POLITECNICO DI TORINO

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# Some recent results on the norm of localization operators

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# Chapter 1

# Introduction

Con il blu indico frasi da mettere a posto o dove manca una citazione.
Con il rosso indico frasi da controllare se sono corrette.

# Chapter 2

# Preliminaries

In this first chapter we briefly recall some basics definition and results about functional analysis and Fourier transform. In section 2.1 basic concepts about operators between Banach spaces are presented. In section 2.2 Fourier transform is defined and essential properties are given.

## 2.1 Basics of Functional Analysis

In this section we focus our attention on linear operators between Banach spaces. Across the section a generic Banach space will be denoted as $X$ (or $Y$) endowed with the norm $\|\cdot\|_X$. In case $X$ is an Hilbert space we will denote its inner product as $\langle\cdot,\cdot\rangle_X$.
A generic linear operator between two Banach spaces $X$ and $Y$ will be denoted as $T : X \to Y$. As a standard notation, the image of $x \in X$ through $T$ will be indicated as $T(x)$, or equivalently as $Tx$.

**Definition 2.1.** *A linear operator $T : X \to Y$ is **bounded** if there exist $C > 0$ such that*

$$\|Tx\|_Y \leq C\|x\|_X \quad \forall x \in X \tag{2.1}$$

For linear operator boundedness is strictly related to continuity as the subsequent theorem states.

**Theorem 2.2.** *For a linear operator $T$ the following statements are equivalent:*

- *$T$ is continuous*
- *$T$ is bounded.*

We denote the set of linear bounded (continuous) operators from $X$ to $Y$ as $\mathscr{B}(X, Y)$, while if $X = Y$ we will just write $\mathscr{B}(X)$.
For the sake of completeness we mention that actually, for linear operators, boundedness is equivalent to uniform continuity.

After this we define the *norm* of an operator

**Definition 2.3.** *Given a linear bounded operator $T$ we define its **norm** as the following number:*

$$\|T\| := \inf\{C > 0 : \|Tx\|_Y \leq C\|x\|_X \ \forall x \in X\} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X \setminus \{0\}\right\}$$

The proof of the equivalence between two definition is straightforward. We see that the norm of an operator is the best constant for which boundedness property (2.1) holds. Sometimes, in order to emphasize the spaces between which $T$ operates, we may write the norm of $T$ as $\|T\|_{X\to Y}$.
In the following we will mostly deal with $X$ and $Y$ being $L^2(\mathbb{R}^d)$, which is an Hilbert space. For operators between Hilbert spaces we can give the norm of an operator by means of the dual norm:

$$\|T\| = \sup\{|\langle Tx, y\rangle_X| : x, y \in X, \ \ \|x\|_X = \|y\|_X = 1\}$$

An important class of operators is the class of *compact operators*.

**Definition 2.4.** *A linear bounded operator $T$ is **compact** if for every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ the sequence of the images $\{Tx_n\}_{n\in\mathbb{N}} \subset Y$ has a converging subsequence.*

The property of compactness can be stated in multiple ways **SERVE SCRIVERLE?**.
Now we suppose $X$ and $Y$ to be Hilbert spaces. Given $T \in \mathscr{B}(X, Y)$ there exist a unique $T^* \in \mathscr{B}(Y, X)$ such that:

$$\langle Tx, y\rangle_X = \langle x, T^*y\rangle_Y \quad \forall x \in X, \, y \in Y$$

$T^*$ is called the **adjoint** operator of $T$. In the particular case in which $T : X \to X$, if $T = T^*$, we say that $T$ is **self-adjoint**.
From now on we suppose that $X$ is over the field $\mathbb{C}$ and that $T \in \mathscr{B}(X)$.

**Definition 2.5.** *The set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$ is called the **spectrum** of $T$.*

For operators between finite-dimensional spaces (matrices) the spectrum is made up of *eigenvalues*, those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. On the other hand, when dealing with infinite-dimensional spaces, this is no more true. Eigenvalues are in the so called *point spectrum*, which in general is just a part of the whole spectrum.
If an operator is compact or self-adjoint its spectrum has some additional properties.

**Theorem 2.6** (Fredholm's alternative)**.** *Let $T \in \mathscr{B}(X)$ be a compact operator. Then for $T - I$ one and only one of the following happens:*

- *$T$ is invertible*
- *$T$ is not injective*

Therefore for compact operators, all the values in the spectrum, except at most for 0, are eigenvalues.
Another fundamental result arises if we study the spectrum of compact and self-adjoint operators.

**Theorem 2.7.** *Let $X$ be a separable Hilbert space and $T \in \mathscr{B}(X)$ a compact and self-adjoint operator. Then there exist an orthonormal basis of $X$ composed of eigenvectors of $T$*

Hence self-adjoint compact operators can always be diagonalized in some suitable basis ([1]).

Now we are going to consider two important classes of operators: *trace class* operators and *Hilbert-Schmidt* operators.
The trace of an operator can be defined as it is for matrices.

**Definition 2.8.** *Let $X$ be an Hilbert space. An operator $T \in \mathscr{B}(X)$ is said* ***positive*** *if*

$$\langle Tx, x\rangle \geq 0 \quad \forall x \in X$$

**Definition 2.9.** *Let $X$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Given $T \in \mathscr{B}(X)$ a positive operator we define the* ***trace*** *of $T$ as*

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$$

Actually one should show that the definition is well posed, namely that is independent of the basis.

**Proposition 2.10.** *The definition of* tr *given by* (2.9) *is independent of the basis.*

*Proof.* contenuto... □

The definition of trace is given only for positive operators. If we want to deal with general ones it is sufficient to consider $|T| = \sqrt{T^*T}$ **LA PARTE SU RADICE QUADRATA E MODULO DI UN OPERATORE VA AGGIUNTA??**

**Definition 2.11.** *An operator $T \in \mathscr{B}(X)$ is called* ***trace class*** *if and only if* $\mathrm{tr}|T| < \infty$

**Theorem 2.12.** *For every trace class operator* $\|T\| \leq \mathrm{tr}(T)$.

*Proof.* contenuto... □

**Theorem 2.13.** *Every trace class operator is compact.*

*Proof.* contenuto... □

**Definition 2.14.** *An operator $T \in \mathscr{B}(X)$ is called* ***Hilbert-Schmidt*** *if and only if* $\mathrm{tr}(T^*T) < \infty$. *We define the* ***Hilbert-Schmidt norm*** *of an operator as* $\|T\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(T^*T)}$.

**Theorem 2.15.** *For every Hilbert-Schmidt operator* $\|T\| \leq \|T\|_{\mathrm{HS}}$.

**Theorem 2.16.** *Every Hilbert-Schmidt operator is compact.*

*Proof.* contenuto... □

The importance of Hilbert-Schmidt operator is related to the following theorem.

**Theorem 2.17.** *Let $X = L^2(\mathbb{R}^d)$. Then $T \in \mathscr{B}(L^2(\mathbb{R}^d))$ is Hilbert-Schmidt if only if there exist a function $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, called integral kernel, such that*

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy \quad \forall f \in L^2(\mathbb{R}^d). \tag{2.2}$$

*Moreover* $\mathrm{tr}(T^*T) = \int_{\mathbb{R}^{2d}} |K(x,y)|^2 dxdy$.

