{X}\right). \tag{4.3.3.1}$$

Now, formal schemes are colimits of affine formal schemes obtained by glueing along open immersions. Furthermore, affine formal schemes are filtered colimits of usual schemes along thickenings. This implies that every formal scheme $\mathfrak{T}$ can be obtained as a colimit of schemes. Therefore, its associated groupoid $\mathfrak{R}_{\mathfrak{L}}(\mathfrak{T})$ is a limit of groupoids. Since the category of groupoids of sets is complete, this is a groupoid. □

The next lemma is necessary in order to apply Construction 3.5.5 (Infinitesimal neighbourhood of immersions).

Lemma 4.3.4 (Diagonal morphism in formal groupoids)**.** *Let $\mathfrak{R}$ be a formal groupoid on a formal scheme $\mathfrak{X}$ over a formal scheme $\mathfrak{L}$. Then $e:\mathfrak{X}\to\mathfrak{R}$ is an immersion of formal schemes.*

Proof. This is the same proof as Lemma 4.2.8 (Diagonal immersion of groupoids) using the corresponding statements for formal schemes given by Lemma 3.3.9 (Diagonal is an immersion) and Lemma 3.3.10 (Factorisation of immersion). □

Now, it may be possible to start the construction of the infinitesimal groupoid associated to a formal groupoid.

Definition 4.3.5 (Infinitesimal groupoids of formal groupoids)**.** Let $\mathfrak{R}$ be a formal groupoid on a formal scheme $\mathfrak{X}$ over $\mathfrak{L}$. The *infinitesimal groupoid* of $\mathfrak{R}$ is an infinitesimal groupoid $\hat{\mathfrak{R}}$ endowed with a morphism $\iota : \hat{\mathfrak{R}} \to \mathfrak{R}$ satisfying the following universal property: for any infinitesimal groupoid $\mathfrak{S}$ on $\mathfrak{X}$ over $\mathfrak{L}$ and any morphism of groupoids $H : \mathfrak{S} \to \mathfrak{R}$, there exists a unique morphism of groupoids $\hat{H} : \mathfrak{S} \to \hat{\mathfrak{R}}$ such that $H = \iota \circ \hat{H}$. By general category theory, if the infinitesimal groupoid of $\mathfrak{R}$ exists, it is unique.

The construction of the infinitesimal groupoid induced by a formal groupoid is fairly straightforward. It is defined as the infinitesimal neighbourhood of the diagonal immersion. In order to show it is a groupoid, it has to be shown that the structural morphisms of the formal groupoid induce morphisms on the infinitesimal neighbourhood. This follows from the universal property of infinitesimal neighbourhoods applied to appropriate diagrams.

Construction 4.3.6 (Infinitesimal groupoids of formal groupoids)**.** Let $\mathfrak{R}$ be a formal groupoid on a formal scheme $\mathfrak{X}$ over a formal scheme $\mathfrak{L}$. By Lemma 4.3.4 (Diagonal morphism in formal groupoids), $e : \mathfrak{X} \to \mathfrak{R}$ is an immersion of formal schemes. Define

$$e : \mathfrak{X} \xrightarrow{\hat{e}} \hat{\mathfrak{R}} \xrightarrow{\iota} \mathfrak{R} \tag{4.3.6.1}$$

to be the infinitesimal neighbourhood of e from Construction 3.5.5 (Infinitesimal neighbourhood of immersions), where *iota* is an immersion.

It is shown that $\hat{\mathfrak{R}}$ is an infinitesimal groupoid. To this end, the structural morphisms $\hat{s}, \hat{t}, \hat{e}, \hat{i}, \hat{c}$ are constructed. The morphisms $\hat{s} = s \circ \iota$ and $\hat{t} = t \circ \iota$ can easily be defined. The morphism $\hat{e}$ is already defined in (4.3.6.1). By construction, this is a thickening satisfying

$$\hat{s} \circ \hat{e} = \mathbb{1}_{\mathfrak{X}} = \hat{t} \circ \hat{e}. \tag{4.3.6.2}$$

To show existence of $\hat{i}$, note that part (4) of Lemma 4.3.2 (Properties of formal groupoids) implies the existence of the solid commutative diagram

$$\begin{tikzcd}
& \hat{\mathfrak{R}} \arrow[r, "\iota"] \arrow[dd, dashed, "\hat{i}"] & \mathfrak{R} \arrow[dd, "i"] \\
\mathfrak{X} \arrow[ur, "\hat{e}"] \arrow[dr, "\hat{e}"'] & & \\
& \hat{\mathfrak{R}} \arrow[r, "\iota"] & \mathfrak{R}.
\end{tikzcd} \tag{4.3.6.3}$$

But then the dashed morphism $\hat{i}$ exists and is unique by Definition 3.5.1 (Infinitesimal neighbourhoods of immersions). Showing existence of $\hat{c}$ is similar. Indeed, it is the same argument applied to the commutative solid diagram

$$\begin{tikzcd}
& \hat{\mathfrak{R}} \times_{(\hat{s},\hat{t})} \hat{\mathfrak{R}} \arrow[r, "\iota\times\iota"] \arrow[dd, dashed, "\hat{c}"] & \mathfrak{R} \times_{(s,t)} \mathfrak{R} \arrow[dd, "c"] \\
\mathfrak{X} \arrow[ur, "\hat{e}\times\hat{e}"] \arrow[dr, "\hat{e}"'] & & \\
& \hat{\mathfrak{R}} \arrow[r, "\iota"] & \mathfrak{R},
\end{tikzcd} \tag{4.3.6.4}$$

where, in addition, it has to be shown that $\hat{e} \times \hat{e}$ is a thickening. To this end, consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\hat{e}"] \arrow[dr, bend right, "\mathbb{1}_{\mathfrak{X}}"'] & \hat{\mathfrak{R}} \arrow[r] \arrow[d, "\hat{t}"] \arrow[rr, bend left, "\mathbb{1}_{\hat{\mathfrak{R}}}"] & \hat{\mathfrak{R}} \times_{(\hat{s},\hat{t})} \hat{\mathfrak{R}} \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & \hat{\mathfrak{R}} \arrow[d, "\hat{t}"] \\
& \mathfrak{X} \arrow[r, "\hat{e}"] \arrow[rr, bend right, "\mathbb{1}_{\mathfrak{X}}"'] & \hat{\mathfrak{R}} \arrow[r, "\hat{s}"] & \mathfrak{X}.
\end{tikzcd} \tag{4.3.6.5}$$

The two squares are Cartesian. Indeed, by construction, the right-most square is Cartesian, and, by (4.3.6.2), the total rectangle is Cartesian. By the pasting lemma for pullback diagrams, the left-most square is also Cartesian. By construction, the morphism $\mathfrak{X} \to \hat{\mathfrak{R}} \times_{(\hat{s},\hat{t})} \hat{\mathfrak{R}}$ is $\hat{e} \times \hat{e}$. Now, combining Lemma 3.4.3 (Composition of thickenings) with Lemma 3.4.4 (Base change of thickenings) implies that $\hat{e} \times \hat{e}$ is a thickening.

Therefore, for any formal scheme $\mathfrak{T}$ over $\mathfrak{L}$, ι expresses $F_{\hat{\mathfrak{R}}}(\mathfrak{T})$ as a subgroupoid of $F_{\mathfrak{R}}(\mathfrak{T})$. This shows that $\hat{\mathfrak{R}}$ is an infinitesimal groupoid.

Furthermore, since ι is an immersion of formal schemes, it is a monomorphism (Lemma 3.3.11 (Immersions are monomorphisms)). Hence, if $\mathfrak{R}$ is a formal equivalence relation, since the composition of monomorphisms is a monomorphism,

$$\hat{\mathfrak{R}} \xrightarrow{\iota} \mathfrak{R} \xrightarrow{j} \mathfrak{X} \times_{\mathfrak{L}} \mathfrak{X} \tag{4.3.6.6}$$

is an infinitesimal equivalence relation.

Lemma 4.3.7 (Universal property of infinitesimal groupoids)**.** *Notation as in Construction 4.3.6 (Infinitesimal groupoids of formal groupoids), $\hat{\mathfrak{R}}$ is the infinitesimal groupoid of $\mathfrak{R}$.*

Proof. The morphism $\hat{H} : \mathfrak{S} \to \hat{\mathfrak{R}}$ exists and is unique by Definition 3.5.1 (Infinitesimal neighbourhoods of immersions). Hence it suffices to show that it is a morphism of groupoids. Let $\mathfrak{T}$ be a formal scheme. There are maps of categories

$$\begin{tikzcd}
\mathfrak{S}_{\mathfrak{L}}(\mathfrak{T}) \arrow[r, "\hat{H}(\mathfrak{T})"] \arrow[dr, "H(\mathfrak{T})"'] & \hat{\mathfrak{R}}_{\mathfrak{L}}(\mathfrak{T}) \arrow[d, "\iota(\mathfrak{T})"] \\
& \mathfrak{R}_{\mathfrak{L}}(\mathfrak{T}),
\end{tikzcd} \tag{4.3.7.1}$$

where $H(\mathfrak{T})$ and $\iota(\mathfrak{T})$ are functors. But since $\iota(\mathfrak{T})$ is injective on objects and is a faithful functor, it is readily checked that $\hat{H}(\mathfrak{T})$ is a functor, and indeed a morphism of groupoids. □

The next example should help in picturing an infinitesimal groupoid. It is taken from [McQ05, Definition II.1.2, page 33]

Example 4.3.8 (Infinitesimal groupoid)**.** An infinitesimal groupoid $\mathfrak{R}$ on a scheme X should be thought as defining, for all $x \in X$, an equivalence relation on an analytic neighbourhood of x together with infinitesimal stabilisers. For instance, let $k := \mathbb{R}$ be the field of real numbers and let $\mathbb{G}_m$ act naturally on $X := \mathbb{A}^1_k$. The infinitesimal groupoid induced by this action is represented in Figure 6. Note the *singularity* of $\mathfrak{R}$ in the origin z. Therein, the infinitesimal

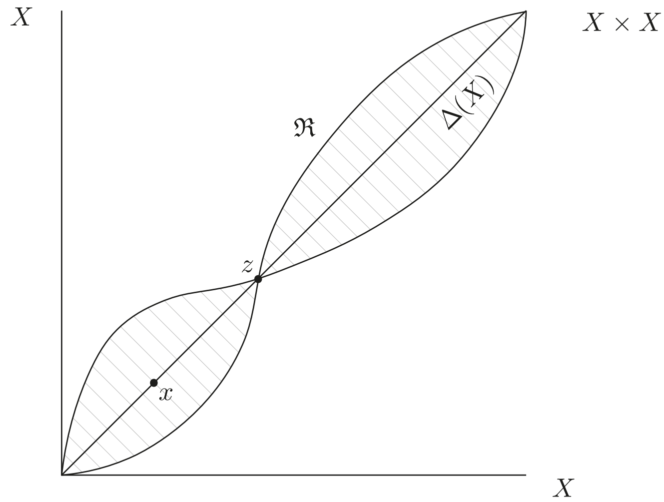


Figure 6: An infinitesimal groupoid

orbit is mapped to a single point.

The remaining material of this section is a formal consequence of the results proved so far.

Definition 4.3.9 (Formal restriction of groupoids)**.** Let $\mathfrak{R}$ be a formal (resp. infinitesimal) groupoid on a formal scheme $\mathfrak{X}$ over a formal scheme $\mathfrak{L}$. Let $g : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of formal schemes over $\mathfrak{L}$. The *formal restriction* (resp. *infinitesimal restriction*) of $\mathfrak{R}$ by g (or simply to $\mathfrak{Y}$ when g is understood) is a formal (resp. infinitesimal) groupoid $\mathfrak{R}|_{\mathfrak{Y}}$ (resp. $\hat{\mathfrak{R}}|_{\mathfrak{Y}}$) on $\mathfrak{Y}$ satisfying the universal property of Definition 4.2.16 (Restriction of groupoids of schemes) in the category of formal (resp. infinitesimal) groupoids. By general category theory, if it exists, it is unique.

Construction 4.3.10 (Formal restriction of groupoids)**.** Notation as in Definition 4.3.9 (Formal restriction of groupoids), the formal groupoid $\mathfrak{R}|_{\mathfrak{Y}}$ is constructed as in Construction 4.2.17 (Restriction of groupoids of schemes) in the category of formal schemes, where fibre products of schemes are replaced by fibre products of formal schemes (Lemma 3.1.10 (Fibre products of formal schemes)). This is a formal groupoid. Define the infinitesimal groupoid $\hat{\mathfrak{R}}|_{\mathfrak{Y}}$ to be the infinitesimal groupoid of $\mathfrak{R}|_{\mathfrak{Y}}$ from Construction 4.3.6 (Infinitesimal groupoids of formal groupoids).

Lemma 4.3.11 (Universal property of formal restriction)**.** *Notation as in Construction 4.3.10 (Formal restriction of groupoids), the groupoid* $\mathfrak{R}|_{\mathfrak{Y}}$ *(resp.* $\hat{\mathfrak{R}}|_{\mathfrak{Y}}$*)*

is the formal (resp. infinitesimal) restriction of $\mathfrak{R}$ to $\mathfrak{Y}$.

Proof. For formal groupoids, the proof is the same as Lemma 4.2.18 (Universal property of restriction in schemes). For infinitesimal groupoids, combine the universal property of formal restriction just proved and the universal property of infinitesimal groupoids (Lemma 4.3.7 (Universal property of infinitesimal groupoids)). □

Definition 4.3.12 (Invariant morphisms of formal schemes)**.** Let $\mathfrak{R}$ be a formal groupoid on a formal scheme $\mathfrak{X}$ over a formal scheme $\mathfrak{L}$. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of formal schemes over $\mathfrak{L}$ is $\mathfrak{R}$-invariant if the diagram

$$\mathfrak{R} \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{X} \xrightarrow{f} \mathfrak{Y} \tag{4.3.12.1}$$

is commutative.

Definition 4.3.13 (Formal pullback of groupoids)**.** Let $\mathfrak{R}$ be a formal (resp. infinitesimal) groupoid on a formal scheme $\mathfrak{X}$ over a formal scheme $\mathfrak{L}$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an $\mathfrak{R}$-invariant morphism of formal schemes over $\mathfrak{L}$ and let $g : \mathfrak{Y}' \to \mathfrak{Y}$ be a morphism of formal schemes over $\mathfrak{L}$. Let $f' : \mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ and $g' : \mathfrak{X}' \to \mathfrak{X}$ denote the base change of f and g respectively. The *formal pullback* (resp. *infinitesimal pullback*) of $\mathfrak{R}$ by g (or to X' when g is understood) is a formal (resp. infinitesimal) groupoid $\mathfrak{R}'$ (resp. $\hat{\mathfrak{R}}'$) on X' satisfying the the universal property of Definition 4.2.21 (Pullback of groupoids of schemes) in the category of formal (resp. infinitesimal) groupoids. By general category theory, if the pullback exists, it is unique.

Construction 4.3.14 (Formal pullback of groupoids)**.** Notation as in Definition 4.3.13 (Formal pullback of groupoids), the formal groupoid $\mathfrak{R}'$ and the infinitesimal groupoid $\hat{\mathfrak{R}}'$ are constructed as in Construction 4.2.22 (Pullback of groupoids of schemes) in the category of formal schemes. This is a formal groupoid. If $\mathfrak{R}$ is an infinitesimal groupoid, then so is $\hat{\mathfrak{R}}'$. This follows from the fact that the left most square of Diagram (4.1.21.1) is Cartesian and Lemma 3.4.4 (Base change of thickenings).

Lemma 4.3.15 (Universal property of formal pullback)**.** *Notation as in Construction 4.3.14 (Formal pullback of groupoids), the formal (resp. infinitesimal) groupoid $\mathfrak{R}'$ (resp. $\hat{\mathfrak{R}}'$) is the formal (resp. infinitesimal) pullback of the formal (resp. infinitesimal) groupoid $\mathfrak{R}$ by g.*

Proof. For both formal and infinitesimal groupoids, the proof is the same as Lemma 4.2.23 (Universal property of pullback in schemes). □

4.3.16 (Formal Pullback and restriction of groupoids)**.** Notation as in Definition 4.3.13 (Formal pullback of groupoids), there is a morphism of formal (resp. infinitesimal) groupoids $\mathfrak{R}' \to \mathfrak{R}|_{\mathfrak{X}'}$ (resp. $\hat{\mathfrak{R}}' \to \hat{\mathfrak{R}}|_{\mathfrak{X}'}$). This comes from the universal property of formal (resp. infinitesimal) restriction of groupoids (Lemma 4.3.15 (Universal property of formal pullback)). If g is a monomorphism, these are isomorphisms. Indeed, after reducing to the case of sets, the result follows from Paragraph 4.1.22 (Pullback and restriction of groupoids of sets).

4.4 Regular infinitesimal groupoids

This section defines regularity for infinitesimal groupoids using the definition of regular thickenings. These are the infinitesimal analogue of smooth groupoids. It is shown that restriction of adic infinitesimal groupoids commutes with localisation and preserves regularity. Finally, infinitesimal stabilisers are defined.

Definition 4.4.1 (Regular infinitesimal groupoids)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a scheme L. Then, $\mathfrak{R}$ is an *adic infinitesimal groupoid* if $\mathfrak{R}$ is an adic formal scheme and the sheaf of ideals of the thickening e is locally finitely generated. Furthermore, $\mathfrak{R}$ is a *regular infinitesimal groupoid* if $\mathfrak{R}$ is an adic locally Noetherian formal scheme and e is a regular thickening. In particular, a regular infinitesimal groupoid is an adic infinitesimal groupoid.

The next paragraph shows that an adic infinitesimal groupoid has a natural filtration, hence it makes sense to speak about first order data and higher order data.

4.4.2 (Filtration on adic infinitesimal groupoids)**.** Let $\mathfrak{R}$ be an adic infinitesimal groupoid on a scheme X over a scheme L and let $\mathscr{K}$ be the kernel ideal corresponding to the thickening $e : X \to \mathfrak{R}$. Since $\mathscr{K}$ is a locally finitely generated sheaf of ideals, locally on $\mathfrak{R}$, $\mathscr{K}$ is an adic ideal of definition (Lemma 2.4.5 (Kernel of thickenings)). Let R_n be the closed subscheme of $\mathfrak{R}$ associated to the sheaf of ideals $\mathscr{K}^{n+1}$ and consider the natural morphism obtained from the universal property of the colimit

$$\operatorname*{colim}_{n\in\mathbb{N}} R_n \to \mathfrak{R}. \tag{4.4.2.1}$$

Since $\mathscr{K}$ is locally an adic ideal of definition, this is an isomorphism locally on the target. Therefore, it is an isomorphism. In other words, $\mathscr{K}$ is a *global* ideal of definition. As a result, $\mathscr{O}_{\mathfrak{R}}$ admits a canonical exhaustive filtration

$$... \subseteq \mathscr{K}^2 \subseteq \mathscr{K} \subseteq \mathscr{O}_{\mathfrak{R}}. \tag{4.4.2.2}$$

The scheme R_n is the *component of order* n of $\mathfrak{R}$.

A groupoid induces an infinitesimal groupoid. If the former is smooth, the latter is regular. Algebraically, this is the completion along the diagonal ideal.

4.4.3 (Infinitesimal groupoids from groupoids of schemes)**.** Let R be a groupoid on a scheme X over a scheme L. By Construction 4.3.6 (Infinitesimal groupoids of formal groupoids), R induces an infinitesimal groupoid $\mathfrak{R}$ on X over L. If $e : X \to R$ is locally of finite presentation, then $\mathfrak{R}$ is an adic groupoid (part (1) of Lemma 3.5.6 (Infinitesimal neighbourhoods of schemes)). In this case, if $X = \operatorname{Spec} B$ and $R = \operatorname{Spf} C$ are affine and K is the kernel ideal of $e^{\#} : C \to B$, the infinitesimal groupoid $\mathfrak{R} = \operatorname{Spf} \hat{C}_K$ where $\hat{C}_K$ is the K-adic completion of C. If furthermore X is locally Noetherian, $\mathfrak{R}$ is an adic locally Noetherian groupoid (part (2) of Lemma 3.5.6 (Infinitesimal neighbourhoods of schemes)). Also, if R is locally Noetherian and e is a quasi-regular immersion, $\mathfrak{R}$ is a regular infinitesimal groupoid (part (3) of Lemma 3.5.6 (Infinitesimal neighbourhoods of schemes)). Finally, if R is a smooth groupoid on a locally Noetherian scheme X, then e is a regular immersion (Lemma 4.2.9 (Regular immersions of groupoids)) and R is locally Noetherian ([Sta23, Lemma 01T6]). Therefore $\mathfrak{R}$ is a regular infinitesimal groupoid. In particular, regular infinitesimal groupoids can be thought as the infinitesimal analogue of algebraic stacks.

Example 4.4.4 (Infinitesimal groupoids of morphisms)**.** If $f : X \to Y$ is a morphism of schemes over L, by Example 4.2.10 (Groupoids induced by morphisms), it induces an equivalence relation D_f on X over L. By Construction 4.3.6 (Infinitesimal groupoids of formal groupoids), $\mathfrak{D}_f$ is an infinitesimal equivalence relation. It is the *relative de Rham space* of f (or of X over Y when f is implicit). If f is smooth and X is locally Noetherian, $D_f \to X$ is smooth, hence $\mathfrak{D}_f$ is a regular infinitesimal equivalence relation (Paragraph 4.4.3 (Infinitesimal groupoids from groupoids of schemes)). When $Y = L$ and f is the structural morphism, $\mathfrak{D}_{X/L} := \mathfrak{D}_f$ is the *de Rham space* of X over L. Since for any infinitesimal groupoid $\mathfrak{R}$ on a scheme X over a scheme L, there is a morphism

$$j = t \times s : \mathfrak{R} \to X \times_L X, \tag{4.4.4.1}$$

by Construction 3.5.5 (Infinitesimal neighbourhood of immersions) applied to the diagonal immersion $\Delta_{X/L} : X \to X \times_L X$, there is a unique morphism of infinitesimal groupoids

$$\hat{j} : \mathfrak{R} \to \mathfrak{D}_{X/L}. \tag{4.4.4.2}$$

Example 4.4.5 (Infinitesimal groupoids of group schemes)**.** If G is a group scheme over L, by Example 4.2.11 (Group schemes), it induces a groupoid G on L over L. Let $\mathfrak{G}$ be the infinitesimal groupoid of G from Paragraph 4.4.3 (Infinitesimal groupoids from groupoids of schemes). Then $\mathfrak{G}$ is a higher order Lie algebra of G on L.

