VECTOR SPACES / SUBSPACES

Def. A vector space V over the numerical field $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) is a nonempty set of objects, called vectors, in which two operations are defined

- addition (+): $\forall v_1, v_2 \in V \Rightarrow v_1+v_2 \in V$
- scalar multiplication: $\forall \alpha \in \mathbb{K}, \forall v \in V \Rightarrow \alpha v \in V$

and the following properties are satisfied

(V1): (+) is commutative: $\forall v_1, v_2 \in V \Rightarrow v_1+v_2 = v_2+v_1$

(V2): (+) is associative: $\forall v_1, v_2, v_3 \in V \Rightarrow (v_1+v_2)+v_3 = v_1+(v_2+v_3)$

(V3): there exists $0 \in V$ (the null/zero vector): $\forall v \in V$ $0+v = v+0 = v$ → (NOT $0 \in \mathbb{K}$!!)

(V4): $\forall v \in V$, there exists its opposite $-v \in V$: $v+(-v)=(-v)+v=0$

(V5): $\forall v \in V \Rightarrow 1v = v$, where $1 \in \mathbb{K}$ is the unity of $\mathbb{K}$

(V6): the distributive property: $\forall \alpha \in \mathbb{K}, \forall v_1, v_2 \in V \Rightarrow \alpha(v_1+v_2) = \alpha v_1 + \alpha v_2$

(V7): — ll — $\forall \alpha_1, \alpha_2 \in \mathbb{K}, \forall v \in V \Rightarrow (\alpha_1+\alpha_2)v = \alpha_1 v + \alpha_2 v$

(V8): the associative property: $\forall \alpha_1, \alpha_2 \in \mathbb{K}, \forall v \in V \Rightarrow (\alpha_1\alpha_2)v = \alpha_1(\alpha_2 v)$

Def. A nonempty subset $W \subseteq V$ is called a vector subspace of $V \Leftrightarrow W$ is a vector space over $\mathbb{K}$.
In particular
(i) $\forall w_1, w_2 \in W \Rightarrow w_1+w_2 \in W$
(ii) $\forall \alpha \in \mathbb{K}, \forall w \in W \Rightarrow \alpha w \in W$
(iii) $0 \in W$

Since (V4)-(V8) are true $\forall v \in V \Rightarrow$ they are valid $\forall w \in W \subseteq V$.

EXAMPLES

VECTOR SPACES
1) $V=\{0\}$ the trivial vector space (addition and scalar multiplication are trivial)
2) $V=\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) the field itself: addition is addition between scalars, scalar multiplication is multiplication between scalars
3) $V=\mathbb{K}^n$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) the set of all n-tuples of real (or complex) numbers
4) $V = \{$the set of all functions $f:[a,b]\to\mathbb{R}\}$ $= X([a,b])$: we can add functions:
$f+g$ s.t. $(f+g)(t) = f(t)+g(t)$, $\forall t \in [a,b]$
cf s.t. $(cf)(t) = c\,f(t)$, $\forall t \in [a,b]$
the null function is f s.t. $f(t)=0$, $\forall t \in [a,b]$
5) $V = C([a,b]) = \{$the set of all continuous functions $f:[a,b]\to\mathbb{R}\}$
6) $V = C^p([a,b]) = \{$the set of all functions $f:[a,b]\to\mathbb{R}$ continuous up to their p-th derivative$\}$
7) $V = C^\infty(\mathbb{R}) = \{$the set of all functions infinite continuously on $\mathbb{R}\}$
8) $V = \mathbb{P}_n = \{p_n$ s.t. $p_n(x) = \sum_{k=0}^{n} a_k x^k$ polynomials of degree less or equal than n, $n \geq 0\}$

VECTOR SUBSPACES
1) $W=\{0\}$ is a vector subspace of any vector space V
2) $C^p([a,b]) \subseteq C([a,b]) \subseteq X([a,b])$ vector subspaces
3) $\mathbb{P}_n \subseteq C^\infty(\mathbb{R})$
4) Given $U \subseteq V$, $W \subseteq V$ vector subspaces
$\Rightarrow$ $U \cap W = \{v \in V$ s.t. $v \in U$ and $v \in W\}$ is a vector subspace
$U+W = \{v \in V$ s.t. $\underbrace{u+w}_{(*)}$, $u \in U, w \in W\}$ — ll — — ll —

If $(*)$ is unique $\forall v \in U+W \Rightarrow$ the sum is called direct sum and it is denoted by $U \oplus W$

SYSTEM OF GENERATORS

Def. Given an arbitrary vector space V and a set of n vectors $\{v_1, \dots, v_n\}$
We call the generated subspace (or span) the set of the linear combinations of $\{v_1, \dots, v_n\}$

$$W = \text{span}\{v_1, \dots, v_n\}$$
$$= \{v \in V : v = \alpha_1 v_1 + \dots + \alpha_n v_n, \ \alpha_1, \dots, \alpha_n \in \mathbb{K}\}$$

- $W \subseteq V$ is a vector subspace
- The system $\{v_1, \dots, v_n\}$ is called a system of generators for V.
- We say that $\{v_1, \dots, v_n\}$ generates V if $\forall v \in V$, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ s.t.

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Def. Given V, $\{v_1, \dots, v_n\}$
We say that $\{v_1, \dots, v_n\}$ is linearly independent if the relation
$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0, \quad \alpha_1, \dots, \alpha_n \in \mathbb{K}$$
implies
$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Otherwise, if there exist $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, not all equal to zero, s.t.
$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0$$
we say that $\{v_1, \dots, v_n\}$ is linearly dependent.

BASIS

Def. We call a basis any set of linearly independent vectors $\{v_1, \dots, v_n\}$ that generate V.
If $\{v_1, \dots, v_n\}$ is a basis for $V \Rightarrow$ any $v \in V$ can be written as
$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, \quad \alpha_1, \dots, \alpha_n \in \mathbb{K} \quad (*)$$
$\alpha_1, \dots, \alpha_n$ are called the components (coefficients, coordinates) of v in the basis $\{v_1, \dots, v_n\}$
$(*)$ is called the composition of v w.r.to the basis $\{v_1, \dots, v_n\}$.
It can be proved that $\alpha_1, \dots, \alpha_n$ are uniquely determined, i.e.
if there exist $\beta_1, \dots, \beta_n \in \mathbb{K}$ s.t.
$$v = \beta_1 v_1 + \dots + \beta_n v_n$$
$$\Rightarrow \alpha_1 = \beta_1, \dots, \alpha_n = \beta_n$$
(This is basically due to the linearly independence).

DIMENSION

Theorem. Let V be a vector space which admits a basis of n elements, say $\{v_1, \dots, v_n\}$.
Then,
1. any set of linearly independent vectors in V has at most n elements
2. any other basis of V has exactly n elements
3. any set of n linearly independent vectors in V must generate V, and hence, form a basis

2.: In short, two basis must have the same number of elements!

Definition. Let V be a vector space having a basis of n elements.
We call n the dimension of V and we write $\dim(V) = n$.
If instead, $\forall n$ there exist n linearly independent vectors of V, then V is called infinite dimensional.

