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Some recent results on the norm of localization operators



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Acknowledgements

Summary

Chapter 1

Introduction

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Chapter 2

Preliminaries

In this first chapter we briefly recall some basic definition and results about functional analysis and Fourier transform. In section 2.1 basic concepts about operators between Banach spaces are presented. In section ?? Fourier transform is defined and essential properties are given.

2.1 Basics of Functional Analysis

In this section we focus our attention on linear operators between Banach spaces. Across the section a generic Banach space will be denoted as X (or Y) endowed with the norm $\|\cdot\|_X$. In case we will deal with an Hilbert space we will denote it with H (or K) and its inner product as $\langle \cdot, \cdot \rangle_H$. **Pedex in the norm and in the scalar product may be dropped in case there is no ambiguity. Moreover, all the theory will be presented assuming we are dealing with infinite dimensional spaces but, if not otherwise stated, everything can be adapted almost directly for finite dimensional spaces.**

A generic linear operator between two Banach spaces X and Y will be denoted as $T : X \rightarrow Y$. As a standard notation, the image of $x \in X$ through T will be indicated as $T(x)$, or equivalently as Tx .

Definition 2.1. A linear operator $T : X \rightarrow Y$ is **bounded** if there exist $C > 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \quad \forall x \in X \quad (2.1)$$

For linear operator boundedness is strictly related to continuity as the subsequent theorem states.

Theorem 2.2. For a linear operator T the following statements are equivalent:

- T is continuous
- T is bounded.

We denote the set of linear bounded (continuous) operators from X to Y as $\mathcal{B}(X, Y)$, while if $X = Y$ we will just write $\mathcal{B}(X)$.

For the sake of completeness we mention that actually, for linear operators, boundedness is equivalent to uniform continuity.

After this we define the *norm* of an operator

Definition 2.3. *Given a linear bounded operator T we define its **norm** as the following number:*

$$\|T\| := \inf\{C > 0 : \|Tx\|_Y \leq C\|x\|_X \ \forall x \in X\} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X \setminus \{0\}\right\}$$

The proof of the equivalence between two definition is straightforward. We see that the norm of an operator is the best constant for which boundedness property (2.1) holds. Sometimes, in order to emphasize the spaces between which T operates, we may write the norm of T as $\|T\|_{X \rightarrow Y}$.

In the following we will mostly deal with X and Y being $L^2(\mathbb{R}^d)$, which is an Hilbert space. For operators between Hilbert spaces we can give the norm of an operator by means of the dual norm:

$$\|T\|_H = \sup\{|\langle Tx, y \rangle_H| : x, y \in H, \ \|x\|_H = \|y\|_H = 1\}$$

An important class of operators is the class of *compact operators*.

Definition 2.4. *A linear bounded operator T is **compact** if for every bounded sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ the sequence of the images $\{Tx_n\}_{n \in \mathbb{N}} \subset Y$ has a converging subsequence.*

The property of compactness can be stated in multiple ways **SERVE SCRIVERLE?**.

Now we suppose H and K to be Hilbert spaces. Given $T \in \mathcal{B}(H, K)$ there exist a unique $T^* \in \mathcal{B}(K, H)$ such that:

$$\langle Tx, y \rangle_H = \langle x, T^*y \rangle_K \quad \forall x \in H, y \in K$$

T^* is called the **adjoint** operator of T . In the particular case in which $T : H \rightarrow H$, if $T = T^*$, we say that T is **self-adjoint**.

From now on we suppose that X is over the field \mathbb{C} and that $T \in \mathcal{B}(X)$.

Definition 2.5. *The set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$ is called the **spectrum** of T .*

For operators between finite-dimensional spaces (matrices) the spectrum is made up of *eigenvalues*, those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. On the other hand, when dealing with infinite-dimensional spaces, this is no more true. Eigenvalues are in the so called *point spectrum*, which in general is just a part of the whole spectrum.

If an operator is compact or self-adjoint its spectrum has some additional properties.

Theorem 2.6 (Fredholm's alternative). *Let $T \in \mathcal{B}(X)$ be a compact operator. Then for $T - I$ one and only one of the following happens:*

- T is invertible
- T is not injective

Therefore for compact operators, all the values in the spectrum, except at most for 0, are eigenvalues.

Another fundamental result arises if we study the spectrum of compact and self-adjoint operators.

Theorem 2.7. *Let H be a separable Hilbert space and $T \in \mathcal{B}(H)$ a compact and self-adjoint operator. Then there exist an orthonormal basis $\{\psi_n\}_{n \in \mathbb{N}}$ with eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ of H composed of eigenvectors of T . Moreover $\lim_{n \rightarrow +\infty} \lambda_n = 0$.*

Hence self-adjoint compact operators can always be diagonalized in some suitable basis ([1]).

From this Theorem immediately follows the next corollary, which relates the eigenvalues of a compact self-adjoint operator with its norm.

Corollary 2.8. *Let $T \in \mathcal{B}(H)$ be a self-adjoint compact operator on a separable Hilbert space H and suppose its eigenvalues are ordered in such a way that $|\lambda_1| \geq |\lambda_2| \geq \dots$. Then $\|T\| = |\lambda_1|$.*

If an operator T is compact but not self-adjoint we can easily construct an operator, related to T , that is compact and also self-adjoint, namely considering T^*T .

Corollary 2.9. *Let $T \in \mathcal{B}(H)$ be a compact operator. There there exist orthonormal sets $\{\psi_n\}_{n \in \mathbb{N}}$ and $\{\phi_n\}_{n \in \mathbb{N}}$ and non-negative real numbers $\{\mu_n\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow +\infty} \mu_n = 0$ so that*

$$T = \sum_{n=1}^{+\infty} \mu_n (\cdot, \psi_n) \phi_n \quad (2.2)$$

where the series converges in norm. These μ_n are called singular values of T and are the square root of the eigenvalues of T^*T or, equivalently, the eigenvalues of $|T|$.

2.1.1 Trace class and Hilbert-Schmidt operators

Now we are going to consider two important classes of operators: *trace class* operators and *Hilbert-Schmidt* operators.

The trace of an operator can be defined as it is for matrices.

Definition 2.10. *Let H be an Hilbert space. An operator $T \in \mathcal{B}(H)$ is said **non-negative** if*

$$\langle Tx, x \rangle \geq 0 \quad \forall x \in H \quad (2.3)$$

Condition (2.3) is sufficient to show that, on Hilbert spaces over the field of complex numbers, non-negative operators are automatically self-adjoint. This result is an immediate corollary of the following form of the polarization identity.

Proposition 2.11. *Let $\mathcal{S} : H \times H \rightarrow \mathbb{C}$ be a sesquilinear form over the complex Hilbert space H . Then, for every $x, y \in H$:*

$$\mathcal{S}(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k \mathcal{S}(x + i^k y, x + i^k y) \quad (2.4)$$

Proof. The proof follows from a direct computation of the right-hand side

$$\begin{aligned} \sum_{k=0}^3 i^k \mathcal{S}(x + i^k y, x + i^k y) &= \mathcal{S}(x, x) \sum_{k=0}^3 i^k + \mathcal{S}(x, y) \sum_{k=0}^3 i^k i^{-k} + \mathcal{S}(y, x) \sum_{k=0}^3 i^k i^k + \\ &+ \mathcal{S}(y, y) \sum_{k=0}^3 i^k i^k i^{-k} = 4\mathcal{S}(x, y) \implies \mathcal{S}(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k \mathcal{S}(x + i^k y, x + i^k y). \end{aligned}$$

□

Proposition 2.12. *Let $T \in \mathcal{B}(H)$ be a non-negative operator over a complex Hilbert space H . Then T is self-adjoint.*

Proof. First of all we notice that if T is non-negative, the quantity $\langle Tx, x \rangle$ is real, therefore $\langle Tx, x \rangle = \overline{\langle Tx, x \rangle} = \langle x, Tx \rangle$. Letting $\mathcal{T}(\cdot, \cdot) = \langle T\cdot, \cdot \rangle$, and using the polarization identity (2.4):

$$\begin{aligned} \overline{\mathcal{T}(y, x)} &= \frac{1}{4} \sum_{k=0}^3 \overline{i^k \mathcal{T}(y + i^k x, y + i^k x)} \stackrel{T \text{ positive}}{=} \frac{1}{4} \sum_{k=0}^3 i^{-k} \mathcal{T}(y + i^k x, y + i^k x) = \\ &= \frac{1}{4} \sum_{k=0}^3 i^{-k} \mathcal{T}(i^k(x + i^{-k}y), i^k(x + i^{-k}y)) = \end{aligned} \tag{2.5}$$

$$= \frac{1}{4} \sum_{k=0}^3 i^{-k} \mathcal{T}(x + i^{-k}, x + i^{-k}y) = \mathcal{T}(x, y) \tag{2.6}$$

□

Non-negative operators are somewhat “special”, since their behaviour resembles, in some sense, the one of complex numbers. In particular, one can define the *square root* of non-negative operator. Before doing so, we need the following Lemma.

Lemma 2.13. *The power series of $\sqrt{1-z}$ about zero converges absolutely for all complex numbers such that $|z| \leq 1$.*

Proof. The power series of $\sqrt{1-z}$ about the origin is given by

$$\sqrt{1-z} = \sum_{n=0}^{+\infty} c_n z^n = 1 + \sum_{n=1}^{+\infty} (-1)^n \binom{1/2}{n} z^n$$

where the binomial $\binom{r}{n}$ is defined by $\frac{r(r-1)\cdots(r-n+1)}{n!}$ for every $r \in \mathbb{R}$ and every $n \in \mathbb{N}$, while if $n = 0$ we have $\binom{r}{0} = 1$. Since $\sqrt{1-z}$ is analytic for $|z| < 1$, the series here converges absolutely, so now we have to consider the case $|z| = 1$. We notice that $c_n < 0$ for every $n \geq 1$ because when n is even $\binom{1/2}{n}$ is negative, while if n is odd $\binom{1/2}{n}$ is positive, therefore

$$1 - \sqrt{1-x} = \sum_{n=1}^{+\infty} (-c_n) x^n$$

is a positive series. Using this fact, given $N \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{n=0}^N |c_n| &= 1 + \sum_{n=1}^N (-c_n) = 1 + \lim_{x \rightarrow 1^-} \sum_{n=1}^N (-c_n)x^n \stackrel{\text{positive series}}{\leq} \\ &\leq 1 + \lim_{x \rightarrow 1^-} \sum_{n=1}^{+\infty} (-c_n)x^n = 1 + \lim_{x \rightarrow 1^-} (1 - \sqrt{1-x}) = 2. \end{aligned}$$

Since this holds for every N , taking the limit $N \rightarrow +\infty$ we obtain $\sum_{n=1}^{+\infty} |c_n| < \infty$, which means exactly that the power series is absolutely convergent. \square

Theorem 2.14. *Let $T \in \mathcal{B}(X)$ be a non-negative operator. Then, there exist a unique non-negative operator $S \in \mathcal{B}(X)$ such that $S^2 = T$. Moreover S commutes with all bounded operators commuting with T .*

*We call S the **square root** of T and we denote it by $S = \sqrt{T}$.*

Proof. If $T = 0$ we let $\sqrt{T} = 0$, otherwise we define $B = I - \|T\|^{-1}T$, where I is the identity operator. Since T is non-negative, for every $x \in X$ such that $\|x\|_X = 1$, we have

$$\langle Bx, x \rangle = \langle (I - \|T\|^{-1}T)x, x \rangle = \|x\|_X^2 - \|T\|^{-1} \langle Tx, x \rangle \leq \|x\|_X^2 = 1$$

which implies, using **polarization identity**, that $\|B\| \leq 1$. Thanks to Lemma 2.13, this means that the series $\sum_{n=0}^{+\infty} c_n B^n$ is absolutely convergent, therefore convergent, in $\mathcal{B}(X)$ to an operator we indicate with $B_{1/2}$. We define

$$S = \|T\|^{1/2} B_{1/2} \tag{2.7}$$

and we want to show that S is non-negative and satisfies $S^2 = T$. We start proving that $B_{1/2}$, hence S , is non-negative. Taking $x \in X$, we have

$$\begin{aligned} \langle B_{1/2}x, x \rangle &= \langle (I + \sum_{n=1}^{+\infty} c_n B^n)x, x \rangle = \|x\|_X^2 + \sum_{n=1}^{+\infty} c_n \langle B^n x, x \rangle \geq \|x\|_X^2 + \sum_{n=1}^{+\infty} c_n \|B\|^n \|x\|_X^2 \geq \\ &\geq \|x\|_X^2 + \sum_{n=1}^{+\infty} c_n \|x\|_X^2 = 0 \end{aligned}$$

where we used the fact that c_n are negative, $\|B\| \leq 1$ and that $1 + \sum_{n=1}^{+\infty} c_n = \sqrt{1-x}|_{x=1} = 0$.

Now we shall prove that $S^2 = T$, which means $\|T\|(B_{1/2})^2 = T$:

$$(B_{1/2})^2 = \left(\sum_{n=0}^{+\infty} c_n B^n \right) \left(\sum_{m=0}^{+\infty} c_m B^m \right) = \sum_{n=0}^{+\infty} \left(\sum_{m=0}^n c_m c_{n-m} \right) B^n = \sum_{n=0}^{+\infty} d_n B^n$$

where the rearrangement is justified since all series are absolutely converging. In order to compute $d_n = \sum_{m=0}^n c_m c_{n-m}$, we notice the following:

$$1 - x = \sqrt{1-x} \sqrt{1-x} = \left(\sum_{n=0}^{+\infty} c_n x^n \right) \left(\sum_{m=0}^{+\infty} c_m x^m \right) = \sum_{n=0}^{+\infty} \left(\sum_{m=0}^n c_m c_{n-m} \right) x^n$$

which implies that $d_0 = 1$, $d_1 = -1$ and $d_n = 0$ for $n \geq 2$, therefore $B_{1/2} = I - B$ and, in the end:

$$S^2 = \|T\|(B_{1/2})^2 = \|T\|(I - B) = \|T\|(I - I + \|T\|^{-1}T) = T.$$

Lastly, since the series that defines $B_{1/2}$, hence S , is absolutely convergent, it commutes with every bounded operator commuting with T .

Up to now we only proved the existence of a square root for T , in particular the one given by the expression (2.7). To prove uniqueness of the square root we start supposing S_0 is another non-negative operator in $\mathcal{B}(X)$ such that $S_0^2 = T$. We notice that S_0 commutes with T , indeed $S_0T = S_0S_0^2 = S_0^2S_0 = TS_0$. Therefore S_0 commutes also with S , thus we have:

$$\begin{aligned} (S - S_0)^2S + (S - S_0)^2S_0 &= (S - S_0)[(S - S_0)S + (S - S_0)S_0] = \\ &= (S - S_0)(S^2 - S_0S + SS_0 - S_0^2) = \\ &= (S - S_0)(S^2 - S_0^2) = (S - S_0)(T - T) = 0. \end{aligned}$$

But $(S - S_0)^2S$ and $(S - S_0)^2S_0$ are non-negative operator, so they must vanish, in particular also their difference $(S - S_0)^2S - (S - S_0)^2S_0 = (S - S_0)^3$ is zero. This implies $(S - S_0)^4 = (S - S_0)(S - S_0)^3 = 0$, but since both S and S_0 are self-adjoint, for every $x \in H$ we have:

$$0 = \langle (S - S_0)^4x, x \rangle = \langle (S - S_0)^2u, (S - S_0)^2u \rangle = \|(S - S_0)^2x\|^2 \implies (S - S_0)^2 = 0$$

and with the same argument we conclude that also $S - S_0 = 0$. \square

Remark. From expression (2.7) it is clear that if T is compact then also \sqrt{T} , since it is a limit of compact operators.

Definition 2.15. Given an operator $T \in \mathcal{B}(X)$ we define its **absolute value** as

$$|T| := \sqrt{T^*T} \quad (2.8)$$

Definition 2.16. A linear operator $U \in \mathcal{B}(X)$ is a **partial isometry** if it is an isometry over $(\text{Ker } U)^\perp$, i.e. $\|Ux\|_X = \|x\|_X$ for every $x \in (\text{Ker } U)^\perp$

Proposition 2.17. Let $U \in \mathcal{B}(H)$ be a partial isometry. Then also U^* is a partial isometry.

Proof. Since U is a partial isometry, from polarization identity it follows that $\langle Ux, Uy \rangle = \langle x, y \rangle$ for every $x, y \in (\text{Ker}(U))^\perp$. Clearly the same equality holds if $x \in (\text{Ker}(U))^\perp$ while $y \in \text{Ker}(U)$, so U^*U is the identity over $(\text{Ker}(U))^\perp$. This implies that U^* is an isometry onto $\overline{\text{Im}(U)} = (\text{Ker}(U^*))^\perp$, hence it is a partial isometry too. \square

Theorem 2.18. Given $T \in \mathcal{B}(X)$ there exist unique a partial isometry U such that

$$T = U|T|, \quad (2.9)$$

which is uniquely determined by the condition $\text{Ker } U = \text{Ker } T$.

Proof. We start defining $\tilde{U} : \text{Im}|T| \rightarrow \text{Im}T$. Every $\phi \in \text{Im}|T|$ can be written as $\phi = |T|\psi$ for some $\psi \in H$ and we define $\tilde{U}(\phi) = \tilde{U}(|T|\psi) := T\psi$. We notice that

$$\|\phi\|^2 = \||T|\psi\|^2 = \langle |T|\psi, |T|\psi \rangle = \langle |T|^2\psi, \psi \rangle = \langle T^*T\psi, \psi \rangle = \|T\psi\|^2 = \|\tilde{U}\phi\|.$$

This computation ensures us that the definition of \tilde{U} is consistent (if $\phi = |T|\psi_1 = |T|\psi_2$ for some $\psi_1, \psi_2 \in H$, then $|T|(\psi_1 - \psi_2) = 0$, but $\||T|(\psi_1 - \psi_2)\| = \|T(\psi_1 - \psi_2)\|$, $T\psi_1 = T\psi_2$) and it implies that \tilde{U} is an isometry over $\text{Im}(|T|)$, thus it can be uniquely extended to an isometry of $\overline{\text{Im}(|T|)}$ over $\overline{\text{Im}T}$. The definition of the map U is straightforward extending \tilde{U} to all H defining it 0 on $(\text{Im}(|T|))^\perp$. Since $|T|$ is self-adjoint $(\text{Im}(|T|))^\perp = \text{Ker}(|T|)$. Furthermore, since $\||T|\psi\| = \|T\psi\|$, $|T|\psi = 0$ if and only if $T\psi = 0$, therefore $\text{Ker}|T| = \text{Ker}T$ which implies, in the end, that $\text{Ker}U = \text{Ker}T$.