Example 4.4.6 (Infinitesimal groupoids of group actions)**.** If G is a group scheme acting on X over L, by Example 4.2.12 (Group actions on schemes), it induces a groupoid of schemes $G \times_L X$ on X over L. Let $\mathfrak{G}$ be the infinitesimal groupoid of $G \times_L X$ from Paragraph 4.4.3 (Infinitesimal groupoids from groupoids of schemes). Suppose furthermore that $G \to L$ is a smooth morphism and X is locally Noetherian. Then $G \times_L X$ is a smooth groupoid on X, hence $\mathfrak{G}$ is a regular infinitesimal groupoid. If furthermore G acts freely on X, then $\mathfrak{G}$ is a regular infinitesimal equivalence relation. The infinitesimal groupoid $\mathfrak{G}$ can be thought as the Lie algebra $\mathfrak{g}$ of G acting infinitesimally on X.

The next lemmata study adic infinitesimal groupoids on local schemes.

Lemma 4.4.7 (Infinitesimal groupoids on local schemes)**.** *Let $\mathfrak{R}$ be an adic infinitesimal groupoid on a local scheme $X = \operatorname{Spec} B$ with closed point x over a scheme L. Then $\mathfrak{R} = \operatorname{Spf} C$, where C is a local ring with maximal ideal corresponding to $e(x)$.*

Proof. Firstly, $\mathfrak{R}$ is affine by Lemma 3.4.9 (Image of thickening is affine). Let K be the ideal corresponding to the thickening e. Since K is, by assumption, finitely generated, Lemma 2.4.6 (Descent of local rings along thickenings) implies that C is a local ring with maximal ideal corresponding to $e(x)$. □

Lemma 4.4.8 (Restriction to local schemes)**.** *Let $\mathfrak{R}$ be an adic infinitesimal groupoid on a scheme X over a scheme L. Let $x \in X$ and let X_x denote $\operatorname{Spec} \mathscr{O}_{X,x}$. Let $\mathfrak{R}_x$ denote the complete localisation of $\mathfrak{R}$ in x. Then, the infinitesimal restriction $\mathfrak{R}$ to X_x is $\mathfrak{R}_x$. Furthermore, if $\mathfrak{R}$ is regular, so is $\mathfrak{R}_x$.*

Proof. Let $(X \times_L X)_{e(x)}$ denote the localisation of $X \times_L X$ in $e(x)$ and consider the following commutative solid diagram

$$\begin{tikzcd}
X_x \arrow[rr] \arrow[d] & & \mathfrak{R}_x \arrow[r] \arrow[d] \arrow[dl, dashed, shift left] & (X \times_L X)_{e(x)} \arrow[d] \\
X_x \arrow[r] \arrow[d] & \hat{\mathfrak{R}}|_{X_x} \arrow[r] \arrow[ur, dashed, shift left] & \mathfrak{R}|_{X_x} \arrow[r] \arrow[d] & X_x \times_L X_x \arrow[d] \\
X \arrow[rr] & & \mathfrak{R} \arrow[r] & X \times_L X,
\end{tikzcd} \tag{4.4.8.1}$$

where $\hat{\mathfrak{R}}|_{X_x}$ and $\mathfrak{R}|_{X_x}$ are the infinitesimal restriction and formal restriction of $\mathfrak{R}$ to X_x respectively. It is shown that the there exist unique dashed morphisms making the diagram commute. By uniqueness, this is enough to show that $\mathfrak{R}_x = \hat{\mathfrak{R}}|_{X_x}$.

It is clear $X = \operatorname{Spec} B$ may be assumed affine. Therefore, $\mathfrak{R} = \operatorname{Spf} C$ and $\hat{\mathfrak{R}}|_{X_x} = \operatorname{Spf} C'$ are also affine (Lemma 3.4.9 (Image of thickening is affine)). Let $\mathfrak{p}$ be the prime ideal of B corresponding to x and let $\mathfrak{q}$ and $\mathfrak{q}'$ be the open prime ideals of C and C' respectively, corresponding to $e(x)$ and $e'(x)$.

It is firstly shown that $\hat{\mathfrak{R}}|_{X_x} = \mathfrak{R}|_{X_x}$. This shows in particular that the morphism $\mathfrak{R}_x \to \hat{\mathfrak{R}}|_{X_x}$ exists. To this end, note that $X_x \to X$ is a monomorphism ([Gro60, Proposition 2.4.2, page 102]), hence, by [Sta23, Lemma 01L3],

$$X \times_{(X \times_L X)} (X_x \times_L X_x) = X_x \times_X X_x = X_x. \tag{4.4.8.2}$$

By the pasting lemma for pullback diagrams, the bottom left square of Diagram (4.4.8.1) is Cartesian. But now Lemma 3.4.4 (Base change of thickenings) implies that $X_x \to \mathfrak{R}|_{X_x}$ is a thickening of formal schemes, hence $\hat{\mathfrak{R}}|_{X_x} = \mathfrak{R}|_{X_x}$ by the universal property of infinitesimal neighbourhoods.

Next, it is now shown that there exists a unique dashed morphism from $\hat{\mathfrak{R}}|_{X_x}$ to $\mathfrak{R}_x$. Note that C' is an adic ring (Paragraph 2.1.21 (Tensor product of admissible rings)). Therefore, Lemma 4.4.7 (Infinitesimal groupoids on local schemes) implies that C' is a local ring with maximal ideal $\mathfrak{q}'$. By the universal property of complete localisation, it suffices to show that an element of C not in $\mathfrak{q}$ is mapped to a unit in C'. Since C' is a local ring with maximal ideal $\mathfrak{q}'$, it suffices to show that an element of C not in $\mathfrak{q}$ is mapped to an element of C' not in $\mathfrak{q}'$. But by construction, the contraction of $\mathfrak{q}'$ under the morphism $C \to C'$ is $\mathfrak{q}$. This shows that the dashed morphism exists.

Finally, suppose that $\mathfrak{R}$ is a regular groupoid. Then C is an adic Noetherian ring and its complete localisation $C_{\{\mathfrak{q}\}}$ is an adic Noetherian ring flat and taut over C (Paragraph 2.1.23 (Complete localisation of adic Noetherian rings)).

Therefore, Lemma 2.4.9 (Flat base change of regular thickenings) applies to show that $X_x \to \mathfrak{R}_x = \hat{\mathfrak{R}}|_{X_x}$, is a regular thickening. This shows that $\mathfrak{R}_x$ is a regular infinitesimal groupoid. $\square$

The previous lemma can be applied to describe the infinitesimal restriction of groupoids to local schemes.

4.4.9 (Complete localisation of groupoids)**.** Let $R := \operatorname{Spec} C$ be an affine groupoid of finite presentation on an affine Noetherian scheme $X = \operatorname{Spec} B$ over a scheme L. Let $x \in X$ be a point corresponding to a prime ideal $\mathfrak{p}$ of B. Let K be the ideal of C corresponding to the immersion $e : X \to R$. Recall that the infinitesimal groupoid $\mathfrak{R}$ induced by R is $\operatorname{Spf} \hat{C}$, where $\hat{C}$ is the K-adic completion of C. Then, Lemma 4.4.8 (Restriction to local schemes) establishes that the restriction of $\mathfrak{R}$ to X_x is $\operatorname{Spf} \hat{C}_{\{\mathfrak{p}\}}$, the complete localisation of $\hat{C}$ in $\mathfrak{p}$. By construction, this is equivalent to localising C in $\mathfrak{p}$ and then completing with respect to the $K_{\mathfrak{p}}$-adic topology.

Finally, the infinitesimal counterpart of groupoid stabiliser is introduced. These are Lie algebras of higher order.

4.4.10 (Infinitesimal stabilisers)**.** Let $\mathfrak{R}$ be an adic infinitesimal groupoid on a scheme X over a scheme L and let $x \in X$ be a point. Motivated by Paragraph 4.2.13 (Stabilisers of groupoids of schemes), the *infinitesimal stabiliser* in x is defined to be the fibre product

$$\mathfrak{G}_x = \mathfrak{R} \times_{(X \times_L X)} (\operatorname{Spec} \kappa(x))\,. \tag{4.4.10.1}$$

The same argument in the above mentioned paragraph shows that $\mathfrak{G}_x$ is a formal groupoid. This fits in a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec} \kappa(x) & \longrightarrow & \mathfrak{G}_x & \longrightarrow & \operatorname{Spec} \kappa(x) \\ \downarrow & & \downarrow & & \downarrow \\ X_x & \longrightarrow & \mathfrak{R}_x & \longrightarrow & X_x \times_L X_x \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathfrak{R} & \longrightarrow & X \times_L X, \end{array} \tag{4.4.10.2}$$

where every square is Cartesian. Since e is a thickening,

$$\operatorname{Spec} \kappa(x) \to \mathfrak{G}_x \tag{4.4.10.3}$$

is a thickening (Lemma 3.4.4 (Base change of thickenings)). Therefore $\mathfrak{G}_x$ is an infinitesimal groupoid over a point. It follows that $\mathfrak{G}_x = \operatorname{Spf} E_x$ is an affine formal scheme (Lemma 3.4.9 (Image of thickening is affine)), where E_x is an admissible $\kappa(x)$-algebra endowed with a $\kappa(x)$-rational point. Note that E_x is a quotient ring of the complete localisation of $\mathfrak{R}$ in x. Therefore, if $\mathfrak{R}$ is a regular infinitesimal groupoid, $\mathfrak{R}_x$ is a regular infinitesimal groupoid (Lemma 4.4.8 (Restriction to local schemes)) and, using Paragraph 2.1.25 (Quotients of adic Noetherian local rings), it follows that E_x is an adic Noetherian ring. Finally, Lemma 4.4.7 (Infinitesimal groupoids on local schemes) shows that E_x is an adic Noetherian local ring.

4.4.11 (Infinitesimal stabilisers and Lie algebras)**.** Let R be a smooth groupoid on a locally Noetherian scheme X over a scheme L. Let $x \in X$ be a point with residue field $\kappa(x)$ and let G_x be the group stabiliser in x. Let $\mathfrak{G}_x$ denote the infinitesimal stabiliser of the induced regular infinitesimal groupoid $\mathfrak{R}$. By construction, every square in the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Spec}\kappa(x) & \longrightarrow & \mathfrak{G}_x & \longrightarrow & G_x & \longrightarrow & \operatorname{Spec}\kappa(x) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathfrak{R} & \longrightarrow & R & \longrightarrow & X \times_L X \end{array} \tag{4.4.11.1}$$

is Cartesian. Hence $\mathfrak{G}_x$ is the completion of G_x at its identity element $e(x)$ (Lemma 3.5.3 (Infinitesimal neighbourhoods and fibre products)). Therefore, the infinitesimal stabiliser can be thought as a higher order Lie algebra of the group stabiliser in $x \in X$.

4.5 Foliations

The final section of this chapter is the devoted to the relation between infinitesimal groupoids and foliations. A foliation can be thought as a truncation of an infinitesimal equivalence relation where all the components of order strictly higher than one are discarded. Being closed under the Lie bracket corresponds to the property of being transitive, i.e. the existence of the morphism c. For a smooth foliation over the complex numbers, by Frobenius' theorem, no information is lost when truncating. Some examples are provided to show how infinitesimal groupoids may help in the study of foliations. Since this section is not used elsewhere, its tone is slightly more relaxed.

Definition 4.5.1 (Foliations)**.** Let X be a smooth scheme over a Noetherian scheme L. A *distribution* $\mathscr{F}$ on X is a subsheaf of the tangent sheaf $\mathscr{T}_{X/L}$. The distribution $\mathscr{F}$ is a *foliation* if it is closed under the Lie bracket. A foliation $\mathscr{F}$ is *smooth* if the quotient sheaf $\mathscr{T}_{X/L}/\mathscr{F}$ is locally free.

The first order component of an infinitesimal groupoid $\mathfrak{R}$ induces a distribution. It will be later shown that, if $\mathfrak{R}$ is an infinitesimal equivalence relation, the induced distribution is in fact a foliation.

Construction 4.5.2 (Distributions of infinitesimal groupoids)**.** Let $\mathfrak{R}$ be an adic Noetherian infinitesimal groupoid on a scheme X over a scheme L. Let $\mathscr{K}$ (resp. $\mathscr{I}$) be the kernel of $e^\# : \mathscr{O}_\mathfrak{R} \to \mathscr{O}_X$ (resp. $\Delta^\#_{X/L} : \mathscr{O}_{X\times_L X} \to \mathscr{O}_X$). Let R_1 and D_1 be the components of order 1 of $\mathfrak{R}$ and $X \times_L X$ respectively (vid. Paragraph 4.4.2 (Filtration on adic infinitesimal groupoids)). These are the subschemes cut out by $\mathscr{K}^2$ and $\mathscr{I}^2$ respectively. The morphisms

$$X \xrightarrow{e} \mathfrak{R} \xrightarrow{j} X \times_L X \tag{4.5.2.1}$$

induce morphisms

$$X \xrightarrow{e_1} R_1 \xrightarrow{j_1} D_1. \tag{4.5.2.2}$$

By construction, $\mathscr{K}/\mathscr{K}^2$ (resp. $\mathscr{I}/\mathscr{I}^2$) is the conormal sheaf of e (resp, $\Delta_{X/L}$). Note that, by definition of sheaf of differentials, $\mathscr{I}/\mathscr{I}^2 = \Omega^1_{X/L}$. Therefore, the Jacobi–Zariski exact sequence ([Sta23, Lemma 0E1Z]) yields an exact sequence of morphisms of sheaves over X

$$\Omega^1_{X/L} \xrightarrow{q} \mathscr{K}/\mathscr{K}^2 \to \Omega^1_{R_1/D_1}|_X \to 0. \tag{4.5.2.3}$$

If j is a monomorphism, [Sta23, Lemma 01UY] shows that q is surjective. Define

$$\mathscr{F}_{\mathfrak{R}} := \mathscr{H}om\left(\operatorname{im} q, \mathscr{O}_X\right). \tag{4.5.2.4}$$

Since $\mathscr{H}om$ is left exact, $\mathscr{F}_{\mathfrak{R}}$ is a subsheaf of $\mathscr{T}_{X/L}$, hence it is a distribution. By definition, if $\mathfrak{R}$ is a regular infinitesimal groupoid, $\mathscr{K}/\mathscr{K}^2$ is locally free. If furthermore $\mathfrak{R}$ is a regular infinitesimal equivalence relation, q is surjective and therefore $\mathscr{T}_{X/L}/\mathscr{F}_{\mathfrak{R}}$ is a locally free sheaf over X, so that $\mathscr{F}_{\mathfrak{R}}$ is a smooth distribution.

The next paragraph compares restriction and pullback for foliations. In the literature, what is here called restriction of a foliation is usually called pullback. Certainly, the author has no pretention of changing the terminology currently in use. In fact, the only reason for adopting this non-standard terminology is to follow the terminology pertaining groupoids used in [KM97].

4.5.3 (Restriction and pullback of foliations)**.** Let $\mathfrak{R}$ be an adic infinitesimal equivalence relation on a scheme X over a Noetherian scheme L and let $g : Y \to X$ be an L-morphism. Let $\mathscr{F}$ and $\mathscr{F}|_Y$ be the foliations on X and Y respectively induced by $\mathfrak{R}$ and $\mathfrak{R}|_Y$ respectively. Using the universal property, it is straightforward to verify that $\mathscr{F}|_Y$ is the dual sheaf of the colimit of the commutative diagram

$$\begin{tikzcd} g^*\Omega^1_{X/L} \arrow[r] \arrow[d] & g^*\left(\mathscr{K}/\mathscr{K}^2\right) \\ \Omega^1_{Y/L}, & \end{tikzcd} \tag{4.5.3.1}$$

where $\mathscr{K}$ is defined as in Construction 4.5.2 (Distributions of infinitesimal groupoids). On the other hand, if $f : X \to Y$ is an $\mathfrak{R}$-invariant L-morphism and $g : Y' \to Y$ is an L-morphism, the pullback of $\mathscr{F}$ by the base change $g' : X' \to X$ of g is simply the dual sheaf of

$$(g')^*\left(\mathscr{K}/\mathscr{K}^2\right). \tag{4.5.3.2}$$

Next, the derivative with respect to an infinitesimal equivalence relation (or foliation) is defined. When $\mathfrak{R}$ is the de Rham space of X, the induced derivative performs differentiation of sections up to any order, even in positive characteristic. As expected, the derivative of the induced foliation is the usual exterior derivative.

4.5.4 (Derivatives)**.** Let $\mathfrak{R}$ be an adic infinitesimal equivalence relation on a scheme X over a scheme L. A function $f : X \to \mathbb{A}^1_L$ is $\mathfrak{R}$-*invariant* if $s^\#(f) = t^\#(f) \in \mathscr{O}_{\mathfrak{R}}$. Set

$$d := s^\# - t^\# : \mathscr{O}_X \to \mathscr{O}_{\mathfrak{R}}. \tag{4.5.4.1}$$

Note that, since s and t are homeomorphisms, $\mathscr{O}_{\mathfrak{R}} = s_* \mathscr{O}_{\mathfrak{R}} = t_* \mathscr{O}_{\mathfrak{R}}$. The operator d measures the extent to which a function is constant along the infinitesimal equivalence classes of $\mathfrak{R}$. Since it is a difference of morphisms of rings, it is neither a morphism of rings, nor $\mathscr{O}_X$-linear. Furthermore, since $e^{\#}(s^{\#}(f) - t^{\#}(f)) = f - f = 0$, its image is contained in $\mathscr{K}$, the sheaf of ideals associated to e. Using Construction 4.5.2 (Distributions of infinitesimal groupoids), it is clear that d induces a morphism

$$\mathscr{O}_X \to \Omega^1_X \to \mathscr{K}/\mathscr{K}^2. \tag{4.5.4.2}$$

This is the usual derivative of a foliation. Hence d should be thought as a higher order derivative with respect to the infinitesimal equivalence classes of $\mathfrak{R}$.

The next example shows how the derivative computes the differentials up to any order. It is taken from [D'A16, §4, page 3].

Example 4.5.5 (Derivative of polynomials)**.** Let k be a field and let $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$. Let $\mathfrak{D}$ be the de Rham space of X (vid. Example 4.4.4 (Infinitesimal groupoids of morphisms)). Then

$$\mathfrak{D} = \operatorname{Spf}\left(\lim_{n\in\mathbb{N}} \frac{k[x,y]}{(y-x)^{n+1}}\right), \tag{4.5.5.1}$$

and the the structural morphisms are given by $s^{\#}(f) = f(y)$ and $t^{\#}(f) = f(x)$. Introduce the variable $\epsilon := y - x$. Then

$$\begin{aligned} d : k[x] &\to \lim_{n\in\mathbb{N}} \left(\frac{k[x,\epsilon]}{(\epsilon)^{n+1}}\right) \\ f(x) &\to (f(x+\epsilon) - f(x)). \end{aligned} \tag{4.5.5.2}$$

If $f(x) = a_0 + a_1 x + a_2 x^2 + ...$, then

$$\begin{aligned} d(f) = &\epsilon\left(a_1 + 2a_2 x + 3a_3 x^2 + ...\right) + \\ &\epsilon^2 (a_2 + 3a_3 x + ...) + \\ &\epsilon^3 (a_3 + ...) + ... \end{aligned} \tag{4.5.5.3}$$

Note that, when k is a field of characteristic zero, the coefficient of ϵ^n is

$$\frac{1}{n!} \cdot \frac{\partial^n f}{\partial x^n}. \tag{4.5.5.4}$$

The first order data of an infinitesimal equivalence relation consists of vector fields. It should not be surprising that the higher order data consists of differential operators of higher order.

4.5.6 (Differential operators)**.** Notation as in Construction 4.5.2 (Distributions of infinitesimal groupoids), suppose in addition that $\mathfrak{R}$ is an adic infinitesimal equivalence relation. A *differential operator* of $\mathfrak{R}$ of degree $n \in \mathbb{N}$ is an endomorphism ∂ of $\mathscr{O}_X$ such that there exists a factorisation

$$\partial : \mathscr{O}_X \xrightarrow{s_n^{\#}} \mathscr{O}_{R_n} \xrightarrow{\psi} \mathscr{O}_X, \tag{4.5.6.1}$$

where ψ is a morphism of $\mathscr{O}_X$-module and $\mathscr{O}_{R_n}$ is an $\mathscr{O}_X$-module via $t_n^\#$. The set of differential operators of order n is an $\mathscr{O}_X$-module, denoted by $\mathrm{Diff}^n(\mathfrak{R})$. The module of all differential operators is

$$\mathrm{Diff}(\mathfrak{R}) := \bigcup_{n\in\mathbb{N}} \mathrm{Diff}^n(\mathfrak{R}) \subseteq \mathrm{End}(\mathscr{O}_X). \tag{4.5.6.2}$$

For any open subset U, a differential operator of order zero over U is simply a section of $\mathscr{O}_X$ over U. A differential operator of order one is a section of $\mathscr{O}_X$ together with a vector field of $\mathscr{F}_{\mathfrak{R}}$ over U. By construction, there is a surjective morphism of $\mathscr{O}_X$-modules

$$\mathscr{H}om(\mathscr{O}_{\mathfrak{R}}, \mathscr{O}_X) = \operatorname*{colim}_{n\in\mathbb{N}} \mathscr{H}om(\mathscr{O}_{R_n}, \mathscr{O}_X) \to \mathrm{Diff}(\mathfrak{R}). \tag{4.5.6.3}$$

This is in fact an isomorphism ([Gro67, Proposition 16.8.4, page 41]).

