Implementation of a Fluid-Solid interaction solver

Numerical methods for partial differential equations

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Strong Formulation

The coupled FSI problem is defined on $\Omega = \Omega_f \cup \Omega_s$:

Governing equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega_f, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_f, \\ -\nabla \cdot \sigma(\mathbf{d}) = \mathbf{0} & \text{in } \Omega_s, \end{cases}$$

with
$\sigma(\mathbf{d}) = \mu(\nabla\mathbf{d} + (\nabla\mathbf{d})^T) + \lambda(\nabla \cdot \mathbf{d})\mathbf{I}$.

Boundary & interface conditions

$$\begin{cases} \mathbf{u} = \boldsymbol{\varphi} & \text{on } \partial\Omega_{f_1}, \\ \nu\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_{f_2}, \\ \mathbf{u} = \mathbf{0}, \quad \mathbf{d} = \mathbf{0} & \text{on } \Sigma, \partial\Omega_s \setminus \Sigma, \\ \sigma(\mathbf{d})\mathbf{n} = \nu\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} & \text{on } \Sigma. \end{cases}$$

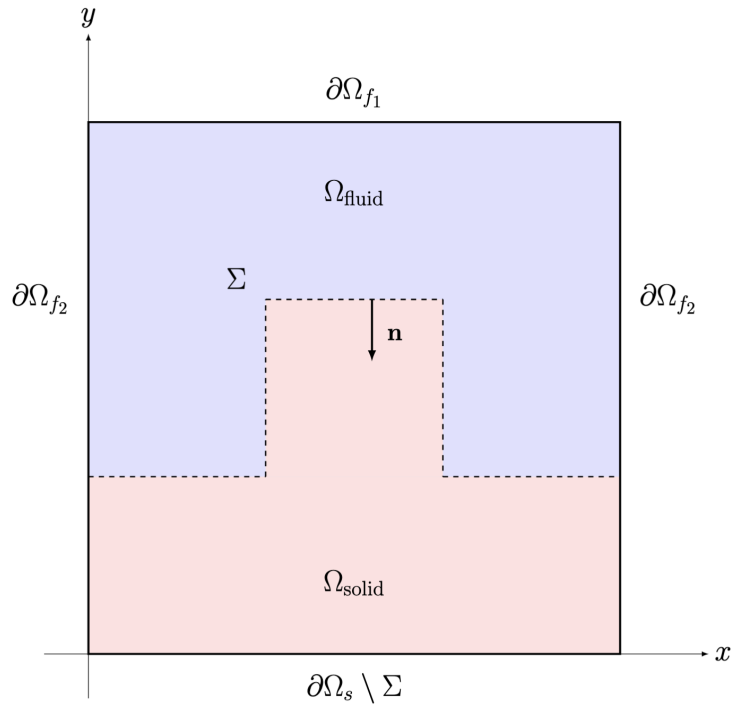


Figure 1: Domain Ω and boundary partition.

Weak Formulation

$$V_0 = \{\mathbf{v} \in [H^1(\Omega_f)]^2 : \mathbf{v}|_\Sigma = \mathbf{0},\ \mathbf{v}|_{\partial\Omega_{f_1}} = \mathbf{0}\}$$

$$\mathbb{Q} = L^2(\Omega_f)$$

$$\mathbb{B} = [H^1_{\Gamma_s}(\Omega_s)]^2$$

$$V = \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathbf{v}|_{\Omega_f} \in V_0,\ \mathbf{v}|_{\Omega_s} = \mathbf{0}\}$$

$$Q = \{q \in L^2(\Omega) : q|_{\Omega_f} \in \mathbb{Q},\ q|_{\Omega_s} = 0\}$$

$$B = \{\mathbf{w} \in [L^2(\Omega)]^2 : \mathbf{w}|_{\Omega_s} \in \mathbb{B},\ \mathbf{w}|_{\Omega_f} = \mathbf{0}\}$$

Find $\mathbf{u}_0 \in V$, $p \in Q$, $\mathbf{d} \in B$ such that:

$$a(\mathbf{u}_0, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}_0, q) + c(\mathbf{d}, \mathbf{b}) + m(p, \mathbf{b}) + n(\mathbf{u}_0, \mathbf{b}) = F(\mathbf{v}) + G(q)$$

for all $\{\mathbf{v}, q, \mathbf{b}\} \in Y$, where the bilinear forms are:

Weak Formulation

Fluid (Stokes)