Lastly we shall prove that U is unique. Suppose V is another partial isometry such that $T = V|T|$ and $\text{Ker}V = \text{Ker}T$. First condition implies that $T = U|T| = V|T|$, so U and V coincide over $\text{Im}|T|$ (and by continuity over its closure), while second condition implies that $\text{Ker}T = \text{Ker}U = \text{Ker}V$, so U and V coincide over $\text{Ker}|T| = (\text{Im}|T|)^\perp$, therefore U and V coincide over $H = \overline{\text{Im}|T|} \oplus \text{Ker}|T|$. \square

Definition 2.19. Let X be a separable Hilbert space with orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. Given $T \in \mathcal{B}(X)$ a non-negative operator we define the **trace** of T as

$$\text{tr}(T) = \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle \quad (2.10)$$

Since T is non-negative the sum in (2.10) is well defined and, in principle, it could depend upon the basis $\{e_n\}_{n \in \mathbb{N}}$. The following theorem states that actually the definition is well posed, i.e. the trace does not depend on the base.

Proposition 2.20. The definition of tr given by (2.10) is independent of the basis.

Proof. Let $T \in \mathcal{B}(H)$ be a non-negative operator and $\{\phi_n\}_{n \in \mathbb{N}}$ and $\{\psi_n\}_{n \in \mathbb{N}}$ be two orthonormal basis of H . We have

$$\begin{aligned} \sum_{n=1}^{+\infty} \langle T \phi_n, \phi_n \rangle &= \sum_{n=1}^{+\infty} \|\sqrt{T} \phi_n\|^2 = \sum_{n=1}^{+\infty} \left(\sum_{m=1}^{+\infty} |\langle \sqrt{T} \phi_n, \psi_m \rangle|^2 \right) = \\ &= \sum_{m=1}^{+\infty} \left(\sum_{n=1}^{+\infty} |\langle \sqrt{T} \psi_m, \phi_n \rangle|^2 \right) = \sum_{m=1}^{+\infty} \|\sqrt{T} \psi_m\|^2 = \\ &= \sum_{m=1}^{+\infty} \langle T \psi_m, \psi_m \rangle \end{aligned}$$

where we used the fact that \sqrt{T} is self-adjoint, while the exchange of series is allowed because all terms are non-negative. \square

Definition 2.21. An operator $T \in \mathcal{B}(X)$ is called **Hilbert-Schmidt** if and only if $\text{tr}(T^*T) < \infty$. We define the **Hilbert-Schmidt norm** of an operator as $\|T\|_{\text{HS}} = \sqrt{\text{tr}(T^*T)}$.

From the definition of the trace we can see that, given an orthonormal basis $\{e_n\}_{n=1}^{+\infty}$,

$$\|T\|_{\text{HS}}^2 = \text{tr}(T^*T) = \sum_{n=1}^{+\infty} \langle T^*T e_n, e_n \rangle = \sum_{n=1}^{+\infty} \langle T e_n, T e_n \rangle = \sum_{n=1}^{+\infty} \|T e_n\|^2 \quad (2.11)$$

so we can say, in an equivalent way, that an operator is Hilbert-Schmidt if and only if $\sum_{n=1}^{+\infty} \|T e_n\|^2 < +\infty$. Thanks to Proposition 2.20 we immediately see that the Hilbert-Schmidt norm is independent on the choice of the basis.

We are now going to show some properties of Hilbert-Schmidt and trace class operators.

Proposition 2.22. *Let $T \in \mathcal{B}(H)$ be a Hilbert-Schmidt operator. Then also $|T|$ and T^* are Hilbert-Schmidt operators and*

$$\||T|\|_{\text{HS}} = \|T^*\|_{\text{HS}} = \|T\|_{\text{HS}}. \quad (2.12)$$

Proof. We already know that, for every $x \in H$, $\|Tx\|^2 = \||T|x\|^2$, therefore from (2.11) we immediately see that $\||T|\|_{\text{HS}} = \|T\|_{\text{HS}}$.

Consider now an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of H . From (2.11):

$$\begin{aligned} \|T\|_{\text{HS}}^2 &= \sum_{n=1}^{+\infty} \|T e_n\|^2 = \sum_{n=1}^{+\infty} \left(\sum_{m=1}^{+\infty} |\langle T e_n, e_m \rangle|^2 \right) = \sum_{m=1}^{+\infty} \left(\sum_{n=1}^{+\infty} |\langle e_n, T^* e_m \rangle|^2 \right) = \\ &= \sum_{m=1}^{+\infty} \|T^* e_m\|^2 = \|T^*\|_{\text{HS}}^2 \end{aligned}$$

where exchange of series is allowed since all terms are non-negative. \square

Theorem 2.23. *Let $T \in \mathcal{B}(H)$ be a Hilbert-Schmidt operator. Then $\|T\| \leq \|T\|_{\text{HS}}$ and T is compact. Moreover, a compact operator T is Hilbert-Schmidt if and only if $\sum_{n=1}^{+\infty} \mu_n^2 < +\infty$, where $\{\mu_n\}_{n=1}^{+\infty}$ are the singular values of T .*

Proof. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of H and let $x \in H$. We have:

$$\begin{aligned} \|Tx\|^2 &= \sum_{n=1}^{+\infty} |\langle Tx, e_n \rangle|^2 = \sum_{n=1}^{+\infty} |\langle x, T^* e_n \rangle|^2 \leq \|x\|^2 \sum_{n=1}^{+\infty} \|T^* e_n\|^2 = \\ &= \|x\|^2 \|T^*\|_{\text{HS}}^2 \stackrel{(2.12)}{=} \|x\|^2 \|T\|_{\text{HS}}^2 \end{aligned}$$

Taking the supremum over all $x \in H$ gives us that $\|T\| \leq \|T\|_{\text{HS}}$.

To prove that T is compact, consider the following sequence of operators $T_N = \sum_{n=1}^N \langle \cdot, e_n \rangle T e_n$. All T_N have finite rank, so they are compact and $T_N \rightarrow T$ in $\mathcal{B}(H)$, indeed:

$$\|T - T_N\| \leq \|T - T_N\|_{\text{HS}}^2 = \sum_{n=N+1}^{+\infty} \|T e_n\|^2 \rightarrow 0 \quad \text{as } N \rightarrow +\infty$$

therefore, since T is limit of compact operators, it is compact.

Lastly, if T is a compact operator also T^*T is compact as well as self-adjoint, hence we can consider as orthonormal basis of H the one given by its eigenvectors $\{e_n\}_{n \in \mathbb{N}}$. Recalling Corollary 2.9, we know that μ_n^2 are exactly the eigenvalues of T^*T . For such basis we have:

$$\|T\|_{\text{HS}}^2 = \sum_{n=1}^{+\infty} \|Te_n\|^2 = \sum_{n=1}^{+\infty} \langle Te_n, Te_n \rangle = \sum_{n=1}^{+\infty} \langle T^*Te_n, e_n \rangle = \sum_{n=1}^{+\infty} \mu_n^2$$

□

We can now turn back to trace class operator. It is evident that trace class operators and Hilbert-Schmidt operators are strictly related. Indeed, as seen in the proof of Proposition 2.20, we have

$$\text{tr}|T| = \sum_{n=1}^{+\infty} \langle |T|e_n, e_n \rangle = \sum_{n=1}^{+\infty} \|\sqrt{|T|}e_n\|^2 = \|T\|_{\text{HS}}^2,$$

so, an operator is trace class if and only if $\sqrt{|T|}$ is Hilbert-Schmidt. By virtue of this link we can exploit properties of Hilbert-Schmidt operators in order to obtain informations about trace class operators. First of all, we are going to prove that expression (2.10) is well-defined for every trace class operator and not only for non-negative ones.

Theorem 2.24. *Let $T \in \mathcal{B}(H)$ be a trace class operator and $\{e_n\}_{n \in \mathbb{N}}$ an orthonormal basis of H . Then $\sum_{n=1}^{+\infty} \langle Te_n, e_n \rangle$ converges absolutely and the limit is independent of the basis.*

Proof. We start proving that the series converges absolutely. Letting $T = U|T|$ be the polar decomposition of T , we want to show that:

$$\sum_{n=1}^{+\infty} |\langle Te_n, e_n \rangle| = \sum_{n=1}^{+\infty} |\langle U\sqrt{|T|}\sqrt{|T|}e_n, e_n \rangle| = \sum_{n=1}^{+\infty} |\langle \sqrt{|T|}e_n, \sqrt{|T|}U^*e_n \rangle| < +\infty. \quad (2.13)$$

From Cauchy-Schwarz inequality, for every term we have that $|\langle \sqrt{|T|}e_n, \sqrt{|T|}U^*e_n \rangle| \leq \|\sqrt{|T|}e_n\| \|\sqrt{|T|}U^*e_n\|$ and, since T is trace class, $\sqrt{|T|}$ is Hilbert-Schmidt, so the sequence of $\|\sqrt{|T|}e_n\|$ is in $\ell^2(\mathbb{N})$. Therefore we need to prove that also $\sqrt{|T|}U^*$ is an Hilbert-Schmidt operator. To do so, we consider an orthonormal basis of $(\text{Ker } U^*)^\perp$, denoted with $\{\phi_i\}_{i \in \mathcal{I}}$, and an orthonormal basis of $\text{Ker } U^*$, denoted with $\{\psi_j\}_{j \in \mathcal{J}}$, where \mathcal{I} and \mathcal{J} are at most countable. Clearly $\{\eta_n\}_{n \in \mathbb{N}} := \{\phi_i\}_{i \in \mathcal{I}} \cup \{\psi_j\}_{j \in \mathcal{J}}$ is an orthonormal basis of H . We notice that, from Proposition 2.17, U^* is a partial isometry over $(\text{Ker } U^*)^\perp$, so the set $\{U^*\phi_i\}_{i \in \mathcal{I}}$ is still an orthonormal basis of $(\text{Ker } U^*)^\perp$, which can be completed to a basis of H with $\{\psi_j\}_{j \in \mathcal{J}}$. For the basis given by η_n we have:

$$\sum_{n=1}^{+\infty} \|\sqrt{|T|}U^*\eta_n\|^2 = \sum_{i \in \mathcal{I}} \|\sqrt{|T|}U^*\phi_i\|^2 \leq \sum_{i \in \mathcal{I}} \|\sqrt{|T|}(U^*\phi_i)\|^2 + \sum_{j \in \mathcal{J}} \|\sqrt{|T|}\psi_j\|^2 = \|\sqrt{|T|}\|_{\text{HS}}^2.$$

Since the Hilbert-Schmidt norm is independent of the basis we can conclude that $\sqrt{|T|}U^*$ is an Hilbert-Schmidt operator. We can now go back to (2.13):

$$\begin{aligned} \sum_{n=1}^{+\infty} |\langle Te_n, e_n \rangle| &= \sum_{n=1}^{+\infty} |\langle \sqrt{|T|}e_n, \sqrt{|T|}U^*e_n \rangle| \leq \sum_{n=1}^{+\infty} \|\sqrt{|T|}e_n\| \|\sqrt{|T|}U^*e_n\| \stackrel{C-S}{\leq} \\ &\leq \left(\sum_{n=1}^{+\infty} \|\sqrt{|T|}e_n\|^2 \right)^{1/2} \left(\sum_{n=1}^{+\infty} \|\sqrt{|T|}U^*e_n\|^2 \right)^{1/2} \leq \|\sqrt{|T|}\|_{\text{HS}}^2. \end{aligned}$$

The proof of the independence of the basis is exactly the same as the one for Proposition 2.20, because now the exchange of series is allowed since the series is convergent. \square

Theorem 2.25. *Let $T \in \mathcal{B}(H)$. Then T is trace class if and only if it is compact and $\sum_{n=1}^{+\infty} \mu_n$, where $\{\mu_n\}_{n \in \mathbb{N}}$ are the singular values of T . Moreover, if T is trace class then $\text{tr}|T| = \sum_{n=1}^{+\infty} \mu_n$ and additionally, if it is also self-adjoint, $\text{tr}T = \sum_{n=1}^{+\infty} \lambda_n$ where $\{\lambda_n\}_{n \in \mathbb{N}}$. Finally, we have $\|T\| \leq \text{tr}|T|$.*

Proof. Let $T = U|T|$ be the polar decomposition of T . If T is trace class $\sqrt{|T|}$ is Hilbert-Schmidt, but from Theorem 2.23 we have that $\sqrt{|T|}$ is compact, therefore also $T = U\sqrt{|T|}\sqrt{|T|}$ and $|T|$ are compact. Not only $|T|$ is compact, but it is also self-adjoint and its eigenvalues are exactly $\{\mu_n\}_{n \in \mathbb{N}}$. Letting $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis made up of eigenvectors of $|T|$, we have

$$\text{tr}|T| = \sum_{n=1}^{+\infty} \langle |T|e_n, e_n \rangle = \sum_{n=1}^{+\infty} \mu_n.$$

Conversely, if T is a compact operator the previous formula still holds and shows that T is also trace-class. With the same reasoning, if T is self-adjoint and trace class, we can write the trace with respect to the basis made up its eigenvectors thus obtaining $\text{tr}T = \sum_{n=1}^{+\infty} \lambda_n$.

For the last part of the Theorem, we notice that $|T|$ is a compact self-adjoint non-negative operator, therefore, assuming its eigenvalues are decreasingly ordered, from Corollary 2.8 we have that $\||T|\| = \mu_1$, hence:

$$\|T\| = \|U|T|\| \leq \||T|\| = \mu_1 \leq \sum_{n=1}^{+\infty} \mu_n = \text{tr}|T|.$$

\square

In the special case $H = L^2(\mathbb{R}^d)$ we are able to give a characterization of Hilbert-Schmidt operators.

Theorem 2.26. *An operator $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ is Hilbert-Schmidt if and only if there exist a function $K_T \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, called integral kernel, such that*

$$(Tf)(x) = \int_{\mathbb{R}^d} K_T(x, y) f(y) dy \quad \forall f \in L^2(\mathbb{R}^d). \quad (2.14)$$

Moreover $\|T\|_{\text{HS}} = \text{tr}(T^*T) = \int_{\mathbb{R}^{2d}} |K(x, y)|^2 dx dy$.

Proof. Letting $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$, then $\{e_n \otimes \bar{e}_m\}_{(n,m) \in \mathbb{N} \times \mathbb{N}}$, where $(e_n \otimes \bar{e}_m)(x, y) := e_n(x)\overline{e_m(y)}$, is an orthonormal basis of $L^2(\mathbb{R}^{2d})$.

We start supposing that $T \in \mathcal{B}(H)$ is a Hilbert-Schmidt operator and we define

$$K_T(x, y) = \sum_{n,m=1}^{+\infty} \langle T e_n, e_m \rangle_{L^2(\mathbb{R}^d)} e_m(x) \overline{e_n(y)} \quad (2.15)$$

Then

$$\begin{aligned} \|K_T\|_{L^2(\mathbb{R}^{2d})}^2 &= \sum_{n,m=1}^{+\infty} |\langle T e_n, e_m \rangle|^2 = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} |\langle T e_n, e_m \rangle|^2 = \\ &= \sum_{n=1}^{+\infty} \|T e_n\|_{L^2(\mathbb{R}^d)}^2 = \|T\|_{\text{HS}}^2 \end{aligned} \quad (2.16)$$

and, given $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \int_{\mathbb{R}^d} K_T(x, y) f(y) dy &= \int_{\mathbb{R}^d} \sum_{n,m=1}^{+\infty} \langle T e_n, e_m \rangle_{L^2(\mathbb{R}^d)} e_m(x) \overline{e_n(y)} f(y) dy = \\ &= \sum_{n,m=1}^{+\infty} \langle T e_n, e_m \rangle_{L^2(\mathbb{R}^d)} \langle f, e_n \rangle_{L^2(\mathbb{R}^d)} e_m(x) = \\ &= \sum_{m=1}^{+\infty} \langle T f, e_m \rangle_{L^2(\mathbb{R}^d)} e_m(x) = (T f)(x) \end{aligned}$$

perché lo scambio di serie e integrale è permesso?

Conversely, if we define T by (2.14) we have to show that this operator is Hilbert-Schmidt, but this is straightforward since (2.16) still holds. \square

In light of this theorem, operators defined by (2.14) are called *Hilbert-Schmidt integral operators*.