On the affine line, differential operators are precisely the higher order derivatives.

Example 4.5.7 (Differential operators on the affine line)**.** Notation as in Example 4.5.5 (Derivative of polynomials), let $\{\epsilon^n\}_{n\in\mathbb{N}}$ be a basis of $\Gamma(X, \mathscr{O}_{\mathfrak{D}})$ over $k[x]$. Let ψ_n be the dual of ϵ^n with respect to this basis and let ∂_n be the associated differential operator. Then, when k is a field of characteristic zero, by (4.5.5.4),

$$\partial_n = \frac{1}{n!} \cdot \frac{\partial^n}{\partial x^n}. \tag{4.5.7.1}$$

When k is a field of positive characteristic, this is used to define a divided power structure on $\mathrm{Diff}(\mathfrak{R})$ by declaring ∂_p to be the p^{th} divided power of ∂_1.

The next paragraph shows that the distribution induced by an infinitesimal equivalence relation is a foliation. It does so by expressing the composition of differential operators in terms of the composition morphism c of the infinitesimal equivalence relation.

4.5.8 (Composition and lie bracket)**.** Let $\mathfrak{R}$ be an adic infinitesimal groupoid on a Noetherian scheme X over a scheme L. Motivated by Paragraph 4.5.6 (Differential operators), it is reasonable to define

$$\mathrm{Diff}^n(\mathfrak{R}) := \mathscr{H}om(\mathscr{O}_{R_n}, \mathscr{O}_X). \tag{4.5.8.1}$$

The morphism c can then be used to define composition of differential operators. Suppose that X is affine and let $\partial : \mathscr{O}_{R_n} \to \mathscr{O}_X$ and $\theta : \mathscr{O}_{R_m} \to \mathscr{O}_X$ be two differential operators of order n and m respectively. By the universal property of tensor product, there is an induced morphism

$$(\partial, \theta) : \mathscr{O}_{R_n} \otimes_{(t_n^\#, t_m^\#)} \mathscr{O}_{R_m} \to \mathscr{O}_X \tag{4.5.8.2}$$

Equivalently, there is an induced morphism

$$(\partial \circ i, \theta) : \mathscr{O}_{R_n} \otimes_{(s_n^\#, t_m^\#)} \mathscr{O}_{R_m} \to \mathscr{O}_X \tag{4.5.8.3}$$

On the other hand, a computation shows that there exists a factorisation by the dashed morphism

$$\begin{tikzcd} R_n \times_{(s_n,t_m)} R_m \arrow[r] \arrow[d, dashed, "c_{n,m}"] & \mathfrak{R} \times_{(s,t)} \mathfrak{R} \arrow[d, "c"] \\ R_{n+m} \arrow[r] & \mathfrak{R}. \end{tikzcd} \tag{4.5.8.4}$$

Therefore, the morphism c induces a morphism

$$\mathscr{O}_{R_{n+m}} \xrightarrow{c^{\#}_{n,m}} \mathscr{O}_{R_n \times_{(s_n,t_m)} R_m} = \mathscr{O}_{R_n} \otimes_{(s_n^{\#}, t_m^{\#})} \mathscr{O}_{R_m}, \tag{4.5.8.5}$$

where the equality follows from the fact that s_n and t_m induce homeomorphisms of topological spaces and isomorphisms of residue fields. Composing with the morphism $(\partial \circ i, \theta)$ gives an element of

$$\mathscr{H}om\left(\mathscr{O}_{R_{n+m}}, \mathscr{O}_X\right), \tag{4.5.8.6}$$

that is a differential operator of order $n+m$ and is defined to be the *composition* of ∂ and θ, denoted by $\partial \circ \theta$. Note the asymmetry in the definition of composition. In particular, $\partial \circ \theta \neq \theta \circ \partial$ necessarily. It can be checked that the difference

$$[\partial, \theta] := \partial \circ \theta - \theta \circ \partial \tag{4.5.8.7}$$

is a differential operator of order $n+m-1$. The operation of composition endows $\mathrm{Diff}(\mathfrak{R})$ with the structure of a non-commutative $\mathscr{O}_X$-algebra. In particular, if $\mathfrak{R}$ is an infinitesimal equivalence relation, and ∂ and θ are differential operators of order one, that is they are elements of $\mathscr{F}_{\mathfrak{R}}$, then $[\partial, \theta]$ is a differential operator of order one. This shows that $\mathscr{F}_{\mathfrak{R}}$ is a foliation.

Conversely, Frobenius' theorem implies that a smooth foliation over the complex numbers induces a regular infinitesimal equivalence relation.

4.5.9 (Graph of a foliation)**.** For a smooth foliation $\mathscr{F}$ on a smooth scheme X of finite type over $\mathbb{C}$, Construction 4.5.2 (Distributions of infinitesimal groupoids) admits an inverse. Let $\mathscr{A}$ be the sub-algebra of $\mathrm{End}\,\mathscr{O}_X$ generated by the vector fields of the foliation. Using Frobenius' theorem, it can be shown that these are the differential operators of a unique adic infinitesimal equivalence relation known as the *graph of a foliation.* This establishes a bijection

$$\left\{ \begin{array}{c} \mathscr{F} \subseteq \mathscr{T}_X \,|\, \mathscr{F} \text{ is a} \\ \text{smooth foliation.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathfrak{R} \rightrightarrows X \,|\, \mathfrak{R} \text{ is a regular} \\ \text{infinitesimal equivalence relation.} \end{array} \right\}. \tag{4.5.9.1}$$

Given this bijective correspondence, the reader may rightfully ask why studying infinitesimal groupoids should be worthwhile.

4.5.10 (Infinitesimal groupoids and foliations)**.** In light of (4.5.9.1), infinitesimal groupoids can be thought as a more flexible version of algebraic foliations. They can be helpful in two ways.

1. In the study of foliation singularities. Let $\mathfrak{R}$ be a regular infinitesimal groupoid which is generically an infinitesimal equivalence relation. This yields a morphism (4.5.2.3) which is generically surjective but might not be surjective. The singularities of the foliation correspond to the locus where the morphism is not surjective, that is when the groupoid has non-trivial stabiliser. At the same time, $\mathfrak{R}$ is a regular infinitesimal groupoid and it can be thought as a resolution of the foliation singularity (vid. Example 4.5.11 (Radial foliation)).

2. In the study of foliations in positive and mixed characteristic. In this case, as observed in Example 4.5.7 (Differential operators on the affine line), the higher order differential operators have strictly more information than the first order differential operators. As a result, several constructions in characteristic zero can only be carried out in this greater generality (vid. Example 5.3.4 (Factorisations of Frobenius morphism)).

Example 4.5.11 (Radial foliation)**.** Let k be a field, let $B := k[x, y]$ and let $X := \mathbb{A}^2_k = \operatorname{Spec} B$. Suppose that $G := \mathbb{G}_m = k^\times$ acts on X with weights $(1, 1)$. This induces a regular infinitesimal groupoid $\mathfrak{G}$ which in turn induces a foliation $\mathscr{F}_G$ on X given by the subsheaf of $\mathscr{T}_{X/k}$ generated by the vector field $\partial := x\,\partial/\partial x + y\,\partial/\partial y$. This is represented in Figure 7. The grey lines are

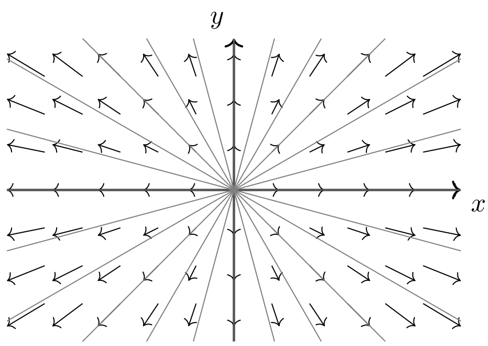


Figure 7: Radial foliation

the subvarieties of X which are everywhere tangent to ∂. Once removed the singularity at the origin, these are precisely the orbits of the action. Note that the foliation has a singularity where the stabiliser is non-trivial.

Example 4.5.12 (Exponential foliation)**.** Let k be a field of characteristic zero, let $B := k[x, y]$ and let $X := \mathbb{A}^2_k = \operatorname{Spec} B$. Consider the infinitesimal equivalence relation

$$(x, y) \sim (s, t) \text{ if and only if } ye^{-x} = te^{-s}. \tag{4.5.12.1}$$

In details, this is the infinitesimal equivalence relation $\mathfrak{R}$ which is the formal subscheme of the de Rham space of X

$$\mathfrak{D}_X = \operatorname{Spf}\left(\lim_{n\in\mathbb{N}} \frac{k[x, y, s, t]}{(x - s, y - t)^{n+1}}\right) \tag{4.5.12.2}$$

cut out by the equation $y = te^{x-s}$. Alternatively, using Paragraph 6.4.1 (Frobenius' theorem), this is the infinitesimal equivalence relation whose differential operators are generated by the vector field $\partial := \partial/\partial x + y\,\partial/\partial y$. This is represented in Figure 8.

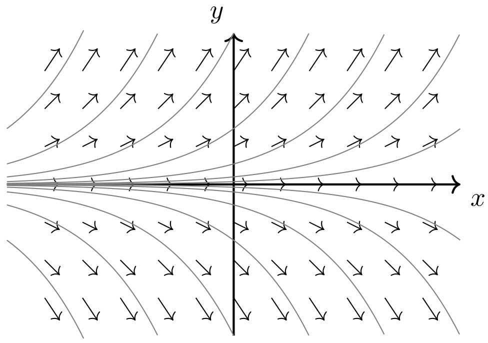


Figure 8: Exponential foliation

The grey lines are cut out by the equation $y = ae^x$, as a varies in k, and are everywhere tangent to ∂. Note that, except for the case $a = 0$, they are not algebraic subvarieties. Such infinitesimal equivalence relation does not come from an equivalence relation of schemes, hence it is not *algebraically integrable*. On the other hand, if the characteristic of the field is $p > 0$, this is not an infinitesimal equivalence relation. In fact, compositing ∂ with itself p times yields

$$\partial \circ \partial \circ \ldots \partial = y\,\partial/\partial y. \qquad (4.5.12.3)$$

This is not a vector field induced by $\mathfrak{R}$, hence the morphism c is not well-defined and $\mathfrak{R}$ cannot be an infinitesimal equivalence relation. The property of being closed under p^{th} powers is known as being *p-closed*. This shows that being an infinitesimal groupoid in positive characteristic seems to be much closer to being algebraically integrable. This is related to a conjecture proposed by Ekedahl and Shepherd-Barron stating that, if a foliation is p-closed for all $p \gg 0$, then it is algebraically integrable [ESBT99, Conjecture F, page 1]. The reader is invited to consult [Bos01, Theorem 2.1, page 168] for further discussion and a partial result.

Chapter 5

First integrals

5.1 First integrals

This section studies invariant sections of infinitesimal groupoids. These are called first integrals. Considering first integrals of an infinitesimal groupoid, as opposed to invariant sections of a groupoid, has at least two advantages. The first one is that, since the the source and target morphisms are thickenings, the first integrals form a sheaf of algebras. The second one is that first integrals behave well with respect to localisation.

Definition 5.1.1 (First integrals)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a scheme L. A *first integral* of $\mathfrak{R}$ over an open subset $U \subseteq X$ is a section of $\mathscr{O}_X$ over U such that its associated function

$$f : U \to \mathbb{A}^1_L \tag{5.1.1.1}$$

is an $\mathfrak{R}|_U$-invariant L-morphism, where $\mathfrak{R}|_U$ is the infinitesimal restriction of $\mathfrak{R}$.

Using the fact that s and t are thickenings, the next paragraph defines the derivative with respect to an infinitesimal groupoid.

5.1.2 (Structure sheaf of groupoid)**.** The morphisms of formal schemes s, t and e induce homeomorphisms of topological spaces. As a result, they induce morphisms of sheaves of topological rings

$$s^\# : \mathscr{O}_X \to s_*\mathscr{O}_\mathfrak{R} = \mathscr{O}_\mathfrak{R} \tag{5.1.2.1}$$

$$t^\# : \mathscr{O}_X \to t_*\mathscr{O}_\mathfrak{R} = \mathscr{O}_\mathfrak{R} \tag{5.1.2.2}$$

$$e^\# : \mathscr{O}_\mathfrak{R} \to e_*\mathscr{O}_X = \mathscr{O}_X. \tag{5.1.2.3}$$

Define

$$d := s^\# - t^\# : \mathscr{O}_X \to \mathscr{O}_\mathfrak{R} \tag{5.1.2.4}$$

to be the difference between $s^\#$ and $t^\#$. Note that this is not a morphism of $\mathscr{O}_X$-modules and should be thought instead as the derivative with respect to the infinitesimal groupoid (vid. Paragraph 4.5.4 (Derivatives)).

The first integrals are precisely the kernel of the derivative.

Lemma 5.1.3 (Characterisation of first integrals)**.** *Let $\mathfrak{R}$ be an infinitesimal groupoid on affine scheme X over a scheme L. Let f be a global section of $\mathscr{O}_X$. Let $\mathscr{K}$ denote the sheaf of ideals associated to $e^\#$ and let $K = \Gamma(X, \mathscr{K})$. Then $d(f) \in K$ and f is a first integral of $\mathfrak{R}$ if and only if $d(f) = 0 \in \Gamma(X, \mathscr{O}_{\mathfrak{R}})$.*

Proof. Since $f = e^\#(s^\#(f)) = e^\#(t^\#(f))$, $e^\#(s^\#(f) - t^\#(f)) = 0$. This means precisely that $d(f) \in K$.

By Lemma 3.4.9 (Image of thickening is affine), $\mathfrak{R} = \operatorname{Spf} C$ is an affine groupoid. By definition, $f : X \to \mathbb{A}^1_L$ is $\mathfrak{R}$-invariant if and only if the diagram

$$\mathfrak{R} \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{f} \mathbb{A}^1_L \tag{5.1.3.1}$$

is commutative. This happens if and only if the diagram of sheaves

$$\mathscr{O}_{\mathbb{A}^1_L} = \mathscr{O}_L[a] \xrightarrow{f^\#} \mathscr{O}_X \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} \mathscr{O}_{\mathfrak{R}} \tag{5.1.3.2}$$

is commutative. Since X and $\mathfrak{R}$ are affine schemes, commutativity of the above diagram can be verified solely on the element a. But by construction, f is the image of a under $f^\#$. Therefore, the diagram is commutative if and only if $s^\#(f) = t^\#(f)$. ☐

The next lemma essentially shows that the pre-sheaf of first integrals is a sheaf.

Lemma 5.1.4 (Being a first integral is a local property)**.** *Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a scheme L. Let f be a global section of $\mathscr{O}_X$.*

1. *Suppose that f is a first integral of $\mathfrak{R}$ over X and let $U \subseteq X$ be an open subset. Then the restriction $f|_U$ is a first integral of $\mathfrak{R}$ over U.*
2. *Suppose that there exists an open cover $\{U_i\}_{i \in I}$ of X such that, for all $i \in I$, the restriction $f|_{U_i}$ is a first integral of $\mathfrak{R}$ over $U_i \subseteq X$. Then f is a first integral of $\mathfrak{R}$ over X.*

Proof. 1. Observe the following solid commutative diagram

$$\begin{tikzcd} \mathfrak{R}|_U \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & U \times_{f|_U} U \arrow[r] \arrow[d] & U \times_L U \arrow[d] \\ \mathfrak{R} \arrow[r] & X \times_f X \arrow[r] & X \times_L X, \end{tikzcd} \tag{5.1.4.1}$$

where the right-most square is Cartesian. Note that $f|_U$ is $\mathfrak{R}|_U$-invariant if and only if the dashed morphism exists. This holds by the universal property of fibre products.

2. Proving that f is a first integral of $\mathfrak{R}$ over X amounts to checking that the dashed morphism in the commutative diagram

$$\begin{tikzcd} \mathfrak{R} \arrow[rr, dashed] \arrow[dr, "j"'] & & X \times_f X \arrow[dl] \\ & X \times_L X & \end{tikzcd} \tag{5.1.4.2}$$

exists. Since $X \times_f X \to X \times_L X$ is a monomorphism, if the dashed morphism exists, it is unique. Note that, since e is a homeomorphism, $\{U_i\}_{i \in I}$ is an open cover of the topological space of $\mathfrak{R}$. Therefore $\{\mathfrak{R}|_{U_i}\}_{i \in I}$ is an open cover of $\mathfrak{R}$. By uniqueness, it suffices to show that a compatible morphism

$$\mathfrak{R}|_{U_i} \to X \times_f X \tag{5.1.4.3}$$

exists for all $i \in I$. This can be constructed as the composition

$$\mathfrak{R}|_{U_i} \to X \times_{f|_{U_i}} X \to X \times_f X, \tag{5.1.4.4}$$

which exists since $f|_{U_i}$ is a first integral of $\mathfrak{R}$ over U_i. $\square$

Definition 5.1.5 (Sheaf of first integrals)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a scheme L. The *sheaf of first integrals* of $\mathfrak{R}$ is a sheaf on X defined as

$$\mathscr{O}_X^{\mathfrak{R}} = \mathrm{Eq}\left(\mathscr{O}_X \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} \mathscr{O}_{\mathfrak{R}}\right) = \mathrm{Ker}\left(\mathscr{O}_X \xrightarrow{d} \mathscr{O}_{\mathfrak{R}}\right) \tag{5.1.5.1}$$

in the category of sheaves of topological rings over X. Since $\mathscr{O}_X^{\mathfrak{R}}$ is the pre-sheaf kernel of a morphism of sheaves, it is already a sheaf ([Sta23, Lemma 00W2]), hence

$$\mathscr{O}_X^{\mathfrak{R}}(U) = \{f \in \mathscr{O}_X(U) \,|\, d(f) = 0\}. \tag{5.1.5.2}$$

5.1.6 (Sheaf of first integrals)**.** In order to justify nomenclature, it has to be shown that, for all open subsets $U \subseteq X$,

$$\mathscr{O}_X^{\mathfrak{R}}(U) = \{f \in \mathscr{O}_X(U) \,|\, f \text{ is a first integral of } \mathfrak{R}\}. \tag{5.1.6.1}$$

Let f be a section of $\mathscr{O}_X$ over U and pick an affine open cover $\{U_i\}_{i \in I}$ of U. By part (1) and (2) of Lemma 5.1.4 (Being a first integral is a local property), f is a first integrals of $\mathfrak{R}$ over U if and only if $f|_{U_i}$ is a first integral of $\mathfrak{R}$ over U_i, for all $i \in I$. By Lemma 5.1.3 (Characterisation of first integrals), this happens if and only if $f \in \mathscr{O}_X^{\mathfrak{R}}(U_i)$, for all $i \in I$. Since $\mathscr{O}_X^{\mathfrak{R}}$ is a sheaf, this happens if and only if $f \in \mathscr{O}_X^{\mathfrak{R}}(U)$.

5.1.7 (First integrals and invariance)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a scheme L and let $f : X \to Y$ be an L-morphism of schemes. It is readily checked that the composition

$$f^{-1}\mathscr{O}_Y \to \mathscr{O}_X \xrightarrow{d} \mathscr{O}_{\mathfrak{R}} \tag{5.1.7.1}$$

is zero. Indeed, this follows from the fact that $f \circ s = f \circ t$. By the universal property of the kernel, there exists a unique morphism

$$f^{-1}\mathscr{O}_Y \to \mathscr{O}_X^{\mathfrak{R}}. \tag{5.1.7.2}$$

At last, it is shown that the first integrals form a sheaf of algebras.

5.1.8 (First integrals form a sheaf of algebras)**.** It is readily checked that, for all open subsets $U \subseteq X$, $\mathscr{O}_X^{\mathfrak{R}}(U)$ is a subring of $\mathscr{O}_X(U)$. Indeed, this is a consequence of the fact that the morphisms $s^{\#}$ and $t^{\#}$ over U are unital morphisms of rings. This shows that $\mathscr{O}_X^{\mathfrak{R}}$ is a sheaf of subrings of $\mathscr{O}_X$. In particular, if X is integral, $\mathscr{O}_X^{\mathfrak{R}}$ is a sheaf of integral domains. Furthermore, since the structural morphism $p : X \to L$ is $\mathfrak{R}$-invariant, Paragraph 5.1.7 (First integrals and invariance) implies the existence of a unique morphism

$$p^{-1}\mathscr{O}_L \to \mathscr{O}_X^{\mathfrak{R}}, \tag{5.1.8.1}$$

endowing $\mathscr{O}_X^{\mathfrak{R}}$ with the structure of a $p^{-1}\mathscr{O}_L$-algebra.

Next, the relation between first integrals and localisation is explored. Firstly, local first integrals are defined as those sections whose derivative locally vanishes. This notion has no counterpart when studying invariant sections of a groupoid or group action.

Definition 5.1.9 (Local first integrals)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a scheme L and let $x \in X$. Let $\mathfrak{R}_x$ the restriction of the groupoid $\mathfrak{R}$ via the localisation $X_x := \operatorname{Spec} \mathscr{O}_{X,x} \to X$. The *local first integrals* of $\mathfrak{R}$ in $x \in X$, denoted by $\mathscr{O}_{X,x}^{\mathfrak{R}}$, are the first integrals of $\mathfrak{R}_x$ over X_x. When X is integral and $x = \eta$ is its generic point, elements of $\mathscr{O}_{X,\eta}^{\mathfrak{R}}$ are the *rational first integrals* of $\mathfrak{R}$.

The next lemma shows that the ring of local first integrals is a local ring.

Lemma 5.1.10 (Structure of local first integrals)**.** *Let $\mathfrak{R}$ be an infinitesimal groupoid on a local scheme $X = \operatorname{Spec} B$ over a scheme L. Then the inclusion*

$$B^{\mathfrak{R}} \to B \tag{5.1.10.1}$$

is a morphism of local rings. In particular, the ring of local first integrals is a local ring.