Theorem. Let V be a vector space having a basis of n elements.
Let $W \subseteq V$ be a vector subspace. Then

$$\dim(W) \leq \dim(V).$$

EXAMPLES
1) $\forall$ integer p, $C^p([a,b])$ is infinite dimensional
2) $\dim(\mathbb{R}^n) = \dim(\mathbb{C}^n) = n$

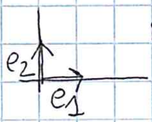
Standard basis for $\mathbb{R}^n$: $\{e_1, \dots, e_n\}$

$$e_1 = (1, 0, \dots, 0)^T$$
$$e_2 = (0, 1, \dots, 0)^T$$
$$\vdots$$
$$e_n = (0, \dots, 0, 1)^T$$

$$\underbrace{(e_i)_j}_{j\text{-th component of } e_i} = \underbrace{\delta_{ij}}_{\text{Kronecker symbol}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad i, j = 1, \dots, n$$

EXERCISES

① In $\mathbb{R}^2$



Standard basis given $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Prove that $(1,1)^T$ and $(-1,2)^T$ form a basis.
Since I have two vectors in $\mathbb{R}^2$ (and we know $\dim(\mathbb{R}^2) = 2$), then it is sufficient to prove that they're linearly independent.
We check by applying the definition:

Given $a, b \in \mathbb{K} \Rightarrow a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0_{\mathbb{R}^2}$

$$\Leftrightarrow \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} -b \\ 2b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

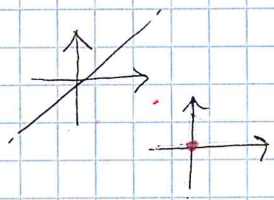
$$\Leftrightarrow \begin{cases} a - b = 0 \\ a + 2b = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = +b \\ a = -2b \end{cases}$$

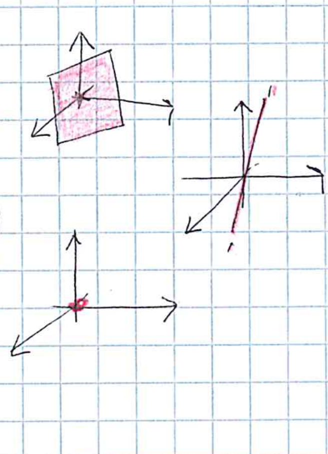
$\Leftrightarrow a = b = 0$ → so ok, linearly independent.

② Check if $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$ are linearly independent.
Answer:
We already know they're not because in $\mathbb{R}^2$ we can find at most $\dim(\mathbb{R}^2) = 2$ lin. ind. vectors

③ $V \subseteq \mathbb{R}^2$ a vector subspace. Which are the possible dimensions and candidates for V.
First, $\dim(V) \leq \dim(\mathbb{R}^2) = 2$.
If $\dim(V) = 2 \Rightarrow V = \mathbb{R}^2$
If $\dim(V) = 1 \Rightarrow V$ is a straight line through the origin
If $\dim(V) = 0 \Rightarrow V = \{0\}$
These are all the possibilities. ↳ we mean here $0_{\mathbb{R}^2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



④ If $V \subseteq \mathbb{R}^3$ vector subspace?
Then, $\dim(V) \leq \dim(\mathbb{R}^3) = 3$
If $\dim(V) = 3 \Rightarrow V = \mathbb{R}^3$
If $\dim(V) = 2 \Rightarrow V$ is a plane through the origin
If $\dim(V) = 1 \Rightarrow V$ is a straight line through the origin
If $\dim(V) = 0 \Rightarrow V$ is the origin $V = \{0\}$ $0_{\mathbb{R}^3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



MATRICES

Def. Given m, n two positive integers.
We call a matrix having m rows and n columns, or a $m \times n$ matrix, or a (m,n) matrix, any array of mn scalars $a_{ij} \in \mathbb{K}$, with $i = 1, \dots, m$, $j = 1, \dots, n$, represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \leftarrow \text{rows} \\ \leftarrow \\ \\ \\ \end{array}$$

columns

We abbreviate the notation and write $A = (a_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, n$.
If $\mathbb{K} = \mathbb{R} \Rightarrow A \in \mathbb{R}^{m \times n}$; if $\mathbb{K} = \mathbb{C} \Rightarrow A \in \mathbb{C}^{m \times n}$.
A has m rows & n columns.

Example:
The matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ 5 & 0 & 3 \end{bmatrix}$ is a 2×3 matrix, $A \in \mathbb{R}^{2 \times 3}$ (an array of $2 \cdot 3 = 6$ scalars)

Row vectors of A $r_1 = [1, 1, -2]$
$r_2 = [5, 0, 3]$
Column vectors of A: $a_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_3 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Particular examples:
Vectors of $\mathbb{R}^n$ / $\mathbb{R}^m$ and scalars can be seen as particular matrices.
① A vector $[x_1, \dots, x_n]$ is a $1 \times n$ matrix.
② A vector $\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ is a $m \times 1$ matrix.
③ A scalar $a \in \mathbb{K}$ is a 1×1 matrix.

Matrix operations
Let $A = (a_{ij})$, $B = (b_{ij})$ be two $m \times n$ matrices over $\mathbb{K}$.
We define the following operations:

- matrix sum: $(A+B) = (a_{ij} + b_{ij})$ $i = 1, \dots, m$, $j = 1, \dots, n$

example: $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

$$\Rightarrow A+B = \begin{bmatrix} 1+0 & -1-1 & 0+3 \\ 2+1 & 3+2 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 5 & 4 \end{bmatrix}$$

(elementwise operation)

<u>important</u>: A and B must have the same size!!

- matrix multiplication by scalar $\alpha \in \mathbb{K}$: $\alpha A = (\alpha a_{ij})$ $i = 1, \dots, m$, $j = 1, \dots, n$

example: $\alpha = 2 \Rightarrow 2A = \begin{bmatrix} 2 \cdot 1 & 2 \cdot (-1) & 2 \cdot 0 \\ 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 4 & 6 & 8 \end{bmatrix}$

(elementwise operation)

Definitions

1. A square matrix: $m=n \Rightarrow$ we will say a square matrix of order n
2. A diagonal matrix: $a_{ij}=0 \; \forall \; i \neq j$:

$$D=\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & \cdots & & \vdots \\ 0 & \cdots & & a_{nn} \end{bmatrix}.$$

$$D=\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$$

3. zero matrix of order n: $a_{ij}=0 \; \forall i,j=1,\ldots,n$

$$0=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Note: $0+A=A+0=A \; \forall A$

4. Symmetrix: $a_{ij}=a_{ji} \; \forall i,j$
5. $-A$ st. $-A=(-a_{ij}) \; \forall i,j=1,\ldots,n$

MULTIPLICATION OF MATRICES

Let $A=(a_{ij})$ $i=1,\ldots,m$, $j=1,\ldots,n$ be a $m\times n$ matrix over $\mathbb{K}$.
Let $B=(b_{jk})$ $j=1,\ldots,n$, $k=1,\ldots,s$ be a $n\times s$ matrix over $\mathbb{K}$.

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n} \qquad B=\begin{bmatrix} b_{11} & \cdots & & b_{1s} \\ b_{21} & \cdots & & b_{2s} \\ \vdots & & & \\ b_{n1} & \cdots & & b_{ns} \end{bmatrix} \in \mathbb{R}^{n\times s}$$

Then the product $C=AB$ is a $m\times s$ matrix over $\mathbb{K}$ st.

$$c_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$$

PROPERTIES OF MATRIX MULTIPLICATION

- Distributive law:

$$A(B+C)=AB+AC$$

$$A(\alpha B)=\alpha(AB)$$

- ASSOCIATIVE LAW:

$$(AB)C=A(BC)$$

- NO COMMUTATIVE LAW! in general $AB \neq BA$
  Example: Try AB and BA with $A=\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$

  Special case where comm. law holds: powers of A commute: $A^rA^s=A^sA^r$

Exercise: Exhibit an example to show that the product of two symmetrix matrices may not be symmetric:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}=\begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}$$

We denote by $\mathbb{K}^{m\times n}$ the set of all $m\times n$ matrices over $\mathbb{K}$.
(or: $\mathcal{M}_{m\times n}(\mathbb{K})$)
Then $\mathbb{K}^{m\times n}$ is a vector space over $\mathbb{K}$.

It can be easily seen that (V1)-(V8) are verified

$\dim(\mathbb{K}^{m\times n}) = mn$

A possible basis is given by the matrices with only one entry $= 1$ and the others $= 0$.

Example:

$$\mathcal{M}_{2\times 2} = \operatorname{span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

DETERMINANTS

We first carry out separately the cases $n=2, n=3$.

① DETERMINANTS OF ORDER 2

Let $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ a 2×2 matrix over $\mathbb{K}$.

