POLITECNICO DI TORINO

**Corso di Laurea Magistrale
in Ingegneria Matematica**

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Some recent results on the norm of localization operators

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Theorem from [**nicolatilli_fk**]

Theorem 3.1. *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2 dx d\omega \leq G(|\Omega|)$$

where $G(s)$ is given by

$$G(s) := \int_0^s e^{(-d!\tau)^{1/d}} d\tau \tag{3.1}$$

Capitolo 4

Localization Operators

4.1 Definition and properties

4.2 Eigenvalues and eigenfunctions

Capitolo 5

Recent results from Nicola-Tilli

5.1 Case $q = +\infty$

5.2 Generic case

Let's now consider the case where both p and q are neither 1 or $+\infty$. The result presented in [**nicolatilli_norm**] include the case ...

$$\|L_F\|_{L_2 \to L_2} \le \min\{\kappa_p^{d\kappa_p} A,\ \kappa_q^{d\kappa_q} B\}$$

Suppose that the minimum is given by $\kappa_p^{d\kappa_p} A$, therefore

$$\kappa_p^{d\kappa_p} A \le \kappa_q^{d\kappa_q} B \iff \frac{B}{A} \ge \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}} \right)^d$$

We can check if the solution of the problem with just the L^p bound solves also the problem with both bounds, that is $F\|_{L^q} \le B$, where F is given by ...

$$\|F\|_{L^q}^q = \int_{\mathbb{R}^{2d}} |F(z)|^q dz = ... = \lambda^q \left(\frac{p-1}{q} \right)^d$$

Since we want F to satisfy the L^q constraint we should have

$$\frac{B}{A} \ge \kappa_p^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \left(\frac{p}{q} \right)^{\frac{d}{q}}$$

It would be nice if this bound was less restrictive than the first one. Unfortunately that's not the case, in fact it's always true that

$$\left(\frac{p'}{q'} \right)^{\frac{1}{q'}} \left(\frac{p}{q} \right)^{\frac{1}{q}} \ge 1$$

Following the path in [**nicolatilli_norm**] we obtain ...

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \implies u(t) = \frac{1}{d!}\left[-\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d, \ t \in (0, M)$$

Our main goal now is to show that multipliers λ_1, λ_2 are unique and both positive.
The easiest fact to prove is that both multipliers are not 0. In fact if one, say λ_2, was 0, we would obtain that the solution of our problem is the same as the one with just the L^p bound. But we already know that this function does not satisfy the L^q constraint hence it is impossible that $\lambda_2 = 0$.
Suppose now that one of the multipliers, say always λ_2, is negative. Consider an interval $[a-l, a+l]$ contained in $(0, M)$ and the variation

$$\eta(t) = \begin{cases} -t^{1-p}, & t \in (a-l, a) \\ \ \ t^{1-p}, & t \in (a, a+l) \end{cases}$$

η is an admissible variation because u is strictly positive in $[a-l, a+l]$, $\int_{a-l}^{a+l} t^{p-1}\eta(t)dt = 0$, and $\int_{a-l}^{a+l} t^{q-1}\eta(t)dt < 0$ if $q < p$ (otherwise we can take $-\eta$ instead of η). The directional derivative of G along η is

$$\begin{aligned}\int_{a-l}^{a+l} G'(u(t))\eta(t)dt &= \int_{a-l}^{a+l} (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \\ &= \lambda_2 \int_{a-l}^{a+l} t^{q-1}\eta(t)dt > 0\end{aligned}$$

which contradicts the fact that u is a maximizer.
PROOF OF CONTINUITY
Lastly we shall prove that multipliers λ_1, λ_2, and hence maximizer, are unique.