Proof. By Lemma 3.4.9 (Image of thickening is affine), $\mathfrak{R} = \operatorname{Spf} C$ is an affine groupoid. Let $\mathfrak{m}$ be the maximal ideal of the local ring B. It is shown that the inverse image $\mathfrak{m}^{\mathfrak{R}} := \mathfrak{m} \cap B^{\mathfrak{R}}$ is the set of non-units of $A^{\mathfrak{R}}$.

Since $\mathfrak{m}^{\mathfrak{R}}$ is an ideal, it cannot contain any units. Conversely, let $u \in B^{\mathfrak{R}} \setminus \mathfrak{m}^{\mathfrak{R}}$. It is shown that it is a unit in $B^{\mathfrak{R}}$. By assumption, $u \notin \mathfrak{m}$. Since B is a local ring, u is a unit in B. Therefore there exists a unique $v \in B$ such that $uv = 1$. To conclude, it suffices to show that $v \in B^{\mathfrak{R}}$. Applying d to the equation $uv = 1$ yields $d(uv) = d(1) = 0$. By definition of d,

$$s^{\#}(u)s^{\#}(v) = t^{\#}(u)t^{\#}(v) \in C \tag{5.1.10.2}$$

Since $d(u) = 0$, $s^{\#}(u) = t^{\#}(u) =: w \in C$. Rearranging (5.1.10.2) yields

$$w(s^{\#}(v) - t^{\#}(v)) = 0. \tag{5.1.10.3}$$

Since $s^{\#}$ is a unital morphism of rings, w, the image of the unit u, is a unit in C. Therefore $s^{\#}(v) - t^{\#}(v) = 0$, hence $v \in B^{\mathfrak{R}}$. But now $\mathfrak{m}^{\mathfrak{R}}$ is an ideal consisting of the non-units of $B^{\mathfrak{R}}$. Hence, [Sta23, Lemma 00E9] implies that $B^{\mathfrak{R}}$ is a local ring and that $B^{\mathfrak{R}} \to B$ is a morphism of local rings. ☐

The next lemma states that, for a regular infinitesimal groupoid, pulling back by a localisation morphism commutes with taking first integrals.

Lemma 5.1.11 (Localisation and first integrals)**.** *Let $\mathfrak{R}$ be a regular infinitesimal groupoid on a affine integral scheme $X = \operatorname{Spec} B$ over a scheme L. Let $A = B^{\mathfrak{R}}$, let S be a multiplicatively closed subset of A and let $\iota : A \to A' := S^{-1}A$ be the localisation morphism. Define $B' := B \otimes_A A'$, $X' := \operatorname{Spec} B'$ and let $\mathfrak{R}'$ be the pullback of $\mathfrak{R}$ to X'. Then $\mathfrak{R}'$ is a regular infinitesimal groupoid on X' over L and*

$$A' = (B')^{\mathfrak{R}'} \tag{5.1.11.1}$$

Proof. Note that, since X is affine, $\mathfrak{R} = \operatorname{Spf} C$ is affine (Lemma 3.4.9 (Image of thickening is affine)). Consider the equaliser diagram

$$A = \operatorname{Eq}\left(B \overset{s^{\#}}{\underset{t^{\#}}{\rightrightarrows}} C\right). \tag{5.1.11.2}$$

Let K be the kernel ideal of C corresponding to the closed immersion $e : X \to \mathfrak{R}$. Since C is adic,

$$C = \lim_{n\in\mathbb{N}} \frac{C}{K^{n+1}}. \tag{5.1.11.3}$$

Without loss of generality, $0 \in A$ may be assumed not in S. Since $A \to A'$ is a localisation, the functor $\_ \otimes_A A'$ is exact. Applying the exact functor to the equaliser gives

$$A' = \operatorname{Eq}\left(B' \overset{s^{\#}}{\underset{t^{\#}}{\rightrightarrows}} \overline{C}\right), \tag{5.1.11.4}$$

where $\overline{C} := C \otimes_A A'$. Let $\overline{K} := K \cdot \overline{C}$. By definition of pullback groupoid, $\mathfrak{R}' = \operatorname{Spf} C'$, where C' is the completion of $\overline{C}$ with respect to $\overline{K}$. Firstly, $\overline{C} \to C'$ is taut and flat and C' is Noetherian (Paragraph 2.1.23 (Complete localisation of adic Noetherian rings)). Therefore Lemma 2.4.9 (Flat base change of regular thickenings) applies to show that $C' \to B'$ is a regular thickening. Furthermore, the image of S in C does not intersect the prime ideal K. Indeed, if that were the case, the image of S would have to contain $0 \in B$. However, this cannot happen since S does not contain $0 \in A$ and $A \to B$ is injective. By Paragraph 2.1.23 (Complete localisation of adic Noetherian rings), the morphism $\overline{C} \to C'$ is injective. Therefore (5.1.11.1) holds. □

The final lemma states that a section which is generically a first integral is actually a first integral. This is a key step in the proof of Theorem 6.3.3 (Finite generation of first integrals).

Lemma 5.1.12 (First integrals at generic point)**.** *Let $\mathfrak{R}$ be a regular infinitesimal groupoid on an integral scheme X over a scheme L. Let η be the generic point of X. Let f be a local section of $\mathscr{O}_X$ over an affine open subset U. Then f is first integral of $\mathfrak{R}$ if and only if it is a rational first integral. That is, its restriction to $\operatorname{Spec}\kappa(\eta)$, $f|_\eta$, is a first integral of $\mathfrak{R}|_\eta$. Equivalently, for any affine open subset U,*

$$\mathscr{O}_X^{\mathfrak{R}}(U) = \mathscr{O}_X(U) \cap \mathscr{O}_{X,\eta}^{\mathfrak{R}} \subseteq \mathscr{O}_{X,\eta}. \tag{5.1.12.1}$$

Proof. Without loss of generality, after replacing U by X, X may be assumed affine. It follows that $\mathfrak{R} = \operatorname{Spf} C$ is affine too (Lemma 3.4.9 (Image of thickening is affine)). Let $X = \operatorname{Spec} B$, $B^{\mathfrak{R}} := \Gamma(X, \mathscr{O}_X^{\mathfrak{R}})$, $M := \mathscr{O}_{X,\eta}^{\mathfrak{R}}$, $F := \mathscr{O}_{X,\eta}$ and $C_\eta := \mathscr{O}_{\mathfrak{R}|_\eta}$. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B^{\mathfrak{R}} \arrow[r] \arrow[d] & B \arrow[r, "d"] \arrow[d] & C \arrow[d] \\ 0 \arrow[r] & M \arrow[r] & F \arrow[r, "d_\eta"] & C_\eta, \end{tikzcd} \tag{5.1.12.2}$$

whose rows are exact. This gives the existence of an injective morphism

$$B^{\mathfrak{R}} \to B \cap M \subseteq F. \tag{5.1.12.3}$$

It has to be shown that it is an isomorphism. Equivalently, it is shown that, if there exists an $f \in B$ such that $d_\eta(f|_\eta) = 0$, then $d(f) = 0$.

Let K be the kernel of the surjection $e^{\#} : C \to B$. Since C is complete with respect to the K-adic topology, it suffices to show that $d(f) \in K^{n+1}$ for all $n \in \mathbb{N}$. This is proved by induction on n. Certainly $d(f) \in K$ (Lemma 5.1.3 (Characterisation of first integrals)). Let now $n \in \mathbb{N}$. Suppose $d(f) \in K^n$. It is shown that $d(f) \in K^{n+1}$. Let K_η be the kernel of $C_\eta \to B_\eta$. Then $K_\eta = K \cdot C_\eta$, hence $K_\eta^{n+1} = K^{n+1} \cdot C_\eta$ for all $n \in \mathbb{N}$. It follows that, since $B \to B_\eta$ is flat,

$$K_\eta^n / K_\eta^{n+1} = K^n / K^{n+1} \otimes_B B_\eta. \tag{5.1.12.4}$$

Since $\mathfrak{R}$ is a regular infinitesimal groupoid, K^n/K^{n+1} is a projective module over B (Paragraph 3.4.11 (Being a regular thickening is a local property)). Since $B \to B_\eta$ is injective, applying $\_ \otimes_B K^n/K^{n+1}$ yields an injective morphism

$$K^n/K^{n+1} \to K_\eta^n / K_\eta^{n+1}. \tag{5.1.12.5}$$

Under this morphism, $d(f)$ is mapped to $d_\eta(f|_\eta)$. By assumption $d_\eta(f|_\eta) \in K_\eta^{n+1}$, hence $d(f) \in K^{n+1}$. The induction is completed. $\square$

5.2 First integrals and invariance

The aim of this section is to prove that, under suitable hypotheses, the invariant sections of a groupoid are precisely the first integrals of the induced infinitesimal groupoid. Intuitively, this amounts to proving that a function whose derivative is identically zero is constant along the equivalence classes of the groupoid. Since the derivative is of higher order, the result is also valid in positive characteristic. A more precise statement is the following.

Proposition 5.2.1 (First integrals and invariance)**.** *Let R be a groupoid on a scheme X over a scheme L and let $\mathfrak{R}$ denote its induced infinitesimal groupoid. Let $f : X \to Y$ be an L-morphism of schemes.*

1. *If f is R-invariant, then it is $\mathfrak{R}$-invariant.*
2. *Suppose furthermore that R is flat and locally of finite presentation with geometrically irreducible fibres and that X is locally Noetherian. Then, if f is $\mathfrak{R}$-invariant, it is R-invariant.*

Having irreducible fibres is necessary in order to avoid this type of example.

Example 5.2.2 (Action of a finite group)**.** Let G be a finite group acting freely on a scheme X. The induced infinitesimal groupoid is zero, hence every section is a first integral. On the other hand, if the action is non-trivial, there must exist a section which is not G-invariant.

In order to provide context and to fix notation, the proof of the proposition is shown on assuming the lemmata proved thereafter.

Proof of Proposition 5.2.1 (First integrals and invariance). Note that there are morphisms

$$\mathfrak{R} \to R \overset{s}{\underset{t}{\rightrightarrows}} X \xrightarrow{f} Y. \tag{5.2.2.1}$$

1. Using (5.2.2.1), it is clear that, if f is R-invariant, it is also $\mathfrak{R}$-invariant.
2. Consider the following commutative diagram

$$\begin{tikzcd}
& & R \arrow[rd, "j"] & \\
X \arrow[r, "e'"] \arrow[rru, bend left, "e"] \arrow[rrd, bend right, "\Delta_{X/Y}"'] & R' \arrow[ru, "\phi"] \arrow[rd] & & X \times_L X, \\
& & X \times_Y X \arrow[ru] &
\end{tikzcd} \tag{5.2.2.2}$$

where R' is defined to be the fibre product of the rhombus. Since R' is a fibre product of groupoids, it is a groupoid ([Sta23, Lemma 0041]). Showing that f is R-invariant amounts to showing that $R' \to R$ is an isomorphism.

Firstly, note that the morphism of groupoids $\phi : R' \to R$ is an immersion. Indeed, it is the base change of $X \times_Y X \to X \times_L X$. This is an immersion by [Sta23, Lemma 01KR].

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be the induced infinitesimal groupoids of R and R' respectively. The morphism $\phi : R' \to R$ induces a unique morphism $\hat{\phi} : \mathfrak{R}' \to \mathfrak{R}$. To conclude, by Lemma 5.2.3 (Lifting isomorphism), it suffices to show that $\hat{\phi}$ is an isomorphism. By assumption, there exists a morphism $\mathfrak{R} \to X \times_Y X$. By the universal property of fibre products (Lemma 3.1.10 (Fibre products of formal schemes)), there exists a unique morphism $\mathfrak{R} \to R'$. By the universal property of infinitesimal neighbourhoods, there exists a unique morphism $\mathfrak{R} \to \mathfrak{R}'$. By uniqueness, it follows that this is the inverse morphism to $\hat{\phi}$. □

The proof relies on showing that an isomorphism of infinitesimal groupoids lifts to an isomorphism of groupoids. This is stated in the next lemma.

Lemma 5.2.3 (Lifting isomorphism)**.** *Let R' and R be groupoids on a locally Noetherian scheme X over a scheme L, where R is a flat groupoid locally of finite presentation with geometrically irreducible fibres. Suppose that there exists a morphism of groupoids $\phi : R' \to R$ which is an immersion and suppose that the induced morphism on infinitesimal neighbourhoods*

$$\hat{\phi} : \mathfrak{R}' \xrightarrow{\sim} \mathfrak{R} \tag{5.2.3.1}$$

of e' and e respectively is an isomorphism. Then ϕ is an isomorphism.

The proof of this lemma has two steps. In the first step, the isomorphism is lifted to a dense open subset of R. Therein, the crux of the argument is that the completion of an adic Noetherian local ring is an injective morphism. In the second step, the transitivity of R is used to translate the isomorphism to the boundaries of the groupoid.

Lemma 5.2.4 (Lifting isomorphism to localisation)**.** *Let R' and R be groupoids on a locally Noetherian scheme X over a scheme L, where R is a flat groupoid locally of finite presentation. Suppose that there exists a morphism of groupoids $\phi : R' \to R$ which is an immersion and suppose that the induced morphism on infinitesimal neighbourhoods $\hat{\phi} : \mathfrak{R}' \to \mathfrak{R}$ of e' and e respectively is an isomorphism. Then, the surjective morphism of rings*

$$\mathscr{O}_{R,e(x)} \to \mathscr{O}_{R',e'(x)} \tag{5.2.4.1}$$

is an isomorphism.

Proof. The assumptions imply that R is locally Noetherian. This follows from the fact that $t : R \to X$ is locally of finite presentation and X is locally Noetherian ([Sta23, Lemma 01T6]). Since R' is a locally closed subscheme of R, R' is also locally Noetherian ([Sta23, Lemma 02IK]).

By assumption $\mathfrak{R}' = \mathfrak{R}$, hence $\mathfrak{R}'|_{X_x} = \mathfrak{R}|_{X_x}$. For ease of notation, let C, C' and B denote the local rings $\mathscr{O}_{R,e(x)}$, $\mathscr{O}_{R',e'(x)}$ and $\mathscr{O}_{X,x}$ respectively. Furthermore, let K and K' be the ideals corresponding to e and e' respectively. By Lemma 4.4.8 (Restriction to local schemes), $\mathfrak{R}|_{X_x}$ is the complete localisation of $\mathfrak{R}$ in x. By Paragraph 4.4.9 (Complete localisation of groupoids), this is isomorphic to $\hat{C}_K$, the K-adic completion C. Ditto for R'. It follows that there is an isomorphism

$$\hat{C}_K \xrightarrow{\sim} \hat{C}'_K. \tag{5.2.4.2}$$

But now there is a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d] \\ \hat{C}_K \arrow[r, "\sim"] & \hat{C'}_{K'}. \end{tikzcd} \tag{5.2.4.3}$$

Since R is locally Noetherian, C is a Noetherian local ring, hence, by [Sta23, Lemma 00MC], the K-adic completion $C \to \hat{C}_K$ is injective. Therefore $C \to C'$ is injective, hence an isomorphism. □

The isomorphism is now lifted to an open subset.

Lemma 5.2.5 (Lifting isomorphism to open set)**.** *Notation and assumptions as in Lemma 5.2.4 (Lifting isomorphism to localisation), there exists an open subset $U \subseteq R'$ containing $e(X)$ such that the composition*

$$\phi|_U : U \to R \xrightarrow{\phi} R \tag{5.2.5.1}$$

is an open immersion.

Proof. Fix an arbitrary $x \in X$. By assumption, (5.2.4.1) is an isomorphism. Therefore, the immersion $R' \to R$ is flat at $e'(x) \in R'$. Since an immersion into a locally Noetherian scheme is locally of finite presentation (combine part (1) of [Sta23, Lemma 01TX] with [Sta23, Lemma 01T5]), the theorem on openness of the flat locus ([Sta23, Theorem 0399]) may be applied to show existence of an open set $U_x \subseteq R'$ such that the composition $U_x \to R$ is flat. Define $U = \bigcup_{x \in X} U_x$. By construction, U is an open subset of R' containing $e(X)$. But now $U \to R$ is a flat monomorphism locally of finite presentation, hence, by [Sta23, Theorem 025G], it is an open immersion. □

This concludes the first step. Now the transitivity property is used, together with the assumption that the fibres are geometrically irreducible, to conclude that ϕ is an isomorphism.

Lemma 5.2.6 (Translation of isomorphism)**.** *Let R' and R be groupoids on a scheme X over a scheme L. Suppose that $t : R \to X$ is a flat morphism with geometrically irreducible fibres and assume there exists a morphism of groupoids $\phi : R' \to R$ and an open subset $U \subseteq R'$ containing $e(X)$ such that the composition $\phi|_U : U \to R$ is an open immersion. Then ϕ is an isomorphism.*

When both R' and R are equivalence relations, the basic idea of the proof is the following: by assumption $x \sim_R y$ and it has to be shown that $x \sim_{R'} y$. Since x and y are related in R, their respective equivalence classes, R_x and R_y, are equal. By assumption, the equivalence classes R'_x and R'_y contain open sets which are open dense subsets of $R_x = R_y$. Therefore, the intersection $R'_x \cap R'_y$ cannot be empty. But now R'_x and R'_y are two equivalence classes which are not disjoint. Hence, they must be equal. This implies that $x \sim_{R'} y$. The proof consists in turning this set-theoretic argument into a scheme-theoretic argument.

Proof of Lemma 5.2.6 (Translation of isomorphism). Note there is a commutative diagram

$$\begin{tikzcd}
R \times_{(s,t)} R \arrow[rr, "\mathrm{pr}_1 \times c", "\sim"'] \arrow[dr, "\mathrm{pr}_1"'] & & R \times_{(t,t)} R \arrow[dl, "\mathrm{pr}_1"] \\
& R &
\end{tikzcd} \tag{5.2.6.1}$$

where the top arrow is an isomorphism (top left of [Sta23, Equation 04LG]). Intuitively, taking the fibres of this diagram over a point $r \in R$ gives an isomorphism of equivalence classes between $s(r)$ and $t(r)$. Now consider the commutative *square*

$$\begin{tikzcd}
& R \times_{(s,t')} R' \arrow[r, "\mathbb{1}_R \times \phi"] & R \times_{(s,t)} R \arrow[dd, "\mathrm{pr}_1 \times c", "\wr"'] \\
R' \times_{(s',t')} R' \arrow[ur, "\phi \times \mathbb{1}_{R'}"] \arrow[dr, "\phi \times \mathbb{1}_{R'}"'] & & \\
& R \times_{(s,t')} R' \arrow[r, "i \times \phi"'] & R \times_{(t,t)} R.
\end{tikzcd} \tag{5.2.6.2}$$

It is shown that this is a Cartesian square. This is due to the existence of the composition morphism c' of R'. In order to understand this part of the proof, it is considerably easier to think of elements of R' and R as paths in X, and to think of the morphism c as composition of paths. Let T be a scheme. Giving a morphism to the fibre product of Diagram (5.2.6.2) is equivalent to giving a morphism

$$h : T \to R \times_{(s,t')} R' \tag{5.2.6.3}$$

such that the composition $c \circ h : T \to R$ factors through R'. This follows from the fact that ϕ is a monomorphism. But h factors uniquely through $R' \times_{(s',t')} R'$. Indeed, it follows from the properties of groupoids that

$$h : T \xrightarrow{(h') \times (\mathrm{pr}_2 \circ h)} R' \times_{(s',t')} R' \xrightarrow{\phi \times \mathbb{1}_{R'}} R \times_{(s,t')} R', \tag{5.2.6.4}$$

where h' is defined to be the composition

$$T \xrightarrow{(c \circ h) \times (i' \circ \mathrm{pr}_2 \circ h)} R' \times_{(s',t')} R' \xrightarrow{c'} R'. \tag{5.2.6.5}$$

Note that $c \circ h$ is well-defined since it factors through R'. This shows that Diagram (5.2.6.2) is Cartesian.

Define U_s to be the fibre product in the diagram

$$
\begin{tikzcd}
U_s \arrow[r] \arrow[d] & R \times_{(s,t')} R' \arrow[r, "\mathbb{1}_R \times \phi"] \arrow[d] & R \times_{(s,t)} R \arrow[r, "\mathrm{pr}_1"] \arrow[d] & R \arrow[d, "s"] \\
U \arrow[r] & R' \arrow[r, "\phi"] & R \arrow[r, "t"] & X,
\end{tikzcd}
\tag{5.2.6.6}
$$

where, by construction, every square is Cartesian. Ditto for replacing s with t. By assumption, $U \to R$ is an open immersion and, since t is flat, $U \to X$ is faithfully flat. Therefore, the base change $U_s \to R \times_{(s,t)} R$ is an open immersion and the base change $U_s \to R$ is faithfully flat. Ditto for U_t.

Let $U_{st} := U_s \times U_t$ be the fibre product over $R \times_{(s,t)} R \cong R \times_{(t,t)} R$. Certainly $U_{st} \to R \times_{(s,t)} R$ is an open immersion, therefore $U_{st} \to R$ is flat. It is shown that it is faithfully flat. Since $U_s \to R$ is surjective, it intersects every fibre of pr_1. Since the fibres of t are geometrically irreducible, so are the fibres of pr_1. Therefore, the intersection of U_s with every fibre is an open dense subset. Ditto for U_t. Since the intersection of open dense subsets is dense, the intersection of the fibre product U_{st} with every fibre of pr_1 is an open dense subset. This shows that the composition $U_{st} \to R$ is faithfully flat.