Proposition 2.27. *An Hilbert-Schmidt integral operator with kernel K is self-adjoint if and only if $K(x, y) = \overline{K(y, x)}$.*

Proof. Let $f, g \in L^2(\mathbb{R}^d)$. We start showing that $K(x, y) f(y) \overline{g(x)} \in L^1(\mathbb{R}^{2d})$:

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |K(x, y)| |f(y)| |g(x)| dx dy &\stackrel{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K(x, y)| |f(y)| dy \right) |g(x)| dx = \\ &= \int_{\mathbb{R}^d} (T_{|K|} |f|)(y) |g(y)| dy \stackrel{\text{C-S}}{\leq} \|T_{|K|}\|_2 \|g\|_2 \leq \|K\|_2 \|f\|_2 \|g\|_2 < +\infty, \end{aligned}$$

where $T_{|K|}$ denotes the Hilbert-Schmidt integral operator with kernel $|K|$ and $\|T_{|K|} |f|\|_2 \leq \|K\|_2 \|f\|_2$ because $T_{|K|}$ is Hilbert-Schmidt, therefore $\|T_{|K|}\| \leq \|T_{|K|}\|_{\text{HS}} = \|K\|_2$. We are now in the position to use Fubini's theorem:

$$\begin{aligned} \langle T f, g \rangle &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(x, y) f(y) dy \right) \overline{g(x)} dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(x, y) \overline{g(x)} dx \right) f(y) dy = \\ &= \int_{\mathbb{R}^d} \overline{\left(\int_{\mathbb{R}^d} \overline{K(x, y)} g(x) dx \right)} f(y) dy = \langle f, T^* g \rangle \end{aligned}$$

therefore the expression of T^* is:

$$T^*f(x) = \int_{\mathbb{R}^d} \overline{K(y, x)} f(y) dy,$$

so T is self-adjoint if and only if $K(x, y) = \overline{K(y, x)}$. \square

2.2 Fourier Transform and its properties

Definition 2.28. Let $f \in L^1(\mathbb{R}^d)$. We define the **Fourier transform** of f the function

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot t} f(t) dt \quad (2.17)$$

It is straightforward to see that the definition is well-posed and that $\mathcal{F}f \in L^\infty(\mathbb{R}^d)$ with $\|\mathcal{F}f\|_\infty \leq \|f\|_1$. Therefore \mathcal{F} can be seen as a linear operator between $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ with $\|\mathcal{F}\| = 1$ (from the previous inequality actually we saw that $\|\mathcal{F}\| \leq 1$ but if we take $f \geq 0$ a.e. we have that $\hat{f}(0) = \|f\|_1$ that gives us the equality). The Fourier transform of an $L^1(\mathbb{R}^d)$ is not only bounded, as stated by the *Riemann-Lebesgue lemma*.

Theorem 2.29 (Riemann-Lebesgue lemma). Let $f \in L^1(\mathbb{R}^d)$. Therefore $\hat{f} \in C_0(\mathbb{R}^d) = \{f : \mathbb{R}^d \rightarrow \mathbb{C} \text{ continuous such that } \lim_{|t| \rightarrow \infty} |f(t)| = 0\}$.

Definition 2.30. Let $f \in L^1(\mathbb{R}^d)$. We define the **inverse Fourier transform** of the function f

$$\mathcal{F}^{-1}f(t) = \check{f}(t) := \int_{\mathbb{R}^d} e^{2\pi i \omega \cdot t} f(\omega) d\omega \quad (2.18)$$

The inverse Fourier transform is denoted with \mathcal{F}^{-1} because it is actually the inverse operator of the Fourier transform as stated by the *inversion theorem*.

Theorem 2.31 (Inversion theorem). Let $f \in L^1(\mathbb{R}^d)$ and suppose that also $\hat{f} \in L^1(\mathbb{R}^d)$. Then

$$f(t) = \mathcal{F}^{-1} \circ \mathcal{F}f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi i \omega \cdot t} d\omega$$

The Fourier transform is intimately related to regularity and decay properties. The duality between these two is stated in the following theorems.

Theorem 2.32. Let $f \in L^1(\mathbb{R}^d)$. If $|t|^k f \in L^1(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ then $\hat{f} \in C_0^k(\mathbb{R}^d)$ and the following holds for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$:

$$\mathcal{F}((-2\pi i t)^\alpha f)(\omega) = \partial^\alpha \mathcal{F}f(\omega). \quad (2.19)$$

Theorem 2.33. Let $f \in C^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$. If $f, \partial^\alpha f \in L^1(\mathbb{R}^d)$ for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ then

$$\mathcal{F}(\partial^\alpha f)(\omega) = (2\pi i \omega)^\alpha \mathcal{F}f(\omega) \quad (2.20)$$

In particular this implies that $\hat{f}(\omega) = o(|\omega|^{-k})$ as $|\omega| \rightarrow \infty$.

To sum up, previous theorems state a duality between regularity and decay: if a function is smooth then its Fourier transform decays rapidly and vice versa.

If f is in $L^2(\mathbb{R}^d)$, the integral in (2.17) in general will not converge. Nevertheless we can define the Fourier transform of an L^2 function through a density argument. For example, one can use $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, which is a dense subspace of $L^2(\mathbb{R}^d)$. On this space one can show that the Fourier transform is an isometry with respect to the L^2 norm and therefore it extends to an isometry on the whole $L^2(\mathbb{R}^d)$. This is stated by the *Plancherel theorem*.

Theorem 2.34 (Plancherel theorem). *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ then $\|f\|_2 = \|\hat{f}\|_2$.*

Thanks to the polarization identity this implies that \mathcal{F} preserves the inner product in $L^2(\mathbb{R}^d)$:

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d)} \quad \forall f, g \in L^2(\mathbb{R}^d) \quad (2.21)$$

therefore the Fourier transform \mathcal{F} is a unitary operator on $L^2(\mathbb{R}^d)$. Result (2.21) is called *Parseval formula*.

Lastly, we introduce some fundamental operators in Fourier analysis. Given $x, \xi \in \mathbb{R}^d$ and $\lambda > 0$ we define the *time-shift* (or translation) operator T_x

$$T_x f(t) = f(t - x) \quad \forall t \in \mathbb{R}^d, \quad (2.22)$$

the *modulation* operator M_ξ

$$M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t) \quad \forall t \in \mathbb{R}^d \quad (2.23)$$

and the *dilation* operator D_λ

$$D_\lambda f(t) = \lambda^d f(\lambda t) \quad \forall t \in \mathbb{R}^d. \quad (2.24)$$

Moreover, time-shift and modulation operators can be combined into a *time-frequency shift* operator

$$\pi(x, \xi) f(t) = M_\xi T_x f(t) \quad \forall t \in \mathbb{R}^d. \quad (2.25)$$

It is easy to check that all these operator are isometric (with respect to the L^1 norm) isomorphism. We show how these operator act under the Fourier transform.

Proposition 2.35. *Let $f \in L^1(\mathbb{R}^d)$. Then the following holds:*

- (i) $\mathcal{F}(T_x f)(\omega) = M_{-x} \hat{f}(\omega);$
- (ii) $\mathcal{F}(M_\xi f)(\omega) = T_\xi \hat{f}(\omega);$
- (iii) $\mathcal{F}(D_\lambda f)(\omega) = \hat{f}\left(\frac{\omega}{\lambda}\right).$

Proof.

$$\begin{aligned} \text{(i)} \quad \mathcal{F}(T_x f)(\omega) &= \int_{\mathbb{R}^d} f(t - x) e^{-2\pi i \omega \cdot t} dt \stackrel{s=t-x}{=} \int_{\mathbb{R}^d} f(s) e^{-2\pi i \omega \cdot (s+x)} ds = \\ &= e^{-2\pi i \omega \cdot x} \int_{\mathbb{R}^d} f(s) e^{-2\pi i \omega \cdot s} ds = M_{-x} \hat{f}(\omega) \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \mathcal{F}(M_\xi f)(\omega) &= \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot t} f(t) e^{-2\pi i \omega \cdot t} dt = \int_{\mathbb{R}^d} f(t) e^{-2\pi i (\omega - \xi) \cdot t} dt = T_\xi \hat{f}(\omega) \\
 \text{(iii)} \quad \mathcal{F}(D_\lambda f)(\omega) &= \int_{\mathbb{R}^d} \lambda^d f(\lambda t) e^{-2\pi i \omega \cdot t} dt \stackrel{s=\lambda t}{=} \int_{\mathbb{R}^d} f(s) e^{-2\pi i \omega \cdot \frac{s}{\lambda}} ds = \hat{f}\left(\frac{\omega}{\lambda}\right)
 \end{aligned}$$

□

We point out that the second property, namely that $\mathcal{F}(M_\xi f) = T_\xi \hat{f}$, is shedding light on the role of modulation operator:

while T_x acts as a translation in the time domain, M_ξ is a translation in the frequency domain. Therefore the time-frequency shift operator $\pi(x, \xi)$ is indeed a shift operator because it acts as a translation in the joint time-frequency domain.

Thanks to these properties and Theorems 2.32 and 2.33 we can state compute the Fourier transform of Gaussians.

Proposition 2.36. *The Fourier transform maps Gaussian into Gaussians. More precisely, for $\lambda > 0$, we have:*

$$\mathcal{F}(e^{-\lambda \pi |\cdot|^2})(\omega) = \frac{1}{\lambda^{d/2}} e^{-\frac{1}{\lambda} \pi |\omega|^2} \quad \forall \omega \in \mathbb{R}^d. \quad (2.26)$$

Proof. We start considering the 1-dimensional case and for the ease of notation we let $\varphi_\lambda(t) = e^{-\lambda \pi t^2}$ for $t \in \mathbb{R}$.

First of all we consider $\varphi = \varphi_1$, for which we have:

$$\frac{d\varphi}{dt}(t) = -2\pi t \varphi(t)$$

If we take the Fourier transform of both members, using (2.20) in the former and (2.19) in the latter we obtain:

$$2\pi i \omega \hat{\varphi}(\omega) = -i \frac{d\hat{\varphi}}{d\omega}(\omega) \implies \frac{d\hat{\varphi}}{d\omega}(\omega) = -2\pi \omega \hat{\varphi}(\omega)$$

which is exactly the same equation satisfied by φ , therefore $\hat{\varphi}(\omega) = C e^{-\pi \omega^2}$ for some $C \in \mathbb{R}$. Since $\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(t) dt = \|\varphi\|_1 = 1$ we obtain that $C = 1$.

The general case can be proved using the dilatation operator:

$$\begin{aligned}
 \varphi_\lambda(t) &= e^{-\lambda \pi t^2} = e^{-\pi (\sqrt{\lambda} t)^2} = \frac{1}{\sqrt{\lambda}} D_{\sqrt{\lambda}} \varphi(t) \implies \\
 \implies \hat{\varphi}_\lambda(\omega) &= \frac{1}{\sqrt{\lambda}} \mathcal{F}(D_{\sqrt{\lambda}} \varphi)(\omega) = \frac{1}{\sqrt{\lambda}} \hat{\varphi}\left(\frac{\omega}{\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{\lambda} \pi \omega^2}
 \end{aligned}$$

The passage to the multidimensional case is almost straightforward since $e^{-\lambda |\cdot|^2} = \prod_{j=1}^d e^{-\lambda t_j^2}$, therefore:

$$\begin{aligned}
 \mathcal{F}(e^{-\lambda \pi |\cdot|^2})(\omega) &= \int_{\mathbb{R}^d} e^{-\lambda \pi |t|^2} e^{-2\pi i \omega \cdot t} dt = \prod_{j=1}^d \int_{\mathbb{R}} e^{-\lambda \pi t_j^2} e^{-2\pi i \omega_j t_j} dt_j = \\
 &= \prod_{j=1}^d \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{\lambda} \pi \omega_j^2} = \frac{1}{\lambda^{d/2}} e^{-\frac{1}{\lambda} \pi |\omega|^2}.
 \end{aligned}$$

□

Theorem 2.37 (Hausdorff-Young). *Let $1 \leq p \leq 2$ and let p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\mathcal{F} : L^p(\mathbb{R}^d) \rightarrow L^{p'}(\mathbb{R}^d)$ and $\|\hat{f}\|_{p'} \leq \|f\|_p$.*

In the following we will need the sharp version of the Hausdorff-Young inequality:

$$\|\hat{f}\|_{p'} \leq \left(\frac{p^{1/p}}{p'^{1/p'}} \right)^{d/2} \|f\|_p = A_p^d \|f\|_p \quad (2.27)$$

where A_p is the so-called Babenko-Bechner constant. This constant appears also in the sharp version of Young's theorem

Theorem 2.38 (Young). *Given $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, suppose that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $r \geq 1$. Then $f * g \in L^r(\mathbb{R}^d)$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.*

Just like the Hausdorff-Young inequality, Young's inequality holds in a sharp version:

$$\|f * g\|_r \leq (A_p A_q A_{r'})^d \|f\|_p \|g\|_q. \quad (2.28)$$

Chapter 3

Short-Time Fourier Transform

3.1 STFT

The *short-time Fourier transform* or *STFT* is a powerful tool, introduced by Gabor in [6], used to study properties of a signal locally both in time and frequency. The main idea behind the STFT is the following: if we want some information of the spectrum of a signal around a specific time, say T , we could choose an interval $(T - \Delta T, T + \Delta T)$ and take the Fourier transform of $f\chi_{(T-\Delta T, T+\Delta T)}$. Usually multiplying by a characteristic function will not give us a regular function (not even continuous) and in light of the duality between regularity and decay, the Fourier transform of $f\chi_{(T-\Delta T, T+\Delta T)}$ will not decay rapidly. Therefore a sharp cutoff in the time domain will result in a “bad” localization in the frequency domain. In order to avoid this kind of problems we could think to multiply the signal f by a smooth function.

Definition 3.1. Fix a function $\phi \neq 0$ called window function. The **short-time Fourier transform** of a function f with window ϕ is defined as

$$\mathcal{V}_\phi f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{\phi(t-x)} e^{-2\pi i \omega \cdot t} dt, \quad (x, \omega) \in \mathbb{R}^{2d} \quad (3.1)$$

In the above definition we did not specify where f and ϕ are chosen. Since we are taking the Fourier transform of the function $fT_x\bar{\phi}$, the STFT is well defined whenever the Fourier transform of this function is. For example if both f and ϕ are in $L^2(\mathbb{R}^d)$ then $fT_x\bar{\phi}$ is in $L^1(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$ and so the integral in (3.1) is defined. In this special case the STFT can be written as a scalar product in $L^2(\mathbb{R}^d)$:

$$\mathcal{V}_\phi f(x, \omega) = \langle f, M_\omega T_x \phi \rangle = \langle f, \pi(x, \omega) \phi \rangle$$

In general, the STFT of f with respect to ϕ will be defined whenever $\langle f, M_\omega T_x \phi \rangle$ is an expression of some sort of duality. For example, if $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ then $M_\omega T_x \bar{\phi} \in \mathcal{S}(\mathbb{R}^d)$, therefore $\langle f, M_\omega T_x \phi \rangle$ can be seen as the usual duality between tempered distributions and functions in the Schwartz space.

Aggiungere scritture equivalenti della STFT??

3.1.1 Properties of STFT

In this section we will present and prove some properties about the STFT. An excellent reference is [8].

Theorem 3.2. *Let $f_1, f_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$. Then $\mathcal{V}_{\phi_i} f_i \in L^2(\mathbb{R}^{2d})$ and the following holds:*

$$\langle \mathcal{V}_{\phi_1} f_1, \mathcal{V}_{\phi_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle \phi_1, \phi_2 \rangle} \quad (3.2)$$

Proof. contenuto... □

Corollary 3.3. *If $f, \phi \in L^2(\mathbb{R}^d)$ then*

$$\|\mathcal{V}_\phi f\|_2 = \|f\|_2 \|\phi\|_2 \quad (3.3)$$

In particular if $\|\phi\|_2 = 1$ we see that \mathcal{V}_ϕ is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$.

Proof. contenuto... □

From the Cauchy-Schwarz inequality we immediately see that $\mathcal{V}_\phi f$ is in $L^\infty(\mathbb{R}^{2d})$:

$$|\mathcal{V}_\phi f(x, \omega)| = |\langle f, M_\omega T_x \phi \rangle| \stackrel{\text{C-S}}{\leq} \|f\|_2 \|M_\omega T_x \phi\|_2 = \|f\|_2 \|\phi\|_2. \quad (3.4)$$

Combing this with (3.3) and using a simple interpolation argument we see that $\mathcal{V}_\phi f \in L^p(\mathbb{R}^{2d})$ for every $p \in [2, \infty)$ and that $\|\mathcal{V}_\phi f\|_p \leq \|f\|_2 \|\phi\|_2$. This result is improved by the following theorem due to Lieb.

Theorem 3.4. *If $f, \phi \in L^2(\mathbb{R}^d)$ and $2 \leq p \leq \infty$, then*

$$\|\mathcal{V}_\phi f\|_p^p = \int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x, \omega)|^p dx d\omega \leq \left(\frac{2}{p}\right)^d \|f\|_2^p \cdot \|g\|_2^p \quad (3.5)$$

Proof. For this proof we will need to use an alternative form of the STFT:

$$\mathcal{V}_\phi f(x, \omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega \cdot t} \overline{\phi(t-x)} dt = \mathcal{F}(f T_x \bar{\phi})(\omega)$$

Using Cauchy-Schwarz inequality it is immediate to see that $f T_x \bar{\phi} \in L^1(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. In addition to that, since $\mathcal{V}_\phi f = \mathcal{F}(f T_x \bar{\phi}) \in L^2(\mathbb{R}^{2d})$, from Fubini's theorem we can say that $\mathcal{F}(f T_x \bar{\phi}) \in L^2(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$ and therefore also $f T_x \bar{\phi} \in L^2(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$. Through an interpolation argument we obtain that $f T_x \bar{\phi} \in L^q(\mathbb{R}^d)$ for every $q \in [1, 2]$.

