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Some recent results on the norm of localization operators

Relatori
Prof. Fabio Nicola
firma dei relatori

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Candidato
Federico Riccardi
firma del candidato

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Acknowledgements

Summary

Chapter 1

Introduction

Chapter 2

Preliminaries

In this first chapter we briefly recall some basics definition and results about functional analysis and Fourier transform. In section 2.1 basic concepts about operators between Banach spaces are presented. In section 2.2 Fourier transform is defined and essential properties are given.

2.1 Basics of Functional Analysis

In this section we focus our attention on linear operator between Banach spaces. Across the section a generic Banach space will be denoted as X (or Y) endowed with the norm $\|\cdot\|_X$. In case X is an Hilbert space we will denote its inner product as $\langle\cdot,\cdot\rangle_X$.
A generic linear operator between two Banach spaces X and Y will be denoted as $T: X \to Y$. As a standard notation, the image of $x \in X$ through T will be denoted as $T(x)$ or equivalently as Tx.

Definition 2.1. *A linear operator $T : X \to Y$ is **bounded** if there exist $C > 0$ such that*

$$\|Tx\|_Y \leq C\|x\|_X \quad \forall x \in X \tag{2.1}$$

For linear operator boundedness is strictly related to continuity as the subsequent theorem states.

Theorem 2.2. *For a linear operator T the following statements are equivalent:*

- *T is continuous*
- *T is bounded.*

We denote the set of linear bounded (continuous) operators from X to Y as $\mathcal{B}(X, Y)$, while if $X = Y$ we will just write $\mathcal{B}(X)$.
For the sake of completeness we mention that actually, for linear operators, boundedness is equivalent to uniform continuity.
After this we define the *norm* of an operator

Definition 2.3. *Given a linear bounded operator T we define its* ***norm*** *as the following number:*

$$\|T\| := \inf\{C > 0 : \|Tx\|_Y \leq C\|x\|_X \ \forall x \in X\} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X \setminus \{0\}\right\}$$

The proof of the equivalence between two definition is straightforward. We see that the norm of an operator is the best constant for which boundedness property (2.1) holds. Sometimes, in order to emphasize the spaces between which T operates, we may write the norm of T as $\|T\|_{X \to Y}$.
In the following we will mostly deal with X and Y being $L^2(\mathbb{R}^d)$, which is an Hilbert space. For operators between Hilbert spaces we can give the norm of an operator by means of the dual norm **CONTROLLARE**:

$$\|T\| = \sup\{\langle Tx, y\rangle_X : x, y \in X\}$$

An important class of operators is the class of *compact operators*.

Definition 2.4. *A linear bounded operator T is* ***compact*** *if for every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ the sequence of the images $\{Tx_n\}_{n\in\mathbb{N}} \subset Y$ has a converging subsequence.*

The property of compactness can be stated in multiple ways **SERVE SCRIVERLE?**
Now we suppose X and Y to be Hilbert spaces. Given $T \in \mathcal{B}(X, Y)$ we know that there exist a unique $T^* \in \mathcal{B}(Y, X)$ such that:

$$\langle Tx, y\rangle_X = \langle x, T^*y\rangle_Y \quad \forall x \in X,\, y \in Y$$

T^* is called the **adjoint** operator of T. In the particular case in which $T : X \to X$, if $T = T^*$ we say that T is **self-adjoint**.
From now on we suppose that X is over the field $\mathbb{C}$ and that $T \in \mathcal{B}(X, Y)$.

Definition 2.5. *The set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$ is called the* ***spectrum*** *of T.*

For operators between finite-dimensional spaces (matrices) the spectrum is made up of *eigenvalues*, those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective, because in this case $T - \lambda I$ is not injective if and only if it is not surjective. On the other hand, when dealing with infinite-dimensional spaces, this is no more true. Eigenvalues are in the so called *point spectrum*, which in general is just a part of the whole spectrum.
If an operator is compact or self-adjoint its spectrum has some additional properties.

Theorem 2.6 (Fredholm's alternative)**.** *Let $T \in \mathcal{B}(X)$ be a compact operator. Then per $T - I$ one and only one of the following happens:*

- *T is invertible*
- *T is not injective*

Therefore for compact operators, all the values in the spectrum, except at most for 0, are eigenvalues.
Another fundamental result arises if we study the spectrum of compact and self-adjoint operators.

Theorem 2.7. *Let X be a separable Hilbert space and $T \in \mathcal{B}(X)$ a compact and self-adjoint operator. Then there exist an orthonormal basis of X composed of eigenvectors of T*

Hence self-adjoint compact operators can always be diagonalized in some suitable basis. **CITARE BREZIS**
Now we are going to consider two important classes of operators: *trace class* operators and *Hilbert-Schmidt* operators.
The trace of an operator can be defined as it is for matrices.