Finally, since Diagram (5.2.6.2) is Cartesian, there is a factorisation

$$U_{st} \to R' \times_{(s',t')} R' \xrightarrow{c'} R' \xrightarrow{\phi} R, \tag{5.2.6.7}$$

where the composition is faithfully flat. This is enough to show that ϕ is an isomorphism. To see this, note that ϕ is an immersion and, by (5.2.6.7), it is surjective. Therefore, it induces a homeomorphism and it suffices to show that

$$\phi^{\#} : \mathscr{O}_R \to \phi_* \mathscr{O}_{R'} \tag{5.2.6.8}$$

is an isomorphism. This is a local property on R. Since ϕ is an immersion, $\phi^{\#}$ is surjective on stalks and it suffices to show it is injective on stalks. But since $U_{st} \to R$ is faithfully flat, $\phi^{\#}$ is injective on stalks. The lemma follows. $\square$

Finally, the results are compiled together.

Proof of Lemma 5.2.3 (Lifting isomorphism). By Lemma 5.2.3 (Lifting isomorphism), ϕ induces an isomorphism between an open subset of R' and an open subset of R. Since the fibres of $t : R \to X$ are geometrically irreducible, Lemma 5.2.6 (Translation of isomorphism) implies that ϕ is an isomorphism. $\square$

This concludes the proof of Proposition 5.2.1 (First integrals and invariance).

The next paragraph shows that the assumptions of the proposition are satisfied in a special case of interest.

5.2.7 (Smooth groupoids with connected fibres)**.** A smooth groupoid with geometrically connected fibres R on a locally Noetherian scheme X over a scheme L satisfies the assumptions of Proposition 5.2.1 (First integrals and invariance). Indeed, by definition, R is flat and locally of finite presentation. Furthermore, its geometric fibres are integral schemes. To see this, note that they are smooth schemes over a field, hence normal schemes ([Sta23, Lemma 038X]). Since they are connected, normal and locally Noetherian, they are integral.([Sta23, Lemma 033N]).

In the following example, it is observed that infinitesimal groupoids are well-suited to the study of actions by geometrically connected groups.

Example 5.2.8 (Group invariance)**.** Let G be a smooth group scheme over a scheme L with geometrically connected fibres acting on a locally Noetherian scheme X over L. By Example 4.2.12 (Group actions on schemes), $G \times_L X$ is a smooth groupoid with geometrically connected fibres. Let $\mathfrak{G}$ denote the regular infinitesimal groupoid associated to the action of G. The hypothesis of Proposition 5.2.1 (First integrals and invariance) are satisfied, hence an L-morphism $f : X \to Y$ is G-invariant if and only if it is $\mathfrak{G}$-invariant.

The final example explores what it means to be a first integral of the de Rham space.

Example 5.2.9 (Locally constant functions)**.** Let X be a scheme of finite type over a field k. A section of X which is invariant with respect to all infinitesimal groupoids should be thought as a locally constant section. Define a section to be *locally constant* if it is invariant with respect to the de Rham space of X. Then Proposition 5.2.1 (First integrals and invariance) implies that, if X is geometrically irreducible, the locally constant sections are precisely the elements of the base field k. In particular, if k is a field of characteristic $p > 0$ and f is a non-constant function, the expression $d\,(f^p) \neq 0$ is meaningful.

5.3 Properties of first integrals

This section studies the algebraic properties of first integrals of a regular infinitesimal groupoid. The objective is to show that the inclusion of the ring of first integrals into the ring of sections is algebraically closed. To begin with, here is a remainder of what it means to be algebraically closed.

Definition 5.3.1 (Ring extensions)**.** A *ring extension* $A \to B$ is an injective morphism of integral domains. Recall that the *characteristic* $p \in \mathbb{N}$ of B is the element generating the kernel of the initial morphism $\mathbb{Z} \to B$. Since B is a domain, p is either 0 or a prime number. Since $A \to B$ is injective, A and B have the same characteristic. A ring extension $A \to B$ of characteristic $p > 0$ is *perfect* if any element $b \in B$ satisfying $b^p \in A$, also satisfies $b \in A$. Every ring extension in characteristic zero is, by definition, perfect. The ring extension $A \to B$ is *algebraically closed* if any element $b \in B$ satisfying $P(b) = 0$, for some non-zero polynomial $P(x) \in A[x]$, also satisfies $b \in A$. By definition, an algebraically closed ring extension is perfect.

The next lemma employs Lemma 2.4.11 (Regular thickenings and Frobenius) to show that, if the derivative of the p^{th} power of a local section is zero, then so is the derivative of the local section.

Lemma 5.3.2 (First integrals are perfect)**.** *Let $\mathfrak{R}$ be a regular infinitesimal groupoid on an integral scheme X over a scheme L. Then, for any affine open subset U of X,*

$$\mathscr{O}_X^{\mathfrak{R}}(U) \to \mathscr{O}_X(U) \tag{5.3.2.1}$$

is perfect.

Proof. Let $U = \operatorname{Spec} B$, then $\mathfrak{R} = \operatorname{Spf} C$ is affine by Lemma 3.4.9 (Image of thickening is affine). Then $C \to B$ is a regular thickening of adic rings. Consider the ring extension $s^{\#} : B \to C$ and assume it is of characteristic $p > 0$. By assumption, $d(b^p) = 0$ and it has to be shown that $x := d(b) = 0$. Then

$$0 = d(b^p) = s^{\#}(b^p) - t^{\#}(b^p) = (s^{\#}(b) - t^{\#}(b))^p = d(b)^p, \tag{5.3.2.2}$$

where the third equality is given by the fresher's dream rule in characteristic p. By Lemma 5.1.3 (Characterisation of first integrals), $x \in \ker e^{\#}$. The lemma amounts to showing that, if $x^p = 0 \in C$, then $x = 0$. This follows from Lemma 2.4.11 (Regular thickenings and Frobenius) and the fact that X is integral. □

Now, the next proposition shows the result.

Proposition 5.3.3 (First integrals are algebraically closed)**.** *Let $\mathfrak{R}$ be a regular infinitesimal groupoid on an integral scheme X over a scheme L. Then, for any affine open subset U of X,*

$$\mathscr{O}_X^{\mathfrak{R}}(U) \to \mathscr{O}_X(U) \tag{5.3.3.1}$$

is algebraically closed.

In the notation of the proof below, let $b \in B$ and let $P(x) \in A[x]$ be a non-zero polynomial such that $P(b) = 0$. It has to be shown that $b \in A$. By definition, this amounts to showing that $d(b) = 0$. Since d is morally a derivative, the chain rule gives

$$0 = d(P(b)) = P'(b) \cdot d(b), \tag{5.3.3.2}$$

where $P'(b)$ is the formal polynomial derivative of $P(x)$ applied to b. By the regularity assumption, it holds that, either $d(b) = 0$ or $P'(b) = 0$ (Lemma 2.4.10 (Trickle down)). In the former case, $b \in A$ and the proposition follows. In the latter case, the polynomial obtained is of lower degree and induction may be used. In positive characteristic, it can happen that the formal derivative of a non-constant polynomial is zero, thus voiding the induction. In such instance, Lemma 5.3.2 (First integrals are perfect) is employed to conclude.

Proof of Proposition 5.3.3 (First integrals are algebraically closed). Since U is affine, so is $\mathfrak{R}$ (3.4.9 (Image of thickening is affine)). Let $U := \operatorname{Spec} B$, $\mathfrak{R} :=$

Spf C and $A := B^{\mathfrak{R}}$. Then $C \to B$ is a regular thickening of adic rings. Let K denote its kernel.

Let $P(x) \in A[x]$ be a non-zero polynomial of degree e given by

$$P(x) = a_0 + a_1 x^1 + a_2 x^2 + \dots + a_e x^e, \tag{5.3.3.3}$$

where $a_e \neq 0$. The statement is proved by strong induction on the degree e. The base case $e = 0$ is trivial since a polynomial is either zero or does not have any roots. For the inductive step, suppose that $e > 0$. By assumption, $P(b) = 0$, hence $d(P(b)) = 0 \in C$. For ease of notation, let $s := s^{\#}(b)$ and $t := t^{\#}(b)$. Since the coefficients of $P(x)$ are in A, $s^{\#}(a_k) = t^{\#}(a_k) =: u_k$. A straightforward computation gives

$$d(P(b)) = \sum_{k=1}^{e} u_k \left(s^k - t^k\right). \tag{5.3.3.4}$$

After recalling the standard factorisation

$$s^k - t^k = (s - t)\left(\sum_{j=0}^{k-1} s^{k-1-j} t^j\right), \tag{5.3.3.5}$$

(5.3.3.4) may be rewritten as

$$d(P(b)) = d(b)\left(\sum_{k=1}^{e} u_k \sum_{j=0}^{k-1} s^{k-1-j} t^j\right). \tag{5.3.3.6}$$

By Lemma 5.1.3 (Characterisation of first integrals), $d(b) \in K$. Hence, by Lemma 2.4.10 (Trickle down), either $d(b) = 0$ or

$$w := \sum_{k=1}^{e} u_k \sum_{j=0}^{k-1} s^{k-1-j} t^j \in K. \tag{5.3.3.7}$$

If $d(b) = 0$, the induction step is completed. Hence suppose that $w \in K$. Consider the formal derivative of $P(x)$ with respect to x evaluated in b

$$P'(b) = \sum_{k=1}^{e} k a_k b^{k-1}. \tag{5.3.3.8}$$

Assuming $w \in K$, it is shown that $P'(b) = 0 \in B$. Compute the difference

$$w - t^{\#}\left(P'(b)\right) = \sum_{k=1}^{e} u_k \left(\left(\sum_{j=0}^{e-1} s^{k-1-j} t^j\right) - k t^{k-1}\right) \tag{5.3.3.9}$$

An elementary computation shows that

$$\left(\sum_{j=0}^{k-1} s^{k-1-j} t^j\right) - k t^{k-1} = d(b)\left(\sum_{j=0}^{k-2} (j+1) s^{k-2-j} t^j\right) \tag{5.3.3.10}$$

As a result, there exists a factorisation

$$\begin{aligned} w - t^{\#}\left(P'(b)\right) &= \sum_{k=1}^{e} u_k d(b) \left(\sum_{j=0}^{k-2} (j+1) s^{k-2-j} t^j \right) \\ &= d(b) \left(\sum_{k=1}^{e} u_k \left(\sum_{j=0}^{k-2} (j+1) s^{k-2-j} t^j \right) \right). \end{aligned} \tag{5.3.3.11}$$

This implies that $w - t^{\#}\left(P'(b)\right) \in K$. By assumption, $w \in K$, hence $t^{\#}\left(P'(b)\right) \in K$. But then, $e^{\#}\left(t^{\#}\left(P'(b)\right)\right) = 0$, so that

$$P'(b) = 0 \in B. \tag{5.3.3.12}$$

Now there are two cases to consider:

1. ($P'(x) \neq 0$). In this case, $P'(x)$ is a non-zero polynomial with coefficients in A, whose degree is strictly less than e. By (5.3.3.12) and strong induction, $b \in A$.

2. ($P'(x) = 0$). In this case, $ka_k = 0$ for all $k \leq e$. Since $k \in \mathbb{N}$, either $a_k = 0$ or the characteristic $p > 0$ divides k. This implies that

$$P(x) = a_0 + a_p x^p + a_{2p} x^{2p} + ... + a_{lp} x^{lp}, \tag{5.3.3.13}$$

   for some $l \in \mathbb{N}$ such that $a_{lp} \neq 0$. Since $e > 0$, $l > 0$. Define a polynomial

$$Q(x) = a_0 + a_p x + a_{2p} x^2 + ... + a_{lp} x^l. \tag{5.3.3.14}$$

   This is a non-zero polynomial with coefficients in A and, by construction, $Q(b^p) = 0$. Note that the degree of $Q(x)$ is strictly less than the degree of $P(x)$. By the inductive hypothesis, $d(b^p) = 0$, hence $b^p \in A$. By Lemma 5.3.2 (First integrals are perfect), $b \in A$. □

The next example shows that, in positive characteristic, regular infinitesimal groupoids behave very differently from foliations.

Example 5.3.4 (Factorisations of Frobenius morphism)**.** Let X be a scheme over a field k of characteristic $p > 0$. It is well-known that the first integrals of a smooth p-closed foliation give a factorisation of the Frobenius morphism ([MP97, Proposition 1.9, page 56]). On the other hand, if $\mathfrak{R}$ is a regular infinitesimal groupoid, Proposition 5.3.3 (First integrals are algebraically closed) implies that, if f^p is a first integral of $\mathfrak{R}$ for a local section f, so is f. Therefore, it cannot be a factorisation of the Frobenius morphism. This behaviour is much closer to the characteristic zero case.

5.4 Rank of infinitesimal groupoids

This section defines the rank of an infinitesimal groupoid. Intuitively, it is the dimension of an infinitesimal fibre of t. When an infinitesimal groupoid is induced by a finitely presented groupoid, the rank is the relative dimension. Subsequently, two important results are shown. The first one is that an immersion between regular infinitesimal groupoids of the same rank is an isomorphism and the second one is an upper bound on the rank of an infinitesimal groupoid in terms of the field of rational first integrals.

Definition 5.4.1 (Rank of infinitesimal groupoids)**.** Let $\mathfrak{R}$ be an adic infinitesimal groupoid on a local scheme $X = \operatorname{Spec} B$ over a scheme L. Let $x \in X$ be the unique closed point and let $\kappa(x)$ be its residue field. Then $\mathfrak{R} = \operatorname{Spf} C$ is the spectrum of a local ring with maximal ideal $e(x)$ (Lemma 4.4.7 (Infinitesimal groupoids on local schemes)) and its *rank* is defined to be

$$\operatorname{rank} \mathfrak{R} = \operatorname{rel\,dim}_{e(x)} t := \dim_{e(x)} \left(C \hat{\otimes}_B \kappa(x)\right) \in \mathbb{N} \cup \{\infty\}. \tag{5.4.1.1}$$

Given an adic infinitesimal groupoid $\mathfrak{R}$ over a general scheme X over a scheme L and a point $x \in X$, let $\mathfrak{R}_x$ be the infinitesimal restriction of $\mathfrak{R}$ by the morphism $X_x := \operatorname{Spec} \mathscr{O}_{X,x} \to X$. Then the *rank* of $\mathfrak{R}$ in $x \in X$ is defined to be

$$\operatorname{rank}_x \mathfrak{R} = \operatorname{rank} \mathfrak{R}_x \in \mathbb{N} \cup \{\infty\}. \tag{5.4.1.2}$$

The next lemma shows that, for a regular infinitesimal groupoid, its rank is determined by its foliation.

Lemma 5.4.2 (Rank of regular infinitesimal groupoids)**.** *Let $\mathfrak{R}$ be a regular infinitesimal groupoid on a local scheme X over a scheme L. Let $x \in X$ be the unique closed point. Let*

$$\mathscr{K} := \ker \left(\mathscr{O}_{\mathfrak{R}} \xrightarrow{e^{\#}} \mathscr{O}_X \right) \tag{5.4.2.1}$$

Then

$$\operatorname{rank} \mathfrak{R} = \dim_{\kappa(x)} \left(\mathscr{K}/\mathscr{K}^2 \otimes_{\mathscr{O}_X} \kappa(x) \right) \in \mathbb{N}. \tag{5.4.2.2}$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of B corresponding to the point $x \in X$ and let K be the ideal of C corresponding to the sheaf of ideals $\mathscr{K}$. The assumptions imply that $\mathfrak{R} := \operatorname{Spf} C$ is an affine groupoid over the local scheme $X := \operatorname{Spec} B$ where C is an adic Noetherian local ring with maximal ideal $\mathfrak{q}$ given by the pre-image of $\mathfrak{m}$ under $e^{\#}$. It has to be shown that

$$\dim_{\mathfrak{q}} \left(C \hat{\otimes}_B \kappa(x)\right) = \dim_{\kappa(x)} \left(K/K^2 \otimes_B \kappa(x)\right) =: r. \tag{5.4.2.3}$$

Note that $D := \left(C \hat{\otimes}_B \kappa(x)\right)$ is a quotient of the adic Noetherian local ring C. As a result, $D = C \hat{\otimes}_B \kappa(x) = C \otimes_B \kappa(x)$ is an adic Noetherian local ring (Paragraph 2.1.25 (Quotients of adic Noetherian local rings)) with maximal ideal $\mathfrak{n} := \mathfrak{q} \cdot D$.

It is shown that $\mathfrak{n} = K \cdot D$. Since e is a section of t, the extension ideal $e^{\#}\left(t^{\#}(\mathfrak{m}) \cdot C\right) \cdot B = \mathfrak{m}$. Since $C \to B$ is a surjection with kernel K and the pre-image of $\mathfrak{m}$ under $e^{\#}$ is $\mathfrak{q}$, $\mathfrak{q} = (K, t^{\#}(\mathfrak{m}) \cdot C)$. Furthermore, since the morphism $B \to C \to D$ factors through $\kappa(x)$, $\left(t^{\#}(\mathfrak{m}) \cdot C\right) \cdot D = 0$. This implies that

$$\mathfrak{n} = \mathfrak{q} \cdot D = (K, t^{\#}(\mathfrak{m}) \cdot C) \cdot D = K \cdot D. \tag{5.4.2.4}$$

By [Sta23, Proposition 00KQ], the dimension of D is equal to one plus the degree of the Hilbert function. Recall that the Hilbert function is defined by

$$\begin{aligned}\chi_D : \mathbb{N} &\to \mathbb{N} \\ n &\to \dim_{\kappa(x)} \mathfrak{n}^n/\mathfrak{n}^{n+1}\end{aligned} \tag{5.4.2.5}$$

as explained in [Sta23, Definition 00KA]. By the previous paragraph, $\mathfrak{n}^n$ is the ideal extension of K^n under the morphism $C \to D$,

$$\mathfrak{n}^n/\mathfrak{n}^{n+1} = K^n/K^{n+1} \otimes_C D = K^n/K^{n+1} \otimes_B \kappa(x). \tag{5.4.2.6}$$

Therefore, $\mathfrak{n}/\mathfrak{n}^2$ is a free $\kappa(x)$-module of rank r. Furthermore, since $\mathfrak{R}$ is a regular infinitesimal groupoid and symmetric powers commute with base change ([Eis95, part (b) of Proposition A2.2, page 576]), it follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\operatorname{Sym}^n_{\kappa(x)} \mathfrak{n}/\mathfrak{n}^2 &= \left(\operatorname{Sym}^n_B K/K^2\right) \otimes_B \kappa(x) \\ &= \left(K^n/K^{n+1}\right) \otimes_B \kappa(x) \\ &= \mathfrak{n}^n/\mathfrak{n}^{n+1}.\end{aligned} \tag{5.4.2.7}$$

By an elementary computation, it follows that

$$\chi_D(n) = \binom{n+r-1}{n}. \tag{5.4.2.8}$$

Hence, the order of growth of χ_D is $r-1$. This proves that the dimension of D is r. $\square$

5.4.3 (Global rank of infinitesimal groupoids)**.** Let $\mathfrak{R}$ be a regular infinitesimal groupoid on a connected scheme X over a scheme L. By Lemma 5.4.2 (Rank of regular infinitesimal groupoids), for all $x \in X$,

$$\operatorname{rank}_x \mathfrak{R} = \dim_{\kappa(x)} \left(\mathscr{K}/\mathscr{K}^2 \otimes_{\mathscr{O}_X} \kappa(x)\right), \tag{5.4.3.1}$$

where $\mathscr{K}$ is the sheaf of ideals associated to e. Since $\mathscr{K}/\mathscr{K}^2$ is locally free and X is connected, $\operatorname{rank}_x \mathfrak{R}$ does not depend on $x \in X$ ([Sta23, Lemma 01C9]). In this case, it is the *global rank* of $\mathfrak{R}$ and is simply denoted by $\operatorname{rank} \mathfrak{R}$.

The next lemma shows that the rank of an infinitesimal groupoid induced by a finitely presented groupoid is simply the relative dimension of the groupoid.

Lemma 5.4.4 (Rank of finitely presented groupoids)**.** *Let R be a groupoid locally of finite presentation on a locally Noetherian scheme X over a scheme L. Suppose that $t : R \to X$ is of relative dimension r in $e(x) \in R$. Then the rank of $\mathfrak{R}$ in x is r.*

Proof. Firstly, since X is locally Noetherian and t is locally of finite presentation, R is locally Noetherian ([Sta23, Lemma 01T6]).

Let $\mathfrak{R}_x$ denote the restriction of $\mathfrak{R}$ to the local scheme $X_x := \operatorname{Spec} \mathscr{O}_{X,x}$. Let $B := \mathscr{O}_{X,x}$, $C := \mathscr{O}_{R,e(x)}$, $\hat{C} := \mathscr{O}_{\mathfrak{R}_x}$ and

$$K := \ker\left(\mathscr{O}_{R,e(x)} \to \mathscr{O}_{X,x}\right). \tag{5.4.4.1}$$

By definition of rank,

$$\operatorname{rank}_x \mathfrak{R} = \dim_{e(x)} \left(\hat{C} \hat{\otimes}_B \kappa(x) \right). \tag{5.4.4.2}$$

By definition of relative dimension,

$$r = \dim_{e(x)} \left(C \otimes_B \kappa(x) \right). \tag{5.4.4.3}$$

By Paragraph 4.4.9 (Complete localisation of groupoids) and the fact that C is Noetherian, $\hat{C}$ is the K-adic completion of C. Furthermore, by Paragraph 2.1.25 (Quotients of adic Noetherian local rings), $\hat{C} \hat{\otimes}_B \kappa(x)$ is the K-adic completion of the Noetherian local ring $C \otimes_B \kappa(x)$, hence, they have the same dimension ([Sta23, Lemma 07NV]). $\square$

For group schemes, the rank is the dimension.

5.4.5 (Dimension of group schemes)**.** Let G be a group scheme locally of finite presentation over a field L and let $\mathfrak{G}$ be the induced infinitesimal groupoid on L. Since G is a group scheme, its local dimension in the identity element e, denoted by $\dim_e G$, is simply the global dimension of G ([Sta23, Lemma 045X]). Then, Lemma 5.4.4 (Rank of finitely presented groupoids) implies that

$$\begin{aligned} \dim \mathfrak{G} &= \dim_e \mathfrak{G} \\ &= \operatorname{rank}_e \mathfrak{G} \\ &= \dim_e G = \dim G. \end{aligned} \tag{5.4.5.1}$$

When an infinitesimal groupoid is not regular, it is not true that its rank is determined by its foliation.