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega_f} \nu \, \nabla \mathbf{u} : \nabla \mathbf{v} \; d\mathbf{x},$$

$$b(\mathbf{v}, p) = -\int_{\Omega_f} p \, (\nabla \cdot \mathbf{v}) \; d\mathbf{x}.$$

Lifting functionals

$$F(\mathbf{v}) = -a(R_\varphi, \mathbf{v}),$$
$$G(q) = -b(R_\varphi, q).$$

Solid (Elasticity)

$$c(\mathbf{d}, \mathbf{b}) = \int_{\Omega_s} \left[\mu(\nabla \mathbf{d} + (\nabla \mathbf{d})^T) : \nabla \mathbf{b} + \lambda(\nabla \cdot \mathbf{d})(\nabla \cdot \mathbf{b}) \right] d\mathbf{x},$$

Interface coupling

$$m(p, \mathbf{b}) = -\int_\Sigma p \, \mathbf{n} \cdot \mathbf{b} \; d\Gamma,$$

$$n(\mathbf{u}, \mathbf{b}) = \int_\Sigma \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{b} \; d\Gamma.$$

Discretization

We introduce a triangulation $\mathcal{T}_h$ of Ω and finite-dimensional subspaces $V_h \subset V_0$, $Q_h \subset Q$, $B_h \subset B$. Using **Taylor-Hood** elements to satisfy the LBB (inf-sup) condition:

Finite element spaces

V_h : vector velocity, Q^d_{p+1} on Ω_f,

Q_h : scalar pressure, Q_p on Ω_f,

B_h : vector displacement, Q^d_r on Ω_s,

with fields extended by zero outside their subdomain. Here Q^d_r denotes d-dimensional polynomials of degree r on each element $K \in \mathcal{T}_h$.

Discrete expansions

$$\mathbf{u}_h = \sum_{i=1}^{N_h^u} u_i \, \boldsymbol{\varphi}_i, \quad p_h = \sum_{i=1}^{N_h^p} p_i \, \varepsilon_i$$

$$\mathbf{d}_h = \sum_{i=1}^{N_h^d} d_i \, \boldsymbol{\psi}_i,$$

where $\{\boldsymbol{\varphi}_i\}$, $\{\varepsilon_i\}$, $\{\boldsymbol{\psi}_i\}$ are the vector/scalar Lagrangian bases for velocity, pressure and displacement respectively.

Algebraic system

Assembling the bilinear forms yields:

$$\underbrace{\begin{bmatrix} A & B^T & 0 \\ B & 0 & 0 \\ N & M & C \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \\ \mathbf{0} \end{bmatrix},$$

where the matrix blocks are:

$$A_{ij} = a(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i), \qquad B_{ij} = b(\boldsymbol{\varphi}_j, \varepsilon_i),$$
$$C_{ij} = c(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i), \qquad M_{ij} = m(\varepsilon_j, \boldsymbol{\psi}_i),$$
$$N_{ij} = n(\boldsymbol{\varphi}_j, \boldsymbol{\psi}_i).$$

$\mathbf{U}, \mathbf{P}, \mathbf{D}$ collect the DoF coefficients for velocity, pressure and displacement; $\mathbf{F}, \mathbf{G}$ arise from the lifting R_φ of the inlet data $\boldsymbol{\varphi}$.

Preconditioner

Reordering DoFs as $(\mathbf{U}, \mathbf{P}, \mathbf{D})$, the global matrix has a **block-triangular** structure:

$$A = \begin{pmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & 0 & 0 \\ R_{du} & R_{dp} & A_{dd} \end{pmatrix} = \begin{pmatrix} A_{\text{fluid}} & 0 \\ R & A_{\text{solid}} \end{pmatrix},$$

Outer (global) preconditioner
Exploiting the block-triangular form, the exact inverse satisfies:

$$\widetilde{A^{-1}} = \begin{pmatrix} \widetilde{A_{\text{fluid}}^{-1}} & 0 \\ -\widetilde{A_{\text{solid}}^{-1}}\, R\, \widetilde{A_{\text{fluid}}^{-1}} & \widetilde{A_{\text{solid}}^{-1}} \end{pmatrix},$$

requiring one approximate solve each with A_{fluid} and A_{solid}.