Definition 2.8. *Let X be an Hilbert space. An operator $T \in \mathcal{B}(X)$ is said* ***positive*** *if*

$$\langle Tx, x \rangle \geq 0 \quad \forall x \in X$$

Definition 2.9. *Let X be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Given $T \in \mathcal{B}(X)$ a positive operator be define the* ***trace*** *of T as*

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle$$

One can show that the definition is well defined since it is independent of the basis. **AGGIUNGERE DIM?**
The definition of trace is given only for positive operators. If we want to deal with general ones it is sufficient to consider $|T| = \sqrt{T^*T}$ **LA PARTE SU RADICE QUADRATA E MODULO DI UN OPERATORE VA AGGIUNTA??**

Definition 2.10. *An operator $T \in \mathcal{B}(X)$ is called* ***trace class*** *if and only if* $\mathrm{tr}|T| < \infty$

Theorem 2.11. *For every trace class operator one has $\|T\| \leq \mathrm{tr}(T)$.*

Proof. contenuto... □

Theorem 2.12. *Every trace class operator is compact.*

Proof. contenuto... □

Definition 2.13. *An operator $T \in \mathcal{B}(X)$ is called* ***Hilbert-Schmidt*** *if and only if* $\mathrm{tr}(T^*T) < \infty$.

Theorem 2.14. *Every Hilbert-Schmidt operator is compact.*

Proof. contenuto... □

The importance of Hilbert-Schmidt operator is related to the following theorem.

Theorem 2.15. *Let $X = L^2(\mathbb{R}^d)$. Then $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ is Hilbert-Schmidt if only if there exist a function $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, called integral kernel, such that*

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy \quad \forall f \in L^2(\mathbb{R}^d).$$

Moreover $\mathrm{tr}(T^*T) = \int_{\mathbb{R}^{2d}} |K(x,y)|^2 dx dy$.

Proof. contenuto... □

- Norma operatoriale (FATTO)
- Operatori autoaggiunti? (FATTO)
- Operatori compatti (FATTO)
- Spettro operatori (FATTO)
- Operatori di classe traccia
- Operatori di Hilbert-Schmidt?

2.2 Fourier Transform and its properties

Definition 2.16. *Let $f \in L^1(\mathbb{R}^d)$. We define the **Fourier transform** of f the function*

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f(t)dt \tag{2.2}$$

It's straightforward to see that the definition is well-posed and that $\mathcal{F}f \in L^\infty(\mathbb{R}^d)$ with $\|\mathcal{F}f\|_\infty \leq \|f\|_1$. Therefore $\mathcal{F}$ can be seen as a linear operator between $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ with $\|\mathcal{F}\| = 1$ (from the previous inequality actually we saw that $\|\mathcal{F}\| \leq 1$ but if we take $f \geq 0$ a.e. we have that $\hat{f}(0) = \|f\|_1$ that gives us the equality).
The Fourier transform of an $L^1(\mathbb{R}^d)$ is not only bounded as stated by the *Riemann-Lebesgue lemma*.

Theorem 2.17 (Riemann-Lebesgue lemma). *Let $f \in L^1(\mathbb{R}^d)$. Therefore $\hat{f} \in C_0(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C}$ continuous such that $\lim_{|t|\to\infty} |f(t)| = 0\}$.*

Definition 2.18. *Let $f \in L^1(\mathbb{R}^d)$. We define the **inverse Fourier transform** of the function f*

$$\mathcal{F}^{-1}f(t) = \check{f}(t) := \int_{\mathbb{R}^d} e^{2\pi i\omega\cdot t} f(\omega)d\omega \tag{2.3}$$

The inverse Fourier transform is denoted with $\mathcal{F}^{-1}$ because it is actually the inverse operator of the Fourier transform as stated by the *inversion theorem*.

Theorem 2.19 (Inversion theorem). *Let $f \in L^1(\mathbb{R}^d)$ and suppose that also $\hat{f} \in L^1(\mathbb{R}^d)$. Then*

$$f(t) = \mathcal{F}^{-1} \circ \mathcal{F}f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega)e^{2\pi i\omega\cdot t}d\omega$$

The Fourier transform is intimately related to regularity and decay properties. The duality between these two is stated in the following theorems.

Theorem 2.20. *Let $f \in L^1(\mathbb{R}^d)$. If $|t|^k f \in L^1(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ then $\hat{f} \in C_0^k(\mathbb{R}^d)$ and the following holds for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$:*

$$\mathcal{F}\left((-2\pi it)^\alpha f\right)(\omega) = \partial^\alpha \mathcal{F}f(\omega).$$

Theorem 2.21. *Let $f \in C^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$. If $f, \partial^\alpha f \in L^1(\mathbb{R}^d)$ for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ then*

$$\mathcal{F}\left(\partial^\alpha f\right)(\omega) = (2\pi i\omega)^\alpha \mathcal{F}f(\omega)$$

In particular this implies that $\hat{f}(\omega) = o\left(|\omega|^{-k}\right)$ as $|\omega| \to \infty$.

To sum up, previous theorems state a duality between regularity and decay: if a function is smooth then it decays rapidly and vice versa.

If f is in $L^2(\mathbb{R}^d)$, the integral in (2.2) in general will not converge. Nevertheless we can define the Fourier transform of an L^2 function through a density argument. For example, $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$. On this space one can show that the Fourier transform is an isometry with respect to the L^2 norm and therefore it extends to an isometry on the whole $L^2(\mathbb{R}^d)$. This is stated by the *Plancherel theorem*.