Example 5.4.6 (Rank of Frobenius morphism)**.** Let X be a scheme of finite type over a field k of characteristic $p > 0$. Let $\mathrm{Fr} : X \to X^{(p)}$ denote the Frobenius morphism and let $\mathfrak{R}_{\mathrm{Fr}}$ denote the infinitesimal groupoid induced by Fr. Since Fr is a morphism of relative dimension 0, Lemma 5.4.4 (Rank of finitely presented groupoids) implies that the rank of $\mathfrak{R}_{\mathrm{Fr}}$ is 0. This is in contrast with the fact that the induced foliation, being the entire tangent sheaf, has maximal rank.

Next, it is shown that an immersion of regular infinitesimal groupoids of the same rank must be an isomorphism. This is proved by induction. The base case is provided by the fact that a surjective morphism of projective modules of the same rank is an isomorphism.

Lemma 5.4.7 (Regular infinitesimal groupoids of equal rank)**.** *Let $\mathfrak{R}'$ and $\mathfrak{R}$ be regular infinitesimal groupoids on a scheme X over a scheme L. Suppose that there exists a morphism of infinitesimal groupoids $\phi : \mathfrak{R}' \to \mathfrak{R}$ which is a thickening of formal schemes and suppose that, for all $x \in X$,*

$$\operatorname{rank}_x \mathfrak{R}' = \operatorname{rank}_x \mathfrak{R}. \tag{5.4.7.1}$$

Then ϕ is an isomorphism.

Proof. Since ϕ is a homeomorphism of topological spaces, it suffices to show that the pullback morphism

$$\phi^{\#} : \mathscr{O}_{\mathfrak{R}} \to \mathscr{O}_{\mathfrak{R}'} \tag{5.4.7.2}$$

is an isomorphism of sheaves of topological rings over X. This can be checked locally on $\mathfrak{R}$. Hence let $\mathfrak{R} = \operatorname{Spf} C$, $\mathfrak{R}' = \operatorname{Spf} C'$ and $X = \operatorname{Spec} B$ be affine open neighbourhoods and let $C \to B$ and $C' \to B$ be the corresponding regular thickenings of adic rings. Let K and K' be the kernel of the surjective morphisms $C \to B$ and $C' \to B$ respectively. By assumption, ϕ induces a weakly surjective morphism of admissible rings $\phi^{\#} : C \to C'$ and, by construction, $K' = K \cdot C'$. It follows that $(K')^{n+1} = K^{n+1} \cdot C'$, for all $n \in \mathbb{N}$. Therefore, $\phi^{\#}$ induces a morphism

$$\phi^{\#}_n : C/K^{n+1} \to C'/\left(K'\right)^{n+1} \tag{5.4.7.3}$$

Since $\phi^{\#}$ is weakly surjective, $\phi^{\#}_n$ is surjective by Lemma 2.3.5 (Weakly surjective morphisms of admissible rings). Since C and C' are complete with respect to the K and K'-adic topology respectively, it suffices to show that $\phi^{\#}_n$ is an isomorphism for all $n \in \mathbb{N}$. This is proved by induction.

(Base case $n = 0$). When $n = 0$, $\phi^{\#}_0$ is the identity morphism of B.

(Base case $n = 1$). When $n = 1$, $\phi^{\#}_1$ induces a surjective morphism of split B-modules

$$B \oplus K/K^2 \to B \oplus K'/\left(K'\right)^2, \tag{5.4.7.4}$$

which is the identity on B. Therefore, there is an induced surjective morphism of B-modules

$$K/K^2 \to K'/\left(K'\right)^2. \tag{5.4.7.5}$$

By assumption, both B-modules are projective. Furthermore, by assumption and Lemma 5.4.2 (Rank of regular infinitesimal groupoids), they have the same rank in every $x \in X$. But then [Sta23, Lemma 089Q] implies that (5.4.7.5) is an isomorphism. Therefore, $\phi^{\#}_1$ is an isomorphism.

(Inductive step). Now let $n \geq 0$. There is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K^n/K^{n+1} \arrow[r] \arrow[d] & C/K^{n+1} \arrow[r] \arrow[d, "\phi^{\#}_n"] & C/K^n \arrow[r] \arrow[d, "\phi^{\#}_{n-1}"] & 0 \\ 0 \arrow[r] & \left(K'\right)^n/\left(K'\right)^{n+1} \arrow[r] & C'/\left(K'\right)^{n+1} \arrow[r] & C'/\left(K'\right)^n \arrow[r] & 0. \end{tikzcd} \tag{5.4.7.6}$$

By the inductive assumption $\phi^{\#}_{n-1}$ is an isomorphism. Since $C \to B$ is a regular thickening,

$$\operatorname{Sym}^n_B K/K^2 = K^n/K^{n+1}. \tag{5.4.7.7}$$

Ditto for K'. Therefore, (5.4.7.5) implies that the morphism

$$K^n/K^{n+1} \xrightarrow{\sim} \left(K'\right)^n/\left(K'\right)^{n+1} \tag{5.4.7.8}$$

is an isomorphism. By the short five lemma, $\phi^{\#}_n$ is an isomorphism. $\square$

It follows that an immersion of two smooth groupoids of equal rank is an isomorphism.

Lemma 5.4.8 (Smooth groupoids of equal rank)**.** *Let R' and R be smooth groupoids on a locally Noetherian scheme X over a scheme L, where $t : R \to X$ has geometrically connected fibres. Suppose that there exists a morphism of groupoids $\phi : R' \to R$ which is an immersion and suppose that, for all $x \in X$, the relative dimensions of $t' : R' \to X$ in $e'(x) \in R'$ and of $t : R \to X$ in $e(x) \in R$ are the same. Then ϕ is an isomorphism.*

Proof. By Lemma 5.2.3 (Lifting isomorphism), it suffices to show that the induced morphism of regular infinitesimal groupoids

$$\hat{\phi} : \mathfrak{R}' \to \mathfrak{R} \tag{5.4.8.1}$$

is an isomorphism. This is an immersion of formal schemes (Lemma 3.3.10 (Factorisation of immersion)). Furthermore, Lemma 5.4.4 (Rank of finitely presented groupoids) implies that $\mathfrak{R}'$ and $\mathfrak{R}$ have the same rank. Therefore Lemma 5.4.7 (Regular infinitesimal groupoids of equal rank) applies to show that $\hat{\phi}$ is an isomorphism. □

The last proposition shows that the rank of an infinitesimal equivalence relation cannot be larger than the dimension of an invariant algebraic subscheme. In the case of groupoids, there is a correction term due to the presence of stabilisers. The bound is obtained by use of Lemma 1.2.3 (Grothendieck–Sharp).

Proposition 5.4.9 (Rank inequality)**.** *Let $\mathfrak{R}$ be an adic Noetherian infinitesimal groupoid on a field F finitely generated as a field over a scheme L. Let $\mathfrak{G}$ be the infinitesimal generic stabiliser and let M denote the field of rational first integrals of $\mathfrak{R}$. Then*

$$\operatorname{rank} \mathfrak{R} \leq \operatorname{tr\,deg}_M F + \dim \mathfrak{G}. \tag{5.4.9.1}$$

In particular, if $\mathfrak{R}$ is an equivalence relation

$$\operatorname{rank} \mathfrak{R} \leq \operatorname{tr\,deg}_M F. \tag{5.4.9.2}$$

Proof. It is firstly shown that L may be assumed a field. Let η be the unique point of $\operatorname{Spec} F$ and let $L(\eta)$ denote the spectrum of the localisation of L in the image of η. Then $L(\eta) \to L$ is a monomorphism and

$$F \times_{L(\eta)} F = F \times_L F. \tag{5.4.9.3}$$

Hence, $\mathfrak{R}$ may be assumed a groupoid on F over $L(\eta)$. Furthermore, $L(\eta) \subseteq F$, hence it is a field and the extension is, by assumption, finitely generated. Replace $L(\eta)$ by L.

Since $L \subseteq F$ is a finitely generated field extension, so is $M \subseteq F$ ([Isa09, Theorem 24.9, page 387]). The groupoid $\mathfrak{R} = \operatorname{Spf} C$ is an affine local ring (Lemma 4.4.7 (Infinitesimal groupoids on local schemes)). By construction, the morphism

$$\operatorname{Spec} F \to \operatorname{Spec} M \tag{5.4.9.4}$$

is $\mathfrak{R}$-invariant. Therefore there is a factorisation

$$\operatorname{Spec} F \to \mathfrak{R} \to \operatorname{Spec}(F \otimes_M F). \tag{5.4.9.5}$$

This corresponds to morphisms of rings

$$D := F \otimes_M F \to C \to F. \tag{5.4.9.6}$$

Note that C and D are Noetherian rings. The former by assumption and the latter by [Sta23, Lemma 045I]. Up to localising the morphisms of rings at the prime ideals corresponding to $e(x)$ (that is the kernel ideals of $D \to F$ and $C \to F$), it may be assumed that $D \to C$ is a morphism of Noetherian local rings. Now, by definition of infinitesimal stabiliser, $\mathfrak{G} = \operatorname{Spf} E$ where

$$E = C \hat{\otimes}_D F. \tag{5.4.9.7}$$

Note that, since C is Noetherian and $D \to F$ is surjective, $E = C \otimes_D F$ (Paragraph 2.1.25 (Quotients of adic Noetherian local rings)). By the dimension formula for Noetherian local rings ([Sta23, Lemma 00OM]) applied to the morphism $D \to C$,

$$\dim C \leq \dim D + \dim E. \tag{5.4.9.8}$$

By definition of rank, $\dim C = \operatorname{rank} \mathfrak{R}$. Furthermore, by definition, $\dim E = \dim \mathfrak{G}$. Hence, (5.4.9.1) follows if it can be shown that

$$\dim D = \operatorname{tr} \deg_M F. \tag{5.4.9.9}$$

This follows from Lemma 1.2.3 (Grothendieck–Sharp).

Finally, if j is a monomorphism, $E = F$. Indeed, there is a splitting

$$\mathbb{1}_F : F \xrightarrow{j^\#} E \xrightarrow{e^\#} F. \tag{5.4.9.10}$$

If j is a monomorphism, $j^\#$ is an epimorphism in the category of rings, hence the composition $E \to F \to E$ also has to be the identity. But then $E = F$ has dimension zero, hence (5.4.9.2) follows. $\square$

Chapter 6

Quotient spaces

6.1 Properties of quotient spaces

This section defines three types of quotients: categorical, effective and infinitesimal. The first two are well-known in the literature. The last one is a type of quotient which can be defined by considering only the infinitesimal properties of an equivalence relation. It is then shown that being an infinitesimal quotient is a local property and that, under suitable hypotheses, an infinitesimal quotient is an effective quotient.

Definition 6.1.1 (Categorical quotients)**.** Let R be a groupoid on a scheme X over a scheme L. An L-morphism $q : X \to Q$ is a *categorical quotient* of X by R if the following properties are satisfied:

1. It is R-invariant.
2. For all R-invariant L-morphisms $f : X \to Y$, there exists a unique $g : Q \to Y$ such that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "f"] & \\ Q \arrow[r, "g"] & Y. \end{tikzcd} \tag{6.1.1.1}$$

It is a *universal categorical quotient* of X by R if, for all morphisms $h : Q' \to Q$, the base change morphism

$$q' : X' := X \times_Q Q' \to Q' \tag{6.1.1.2}$$

is a categorical quotient for the pullback groupoid R' of R by h. By general category theory, if a categorical quotient exists, it is unique.

Definition 6.1.2 (Effective quotients)**.** Let R be an equivalence relation on a scheme X over a scheme L. An L-morphism $q : X \to Q$ is an *effective quotient* of X by R if the following properties are satisfied:

(i) It induces an isomorphism

$$R \xrightarrow{\sim} X \times_Q X \tag{6.1.2.1}$$

of schemes over $X \times_L X$.

(ii) q is faithfully flat and quasi-compact.

The following notion of infinitesimal quotient relaxes Property (i) to an isomorphism between the infinitesimal equivalence relations induced by R and by $X \times_Q X$. Recall that the latter infinitesimal equivalence relation is the relative de Rham space of X over Q and is denoted by $\mathfrak{D}_{X/Q}$.

Definition 6.1.3 (Infinitesimal quotients)**.** Let R be an equivalence relation on a scheme X over a scheme L. Let $\mathfrak{R}$ be the induced infinitesimal equivalence relation. An L-morphism $q : X \to Q$ is an *infinitesimal quotient* if the following properties are satisfied:

(a) It induces an isomorphism

$$\mathfrak{R} \xrightarrow{\sim} \mathfrak{D}_{X/Q} \tag{6.1.3.1}$$

of formal schemes over $X \times_L X$.

(b) It is smooth and surjective.

(c) For all generic points $\eta \in X$,

$$\mathscr{O}_{Q,q(\eta)} = \mathscr{O}^{\mathfrak{R}}_{X,\eta}. \tag{6.1.3.2}$$

Lemma 6.1.4 (Effective quotients are universal)**.** *Let R be an equivalence relation on a scheme X over a scheme L. Suppose that an effective quotient $q : X \to Q$ of X by R exists. Then it is the universal categorical quotient of X by R. In particular, it is unique.*

Proof. By [Sta23, Lemma 023Q], a faithfully flat and quasi-compact morphism is a universal effective epimorphism (vid. [Sta23, Definition 00WP]). Since $R = X \times_Q X$, q is the universal categorical quotient of X by R. ☐

The next lemma shows that infinitesimal quotients have the same local properties of geometric quotients.

Lemma 6.1.5 (Infinitesimal quotient is local on the target)**.** *Let R be an equivalence relation on a scheme X over a scheme L. Let $\mathfrak{R}$ be the induced infinitesimal equivalence relation. For an indexing set I, let $\{X_i\}_{i\in I}$ be an open cover of X and let $q : X \to Q$ be a surjective L-morphism.*

1. *If q is an infinitesimal quotient of X by $\mathfrak{R}$, then, for all $i \in I$, $q_i : X_i \to Q_i =: f(X_i)$ is an infinitesimal quotient of X by $\mathfrak{R}|_{X_i}$.*
2. *If the restriction $q_i : X_i \to Q_i =: f(X_i)$ is an infinitesimal quotient of X by $\mathfrak{R}|_{X_i}$ for all $i \in I$, then $q : X \to Q$ is an infinitesimal quotient.*

Proof. 1. For ease of notation, let $\iota : U := X_i \to X$ be an open immersion and let $q|_U : U \to f(U) =: V$ be the restriction of q to U. Note that, since q is smooth, $V \subseteq X$ is an open subset ([Sta23, Lemma 056G]).

It is shown that Property (a) holds. This amounts to checking that pulling back the isomorphism

$$\mathfrak{R} \xrightarrow{\sim} \mathfrak{D}_{X/Q} \tag{6.1.5.1}$$

by ι gives an isomorphism

$$\mathfrak{R}|_U \xrightarrow{\sim} \mathfrak{D}_{U/V} \tag{6.1.5.2}$$

By the universal property of infinitesimal neighbourhoods, it suffices to check that the restriction of $X \times_Q X$ by ι is $U \times_V U$. By definition, the restriction is $U \times_Q U$. This is isomorphic to $U \times_V U$ since $V \to Q$ is a monomorphism.

Next, Property (b) is shown. This is easy to see using the fact that being smooth is a property local on the target ([Sta23, Lemma 01V6]) and $q|_U$ is surjective by assumption.

Finally Property (c) is shown. Let η be a generic point of U. Since ι is an open immersion, η is also a generic point of X and there is a factorisation

$$\operatorname{Spec} \kappa(\eta) \to U \to X. \tag{6.1.5.3}$$

This shows that $\mathscr{O}_{U,\eta} = \mathscr{O}_{X,\eta}$. Also, by definition of local first integrals

$$\mathscr{O}_{U,\eta}^{\mathfrak{R}|_U} = \mathscr{O}_{X,\eta}^{\mathfrak{R}}. \tag{6.1.5.4}$$

2. As in the previous paragraph, $f(X_i)$ is open in Q for all $i \in I$.

   Property (a) holds. Indeed $\mathfrak{R} \to \mathfrak{D}_{X/Q}$ is a homeomorphism of topological spaces, hence, whether a morphism is an isomorphism can be checked on the structure sheaves. This can be checked on an open cover of X. By assumption, $\{X_i\}_{i \in I}$ is an open where the morphism of sheaves is an isomorphism.

   Property (b) holds since being smooth is a property local on the target ([Sta23, Lemma 01V6]).

   Property (c) holds. Indeed, if η is any generic point of X, there exists an $i \in I$ such that $\eta \in X_i$. Now the same argument in the corresponding paragraph of part (1) shows the result. ∎

Recall that a groupoid (or equivalence relation) is smooth with geometrically connected fibres if the morphism $s : R \to X$ (or equivalently $t : R \to X$) is smooth with geometrically connected fibres.

Lemma 6.1.6 (Restriction of smooth equivalence relations)**.** *Let $j : R \to X \times_L X$ be a smooth equivalence relation with geometrically connected fibres on a scheme X over a scheme L, where j is a closed immersion. Let U be an open subset of X. Then the restriction $R|_U$ to U is a smooth equivalence relation with geometrically connected fibres such that $j_U : R \to U \times_L U$ is a closed immersion.*

Proof. Since an open immersion is smooth and R is smooth over X, $R|_U$ is smooth over U (Lemma 4.2.19 (Properties of restrictions of groupoids)). Furthermore, a fibre of $R|_U \to U$ is an open subset of a fibre of $R \to X$. Since the latter is geometrically irreducible, so is the former. Finally, j_U is a closed immersion since it is the base change of j. ☐

This is the key proposition of the section. A proof is outlined in §0.5 (Main results).

Proposition 6.1.7 (Infinitesimal quotients are effective)**.** *Let $j : R \to X \times_L X$ be a smooth equivalence relation with geometrically connected fibres on an integral scheme X of finite type over a Noetherian scheme L, where j is a closed immersion. Suppose that an infinitesimal quotient $q : X \to Q$ of X by R exists. Then it is the effective quotient of X by R and Q is separated and of finite type over L. In particular, it is unique.*

Proof. Property (i) is shown. By Property (a), q is $\mathfrak{R}$-invariant. Since X is a Noetherian scheme and R is smooth with geometrically connected fibres, part (2) of Proposition 5.2.1 implies that q is R-invariant. Therefore there is a morphism

$$\phi : R \to X \times_Q X \tag{6.1.7.1}$$

Moreover, since j is a closed immersion and $X \times_Q X \to X \times_L X$ is an immersion, ϕ is a closed immersion (part (2) of [Sta23, Lemma 07RK]). It has to be shown that ϕ is an isomorphism.

Firstly, it is shown that the generic fibre of q is geometrically irreducible. Let η be the generic point of X and let F be its residue field. Furthermore, let M be the residue field of $q(\eta) \in Q$. It is shown that the generic fibre of q

$$\operatorname{Spec} W := X \times_Q (\operatorname{Spec} M) \tag{6.1.7.2}$$

is geometrically irreducible over $\operatorname{Spec} M$. By construction, η is the generic point of $\operatorname{Spec} W$ and its residue field is also F. By [Sta23, Lemma 054Q], $\operatorname{Spec} W$ is geometrically irreducible over $\operatorname{Spec} M$ if and only if

$$M \subseteq F \tag{6.1.7.3}$$

is a geometrically irreducible field extension. By [Sta23, Lemma 037P], it suffices to show that it is an algebraically closed field extension. By Property (c), $M = \mathscr{O}_{Q,q(\eta)}$ is the field of rational first integrals of $\mathfrak{R}$. Since $\mathfrak{R}$ is a regular groupoid, Lemma 4.4.8 (Restriction to local schemes) implies that the restriction of $\mathfrak{R}$ to $\operatorname{Spec} F$ is a regular groupoid. By Proposition 5.3.3 (First integrals are algebraically closed), $M \subseteq F$ is an algebraically closed field extension. Therefore $\operatorname{Spec} W$ is geometrically irreducible over $\operatorname{Spec} M$.

Next, it is shown that there exists an open subset U of X such that ϕ restricted to $U \times_Q U$ is an isomorphism. Note that q is a surjective morphism from a Noetherian scheme X. Therefore q is quasi-compact ([Sta23, Lemma 01P0]), hence of finite type. Furthermore, since q is surjective and X is irreducible, Q is irreducible ([Sta23, Lemma 0379]). Now [Sta23, Lemma 0559] implies that there exists a non-empty open subset $V \subseteq Q$ such that the base change of q by $V \to Q$, $q|_V : U \to V$, has geometrically irreducible fibres. Since X is integral, U is dense in X. Furthermore, $q|_U$ is smooth. Now consider the equivalence relation $U \times_Q U$ on U. Since being smooth and having geometrically connected fibres is invariant by open base change (Lemma 6.1.6 (Restriction of smooth equivalence relations)), this is a smooth equivalence relation with geometrically connected fibres. Consider the base change of $\phi : R \to X \times_Q X$ by the open immersion $U \times_Q U \to X \times_Q X$. Denote this morphism by

$$\phi|_U : R|_U \to U \times_Q U \tag{6.1.7.4}$$

Since $U \times_Q U$ is a smooth equivalence relation with geometrically connected fibres and X is locally Noetherian, Lemma 5.2.3 (Lifting isomorphism) implies that $\phi|_U$ is an isomorphism if it induces an isomorphism on the associated infinitesimal equivalence relations. This holds true by Property (a) on X. By part (1) of Lemma 6.1.5 (Infinitesimal quotient is local on the target), this also holds true on U. Hence $\phi|_U$ is an isomorphism.