Inner (Stokes) preconditioner
The fluid block is approximated via:

$$P = \begin{bmatrix} A_{uu} & 0 \\ A_{pu} & \frac{1}{\nu} M_p \end{bmatrix},$$

where ν is the fluid viscosity and $(M_p)_{ij} = \int_{\Omega_f} \varepsilon_i \varepsilon_j \, d\mathbf{x}$ is the **pressure mass matrix**.

$$A_{uu}\,\tilde{\mathbf{u}} = \mathbf{x}_u \;\longrightarrow\; \tilde{\mathbf{p}} = \nu M_p^{-1}(\mathbf{x}_p - A_{pu}\tilde{\mathbf{u}}) \;\longrightarrow\; A_{dd}\,\tilde{\mathbf{d}} = \mathbf{x}_d - R_{du}\tilde{\mathbf{u}} - R_{dp}\tilde{\mathbf{p}}.$$

Adaptive Mesh Refinement

After each solve, the mesh is locally refined to concentrate resolution where it is most needed.

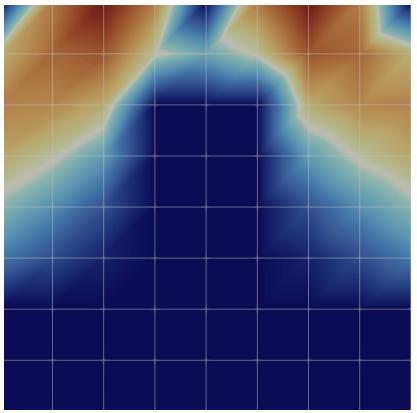
Kelly error estimator

For each cell $K \in \mathcal{T}_h$, the local indicator is:

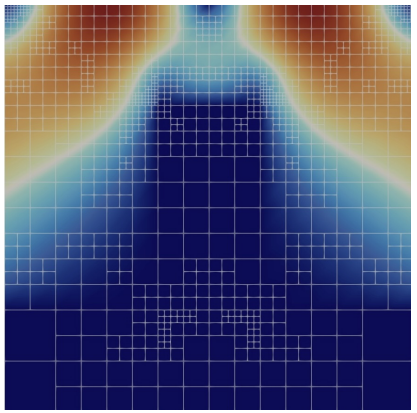
$$\eta_K^2 = \sum_{F \in \partial K} c_F \int_{\partial K_F} \left[\left[a \frac{\partial u_h}{\partial n} \right]\right]^2 d\Gamma,$$

where $[\![\cdot]\!]$ denotes the jump across face F, c_F is a face-geometry factor and a a coefficient depending on the specific pde. For each subdomain, the global estimate is:

$$\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2.$$



Initial mesh



Final refined mesh

Tests

Two test cases were run, sharing the following parameters:

Parameter	Symbol	Value
Kinematic viscosity	ν	2.0
Lamé parameter (shear)	μ	$1.0\,\mathrm{Pa}$
Lamé parameter (bulk)	λ	$10.0\,\mathrm{Pa}$
Kelly estimator weights	$w_{\mathrm{Stokes}}/w_{\mathrm{elas}}$	$4.0\,/\,1.0$
Refinement target		top 30% of cells
Refinement cycles		5 (2D) - 3 (3D)

Test 1 — 2D

Inlet velocity on $\partial\Omega_{f_1}$:

$$v_{\mathrm{in}} = \sin(\pi x)$$

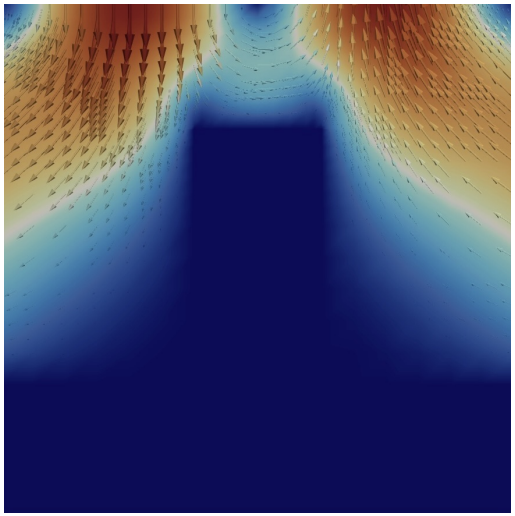
Test 2 — 3D

Inlet velocity on $\partial\Omega_{f_1}$:

$$v_{\mathrm{in}} = \sin(\pi x)\sin(\pi y)$$

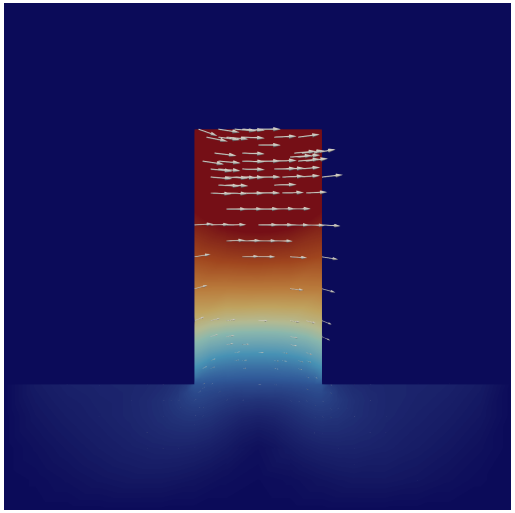
In both cases, zero-displacement conditions are imposed on $\partial\Omega_s \setminus \Sigma$, and Gauss quadrature of order $p+2$ is used.