*Proof.* contenuto... □

In light of this theorem, operators defined by (2.2) are called *Hilbert-Schmidt integral operators.*

**Proposition 2.18.** *An Hilbert-Schmidt integral operator with kernel $K$ is self-adjoint if and only if $K(x,y) = \overline{K(y,x)}$.*

*Proof.* contenuto... □

- Norma operatoriale (FATTO)
- Operatori autoaggiunti? (FATTO)
- Operatori compatti (FATTO)
- Spettro operatori (FATTO)
- Operatori di classe traccia
- Operatori di Hilbert-Schmidt?

## 2.2 Fourier Transform and its properties

**Definition 2.19.** *Let $f \in L^1(\mathbb{R}^d)$. We define the* ***Fourier transform*** *of $f$ the function*

$$\mathscr{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f(t)dt \tag{2.3}$$

It is straightforward to see that the definition is well-posed and that $\mathscr{F}f \in L^\infty(\mathbb{R}^d)$ with $\|\mathscr{F}f\|_\infty \leq \|f\|_1$. Therefore $\mathscr{F}$ can be seen as a linear operator between $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ with $\|\mathscr{F}\| = 1$ (from the previous inequality actually we saw that $\|\mathscr{F}\| \leq 1$ but if we take $f \geq 0$ a.e. we have that $\hat{f}(0) = \|f\|_1$ that gives us the equality).
The Fourier transform of an $L^1(\mathbb{R}^d)$ is not only bounded, as stated by the *Riemann-Lebesgue lemma.*

**Theorem 2.20** (Riemann-Lebesgue lemma). *Let $f \in L^1(\mathbb{R}^d)$. Therefore $\hat{f} \in C_0(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C}$ continuous such that $\lim_{|t|\to\infty} |f(t)| = 0\}$.*

**Definition 2.21.** *Let $f \in L^1(\mathbb{R}^d)$. We define the **inverse Fourier transform** of the function $f$*

$$\mathscr{F}^{-1} f(t) = \check{f}(t) := \int_{\mathbb{R}^d} e^{2\pi i \omega \cdot t} f(\omega) d\omega \tag{2.4}$$

The inverse Fourier transform is denoted with $\mathscr{F}^{-1}$ because it is actually the inverse operator of the Fourier transform as stated by the *inversion theorem*.

**Theorem 2.22** (Inversion theorem)**.** *Let $f \in L^1(\mathbb{R}^d)$ and suppose that also $\hat{f} \in L^1(\mathbb{R}^d)$. Then*

$$f(t) = \mathscr{F}^{-1} \circ \mathscr{F} f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi i \omega \cdot t} d\omega$$

The Fourier transform is intimately related to regularity and decay properties. The duality between these two is stated in the following theorems.

**Theorem 2.23.** *Let $f \in L^1(\mathbb{R}^d)$. If $|t|^k f \in L^1(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ then $\hat{f} \in C_0^k(\mathbb{R}^d)$ and the following holds for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$:*

$$\mathscr{F}\left((-2\pi i t)^{\alpha} f\right)(\omega) = \partial^{\alpha} \mathscr{F} f(\omega).$$

**Theorem 2.24.** *Let $f \in C^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$. If $f, \partial^{\alpha} f \in L^1(\mathbb{R}^d)$ for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ then*

$$\mathscr{F}\left(\partial^{\alpha} f\right)(\omega) = (2\pi i \omega)^{\alpha} \mathscr{F} f(\omega)$$

*In particular this implies that $\hat{f}(\omega) = o\left(|\omega|^{-k}\right)$ as $|\omega| \to \infty$.*

To sum up, previous theorems state a duality between regularity and decay: if a function is smooth then its Fourier transform decays rapidly and vice versa.

If $f$ is in $L^2(\mathbb{R}^d)$, the integral in (2.3) in general will not converge. Nevertheless we can define the Fourier transform of an $L^2$ function through a density argument. For example, one can use $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, which is a dense subspace of $L^2(\mathbb{R}^d)$. On this space one can show that the Fourier transform is an isometry with respect to the $L^2$ norm and therefore it extends to an isometry on the whole $L^2(\mathbb{R}^d)$. This is stated by the *Plancherel theorem*.

**Theorem 2.25** (Plancherel theorem)**.** *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ then $\|f\|_2 = \|\hat{f}\|_2$.*

Thanks to the polarization identity this implies that $\mathscr{F}$ preserves the inner product in $L^2(\mathbb{R}^d)$:

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d)} \quad \forall f, g \in L^2(\mathbb{R}^d) \tag{2.5}$$

therefore the Fourier transform $\mathscr{F}$ is a unitary operator on $L^2(\mathbb{R}^d)$. Result (2.5) is called *Parseval formula*.

Lastly, we introduce two fundamental operators in Fourier analysis. Given $x, \omega \in \mathbb{R}^d$ we define the *time-shift* (or translation) operator $T_x$

$$T_x f(t) = f(t - x) \tag{2.6}$$

and the *modulation* operator $M_\omega$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \tag{2.7}$$

Va aggiunta anche la composizione e le loro proprietà? These can be combined into a *time-frequency shift* operator

$$\pi(x, \omega)f(t) = M_\omega T_x f(t) \tag{2.8}$$

**Theorem 2.26** (Hausdorff-Young)**.** *Let $1 \leq p \leq 2$ and let $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\mathscr{F} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ and $\|\hat{f}\|_{p'} \leq \|f\|_p$.*

In the following we will need the sharp version of the Hausdorff-Young inequality:

$$\|\hat{f}\|_{p'} \leq \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{d/2} \|f\|_p = A_p^d \|f\|_p \tag{2.9}$$

where $A_p$ is the so-called Babenko-Bechner constant. This constant appears also in the sharp version of Young's theorem

**Theorem 2.27** (Young)**.** *Given $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, suppose that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $r \geq 1$. Then $f * g \in L^r(\mathbb{R}^d)$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.*

Just like the Hausdorff-Young inequality, Young's inequality holds in a sharp version:

$$\|f * g\|_r \leq (A_p A_q A_{r'})^d \|f\|_p \|g\|_q. \tag{2.10}$$

- Defizione trasformata (FATTO)
- Teorema di Plancherel (FATTO)
- Formula di inversione(FATTO)
- Proprietà di decadimento e regolarità (FATTO)
- Proprietà operatori di traslazione e modulazione??
- Disuguaglianza di Hausdorff-Young ??

# Chapter 3

# Short-Time Fourier Transform

## 3.1 STFT

The *short-time Fourier transform* or *STFT* is a powerful tool, introduced by Gabor in [6], used to study properties of a signal locally both in time and frequency. The main idea behind the STFT is the following: if we want some information of the spectrum of a signal around a specific time, say $T$, we could choose an interval $(T-\Delta T, T+\Delta T)$ and take the Fourier transform of $f\chi_{(T-\Delta T,T+\Delta T)}$. Usually multiplying by a characteristic function will not give us a regular function (not even continuous) and in light of the duality between regularity and decay, the Fourier transform of $f\chi_{(T-\Delta T,T+\Delta T)}$ will not decay rapidly. Therefore a sharp cutoff in the time domain will result in a "bad" localization in the frequency domain. In order to avoid this kind of problems we could think to multiply the signal $f$ by a smooth function.