Theorem 2.22 (Plancherel theorem)**.** *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ then $\|f\|_2 = \|\hat{f}\|_2$.*

Thanks to the polarization identity this implies that $\mathcal{F}$ preserves the inner product in $L^2(\mathbb{R}^d)$:

$$\langle f, g\rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R}^d)} \quad \forall f, g \in L^2(\mathbb{R}^d) \tag{2.4}$$

therefore the Fourier transform F is a unitary operator on $L^2(\mathbb{R}^d)$. Result (2.4) is called *Parseval formula*.

Lastly we introduce two fundamental operators in Fourier analysis. Given $x, \omega \in \mathbb{R}^d$ we define the *time-shift* (or translation) operator T_x

$$T_x f(t) = f(t - x) \tag{2.5}$$

and the *modulation* operator M_ω

$$M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t) \tag{2.6}$$

- Defizione trasformata (FATTO)
- Teorema di Plancherel (FATTO)
- Formula di inversione(FATTO)
- Proprietà di decadimento e regolarità (FATTO)
- Proprietà operatori di traslazione e modulazione??
- Disuguaglianza di Hausdorff-Young ??

Chapter 3

Short-Time Fourier Transform

3.1 STFT

The *short-time Fourier transform* or *STFT* is a powerful tool, introduced by Gabor in **AGGIUNGERE CITAZIONE E CONTROLLARE CHE SIA CORRETTA**, used to study properties of a signal locally both in time and frequency. The main idea behind the STFT is the following: if we want some information of the spectrum of a signal around a specific time, say T, we could choose an interval $(T-\Delta T, T+\Delta T)$ and take the Fourier transform of $f\chi_{(T-\Delta T, T+\Delta T)}$. Usually multiplying by a characteristic function will not give us a regular function (not even continuous) and in light of the duality between regularity and decay, the Fourier transform of $f\chi_{(T-\Delta T, T+\Delta T)}$ will not decay rapidly. Therefore a sharp cutoff in the time domain will result in a "bad" localization in the frequency domain. In order to avoid this kind of problems we could think to multiply the signal f by a smooth function.

Definition 3.1. *Fix a function $\phi \neq 0$ called* window function*. The **short-time Fourier transform** of a function f with window ϕ is defined as*

$$\mathcal{V}_\phi f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{\phi(t-x)}e^{-2\pi i\omega\cdot t}dt, \quad (x,\omega)\in\mathbb{R}^{2d} \tag{3.1}$$

In the above definition did not specify where f and *phi* are chosen. Since we are taking the Fourier transform of the function $fT_x\overline{\phi}$, the STFT is well defined whenever the Fourier transform of this function is. For example if both f and ϕ are in $L^2(\mathbb{R}^d)$ then $fT_x\overline{\phi}$ is in $L^1(\mathbb{R}^d)$ for every $x\in\mathbb{R}^d$ and so the integral in (3.1) is defined. In this special case the STFT can be written as a scalar product:

$$\mathcal{V}_\phi f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{\phi(t-x)}e^{-2\pi i\omega\cdot t}dt = \langle f, M_\omega T_x\phi\rangle$$

In general, the STFT of f with respect to ϕ will be defined whenever $\langle f, M_\omega T_x\phi\rangle$ is an expression of some sort of duality. For example, if $f\in\mathcal{S}'(\mathbb{R}^d)$ and $\phi\in\mathcal{S}(\mathbb{R}^d)$ then $M_\omega T_x\overline{\phi}\in\mathcal{S}(\mathbb{R}^d)$, therefore $\langle f, M_\omega T_x\phi\rangle$ can be seen as the usual duality between tempered distributions and functions in the Schwartz space.

AGGIUNGERE ALTRE DEFINIZIONI EQUIVALENTI DELLA STFT?

3.1.1 Properties of STFT

In this section we will present and prove some properties about the STFT. An excellent reference is [**grochenig**].

Theorem 3.2. *Let $f_1, f_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$. Then $\mathcal{V}_{\phi_i} f_i \in L^2(\mathbb{R}^{2d})$ and the following holds:*

$$\langle \mathcal{V}_{\phi_1} f_1, \mathcal{V}_{\phi_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle \phi_1, \phi_2 \rangle} \tag{3.2}$$

Proof. **VA MESSA??** □

Corollary 3.3. *If $f, \phi \in L^2(\mathbb{R}^d)$ then*

$$\|\mathcal{V}_\phi f\|_2 = \|f\|_2 \|\phi\|_2$$

In particular if $\|\phi\|_2 = 1$ we see that $\mathcal{V}_\phi$ is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$.

Proof. **VA MESSA??** □

From a direct computation one can see that the adjoint operator of the STFT operator $\mathcal{V}_\phi$ is given by the following expression:

$$\mathcal{V}_\phi^* g(t) = \int_{\mathbb{R}^{2d}} g(x,\omega)\phi(t-x)e^{2\pi i\omega\cdot t} dx d\omega = \int_{\mathbb{R}^{2d}} g(x,\omega) M_\omega T_x \phi(t) dx d\omega \quad \forall g \in L^2(\mathbb{R}^{2d}) \tag{3.3}$$

This adjoint operator appears in the following nice property

Theorem 3.4. *Let $f \in L^2(\mathbb{R}^d)$ and $\phi, \gamma \in L^2(\mathbb{R}^{2d})$ such that $\langle \phi, \gamma \rangle \neq 0$. Then:*

$$f(t) = \frac{1}{\langle \phi, \gamma \rangle} \mathcal{V}_\gamma^* \mathcal{V}_\phi f(t) = \frac{1}{\langle \phi, \gamma \rangle} \int_{\mathbb{R}^{2d}} \mathcal{V}_\phi f(x,\omega) M_\omega T_x \gamma(t) dx d\omega \quad \forall t \in \mathbb{R}^d \tag{3.4}$$

Proof. **VA MESSA??** □

Therefore the adjoint operator $\mathcal{V}_\gamma^*$ acts, in some sense, as an inverse operator. This will be of paramount importance in the following.