Simulation Results, 2D

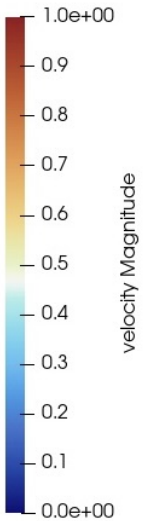


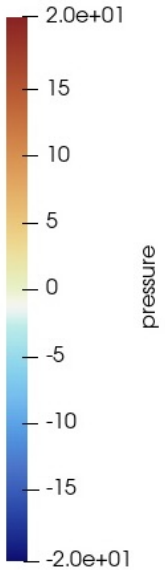
Velocity vectors

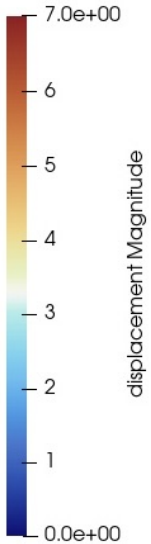
Pressure field



Displacement field







Simulation results, 3D

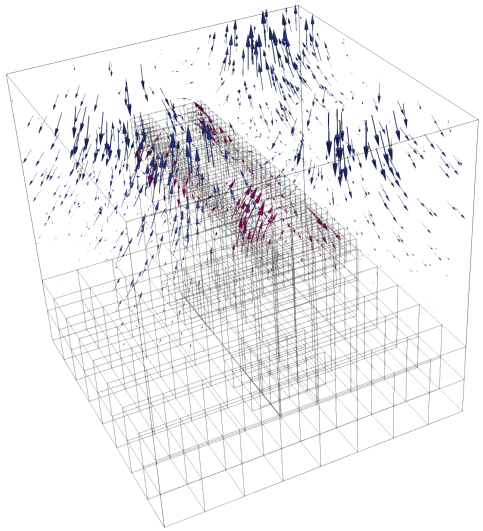
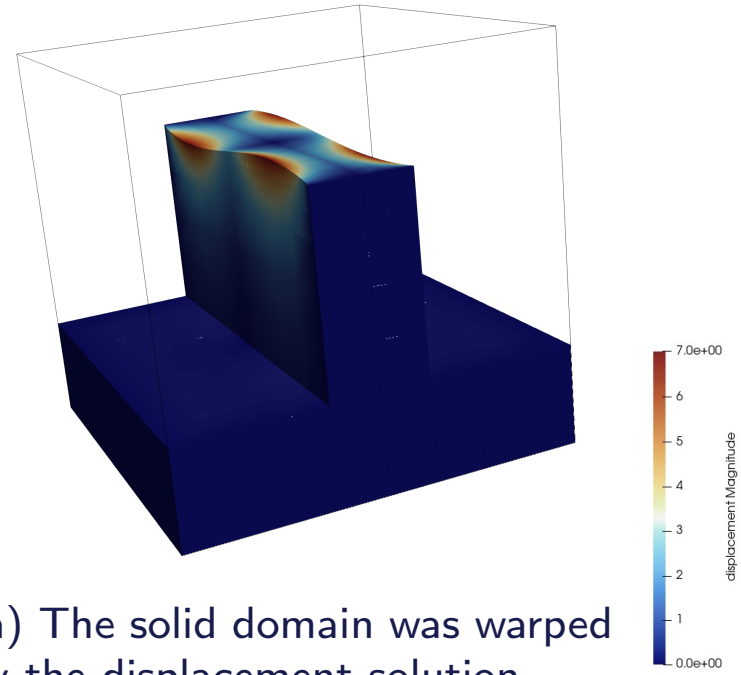


Figure 2: Vector plot for velocity (in blue) and for solid displacement (in red).