We start considering the L^p norm of $\mathcal{V}_\phi f$:

$$\begin{aligned} \|\mathcal{V}_\phi f\|_p &= \left(\int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x, \omega)|^p dx d\omega \right)^{1/p} \stackrel{\text{Tonelli}}{=} \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathcal{V}_\phi f(x, \omega)|^p d\omega \right) dx \right]^{1/p} = \\ &= \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathcal{F}(f T_x \bar{\phi})(\omega)|^p d\omega \right) dx \right]^{1/p} \stackrel{(2.28)}{\leq} \\ &\leq A_{p'}^d \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(t) \overline{\phi(t-x)}|^{p'} dt \right)^{p/p'} dx \right]^{1/p} \end{aligned} \quad (3.6)$$

where the use of Young's inequality (2.28) is justified since we noticed that $fT_x\bar{\phi}$ is in $L^q(\mathbb{R}^d)$ for every $q \in [1, 2]$, so in particular it is in $L^{p'}(\mathbb{R}^d)$. Letting $\phi^*(t) = \overline{\phi(-t)}$ and considering the inner integral we have

$$\int_{\mathbb{R}^d} |f(t)\overline{\phi(t-x)}|^{p'} dt = \int_{\mathbb{R}^d} |f(t)|^{p'} |\phi^*(x-t)|^{p'} dt = (|f|^{p'} * |\phi^*|^{p'})(x),$$

so the expression in (3.6) is the $L^{p/p'}(\mathbb{R}^d)$ norm of $|f|^{p'} * |\phi^*|^{p'}$. Since both f and ϕ are in $L^2(\mathbb{R}^d)$ and $p' \leq 2$ we have that $|f|^{p'}, |\phi^*|^{p'} \in L^{2/p'}(\mathbb{R}^d)$. Thanks to Young's theorem 2.38 $|f|^{p'} * |\phi^*|^{p'}$ belongs to $L^r(\mathbb{R}^d)$, where r is given by:

$$\frac{1}{(2/p')} + \frac{1}{(2/p')} = 1 + \frac{1}{r} \implies r = \frac{1}{p' - 1} = \frac{1}{\frac{p}{p-1} - 1} = p - 1 = \frac{p}{p'}$$

therefore, using the sharp version of Young's inequality (2.28) in (3.6) we obtain:

$$\|\mathcal{V}_\phi f\|_p \leq A_{p'}^d \left(A_{2/p'}^d A_{2/p'}^d A_{(p/p')'}^d \| |f|^{p'} \|_{2/p'} \| |\phi^*|^{p'} \|_{2/p'} \right)^{1/p'}.$$

However $\| |f|^{p'} \|_{2/p'} = (\int_{\mathbb{R}^d} (|f(x)|^{p'})^{2/p'} dx)^{p'/2} = \|f\|_2^{p'}$ and from a direct calculation (which can be found in A.1) one can see that $A_{p'}^d A_{2/p'}^{2d/p'} A_{(p/p')'}^{d/p'} = (2/p)^{d/p}$, which corresponds to the desired result. \square

From a direct computation one can see that the adjoint operator of the STFT operator \mathcal{V}_ϕ is given by the following expression:

$$\mathcal{V}_\phi^* g(t) = \int_{\mathbb{R}^{2d}} g(x, \omega) \phi(t-x) e^{2\pi i \omega \cdot t} dx d\omega = \int_{\mathbb{R}^{2d}} g(x, \omega) M_\omega T_x \phi(t) dx d\omega \quad \forall g \in L^2(\mathbb{R}^{2d}) \quad (3.7)$$

This adjoint operator appears in the following nice property

Theorem 3.5. *Let $f \in L^2(\mathbb{R}^d)$ and $\phi, \gamma \in L^2(\mathbb{R}^{2d})$ such that $\langle \phi, \gamma \rangle \neq 0$. Then:*

$$f(t) = \frac{1}{\langle \phi, \gamma \rangle} \mathcal{V}_\gamma^* \mathcal{V}_\phi f(t) = \frac{1}{\langle \phi, \gamma \rangle} \int_{\mathbb{R}^{2d}} \mathcal{V}_\phi f(x, \omega) M_\omega T_x \gamma(t) dx d\omega \quad \forall t \in \mathbb{R}^d \quad (3.8)$$

Proof. **VA MESSA??** \square

Therefore the adjoint operator \mathcal{V}_γ^* acts, in some sense, as an inverse operator. This will be of paramount importance in the following.

- Relazione di ortogonalità
- Formula di inversione

3.2 Bargmann Transform and Fock Space

Throughout this section we will consider the STFT with Gaussian window. We choose

$$\varphi(x) = 2^{d/4} e^{-\pi|x|^2} \quad (3.9)$$

where $|x|^2 = \sum_{k=1}^d x_k^2$ is the Euclidean norm of x in \mathbb{R}^d . The factor $2^{d/4}$ is chosen so that $\|\varphi\|_2 = 1$. The STFT with Gaussian window becomes

$$\mathcal{V}_\varphi f(x, \omega) = 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi|t-x|^2} e^{-2\pi i \omega \cdot t} dt \quad (3.10)$$

Our aim now is to rearrange the terms in the above expression in order to make $z = x + i\omega \in \mathbb{C}^d$ appear. We want to highlight the fact that when talking about complex quantities $|z|^2 = z\bar{z} = |x|^2 + |\omega|^2$. [Aggiungere remark sulla notazione usata.](#)

$$\begin{aligned} \mathcal{V}_\varphi f(x, \omega) &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi|t|^2 + 2\pi x \cdot t - \pi|\omega|^2} e^{-2\pi i \omega \cdot t} dt \\ &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi|t|^2} e^{2\pi(x-i\omega) \cdot t} e^{-\frac{\pi}{2}(|x|^2 - 2ix \cdot \omega - |\omega|^2)} e^{-\frac{\pi}{2}(|x|^2 + |\omega|^2 + 2ix \cdot \omega)} dt \\ &= 2^{d/4} e^{-\pi ix \cdot \omega} e^{-\frac{\pi}{2}(|x|^2 + |\omega|^2)} \int_{\mathbb{R}^d} f(t) e^{-\pi|t|^2} e^{2\pi(x-i\omega) \cdot t} e^{-\frac{\pi}{2}(x-i\omega)^2} dt \end{aligned}$$

The rearrangement may seem arbitrary but actually is done in such a way that inside the integral x and ω enter only via \bar{z} .

Definition 3.6. The **Bargmann transform** of a function f on \mathbb{R}^d is the function $\mathcal{B}f$ on \mathbb{C}^d given by

$$\mathcal{B}f(z) = 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{2\pi t \cdot z - \pi|t|^2 - \frac{\pi}{2}z^2} dt \quad (3.11)$$

We recall that a function defined over \mathbb{C}^d is *entire* if it is holomorphic over all \mathbb{C}^d .

Definition 3.7. The **Fock space** $\mathcal{F}^2(\mathbb{C}^d)$ is the Hilbert space of all entire functions F on \mathbb{C}^d for which the norm

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} dz \quad (3.12)$$

is finite.

Clearly the norm of the Fock space is induced by the following scalar product

$$\langle F, G \rangle_{\mathcal{F}} = \int_{\mathbb{C}^d} F(z) \overline{G(z)} e^{-\pi|z|^2} dz \quad (3.13)$$

Proposition 3.8. If f is a function on \mathbb{R}^d with polynomial growth then its Bargmann transform $\mathcal{B}f$ is an entire function on \mathbb{C}^d . Moreover, letting $z = x + i\omega$, the Bargmann transform of f is related to its STFT through the following

$$\mathcal{V}_\varphi f(x, -\omega) = e^{\pi ix \cdot \omega} \mathcal{B}f(z) e^{-\pi|z|^2/2} \quad (3.14)$$

Proof. contenuto...

□

Proposition 3.9. *If $f \in L^2(\mathbb{R}^d)$ then*

$$\|f\|_2 = \left(\int_{\mathbb{C}^d} |\mathcal{B}f(z)|^2 e^{-\pi|z|^2} dz \right)^{1/2} = \|\mathcal{B}f\|_{\mathcal{F}}. \quad (3.15)$$

Thus \mathcal{B} is an isometry from $L^2(\mathbb{R}^d)$ into $\mathcal{F}^2(\mathbb{C}^d)$.

Chapter 4

Localization Operators

One of the main problems in signal analysis or, in general, time-frequency analysis is to extract some informations about signal in order to analyse it. **We saw that STFT can be a tool for this purpose, however it has some practical problems: it doubles the dimension of the output and it is highly redundant (SE AVANZA TEMPO).**

In this chapter we see two possible ways to deal with the problem of creating operators able to localize a signal both in time and frequency.

4.1 Localization with projections

The most straightforward way to localize a signal, say in the time domain, is to use a sharp cut-off, which means a characteristic function. If we suppose to have a signal $f \in L^2(\mathbb{R}^d)$ and we want to localize it in a measurable subset $T \subseteq \mathbb{R}^d$ of the time domain we can consider the natural projection operator

$$P_T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \quad P_T f(t) = \chi_T(t) f(t) \quad (4.1)$$

This is clearly a projection operator, which means that $P_T^2 = P_T = P_T^*$.

In the same fashion we can define an operator able to localize on a measurable subset $\Omega \subseteq \mathbb{R}^d$ in the frequency domain. Their definition it is not as direct as the one for time projections but it is still easy to understand

$$Q_\Omega : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \quad Q_\Omega f(t) = \mathcal{F}^{-1}(\chi_\Omega \mathcal{F} f)(t) = \int_\Omega \hat{f}(\omega) e^{2\pi i \omega \cdot t} d\omega \quad (4.2)$$

It is also quite simple to show that this is a projection operator

$$\begin{aligned} Q_\Omega^2 &= \mathcal{F}^{-1} \chi_\Omega \mathcal{F} \mathcal{F}^{-1} \chi_\Omega \mathcal{F} = \mathcal{F}^{-1} \chi_\Omega \chi_\Omega \mathcal{F} = \mathcal{F}^{-1} \chi_\Omega \mathcal{F} = Q_\Omega \\ Q_\Omega^* &= \left(\mathcal{F}^{-1} \chi_\Omega \mathcal{F} \right)^* = \mathcal{F}^* \chi_\Omega^* \left(\mathcal{F}^{-1} \right)^* = \mathcal{F}^{-1} \chi_\Omega \mathcal{F} = Q_\Omega \end{aligned}$$

where we used the fact that the Fourier transform is a unitary operator on $L^2(\mathbb{R}^d)$, namely that $\mathcal{F}^* = \mathcal{F}^{-1}$.

Moreover, both operators have norm less or equal than 1, independently of T and Ω :

$$\|P_T f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(T)} \leq \|f\|_{L^2(\mathbb{R}^d)}$$

$$\|Q_\Omega f\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}^{-1} \chi_\Omega \mathcal{F} f\|_{L^2(\mathbb{R}^d)} \stackrel{\text{Plancherel}}{=} \|\mathcal{F} f\|_{L^2(\Omega)} \leq \|\mathcal{F} f\|_{L^2(\mathbb{R}^d)} \stackrel{\text{Plancherel}}{=} \|f\|_{L^2(\mathbb{R}^d)}$$

After defining these projection operators we may think to combine them into a single operator

$$Q_\Omega P_T, P_T Q_\Omega : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

which hopefully is able to localize a signal both in time and frequency “near” to the set $T \times \Omega$.

It is clear that these operators are linear and bounded, in particular their norms are less or equal than 1.

Up to now the only (essential) hypothesis on T and Ω is that they are measurable. Clearly, by adding some requirements on T and Ω we expect $Q_\Omega P_T$ and $P_T Q_\Omega$ to gain some properties.

Proposition 4.1. *Let $T, \Omega \subset \mathbb{R}^d$ with finite measure. Then $Q_\Omega P_T$ and $P_T Q_\Omega$ are Hilbert-Schmidt integral operators of the form*

$$Q_\Omega P_T f(x) = \int_{\mathbb{R}^d} K(x, t) f(t) dt \quad (4.3)$$

$$P_T Q_\Omega f(x) = \int_{\mathbb{R}^d} \overline{K(t, x)} f(t) dt \quad (4.4)$$

where

$$K(x, t) = \chi_T(t) \int_{\Omega} e^{2\pi i \omega \cdot (x-t)} d\omega \quad (4.5)$$

which has $\|K\|_{L^2(\mathbb{R}^{2d})} = \sqrt{|T||\Omega|}$.

Proof. contenuto... □

If we compare the integral kernels of $Q_\Omega P_T$ and $P_T Q_\Omega$ we see that $K(x, t) \neq \overline{K(t, x)}$, hence, by proposition 2.27, we immediately conclude that both operator are not self-adjoint. Since it is better to deal with self-adjoint operators when it comes to spectral properties, it would be nice if we could construct those starting from $Q_\Omega P_T$ and $P_T Q_\Omega$. This is easily done by considering

$$(Q_\Omega P_T)^* Q_\Omega P_T = P_T^* Q_\Omega^* Q_\Omega P_T = P_T Q_\Omega P_T \quad (4.6)$$

$$(P_T Q_\Omega)^* P_T Q_\Omega = Q_\Omega^* P_T^* P_T Q_\Omega = Q_\Omega P_T Q_\Omega \quad (4.7)$$

These are, by construction, self-adjoint operators, and since both $Q_\Omega P_T$ and $P_T Q_\Omega$ are compact operators (thanks to Proposition 4.1 and Theorem 2.23) they are also compact (we recall that composition between a general operator and a compact one is compact). Hence, by Theorem 2.7 they can be diagonalized.

Aggiungere qualcosa sulle prolata spheroidal wave functions?

4.2 Daubechies' localization operators

Projection operators considered in the previous section are powerful tools in signal analysis and quantum mechanics. Nevertheless they treat time and frequency in a separate way. We already saw that a good tool to simultaneously study a signal in time and frequency is the STFT but we also pointed out its limits. However we can think to construct localization operators using the STFT instead of the Fourier transform. This is exactly what was done by Ingrid Daubechies in 1988 in [3]. From Theorem 3.5 we know that the adjoint operator of \mathcal{V}_ϕ acts as inverse operator. If we choose a window $\phi \in L^2(\mathbb{R}^{2d})$ normalized, 3.8 becomes

$$f(t) = \mathcal{V}_\phi^* \mathcal{V}_\phi f(t)$$

The key idea is to multiply $\mathcal{V}_\phi f$ by a *weight function* $F(\omega, t)$, which logically should highlight some features of $\mathcal{V}_\phi f$:

$$L_{F,\phi} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \quad L_{F,\phi} f(t) = \mathcal{V}_\phi^* F \mathcal{V}_\phi f(t) \quad (4.8)$$

Related to this localization operator is the sesquilinear form $\mathcal{L}_{F,\phi} : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \rightarrow \mathbb{R}$ defined by the expression

$$\mathcal{L}_{F,\phi}(f, g) = \int_{\mathbb{R}^{2d}} F(x, \omega) \mathcal{V}_\phi f(x, \omega) \overline{\mathcal{V}_\phi g(x, \omega)} dx d\omega \quad (4.9)$$

Indeed, assuming $\mathcal{L}_{F,\phi}$ is bounded, we could define $L_{F,\phi} f$ through Riesz' representation theorem as the only element of $L^2(\mathbb{R}^d)$ such that

$$\mathcal{L}_{F,\phi}(f, g) = \langle L_{F,\phi} f, g \rangle = \int_{\mathbb{R}^d} L_{F,\phi} f(t) \overline{g(t)} \quad \forall g \in L^2(\mathbb{R}^d) \quad (4.10)$$

and therefore $L_{F,\phi}$ as the function which maps f into its representation.

QUESTA PARTE VA MESSA A POSTO IN MODO DA UNIFORMARE I RISULTATI

Proposition 4.2. *If $F \in L^\infty(\mathbb{R}^{2d})$ then $\|L_{F,\phi}\| \leq \|F\|_\infty$.*

Proof. contenuto...

□

Proposition 4.3. *If $F \in L^1(\mathbb{R}^{2d})$ then $\|L_{F,\phi}\| \leq \|F\|_1$.*

Proof. contenuto...

□

Thanks to these result and an interpolation argument one can show that $\mathcal{L}_{F,\phi}$, and hence $L_{F,\phi}$, is bounded also when $F \in L^p(\mathbb{R}^{2d})$ for $1 < p < \infty$. A proof can be found in [21], Proposition 12.3.

Proposition 4.4. *Let $F \in L^p(\mathbb{R}^{2d})$ for $1 < p < \infty$. Then there exist a unique bounded linear operator $L_{F,\phi} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ such that $L_{F,\phi} f$ is given by (4.10) for all $f \in L^2(\mathbb{R}^d)$ and all simple functions F for which $|\{F(x) \neq 0\}| < \infty$. Moreover, the application that maps F into $L_{F,\phi}$ is bounded.*

Since simple functions supported on a set of finite measure are dense in $L^p(\mathbb{R}^{2d})$, through dominated convergence we can conclude that (4.9) holds for every $F \in L^p(\mathbb{R}^{2d})$.

- Definizione
- Operatori di proiezione
- Operatori di localizzazione di Daubechies
- Proprietà di limitatezza e compattezza
- Autovalori e autofunzioni

4.2.1 Spherically Symmetric Weights

In this section we will consider the special case in which the window for the STFT is a Gaussian (3.9) and the weight F is spherically symmetric. Letting $r_j^2 = x_j^2 + \omega_j^2$ for $j = 1, \dots, d$ and $r^2 = (r_1^2, \dots, r_d^2) \in \mathbb{R}^d$, the hypothesis about F can be rephrased in the following way

$$F(x, \omega) = \mathcal{F}(r^2). \quad (4.11)$$

In order to highlight the dependence of F through \mathcal{F} the corresponding localization operator will be denoted as $L_{\mathcal{F}, \varphi}$. For this operators a complete characterization of the spectrum and eigenspaces is given in [3]. Before stating and proving the theorem we need to introduce some special function, namely *Hermite functions*. In dimension $d = 1$ Hermite functions are given by

$$H_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(-\frac{1}{2\sqrt{\pi}} \right)^k e^{\pi t^2} \frac{d^k}{dt^k} \left(e^{-2\pi t^2} \right) \quad (4.12)$$

where $k \in \mathbb{N}$ (\mathbb{N} is supposed to contain also 0). Hermite functions have lots of interesting and useful properties which can be found in [4, Section 1.7]. We cite some of them which will be useful in the following.