- Relazione di ortogonalità
- Formula di inversione

3.2 Fock Space and Bargmann Transform

Chapter 4

Localization Operators

- Definizione
- Operatori di proiezione
- Operatori di localizzazione di Daubechies
- Proprietà di limitatezza e compattezza
- Autovalori e autofunzioni

Chapter 5

Uncertainty principles

5.1 Heisenberg's uncertainty principle

5.2 Donoho-Stark's uncertainty principle

5.3 Lieb's uncertainty principle

5.4 Nicola-Tilli's uncertainty principle or Faber-Krahn Inequality for the STFT

Theorem from [**nicolatilli_fk**]

Theorem 5.1. *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2 dx d\omega \leq G(|\Omega|)$$

where $G(s)$ is given by

$$G(s) \coloneqq \int_0^s e^{(-d!\tau)^{1/d}} d\tau \tag{5.1}$$

Moreover, equality occurs if and only if f is a Gaussian ***METTERE A POSTO*** *and Ω is equivalent, in measure, to a ball of centre (x_0, ω_0).*

Chapter 6

Recent results

6.1 Norm of localization operators: results from Nicola-Tilli

6.2 Generic case

Theorem 6.1. *Assume* $F \in L^p(\mathbb{R}^{2d})$ *for some* $p \in [1, +\infty)$ *and let* $\mu(t) = |\{|F| > t\}|$ *be the distribution function of* $|F|$. *Then*

$$\|L_F\| \leq \int_0^\infty G(\mu(t))dt \tag{6.1}$$

Equality occurs if and only if $F(z) = e^{i\theta}\rho(|z - z_0|)$ *for some* $\theta \in \mathbb{R}$, $z_0 \in \mathbb{R}^{2d}$ *and some nonincreasing function* $\rho : [0, +\infty) \to [0, +\infty)$

Proof. For the sake of brevity we denote the variable $(x, \omega) \in R^{2d}$ as z and therefore $dxd\omega$ as dz. Let $f, g \in L^2(\mathbb{R}^d)$ such that $\|f\|_2 = \|g\|_2 = 1$. Since we are in a Hilbert space $\|L_F\|$ can be computed as the supremum of $|\langle L_F f, g\rangle|$ over all normalized f and g. Therefore we are interested in estimating the previous scalar product

$$|\langle L_F f, g\rangle| = |\langle \mathcal{V}^* F \mathcal{V} f, g\rangle| = |\langle F\mathcal{V}f, \mathcal{V}g\rangle| \leq \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)| \cdot |\mathcal{V}g(z)|dz \overset{\text{C-S}}{\leq} \tag{6.2}$$

$$\leq \left(\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}g(z)|^2 dz\right)^{1/2} \tag{6.3}$$

Since the result is symmetric in f and g we can study just one of the terms. Letting $m = \operatorname{ess\,sup} |F(z)|$ and assuming $m > 0$ (otherwise every result is trivial) we can use the "layer cake" representation **CITARE CON PAGINA**

$$|F(z)| = \int_0^m \chi_{|F|>t}(z)dt$$

we have

$$\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz = \int_{\mathbb{R}^{2d}} \left(\int_0^m \chi_{|F|>t}(z) dt \right) |\mathcal{V}f(z)|^2 dz \overset{\text{Fubini}}{=}$$
$$= \int_0^m \left(\int_{\mathbb{R}^{2d}} \chi_{|F|>t}(z) |\mathcal{V}f(z)|^2 dz \right) dt = \int_0^m \left(\int_{\{|F|>t\}} |\mathcal{V}f(z)|^2 dz \right) dt$$

We notice that the quantity in the inner integral is exactly the one in the theorem 5.1, hence

$$\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz \leq \int_0^m G(\mu(t)) dt \tag{6.4}$$

We point out that since $\mu(t) = 0$ and $G(0) = 0$ the previous expression is equivalent to (6.1).
Because $p < \infty$, L_F is a compact operator on L_2 therefore there exist normalized f and g which achieve equality in the supremum of the norm: $\langle L_F f, g \rangle = \|L_F\|$. Therefore equality in (6.1) occurs if and only if all the previous inequalities become equalities. Equality in (6.4) occurs if and only if