**Definition 3.1.** *Fix a function $\phi \neq 0$ called* window function*. The* ***short-time Fourier transform*** *of a function $f$ with window $\phi$ is defined as*

$$\mathcal{V}_\phi f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{\phi(t-x)}e^{-2\pi i\omega\cdot t}dt, \quad (x,\omega) \in \mathbb{R}^{2d} \tag{3.1}$$

In the above definition we did not specify where $f$ and $\phi$ are chosen. Since we are taking the Fourier transform of the function $fT_x\overline{\phi}$, the STFT is well defined whenever the Fourier transform of this function is. For example if both $f$ and $\phi$ are in $L^2(\mathbb{R}^d)$ then $fT_x\overline{\phi}$ is in $L^1(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$ and so the integral in (3.1) is defined. In this special case the STFT can be written as a scalar product in $L^2(\mathbb{R}^d)$:

$$\mathcal{V}_\phi f(x,\omega) = \langle f, M_\omega T_x \phi\rangle = \langle f, \pi(x,\omega)\phi\rangle$$

In general, the STFT of $f$ with respect to $\phi$ will be defined whenever $\langle f, M_\omega T_x\phi\rangle$ is an expression of some sort of duality. For example, if $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ then $M_\omega T_x\overline{\phi} \in \mathcal{S}(\mathbb{R}^d)$, therefore $\langle f, M_\omega T_x\phi\rangle$ can be seen as the usual duality between tempered distributions and functions in the Schwartz space.
Aggiungere scritture equivalenti della STFT??

### 3.1.1 Properties of STFT

In this section we will present and prove some properties about the STFT. An excellent reference is [8].

**Theorem 3.2.** *Let $f_1, f_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$. Then $\mathcal{V}_{\phi_i} f_i \in L^2(\mathbb{R}^{2d})$ and the following holds:*

$$\langle \mathcal{V}_{\phi_1} f_1, \mathcal{V}_{\phi_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle \phi_1, \phi_2 \rangle} \tag{3.2}$$

*Proof.* contenuto... □

**Corollary 3.3.** *If $f, \phi \in L^2(\mathbb{R}^d)$ then*

$$\|\mathcal{V}_\phi f\|_2 = \|f\|_2 \|\phi\|_2 \tag{3.3}$$

*In particular if $\|\phi\|_2 = 1$ we see that $\mathcal{V}_\phi$ is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$.*

*Proof.* contenuto... □

From the Cauchy-Schwarz inequality we immediately see that $\mathcal{V}_\phi f$ is in $L^\infty(\mathbb{R}^{2d})$:

$$|\mathcal{V}_\phi f(x,\omega)| = |\langle f, M_\omega T_x \phi \rangle| \overset{\text{C}-\text{S}}{\leq} \|f\|_2 \|M_\omega T_x \phi\|_2 = \|f\|_2 \|\phi\|_2. \tag{3.4}$$

Combing this with (3.3) and using a simple interpolation argument we see that $\mathcal{V}_\phi f \in L^p(\mathbb{R}^{2d})$ for every $p \in [2, \infty)$ and that $\|\mathcal{V}_\phi f\|_p \leq \|f\|_2 \|\phi\|_2$. This result is improved by the following theorem due to Lieb.

**Theorem 3.4.** *If $f, \phi \in L^2(\mathbb{R}^d)$ and $2 \leq p \leq \infty$, then*

$$\int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi(x,\omega)|^p dx d\omega \leq \left(\frac{2}{p}\right)^d \|f\|_2^p \cdot \|g\|_2^p \tag{3.5}$$

*Proof.* For this proof we will need to use an alternative form of the STFT:

$$\mathcal{V}_\phi f(x,\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega \cdot t} \overline{\phi(t-x)} dt = \mathscr{F}(f T_x \overline{\phi})(\omega)$$

Using Cauchy-Schwarz inequality it is immediate to see that $f T_x \overline{\phi} \in L^1(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. In addition to that, since $\mathcal{V}_\phi f = \mathscr{F}(f T_x \overline{\phi}) \in L^2(\mathbb{R}^{2d})$, from Fubini's theorem we can say that $\mathscr{F}(f T_x \overline{\phi}) \in L^2(\mathbb{R}^d)$ for almost every $x \in R^d$ and therefore also $f T_x \overline{\phi} \in L^2(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$. Through an interpolation argument we obtain that $f T_x \overline{\phi} \in L^q(\mathbb{R}^d)$ for every $q \in$ [1,2].

We start considering the $L^p$ norm of $\mathcal{V}_\phi f$:

$$\begin{aligned}
\|\mathcal{V}_\phi f\|_p = \left(\int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x,\omega)|^p dx d\omega\right)^{1/p} &\overset{\text{Tonelli}}{=} \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathcal{V}_\phi f(x,\omega)|^p d\omega\right) dx\right]^{1/p} = \\
= \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathscr{F}(f T_x \overline{\phi})(\omega)|^p d\omega\right) dx\right]^{1/p} &\overset{(2.10)}{\leq} \\
\leq A_{p'}^d \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(t)\overline{\phi(t-x)}|^{p'} dt\right)^{p/p'} dx\right]^{1/p} & 
\end{aligned} \tag{3.6}$$

where the use of Young's inequality (2.10) is justified since we noticed that $fT_x\overline{\phi}$ is in $L^q(\mathbb{R}^d)$ for every $q \in [1,2]$, so in particular it is in $L^{p'}(\mathbb{R}^d)$. Letting $\phi^*(t) = \overline{\phi(-t)}$ and considering the inner integral we have

$$\int_{\mathbb{R}^d} |f(t)\overline{\phi(t-x)}|^{p'} dt = \int_{\mathbb{R}^d} |f(t)|^{p'} |\phi^*(x-t)|^{p'} dt = \left(|f|^{p'} * |\phi^*|^{p'}\right)(x),$$

so the expression in (3.6) is the $L^{p/p'}(\mathbb{R}^d)$ norm of $|f|^{p'} * |\phi^*|^{p'}$. Since both $f$ and $\phi$ are in $L^2(\mathbb{R}^d)$ and $p' \leq 2$ we have that $|f|^{p'}, |\phi^*|^{p'} \in L^{2/p'}(\mathbb{R}^d)$. Thanks to Young's theorem 2.27 $|f|^{p'} * |\phi^*|^{p'}$ belongs to $L^r(\mathbb{R}^d)$, where $r$ is given by:

$$\frac{1}{(2/p')} + \frac{1}{(2/p')} = 1 + \frac{1}{r} \implies r = \frac{1}{p'-1} = \frac{1}{\frac{p}{p-1} - 1} = p - 1 = \frac{p}{p'}$$

therefore, using the sharp version of Young's inequality (2.10) in (3.6) we obtain:

$$\|\mathcal{V}_\phi f\|_p \leq A^d_{p'} \left( A^d_{2/p'} A^d_{2/p'} A^d_{(p/p')'} \| \, |f|^{p'} \|_{2/p'} \| \, |\phi^*|^{p'} \|_{2/p'} \right)^{1/p'}.$$

However $\| \, |f|^{p'} \|_{2/p'} = (\int_{\mathbb{R}^d} (|f(x)|^{p'})^{2/p'} dx)^{p'/2} = \|f\|_2^{p'}$ and from a direct calculation QUALE?? one can see that $A^d_{p'} A^{2d/p'}_{2/p'} A^{d/p'}_{(p/p')'} = (2/p)^d$, which corresponds to the desired result. □

From a direct computation one can see that the adjoint operator of the STFT operator $\mathcal{V}_\phi$ is given by the following expression:

$$\mathcal{V}^*_\phi g(t) = \int_{\mathbb{R}^{2d}} g(x,\omega)\phi(t-x)e^{2\pi i \omega \cdot t} dx d\omega = \int_{\mathbb{R}^{2d}} g(x,\omega) M_\omega T_x \phi(t) dx d\omega \quad \forall g \in L^2(\mathbb{R}^{2d}) \tag{3.7}$$

This adjoint operator appears in the following nice property

**Theorem 3.5.** *Let $f \in L^2(\mathbb{R}^d)$ and $\phi, \gamma \in L^2(\mathbb{R}^{2d})$ such that $\langle \phi, \gamma \rangle \neq 0$. Then:*

$$f(t) = \frac{1}{\langle \phi, \gamma \rangle} \mathcal{V}^*_\gamma \mathcal{V}_\phi f(t) = \frac{1}{\langle \phi, \gamma \rangle} \int_{\mathbb{R}^{2d}} \mathcal{V}_\phi f(x,\omega) M_\omega T_x \gamma(t) dx d\omega \quad \forall t \in \mathbb{R}^d \tag{3.8}$$

*Proof.* **VA MESSA??** □

Therefore the adjoint operator $\mathcal{V}^*_\gamma$ acts, in some sense, as an inverse operator. This will be of paramount importance in the following.