We define the determinant of A to be the scalar $ad-bc$.
We denote it by

$$\det(A)=\begin{vmatrix} a & b \\ c & d\end{vmatrix}= ad-bc$$

For example, if $A=\begin{bmatrix} 1 & 2 \\ 4 & 1\end{bmatrix} \Rightarrow \det(A)=1\cdot 1-2\cdot 4=1-8=7$

if $B=\begin{bmatrix} -2 & -3 \\ 4 & 5\end{bmatrix} \Rightarrow \det(B)=-2\cdot 5-(-3)4=-10+12=2$

Properties (i) if the two columns of A are equal $\Rightarrow \det(A)=0$

Proof: $\det(A)=\begin{vmatrix} a & a \\ b & b\end{vmatrix}= ab-ab=0$

(ii) $\det(I_2)=1$

Proof: $\det(I_2)=\begin{vmatrix} 1 & 0 \\ 0 & 1\end{vmatrix}=1$

(iii) $\det(A^T)=\det(A)$

Proof: $\left.\begin{array}{l}\det(A)=\begin{vmatrix} a & b \\ c & d\end{vmatrix}=ad-bc \\ \det(A^T)=\begin{vmatrix} a & c \\ b & d\end{vmatrix}= ad-bc\end{array}\right\}$ are equal

(iv) if the two columns of A are interchanged $\Rightarrow$ the determinant changes by a sign.

Proof: $\det(A)=\begin{vmatrix} a & b \\ c & d\end{vmatrix}=ad-bc$

$$\det(A^T)=\begin{vmatrix} b & a \\ d & c\end{vmatrix}= bc-ad=-(ad-bc)$$

(v) The columns of A are linearly independent $\Longleftrightarrow \det(A)\neq 0$.
{ " " dependent $\Longleftrightarrow \det(A)=0$ }

Proof: Assume that the colums of A are linearly dependent,
i.e. there exist scalars α, β, not both equa to zero, s.t.
(say $\alpha\neq 0$)

$$\alpha\begin{bmatrix} a \\ b\end{bmatrix}+\beta\begin{bmatrix} c \\ d\end{bmatrix}=\begin{bmatrix} 0 \\ 0\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \alpha a+\beta c \\ \alpha b+\beta d\end{bmatrix}=\begin{bmatrix} 0 \\ 0\end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \alpha a+\beta c=0 \\ \alpha b+\beta d=0\end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha=-\beta\dfrac{c}{a} \\ -\beta\dfrac{c}{a}b+\beta d=0\end{cases} \Rightarrow (\beta\neq 0, \text{ unless } c=0)$$

$$\Leftrightarrow \begin{cases} \alpha=-\beta\dfrac{c}{a} \\ \beta\left(-\dfrac{c}{a}b+d\right)=0\end{cases} \to \begin{array}{l}\beta=0 \quad \text{No} \\ \underbrace{-\dfrac{c}{a}b+d=0}_{(*)}\end{array}$$

$(*)$ means $-\dfrac{c}{a}b=-d \Leftrightarrow \underbrace{ad-bc}_{\det(A)}=0$

DETERMINANTS OF ORDER 3

Let $A=(a_{ij})=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ be a 3x3 matrix over $\mathbb{K}$.

We define its determinant according to the following "Expansion by a row rule", say first row:

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$$

$$=a_{11}\det(A_{11})-a_{12}\det(A_{12})+a_{13}\det(A_{13}), \quad (*)$$

where A_{ij} is the matrix obtained by deleting the ith row and j-th column of A.

Example: $A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5\end{bmatrix} \Rightarrow A_{11}=\begin{bmatrix} 1 & 4 \\ 2 & 5\end{bmatrix}, A_{12}=\begin{bmatrix} 1 & 4 \\ -3 & 5\end{bmatrix}, A_{13}=\begin{bmatrix} 1 & 1 \\ -3 & 2\end{bmatrix}$

$\Rightarrow \det(A)=2(5-8)-1(5+12)+0=-23$

- There is no particular reason why we should expand according to the first row. Let's expand by the second row:

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}=-a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{vmatrix}+a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix}-a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix}$$

$$=-a_{21}\det(A_{21})+a_{22}\det(A_{22})-a_{23}\det(A_{23})$$

Try the example!

- We can expand according to any row and column. Signs follow this pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Example: Compute the determinant $\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 5 \\ -1 & 4 & 2\end{vmatrix}$ by expanding according to the 2nd column

$\Rightarrow \det(A)=-0\begin{vmatrix} & \\ & \end{vmatrix}+2\begin{vmatrix} 3 & 1 \\ -1 & 2\end{vmatrix}-4\begin{vmatrix} 3 & 1 \\ 1 & 5\end{vmatrix}=2(6-(-1))-4(15-1)=-42$

according to the 3rd column:

$\Rightarrow \det(A)=+1\begin{vmatrix} 1 & 2 \\ -1 & 4\end{vmatrix}-5\begin{vmatrix} 3 & 0 \\ -1 & 4\end{vmatrix}+2\begin{vmatrix} 3 & 0 \\ 1 & 2\end{vmatrix}=-42$

we observe we get the same result!

Note: Same properties as in $n=2$ hold true: $\det(A^T)=\det(A)$, $\det(I_3)=1$

Note: If we expand $(*)$:

$\det(A)=a_{11}\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}+\dots$

$=\dots$

$=a_{11}a_{22}a_{33}-a_{11}a_{32}a_{23}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31} \quad (**)$

We can expand according to any column/row and we'd get the same result as in $(**)$. This can be generalized in n dimensions.

DETERMINANTS OF ORDER n

Let $A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$.

We define its determinant to be the scalar

$$\det(A) = \sum_{\pi \in P} \text{sign}(\pi)\, a_{1\pi_1} a_{2\pi_2} \dots a_{n\pi_n},$$

where $P = \{ \underline{\pi} = (\pi_1, \dots, \pi_n)^T \}$ is the set of the $n!$ vectors that are obtained by permuting the index vector $i = (1, \dots, n)^T$ and $\text{sign}(\pi) = \begin{cases} 1 & \text{if an even number of exchanges are needed to obtain } \underline{\pi} \text{ from } \underline{i} \\ -1 & \text{if an odd } \quad '' \quad '' \quad '' \quad '' \quad '' \end{cases}$

Laplace Rule: $\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ \sum_{j=1}^{n} \underbrace{(-1)^{i+j} \underbrace{\det(A_{ij})}_{\text{the complementary minor associated with the entry } a_{ij}}}_{\text{the cofactor of the entry } a_{ij}} a_{ij} & \text{for } n \geq 2 \end{cases}$

the complementary minor associated with the entry a_{ij}

the cofactor of the entry a_{ij}
sometimes: $\Delta_{ij} = (-1)^{i+j} \det(A_{ij})$
or C_{ij}

Same properties are valid also in the generic case:

(i) If two entries are equal $\Rightarrow \det(A) = 0$

(ii) $\det(I_n) = 1$

(iii) If two columns are interchanged $\Rightarrow$ the determinant changes by a sign.

(iv) $\det(A) = \det(A^T)$

(v) $\det(\alpha A) = \alpha^n \det(A)$

(vi) $\det(AB) = \det(A)\det(B)$ (BINET'S THEOREM)

$\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$ (for A inv.)

ADJOINT METHOD (RULE) FOR COMPUTING THE INVERSE OF A MATRIX

$$A^{-1} = \frac{1}{\det(A)} \underbrace{C^T}_{\text{cofactor matrix}}$$

Note: We see that A is invertible $\Leftrightarrow \det(A) \neq 0$.