(a) The solid domain was warped by the displacement solution

Strong Scalability

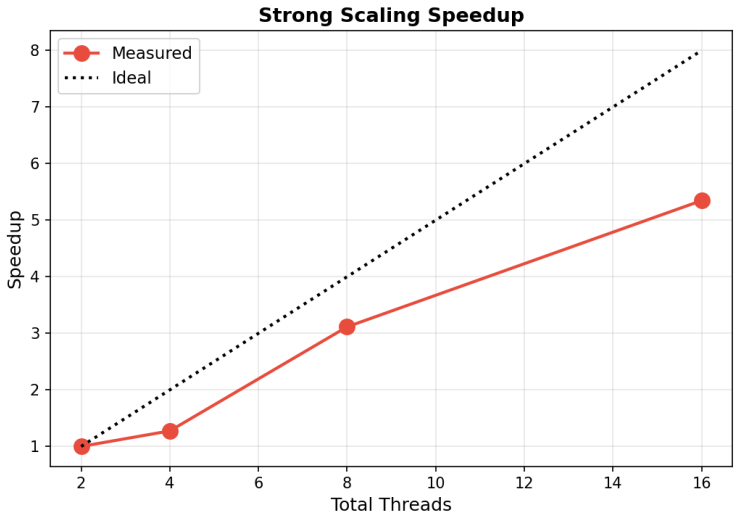


Figure 4: Speedup on the unrefined mesh: nearly linear trend confirms good parallel efficiency.

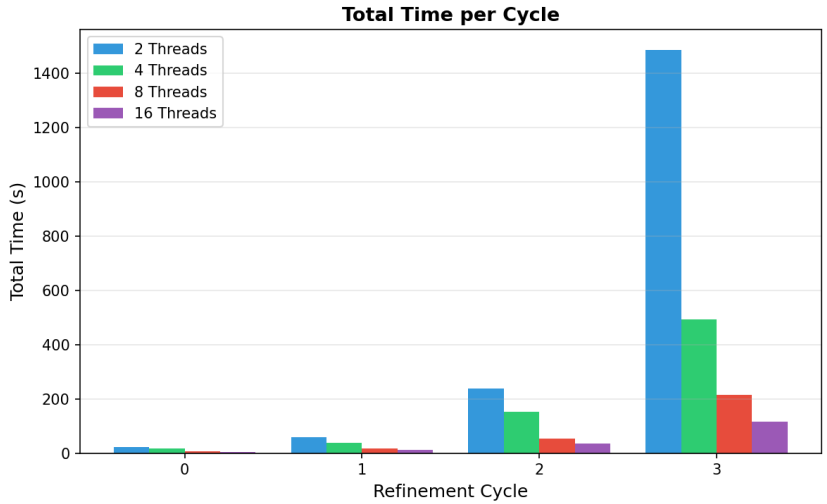


Figure 5: Total time per cycle: doubling resources approximately halves execution time within each cycle.

Within the same refinement cycle, a $2\times$ increase in cores yields a $\approx 2\times$ reduction in wall time.

Weak Scalability & Preconditioner Effectiveness

Weak Scalability
Problem size grows proportionally with the thread count.

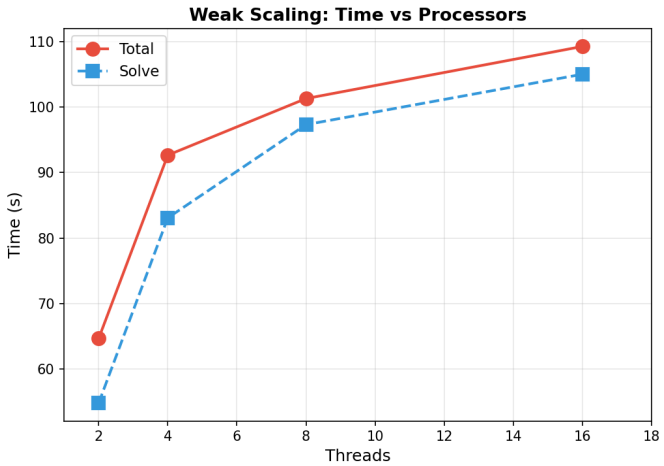


Figure 6: Execution time vs. threads. Sub-linear growth that flattens confirms solid weak scalability.

Preconditioner vs. Direct Solver
GMRES + block preconditioner compared against `SparseDirectUMFPACK`.

Cycle	Iters	Prec [s]	UMFPACK [s]
0	48	20.9	4.1
1	51	46.7	19.7
2	51	196.3	96.7
3	73	853.1	**1198.2**

- GMRES iteration count stays stable, proving the preconditioner controls the condition number effectively.