- Relazione di ortogonalità

- Formula di inversione

## 3.2 Bargmann Transform and Fock Space

Throughout this section we will consider the STFT with Gaussian window. We choose

$$\varphi(x) = 2^{d/4} e^{-\pi|x|^2} \tag{3.9}$$

where $|x|^2 = \sum_{k=1}^d x_k^2$ is the Euclidean norm of $x$ in $\mathbb{R}^d$. The factor $2^{d/4}$ is chosen so that $\|\varphi\|_2 = 1$. The STFT with Gaussian window becomes

$$\mathcal{V}_\varphi f(x,\omega) = 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi|t-x|^2} e^{-2\pi i\omega\cdot t} dt \tag{3.10}$$

Our aim now is to rearrange the terms in the above expression in order to make $z = x + i\omega \in \mathbb{C}^d$ appear. We want to highlight the fact that when talking about complex quantities $|z|^2 = z\overline{z} = |x|^2 + |\omega|^2$. Aggiungere remark sulla notazione usata.

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi|t|^2 + 2\pi x\cdot t - \pi|\omega|^2} e^{-2\pi i\omega\cdot t} dt \\
&= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi|t|^2} e^{2\pi(x-i\omega)\cdot t} e^{-\frac{\pi}{2}\left(|x|^2 - 2ix\cdot\omega - |\omega|^2\right)} e^{-\frac{\pi}{2}\left(|x|^2 + |\omega|^2 + 2ix\cdot\omega\right)} dt \\
&= 2^{d/4} e^{-\pi i x\cdot\omega} e^{-\frac{\pi}{2}\left(|x|^2 + |\omega|^2\right)} \int_{\mathbb{R}^d} f(t) e^{-\pi|t|^2} e^{2\pi(x-i\omega)\cdot t} e^{-\frac{\pi}{2}(x-i\omega)^2} dt
\end{aligned}$$

The rearrangement may seem arbitrary but actually is done is such a way that inside the integral $x$ and $\omega$ enter only via $\overline{z}$.

**Definition 3.6.** *The **Bargmann transform** of a function $f$ on $\mathbb{R}^d$ is the function $\mathcal{B}f$ on $\mathbb{C}^d$ given by*

$$\mathcal{B}f(z) = 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{2\pi t\cdot z - \pi|t|^2 - \frac{\pi}{2}z^2} dt \tag{3.11}$$

We recall that a function defined over $C^d$ is *entire* if it is holomorphic over all $\mathbb{C}^d$.

**Definition 3.7.** *The **Fock space** $\mathcal{F}^2(\mathbb{C}^d)$ is the Hilbert space of all entire functions $F$ on $\mathbb{C}^d$ for which the norm*

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} dz \tag{3.12}$$

*is finite.*

Clearly the norm of the Fock space is induced by the following scalar product

$$\langle F, G\rangle_{\mathcal{F}} \int_{\mathbb{C}^d} F(z)\overline{G(z)} e^{-\pi|z|^2} dz \tag{3.13}$$

**Proposition 3.8.** *If $f$ is a function on $\mathbb{R}^d$ with polynomial growth then its Bargmann transform $\mathcal{B}f$ is an entire function on $\mathbb{C}^d$. Moreover, letting $z = x + i\omega$, the Bargmann transform of $f$ is related to its STFT through the following*

$$\mathcal{V}_\varphi f(x,-\omega) = e^{\pi i x\cdot\omega} \mathcal{B}f(z) e^{-\pi|z|^2/2} \tag{3.14}$$

*Proof.* contenuto... □

**Proposition 3.9.** *If $f \in L^2(\mathbb{R}^d)$ then*

$$\|f\|_2 = \left( \int_{\mathbb{C}^d} |\mathcal{B}f(z)|^2 e^{-\pi|z|^2} dz \right)^{1/2} = \|\mathcal{B}f\|_{\mathcal{F}}. \tag{3.15}$$

*Thus $\mathcal{B}$ is an isometry from $L^2(\mathbb{R}^d)$ into $\mathcal{F}^2(\mathbb{C}^d)$.*

# Chapter 4

# Localization Operators

One of the main problems in signal analysis or, in general, time-frequency analysis is to extract some informations about signal in order to analyse it. We saw that STFT can be a tool for this purpose, however it has some practical problems: it doubles the dimension of the output and it is highly redundant (SE AVANZA TEMPO).
In this chapter we see two possible ways to deal with the problem of creating operators able to localize a signal both in time and frequency.

## 4.1 Localization with projections

The most straightforward way to localize a signal, say in the time domain, is to use a sharp cut-off, which means a characteristic function. If we suppose to have a signal $f \in L^2(\mathbb{R}^d)$ and we want to localize it in a measurable subset $T \subseteq \mathbb{R}^d$ of the time domain we can consider the natural projection operator

$$P_T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad P_T f(t) = \chi_T(t) f(t) \tag{4.1}$$

This is clearly a projection operator, which means that $P_T^2 = P_T = P_T^*$.
In the same fashion we can define an operator able to localize on a measurable subset $\Omega \subseteq \mathbb{R}^d$ in the frequency domain. Their definition it is not as direct as the one for time projections but it is still easy to understand

$$Q_\Omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad Q_\Omega f(t) = \mathscr{F}^{-1} \left( \chi_\Omega \mathscr{F} f \right)(t) = \int_\Omega \hat{f}(\omega) e^{2\pi i \omega \cdot t} d\omega \tag{4.2}$$

It is also quite simple to show that this is a projection operator

$$\begin{aligned} Q_\Omega^2 &= \mathscr{F}^{-1}\chi_\Omega \mathscr{F} \mathscr{F}^{-1} \chi_\Omega \mathscr{F} = \mathscr{F}^{-1}\chi_\Omega \chi_\Omega \mathscr{F} = \mathscr{F}^{-1}\chi_\Omega \mathscr{F} = Q_\Omega \\ Q_\Omega^* &= \left( \mathscr{F}^{-1}\chi_\Omega \mathscr{F} \right)^* = \mathscr{F}^* \chi_\Omega^* \left( F^{-1} \right)^* = \mathscr{F}^{-1}\chi_\Omega \mathscr{F} = Q_\Omega \end{aligned}$$

where we used the fact that the Fourier transform is a unitary operator on $L^2(\mathbb{R}^d)$, namely that $\mathscr{F}^* = \mathscr{F}^{-1}$.