EXERCISE

Compute the following determinants:

a) $\begin{vmatrix} 2 & 0 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{vmatrix}$, b) $\begin{vmatrix} 3 & -1 & 5 \\ -1 & 2 & 1 \\ -2 & 4 & 3 \end{vmatrix}$, c) $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & 7 \end{vmatrix}$ d) $\begin{vmatrix} -1 & 5 & 3 \\ 4 & 0 & 0 \\ 2 & 7 & 8 \end{vmatrix}$

LINEAR MAPPINGS

Def. Let V, W be two vector spaces.
A linear mapping $L: V \to W$ is a mapping that satisfies the following properties:
(i) $\forall u, v \in V$: $L(u+v) = L(u) + L(v)$
(ii) $\forall u \in V$, $\forall \alpha \in \mathbb{K}$: $L(\alpha u) = \alpha L(u)$

Example: The linear map associated with a matrix.
Given $A \in \mathbb{R}^{m \times n}$
Define $L_A: \mathbb{R}^n \to \mathbb{R}^m$, $v \mapsto L_A(v) := Av$

Then L_A is a linear map.
Proof. (i) $\forall u, v \in V$, $L_A(u+v) = A(u+v) \overset{\star}{=} Au + Av = L_A(u) + L_A(v)$ OK
(ii) $\forall u \in V$, $\forall c \in \mathbb{K}$, $L_A(cu) = A(cu) \underset{(*)}{=} c(Au) = c L_A(u)$ OK

Recall:
$\star$ Distributive law of matrix multiplication:
$A(B+C) = AB + BC$
$(*)$ Property of matrix multiplication:
$A(\alpha B) = \alpha(AB)$

Example: V vector space.
- Define $I: V \to V$, $v \mapsto I(v) = v$ the identity mapping. Trivially satisfies (i) and (ii).
- Define $O: V \to V$, $v \mapsto O(v) = 0$, the zero mapping. " "

Properties of linear mappings
$L: V \to W$ a linear map, where V, W are vector spaces.
Then, 1) $L(\alpha_1 u_1 + \ldots + \alpha_m u_m) = \alpha_1 L(u_1) + \ldots + \alpha_m L(u_m)$

$$\Updownarrow$$

$$L\left(\sum_{j=1}^{m} \alpha_j u_j\right) = \sum_{j=1}^{m} \alpha_j L(u_j)$$

2) $L(0) = 0$
3) $L(-u) = -L(u)$

Let $L: V \to W$ and $F: V \to W$ be two linear mappings, where V, W are vector spaces.
Define $\mathcal{L}(V, W) = \{L: V \to W\}$ as the set of all linear mappings from V to W.
We want to define the operations of addition and scalar multiplication in such a way to make $\mathcal{L}(V, W)$ a vector space.
Define the following operations
(i) SUM OF LINEAR MAPPINGS: $\forall F, L \in \mathcal{L}(V, W)$, define $L + F: V \to W$
$u \mapsto (L+F)(u) = L(u) + F(u)$
(ii) Scalar multiplication of a lin. map: $\forall F \in \mathcal{L}(V, W), \forall \alpha$, define $\alpha F: V \to W$
$u \mapsto (\alpha F)(u) = \alpha F(u)$

It can be easily seen that $(L+F)$ and (αF) are linear maps.
- We've defined the zero map
- $\forall L$, define $(-L): V \to W$, $u \mapsto (-L)(u) := -1 L(u)$
- It can be easily seen that (V1) – (V8) are verified.

The Kernel and the image of a linear map.
Let $F: V \to W$ a linear map (V, W vector spaces)

- We define the image of F as the set
$$Im(F) = \{w \in W \text{ st. } \exists v \in V \text{ st. } Fv = w\}$$
$Im(F) \subseteq W$ is a vector subspace.
- We say that F is surjective if $Im(F) = W$
- We define the kernel of F as the set
$$Ker(F) = \{v \in V \text{ st. } F(v) = 0_W\}$$
$Ker(F) \subseteq V$ is a vector subspace.
- We say that F is injective if $\forall v_1, v_2 \in V$ st. $F(v_1) = F(v_2) \Rightarrow v_1 = v_2$.

Theorem: F is injective $\Leftrightarrow$ $Ker(F) = \{0\}$
Theorem: $\dim(V) = \dim(Ker(F)) + \dim(Im(F))$.

Exercise: Find the kernel and images of the following linear maps
1. $L: \mathbb{R}^3 \to \mathbb{R}$ st. $L(x,y,z) = 3x - 2y + z$
2. $P: \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $P(x,y,z) = (x,y)$ the projection map

MATRICES ASSOCIATED TO LINEAR MAPS
Before: Given $A \in \mathbb{R}^{m \times n} \rightsquigarrow L_A: \mathbb{R}^n \to \mathbb{R}^m$ st. $L_A(v) = Av, \forall v \in \mathbb{R}^n$
Now we will see that given $L: \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow$ there is some associated matrix $A \in \mathbb{R}^{m \times n}$ st. $L = L_A$.
Define $L: \mathbb{R}^n \to \mathbb{R}^m$ st. $L(e_j) = A_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$ j-th column of A $\forall j = 1, \dots, n$

Take $v = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$, $v = \sum_{j=1}^{n} \alpha_j v_j$ decomposition in the standard basis of $\mathbb{R}^n$.
$$\Rightarrow L(v) = L\left(\sum_{j=1}^{n} \alpha_j v_j\right) = \sum_{j=1}^{n} \alpha_j L(v_j) = \sum_{j=1}^{n} \alpha_j v_j = Av = L_A(v)$$
$\Rightarrow L = L_A$.
We can conclude the following important:
Theorem: There is a one to one correspondence between
$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \rightleftarrows \mathcal{M}_{m \times n}(\mathbb{R})$$

<u>Note</u>: The above procedure shows how a linear map is defined once we define it in the basis chosen!
<u>Note</u>: Different choices of basis lead to different matrices!

Example 1: let $F: \mathbb{R}^3 \to \mathbb{R}^2$ st. $F((x_1,x_2,x_3)^T) = (x_1,x_2)^T$ (the projection map.
Find the matrix associated to F.
Solution: $F(e_1) = F((1,0,0)^T) = (1,0)^T =: A_1$
$F(e_2) = F((0,1,0)^T) = (0,1)^T =: A_2$
$F(e_3) = F((0,0,1)^T) = (0,0) =: A_3$
$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Example 2: let $I: \mathbb{R}^n \to \mathbb{R}^n$ the identity map $\Rightarrow I = \begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & \cdots & \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$

Example 3: let $L: \mathbb{R}^4 \to \mathbb{R}^2$ be the map s.t.
$L(e_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $L(e_2) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $L(e_3) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$, $L(e_4) = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.
$\Rightarrow$ the matrix associated to L is $A = \begin{bmatrix} 2 & 3 & -5 & 1 \\ 1 & -1 & 4 & 7 \end{bmatrix}$.

CHANGE OF BASIS/CHANGE OF COORDINATES

V an arbitrary a vector space.
$\mathcal{B}=\{v_1,..,v_n\}$ and $\mathcal{B}'=\{w_1,..,w_n\}$ two basis for V.
On one hand, since $\mathcal{B}$ is a basis $\Rightarrow$ any $v\in V$ can be uniquely decomposed as

$$v=\alpha_1 v_1+..+\alpha_n v_n=\sum_{j=1}^{n}\alpha_j v_j \quad \alpha_j\in\mathbb{K},\ j=1..n$$

Notation:
We denote by $[v]_{\mathcal{B}}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}$ the coordinates of v in the basis $\mathcal{B}$, viewed as a column vector.

On the other hand, since $\mathcal{B}'$ is a basis $\Rightarrow$ v can also uniquely decomposed as

$$v=\beta_1 w_1+..+\beta_n w_n=\sum_{j=1}^{n}\beta_j v_j, \qquad \beta_j\in\mathbb{K}, j=1,..,n$$

Similarly, let $[v]_{\mathcal{B}'}=\begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}$.

AIM: For a generic $v\in V$, given its coordinates $[v]_{\mathcal{B}}$, we would like to be able to express $[v]_{\mathcal{B}'}$ and viceversa.