- (i) $\{H_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$;
- (ii) $H_0(t) = \varphi(t)$ where φ is the normalized Gaussian given by (3.9);
- (iii) Setting $H_{-1} = 0$, the following recursive relation holds

$$2\sqrt{\pi}tH_k = \sqrt{k+1}H_{k+1} + \sqrt{k}H_{k-1} \quad \text{for } k = 0, 1, \dots; \quad (4.13)$$

- (iv) Hermite functions are eigenfunctions of \mathcal{F} , specifically

$$\mathcal{F}H_k = (-i)^k H_k. \quad (4.14)$$

Hermite functions in generic dimension are given through multiplication of 1-dimensional Hermite functions. Explicitly, given a multi-index $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ the corresponding Hermite function is given by

$$H_k(t) = \prod_{j=1}^d H_{k_j}(t_j). \quad (4.15)$$

It is still true that d -dimensional Hermite functions (now ranging between all possible multi-indices) form an orthonormal basis of $L^2(\mathbb{R}^d)$ and using (4.14) it is easy to see that d -dimensional Hermite functions are still eigenfunction of the Fourier transform, namely

$$\mathcal{F}H_k = (-i)^{|k|}H_k \quad (4.16)$$

where $|k| = k_1 + \dots + k_d$ is the length of the multi-index.

Theorem 4.5. *Eigenfuctions of $L_{\mathcal{F},\varphi}$ are the d -dimensional Hermite functions H_k , with corresponding eigenvalues*

$$\lambda_k = \frac{\pi^{|k|}}{k!} \int_0^{+\infty} \dots \int_0^{+\infty} \mathcal{F}\left(\frac{s_1}{\pi}, \dots, \frac{s_d}{\pi}\right) \left(\prod_{j=1}^d s^{k_j}\right) e^{-(s_1+\dots+s_d)} ds_1 \dots ds_d \quad (4.17)$$

Before proving the theorem we need the following lemma:

Lemma 4.6. *Given $z \in \mathbb{C}$ we have*

$$\int_{\mathbb{R}} e^{-2\pi(t^2+zt)} dt = \frac{1}{2^{1/2}} e^{\pi z^2/2} \quad (4.18)$$

Proof. Letting $z = \operatorname{Re} z + i\operatorname{Im} z = u + iv$ we have:

$$\begin{aligned} \int_{\mathbb{R}} e^{-2\pi(t^2+zt)} dt &= \int_{\mathbb{R}} e^{-2\pi(t^2+ut)} e^{-2\pi ivt} dt = e^{\pi u^2/2} e^{-2\pi(t^2+ut+u^2/4)} e^{-2\pi ivt} dt \\ &= e^{\pi u^2/2} \int_{\mathbb{R}} e^{-2\pi(t+u/2)^2} e^{-2\pi ivt} dt = e^{\pi u^2/2} \mathcal{F}(T_{-u/2} e^{-2\pi(\cdot)^2})(v) \stackrel{2.35(i)+2.36}{=} \\ &= e^{\pi u^2/2} e^{2\pi i(u/2)v} \frac{1}{2^{1/2}} e^{-\pi v^2/2} = \frac{1}{2^{1/2}} e^{\pi(u^2+2iuv-v^2)/2} = \frac{1}{2^{1/2}} e^{\pi z^2/2} \end{aligned}$$

□

Proof of Theorem 4.5. Since Hermite functions are an orthonormal basis of $L^2(\mathbb{R}^n)$ it is sufficient to prove that $\langle L_{\mathcal{F},\varphi} H_k, H_l \rangle = \langle \mathcal{F} \mathcal{V}_{\varphi} H_k, \mathcal{V}_{\varphi} H_l \rangle = \lambda_k \prod_{j=1}^d \delta_{k_j, l_j}$, which means that the scalar product is different from zero if and only if $k = l$. We start by computing

the STFT of an Hermite function

$$\begin{aligned}
 \mathcal{V}_\varphi H_k(x, \omega) &= \int_{\mathbb{R}^d} H_k(t) e^{-2\pi i \omega \cdot t} 2^{d/4} e^{-\pi |t-x|^2} dt = \int_{\mathbb{R}^d} \prod_{j=1}^d H_{k_j}(t_j) e^{-2\pi i \omega \cdot t} 2^{d/4} e^{-\pi |t-x|^2} dt = \\
 &= \prod_{j=1}^d 2^{1/4} \int_{\mathbb{R}} H_{k_j}(t_j) e^{-2\pi i \omega_j t_j} e^{-\pi (t_j - x_j)^2} dt_j = \\
 &= \prod_{j=1}^d 2^{1/4} \int_{\mathbb{R}} \frac{2^{1/4}}{\sqrt{k_j!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^{k_j} e^{\pi t_j^2} \frac{d^{k_j}}{dt^{k_j}} \left(e^{-2\pi t_j^2}\right) e^{-2\pi i \omega_j t_j} e^{-\pi (t_j - x_j)^2} dt_j = \\
 &= \prod_{j=1}^d \frac{2^{1/2}}{\sqrt{k_j!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^{k_j} \int_{\mathbb{R}} \frac{d^{k_j}}{dt^{k_j}} \left(e^{-2\pi t_j^2}\right) e^{\pi (t_j^2 - 2i\omega_j t_j - t_j^2 + 2t_j x_j - x_j^2)} dt_j = \\
 &= \prod_{j=1}^d \frac{2^{1/2}}{\sqrt{k_j!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^{k_j} e^{-\pi x_j^2} \int_{\mathbb{R}} \frac{d^{k_j}}{dt^{k_j}} \left(e^{-2\pi t_j^2}\right) e^{2\pi (x_j - i\omega_j) t_j} dt_j \stackrel{\text{integration by parts}}{=} \\
 &= \prod_{j=1}^d \frac{2^{1/2}}{\sqrt{k_j!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^{k_j} e^{-\pi x_j^2} [2\pi (x_j - i\omega_j)]^{k_j} (-1)^{k_j} \int_{\mathbb{R}} e^{-2\pi [t_j^2 - (x_j - i\omega_j) t_j]} dt_j = \\
 &= \prod_{j=1}^d \sqrt{\frac{\pi^{k_j}}{k_j!}} 2^{1/2} (x_j - i\omega_j)^{k_j} e^{-\pi x_j^2} \frac{1}{2^{1/2}} e^{\pi (x_j - i\omega_j)^2 / 2} = \\
 &= \prod_{j=1}^d \sqrt{\frac{\pi^{k_j}}{k_j!}} (x_j - i\omega_j)^{k_j} e^{-\pi i \omega_j x_j} e^{-\pi (x_j^2 + \omega_j^2) / 2} = \\
 &= \left(\prod_{j=1}^d \sqrt{\frac{\pi^{k_j}}{k_j!}} (x_j - i\omega_j)^{k_j} \right) e^{-\pi i \omega \cdot x} e^{-\pi (r_1^2 + \dots + r_d^2) / 2}
 \end{aligned}$$

Before computing the scalar product between $L_{\mathcal{F}, \varphi} H_k$ and H_l , we introduce the angular coordinate θ_j such that $x_j + i\omega_j = r_j e^{i\theta_j}$. Therefore we have:

$$\begin{aligned}
 \langle L_{\mathcal{F}, \varphi} H_k, H_l \rangle &= \int_{\mathbb{R}^{2d}} F(x, \omega) \left(\prod_{j=1}^d \sqrt{\frac{\pi^{k_j}}{k_j!}} r_j^{k_j} e^{-ik_j \theta_j} \right) e^{-\pi i \omega \cdot x} e^{-\pi (r_1^2 + \dots + r_d^2) / 2} \\
 &\quad \overline{\left(\prod_{m=1}^d \sqrt{\frac{\pi^{l_m}}{l_m!}} r_m^{l_m} e^{-il_m \theta_m} \right) e^{-\pi i \omega \cdot x} e^{-\pi (r_1^2 + \dots + r_d^2) / 2}} dx d\omega = \\
 &= \int_{\mathbb{R}^{2d}} F(x, \omega) \left(\prod_{j=1}^d \sqrt{\frac{\pi^{k_j + l_j}}{k_j! l_j!}} r_j^{k_j + l_j} e^{i(l_j - k_j) \theta_j} \right) e^{-\pi (r_1^2 + \dots + r_d^2) / 2} dx d\omega. \quad (4.19)
 \end{aligned}$$

For every pair of coordinated (x_j, ω_j) we can switch to polar coordinates (r_j, θ_j) . Since $F(x, \omega) = \mathcal{F}(r^2)$ is independent of angular coordinates, only functions depending on those

are $e^{i(l_j-k_j)\theta_j}$, for which:

$$\int_0^{2\pi} e^{i(l_j-k_j)\theta_j} d\theta_j = \begin{cases} \left. \frac{e^{i(l_j-k_j)\theta_j}}{i(l_j-k_j)} \right|_0^{2\pi} = 0 & \text{if } k_j \neq l_j \\ \int_0^{2\pi} d\theta_j = 2\pi & \text{if } k_j = l_j \end{cases}$$

Therefore, if $k \neq l$, the whole integral is 0, otherwise, letting $k_j = l_j$ in (4.19):

$$\begin{aligned} \langle L_{\mathcal{F},\varphi} H_k, H_k \rangle &= (2\pi)^d \frac{\pi^{|k|}}{k!} \int_0^{+\infty} \cdots \int_0^{+\infty} \mathcal{F}(r_1^2, \dots, r_d^2) e^{-\pi(r_1^2 + \cdots + r_d^2)} \left(\prod_{j=1}^d r_j^{2k_j+1} \right) dr_1 \cdots dr_d = \\ &= \frac{1}{k!} \int_0^{+\infty} \cdots \int_0^{+\infty} \mathcal{F}(r_1^2, \dots, r_d^2) e^{-\pi(r_1^2 + \cdots + r_d^2)} \left(\prod_{j=1}^d (\pi r_j^2)^{k_j} \right) \pi r_1 dr_1 \cdots \pi r_d dr_d \end{aligned}$$

where $k! = k_1! \cdots k_d!$. With the change of variable $s_j = \pi r_j^2$, we finally obtain:

$$\langle L_{\mathcal{F},\varphi} H_k, H_k \rangle = \frac{1}{k!} \int_0^{+\infty} \cdots \int_0^{+\infty} \mathcal{F}\left(\frac{s_1}{\pi}, \dots, \frac{s_d}{\pi}\right) \left(\prod_{j=1}^d s_j^{k_j} \right) e^{-(s_1 + \cdots + s_d)} ds_1 \cdots ds_d$$

which is exactly the expression (4.17). \square

Chapter 5

Uncertainty principles

Lemma 5.1. *Let $f \in L^2(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt + \int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi} \sum_{k=0}^{+\infty} (2k+1) |\langle f, H_k \rangle|^2 \quad (5.1)$$

where H_k is the k -th Hermite function. In particular we have

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt + \int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 d\omega \geq \frac{\|f\|_2^2}{2\pi} \quad (5.2)$$

with equality if and only if f is a multiple of H_0 .

Proof. Our aim is to exploit the fact that Hermite functions are an orthonormal basis of $L^2(\mathbb{R})$ by computing $\langle tf, H_k \rangle$. **Before going on we remark that the previous expression is not an L^2 scalar product but is a duality between a tempered distribution and a function in the Schwartz class.** To do so, we just need to use the recursive relation (4.13):

$$\langle tf, H_k \rangle = \langle f, tH_k \rangle = \frac{1}{2\sqrt{\pi}} \left(\sqrt{k+1} \langle f, H_{k+1} \rangle + \sqrt{k} \langle f, H_{k-1} \rangle \right).$$

To compute the similar quantity for \hat{f} we also need to recall that Hermite functions are eigenfunction of the Fourier transform (4.14):

$$\begin{aligned} \langle \omega \hat{f}, H_k \rangle &= \langle \hat{f}, \omega H_k \rangle = \frac{1}{2\sqrt{\pi}} \left(\sqrt{k+1} \langle \hat{f}, H_{k+1} \rangle + \sqrt{k} \langle \hat{f}, H_{k-1} \rangle \right) \stackrel{(4.14) + \mathcal{F} \text{ unitary}}{=} \\ &= \frac{1}{2\sqrt{\pi}} \left(\sqrt{k+1} (-i)^{k+1} \langle f, H_{k+1} \rangle + \sqrt{k} (-i)^{k-1} \langle f, H_{k-1} \rangle \right) = \\ &= \frac{1}{2\sqrt{\pi}} (-i)^{k-1} \left(\sqrt{k} \langle f, H_{k-1} \rangle - \sqrt{k+1} \langle f, H_{k+1} \rangle \right) \end{aligned}$$

Now, since H_k form an orthonormal basis of $L^2(\mathbb{R})$ we can use Parseval's identity:

$$\begin{aligned} \int_{\mathbb{R}} t^2 |f(t)|^2 dt + \int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 d\omega &= \sum_{k=0}^{+\infty} \left(|\langle tf, H_k \rangle|^2 + |\langle \omega \hat{f}, H_k \rangle|^2 \right) = \\ &= \frac{1}{2\pi} \sum_{k=0}^{+\infty} \left[(k+1) |\langle f, H_{k+1} \rangle|^2 + k |\langle f, H_{k-1} \rangle|^2 \right] = \\ &= \frac{1}{2\pi} \sum_{k=0}^{+\infty} (2k+1) |\langle f, H_k \rangle|^2. \end{aligned}$$

Since $(2k+1) \geq 1$ inequality (5.2) immediately proven, and equality hold if and only if $\langle f, H_k \rangle = 0$ for every $k > 1$, which means exactly that f is a multiple of H_0 . \square

Lemma 5.2. *Let $f \in L^2(\mathbb{R}^d)$. Then, for every $j = 1, \dots, d$:*

$$\int_{\mathbb{R}^d} t_j^2 |f(t)|^2 dt + \int_{\mathbb{R}^d} \omega_j^2 |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi} \sum_{k \in \mathbb{N}_0^d} (2k_j + 1) |\langle f, H_k \rangle|^2 \quad (5.3)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and H_k is the d -dimensional Hermite functions related to the multi-index $k \in \mathbb{N}_0^d$.

Proof. contenuto... \square

If one of the integrals in (5.2) is infinite the inequality is trivially satisfied, therefore we can focus our attention for those function for which the integrals are finite. This lemma is sufficient in order to prove Heisenberg's uncertainty principle.

Theorem 5.3. *Let $f \in L^2(\mathbb{R}^d)$ and $a, b \in \mathbb{R}^d$. Then:*

$$\left(\int_{\mathbb{R}^d} |t - a|^2 |f(t)|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}^d} |\omega - b|^2 |\hat{f}(\omega)|^2 d\omega \right)^{1/2} \geq \frac{d \|f\|_2^2}{4\pi} \quad (5.4)$$

where $|t - a| = \sum_{j=1}^d (t_j - a_j)^2$ is the Euclidean distance.

Proof. Firstly, we notice that it is sufficient to prove the inequality when $a = b = 0$, since the generic case can be recovered from this special one by means of a phase-space translation. Indeed, given $f \in L^2(\mathbb{R}^d)$ we can consider $g = M_{-b} T_{-a} f$ for which we have:

$$\begin{aligned} |f(t)|^2 &= |(T_a M_b g)(t)|^2 = |e^{2\pi i b \cdot (t-a)} g(t-a)|^2 = |g(t-a)|^2 \\ |\hat{f}(\omega)|^2 &= |\mathcal{F}(T_a M_b g)(\omega)|^2 \stackrel{2.35}{=} |M_{-a} T_b \hat{g}(\omega)|^2 = |e^{-2\pi i a \cdot \omega} \hat{g}(\omega - b)|^2 = |\hat{g}(\omega - b)|^2 \end{aligned}$$

and $\|f\|_2 = \|g\|_2$. In light of we will consider $a = b = 0$.

We start considering the 1-dimensional case. We observe that in the left-hand side of (5.4) we have the product of two integrals, while in (5.2) we have an estimate for the sum of these. The transition from the latter to the former estimate can be done through a dilation argument. Precisely, given $f \in L^2(\mathbb{R})$, we consider the following dilation:

$$g(t) = \lambda^{-1/2} f(t/\lambda) \implies \hat{g}(\omega) = \lambda^{1/2} \mathcal{F}(D_{1/\lambda} g) \stackrel{2.35(iii)}{=} \lambda^{1/2} \hat{f}(\lambda \omega).$$

We remark that this dilation is different from the one considered in Section 2.2, expression (2.24). Indeed, now we chose the dilation so that $\|g\|_2 = \|f\|_2$, while previously the dilation was chosen in order to preserve the L^1 norm. Putting g in (5.2) provides us:

$$\begin{aligned} \frac{\|f\|_2^2}{2\pi} &\leq \int_{\mathbb{R}} t^2 |g(t)|^2 dt + \int_{\mathbb{R}} \omega^2 |\hat{g}(\omega)|^2 d\omega = \\ &= \frac{1}{\lambda} \int_{\mathbb{R}} t^2 |f(t/\lambda)|^2 dt + \lambda \int_{\mathbb{R}} \omega^2 |\hat{f}(\lambda\omega)|^2 d\omega = \\ &= \lambda^2 \int_{\mathbb{R}} t^2 |f(t)|^2 dt + \frac{1}{\lambda^2} \int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 d\omega. \end{aligned}$$

We can choose λ in order to minimize the last expression. Thus, deriving with respect to λ^2 and putting the derivative to 0 we obtain that the minimum is achieved when $\lambda^2 = (\int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 d\omega)^{1/2} (\int_{\mathbb{R}} t^2 |f(t)|^2 dt)^{-1/2}$. If we substitute this λ into the last expression we obtain the following inequality

$$\left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 d\omega \right)^{1/2} \geq \frac{\|f\|_2^2}{4\pi} \quad (5.5)$$

which is exactly (5.4) for $d = 1$. \square

For a generic function $f \in L^2(\mathbb{R}^d)$ the left-hand side of (5.4) may be infinite, in which case the statement is trivially satisfied. We shall comment a mathematical interpretation of Heisenberg's uncertainty principle. This can be written in the following form

$$\left(\int_{\mathbb{R}^d} |t - a|^2 \frac{|f(t)|^2}{\|f\|_2^2} dt \right)^{1/2} \left(\int_{\mathbb{R}^d} |\omega - b|^2 \frac{|\hat{f}(\omega)|^2}{\|\hat{f}\|_2^2} d\omega \right)^{1/2} \geq \frac{d}{4\pi}$$

so we may directly assume that f is normalized. In such a case $|f|^2$ can be seen as a probability distribution. If these integrals are finite for some a and b are always finite and their minimum is achieved when

$$a = \bar{t} = \int_{\mathbb{R}^d} t |f(t)|^2 dt, \quad b = \bar{\omega} = \int_{\mathbb{R}^d} \omega |\hat{f}(\omega)|^2 d\omega$$

which are the mean of $|f|^2$ and $|\hat{f}|^2$, respectively. In this case previous integrals represent the standard deviation of $|f|^2$ and $|\hat{f}|^2$, which we indicate with $\Delta_x f$ and $\Delta_\omega f$. It is fair to believe that a function $|f|^2$ is mostly concentrated around its mean and that its standard deviation is a measure of how spread it is. In light of these arguments, Heisenberg's uncertainty principle can be written as

$$\Delta_x f \cdot \Delta_\omega f \geq \frac{d}{4\pi}$$

In this form the uncertainty principle has a heuristic yet meaningful interpretation: *a function and its Fourier transform can not be simultaneously too concentrated.*

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5.1 Donoho-Stark's uncertainty principle

Definition 5.4. A function $f \in L^2(\mathbb{R}^d)$ is ε -**concentrated** on a measurable set $T \subseteq \mathbb{R}^d$ if

$$\left(\int_{T^c} |f(t)|^2 dt \right)^{1/2} \leq \varepsilon \|f\|_2$$

where $T^c = \mathbb{R}^d \setminus T$ denotes the complement set of T .