Moreover, both operators have norm less or equal than 1, independently of $T$ and $\Omega$:

$$\|P_T f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(T)} \leq \|f\|_{L^2(\mathbb{R}^d)}$$
$$\|Q_\Omega f\|_{L^2(\mathbb{R}^d)} = \|\mathscr{F}^{-1}\chi_\Omega \mathscr{F} f\|_{L^2(\mathbb{R}^d)} \overset{\text{Plancherel}}{=} \|\mathscr{F} f\|_{L^2(\Omega)} \leq \|\mathscr{F} f\|_{L^2(\mathbb{R}^d)} \overset{\text{Plancherel}}{=} \|f\|_{L^2(\mathbb{R}^d)}$$

After defining these projection operators we may think to combine them into a single operator

$$Q_\Omega P_T,\ P_T Q_\Omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

which hopefully is able to localize a signal both in time and frequency "near" to the set $T \times \Omega$.
It is clear that these operators are linear and bounded, in particular their norms are less or equal than 1.

Up to now the only (essential) hypothesis on $T$ and $\Omega$ is that they are measurable. Clearly, by adding some requirements on $T$ and $\Omega$ we expect $Q_\Omega P_T$ and $P_T Q_\Omega$ to gain some properties.

**Proposition 4.1.** *Let $T, \Omega \subset \mathbb{R}^d$ with finite measure. Then $Q_\Omega P_T$ and $P_T Q_\Omega$ are Hilbert-Schmidt integral operators of the form*

$$Q_\Omega P_T f(x) = \int_{\mathbb{R}^d} K(x,t) f(t) dt \tag{4.3}$$
$$P_T Q_\Omega f(x) = \int_{\mathbb{R}^d} \overline{K(t,x)} f(t) dt \tag{4.4}$$

*where*

$$K(x,t) = \chi_T(t) \int_\Omega e^{2\pi i \omega\cdot(x-t)} d\omega \tag{4.5}$$

*which has* $\|K\|_{L^2(\mathbb{R}^{2d})} = \sqrt{|T||\Omega|}$.

*Proof.* contenuto... □

If we compare the integral kernels of $Q_\Omega P_T$ and $P_T Q_\Omega$ we see that $K(x,t) \neq \overline{K(t,x)}$, hence, by proposition 2.18, we immediately conclude that both operator are not self-adjoint. Since it is better to deal with self-adjoint operators when it comes to spectral properties, it would be nice if we could construct those starting from $Q_\Omega P_T$ and $P_T Q_\Omega$. This is easily done by considering

$$(Q_\Omega P_T)^* Q_\Omega P_T = P_T^* Q_\Omega^* Q_\Omega P_T = P_T Q_\Omega P_T \tag{4.6}$$
$$(P_T Q_\Omega)^* P_T Q_\Omega = Q_\Omega^* P_T^* P_T Q_\Omega = Q_\Omega P_T Q_\Omega \tag{4.7}$$

These are, by construction, self-adjoint operators, and since both $Q_\Omega P_T$ and $P_T Q_\Omega$ are compact operators (thanks to Proposition 4.1 and Theorem 2.16) they are also compact (we recall that composition between a general operator and a compact one is compact). Hence, by Theorem 2.7 they can be diagonalized.
Aggiungere qualcosa sulle prolate spheroidal wave functions?

## 4.2 Daubechies' localization operators

Projection operators considered in the previous section are powerful tools in signal analysis and quantum mechanics. Nevertheless they treat time and frequency in a separate way. We already saw that a good tool to simultaneously study a signal in time and frequency is the STFT but we also pointed out its limits. However we can think to construct localization operators using the STFT instead of the Fourier transform. This is exactly what was done by Ingrid Daubechies in 1988 in [3]. From Theorem 3.5 we know that the adjoint operator of $\mathcal{V}_\phi$ acts as inverse operator. If we choose a window $\phi \in L^2(\mathbb{R}^{2d})$ normalized, 3.8 becomes

$$f(t) = \mathcal{V}_\phi^* \mathcal{V}_\phi f(t)$$

The key idea is to multiply $\mathcal{V}_\phi f$ by a *weight function* $F(\omega, t)$, which logically should highlight some features of $\mathcal{V}_\phi f$ :

$$L_{F,\phi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad L_{F,\phi} f(t) = \mathcal{V}_\phi^* F \mathcal{V}_\phi f(t) \tag{4.8}$$

Related to this localization operator is the sesquilinear form $\mathscr{L}_{F,\phi} : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to \mathbb{R}$ defined by the expression

$$\mathscr{L}_{F,\phi}(f, g) = \int_{\mathbb{R}^{2d}} F(x, \omega) \mathcal{V}_\phi f(x, \omega) \overline{\mathcal{V}_\phi g(x, \omega)} dx d\omega \tag{4.9}$$

Indeed, assuming $\mathscr{L}_{F,\phi}$ is bounded, we could define $L_{F,\phi} f$ through Riesz' representation theorem as the only element of $L^2(\mathbb{R}^d)$ such that

$$\mathscr{L}_{F,\phi}(f, g) = \langle L_{F,\phi} f, g \rangle = \int_{\mathbb{R}^d} L_{F,\phi} f(t) \overline{g(t)} \quad \forall g \in L^2(\mathbb{R}^d) \tag{4.10}$$

and therefore $L_{F,\phi}$ as the function which maps $f$ into its representation.
QUESTA PARTE VA MESSA A POSTO IN MODO DA UNIFORMARE I RISULTATI

**Proposition 4.2.** *If $F \in L^\infty(\mathbb{R}^{2d})$ then $\|L_{F,\phi}\| \leq \|F\|_\infty$.*

*Proof.* contenuto... □

**Proposition 4.3.** *If $F \in L^1(\mathbb{R}^{2d})$ then $\|L_{F,\phi}\| \leq \|F\|_1$.*

*Proof.* contenuto... □

Thanks to these result and an interpolation argument one can show that $\mathscr{L}_{F,\phi}$, and hence $L_{F,\phi}$, is bounded also when $F \in L^p(\mathbb{R}^{2d})$ for $1 < p < \infty$. A proof can be found in [21], Proposition 12.3.

**Proposition 4.4.** *Let $F \in L^p(\mathbb{R}^{2d})$ for $1 < p < \infty$. Then there exist a unique bounded linear operator $L_{F,\phi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $L_{F,\phi} f$ is given by (4.10) for all $f \in L^2(\mathbb{R}^d)$ and all simple functions $F$ for which $|\{F(x) \neq 0\}| < \infty$. Moreover, the application that maps $F$ into $L_{F,\phi}$ is bounded.*

Since simple functions supported on a set of finite measure are dense in $L^p(\mathbb{R}^{2d})$, through dominated convergence we can conclude that (4.9) holds for every $F \in L^p(\mathbb{R}^{2d})$.

- Definizione
- Operatori di proiezione
- Operatori di localizzazione di Daubechies
- Proprietà di limitatezza e compattezza
- Autovalori e autofunzioni

### 4.2.1 Spherically Symmetric Weights

In this section we will consider the special case in which the window for the STFT is a Gaussian (3.9) and the weight $F$ is spherically symmetric. Letting $r_j^2 = x_j^2 + \omega_j^2$ for $j = 1, \dots, d$ and $r^2 = (r_1^2 \dots, r_d^2) \in \mathbb{R}^d$, the hypothesis about $F$ can be rephrased in the following way

$$F(x, \omega) = \mathcal{F}(r^2). \tag{4.11}$$

In order to highlight the dependence of $F$ through $\mathcal{F}$ the corresponding localization operator will be denoted as $L_{\mathcal{F},\varphi}$. For this operators a complete characterization of the spectrum and eigenspaces is given in [3]. Before stating and proving the theorem we need to introduce some special function, namely *Hermite functions*. In dimension $d = 1$ Hermite functions are given by

$$H_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^k e^{\pi t^2} \frac{d^k}{dt^k} \left(e^{-2\pi t^2}\right) \tag{4.12}$$

where $k \in \mathbb{N}$ ($\mathbb{N}$ is supposed to contain also 0). Hermite functions have lots of interesting and useful properties which can be found in [4, Section 1.7]. We cite some of them which will be useful in the following.