In particular,
$$\left.\begin{array}{l} w_1=\sum_{j=1}^{n} b_{1j}v_j\\ \vdots\\ w_n=\sum_{j=1}^{n} b_{nj}v_j\end{array}\right\}(*)$$

We collect in a matrix M the $n\times n=n^2$ coefficients b_{ij}, $i,j=1,..n$
Then $(*)\Leftrightarrow B=M_{\mathcal{B}'\to\mathcal{B}}B'$

The matrix M is called the change of coordinates matrix from $\mathcal{B}'$ to $\mathcal{B}$
(or change of basis / transition matrix from $\mathcal{B}'$ to $\mathcal{B}$)
Then the following formulas for the change of coordinates for v hold true:
(Note that it can be proved that M is invertible, i.e. $\exists\ M^{-1}_{\mathcal{B}\to\mathcal{B}'}$ s.t. $B'=M^{-1}B$)

$$[v]_{\mathcal{B}}=M_{\mathcal{B}'\to\mathcal{B}}[v]_{\mathcal{B}'}$$

$$[v]_{\mathcal{B}'}=M^{-1}_{\mathcal{B}\to\mathcal{B}'}[v]_{\mathcal{B}}$$

Example1 Let $\mathcal{B}=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ (standard basis in $\mathbb{R}^2$) and $\mathcal{B}'=\left\{\underbrace{\begin{bmatrix}3\\1\end{bmatrix}}_{w_1},\underbrace{\begin{bmatrix}-2\\1\end{bmatrix}}_{w_2}\right\}$ in $\mathbb{R}^2$

a) verify that $\mathcal{B}'$ forms a basis for $\mathbb{R}^2$.

Solution:

Since $\dim(\mathbb{R}^2)=2 \Rightarrow$ any two linearly independent vectors, also generate it, and hence form a basis.
So it is sufficient to prove that w_1 and w_2 are linearly independent.

Check: $\begin{vmatrix}3 & -2\\1 & 1\end{vmatrix}=3-(-2)=3+2=5\neq 0 \Rightarrow$ its column vectors (w_1 and w_2) are lin. independent.

b) Write the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$

$w_1=3\begin{bmatrix}1\\0\end{bmatrix}+1\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow [w_1]_{\mathcal{B}}=\begin{bmatrix}3\\1\end{bmatrix}$

$w_2=-2\begin{bmatrix}1\\0\end{bmatrix}+1\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow [w_2]_{\mathcal{B}}=\begin{bmatrix}-2\\1\end{bmatrix}$

$\Rightarrow M_{\mathcal{B}'\to\mathcal{B}}=\begin{bmatrix}3 & -2\\1 & 1\end{bmatrix}$

c) Given v s.t. $[v]_{\mathcal{B}'}=\begin{bmatrix}2\\1\end{bmatrix}$. Find $[v]_{\mathcal{B}}$

Solution:

Apply the formula $[v]_{\mathcal{B}}=M_{\mathcal{B}'\to\mathcal{B}}[v]_{\mathcal{B}'}$

$$\Rightarrow [v]_{\mathcal{B}}=M_{\mathcal{B}'\to\mathcal{B}}[v]_{\mathcal{B}'}=\begin{bmatrix}3 & -2\\1 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$$

d) Find $M^{-1}_{\mathcal{B}'\to\mathcal{B}}$ and verify $[v]_{\mathcal{B}'}=M^{-1}_{\mathcal{B}'\to\mathcal{B}}[v]_{\mathcal{B}}$

$M=\begin{bmatrix}3 & -2\\1 & 1\end{bmatrix}$. Apply adjoint cofactor method: $C=\begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$ will be the matrix of cofactors

$C^T=\begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}$ " " transpose

$M^{-1}=\dfrac{1}{\det(M)}\begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}=\dfrac{1}{5}\begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}$

CHECK: $M^{-1}_{\mathcal{B}'\to\mathcal{B}}[v]_{\mathcal{B}}=\begin{bmatrix}1/5 & 2/5\\-1/5 & 3/5\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}4/5+6/5\\-4/5+9/5\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ OK.

Example2 Apply same requirements for $\mathcal{B}=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ and $\mathcal{B}'=\left\{\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix}\right\}$ in $\mathbb{R}^2$

Example3 " " $\mathcal{B}'=\left\{\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix}\right\}$ and $\mathcal{B}''=\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}-1\\4\end{bmatrix}\right\}$ in $\mathbb{R}^2$

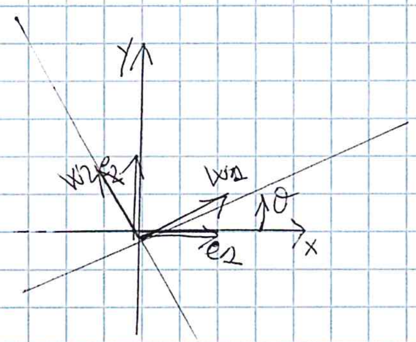
EXAMPLE 4: ROTATION OF THE COORDINATE AXES

Suppose we rotate counterclockwise by an angle θ the x- and y- axes of the standard coordinate system $\mathcal{B} = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$

Then, the new basis $\mathcal{B}' = \{w_1, w_2\}$ of unit vectors along the x'- and y'- axes, has coordinates in the original coordinate system

$$[w_1]_{\mathcal{B}} = \begin{bmatrix} \cos\theta \\ \operatorname{sen}\theta \end{bmatrix}$$

$$[w_2]_{\mathcal{B}} = \begin{bmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{bmatrix}$$



a) Write $M_{\mathcal{B}' \to \mathcal{B}}$

Solution: $M_{\mathcal{B}' \to \mathcal{B}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix}$

b) Prove that it's an orthogonal matrix.

Solution: $MM^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \overbrace{\cos^2\theta + \operatorname{sen}^2\theta}^{=1} & \overbrace{\cos\theta\operatorname{sen}\theta - \operatorname{sen}\theta\cos\theta}^{=0} \\ \underbrace{\operatorname{sen}\theta\cos\theta + \cos\theta(-\operatorname{sen}\theta)}_{=0} & \underbrace{\operatorname{sen}\theta\operatorname{sen}\theta + \cos^2\theta}_{1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Longrightarrow M^{-1} = M^T$$

c) Example: let v s.t. $[v]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. let $\theta = 45^\circ$

$$\Rightarrow [v]_{\mathcal{B}'} = M^{-1}[v]_{\mathcal{B}} = \begin{bmatrix} \cos 45^\circ & \operatorname{sen} 45^\circ \\ -\operatorname{sen} 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

SCALAR PRODUCT AND NORMS IN VECTOR SPACES

Def. Let V be a vector space over $\mathbb{R}$.
A scalar product on V is any map $(\cdot,\cdot): V \times V \to \mathbb{R}$ satisfying the following properties:

1. it is linear w.r.t. vectors on V: $(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 (v_1, w) + \alpha_2 (v_2, w)$ $\forall v_1, v_2 \in V$, $\forall \alpha_1, \alpha_2 \in \mathbb{R}$
2. it is symmetric: $(v_1, v_2) = (v_2, v_1)$ $\forall v_1, v_2 \in V$
3. it is positive definite: $(v,v) \geq 0$, $\forall v \in V$ and $(v,v) = 0 \iff v = 0$

Equivalent terminology: scalar / inner / dot product
Equivalent notation: $(\cdot,\cdot)$ or $\langle \cdot,\cdot \rangle$ or $\cdot$

BASIC TERMINOLOGY FOR THE CASE $\mathbb{K} = \mathbb{C}$

- $A \in \mathbb{C}^{m \times m} \Rightarrow B = A^H$ is called the conjugate transpose (or adjoint) of A if $b_{ij} = \overline{a_{ji}}$, where $\overline{a_{ji}}$ is the complex conjugate of a_{ji}.

Example: $A = \begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix} \Rightarrow A^H = \begin{bmatrix} 1 & 1+i \\ -2-i & i \\ 5 & 4-2i \end{bmatrix}$

- $A \in \mathbb{C}^{n \times n}$ is called hermitian if $A = A^H$
- $A \in \mathbb{C}^{n \times n}$ is called skew-hermitian or antihermitian if $A = -A^H$.
- $A \in \mathbb{C}^{n \times n}$ is called unitary if $AA^H = A^H A = I \iff A^{-1} = A^H$

so in the case V on $\mathbb{K} = \mathbb{C} \Rightarrow$ 2. it is hermitian: $(v_1, v_2) = \overline{(v_2, v_1)}$ $\forall v_1, v_2 \in V$.