$$\int_{\{|F|>t\}} |\mathcal{V}f(z)|^2 dz = G(\mu(t)) \tag{6.5}$$

for a.e. $t \in (0, m)$. Thanks to Theorem 5.1 for just one $t_0 \in (0, m)$ infer that f is a Gaussian **METTERE A POSTO** and that $\{|F| > t\}$ is a ball centred in $z_0 = (x_0, \omega_0)$. Then, still by theorem 5.1, since (6.5) holds a.e in $(0, m)$ and that f is always the same we have that also the other levels sets $\{\||F| > t\}$ are equivalent to balls centred at the same z_0. Finally, we can extend the result to every $t \in (0, m)$ because $|F| > t = \bigcup_{s>t} \{|F| > s\}$. Since theorem 5.1 is a "if and only if", these conditions on F and f are also sufficient to guarantee equality in (6.4). Clearly the same result holds for g which can be a Gaussian with different coefficient but the same centre **METTERE A POSTO**.
In the end it turns that $|F|$ is spherically symmetric and radially decreasing as claimed in theorem's statement.
Conditions for f and g imply that $\mathcal{V}g = e^{i\alpha}\mathcal{V}f$ for some $\alpha \in \mathbb{R}$. This provides equality in using Cauchy-Schwartz inequality in (6.2). Lastly we shall prove that also the first inequality in (6.2), that is

$$\left| \int_{\mathbb{R}^{2d}} F(z) \mathcal{V}f(z) \overline{\mathcal{V}g(z)} dz \right| \leq \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)| \cdot |\mathcal{V}g(z)| dz$$

becomes an equality, which is true if and only if

$$e^{-i\theta} \int_{\mathbb{R}^{2d}} F(z) \cdot |\mathcal{V}f(z)|^2 dz = \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz$$

for some $\theta \in \mathbb{R}$. This, in turn, is equivalent to the condition

$$e^{-i\theta} F(z) \cdot |\mathcal{V}f(z)|^2 = |F(z)| \cdot |\mathcal{V}f(z)|^2 \quad \text{for a.e. } z \in \mathbb{R}^{2d}$$

but since $|\mathcal{V}f(z)|^2 > 0$, equality in 6.2 with f and g as **METTERE A POSTO** occurs if and only if $F(z) = e^{i\theta}|F(z)|$. $\square$

Let's now consider the case where both p and q are neither 1 or $+\infty$. The result presented in [**nicolatilli_norm**] include the case ...

$$\|L_F\|_{L_2\to L_2} \leq \min\{\kappa_p^{d\kappa_p}A,\, \kappa_q^{d\kappa_q}B\}$$

Suppose that the minimum is given by $\kappa_p^{d\kappa_p}A$, therefore

$$\kappa_p^{d\kappa_p}A \leq \kappa_q^{d\kappa_q}B \iff \frac{B}{A} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d$$

We can check if the solution of the problem with just the L^p bound solves also the problem with both bounds, that is $F\|_{L^q} \leq B$, where F is given by ...

$$\|F\|_{L^q}^q = \int_{\mathbb{R}^{2d}} |F(z)|^q dz = ... = \lambda^q \left(\frac{p-1}{q}\right)^d$$

Since we want F to satisfy the L^q constraint we should have

$$\frac{B}{A} \geq \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{\frac{d}{q}}$$

It would be nice if this bound was less restrictive than the first one. Unfortunately that's not the case, in fact it's always true that

$$\left(\frac{p'}{q'}\right)^{\frac{1}{q'}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \geq 1$$

Following the path in [**nicolatilli_norm**] we obtain ...

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \implies u(t) = \frac{1}{d!}\left[-\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d,\ t \in (0, M)$$

Our main goal now is to show that multipliers λ_1, λ_2 are unique and both positive.

The easiest fact to prove is that both multipliers are not 0. In fact if one, say λ_2, was 0, we would obtain that the solution of our problem is the same as the one with just the L^p bound. But we already know that this function does not satisfy the L^q constraint hence it is impossible that $\lambda_2 = 0$.

Suppose now that one of the multipliers, say always λ_2, is negative. Consider an interval $[a, b] \subset (0, M)$ and a variation $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Thanks to the Gram-Schmidt process we can construct a variation orthogonal to t^{p-1}. Since η is arbitrary we can suppose that it is not orthogonal to t^{q-1}, in particular we can suppose that $\int_a^b t^{q-1}\eta(t)dt < 0$. Therefore the directional derivative of G along η is:

$$\begin{aligned}\int_a^b G'(u(t))\eta(t)dt &= \int_a^b (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \\ &= \lambda_2 \int_a^b t^{q-1}\eta(t)dt > 0\end{aligned}$$

which contradicts the fact that u is a maximizer.

Now that we now that both multipliers are positive we can prove that u is continuous, which is equivalent to say that $M = T$, where T is the unique positive number such that $\lambda_1 T^{p-1} + \lambda_2 T^{q-1} = 1$ (uniqueness of T follows from the positivity of multipliers).
We start supposing that $M < T$ which means that $\lim_{t\to M^-} u(t) > 0$. Consider the following variation

$$\eta(t) = \begin{cases} -1 + \alpha\dfrac{t}{M} + \beta, & t \in (M - M\delta, M) \\ 1, & t \in (M, M + M\delta) \\ 0, & \text{otherwise} \end{cases}$$

where $\delta > 0$ is small enough so that $M - M\delta > 0$ and $M + M\delta < T$ while α and β are constants, depending on δ, to be found. Since we want this to be an admissible variation we need to impose that η is orthogonal to t^{p-1} and t^{q-1}. For example, the first condition is:

$$0 = \int_{M-M\delta}^{M+M\delta} t^{p-1}\eta(t)dt = -\int_{M-M\delta}^{M} t^{p-1}dt + \int_{M-M\delta}^{M} t^{p-1}\left(\alpha\frac{t}{M} + \beta\right)dt + \int_{M}^{M+M\delta} t^{p-1}dt \overset{\tau=t/M}{=}$$
$$= M^p\int_{1-\delta}^{1} \tau^{p-1}(\alpha\tau+\beta)d\tau - M^p\int_{1-\delta}^{1}\tau^{p-1}d\tau + M^p\int_{1}^{1+\delta}\tau^{p-1}d\tau \overset{1/\delta}{\Longrightarrow}$$
$$\Longrightarrow \fint_{1-\delta}^{1} \tau^{p-1}(\alpha\tau+\beta)d\tau = \alpha\fint_{1-\delta}^{1}\tau^{p}d\tau + \beta\fint_{1-\delta}^{1}\tau^{p-1}d\tau = \fint_{1-\delta}^{1}\tau^{p-1}d\tau - \fint_{1}^{1+\delta}\tau^{p-1}d\tau$$

The equation stemming from the orthogonality with t^{q-1} is analogous. Therefore we obtained a nonhomogeneous linear system for α and β **VA RESO MEGLIO**

$$\begin{pmatrix} \fint_{1-\delta}^{1}\tau^{p}d\tau & \fint_{1-\delta}^{1}\tau^{p-1}d\tau \\ \fint_{1-\delta}^{1}\tau^{q}d\tau & \fint_{1-\delta}^{1}\tau^{q-1}d\tau \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} \fint_{1-\delta}^{1}\tau^{p-1}d\tau - \fint_{1}^{1+\delta}\tau^{p-1}d\tau \\ \fint_{1-\delta}^{1}\tau^{q-1}d\tau - \fint_{1}^{1+\delta}\tau^{q-1}d\tau \end{pmatrix} \tag{6.6}$$

This system has a unique solution if and only if the determinant of the matrix is not 0. We can show this directly:

$$\fint_{1-\delta}^{1}\tau^{p}d\tau\fint_{1-\delta}^{1}\tau^{q-1}d\tau - \fint_{1-\delta}^{1}\tau^{q}d\tau\fint_{1-\delta}^{1}\tau^{p-1}d\tau =$$
$$= \frac{1}{\delta^2}\int_{(1-\delta,1)^2}\left(\tau^p\sigma^{q-1} - \tau^{p-1}\sigma^q\right)d\tau d\sigma = \frac{1}{\delta^2}\int_{(1-\delta,1)^2}\tau^{p-1}\sigma^{q-1}(\tau-\sigma)\,d\tau d\sigma =$$
$$= \frac{1}{\delta^2}\left(\int_{Q_1}\tau^{p-1}\sigma^{q-1}(\tau-\sigma)\,d\tau d\sigma + \int_{Q_2}\tau^{p-1}\sigma^{q-1}(\tau-\sigma)\,d\tau d\sigma\right) =$$

where $Q_1 = (1-\delta,1)^2 \cap \{\tau > \sigma\}$ and $Q_2 = (1-\delta,1)^2 \cap \{\tau < \sigma\}$. In the second integral we can consider the change of variable that swaps τ and σ. In this case the new domain is Q_1, hence:

$$= \frac{1}{\delta^2}\int_{Q_1}\left(\tau^{p-1}\sigma^{q-1} - \tau^{q-1}\sigma^{p-1}\right)(\tau-\sigma)\,d\tau d\sigma$$

In Q_1 $\tau-\sigma>0$ and the sign of $\tau^{p-1}\sigma^{q-1}-\tau^{q-1}\sigma^{p-1}$ is constant, in fact:

$$\tau^{p-1}\sigma^{q-1}-\tau^{q-1}\sigma^{p-1}>0 \iff \left(\frac{\tau}{\sigma}\right)^{p-q}>1 \overset{\tau>\sigma}{\iff} p>q$$

Therefore the determinant of the matrix is always non 0.
The derivative of G along η is nonpositive because u is supposed to be a maximizer, therefore

$$\begin{aligned}0 \geq \int_{M-M\delta}^{M+M\delta} G'(u(t))\eta(t)dt = -\int_{M-M\delta}^{M}\left(\lambda_1 t^{p-1}+\lambda_2 t^{q-1}\right)dt +\\ +\int_{M-M\delta}^{M}\left(\lambda_1 t^{p-1}+\lambda_2 t^{q-1}\right)\left(\alpha\frac{t}{M}+\beta\right)dt+\int_{M}^{M+M\delta} dt =\\ = -\int_{M-M\delta}^{M}\left(\lambda_1 t^{p-1}+\lambda_2 t^{q-1}\right)dt+\lambda_1 M^p\int_{1-\delta}^{1} t^{p-1}(\alpha t+\beta)dt+\\ +\lambda_2 M^q\int_{1-\delta}^{1} t^{q-1}(\alpha t+\beta)dt+M\delta\end{aligned}$$

Dividing by $M\delta$ and rearranging we obtain:

$$⨍_{M-M\delta}^{M}\left(\lambda_1 t^{p-1}+\lambda_2 t^{q-1}\right)dt \geq 1+\lambda_1 M^{p-1}⨍_{1-\delta}^{1} t^{p-1}(\alpha t+\beta)dt+\lambda_2 M^{q-1}⨍_{1-\delta}^{1} t^{q-1}(\alpha t+\beta)dt \tag{6.7}$$