(i) $\{H_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$;

(ii) $H_0(t) = \varphi(t)$ where $\varphi$ is the normalized Gaussian given by (3.9);

(iii) Setting $H_{-1} = 0$, the following recursive relation holds

$$2\sqrt{\pi} t H_k = \sqrt{k+1} H_{k+1} + \sqrt{k} H_{k-1} \quad \text{for } k = 0,1,\dots; \tag{4.13}$$

(iv) Hermite functions are eigenfunctions of $\mathscr{F}$, specifically

$$\mathscr{F} H_k = (-i)^k H_k. \tag{4.14}$$

Hermite functions in generic dimension are given through multiplication of 1-dimensional Hermite functions. Explicitly, given a multi-index $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ the corresponding Hermite function is given by

$$H_k(t) = \prod_{j=1}^{d} H_{k_j}(t_j). \tag{4.15}$$

It is still true that $d$-dimensional Hermite functions (now ranging between all possible multi-indices) form an orthonormal basis of $L^2(\mathbb{R}^d)$ and using (4.14) it is easy to see that $d$-dimensional Hermite functions are still eigenfunction of the Fourier transform, namely

$$\mathscr{F} H_k = (-i)^{|k|} H_k \tag{4.16}$$

where $|k| = k_1 + \cdots k_d$ is the length of the multi-index.

**Theorem 4.5.** *Eigenfuctions of $L_{\mathcal{F},\varphi}$ are the d-dimensional Hermite functions*

# Chapter 5

# Uncertainty principles

In previous chapters sometimes we wrote about "good" and "bad" localization of a signal. Although intuitively this can be understood in these sloppy terms, we can specify and quantify the localization of a function. The notion of localization can be stated in multiple ways and in various contexts. Here we present some *uncertainty principles*.

## 5.1 Heisenberg's uncertainty principle

**Lemma 5.1.** *Let $A, B : D \subset X \to X$ be two linear symmetric (i.e. formally self-adjoint) operators defined on an invariant subspace $D$ (i.e. $A(D), B(D) \subset D$) of an Hilbert space $X$. Then, for every $a, b \in \mathbb{R}$ and for every $f$ in $D$*

$$\|(A-a)f\| \cdot \|(B-b)f\| \geq \frac{1}{2}|\langle [A,B]f, f\rangle| \tag{5.1}$$

*where $[A,B] = AB - BA$ is the commutator of $A$ and $B$. Equality holds if and only if $(A-a)f = ic(B-b)f$ for some $c \in \mathbb{R}$.*

*Proof.* By Cauchy-Schwarz inequality we have

$$\|(A-a)f\| \cdot \|(B-b)f\| \geq |\langle (A-a)f, (B-b)f\rangle| \tag{5.2}$$

The right-hand side is the modulus of a complex number. For every $z \in \mathbb{C}$ we have

$$|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \geq \mathrm{Im}(z)^2 \implies |z| \geq \left|\frac{z-\overline{z}}{2i}\right| = \frac{1}{2}|z-\overline{z}|.$$

Using this inequality we have

$$\begin{aligned} |\langle (A-a)f, (B-b)f\rangle| &\geq \frac{1}{2}\left|\langle (A-a)f, (B-b)f\rangle - \langle (B-b)f, (A-a)f\rangle\right| = \qquad (5.3) \\ &= \frac{1}{2}|\langle Af, Bf\rangle - \langle Af, bf\rangle - \langle af, Bf\rangle + \langle af, bf\rangle + \\ &\quad - \langle Bf, Af\rangle + \langle Bf, af\rangle + \langle bf, Af\rangle - \langle bf, af\rangle| = \\ &= \frac{1}{2}|\langle (AB-BA)f, f\rangle| = \frac{1}{2}|\langle [A,B]f, f\rangle| \end{aligned}$$

When using Cauchy-Schwarz in (5.2), we have equality if and only if

(i) $(A-a)f = \lambda(B-b)f$ for some $\lambda \in \mathbb{C}$;

(ii) $(B-b)f = 0$.

In the latter case we have equality also in (5.3), while in the former equality occurs if and only if $|\langle (A-a)f, (B-b)f\rangle| = \lambda \|(B-b)f\|^2$ is purely imaginary, which means that $\lambda = ic$ for some $c \in \mathbb{R}$. ☐

**Theorem 5.2** (Heisenberg's uncertainty principle)**.** *If $f \in L^2(\mathbb{R}^d)$, for every $a, b \in \mathbb{R}^d$*

$$\left(\int_{\mathbb{R}^d} |t-a|^2 |f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}^d} |\omega - b|^2 |\hat{f}(\omega)|^2 d\omega\right)^{1/2} \geq \frac{d}{4\pi}\|f\|_2^2 \tag{5.4}$$

*Equality holds if and only if $f$ is a multiple of a Gaussian, i.e. $e^{2\pi i b\cdot t}e^{-\pi c|x-a|^2}$ for some $c > 0$.*

*Proof.* We will first consider $f$ is a function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, in which case both integrals in (5.4) are finite. Since $|t-a|^2 = \sum_1^d (t_j - i_j)^2$ (and the same holds for $|\omega - b|^2$), we will start considering

$$\left(\int_{\mathbb{R}^d} (t_j - a_j)^2 |f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}^d} (\omega_k - b_k)^2 |\hat{f}(\omega)|^2 d\omega\right)^{1/2} = \|(t_j - a_j)f\|_2 \cdot \|(\omega_k - b_k)\hat{f}\|_2.$$

for $1 \leq j, k \leq d$. The first term is exactly as the first one appearing in (5.1), where the operator $A = A_j$ is now the multiplication operator $f \in \mathcal{S}(\mathbb{R}^d) \mapsto A_j f = t_j f \in \mathcal{S}(\mathbb{R}^d)$, which is (formally) self-adjoint over $\mathcal{S}(\mathbb{R}^d)$. So, in order to apply 5.1, we have to work on the second term:

$$\begin{aligned}(\omega_k - b)\hat{f}(\omega) = \omega_k \hat{f}(\omega) - \mathscr{F}(b_k f)(\omega) &\overset{2.24}{=} \mathscr{F}\left(\frac{1}{2\pi i}\frac{\partial f}{\partial t_k}\right) - \mathscr{F}(b_k f)(\omega) = \\ &= \mathscr{F}\left(\left(\frac{1}{2\pi i}\frac{\partial}{\partial t_k} - b_k\right) f\right)(\omega)\end{aligned}$$

and thanks to Plancherel's theorem we have that $\|(\omega_k - b_k)\hat{f}\|_2 = \left\|\left(\frac{1}{2\pi i}\frac{\partial}{\partial t_k} - b_k\right) f\right\|_2$, so that now the operator $B = B_k$ appearing in (5.1) is $f \in \mathcal{S}(\mathbb{R}^d) \mapsto B_k f = \frac{1}{2\pi i}\frac{\partial f}{\partial t_k} \in \mathcal{S}(\mathbb{R}^d)$. Again, this is (formally) self-adjoint over $\mathcal{S}(\mathbb{R}^d)$, so we can apply Lemma 5.1:

$$\left(\int_{\mathbb{R}^d} (t_j - a_j)^2 |f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}^d} (\omega_k - b_k)^2 |\hat{f}(\omega)|^2 d\omega\right)^{1/2} \geq \frac{1}{2}|\langle [A_j, B_k] f, f\rangle|$$