Next, it is shown that $U \times_Q U$ is dense in $X \times_Q X$. Note that q is smooth, hence its base change $q' : X \times_Q X \to X$ is smooth, therefore universally open [Sta23, Lemma 056G]. Furthermore X is irreducible and, by construction, the fibres W_x of q' are geometrically irreducible for all $x \in U$. Indeed, they are geometric fibres of the morphism $U \to V$. Since U is an open dense subset of X, [Sta23, Lemma 004Z] applied to q implies that $X \times_Q X$ is irreducible. This shows that $U \times_Q U$ is dense in $X \times_Q X$.

Next, it is shown that ϕ is an isomorphism. Consider the following commutative diagram

$$\begin{tikzcd} R|_U \arrow[r] \arrow[d, "\phi|_U"] & R \arrow[d, "\phi"] \\ U \times_Q U \arrow[r] & X \times_Q X. \end{tikzcd} \tag{6.1.7.5}$$

Since $\phi|_U$ is an isomorphism and $U \times_Q U$ is dense in $X \times_Q X$, $R|_U$ is dense in $X \times_Q X$. This implies that R is dense in $X \times_Q X$. On the other hand, ϕ is a closed immersion, therefore it has to be surjective, hence it is a thickening. By the fibre-wise criterion of flatness ([Sta23, Lemma 039E]) applied to the diagram

$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, "t"'] & & X \times_Q X \arrow[dl, "\mathrm{pr}_1"] \\ & X, & \end{tikzcd} \tag{6.1.7.6}$$

it suffices to show that the base change of ϕ by the inclusion $x \in X$, ϕ_x, is an isomorphism. But ϕ_x is a thickening whose codomain is smooth, therefore it is an isomorphism (Lemma 1.1.6 (Thickenings from reduced rings)). Therefore, (a) holds.

Property (ii) is shown. By definition, q is smooth, hence it is an faithfully flat. Furthermore, it has been already observed that q is quasi-compact.

Finally, it is shown that $Q \to L$ is separated and of finite type. Consider the following Cartesian diagram

$$\begin{tikzcd} X \times_Q X \arrow[r, "j"] \arrow[d] & X \times_L X \arrow[d, "q \times q"] \\ Q \arrow[r, "\Delta_{Q/L}"] & Q \times_L Q. \end{tikzcd} \tag{6.1.7.7}$$

In order to show that Q is separated over L, it has to be shown that $\Delta_{Q/L}$ is a closed immersion. Since j is a closed immersion, $R = X \times_Q X \to X \times_L X$ is a closed immersion. But $q \times q$ is faithfully flat and quasi-compact, hence $\Delta_{Q/L}$ is a closed immersion by faithfully flat descent ([Sta23, Lemma 02L6]). By [Sta23, Lemma 02KL], Q is locally of finite type over L and, since X is quasi-compact over L and q is surjective, Q is quasi-compact over L. Therefore Q is of finite type over L. □

6.2 Construction of quotient spaces

In this section, a quotient space is constructed on the open set of algebraically smooth points. Firslty, in order to motivate the techniques involved, it is shown how to construct the projective line using the proof of the theorem. Subsequently, it is shown that the construction of quotients can be carried out locally around a point. Then, the definition of an algebraically smooth point is presented and it is shown that such a point admits a quotient in an open neighbourhood. The main theorem is stated and a further example shows when not to apply it.

Example 6.2.1 (Glueing first integrals)**.** Recall the setting of Example 4.5.11 (Radial foliation). Let k be a field, let $B := k[x, y]$ and let $X := \mathbb{A}^2_k = \operatorname{Spec} B$. Suppose that $G := \mathbb{G}_m = k^\times$ acts on X with weights $(1, 1)$. This induces a groupoid $G \times_k X$ which in turn induces an infinitesimal groupoid $\mathfrak{G}$ on X described by the vector field $x\,\partial/\partial x + y\,\partial/\partial y$.

The first integrals of $\mathfrak{G}$ (or the $\mathbb{G}_m$-invariant sections) are only the constant functions in k. The same is true when restricting to $X \setminus \{0\}$. However, when restricting to the distinguished affine open set $D(x)$, where x is non-zero, one can compute that

$$d\left(\frac{y}{x}\right) = \frac{1}{x^2}(y \cdot dx - x \cdot dy) = 0 \in \mathscr{O}_{\mathfrak{G}}. \tag{6.2.1.1}$$

Hence y/x is a first integral of $\mathfrak{G}$ over $D(x)$. More precisely, $k\,[y/x]$ is the ring of first integrals of $\mathfrak{G}$ over $D(x)$. The morphism induced by the inclusion of the first integrals is

$$\mathbb{A}^2_k \setminus \{x = 0\} = \operatorname{Spec} k[x, y]_x \to \operatorname{Spec} k\,[y/x] = \mathbb{A}^1_k \setminus \{0\}. \tag{6.2.1.2}$$

This is a smooth morphism of rank one, hence all the points of $D(x)$ are *algebraically smooth.* Observe that the ring of first integrals over $D(x)$ is not a localisation of the ring of first integrals over X. Similarly, $k[x/y]$ is the ring of first integrals of $\mathfrak{G}$ over $D(y)$ and all the points in $D(y)$ are algebraically smooth. Therefore, $X(\mathrm{AlgSmooth}) = X \setminus \{0\}$. Notice that $X \setminus \{0\}$ is also the locus when the action of $\mathbb{G}_m$ is free. On the intersection $D(xy)$, the ring of first integrals is $k[y/x, x/y]$. The first integrals can be glued together along their intersection to obtain a quotient morphism

$$q : \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k. \tag{6.2.1.3}$$

This is an infinitesimal quotient, an effective quotient and a universal categorical quotient. The fibres are precisely the orbits of the action.

Lemma 6.2.2 (Localisation of construction of quotients)**.** *Let $j : R \to X \times_L X$ be a smooth equivalence relation with geometrically connected fibres on a scheme X of finite type over a Noetherian scheme L, where j is a closed immersion. Let $\mathfrak{R}$ denote the induced infinitesimal equivalence relation. Suppose there exists an open cover $\{X_i\}_{i\in I}$ of X and a collection of infinitesimal quotients*

$$q_i : X_i \to Q_i \tag{6.2.2.1}$$

of X_i by $\mathfrak{R}|_{X_i}$. Then there exists an infinitesimal quotient, effective quotient and universal categorical quotient

$$q : X \to Q \tag{6.2.2.2}$$

of X by $\mathfrak{R}$, where Q is separated and of finite type over L.

Proof. Since X is quasi-compact, the covering can be refined to a finite covering. By induction, it can be assumed that the covering consists of two open sets X_1 and X_2 with respective quotients $q_1 : X_1 \to Q_1$ and $q_2 : X_2 \to Q_2$.

Let $X_{12} = X_1 \cap X_2 \subseteq X$. Since X_{12} is open in both X_1 and X_2, part (1) of Lemma 6.1.5 (Infinitesimal quotient is local on the target) implies that both

$$q_1|_{X_{12}} : X_{12} \to q_1(X_{12}) \subseteq Q_1 \text{ and} \tag{6.2.2.3}$$
$$q_2|_{X_{12}} : X_{12} \to q_1(X_{12}) \subseteq Q_2 \tag{6.2.2.4}$$

are infinitesimal quotients of X_{12} by $\mathfrak{R}|_{X_{12}}$. Note that $\mathfrak{R}|_{X_{12}}$ is the infinitesimal equivalence relation induced by $R|_{X_{12}}$. This is a smooth equivalence relation with geometrically connected fibres such that $j : R_{12} \to X_{12} \times_L X_{12}$ is closed (Lemma 6.1.6 (Restriction of smooth equivalence relations)). By Proposition 6.1.7 (Infinitesimal quotients are effective), they are effective quotients, hence categorical quotients (Lemma 6.1.4 (Effective quotients are universal)). In particular, they are unique, hence $q_{12} := q_1|_{X_{12}} = q_2|_{X_{12}}$ and $Q_{12} := q_1(X_{12}) = q_2(X_{12})$ is an open subset of both Q_1 and Q_2.

Let Q be the pushout

$$Q_1 \amalg_{Q_{12}} Q_2. \tag{6.2.2.5}$$

Then, there is a unique morphism of schemes $q : X \to Q$ which restricts to q_1 and q_2 on X_1 and X_2 respectively. Since q is, by construction, surjective, part (2) of Lemma 6.1.5 (Infinitesimal quotient is local on the target) implies that q is an infinitesimal quotient. Therefore, employing again Proposition 6.1.7 (Infinitesimal quotients are effective) and Lemma 6.1.4 (Effective quotients are universal) gives that q is an infinitesimal quotient, an effective quotient and a universal categorical quotient, where Q is separated and of finite type over L. $\square$

Algebraically smooth points are points such that the morphism induced by the inclusion of the first integrals is smooth of relative dimension equal to the rank of the equivalence relation.

Definition 6.2.3 (Algebraically smooth points)**.** Let $\mathfrak{R}$ be a regular infinitesimal equivalence relation of global rank r on a connected scheme X of finite type over a Noetherian scheme L. A point $x \in X$ is an *algebraically smooth point* of $\mathfrak{R}$ if there exists an affine open neighbourhood $U \ni x$ such that the induced morphism

$$U = \operatorname{Spec} \mathscr{O}_X(U) \to \operatorname{Spec} \mathscr{O}_X^{\mathfrak{R}}(U) \tag{6.2.3.1}$$

is smooth of relative dimension r. Let $X(\text{AlgSmooth})$ denote the open subset of X of algebraically smooth points.

The proof of the following proposition is outlined in §0.5 (Main results).

Proposition 6.2.4 (Quotient around algebraically smooth points)**.** *Let $j : R \to X \times_L X$ be a smooth equivalence relation with geometrically connected fibres on a normal integral scheme X of finite type over a Noetherian scheme L, where j is a closed immersion. Let $x \in X$ be an algebraically smooth point. Then, there exists an affine open neighbourhood $U_x \ni x$ such that the associated morphism*

$$q_x : U_x = \operatorname{Spec} \mathscr{O}_X(U_x) \to \operatorname{Spec} \mathscr{O}_X^{\mathfrak{R}}(U_x). \tag{6.2.4.1}$$

is an infinitesimal quotient of U_x by $R|_{U_x}$.

Proof. By definition, there exists an affine open neighbourhood $U \ni x$ such that the morphism

$$q : U = \operatorname{Spec} \mathscr{O}_X(U) \to \operatorname{Spec} \mathscr{O}_X^{\mathfrak{R}}(U) \tag{6.2.4.2}$$

is smooth of relative dimension r. Let $A = \mathscr{O}_X^{\mathfrak{R}}(U)$ and $B = \mathscr{O}_X(U)$. This need not be surjective, however since the morphism is smooth, the image $q(U)$ is open in $\operatorname{Spec} A$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ containing $q(x)$. Let

$$U_x := X \times_{\operatorname{Spec} A} \operatorname{Spec} A_f \tag{6.2.4.3}$$

Note that $B_f = B \otimes_A A_f$ and $U_x = \operatorname{Spec} B_f$. By Lemma 5.1.11 (Localisation and first integrals) and Paragraph 4.3.16 (Formal Pullback and restriction of groupoids),

$$A_f = \mathscr{O}_X^{\mathfrak{R}|_{U_x}}(U_x). \tag{6.2.4.4}$$

Since being smooth of relative dimension r is invariant under base change, its base change

$$\operatorname{Spec} B_f \to \operatorname{Spec} A_f \tag{6.2.4.5}$$

is smooth of relative dimension r. By construction, it is also surjective. This verifies Property (b). Up to replacing A_f with A and B_f with B and employing Lemma 5.1.11 (Localisation and first integrals) to preserve regularity of the equivalence relation, it may be assumed that

$$q : X := \operatorname{Spec} B \to \operatorname{Spec} A =: Y \tag{6.2.4.6}$$

is a smooth surjective morphism or relative dimension r and

$$A = \Gamma\left(X, \mathscr{O}_X^{\mathfrak{R}}\right). \tag{6.2.4.7}$$

Now, Property (a) is verified. Since R and $X \times_Q X$ are smooth equivalence relations everywhere of the same relative dimension over X, Lemma 5.4.8 (Smooth groupoids of equal rank) implies that

$$\hat{\jmath} : \mathfrak{R} \to \mathfrak{D}_{X/Q}. \tag{6.2.4.8}$$

is an isomorphism.

Finally, Property (c) is verified. Let η be the generic point of the normal domain B. Let M and F denote the field of fractions of A and B respectively. Note that $M = A_{q(\eta)}$ and $F = B_\eta$. Let $M' = \mathscr{O}_{X,\eta}^{\mathfrak{R}}$. There are field extensions

$$M \subseteq M' \subseteq F. \tag{6.2.4.9}$$

It has to be shown that $M = M'$. It is firstly shown that $M \subseteq M'$ is an algebraic field extension. Equivalently, it is shown that

$$\operatorname{tr} \deg_M M' = 0. \tag{6.2.4.10}$$

Since $A \to B$ is of relative dimension r, the dimension of the generic fibre

$$\dim\left(B \otimes_A M\right) = r. \tag{6.2.4.11}$$

Since $B \otimes_A M$ is an integral domain of finite type over M and its field of fractions is F, [Sta23, Lemma 00P0] implies that

$$\operatorname{tr} \deg_M F = r. \tag{6.2.4.12}$$

Furthermore, by Proposition 5.4.9 (Rank inequality), it follows that

$$\operatorname{tr} \deg_{M'} F \geq r. \tag{6.2.4.13}$$

Therefore, by [Sta23, Lemma 030H],

$$\operatorname{tr} \deg_M M' = \operatorname{tr} \deg_M F - \operatorname{tr} \deg_{M'} F \leq r - r = 0. \tag{6.2.4.14}$$

Hence $M \subseteq M'$ is an algebraic field extension.

It is finally shown that $M \subseteq M'$ is an algebraically closed field extension. This implies that $M = M'$. It suffices to show that $M \subseteq F$ is an algebraically closed field extension. By Proposition 5.3.3 (First integrals are algebraically closed), $A \to B$ is an integrally closed ring extension. Localising this morphism by $A \to M$ yields an integrally closed morphism $M \to B \otimes_A M$ ([Sta23, Lemma 0307]). Since B is normal, and $B \otimes_A M$ is a localisation of B, it is also normal ([Sta23, Lemma 00GY]). But now the field of fractions of both B and $B \otimes_A M$ is F. Therefore $B \otimes_A M$ is integrally closed in F. Since the composition of integrally closed morphisms is integrally closed, M is integrally closed in F. Since M is a field, this implies that M is algebraically closed in F. Indeed, multiplying any polynomial with coefficients in M by the inverse of the leading coefficient makes the polynomial monic. In particular, M is algebraically closed in M'. □

The main theorem follows easily by combining the last three statements.

Theorem 6.2.5 (Existence of quotients)**.** *Let $j : R \to X \times_L X$ be a smooth equivalence relation with geometrically connected fibres on a normal integral scheme X of finite type over a Noetherian scheme L, where j is a closed immersion. Then, there exists an infinitesimal, effective and universal categorical quotient*

$$q : X(\mathrm{AlgSmooth}) \to Q \tag{6.2.5.1}$$

of X by R, where Q is separated and of finite type over L.

Proof. Let $x \in X(\mathrm{AlgSmooth})$. By Proposition 6.2.4 (Quotient around algebraically smooth points), there exists a neighbourhood $U_x \ni x$ with an infinitesimal quotient

$$q_x : U_x \to V_x. \tag{6.2.5.2}$$

Certainly $\{U_x\}_{x \in X}$ is an open cover of X, hence, by Lemma 6.2.2 (Localisation of construction of quotients), there exists an infinitesimal, effective and universal categorical quotient

$$q : X \to Q, \tag{6.2.5.3}$$

where Q is separated and of finite type over L. □

More generally, the theorem is true when j is a locally closed immersion, however in this case, the quotient space need not be separated. This follows easily by restricting to a sufficiently small open neighbourhood and applying the same proof. On the other hand, when j is not closed, Proposition 6.1.7 (Infinitesimal quotients are effective) fails. The final example shows why.

Example 6.2.6 (Non-closed equivalence relations)**.** Recall the setting of Example 0.4.5 (Hyperbolic foliation). Let k be a field, let $B := k[x, y]$ and let $X := \operatorname{Spec} B$. Suppose that $G := \mathbb{G}_m = k^\times$ acts on X with weights $(1, -1)$. This induces a groupoid $G \times_k X$ which in turn induces an infinitesimal groupoid $\mathfrak{G}$ on X described by the vector field $x\,\partial/\partial x - y\,\partial/\partial y$.

Now let X' be X without the diagonal $y = x$. Let R be the restriction of $G \times_k X$ to X' and let $\mathfrak{R}$ be the induced infinitesimal groupoid. Note that R is a smooth equivalence relation with geometrically connected fibres, however j is not a closed immersion. This comes from the fact that the action of G on X is not closed. The subring of R-invariant sections, or equivalently the subring of first integrals of $\mathfrak{R}$, is given by

$$k[xy] \subseteq k[x, y]_{y-x} \tag{6.2.6.1}$$

It is straightforward to verify that this is an infinitesimal quotient, however Proposition 6.1.7 (Infinitesimal quotients are effective) may not be applied to conclude that (6.2.6.1) is a universal categorical quotient. In fact, it is not a categorical quotient as there exists a factorisation by an R-invariant morphism

$$X' \to \mathbb{A}^1_\infty \to \mathbb{A}^1 = \operatorname{Spec} k[xy], \tag{6.2.6.2}$$

where $\mathbb{A}^1_\infty$ is the affine line with the double origin and the morphism $X' \to \mathbb{A}^1_\infty$ maps the x-axis and the y-axis to two distinct double origins.

6.3 Finite generation

By Theorem 6.2.5 (Existence of quotients), in order to show existence of a global quotient, it suffices to show that the inclusion of the first integrals is a smooth morphism of relative dimension equal to the rank of the equivalence relation. Showing that a morphism is smooth typically involves showing that a morphism is formally smooth and that it is finitely presented. For the latter, it suffices to show that the rings of first integrals are finitely generated. Using a theorem of Oscar Zariski and Masayoshi Nagata, this is accomplished in this section in a special case. Finally, an example shows when not to apply the theorem and a potential connection to the theory of foliation singularities is discussed.

Definition 6.3.1 (Finite generation of first integrals)**.** Let $p : X \to L$ be a morphism of schemes locally of finite presentation. Let $\mathfrak{R}$ be an infinitesimal groupoid on X over L. Recall from Paragraph 5.1.8 (First integrals form a sheaf of algebras) the existence of a unique morphism of sheaves over X

$$p^{-1}\mathscr{O}_L \to \mathscr{O}_X^{\mathfrak{R}} \tag{6.3.1.1}$$

The first integrals $\mathfrak{R}$ are *finitely generated* if, for all affine open subsets $V \subseteq L$ and $U \subseteq X$ such that $p(U) \subseteq V$, the composition of morphism of rings

$$\mathscr{O}_L(V) \to p^{-1}\mathscr{O}_L(U) \to \mathscr{O}_X^{\mathfrak{R}}(U) \tag{6.3.1.2}$$

is finitely presented.

Theorem 6.3.2 (Zariski–Nagata finite generation)**.** *Let B be a normal domain finitely presented over a ring A. Let F be the fraction field of B and let $M \subseteq F$ be a subfield. Suppose that either*

1. *A is field and* $\operatorname{tr}\deg_A M \leq 2$, *or*

2. *A is a Dedekind domain and* $\operatorname{tr\,deg}_A M \leq 1$.

Then $B \cap M \subseteq F$ *is finitely generated over* A.

Proof. The above statement is a translation of the main result of [Nag56b]. The terminology used by the author is non-standard and it is explained in [Nag56a]. In particular, an *affine domain* is defined as an integral domain over a field or a Dedekind domain ([Nag56a, §2, page 86]). Furthermore, if M is a field over A, where A is either a field or a Dedekind domain, the *transcendence degree* of M over A is defined to be

$$\begin{cases} \operatorname{tr\,deg}_A M & \text{if } A \text{ is a field} \\ \operatorname{tr\,deg}_A M + 1 & \text{if } A \text{ is a Dedekind domain.} \end{cases} \tag{6.3.2.1}$$

This is explained in [Nag56a, §3, page 87]. □

The above theorem is applied with $\operatorname{Spec} B$ an affine open subset of X, F the field of fractions of X and M the field of rational first integrals. This can be applied since, for regular infinitesimal groupoids, a section which is generically a first integral is a first integral. Finally, the rank inequality is applied to deduce the bound on the transcendence degree and conclude.

Theorem 6.3.3 (Finite generation of first integrals). *Let* $p : X \to L$ *be a morphism of schemes locally of finite presentation. Let* $\mathfrak{R}$ *be a regular infinitesimal groupoid on* X *over* L*, where* X *is a normal integral scheme with generic point* η *and let* $\mathfrak{G}_\eta$ *be the generic infinitesimal stabiliser. Let*

$$c := \operatorname{tr\,deg}_L F - \operatorname{rank}_\eta \mathfrak{R} + \dim \mathfrak{G}_\eta, \tag{6.3.3.1}$$

where F *is the residue field of* η*, and suppose that either*

1. *L is a field and* $c \leq 2$*, or*

2. *L is a Dedekind domain and* $c \leq 1$.

Then, the first integrals of $\mathfrak{R}$ *are finitely generated.*

Proof. Let $\mathfrak{R}_\eta$ be the restriction of $\mathfrak{R}$ to $\operatorname{Spec} F$. By Lemma 4.4.8 (Restriction to local schemes), $\mathfrak{R}_\eta$ is a regular groupoid. Let M be the field of rational first integrals.