Theorem 5.5 (Donoho-Stark's uncertainty principle). Let $f \in L^2(\mathbb{R}^d) \setminus \{0\}$, suppose that f is ε_T -concentrated on $T \subseteq \mathbb{R}^d$ while \hat{f} is ε_Ω -concentrated on $\Omega \subseteq \mathbb{R}^d$. Then

$$|T| |\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2 \quad (5.6)$$

Proof. The result is trivial if T or Ω have infinite measure. Hence we will suppose that they both have finite measure.

Concentration can be stated in an equivalent way through projection operators introduced in section 4.1, indeed:

$$\begin{aligned} \left(\int_{T^c} |f(t)|^2 dt \right)^{1/2} &= \|f - \chi_T f\|_2 = \|f - P_T f\|_2 \leq \varepsilon_T \|f\|_2 \\ \left(\int_{\Omega^c} |\hat{f}(\omega)|^2 d\omega \right)^{1/2} &= \|\hat{f} - \chi_\Omega \hat{f}\|_2 = \|f - \mathcal{F}^{-1}(\chi_\Omega \hat{f})\|_2 = \|f - Q_\Omega f\|_2 \leq \varepsilon_\Omega \|f\|_2 \end{aligned}$$

In section 4.1 we also noticed that $\|Q_\Omega\| \leq 1$, hence

$$\begin{aligned} \|f - Q_\Omega P_T f\|_2 &= \|f - Q_\Omega f + Q_\Omega f - Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega f\|_2 + \|Q_\Omega(f - P_T f)\|_2 \leq \\ &\leq \|f - Q_\Omega f\|_2 + \|(f - P_T f)\|_2 \leq (\varepsilon_\Omega + \varepsilon_T) \|f\|_2. \end{aligned}$$

and consequently

$$\begin{aligned} \|f\|_2 &= \|f - Q_\Omega P_T f + Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega P_T f\|_2 + \|Q_\Omega P_T f\|_2 \implies \\ \implies \|Q_\Omega P_T f\|_2 &\geq \|f\|_2 - \|f - Q_\Omega P_T f\|_2 \geq (1 - \varepsilon_\Omega - \varepsilon_T) \|f\|_2 \end{aligned}$$

Thanks to Proposition 4.1 we know that $\|Q_\Omega P_T\|_{\text{HS}} = \sqrt{|T| |\Omega|}$ and from Theorem ?? we know that $\|Q_\Omega P_T\| \leq \|Q_\Omega P_T\|_{\text{HS}}$, therefore

$$(1 - \varepsilon_\Omega - \varepsilon_T) \|f\|_2 \leq \|Q_\Omega P_T f\|_2 \leq \sqrt{|T| |\Omega|} \|f\|_2.$$

□

5.2 Lieb's uncertainty principle

Up to now we presented two uncertainty principles related to the Fourier transform. However uncertainty principles can be stated for every time of time-frequency analysis. In this and in the following section we present some uncertainty principles for the STFT.

We start considering a simple form uncertainty principle for the STFT. After this we will see how to turn Lieb's inequality 3.4 into an uncertainty principle.

Proposition 5.6. *Let $f, \phi \in L^2(\mathbb{R}^d)$ normalized, $U \subseteq \mathbb{R}^{2d}$ and $\varepsilon \geq 0$. Suppose that*

$$\int_U |\mathcal{V}_\phi f(x, \omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

Then $|U| \geq 1 - \varepsilon$.

Proof. From (3.4) we see that $|\mathcal{V}_\phi f(x, \omega)| \leq 1$ for all $(x, \omega) \in \mathbb{R}^{2d}$, therefore

$$1 - \varepsilon \leq \int_U |\mathcal{V}_\phi f(x, \omega)|^2 dx d\omega \leq \|\mathcal{V}_\phi f\|_\infty^2 |U| \leq |U|. \quad (5.7)$$

□

Theorem 5.7 (Lieb's uncertainty principle). *Suppose that $\|f\|_2 = \|\phi\|_2 = 1$. If $U \subseteq \mathbb{R}^{2d}$ and $\varepsilon \geq 0$ are such that*

$$\int_U |\mathcal{V}_\phi f(x, \omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

Then

$$|U| \geq (1 - \varepsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-1}} \quad \text{for every } p > 2.$$

Proof. If $|U| = \infty$ the result is trivial hence we can suppose that U has finite measure. We start using Hölder's inequality with exponents $p/2$ and $(p/2)' = p/(p-2)$

$$\begin{aligned} 1 - \varepsilon &\leq \int_U |\mathcal{V}_\phi f(x, \omega)|^2 dx d\omega = \int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x, \omega)|^2 \chi_U(x, \omega) dx d\omega \stackrel{\text{Hölder}}{\leq} \\ &\leq \left(\int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x, \omega)|^{2\frac{p}{2}} dx d\omega \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^{2d}} \chi_U(x, \omega)^{\frac{p}{p-2}} dx d\omega \right)^{\frac{p-2}{p}} \stackrel{(3.5)}{\leq} \\ &\leq \left(\frac{2}{p}\right)^{\frac{2d}{p}} \|f\|_2^2 \|g\|_2^2 |U|^{\frac{p-2}{p}} = \left(\frac{2}{p}\right)^{\frac{2d}{p}} |U|^{\frac{p-2}{p}}. \end{aligned}$$

We point out that the use of Hölder's inequality is justified because U has finite measure and, since $\mathcal{V}_\phi f \in L^q(\mathbb{R}^{2d})$ for every $q \geq 2$, $|\mathcal{V}_\phi f|^2 \in L^q(\mathbb{R}^{2d})$ for every $q \geq 1$. □

5.3 Nicola-Tilli's uncertainty principle or Faber-Krahn Inequality for the STFT

Theorem from [15]

Theorem 5.8. *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V}f(x, \omega)|^2 dx d\omega \leq G(|\Omega|)$$

where $G(s)$ is given by

$$G(s) := \int_0^s e^{(-d!\tau)^{1/d}} d\tau \quad (5.8)$$

Moreover, equality occurs if and only if f is a Gaussian of the kind

$$f(x) = ce^{2\pi x \cdot \omega_0} \varphi(x - x_0) \quad x \in \mathbb{R}^d \quad (5.9)$$

for some unimodular $c \in \mathbb{C}$ and some $(x_0, \omega_0) \in \mathbb{R}^{2d}$ and Ω is equivalent, in measure, to a ball of centre (x_0, ω_0) .

Chapter 6

Recent results

6.1 Norm of localization operators: results from Nicola-Tilli

In Section 4.2 we obtained some basic results for the norm of Daubechies' localization operators, independently of the choice of the window ϕ for the STFT. It is reasonable to think that, for specific windows those estimates can be improved and, hopefully, find some sharp bounds. Thanks to 5.8 Nicola and Tilli accomplished this task in the case the window of the STFT is a normalized Gaussian. As done previously, since the windows is fixed one for all we will drop the pedex ϕ .

As for the results in Section 4.2, assumptions on the weight function F are related to its integrability and boundedness. The problem we are consider is, in fact, finding an optimal estimate of the type

$$\|L_F\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq C \quad (6.1)$$

where F satisfies the following constraints:

$$\|F\|_\infty \leq A \quad \text{and} \quad \|F\|_p \leq B. \quad (6.2)$$

Clearly the constant C will depend on p , A and B . In [16] this problem is completely solved: the constant C is computed (explicitly in some cases), weight functions F which achieve this bound are explicitly found and also function f and g such that $|\langle L_f f, g \rangle| = \|L_F\| = C$ are found. Before reporting the main Theorem in [16], we define the following number which will appear many times in the following

$$\kappa_p := \frac{p-1}{p}. \quad (6.3)$$

Theorem 6.1. *Assume $p \in [1, \infty)$, $A \in (0, \infty]$ and $B \in (0, \infty)$ with the additional condition that $A < \infty$ when $p = 1$. Let F satisfy the constraints in (6.2).*

(i) *If $p = 1$, then*

$$\|L_F\| \leq A G(B/A) \quad (6.4)$$

and equality occurs if and only if, for some $\theta \in \mathbb{R}$ and some $z_0 \in \mathbb{R}^{2d}$

$$F(z) = Ae^{i\theta} \chi_{\mathcal{B}}(z - z_0) \quad \forall z \in \mathbb{R}^{2d} \quad (6.5)$$

where $\mathcal{B} \subset \mathbb{R}^{2d}$ is the ball of measure B/A centred at the origin.

(ii) If $p > 1$ and $\frac{B}{A} \leq \kappa_p^{d/p}$, then

$$\|L_F\| \leq \kappa_p^{d\kappa_p} B, \quad (6.6)$$

with equality if and only if, for some $\theta \in \mathbb{R}$ and some $z_0 \in \mathbb{R}^{2d}$,

$$F(z) = e^{i\theta} \lambda e^{\frac{\pi}{p-1}|z-z_0|^2} \quad \forall z \in \mathbb{R}^{2d} \quad (6.7)$$

where $\lambda = \kappa_p^{-d/p} B$.

(iii) If $p > 1$ and $\frac{B}{A} > \kappa_p^{d/p}$, then

$$\|L_F\| \leq \int_0^A G(u_\lambda(t)) dt, \quad (6.8)$$

where $u_\lambda(t) = \left[-\log \left(\left(\frac{t}{\lambda} \right)^{p-1} \right) \right]^d$ and $\lambda > A$ is uniquely determined by the condition $p \int_0^A t^{p-1} u_\lambda(t) dt = B^p$. Equality in (6.8) if and only if, for some $\theta \in \mathbb{R}$ and some $z_0 \in \mathbb{R}^{2d}$,

$$F(z) = e^{i\theta} \min \{ \lambda e^{-\frac{\pi}{p-1}|z-z_0|^2}, A \} \quad (6.9)$$

Finally, in all the cases, condition $|\langle L_F f, g \rangle| = \|L_F\|$ holds for some, $f, g \in L^2(\mathbb{R}^d)$ such that $\|f\|_2 = \|g\|_2 = 1$, if and only if both f and g are of the kind (5.9), possibly with different c 's, but with the same $(x_0, \omega_0) \in \mathbb{R}^{2d}$ which coincides with the centre of F .

We will not give the proof of these results since some of its parts are similar to the one we will see in the following section. Moreover, we point out that the case $A = \infty$ means we are dropping the L^∞ constraint.

6.2 Generic case

In this Section we will deal with a generalized version of the problem considered in [16]. We want to find the optimal constant C such that

$$\|L_F\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq C$$

under the following constraints on F :

$$\|F\|_p \leq A \quad \text{and} \quad \|F\|_q \leq B. \quad (6.10)$$

where $p, q \in (1, \infty)$ and $A, B \in (0, \infty)$. In this setting it is no more possible to find an explicit expression for C and F , **although they can be easily computed numerically**.

Theorem 6.1(ii) includes the case when F satisfies just an L^p constraint by taking A (L^∞ constraint) equal to ∞ . In the current setting we have an L^p and an L^q bound, hence, thanks to 6.6, it is straightforward to see that

$$\|L_F\|_{L_2 \rightarrow L_2} \leq \min\{\kappa_p^{d\kappa_p} A, \kappa_q^{d\kappa_q} B\}.$$

Suppose that the first term is smaller than the second, which means:

$$\kappa_p^{d\kappa_p} A \leq \kappa_q^{d\kappa_q} B \iff \frac{B}{A} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d \quad (6.11)$$

Clearly, for B sufficiently large we expect that the solution of current problem is the same as the one with just an L^p constraint, namely the one given by (6.7). Therefore, we want to compare its L^q norm with the bound given by B :

$$\begin{aligned} \|F\|_q^q &= \int_{\mathbb{R}^{2d}} |F(z)|^q dz = \lambda^q \int_{\mathbb{R}^{2d}} e^{-\frac{q\pi}{p-1}|z-z_0|^2} dz \stackrel{z' = \left(\frac{q\pi}{p-1}\right)^{1/2}(z-z_0)}{=} \\ &= \lambda^q \left(\frac{p-1}{q\pi}\right)^d \int_{\mathbb{R}^{2d}} e^{-|z'|^2} dz' = \lambda^q \left(\frac{p-1}{q\pi}\right)^d \pi^d = \lambda^q \left(\frac{p-1}{q}\right)^d. \end{aligned}$$

Since we want F to satisfy the L^q constraint we should have

$$\lambda \left(\frac{p-1}{q}\right)^{d/q} \leq B \stackrel{\lambda = \kappa_p^{-d/p} A}{\implies} \left(\frac{p}{p-1}\right)^{d/p} \left(\frac{p-1}{q}\right)^{d/q} A \leq B$$

which is equivalent to

$$\frac{B}{A} \geq \kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{q}}.$$

If this condition were less restrictive than the one given by (6.11) we would have solved the problem. Unfortunately, this is not the case. Indeed it is always true, regardless of p and q , that

$$\kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{q}} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d \quad (6.12)$$

The proof of this inequality can be found in A.2.

Despite this fact, at least we can say that, if $B/A \geq \kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{q}}$ or $B/A \leq \kappa_q^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{p}}$, the problem is already solved and the solution is given by Theorem 6.1. Therefore, from now on, we will suppose to be in the intermediate case, that is

$$\kappa_q^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{p}} < \frac{B}{A} < \kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{q}} \quad (6.13)$$

We notice that the condition is well-posed, since it is actually true that

$$\kappa_q^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{p}} < \kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{q}} \quad (6.14)$$

whenever $p \neq q$ (proof is in A.3).

Before tackling the problem in this intermediate regime, we prove a Theorem from [16] which gives a bound for $\|L_F\|$ in terms of the distribution function of $|F|$.

Theorem 6.2. Assume $F \in L^p(\mathbb{R}^{2d})$ for some $p \in [1, +\infty)$ and let $\mu(t) = |\{|F| > t\}|$ be the distribution function of $|F|$. Then

$$\|L_F\| \leq \int_0^\infty G(\mu(t))dt \quad (6.15)$$

Equality occurs if and only if $F(z) = e^{i\theta}\rho(|z - z_0|)$ for some $\theta \in \mathbb{R}$, $z_0 \in \mathbb{R}^{2d}$ and some nonincreasing function $\rho : [0, +\infty) \rightarrow [0, +\infty)$

Proof. For the sake of brevity we denote the variable $(x, \omega) \in \mathbb{R}^{2d}$ as z and therefore $dx d\omega$ as dz . Let $f, g \in L^2(\mathbb{R}^d)$ such that $\|f\|_2 = \|g\|_2 = 1$. Since we are in a Hilbert space $\|L_F\|$ can be computed as the supremum of $|\langle L_F f, g \rangle|$ over all normalized f and g . Therefore we are interested in estimating the previous scalar product

$$\begin{aligned} |\langle L_F f, g \rangle| &= |\mathcal{L}_F(f, g)| \leq \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)| \cdot |\mathcal{V}g(z)| dz \stackrel{\text{C-S}}{\leq} \\ &\leq \left(\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz \right)^{1/2} \left(\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}g(z)|^2 dz \right)^{1/2} \end{aligned} \quad (6.16)$$

Since the result is symmetric in f and g we can study just one of the terms. Letting $m = \text{ess sup } |F(z)|$ and assuming $m > 0$ (otherwise every result is trivial) we can use the “layer cake” representation [13, Theorem 1.13]:

$$|F(z)| = \int_0^m \chi_{\{|F| > t\}}(z) dt$$

in order to find

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz &= \int_{\mathbb{R}^{2d}} \left(\int_0^m \chi_{\{|F| > t\}}(z) dt \right) |\mathcal{V}f(z)|^2 dz \stackrel{\text{Tonelli}}{=} \\ &= \int_0^m \left(\int_{\mathbb{R}^{2d}} \chi_{\{|F| > t\}}(z) |\mathcal{V}f(z)|^2 dz \right) dt = \int_0^m \left(\int_{\{|F| > t\}} |\mathcal{V}f(z)|^2 dz \right) dt \end{aligned}$$

We notice that the quantity in the inner integral is exactly the one in the theorem 5.8, hence

$$\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz \leq \int_0^m G(|\{|F| > t\}|) dt = \int_0^m G(\mu(t))dt \quad (6.17)$$

We point out that since $\mu(t) = 0$ for $t > m$ and that $G(0) = 0$, the previous expression is equivalent to (6.15).