Example: Classical Euclidean Scalar Product

$V = \mathbb{R}^n \Rightarrow (x,y) = y^T x = \sum_{i=1}^{n} x_i y_i$ $\quad x, y \in \mathbb{R}^n$, $x = \sum_{i=1}^{n} x_i e_i$, $y = \sum_{i=1}^{n} y_i e_i$

$V = \mathbb{C}^n \Rightarrow (x,y) = y^H x = \sum_{i=1}^{n} x_i \overline{y_i}$ $\quad x, y \in \mathbb{C}^n$, -//-

Property: $A \in \mathbb{R}^{n \times n} \Rightarrow (Ax, y) = (x, A^T y)$ $\forall x, y \in \mathbb{R}^n$ $(*)$
$A \in \mathbb{C}^{n \times n} \Rightarrow (Ax, y) = (x, A^H y)$ $\forall x, y \in \mathbb{C}^n$ $(**)$

Property: $(*)$ implies that orthogonal matrices preserve the scalar product, i.e.
$(Qx, Qy) = (x,y)$ $\forall x, y \in \mathbb{R}^n$, $\forall Q \in \mathbb{R}^{n \times n}$ orthogonal

Proof: $(Qx, Qy) = (x, Q^T Q y) = (x, Iy) = (x,y)$

Similarly, $(**)$ imply that unitary matrices preserve the scalar product.

Def. Let V be a vector space over $\mathbb{K}$.
A norm on V is any map: $\|\cdot\|: V \to \mathbb{K}$ satisfying the following axioms
1. (i) $\|v\| \geq 0$ $\forall v \in V$ and (ii) $\|v\| = 0 \iff v = 0$
2. $\|\alpha v\| = |\alpha| \|v\|$, $\forall \alpha \in \mathbb{K}$, $\forall v \in V$, where $|\alpha|$ is the absolute value in $\mathbb{R}$ and the module in $\mathbb{C}$.
   $\hookrightarrow$ (homogeneity property)
3. $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$, $\forall v_1, v_2 \in V$ (triangular inequality)

Definitions
1. The pair $(V, \|\cdot\|)$ is called a normed space
2. A map $|\cdot|: V \to \mathbb{K}$ satisfying 1(i), 2 and 3. is called a seminorm
3. If $v \in V$ is s.t. $\|v\| = 1 \implies v$ is called a unit vector.

Example of a normed space: $(\mathbb{R}^n, \|\cdot\|_p)$,
where, $\|\cdot\|_p$ is the p-norm or the Hölder norm, defined as
$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$
(the limit as $p \to \infty$ of $\|x\|_p$ exists and it is:)
$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

<u>Note:</u> for $p=2$, $\|x\|_2^2 = \left[\left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}\right]^2 = \sum_{i=1}^{n} |x_i|^2 = (x_i, x_i)$

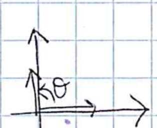
we find the standard definition of the Euclidean norm induced by the standard Euclidean scalar product $(x,x) =: \|x\|^{1/2}$.

Cauchy Schwarz inequality: $\forall x, y \in \mathbb{R}^n$, $|(x,y)| = |y^T x| \leq \|x\|_2 \|y\|_2$

and $|(x,y)| = \|x\|_2 \|y\|_2 \iff y = \alpha x$, for some $\alpha \in \mathbb{R}$

Hölder inequality: $\forall x, y \in \mathbb{R}^n$, $|(x,y)| \leq \|x\|_p \|y\|_q$, $\forall p, q$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

Def. $v_1, v_2 \in V$ are orthogonal, or perpendicular if $(v_1, v_2) = 0$.
Given $S \subset V$ subset, we define
$$S^\perp = \{w \in V \text{ st. } (w, v) = 0, \forall v \in S\} \subset V$$
$S^\perp$ is a vector subspace of V and it is called the orthogonal space of S.
If $V = \mathbb{R}^2$, geometric interpretation:
$(x,y) = \|x\|_2 \|y\|_2 \cos\theta$
$x \perp y \iff (x,y) = 0$
$\iff \cos\theta = 0$ (θ right angle)



Exercise: Prove the Pythagoras theorem:

If v_1, v_2 are orthogonal $\Rightarrow \|v_1+v_2\|_2^2 = \|v_1\|_2^2 + \|v_2\|_2^2$.

Proof: $\|v_1+v_2\|_2^2 \overset{\circ}{=} (v_1+v_2, v_1+v_2)$

$$= (v_1,v_1)+(v_1,v_2)+(v_2,v_1)+(v_2,v_2)$$
$$= \|v_1\|_2^2 + (v_1,v_2)+(v_1,v_2)+\|v_2\|_2^2$$
$$= \|v_1\|_2^2 + \underbrace{2(v_1,v_2)}_{=0 \text{ since } v_1 \perp v_2} + \|v_2\|_2^2$$
$$= \|v_1\|_2^2 + \|v_2\|_2^2 \qquad \text{OK}\checkmark$$

EXERCISE. Which of the following vectors form an orthonormal basis of $\mathbb{C}^3$?

$w=\left[\frac{i}{\sqrt3}, \frac{i}{\sqrt3}, \frac{i}{\sqrt3}\right]$, $x=\left[-\frac{2i}{\sqrt6}, \frac{i}{\sqrt6}, \frac{i}{\sqrt6}\right]$, $y=\left[\frac{i}{\sqrt6}, \frac{i}{\sqrt6}, -\frac{2i}{\sqrt6}\right]$ and $z=\left[0, \frac{-i}{\sqrt2}, \frac{i}{\sqrt2}\right]$

Answer w, x, y.

NORM EQUIVALENCE

Definition: Two norms $\|\cdot\|_p$ and $\|\cdot\|_q$ on V are equivalent if there exist two positive constants c and C s.t.

$$c\|x\|_q \le \|x\|_p \le C\|x\|_q, \ \forall x \in V$$

Theorem: If $\dim(V)=\infty \Rightarrow$ all norms are equivalent.

Example: If $V=\mathbb{R}^n$, all p norms are equivalent.

We show some value of the constants:

c	$q=1$	$q=2$	$q=\infty$
$p=1$	1	1	1
$p=2$	$n^{-1/2}$	1	1
$p=\infty$	n^{-1}	$n^{-1/2}$	1

C	$q=1$	$q=2$	$q=\infty$
$p=1$	1	$n^{1/2}$	n
$p=2$	1	1	$n^{1/2}$
$p=\infty$	1	1	1

Equivalence constants for the main norms in $\mathbb{R}^n$.

For example: $n^{-1/2}\|x\|_2 \le \|x\|_\infty \le \|x\|_2$.

Def. A matrix norm is a map $\|\cdot\|: \mathbb{R}^{m\times n} \to \mathbb{R}$ s.t.
1. $\|A\| \geq 0$, $\forall A \in \mathbb{R}^{m\times n}$ and $\|A\| = 0 \iff A = 0$
2. $\|\alpha A\| = |\alpha| \|A\|$, $\forall \alpha \in \mathbb{R}, \forall A \in \mathbb{R}^{m\times n}$ (homogeneity)
3. $\|A+B\| \leq \|A\| + \|B\|$, $\forall A, B \in \mathbb{R}^{m\times n}$ (triangular inequality)

We shall employ the same symbol to denote vector norms and matrix norms and we leave to the context the understanding.

Def. We say that a matrix norm is compatible or consistent with a vector norm if

$$\|Ax\| \leq \|A\| \|x\| \quad \forall x \in \mathbb{R}^n$$

Def. We say that a matrix norm is submultiplicative if

$$\|AB\| \leq \|A\| \|B\|, \quad \forall A \in \mathbb{R}^{m\times n}, \forall B \in \mathbb{R}^{n\times p}$$

Example of a non submultiplicative norm: $\|A\|_\Delta = \max_{i,j=1,\dots,n} |a_{ij}|$

If $A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow \|AB\|_\Delta = 2 \not\leq \|A\|_\Delta \|B\|_\Delta = 1\cdot 1 = 1$

EXAMPLES OF MATRIX NORMS

① The Frobenius norm (or Euclidean norm in $\mathbb{C}^{n^2}$)

$$\|A\|_F := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = tr(AA^H), \text{ where } tr(M) = \sum_{i=1}^{n} m_{ii}$$

Example:
$M = \begin{bmatrix} 3 & 12 \\ 9 & 74 \\ 4 & -11 \end{bmatrix}$
$\Rightarrow tr(M) = 3 + 7 - 11 = 1$

- $\|\cdot\|_F$ is compatible with the standard Euclidean norm $\|\cdot\|_2$ i.e.
$$\|Ax\|_2 \leq \|A\|_F \|x\|_2$$
- $\|I_n\|_F = \sqrt{\sum_{i=1}^{n} a_{ii}^2} = \sqrt{\sum_{i=1}^{n} 1^2} = \sqrt{1 + \dots + 1} = \sqrt{n}$

② The induced matrix norm from a vector norm

Given $\|\cdot\|$ a vector norm. Define

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\left\{= \sup_{\|x\|=1} \|Ax\|\right\}$$

In particular: $\|A\|_1 = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$ column sum norm

$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$ row sum norm

$\Rightarrow \|A\|_1 = \|A^T\|_\infty$

and if A is self adjoint (or real and symmetric), then $\|A\|_1 = \|A\|_\infty$.