Now we just need to compute the commutator of $A_j$ and $B_k$. Let $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\begin{aligned}[A_j, B_k]f(t) = B_k A_j f(t) - A_j B_k f(t) &= \frac{1}{2\pi i}\frac{\partial}{\partial t_k}(t_j f(t)) - t_j \frac{1}{2\pi i}\frac{\partial f}{\partial t_k}(t) = \\ &= \frac{1}{2\pi i}\delta_{jk} f(t) + \frac{1}{2\pi i} t_j \frac{\partial f}{\partial t_k}(t) - t_j \frac{1}{2\pi i}\frac{\partial f}{\partial t_k}(t) = \frac{1}{2\pi i}\delta_{jk} f(t)\end{aligned}$$

therefore $[A_j, B_k] = \frac{1}{2\pi i}\delta_{jk}$, where $\delta_{jk}$ has to be intended as a function identically equal to 1 if $j = k$ or identically equal to 0 if $j \neq k$. Since $A_j$ and $B_k$ commute when $j \neq k$ from now on we will only consider the case $j = k$, in which Lemma 5.1 allows us to conclude that

$$\left(\int_{\mathbb{R}^d}(t_j - a_j)^2|f(t)|^2dt\right)^{1/2}\left(\int_{\mathbb{R}^d}(\omega_j - b_j)^2|\hat{f}(\omega)|^2d\omega\right)^{1/2} \geq \frac{\|f\|_2^2}{4\pi} \tag{5.5}$$

If we sum over $j$ we can see the left-hand side as a scalar product in $\mathbb{R}^n$ and use Cauchy-Schwarz inequality:

$$\begin{aligned}\frac{d\|f\|_2^2}{4\pi} &\leq \sum_{j=1}^{d}\left(\int_{\mathbb{R}^d}(t_j - a_j)^2|f(t)|^2dt\right)^{1/2}\left(\int_{\mathbb{R}^d}(\omega_j - b_j)^2|\hat{f}(\omega)|^2d\omega\right)^{1/2} \leq \\ &\leq \left(\sum_{j=1}^{d}\int_{\mathbb{R}^d}(t_j - a_j)^2|f(t)|^2dt\right)^{1/2}\left(\sum_{j=1}^{d}\int_{\mathbb{R}^d}(\omega_j - b_j)^2|\hat{f}(\omega)|^2d\omega\right)^{1/2} = \\ &= \left(\int_{\mathbb{R}^d}|t-a|^2|f(t)|^2dt\right)^{1/2}\left(\int_{\mathbb{R}^d}|\omega - b|^2|\hat{f}(\omega)|^2d\omega\right)^{1/2}\end{aligned} \tag{5.6}$$

which is the desired result.

From Lemma 5.1 we also know that in (5.5) equality holds if and only if

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial t_j} - b_j\right)f = ic_j(t_j - a_j)f \tag{5.7}$$

for some $c_j \in \mathbb{R}$. In principle $c_j$ could be different but the second inequality in (5.6) becomes an equality if and only if $|c_j| = c \geq 0$, that is all $c_j$ have the same modulus (but possibly different sign). PERCHé C NON PUò ESSERE 0???.
We start by considering the case when $a = b = 0$.

$$\frac{1}{2\pi i}\frac{\partial f}{\partial t_j}(t) = ic_jt_jf(t) \quad j = 1, \dots, d. \tag{5.8}$$

We notice that if $f$ is a solution it cannot become 0, because in this case there would be some $t_0 \in \mathbb{R}^d$ such that $f(t_0) = 0, \nabla f(t_0) = 0$ which means that $f$ is identically 0 (**Perché??**) . Moreover, if $f$ is a solution also $-f$ is, hence we can suppose that $f$ is always strictly positive and divide by $f$:

$$\frac{1}{2\pi i}\frac{1}{f}\frac{\partial f}{\partial t_j}(t) = ic_jt_jf(t) \implies \frac{\partial}{\partial t_j}[\log(f(t))] = -2\pi c_jt_j$$

Letting $g(t) = \log(f(t))$ and considering the equation for $j = 1$ we get

$$\frac{\partial g}{\partial t_1}(t) = -2\pi c_1t_1 \implies g(t) = -\pi c_1t_1^2 + g_1(t_2, \dots, t_d).$$

The equation for $j = 2$ becomes

$$\frac{\partial g}{\partial t_2}(t) = \frac{\partial g_1}{\partial t_2}(t) = -\pi c_2t_2^2 \implies g_1(t_2, \dots, t_d) = -\pi c_2t_2^2 + g_2(t_3, \dots, t_d).$$

At the end we will have

$$g(t) = -\sum_{j=1}^{d} \pi c_j t_j^2 + K \implies f(t) = e^K e^{-\pi \sum_{j=1}^{d} c_j t_j^2} = C e^{-\pi \sum_{j=1}^{d} c_j t_j^2}$$

for some $K \in \mathbb{R}$. We already noticed that all $c_j$ have same modulus. Now that we have the explicit expression of $f$ we can see that they are all positive, otherwise $f$ would not belong to $L^2(\mathbb{R}^d)$. Thanks to this observation we can finally write $f(t) = Ce^{-\pi c|t|^2}$.
Suppose now that $a$ and $b$ are not 0. We define $\tilde{f}(t) = M_b T_a f(t)$ and we will show that $\tilde{f}$ solves (5.7):

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial t_j} - b_j\right)\tilde{f}(t) = \frac{1}{2\pi i} e^{2\pi i b\cdot t}\left(2\pi i b_j f(t-a) + \frac{\partial f}{\partial t_j}(t-a)\right) - b_j e^{2\pi i b\cdot t} f(t-a) =$$
$$= e^{2\pi i b\cdot t}\frac{1}{2\pi i} f(t-a) \overset{(5.8)}{=} e^{2\pi i b\cdot t} ic(t_j - a_j) f(t-a) = ic(t_j - a_j)\tilde{f}(t).$$

□

For a generic function $f \in L^2(\mathbb{R}^d)$ the left-hand side of (5.4) may be infinite, in which case the statement is trivially satisfied. We shall comment a mathematical interpretation of Heisenberg's uncertainty principle. This can be written in the following form

$$\left(\int_{\mathbb{R}^d} |t-a|^2 \frac{|f(t)|^2}{\|f\|_2^2} dt\right)^{1/2} \left(\int_{\mathbb{R}^d} |\omega - b|^2 \frac{|\hat{f}(\omega)|^2}{\|\hat{f}\|_2^2} d\omega\right)^{1/2} \geq \frac{d}{4\pi}$$

so we may directly assume that $f$ is normalized. In such a case $|f|^2$ can be seen as a probability distribution. If these integrals are finite for some $a$ and $b$ are always finite and their minimum is achieved when

$$a = \bar{t} = \int_{\mathbb{R}^d} t|f(t)|^2 dt, \quad b = \bar{\omega} = \int_{\mathbb{R}^d} \omega |\hat{f}(\omega)|^2 d\omega$$

which are the mean of $|f|^2$ and $|\hat{f}|^2$, respectively. In this case previous integrals represent the standard deviation of $|f|^2$ and $|\hat{f}|^2$, which we indicate with $\Delta_x f$ and $\Delta_\omega f$. It is fair to believe that a function $|f|^2$ is mostly concentrated around its mean and that its standard deviation is a measure of how spread it is. In light of these arguments, Heisenberg's uncertainty principle can written as

$$\Delta_x f \cdot \Delta_\omega f \geq \frac{d}{4\pi}$$

In this form the uncertainty principle has a heuristic yet meaningful interpretation:
*a function and its Fourier transform can not be simultaneously too concentrated.*
**AGGIUNGERE INTERPRETAZIONI IN MECCANICA QUANTISTICA E ANALISI DEI SEGNALI.**