We notice that the last two terms are exactly the one that appear in the orthogonality condition, therefore to understand their behavior as δ approaches 0 we need to study the right-hand side of the system (6.6). If we expand the first term in its Taylor series with respect to δ we have:

$$\left(1-\frac{p-1}{2}\delta+o(\delta)\right)-\left(1+\frac{p-1}{2}\delta+o(\delta)\right)=-(p-1)\delta+o(\delta)$$

and the same is for the other term. Since they both are of order δ if we let $\delta\to 0^+$ in (6.7) we obtain

$$\lambda_1 M^{p-1}+\lambda_2 M^{q-1}\geq 1$$

Since the function $\lambda_1 t^{p-1}+\lambda_2 t^{q-1}$ is strictly increasing (because λ_1 and λ_2 are both positive) $M\geq T$ which is absurd because we supposed that $M<T$.

Lastly we shall prove that multipliers λ_1,λ_2, and hence maximizer, are unique. For this proof it is convenient to express u in a slightly different way:

$$u(t)=\frac{1}{d!}\left[\mathrm{Log}_-\left((c_1 t)^{p-1}+(c_2 t)^{q-1}\right)\right]^d$$

To emphasize that u is parametrized by c_1,c_2 we may write $u(t;c_1,c_2)$. Now we define

$$f(c_1,c_2)=p\int_0^T t^{p-1}u(t;c1,c2)dt,\quad g(c_1,c_2)=q\int_0^T t^{q-1}u(t;c1,c2)dt$$

We want to highlight that, even if not explicit, also T depends on c_1 and c_2. Nevertheless these functions are differentiable since both T and u are differentiable with respect to (c_1, c_2), functions $t^{p-1}u$ and $t^{q-1}u$ and their derivatives are bounded in $(0, T)$. Our maximizer u satisfies the constraints only if $f(c_1, c_2) = A^p$, $g(c_1, c_2) = B^q$. Therefore to prove uniqueness of the maximizer we need to show that level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.
First of all we are studying endpoints. For example, if $c_2 = 0$:

$$\begin{aligned} f(c_1{,}0) &= p\int_0^{1/c_1} t^{p-1}\frac{1}{d!}\left[-\log(c_1 t)^{p-1}\right]^d dt \overset{\tau=c_1 t}{=} \\ &= \frac{p(p-1)^d}{c_1^p d!}\int_0^1 \tau^{p-1}\left[-\log(\tau)\right]^d d\tau = \frac{\kappa_p^d}{c_1^p} = A^p \implies c_{1,f} = \frac{\kappa_p^{d/p}}{A} \end{aligned}$$

The same can be done for g and setting $c_1 = 0$ thus we obtain four points

$$c_{1,f} = \frac{\kappa_p^{d/p}}{A},\ c_{1,g} = \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B},\ c_{2,f} = \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A},\ c_{2,g} = \frac{\kappa_q^{d/q}}{B}$$

In the regime we are considering one has that $c_{1,f} < c_{1,g}$ and $c_{2,f} > c_{2,g}$, indeed

$$\begin{aligned} c_{1,f} < c_{1,g} &\iff \frac{\kappa_p^{d/p}}{A} < \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B} \iff \frac{B}{A} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{d/q} \\ c_{2,f} > c_{2,g} &\iff \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A} > \frac{\kappa_q^{d/q}}{B} \iff \frac{B}{A} > \kappa_q^{d\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{q}{p}\right)^{d/p} \end{aligned}$$

Since there is this dispositions of these points we expect there is an intersection between the level sets. Firstly we notice that, for every $c_1 \in (0, c_{1,f})$ there exist a unique value of c_2 for which $f(c_1, c_2) = A^p$. Indeed, from previous computations we notice that $f(c_1{,}0)$ is a decreasing function hence $f(c_1{,}0) > A^p$, while $\lim_{c_2\to+\infty} f(c_1, c_2) = 0$. The uniqueness of this value follows from strict monotonicity of $f(c_1, \cdot)$, in fact:

$$\frac{\partial f}{\partial c_1}(c_1, c_2) = -\frac{p(p-1)}{(d-1)!}c_1^{p-2}\int_0^T \frac{t^{2(p-1)}}{(c_1 t)^{p-1} + (c_2 t)^{q-1}}\left[-\log\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^{d-1} dt \tag{6.8}$$

is always strictly negative. We point out that the term $\frac{\partial T}{\partial c_1}(c_1, c_2)u(T; c_1, c_2)$ is zero because u is 0 in T. The same is true for g, therefore on the interval $(0, c_{1,f})$ the level sets of f and g can be seen as the graph of two functions φ, γ. Since $\frac{\partial f}{\partial c_2}, \frac{\partial g}{\partial c_2} < 0$ for every (c_1, c_2) from the implicit function theorem we have that φ and γ are differentiable with respect to c_1.

Possibile disegno??