Example: $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$. Compute $\|A\|_1$, $\|A\|_\infty$ and $\|A\|_F$

$\|A\|_1$: solve for column $j=1$: $\sum_{i=1}^{3}|a_{i1}|=|1|+|4|+|7|=12$

solve for column $j=2$: $\sum_{i=1}^{3}|a_{i2}|=|2|+|5|+|8|=15$

solve for column $j=3$: $\sum_{i=1}^{3}|a_{i3}|=|3|+|6|+|9|=18$

$\Longrightarrow \|A\|_1=\max\{12,15,18\}=18$

$\|A\|_\infty$: solve for row $i=1$: $\sum_{j=1}^{3}|a_{1j}|=|1|+|2|+|3|=6$

solve for row $i=2$: $\sum_{j=1}^{3}|a_{2j}|=|4|+|5|+|6|=15$

solve for row $i=3$: $\sum_{j=1}^{3}|a_{3j}|=|7|+|8|+|9|=24$

$\Longrightarrow \|A\|_\infty=\max\{6,15,24\}=24$

$\|A\|_F=\sqrt{\sum_{i,j=1}^{n}|a_{ij}|^2}=\sqrt{|a_{11}|^2+|a_{12}|^2+\ldots+|a_{33}|^2}=\ldots\sqrt{285}\approx 16,9.$

Def. Let $A\in\mathbb{C}^{n\times n}$. Then $\lambda\in\mathbb{C}$ is called an eigenvalue of A if there exists a nonnull vector $x\in\mathbb{C}^n$ s.t. $Ax=\lambda x$

Def. The spectrum of A is the set of all eigenvalues of A, denoted by $\sigma(A)$

Def. The spectral radius of A is the maximum module of the eigenvalues of A:

$$\rho(A)=\max_{\lambda\in\sigma(A)}\{|\lambda|\}$$

(P1) If A is hermitian, then $\|A\|_2=\rho(A)$

(P2) If A is unitary, then $\|A\|_2=1$

Proof: $\|A\|_2:=\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}=\sup_{x\neq 0}\frac{\|x\|_2}{\|x\|_2}=1$

$\|Ax\|_2=(Ax,Ax)=(x,A^HAx)=(x,Ix)=(x,x)=\|x\|_2$

Relation between norms and spectral radius of a matrix:

1. Let $\|\cdot\|$ be a consistent matrix norm $\Rightarrow$ $\rho(A)\leq\|A\|$, $\forall A\in\mathbb{C}^{n\times n}$
2. $\rho(A)=\inf_{\|\cdot\|}\|A\|$ (infimum on the set of all consistent norms)

Def. A matrix $A \in \mathbb{C}^{n\times n}$ is positive definite in $\mathbb{C}^n$ if the number (Ax, x) is real and positive $\forall x \in \mathbb{C}^n, x \neq 0$.
Def. It is positive definite is $\mathbb{R}^n$, if (Ax, x) is positive, $\forall x \in \mathbb{R}^n, x \neq 0$.
Note. A positive definite matrix in $\mathbb{R}^n$ is not necessarily symmetric.
Note A positive definite matrix in $\mathbb{R}^n$ is not necessarily pos. def. in $\mathbb{C}^n$.

Example: $A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ -3x_1 + 2x_2 \end{bmatrix}$

$$\begin{aligned} \Rightarrow (Ax, x) &= (2x_1 + x_2)x_1 + (-3x_1 + 2x_2)x_2 \\ &= 2x_1^2 + x_1x_2 - 3x_1x_2 + 2x_2^2 \\ &= 2(x_1^2 - x_1x_2 + x_2^2) \\ &> 0 \quad (\text{if } x_1, x_2 \neq 0) \text{ in } \mathbb{R}^n \end{aligned}$$

Still in $\mathbb{C}^n$ it can be complex valued $\Rightarrow$ not a real number
$\Rightarrow A$ not pos. def. in $\mathbb{C}^n$

Property $A \in \mathbb{C}^{n\times n}$ is positive definite iff it is hermitian and has positive eigenvalues.
Theorem: Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then A is positive definite iff one of the following properties is satisfied:
1. $(Ax, x) > 0$, $\forall x \neq 0$, $x \in \mathbb{R}^n$
2. The eigenvalues of the principal submatrices of A are all positive.
3. The dominant principal minors of A are all positive, i.e.
   $\det(A_k) > 0$,
   where A_k is obtained by deleting the $(n-k)$ rows and columns of A.
   (SYLVESTER CRITERION)
4. There exists a nonsingular matrix H st. $A = H^T H$

Observation: All the diagonal entries of a positive definite matrix are positive:

In fact,
$$\underbrace{e_i^T A e_i}_{= (Ae_i, e_i) = a_{ii}} > 0 \quad \forall i = 1, \ldots, n$$

Example. The matrix $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ is not positive definite, since $a_{11} = -1 < 0$.

Exercise: Is the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ positive definite?

Check: $|A_1| = 2 > 0$
$|A_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$
$A_3 = \det(A) = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - - \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0 = 6 - 2 = 4 > 0$

Since $A_1, A_2, A_3 > 0 \Rightarrow$ use criterion 3. to conclude that A is positive definite

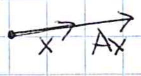
EIGENVALUES/EIGENVECTORS

Def. Let $A \in \mathbb{C}^{n\times n}$. Then, $\lambda \in \mathbb{C}$ is called an eigenvalue of A if there exists a nonzero vector $x \in \mathbb{C}^n$ s.t.

$$Ax = \lambda x$$

x is called the associated eigenvector.
The eigenvalue λ tells us whether x is stretched, shrunk, reversed or left unchanged when A is applied to it.

We may find $\lambda = 2 \Rightarrow Ax = 2x$: x is stretched



We may find $\lambda = \frac{1}{3} \Rightarrow Ax = \frac{1}{3}x$; x is shrunk

We may find $\lambda = -1 \Rightarrow Ax = -x$: x is reversed

Now we address the question "How to compute eigenvalues/eigenvectors."

① If (λ, x) is an eigenpair $\Rightarrow \lambda$ can be computed by the Rayleigh quotient:

$$\lambda = \frac{x^H A x}{x^H x} = \frac{(Ax,x)}{(x,x)} = \frac{(Ax,x)}{\|x\|}$$

Recall: $(x,y) = y^T x$.
in fact: $(x,y) = \sum_{i=1}^{n} x_i y_i$

while: $y^T x = [y_1, \dots, y_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 y_1 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

Remark:
If A is SPD $\Rightarrow$ The norm $\|x\|_A = (Ax,x) = x^T A x$ defines a vector norm, called the energy norm of x and the energy scalar product is given by:

$$(x,y)_A = (Ax,y) = y^T A x.$$

② λ is the solution of the "characteristic equation" $\det(A - \lambda I) = 0$

$$|A-\lambda I| = \left| \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & 0 & 0 \\ 0 & a_{22}-\lambda & \cdots & 0 & 0 \\ \vdots & & & & \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix} \right|$$

$$= (-1)^n \lambda^n + (\text{low order terms in } \lambda)$$

by the definition of the determinant it can be easily proved

$\Rightarrow \det(A - \lambda I)$ is a polynomial of degree n w.r.t. λ and we call it the characteristic polynomial and denote it by $p_A(\lambda)$.
$\Rightarrow$ Use "Fundamental Theorem of Algebra": "Every non zero, single variable, degree n polynomial with complex coefficients has exactly n complex roots, counted with multiplicity (Eg. $(x-1)^3 \Rightarrow x=1$ has algebraic mult $=3$)
$\Rightarrow$ There exist n complex eigenvalues of A not necessarily distinct

Remark ① λ can be complex even when A is real

Remark ② Eigenvectors can be scaled arbitrarily, i.e. if (λ, x) is an eigenpair, then also $(\lambda, \alpha x)$, $\alpha \neq 0$ is an eigenpair.