Because $p < \infty$, L_F is a compact operator, there exist normalized f and g which achieve equality in the supremum of the norm, namely $\langle L_F f, g \rangle = \|L_F\|$. Therefore equality in (6.15) occurs if and only if all the previous inequalities become equalities. Equality in (6.17) occurs if and only if

$$\int_{\{|F| > t\}} |\mathcal{V}f(z)|^2 dz = G(\mu(t)) \quad (6.18)$$

for a.e. $t \in (0, m)$. Thanks to Theorem 5.8, for just one $t_0 \in (0, m)$ we can infer that $\{|F| > t_0\}$ is (equivalent to) a ball centred in $z_0 = (x_0, \omega_0)$ and that f is a Gaussian of the kind (5.9) with the same centre z_0 . Then, still by theorem 5.8, since (6.18) holds a.e. in $(0, m)$ and that f is always the same we have that also the other levels sets $\{|F| > t\}$ are equivalent to balls centred at the same z_0 . Finally, we can extend the result to every $t \in (0, m)$ because $\{|F| > t\} = \bigcup_{s>t} \{|F| > s\}$. Since Theorem 5.8 is a “if and only if”, these conditions on F and f are also sufficient to guarantee equality in (6.17). Clearly the same result holds for g which has to be a Gaussian, possibly with different coefficient c but the same centre.

In the end it turns that $|F|$ is spherically symmetric and radially decreasing as claimed in theorem’s statement.

Conditions for f and g imply that $\mathcal{V}g = e^{i\alpha}\mathcal{V}f$ for some $\alpha \in \mathbb{R}$. This provides equality in (6.16) when using Cauchy-Schwarz inequality. Lastly we shall prove that also the first inequality in (6.16), that is

$$\left| \int_{\mathbb{R}^{2d}} F(z) \mathcal{V}f(z) \overline{\mathcal{V}g(z)} dz \right| \leq \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)| \cdot |\mathcal{V}g(z)| dz$$

becomes an equality, which is true if and only if

$$e^{-i\theta} \int_{\mathbb{R}^{2d}} F(z) \cdot |\mathcal{V}f(z)|^2 dz = \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz$$

for some $\theta \in \mathbb{R}$. This, in turn, is equivalent to the condition

$$e^{-i\theta} F(z) \cdot |\mathcal{V}f(z)|^2 = |F(z)| \cdot |\mathcal{V}f(z)|^2 \quad \text{for a.e. } z \in \mathbb{R}^{2d}$$

but since $|\mathcal{V}f(z)|^2 > 0$, equality in 6.16 with f and g as (5.9) occurs if and only if $F(z) = e^{i\theta}|F(z)|$. \square

In light of the previous Theorem it is natural to seek for sharp upper bound for the right-hand side of (6.15). Since this involves the distribution function $|F|$ we shall search this bound between all the possible distribution functions. In order to do so we need to rephrase constraints (6.10) in terms of μ . This can be easily done one more time thanks to a more general version of the “layer cake” representation (see [13, Theorem 1.13] or [7, Proposition 1.1.4]):

$$\|F\|_p^p = p \int_0^\infty t^{p-1} |\{|F| > t\}| dt.$$

Hence, constraints (6.10) become

$$p \int_0^\infty t^{p-1} u(t) dt \leq A^p \quad \text{and} \quad q \int_0^\infty t^{q-1} u(t) dt \leq B^q \quad (6.19)$$

and we can define the proper space of possible distribution functions

$$\mathcal{C} = \{u : (0, +\infty) \rightarrow [0, +\infty) \text{ such that } u \text{ is decreasing and satisfies (6.19)}\}. \quad (6.20)$$

Up to now we have rephrased our original question in terms of the following variational problem:

$$\sup_{v \in \mathcal{C}} I(v) \quad \text{where} \quad I(v) := \int_0^{+\infty} G(v(t))dt \quad (6.21)$$

Firstly, we shall prove existence of maximizers.

Proposition 6.3. *The supremum in (6.21) is finite and it is attained by at least one function $u \in \mathcal{C}$. Moreover, every extremal function u achieves equality in at least one of the constraints (6.19).*

Proof. Considering, for example, the first constraint in (6.19), we see that

$$t^p u(t) = p \int_0^t \tau^{p-1} u(\tau) d\tau \stackrel{u \text{ decreasing}}{\leq} p \int_0^t \tau^{p-1} u(\tau) d\tau \leq A^p$$

hence functions in \mathcal{C} are pointwise bounded by A^p/t^p . It is straightforward to verify that G in (5.8) is increasing, that $G(s) \leq s$ and that $G(s) \leq 1$. Using these properties we have:

$$\begin{aligned} I(u) &= \int_0^{+\infty} G(u(t))dt = \int_0^1 G(u(t))dt + \int_1^{+\infty} G(u(t))dt \stackrel{G(s) \leq 1}{\leq} 1 + \int_1^{+\infty} G(u(t))dt \stackrel{G \text{ increasing}}{\leq} \\ &\leq 1 + \int_1^{+\infty} G(A^p/t^p)dt \stackrel{G(s) \leq s}{\leq} 1 + \int_1^{+\infty} \frac{A^p}{t^p} dt < \infty \end{aligned}$$

therefore the supremum in (6.21) is finite.

Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ be a maximizing sequence. Since every u_n is pointwise bounded by A^p/t^p , thanks to **Helly's selection theorem** we can say that, up to a subsequence, u_n converges pointwise to a decreasing function u , which is still in \mathcal{C} , indeed:

$$\int_0^{+\infty} t^{p-1} u(t) dt = \int_0^{+\infty} \lim_{n \rightarrow \infty} t^{p-1} u_n(t) dt \stackrel{\text{Fatou's lemma}}{\leq} \liminf_{n \rightarrow \infty} \int_0^{+\infty} t^{p-1} u_n(t) dt \leq \frac{A^p}{p}$$

and the same holds for q instead of p .

Now we have to prove that u is actually achieving the supremum. We already saw that the following holds:

$$|G(u_n(t))| \leq \chi_{(0,1)}(t) + \frac{A^p}{t^p} \chi_{(1,+\infty)}(t)$$

and that the left-hand side is a function in $L^1(0, +\infty)$. This allows us to use dominated convergence theorem to conclude that

$$I(u) = \int_0^{+\infty} G(u(t))dt = \lim_{n \rightarrow \infty} \int_0^{+\infty} G(u_n(t))dt = \lim_{n \rightarrow \infty} I(u_n) = \sup_{v \in \mathcal{C}} I(v)$$

Lastly we need to show that u achieves equality at least in one of the constraints (6.19). Suppose that this is not true. If we let $u_\varepsilon(t) = (1 + \varepsilon)u(t)$, then for $\varepsilon > 0$ sufficiently small constraints are still satisfied and since G is strictly increasing $I(u_\varepsilon) > I(u)$, which contradicts the hypothesis that u is a maximizer. \square

In order to do some “meaningful” calculus of variations over I we need to enlarge \mathcal{C} , because the monotonicity assumption is quite strict. We will show that removing this hypothesis leaves the supremum unchanged and that maximizers are indeed monotonic.

Proposition 6.4. *Let $\mathcal{C}' = \{u : (0, +\infty) \rightarrow [0, +\infty)\}$ such that u is measurable and satisfies (6.19). Then*

$$\sup_{v \in \mathcal{C}} I(v) = \sup_{v \in \mathcal{C}'} I(v). \quad (6.22)$$

In particular, any function $u \in \mathcal{C}$ achieving the supremum on the left-hand side also achieves it on the right-hand side.

Proof. Let $u \in \mathcal{C}'$. We define its *decreasing rearrangement* as

$$u^*(s) = \sup\{t \geq 0 : |\{u > t\}| > s\} \quad (6.23)$$

AGGIUNGERE DIMOSTRAZIONE IN APPENDICE?? □

Before stating and proving the theorem that gives the only maximal function of (6.21), we introduce the following notation:

$$\text{Log}_-(x) = \max\{-\log(x), 0\}, \quad x > 0.$$

Theorem 6.5. *There exist a unique function $u \in \mathcal{C}$ achieving the supremum in (6.21) that is:*

$$u(t) = \frac{1}{d!} \left[\text{Log}_- \left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1} \right) \right]^d, \quad t > 0 \quad (6.24)$$

where λ_1, λ_2 are both positive and uniquely determined by

$$p \int_0^{+\infty} t^{p-1} u(t) dt = A^p, \quad q \int_0^{+\infty} t^{q-1} u(t) dt = B^q$$

Proof. We will split the proof in several parts. Firstly we will show that maximizers are given by (6.24). Then we will show that multipliers λ_1 and λ_2 are both strictly positive and unique.

Let $M = \sup\{t \in (0, +\infty) : u(t) > 0\}$. From Proposition 6.3 we know that u has to achieve at least one of the constraints, therefore $M > 0$. Consider now a closed interval $[a, b] \subset (0, M)$ and a function $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Without loss of generality we can suppose that η is orthogonal, in the L^2 sense, to t^{p-1} and t^{q-1} , explicitly

$$\int_a^b t^{p-1} \eta(t) dt = 0, \quad \int_a^b t^{q-1} \eta(t) dt = 0. \quad (6.25)$$

On $[a, b]$ we have that $u(t) \geq u(b) > 0$, hence, for $|\varepsilon|$ sufficiently small, $u + \varepsilon\eta$ is still a nonnegative function which satisfies (6.19), therefore $u + \varepsilon\eta \in \mathcal{C}'$. Since we are supposing that u is a maximizer, the function $\varepsilon \mapsto I(u + \varepsilon\eta)$ has a maximum for $\varepsilon = 0$. Since η is supported in a compact interval we can differentiate under the integral sign and obtain

$$0 = \frac{d}{d\varepsilon} I(u + \varepsilon\eta)|_{\varepsilon=0} = \int_a^b G'(u(t)) \eta(t) dt.$$

We would like to extend this result to every η in $L^2(a, b)$ satisfying (6.25). Since $L^\infty(a, b)$ is dense in $L^2(a, b)$, there exist a sequence $\{\eta_k\}_{k \in \mathbb{N}} \subset L^\infty(a, b)$ such that $\eta_k \rightarrow \eta$ in $L^2(a, b)$. We can consider the projection operator P such that, given $\psi \in L^2(a, b)$, $P\psi$ is the orthogonal projection of ψ onto $X = \text{span}\{t^{p-1}, t^{q-1}\}^\perp \subset L^2(a, b)$. Since P is continuous we have that $P\eta_k \rightarrow P\eta = \eta$, hence

$$\begin{aligned} 0 &= \int_a^b G'(u(t))P\eta_k(t)dt = \langle G'(u), P\eta_k \rangle_{L^2(a,b)} \rightarrow \langle G'(u), \eta \rangle_{L^2(a,b)} = \int_a^b G'(u(t))\eta(t)dt \implies \\ &\implies \int_a^b G'(u(t))\eta(t)dt = 0. \end{aligned} \quad (6.26)$$

Since (6.26) holds for every $\eta \in X$ it must be that

$$G'(u) \in X^\perp = \left(\text{span}\{t^{p-1}, t^{q-1}\}^\perp \right)^\perp = \text{span}\{t^{p-1}, t^{q-1}\} \quad \text{in } (a, b).$$

By letting $a \rightarrow 0^+$ and $b \rightarrow M^-$ we then obtain

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \quad \text{for a.e. } t \in (0, M) \quad (6.27)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Since u is decreasing actually (6.27) holds for every $t \in (0, M)$. We point out that this argument enables us to say that maximizers have to achieve equality in both constraints in (6.19). Indeed, if, for example, we had that $q \int_0^\infty t^{q-1} u(t) dt < B^q$, the second condition of orthogonality in (6.25) could be removed (since, for sufficiently small ε , a variation non-orthogonal to t^{q-1} would be admissible), providing us with the solution of the same variational problem but without the L^q constraint. Since we already know that actually this solution does not satisfy the L^q constraint we conclude that u has to achieve equality in both constraints. With the very same thinking we can say that neither λ_1 nor λ_2 can be 0.

Recalling the expression of (5.8) we see that $G'(s) = e^{-(d!s)^{1/d}}$ and easily invert (6.27), thus obtaining

$$u(t) = \begin{cases} \frac{1}{d!} [-\log(\lambda_1 t^{p-1} + \lambda_2 t^{q-1})]^d & t \in (0, M) \\ 0 & t \in (M, +\infty) \end{cases} \quad (6.28)$$

We remark that a priori it was possible that $M = +\infty$, but from the explicit expression of maximizers we see that this is not possible since u has to be nonnegative.

Our main goal now is to show that multipliers λ_1, λ_2 are both positive and unique since this will give us uniqueness for the maximizer.

We start by proving that both multipliers are positive. Suppose that one of them, for example λ_2 , is negative. Consider an interval $[a, b] \subset (0, M)$ and a variation $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Thanks to the Gram-Schmidt process we can construct a variation orthogonal to t^{p-1} . Since η is arbitrary we can also suppose that it is not orthogonal to t^{q-1} , in particular we can ask that $\int_a^b t^{q-1} \eta(t) dt < 0$. Therefore, the directional derivative of G along η is:

$$\int_a^b G'(u(t))\eta(t)dt = \int_a^b (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \lambda_2 \int_a^b t^{q-1} \eta(t)dt > 0$$

which contradicts the fact that u is a maximizer.

Now that we now that both multipliers are positive we can prove that u is continuous, which is equivalent to say that $M = T$, where T is the unique positive number such that $\lambda_1 T^{p-1} + \lambda_2 T^{q-1} = 1$ (uniqueness of T follows from the positivity of multipliers).

We start supposing that $M < T$, which means that $\lim_{t \rightarrow M^-} u(t) > 0$. Consider the following variation

$$\eta(t) = \begin{cases} -1 + \alpha \frac{t}{M} + \beta, & t \in (M - M\delta, M) \\ 1, & t \in (M, M + M\delta) \\ 0, & \text{otherwise} \end{cases}$$

where $\delta > 0$ is small enough so that $M - M\delta > 0$ and $M + M\delta < T$, while α and β are constants, depending on δ , to be determined. Since we want this to be an admissible variation we need to impose that η is orthogonal to t^{p-1} and t^{q-1} . For example, the first condition is:

$$\begin{aligned} 0 &= \int_{M-M\delta}^{M+M\delta} t^{p-1} \eta(t) dt = - \int_{M-M\delta}^M t^{p-1} dt + \int_{M-M\delta}^M t^{p-1} \left(\alpha \frac{t}{M} + \beta \right) dt + \int_M^{M+M\delta} t^{p-1} dt \stackrel{\tau=t/M}{=} \\ &= M^p \int_{1-\delta}^1 \tau^{p-1} (\alpha \tau + \beta) d\tau - M^p \int_{1-\delta}^1 \tau^{p-1} d\tau + M^p \int_1^{1+\delta} \tau^{p-1} d\tau \stackrel{1/\delta}{=} \\ &\implies \int_{1-\delta}^1 \tau^{p-1} (\alpha \tau + \beta) d\tau = \alpha \int_{1-\delta}^1 \tau^p d\tau + \beta \int_{1-\delta}^1 \tau^{p-1} d\tau = \int_{1-\delta}^1 \tau^{p-1} d\tau - \int_1^{1+\delta} \tau^{p-1} d\tau \end{aligned}$$

The equation stemming from the orthogonality with t^{q-1} is analogous. Therefore we obtained a nonhomogeneous linear system for α and β **VA RESO MEGLIO**

$$\begin{pmatrix} \int_{1-\delta}^1 \tau^p d\tau & \int_{1-\delta}^1 \tau^{p-1} d\tau \\ \int_{1-\delta}^1 \tau^q d\tau & \int_{1-\delta}^1 \tau^{q-1} d\tau \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \int_{1-\delta}^1 \tau^{p-1} d\tau - \int_1^{1+\delta} \tau^{p-1} d\tau \\ \int_{1-\delta}^1 \tau^{q-1} d\tau - \int_1^{1+\delta} \tau^{q-1} d\tau \end{pmatrix} \quad (6.29)$$

This system has a unique solution if and only if the determinant of the matrix is not 0. We can show this directly:

$$\begin{aligned} &\int_{1-\delta}^1 \tau^p d\tau \int_{1-\delta}^1 \tau^{q-1} d\tau - \int_{1-\delta}^1 \tau^q d\tau \int_{1-\delta}^1 \tau^{p-1} d\tau = \\ &= \frac{1}{\delta^2} \int_{(1-\delta,1)^2} (\tau^p \sigma^{q-1} - \tau^{p-1} \sigma^q) d\tau d\sigma = \frac{1}{\delta^2} \int_{(1-\delta,1)^2} \tau^{p-1} \sigma^{q-1} (\tau - \sigma) d\tau d\sigma = \\ &= \frac{1}{\delta^2} \left(\int_{Q_1} \tau^{p-1} \sigma^{q-1} (\tau - \sigma) d\tau d\sigma + \int_{Q_2} \tau^{p-1} \sigma^{q-1} (\tau - \sigma) d\tau d\sigma \right) = \end{aligned}$$

where $Q_1 = (1 - \delta, 1)^2 \cap \{\tau > \sigma\}$ and $Q_2 = (1 - \delta, 1)^2 \cap \{\tau < \sigma\}$. In the second integral we can consider the change of variable that swaps τ and σ . In this case the new domain is Q_1 , hence:

$$= \frac{1}{\delta^2} \int_{Q_1} (\tau^{p-1} \sigma^{q-1} - \tau^{q-1} \sigma^{p-1}) (\tau - \sigma) d\tau d\sigma$$

In Q_1 $\tau - \sigma > 0$ and the sign of $\tau^{p-1}\sigma^{q-1} - \tau^{q-1}\sigma^{p-1}$ is constant, in fact:

$$\tau^{p-1}\sigma^{q-1} - \tau^{q-1}\sigma^{p-1} > 0 \iff \left(\frac{\tau}{\sigma}\right)^{p-q} > 1 \iff_{\tau > \sigma} p > q$$

Therefore the determinant of the matrix is always not 0.