After defining φ and γ we want to prove that $(\varphi - \gamma)' < 0$. Still from the implicit function theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}c_1}(\varphi - \gamma)(c_1) = -\frac{\frac{\partial f}{\partial c_1}(c_1, \varphi(c_1))}{\frac{\partial f}{\partial c_2}(c_1, \varphi(c_1))} + \frac{\frac{\partial g}{\partial c_1}(c_1, \gamma(c_1))}{\frac{\partial g}{\partial c_2}(c_1, \gamma(c_1))} < 0 \iff$$

$$\frac{\partial f}{\partial c_1}(c_1, \varphi(c_1))\frac{\partial g}{\partial c_2}(c_1, \gamma(c_1)) - \frac{\partial f}{\partial c_2}(c_1, \varphi(c_1))\frac{\partial g}{\partial c_1}(c_1, \gamma(c_1)) > 0$$

As for (6.8) the other derivatives are computed. To simplify the notation we define $h(t;c_1,c_2)=\frac{1}{(d-1)!}\frac{1}{(c_1t)^{p-1}+(c_2t)^{q-1}}\left[-\log\left((c_1t)^{p-1}+(c_2t)^{q-1}\right)\right]^{d-1}$. From Fubini's theorem we can write the product of the integrals as a double integral

$$p(p-1)q(q-1)c_1^{p-2}\gamma(c_1)^{q-2}\iint_{[0,T]^2} h(t;c_1,\varphi(c_2))h(s;c_1,\gamma(c_2))t^{2(p-1)}s^{2(q-1)}dtds-$$
$$p(q-1)q(p-1)c_1^{p-2}\varphi(c_1)^{q-2}\iint_{[0,T]^2} h(t;c_1,\varphi(c_2))h(s;c_1,\gamma(c_2))t^{p+q-2}s^{p+q-2}dtds$$

At the intersection point $\varphi(c_1)=\gamma(c_1)$ hence the sign of the previous expression depends only on the sign of

$$\iint_{[0,T]^2} h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))\left(t^{2(p-1)}s^{2(q-1)}-t^{p+q-2}s^{p+q-2}\right)dtds=$$
$$=\iint_{[0,T]^2} h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))t^{p-2}s^{q-2}\left(t^ps^q-t^qs^p\right)dtds$$

In order to simplify the notation once again we set $H(t,s;c_1)=h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))$. Let $T_1=[0,T]^2\cap\{t>s\}$ and $T_2=[0,T]^2\cap\{t<s\}$. We can split the above integral in two parts

$$\iint_{T_1} H(t,s;c_1)t^{p-2}s^{q-2}\left(t^ps^q-t^qs^p\right)dtds+\iint_{T_2} H(t,s;c_1)t^{p-2}s^{q-2}\left(t^ps^q-t^qs^p\right)dtds$$

We can exchange t with s in the second integral. With this change of variables the domain of integration becomes T_1 and since H is symmetric in t and s we have that the previous quantity is equal to

$$\iint_{T_1} H(t,s;c_1)\left(t^{p-2}s^{q-2}-t^{q-2}s^{p-2}\right)\left(t^ps^q-t^qs^p\right)dtds=$$
$$\iint_{T_1} H(t,s;c_1)\frac{1}{t^2s^2}\left(t^ps^q-t^qs^p\right)^2dtds$$

which is strictly positive.

Now we are able to prove the uniqueness of multipliers.
First of all, since $(\varphi-\gamma)'<0$ whenever $\varphi(c_1)=\gamma(c_1)$, for every point of intersection there exist $\delta>0$ such that $\varphi(t)>\gamma(t)$ for $t\in(c_1-\delta,c_1)$ while $\varphi(t)<\gamma(t)$ for $t\in(c_1,c_1+\delta)$. Define $c_1^*:=\sup\{c_1\in[0,c_{1,f}]:\forall t\in[0,c_1]\ \varphi(t)\geq\gamma(t)\}$. This is an intersection point between φ and γ (if $\varphi(c_1^*)>\gamma(c_1^*)$ due to continuity there would be $\varepsilon>0$ such that $\varphi(c_1^*+\varepsilon)>\gamma(c_1^*+\varepsilon)$ which contradicts the definition of c_1^*) and it is the first one, because we saw that after every intersection point there is an interval where $\varphi<\gamma$. Lastly, since $\varphi(0)>\gamma(0)$ and $\varphi(c_{1,f})=0<\gamma(c_1,f)$ we have that $0<c_1^*<c_{1,f}$.
Suppose now that there is a second point of intersection $\tilde{c}_1$ after the first one. Since immediately after c_1^* we have that φ becomes smaller than γ this second point of intersection is given by $\tilde{c}_1=\sup\{c_1\in[c_1^*,c_{1,f}]:\forall t\in[c_1^*,c_1]\ \varphi(t)\leq\gamma(t)\}$. Considering that this is an intersection point, there exist an interval before $\tilde{c}_1$ where φ is strictly grater than γ which

is absurd, hence c_1^* is the only intersection point between φ and γ.
Therefore $(c_1^*, \varphi(c_1^*) = c_2^*)$ is the unique pair of multipliers for which $p \int_0^T t^{p-1} u(t; c_1^*, c_2^*) dt = A^p,\ q \int_0^T t^{q-1} u(t; c_1^*, c_2^*) dt = B^q$ and in the end $u(t; c_1^*, c_2^*)$ is the unique maximizer for **CITARE PROBLEMA VARIAZIONALE**