In fact, from (λ, x) eigenpair $\Rightarrow Ax = \lambda x$

Check: $A(\alpha x) = \alpha (Ax) = \alpha(\lambda x) = \lambda(\alpha x)$, which means $(\lambda, \alpha x)$ is an eigenpair, too.

PROPERTIES ① The set of eigenvalues of A is the same as the set of the eigenvalues of A^T i.e. $\sigma(A) = \sigma(A^T)$.

Let $\lambda \in \sigma(A^T) \Rightarrow \det(A^T - \lambda I) = 0$

But $\det(A^T - \lambda I) \underset{(*)}{=} \det((A-\lambda I)^T) \underset{(**)}{=} \det(A - \lambda I)$

$(*)$ $(A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$(**)$ $\det(M) = \det(M^T)$

So, $\det(A^T - \lambda I) = \det(A - \lambda I)$.

② $\det(A) = \prod_{j=1}^{n} \lambda_j$

Consequence: A is singular (not invertible) $\Leftrightarrow \det(A) = 0$

$\Leftrightarrow \prod_{j=1}^{n} \lambda_j = 0$

$\Leftrightarrow$ A has at least one eigenvalue $= 0$.

③ If $A \in \mathbb{R}^{n \times n} \Rightarrow p_A(\lambda)$ has real coefficients, so complex numbers should appear in complex conjugate pairs.

④ $tr(A) = \sum_{j=1}^{n} \lambda_j$

⑤ CAYLAY-HAMILTON THEOREM: $p_A(A) = 0$, where $p_A(A)$ denotes a matrix polynomial.

E.g. If $p_A(\lambda) = \lambda^2 - 2\lambda + 3 \Rightarrow$ it holds true that $A^2 - 2A + 3 = 0$.

Example: Computation of Eigenvalues/Eigenvectors

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Impose $p_A(\lambda) = \det(A - \lambda I) = 0$.

But $p_A(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \underbrace{\lambda^2 - 4\lambda + 3 = 0}_{(*)}$

$(*)$ factors to $(\lambda - 1)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 3$

Check: $tr(A) = 2 + 2 = 3 + 1$ OK ✓

$\det(A) = (4 - 1) = 3 = 3 \cdot 1$ OK ✓

We find the associated eigenvectors by solving separately $(A - \lambda I)x = 0$

For $\lambda_1 = 1$: $(A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 3$: $(A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

KERNEL / IMAGE / RANK OF A MATRIX

let $A \in \mathbb{R}^{m \times n}$.

$Ker(A) = \{v \in \mathbb{R}^n \text{ s.t. } Av = 0\} \subseteq \mathbb{R}^n$ vector subspace.

$Im(A) = range(A) = \{w \in \mathbb{R}^m \text{ s.t. } \exists v \in \mathbb{R}^n \text{ s.t. } Av = w\} \subseteq \mathbb{R}^m$ vector subspace.

Recall: We previously defined the rank of A as the maximum order of nonvanishing determinants extracted from A.

let's analyse this notion better.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \leftarrow r_1 \\ \vdots \\ \leftarrow r_m \end{matrix}$$

$\uparrow a_1 \qquad \uparrow a_n$

$a_j = \begin{bmatrix} a_{1j} \\ a_{mj} \end{bmatrix}$ the j-th column of A, $j = 1, \dots, n$

$r_i = [a_{i1}, \dots, a_{in}]$ the i-th row of A, $i = 1, \dots, m$

The columns of A generate a space $span\{a_1, \dots, a_n\} \subseteq \mathbb{R}^m$ vector subspace.

We define the column rank of A as $=: dim(span\{a_1, \dots, a_n\}) \leq n$

$\Rightarrow$ the column rank of A is the maximum number of linearly indep. columns of A.

Similarly, the rows of A generate a space $span\{r_1, \dots, r_m\}$.

We define the row rank of A as $= dim(span\{r_1, \dots, r_m\}) \leq m$

$\Rightarrow$ the row rank of A is the maximum number of linearly independent rows of A.

Theorem: column rank of A = Row rank of A

$\Rightarrow$ we can omit the word column or row and just call it the rank of A : rank(A).

Notice: $rank(A) \leq min(m, n)$

Def The matrix has full rank (or complete rank) if $rank(A) = min(m,n)$

Example:

let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ 2x3 matrix

$\Rightarrow rank(A) \leq min(2,3) = 2$

On the other hand, the two columns are linearly independent: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$:

$\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = -2 - 0 = -2 \neq 0.$

So $rank(A) = 2$

Recall the linear application associated with a matrix $A \in \mathbb{R}^{m \times n}$.

$L_A : \mathbb{R}^n \to \mathbb{R}^m, \ v \mapsto L_A(v) = Av$

$Im(L_A) = span\{a_1, \dots, a_n\}$

$\Rightarrow rank(A) = dim(Im(L_A))$.

Recall the formula for $L : V \to W$ linear map (V, W vector spaces):

$$dim(V) = dim(Ker(L)) + dim(Im(L))$$

For $L = L_A \Rightarrow \quad dim(\mathbb{R}^n) = dim(Ker(L_A)) + dim(Im(L_A))$

$\iff \quad n = dim(Ker(L_A)) + rank(A)$

If in particular A has full rank: $\quad n = 0 + n \quad \Rightarrow dim(Ker(L_A)) = 0$

$\iff \quad Ker(L_A) = \{0\}$

$\qquad \qquad \parallel$

$\qquad Ker(A)$

We're able to write the following:

Theorem

The following are equivalent: $A \in \mathbb{R}^{n \times n}$ is nonsingular (invertible) $\iff \det(A) \neq 0$

$\iff$ A has n linearly independent columns

$\iff \text{rank}(A) = n$

$\iff \dim(\text{Ker}(A)) = 0$

$\iff \text{Ker}(A) = \{0\}$

As a consequence, the equation $Ax = 0$ can only mean two things:

① A is nonsingular: $x = 0$ (since $\text{Ker}(A) = \{0\}$)

② A is singular and $x \in \text{Ker}(A)$.

Let's go back to eigenvalues/eigenvectors):

Def. The eigenspace associated with λ as the set of the eigenvectors associated with λ $\cup \{0\}$:

$$E(\lambda) := \text{Ker}(A - \lambda I).$$

In particular $\underbrace{\dim(\text{Ker}(A - \lambda I))}_{\text{geometric multiplicity of } \lambda} = n - \text{rank}(A - \lambda I)$

$\text{geom.mult.}(\lambda) \leq \text{alg.mult.}(\lambda)$

↳ multiplicity of λ as a root of the characteristic polynomial

Def. A and B are called similar if $B = C^{-1}AC$, C invertible.

Property: • $\sigma(A) = \sigma(C^{-1}AC)$

• $f(A) = f(C^{-1}AC)$

Notation:

• triangular matrix (upper) $\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & & & \\ \vdots & & \ddots & & \\ 0 & \cdots & & 0 & * \end{bmatrix}$ if strict: $\begin{bmatrix} 0 & * & * & * \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

• triangular matrix (lower) $\begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & & & \vdots \\ \vdots & & \ddots & & \\ * & \cdots & & & * \end{bmatrix}$ if strict: $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & & \vdots \\ * & * & \cdots & 0 \end{bmatrix}$

• $A(i_1 : i_2, j_1 : j_2)$

E.g. $A \in \mathbb{R}^{n \times n} \Rightarrow A(2:4, 3:6) \Rightarrow$
$n = 7$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} \end{bmatrix}$$

$A(2:4, 3:6)$