The derivative of G along η is nonpositive because u is supposed to be a maximizer, therefore

$$\begin{aligned} 0 &\geq \int_{M-M\delta}^{M+M\delta} G'(u(t))\eta(t)dt = - \int_{M-M\delta}^M (\lambda_1 t^{p-1} + \lambda_2 t^{q-1}) dt + \\ &+ \int_{M-M\delta}^M (\lambda_1 t^{p-1} + \lambda_2 t^{q-1}) \left(\alpha \frac{t}{M} + \beta\right) dt + \int_M^{M+M\delta} dt = \\ &= - \int_{M-M\delta}^M (\lambda_1 t^{p-1} + \lambda_2 t^{q-1}) dt + \lambda_1 M^p \int_{1-\delta}^1 t^{p-1}(\alpha t + \beta) dt + \\ &+ \lambda_2 M^q \int_{1-\delta}^1 t^{q-1}(\alpha t + \beta) dt + M\delta \end{aligned}$$

Dividing by $M\delta$ and rearranging we obtain:

$$\begin{aligned} \int_{M-M\delta}^M (\lambda_1 t^{p-1} + \lambda_2 t^{q-1}) dt &\geq 1 + \lambda_1 M^{p-1} \int_{1-\delta}^1 t^{p-1}(\alpha t + \beta) dt \\ &+ \lambda_2 M^{q-1} \int_{1-\delta}^1 t^{q-1}(\alpha t + \beta) dt \end{aligned} \quad (6.30)$$

We notice that the last two terms are exactly the one that appear in the orthogonality condition, therefore, to understand their behavior as δ approaches 0, we need to study the right-hand side of the system (6.29). If we expand the first term in its Taylor series with respect to δ we have:

$$\left(1 - \frac{p-1}{2}\delta + o(\delta)\right) - \left(1 + \frac{p-1}{2}\delta + o(\delta)\right) = -(p-1)\delta + o(\delta)$$

and the same is for the other term. Since they both are of order δ , if we let $\delta \rightarrow 0^+$ in (6.30) we obtain

$$\lambda_1 M^{p-1} + \lambda_2 M^{q-1} \geq 1$$

Since the function $\lambda_1 t^{p-1} + \lambda_2 t^{q-1}$ is strictly increasing (because λ_1 and λ_2 are both positive) $M \geq T$ which is absurd because we supposed that $M < T$. This allows us to write u as in (6.24).

Lastly we shall prove that multipliers λ_1, λ_2 , and hence maximizer, are unique. For this proof it is convenient to express u in a slightly different way:

$$u(t) = \frac{1}{d!} \left[\text{Log}_- \left((c_1 t)^{p-1} + (c_2 t)^{q-1} \right) \right]^d$$

To emphasize that u is parametrized by c_1, c_2 we may write $u(t; c_1, c_2)$. Now we define

$$f(c_1, c_2) = p \int_0^T t^{p-1} u(t; c_1, c_2) dt, \quad g(c_1, c_2) = q \int_0^T t^{q-1} u(t; c_1, c_2) dt$$

We want to highlight that, even if it is not explicit, also T depends on c_1 and c_2 . Nevertheless these functions are differentiable since both T and u are differentiable with respect to (c_1, c_2) , functions $t^{p-1}u$ and $t^{q-1}u$ and their derivatives are bounded in $(0, T)$. Our maximizer u satisfies the constraints only if $f(c_1, c_2) = A^p$, $g(c_1, c_2) = B^q$. Therefore to prove uniqueness of the maximizer we need to show that level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.

First of all we are studying endpoints. For example, if $c_2 = 0$:

$$\begin{aligned} f(c_1, 0) &= p \int_0^{1/c_1} t^{p-1} \frac{1}{d!} [-\log(c_1 t)^{p-1}]^d dt \stackrel{\tau=c_1 t}{=} \\ &= \frac{p(p-1)^d}{c_1^p d!} \int_0^1 \tau^{p-1} [-\log(\tau)]^d d\tau = \frac{\kappa_p^d}{c_1^p} = A^p \implies c_{1,f} = \frac{\kappa_p^{d/p}}{A} \end{aligned}$$

The same can be done for g and setting $c_1 = 0$ thus we obtain four points

$$c_{1,f} = \frac{\kappa_p^{d/p}}{A}, \quad c_{1,g} = \left(\frac{p-1}{q}\right)^{d/q} \frac{1}{B}, \quad c_{2,f} = \left(\frac{q-1}{p}\right)^{d/p} \frac{1}{A}, \quad c_{2,g} = \frac{\kappa_q^{d/q}}{B}$$

In the regime we are considering one has that $c_{1,f} < c_{1,g}$ and $c_{2,f} > c_{2,g}$, indeed

$$\begin{aligned} c_{1,f} < c_{1,g} &\iff \frac{\kappa_p^{d/p}}{A} < \left(\frac{p-1}{q}\right)^{d/q} \frac{1}{B} \iff \frac{B}{A} < \kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{d/q} \\ c_{2,f} > c_{2,g} &\iff \left(\frac{q-1}{p}\right)^{d/p} \frac{1}{A} > \frac{\kappa_q^{d/q}}{B} \iff \frac{B}{A} > \kappa_q^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{d/p} \end{aligned}$$

which are exactly conditions in (6.13). Since there is this dispositions of these points we expect there is an intersection between level sets. Firstly we notice that, for every $c_1 \in (0, c_{1,f})$, there exist a unique value of c_2 for which $f(c_1, c_2) = A^p$. Indeed, from previous computations we notice that $f(c_1, 0)$ is a decreasing function hence $f(c_1, 0) > A^p$, while $\lim_{c_2 \rightarrow +\infty} f(c_1, c_2) = 0$. The uniqueness of this value follows from strict monotonicity of $f(c_1, \cdot)$, in fact:

$$\frac{\partial f}{\partial c_1}(c_1, c_2) = -\frac{p(p-1)}{(d-1)!} c_1^{p-2} \int_0^T \frac{t^{2(p-1)}}{(c_1 t)^{p-1} + (c_2 t)^{q-1}} [-\log((c_1 t)^{p-1} + (c_2 t)^{q-1})]^{d-1} dt \quad (6.31)$$

is always strictly negative. We point out that the term $\frac{\partial T}{\partial c_1}(c_1, c_2)u(T; c_1, c_2)$, that should appear since T depends on c_1 , is zero because u is 0 in T . The same is true for g , therefore on the interval $(0, c_{1,f})$ the level sets of f and g can be seen as the graph of two functions φ, γ . Since $\frac{\partial f}{\partial c_2}, \frac{\partial g}{\partial c_2} < 0$ for every (c_1, c_2) , from the implicit function theorem we have that φ and γ are differentiable with respect to c_1 .

After defining φ and γ we want to prove that $(\varphi - \gamma)' < 0$. Still from the implicit function theorem we have

$$\begin{aligned} \frac{d}{dc_1}(\varphi - \gamma)(c_1) &= -\frac{\frac{\partial f}{\partial c_1}(c_1, \varphi(c_1))}{\frac{\partial f}{\partial c_2}(c_1, \varphi(c_1))} + \frac{\frac{\partial g}{\partial c_1}(c_1, \gamma(c_1))}{\frac{\partial g}{\partial c_2}(c_1, \gamma(c_1))} < 0 \iff \\ \mathcal{I}(c_1) &= \frac{\partial f}{\partial c_1}(c_1, \varphi(c_1)) \frac{\partial g}{\partial c_2}(c_1, \gamma(c_1)) - \frac{\partial f}{\partial c_2}(c_1, \varphi(c_1)) \frac{\partial g}{\partial c_1}(c_1, \gamma(c_1)) > 0 \end{aligned}$$

As for (6.31) the other derivatives are computed. To simplify the notation we define $h(t; c_1, c_2) = \frac{1}{(d-1)!} \frac{1}{(c_1 t)^{p-1} + (c_2 t)^{q-1}} [-\log((c_1 t)^{p-1} + (c_2 t)^{q-1})]^{d-1}$. From Fubini's theorem we can write the product of the integrals as a double integral

$$\begin{aligned} \mathcal{I}(c_1) &= p(p-1)q(q-1)c_1^{p-2}\gamma(c_1)^{q-2} \int_{[0,T]^2} h(t; c_1, \varphi(c_2))h(s; c_1, \gamma(c_2))t^{2(p-1)}s^{2(q-1)}dtds + \\ &\quad - p(q-1)q(p-1)c_1^{p-2}\varphi(c_1)^{q-2} \int_{[0,T]^2} h(t; c_1, \varphi(c_2))h(s; c_1, \gamma(c_2))t^{p+q-2}s^{p+q-2}dtds \end{aligned}$$

When level sets intersect we have $\varphi(c_1) = \gamma(c_1)$. In this situation we can factorize the terms outside the integral and notice that the sign of \mathcal{I} depends on the sign of:

$$\begin{aligned} &\int_{[0,T]^2} h(t; c_1, \varphi(c_1))h(s; c_1, \gamma(c_1)) \left(t^{2(p-1)}s^{2(q-1)} - t^{p+q-2}s^{p+q-2} \right) dtds = \\ &= \int_{[0,T]^2} h(t; c_1, \varphi(c_1))h(s; c_1, \gamma(c_1))t^{p-2}s^{q-2} (t^p s^q - t^q s^p) dtds \end{aligned}$$

In order to simplify the notation once again we set $H(t, s; c_1) = h(t; c_1, \varphi(c_1))h(s; c_1, \gamma(c_1))$. Let $T_1 = [0, T]^2 \cap \{t > s\}$ and $T_2 = [0, T]^2 \cap \{t < s\}$. We can split the above integral in two parts:

$$\int_{T_1} H(t, s; c_1)t^{p-2}s^{q-2} (t^p s^q - t^q s^p) dtds + \int_{T_2} H(t, s; c_1)t^{p-2}s^{q-2} (t^p s^q - t^q s^p) dtds$$

Then, considering the change of variables that swaps t and s , the domain of integration becomes T_1 and since H is symmetric in t and s we have that the previous quantity is equal to

$$\begin{aligned} &\int_{T_1} H(t, s; c_1) \left(t^{p-2}s^{q-2} - t^{q-2}s^{p-2} \right) (t^p s^q - t^q s^p) dtds = \\ &= \int_{T_1} H(t, s; c_1) \frac{1}{t^2 s^2} (t^p s^q - t^q s^p)^2 dtds \end{aligned}$$

which is strictly positive.

Now we are able to prove the uniqueness of multipliers.

First of all, since $(\varphi - \gamma)' < 0$ whenever $\varphi(c_1) = \gamma(c_1)$, for every point of intersection there exist $\delta > 0$ such that $\varphi(t) > \gamma(t)$ for $t \in (c_1 - \delta, c_1)$ while $\varphi(t) < \gamma(t)$ for $t \in (c_1, c_1 + \delta)$. Define $c_1^* := \sup\{c_1 \in [0, c_{1,f}] : \forall t \in [0, c_1] \varphi(t) \geq \gamma(t)\}$. This is an intersection point between φ and γ (if $\varphi(c_1^*) > \gamma(c_1^*)$ due to continuity there would be $\varepsilon > 0$ such that $\varphi(c_1^* + \varepsilon) > \gamma(c_1^* + \varepsilon)$ which contradicts the definition of c_1^*) and it is the first one, because we saw that after every intersection point there is an interval where $\varphi < \gamma$. Lastly, since $\varphi(0) > \gamma(0)$ and $\varphi(c_{1,f}) = 0 < \gamma(c_{1,f})$ we have that $0 < c_1^* < c_{1,f}$.

Suppose now that there is a second point of intersection \tilde{c}_1 after the first one. Since immediately after c_1^* we have that φ becomes smaller than γ , this second point of intersection is given by $\tilde{c}_1 = \sup\{c_1 \in [c_1^*, c_{1,f}] : \forall t \in [c_1^*, c_1] \varphi(t) \leq \gamma(t)\}$. Considering that this is an intersection point, there exist an interval before \tilde{c}_1 where φ is strictly greater than γ

which is absurd, hence c_1^* is the only intersection point between φ and γ . Therefore $(c_1^*, \varphi(c_1^*) = c_2^*)$ is the unique pair of multipliers for which

$$p \int_0^T t^{p-1} u(t; c_1^*, c_2^*) dt = A^p, \quad q \int_0^T t^{q-1} u(t; c_1^*, c_2^*) dt = B^q$$

and, in the end, $u(t; c_1^*, c_2^*)$ is the unique maximizer for (6.21). □

Appendix A

Calculations

A.1 Constant in Lieb's inequality

In the last part of the proof of Lieb's inequality [3.4](#) we used the following equality $A_{p'}^d A_{2/p'}^{2d/p'} A_{(p/p')'}^{d/p'} = (2/p)^{d/p}$ without proving it. We recall that the Babenko-Bechner constant A_p is given by:

$$A_p = \left(\frac{p^{1/p}}{p'^{1/p'}} \right)^{1/2}.$$

Since there is an exponent $1/2$ in the Babenko-Bechner constant it is better to compute the square of $A_{p'}^d A_{2/p'}^{2d/p'} A_{(p/p')'}^{d/p'}$. For the sake of clarity we are going to compute every single term and then we are going to multiply them.

- $A_{p'}^2 = \frac{p^{1/p'}}{p^{1/p}};$
- In order to compute $A_{2/p'}^{2 \cdot 2/p'}$ we start computing $(2/p')'$:

$$\left(\frac{2}{p'} \right)' = \frac{2/p'}{2/p' - 1} = \frac{2}{2 - p'},$$

therefore

$$\begin{aligned} A_{2/p'}^{2 \cdot 2/p'} &= \left[\left(\frac{2}{p'} \right)^{p'/2} \left(\frac{2-p'}{2} \right)^{(2-p')/2} \right]^{2/p'} = \frac{2}{p'} \left(\frac{2-p'}{2} \right)^{(2-p')/(p')} = \\ &= \frac{2^{2(1-1/p')}}{p'} (2-p')^{2/p'-1} = \frac{2^{2/p}}{p'} (2-p')^{1/p'-1/p}; \end{aligned}$$

- Like the previous case, we start computing $(p/p')'$:

$$\left(\frac{p}{p'} \right)' = \frac{p/p'}{p/p' - 1} = \frac{p}{p - p'},$$

hence

$$\begin{aligned} A_{(p/p')'}^{2/p'} &= \left[\left(\frac{p}{p-p'} \right)^{(p-p')/p} \left(\frac{p'}{p} \right)^{p'/p} \right]^{1/p'} = \left(\frac{p}{p-p'} \right)^{1/p'-1/p} \left(\frac{p'}{p} \right)^{1/p} = \\ &= \left(1 - \frac{p'}{p} \right)^{1/p-1/p'} \left(\frac{p'}{p} \right)^{1/p} = (2-p')^{1/p-1/p'} \left(\frac{p'}{p} \right)^{1/p}. \end{aligned}$$

We are now ready to calculate the product of the three constant:

$$\begin{aligned} A_{p'}^2 A_{2/p'}^{2:2/p'} A_{(p/p')'}^{2/p'} &= \frac{p^{1/p'}}{p^{1/p}} \frac{2^{2/p}}{p'} (2-p')^{1/p'-1/p} (2-p')^{1/p-1/p'} \left(\frac{p'}{p} \right)^{1/p} = \\ &= 2^{2/p} p^{-2/p} p^{1/p'+1/p-1} = \left(\frac{2}{p} \right)^{2/p} \end{aligned}$$

which is the desired result.

A.2 Curious inequality between conjugate exponent

We consider inequality (6.12), which we rewrite for the sake of clarity:

$$\kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q} \right)^{\frac{d}{q}} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}} \right)^d$$

Firstly we want to restate in a more concise way. Recalling that $\kappa_p = \frac{1}{p'}$, where p' is the conjugate exponent of p , we have:

$$\left(\frac{1}{p'} \right)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{p}{q} \right)^{\frac{1}{q}} \geq \left(\frac{1}{p'} \right)^{\frac{1}{p'}} \left(\frac{1}{q'} \right)^{-\frac{1}{q'}} \iff \left(\frac{1}{p'} \right)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{p'}} \left(\frac{1}{q'} \right)^{\frac{1}{q'}} \left(\frac{p}{q} \right)^{\frac{1}{q}} \geq 1$$

but, since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} - 1 = \frac{1}{q'}$, in conclusion we have:

$$\left(\frac{p'}{q'} \right)^{\frac{1}{q'}} \left(\frac{p}{q} \right)^{\frac{1}{q}} \geq 1$$

In order to prove that this inequality holds for every pair of $p, q > 1$ we consider the left-hand side as function of $x = \frac{1}{p}$ and $y = \frac{1}{q}$ (therefore $\frac{1}{p'} = 1 - x$ and $\frac{1}{q'} = 1 - y$). If we take the logarithm of this quantity we want to show that:

$$f(x, y) = (1 - y) [\log(1 - y) - \log(1 - x)] + y [\log(y) - \log(x)] \geq 0.$$

The partial derivative of f with respect to x is:

$$\frac{\partial f}{\partial x}(x, y) = \frac{1 - y}{1 - x} - \frac{y}{x} = \frac{x - y}{x(1 - x)}.$$

Since $x \in (0, 1)$, $\frac{\partial f}{\partial x}(x, y)$ is negative for $x < y$ and positive for $x > y$, so f has a minimum for $x = y$ where $f(x, x) = 0$.

A.3 Another inequality between conjugate exponents

We want to prove that inequality (6.14) holds, namely that:

$$\kappa_q^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{p}} < \kappa_p^{d(\frac{1}{q}-\frac{1}{p})} \left(\frac{p}{q}\right)^{\frac{d}{q}}$$

whenever $p \neq q$. As for the previous section, since $\kappa_p = \frac{1}{p'}$ and $\kappa_q = \frac{1}{q'}$ we can write the inequality in a more concise way:

$$\left(\frac{1}{q'}\right)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{p}{q}\right)^{\frac{1}{p}} < \left(\frac{1}{p'}\right)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \iff \left(\frac{q}{q'p}\right)^{\frac{1}{q}-\frac{1}{p}} < 1 \iff \left(\frac{q-1}{p-1}\right)^{\frac{1}{q}-\frac{1}{p}} < 1.$$

If $q > p$ the base is greater than 1 while the exponent is less than 1, whereas if $q < p$ the converse happens, which proves that inequality holds whenever $p \neq q$.

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