Now let $V \subseteq L$ and $U \subseteq X$ be affine open subsets such that $p(U) \subseteq V$. Let $A := \mathscr{O}_L(V)$ and $B := \mathscr{O}_X(U)$. By [Sta23, Lemma 01T2] and the assumptions, B is a normal domain finitely generated over A. Let $B^{\mathfrak{R}}$ be the ring of first integrals of $\mathfrak{R}$. By Lemma 5.1.12 (First integrals at generic point),

$$B^{\mathfrak{R}} = B \cap M \subseteq F. \tag{6.3.3.2}$$

Finally, by Proposition 5.4.9 (Rank inequality),

$$\begin{aligned} \operatorname{tr}\deg_L M &= \operatorname{tr}\deg_L F - \operatorname{tr}\deg_M F \\ &\leq \operatorname{tr}\deg_L F - \operatorname{rank}\mathfrak{R}_\eta + \dim \mathfrak{G}_\eta \\ &= c \leq 2. \end{aligned} \tag{6.3.3.3}$$

The assumptions of Theorem 6.3.2 (Zariski–Nagata finite generation) are fully satisfied and the theorem implies that $B^{\mathfrak{R}}$ is finitely generated over A. $\square$

Corollary 6.3.4 (Finite generation of invariant sections)**.** *Let X be an integral normal affine scheme of finite type over a field k with generic point η. Let R be a smooth groupoid with geometrically connected fibres on X over k and let G_η be the generic stabiliser of R. Assume that*

$$\dim X - \operatorname{rel}\dim_\eta R + \dim G_\eta \leq 2. \tag{6.3.4.1}$$

Then, the R-invariant sections are finitely generated over k.

Proof. Let $\mathfrak{R}$ be the regular infinitesimal groupoid induced by R and let $\mathfrak{G}_\eta$ be the infinitesimal generic stabiliser induced by G_η, the generic stabiliser of R. If X is an integral affine scheme locally of finite type over a field k, then $k \subseteq \operatorname{Frac}(X)$ is a field extension of transcendence degree $\dim X$ ([Sta23, Lemma 00P0]). By Lemma 5.4.4 (Rank of finitely presented groupoids), the rank of $\mathfrak{R}$ in η is $\operatorname{rel}\dim_\eta R$. Furthermore, Paragraph 5.4.5 (Dimension of group schemes) implies that the dimension of the infinitesimal stabiliser $\mathfrak{G}_\eta$ is $\dim G_\eta$. Now it suffices to combine part (2) of Proposition 5.2.1 (First integrals and invariance) and Theorem 6.3.3 (Finite generation of first integrals). $\square$

The next example shows that the corollary fails when R is not smooth. Indeed, the ring of invariant sections in the following example is the standard example of a sub-algebra of a finitely generated algebra which is not finitely generated.

Example 6.3.5 (Finite generation of blowing up)**.** Let k be a field, let $B = k[x, y]$ and let $X = \operatorname{Spec} B$. Define an equivalence relation R on X by declaring $(x, y) \sim (x', y')$ if and only if $x = x'$ and $y = y'$ or if $x = x' = 0$. In other words, the equivalence classes are given by the y-axis E and all the single points. This is the equivalence relation induced by the blowing up morphism

$$\operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x, xy]. \tag{6.3.5.1}$$

More formally, the equivalence relation is described by the morphism of schemes

$$j : R = \operatorname{Spec}\left(\frac{k[x, y, s, t]}{(x - s, xy - st)}\right) \to \operatorname{Spec} k[x, y, s, t] = X \times_k X. \tag{6.3.5.2}$$

This is an equivalence relation with geometrically integral fibres which is not smooth. Therefore Corollary 6.3.4 (Finite generation of invariant sections) does not apply. Indeed, the subring of first integrals is not finitely generated. Pictorially, such forceful contraction crashes the pleasant geometric properties of the affine plane. This is shown in Figure 9.

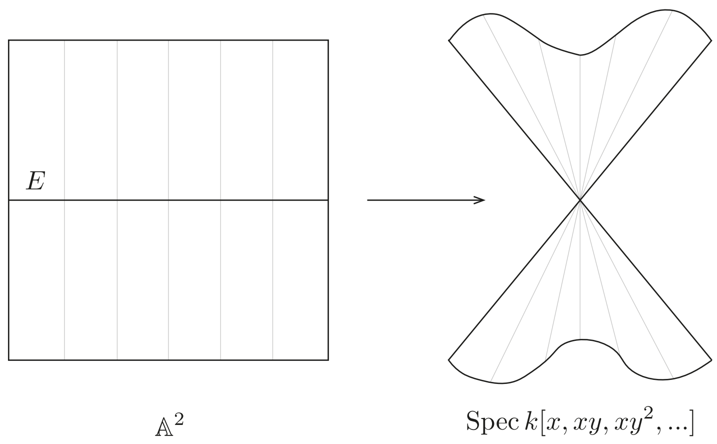


Figure 9: A forceful contraction

It is shown that

$$B^R = k[x, xy, xy^2, xy^3, ...]. \tag{6.3.5.3}$$

To this end, note that

$$\begin{aligned} d(xy^n) &= xy^n - st^n \\ &= (xy - st)(y^{n-1} + y^{n-2}t + ... + t^{n-1}) \\ &\quad - (x - s)(y^{n-1}t + y^{n-2}t^2 + ... + yt^{n-1}). \end{aligned} \tag{6.3.5.4}$$

Therefore $d(xy^n) \in (x - s, xy - st)$, hence $d(xy^n) = 0$.

The previous example shows that the ring of first integrals may not be the right object to consider when constructing quotients by equivalence relations which are not flat, such as those induced by blowing ups. In these cases, a more global approach is needed such as Artin's theorem on contractions ([Art70, Theorem 3.1, page 99]).

Finite generation may be linked to the singularities of the induced foliation.

6.3.6 (Finite generation and singularities)**.** In general, the first integrals of an adic infinitesimal equivalence relation (or foliation) need not be finitely generated. The survey [Fre01, §3.1, page 179] gives an example of a rank one foliation on $\mathbb{A}^7_k$ whose first integrals are not finitely generated. Further examples are given in [Fre01, §3.2, page 181]. The foliations given as counterexamples are all induced by a nilpotent vector field. According to [MP13, Fact I.ii.4, page 285], a rank one foliation has this property if and only if it does not have log-canonical singularities. It seems plausible to conjecture that the first integrals of a log-canonical algebraically integrable foliation are finitely generated.

6.4 Local quotient spaces

This section proposes a strategy to prove that the inclusion of the first integrals of a regular infinitesimal equivalence relation of rank r is formally smooth of relative dimension r. This is based on lifting formal first integrals, whose existence and smoothness properties are guaranteed by Frobenius' theorem. An example is provided to appreciate why the statement is non-trivial. The content of this section is purely speculative.

6.4.1 (Frobenius' theorem)**.** Let $\mathfrak{R}$ be a regular infinitesimal equivalence relation of rank r on a scheme X smooth over $\mathbb{C}$. Frobenius' theorem states that, for every point $x \in X$, there exists an analytic neighbourhood U of x and a smooth morphism of complex analytic varieties

$$q : U \to V \tag{6.4.1.1}$$

of relative dimension r such that $\mathfrak{R}|_U = U \times_V U$, as complex analytic varieties. In other words, every point admits an effective quotient in an analytic neighbourhood.

6.4.2 (Lifting local quotients)**.** Let R be a groupoid on a scheme X over $\mathbb{C}$ and let $x \in X$ be a closed point. Let $\mathfrak{R}$ denote the induced infinitesimal groupoid. By Theorem 6.2.5 (Existence of quotients), showing existence of a global quotient of X by R amounts to showing that every point is algebraically smooth. A strategy to accomplish this is described hereafter. By Frobenius' theorem, formal locally, there are enough invariant sections (formal first integrals). The objective is to lift invariant sections to a neighbourhood of $x \in X$ and conclude there are enough first integrals. Recall that there are different types of localisation depending on the desired degree of magnification. In increasing order of precision, one can look at the functions in a sufficiently small affine open neighbourhood U, the functions on the Zariski localisation, the Hensel localisation and the completion. In symbols,

$$\mathscr{O}_X(U) \subseteq \mathscr{O}_{X,x} \subseteq \mathscr{O}^h_{X,x} \subseteq \hat{\mathscr{O}}_{X,x}. \tag{6.4.2.1}$$

Since these are all formal schemes, one can consider the restriction of $\mathfrak{R}$ to each of them and study their first integrals

$$\mathscr{O}^{\mathfrak{R}}_X(U) \subseteq \mathscr{O}^{\mathfrak{R}}_{X,x} \subseteq \left(\mathscr{O}^h_{X,x}\right)^{\mathfrak{R}} \subseteq \left(\hat{\mathscr{O}}_{X,x}\right)^{\mathfrak{R}}. \tag{6.4.2.2}$$

If $\mathfrak{R}$ is regular, in addition to the fact that the second, third and fourth objects are local rings (Lemma 5.1.10 (Structure of local first integrals)), by use of Proposition 5.3.3 (First integrals are algebraically closed), it can be shown that the third object is an Henselian local ring and the fourth object is a complete local ring. It has to be established whether

1. $\left(\hat{\mathscr{O}}_{X,x}\right)^{\mathfrak{R}}$ is the completion of $\left(\mathscr{O}^h_{X,x}\right)^{\mathfrak{R}}$.

2. $\left(\mathscr{O}^h_{X,x}\right)^{\mathfrak{R}}$ is the Hensel localisation of $\mathscr{O}^{\mathfrak{R}}_{X,x}$.

3. $\mathscr{O}^{\mathfrak{R}}_{X,x}$ is the Zariski localisation of $\mathscr{O}^{\mathfrak{R}}_X(U)$,

for a sufficiently small affine open subset $U \subseteq X$. When all of the above steps hold true, $\mathfrak{R}$ is *algebraically integrable* in $x \in X$. This section presents a strategy to prove the first step.

Example 6.4.3 (Family of parabolae)**.** This example shows that an infinitesimal equivalence relation $\mathfrak{R}$ which is algebraically integrable in the generic point of a scheme X need not be algebraically integrable in every point $x \in X$.

Let $k := \mathbb{C}$ be the field of complex numbers and consider the family of parabolae

$$X := \operatorname{Spec}\left(\frac{k[x,y,t]}{(x^2 - ty)}\right) \to \operatorname{Spec} k[t] =: Y. \tag{6.4.3.1}$$

The fibre over $t \neq 0$ is a smooth parabola whereas the fibre over $t = 0$ is a double line D. Consider the induced equivalence relation R on X. Let $\mathscr{F}_R$ be the induced foliation. By Construction 4.5.2 (Distributions of infinitesimal groupoids), $\mathscr{F}_R$ is a smooth foliation on $X \setminus D$. Let $\overline{\mathscr{F}}_R$ be the saturation of $\mathscr{F}_R$. This is a smooth foliation on $X \setminus \{0\}$. Let $\overline{\mathfrak{R}}$ be the graph of the foliation $\overline{\mathscr{F}}_R$ (Paragraph 4.5.9 (Graph of a foliation)). On the open set $X \setminus D$, $\overline{\mathfrak{R}}$ is the infinitesimal groupoid induced by R. Therefore, $\overline{\mathfrak{R}}$ is generically algebraically integrable. However, $\mathscr{F}_R$ and $\overline{\mathscr{F}}_R$ are different on the set $D \setminus \{0\}$. Indeed, the former is singular but the latter is smooth. Let $x \in D \setminus \{0\}$. Locally analytically in x, by Frobenius' theorem, $\overline{\mathfrak{R}}$ is described by a trivial fibration. However, étale locally in x, it is not algebraically integrable since étale functions cannot split the parabolae encircling the double line. This is shown in Figure 10.

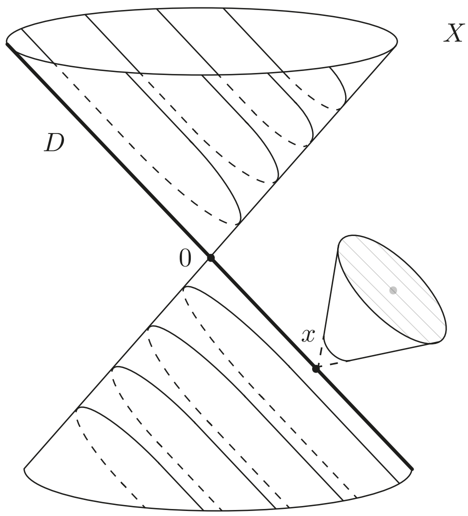


Figure 10: A family of parabolae

Next is the definition of étale first integrals and formal first integrals.

6.4.4 (Étale first integrals)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a field k and let $x \in X$. Let $\mathscr{O}^h_{X,x}$ be the Hensel localisation of X in x and let $\mathfrak{R}^h$ be the restriction by this morphism. An *étale first integral* of $\mathfrak{R}$ in x is a first integral of $\mathfrak{R}^h$ over $\mathscr{O}^h_{X,x}$. By definition of Hensel localisation, an element of $\mathscr{O}^h_{X,x}$ is given by an étale morphism $(U,u) \to (X,x)$ where $u \in U$ and $u \to x$ together with a function on U. Therefore, an element of

$$\left(\mathscr{O}^h_{X,x}\right)^{\mathfrak{R}} \tag{6.4.4.1}$$

is given by a function on U which is $\mathfrak{R}|_U$ invariant.

6.4.5 (Formal first integrals)**.** Let $\mathfrak{R}$ be an infinitesimal groupoid on a scheme X over a field k and let $x \in X$. Let $\hat{\mathfrak{X}} = \operatorname{Spf} \hat{\mathscr{O}}_{X,x}$ be the completion in $x \in X$ and let $\hat{\mathfrak{R}}$ be the restriction of $\mathfrak{R}$ to $\mathfrak{X}$. A *formal first integral* of $\mathfrak{R}$ in x is a first integral of $\hat{\mathfrak{R}}$ over $\hat{\mathfrak{X}}$. Frobenius' theorem shall hold on $\hat{\mathfrak{X}}$.

The basic idea is to apply Artin's approximation theorems.

6.4.6 (Artin approximation theorems)**.** Let X be a scheme of finite type over a field k and let $x \in X$. Let A and $\hat{A}$ be the Hensel localisation of X in x and the completion of X in x respectively. Let F be a functor on the category of A-algebras valued in the category of sets and suppose it preserves filtered colimits. In details, for all cofiltered systems of A-algebras $\{B_\lambda\}_{\lambda\in\Lambda}$, the natural morphism

$$\operatorname*{colim}_{\lambda\in\Lambda} F\left(B_\lambda\right) \xrightarrow{\sim} F\left(\operatorname*{colim}_{\lambda\in\Lambda} B_\lambda\right) \tag{6.4.6.1}$$

is an isomorphism. Then, for any $\hat{\xi} \in F(\hat{A})$ and any natural number $c \in \mathbb{N}$, there exists a $\xi_c \in F\left(A\right)$ such that the image of ξ_c is equal to the image of $\hat{\xi}$ in the diagram

$$\begin{tikzcd} F\left(A\right) \arrow[rr] \arrow[dr] & & F(\hat{A}) \arrow[dl] \\ & F\left(A/\mathfrak{m}_x^{c+1}\right), & \end{tikzcd} \tag{6.4.6.2}$$

where $\mathfrak{m}_x$ is the maximal ideal of $\hat{A}$. This is the content of the approximation theorems of Artin ([Art69, Theorem 1.12, page 26]).

In order to apply the theorem, two candidate functors are defined. It is then shown that only one of them satisfies the hypotheses of the approximation theorems.

6.4.7 (Functor of first integrals)**.** Let X be a scheme of finite type over a field k and let $x \in X$. Let $\mathfrak{R}$ be an infinitesimal groupoid on X over k. Let A be the Hensel localisation of X in x. There is a functor of first integrals of $\mathfrak{R}$ defined as

$$\begin{aligned} F^{\mathfrak{R}} : \{A\text{-algebras}\} &\to \{A^{\mathfrak{R}}\text{-algebras}\} \\ B &\to B^{\mathfrak{R}|_B}, \end{aligned} \tag{6.4.7.1}$$

where $\mathfrak{R}|_B$ is the restriction of $\mathfrak{R}$ by $A \to B$. If $B \to C$ is a morphism of A-algebras, then, by the universal property of equalisers, there is a morphism

$$B^{\mathfrak{R}|_B} \to C^{\mathfrak{R}|_C}. \tag{6.4.7.2}$$

It is straightforward to verify this is a functor.

6.4.8 (Functor of invariant sections)**.** Let X be a scheme of finite type over a field k and let $x \in X$. Let R be a groupoid locally of finite presentation on X over k. Let A be the Hensel localisation of X in x. Similarly to Paragraph 6.4.7 (Functor of first integrals), there is a functor of invariant sections of R defined as

$$\begin{aligned} F^R : \{A\text{-algebras}\} &\to \{A^R\text{-algebras}\} \\ B &\to B^{R|_B}. \end{aligned} \tag{6.4.8.1}$$

Lemma 6.4.9 (Functor of invariant sections preserves colimits)**.** *Notation as in Paragraph 6.4.6 (Artin approximation theorems), the functor of invariant sections preserves colimits.*

Proof. This follows from the fact that, in any concrete category, filtered colimits commute with finite limits ([Sta23, Lemma 002W]). In details, let $R = \operatorname{Spec} C$ be an affine groupoid of finite presentation on A over k and let $\{B_\lambda\}_{\lambda\in\Lambda}$ be a cofiltered system of A-algebras. Let B be the colimit of the cofiltered system, let $R_\lambda = \operatorname{Spec} C_\lambda$ denote the restriction of R by $A \to B_\lambda$ and let $R|_B := \operatorname{Spec} C|_B$ be the restriction of R by $A \to B$. Then

$$\begin{aligned} \operatorname*{colim}_{\lambda\in\Lambda} F(B_\lambda) &= \operatorname*{colim}_{\lambda\in\Lambda} \operatorname{Eq}\left(B_\lambda \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} C_\lambda\right) \\ &= \operatorname{Eq}\left(\left(\operatorname*{colim}_{\lambda\in\Lambda} B_\lambda\right) \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} \left(\operatorname*{colim}_{\lambda\in\Lambda} C_\lambda\right)\right) \\ &= \operatorname{Eq}\left(B \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} \operatorname*{colim}_{\lambda\in\Lambda}\left(C \underset{A\otimes_k A}{\otimes} (B_\lambda \otimes_k B_\lambda)\right)\right) \\ &= \operatorname{Eq}\left(B \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} \left(C \underset{A\otimes_k A}{\otimes} \operatorname*{colim}_{\lambda\in\Lambda} (B_\lambda \otimes_k B_\lambda)\right)\right) \\ &= \operatorname{Eq}\left(B \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} \left(C \underset{A\otimes_k A}{\otimes} (B \otimes_k B)\right)\right) \\ &= \operatorname{Eq}\left(B \underset{t^\#}{\overset{s^\#}{\rightrightarrows}} C|_B\right) = F(B). \end{aligned} \tag{6.4.9.1}$$

Therefore, the functor of invariant sections preserves filtered colimits. $\square$

6.4.10 (Functor of first integrals does not preserve colimits)**.** Notation as in Paragraph 6.4.6 (Artin approximation theorems), the functor of first integrals does not preserves colimits. To see this, note that, in the construction of infinitesimal restriction, a filtered limit (completion) is employed. It is well-known that filtered limits do not in general commute with filtered colimits.

The next paragraph suggests that, on applying the content of Artin's approximation theorems, the completion of the ring of étale first integrals should be the ring of formal first integrals.

6.4.11 (Strategy of approximation)**.** Notation as in Paragraph 6.4.6 (Artin approximation theorems), consider $X = \operatorname{Spec} A$, $\overline{X} = \operatorname{Spec} \hat{A}$ and $\mathfrak{X} = \operatorname{Spf} \hat{A}$. Consider the usual restriction and the infinitesimal restriction of R to $\overline{X}$ and $\mathfrak{X}$.

$$\begin{tikzcd}
\hat{\mathfrak{X}} \arrow[r] \arrow[d] & \hat{\mathfrak{R}} \arrow[r] \arrow[d] & \hat{R} \arrow[r] \arrow[d] & \hat{\mathfrak{X}} \times_k \hat{\mathfrak{X}} \arrow[d] \\
\overline{X} \arrow[r] \arrow[d] & \overline{\mathfrak{R}} \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & \overline{X} \times_k \overline{X} \arrow[d] \\
X \arrow[r] & \mathfrak{R} \arrow[r] & R \arrow[r] & X \times_k X.
\end{tikzcd} \tag{6.4.11.1}$$

The diagram implies the existence of inclusions of rings

$$\begin{tikzcd}
\mathscr{O}_X^R \arrow[r, "1^{\mathrm{st}}"] \arrow[d] & \mathscr{O}_{\overline{X}}^{\overline{R}} \arrow[r] \arrow[d, "2^{\mathrm{nd}}"] & \mathscr{O}_{\hat{\mathfrak{X}}}^{\hat{R}} \arrow[d] \\
\mathscr{O}_X^{\mathfrak{R}} \arrow[r] & \mathscr{O}_{\overline{X}}^{\overline{\mathfrak{R}}} \arrow[r, "3^{\mathrm{rd}}"'] & \mathscr{O}_{\hat{\mathfrak{X}}}^{\hat{\mathfrak{R}}}.
\end{tikzcd} \tag{6.4.11.2}$$

Frobenius' theorem is valid on $\mathscr{O}_{\hat{\mathfrak{X}}}^{\hat{\mathfrak{R}}}$ and it is desirable to approximate its sections by sections in $\mathscr{O}_X^R$. The *first* arrow is a completion morphism by Artin approximation theorem and Lemma 6.4.9 (Functor of invariant sections preserves colimits). It is then required to show that the *second* arrow and the *third* arrow are isomorphisms. This follows if the respective morphisms of rings $\mathscr{O}_{\overline{R}} \to \mathscr{O}_{\overline{\mathfrak{R}}}$ and $\mathscr{O}_{\overline{\mathfrak{R}}} \to \mathscr{O}_{\hat{\mathfrak{R}}}$ are injective. It is straightforward to see that the latter is injective since it is the completion of an adic Noetherian local ring. On the other hand, it seems hard to verify whether the former morphism is injective since it is the completion of a ring which is rarely Noetherian.

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