## 5.2 Donoho-Stark's uncertainty principle

**Definition 5.3.** *A function $f \in L^2(\mathbb{R}^d)$ is $\varepsilon$-**concetrated** on a measurable set $T \subseteq \mathbb{R}^d$ if*

$$\left(\int_{T^c} |f(t)|^2 dt\right)^{1/2} \leq \varepsilon \|f\|_2$$

*where $T^c = \mathbb{R}^d \setminus T$ denotes the complement set of $T$.*

**Theorem 5.4** (Donoho-Stark's uncertainty principle)**.** *Let $f \in L^2(\mathbb{R}^d) \setminus \{0\}$, suppose that $f$ is $\varepsilon_T$-concentrated on $T \subseteq \mathbb{R}^d$ while $\hat{f}$ is $\varepsilon_\Omega$-concentrated on $\Omega \subseteq \mathbb{R}^d$. Then*

$$|T|\,|\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2 \tag{5.9}$$

*Proof.* The result is trivial if $T$ or $\Omega$ have infinite measure. Hence we will suppose that they both have finite measure.
Concentration can be stated in an equivalent way through projection operators introduced in section 4.1, indeed:

$$\left(\int_{T^c} |f(t)|^2 dt\right)^{1/2} = \|f - \chi_T f\|_2 = \|f - P_T f\|_2 \leq \varepsilon_T \|f\|_2$$
$$\left(\int_{\Omega^c} |\hat{f}(\omega)|^2 d\omega\right)^{1/2} = \|\hat{f} - \chi_\Omega \hat{f}\|_2 = \|f - \mathscr{F}^{-1}(\chi_\Omega \hat{f})\|_2 = \|f - Q_\Omega f\|_2 \leq \varepsilon_\Omega \|f\|_2$$

In section 4.1 we also noticed that $\|Q_\Omega\| \leq 1$, hence

$$\begin{aligned}\|f - Q_\Omega P_T f\|_2 = \|f - Q_\Omega f + Q_\Omega f - Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega f\|_2 + \|Q_\Omega(f - P_T f)\|_2 \leq \\ \leq \|f - Q_\Omega f\|_2 + \|(f - P_T f)\|_2 \leq (\varepsilon_\Omega + \varepsilon_T)\|f\|_2.\end{aligned}$$

and consequently

$$\begin{aligned}\|f\|_2 = \|f - Q_\Omega P_T f + Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega P_T f\|_2 + \|Q_\Omega P_T f\|_2 \implies \\ \implies \|Q_\Omega P_T f\|_2 \geq \|f\|_2 - \|f - Q_\Omega P_T f\|_2 \geq (1 - \varepsilon_\Omega - \varepsilon_T)\|f\|_2\end{aligned}$$

Thanks to Proposition 4.1 we know that $\|Q_\Omega P_T\|_{\mathrm{HS}} = \sqrt{|T|\,|\Omega|}$ and from Theorem 2.16 we know that $\|Q_\Omega P_T\| \leq \|Q_\Omega P_T\|_{\mathrm{HS}}$, therefore

$$(1 - \varepsilon_\Omega - \varepsilon_T)\|f\|_2 \leq \|Q_\Omega P_T f\|_2 \leq \sqrt{|T|\,|\Omega|}\|f\|_2.$$

□

## 5.3 Lieb's uncertainty principle

Up to now we presented two uncertainty principles related to the Fourier transform. However uncertainty principles can be stated for every time of time-frequency analysis. In this and in the following section we present some uncertainty principles for the STFT.

We start considering a simple form uncertainty principle for the STFT. After this we will see how to turn Lieb's inequality 3.4 into an uncertainty principle.

**Proposition 5.5.** *Let $f, \phi \in L^2(\mathbb{R}^d)$ normalized, $U \subseteq \mathbb{R}^{2d}$ and $\varepsilon \geq 0$. Suppose that*

$$\int_U |\mathcal{V}_\phi f(x,\omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

*Then $|U| \geq 1 - \varepsilon$.*

*Proof.* From (3.4) we see that $|\mathcal{V}_\phi f(x,\omega)| \leq 1$ for all $(x,\omega) \in \mathbb{R}^{2d}$, therefore

$$1 - \varepsilon \leq \int_U |\mathcal{V}_\phi f(x,\omega)|^2 dx d\omega \leq \|\mathcal{V}_\phi f\|_\infty^2 |U| \leq |U|. \tag{5.10}$$

□

**Theorem 5.6** (Lieb's uncertainty principle)**.** *Suppose that $\|f\|_2 = \|\phi\|_2 = 1$. If $U \subseteq \mathbb{R}^{2d}$ and $\varepsilon \geq 0$ are such that*

$$\int_U |\mathcal{V}_\phi f(x,\omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

*Then*

$$|U| \geq (1-\varepsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-1}} \quad \textit{for every } p > 2.$$

*Proof.* If $|U| = \infty$ the result is trivial hence we can suppose that $U$ has finite measure. We start using Hölder's inequality with exponents $p/2$ and $(p/2)' = p/(p-2)$

$$\begin{aligned} 1 - \varepsilon &\leq \int_U |\mathcal{V}_\phi f(x,\omega)|^2 dx d\omega = \int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x,\omega)|^2 \chi_U(x,\omega) dx d\omega \overset{\text{Hölder}}{\leq} \\ &\leq \left( \int_{\mathbb{R}^{2d}} |\mathcal{V}_\phi f(x,\omega)|^{2\frac{p}{2}} dx d\omega \right)^{\frac{2}{p}} \left( \int_{\mathbb{R}^{2d}} \chi_U(x,\omega)^{\frac{p}{p-2}} dx d\omega \right)^{\frac{p-2}{p}} \overset{(3.5)}{\leq} \\ &\leq \left(\frac{2}{p}\right)^{\frac{2d}{p}} \|f\|_2^2 \, \|g\|_2^2 \, |U|^{\frac{p-2}{p}} = \left(\frac{2}{p}\right)^{\frac{2d}{p}} |U|^{\frac{p-2}{p}}. \end{aligned}$$

We point out that the use of Hölder's inequality is justified because $U$ has finite measure and, since $\mathcal{V}_\phi f \in L^q(\mathbb{R}^{2d})$ for every $q \geq 2$, $|\mathcal{V}_\phi f|^2 \in L^q(\mathbb{R}^{2d})$ for every $q \geq 1$. □

## 5.4 Nicola-Tilli's uncertainty principle or Faber-Krahn Inequality for the STFT

Theorem from [15]

**Theorem 5.7.** *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V} f(x,\omega)|^2 dx d\omega \leq G(|\Omega|)$$

*where $G(s)$ is given by*

$$G(s) := \int_0^s e^{(-d!\tau)^{1/d}} d\tau \tag{5.11}$$

*Moreover, equality occurs if and only if $f$ is a Gaussian of the kind*

$$f(x) = ce^{2\pi x \cdot \omega_0} \varphi(x - x_0) \quad x \in \mathbb{R}^d \tag{5.12}$$

*for some unimodular $c \in \mathbb{C}$ and some $(x_0, \omega_0) \in \mathbb{R}^{2d}$ and $\Omega$ is equivalent, in measure, to a ball of centre $(x_0, \omega_0)$.*

# Chapter 6

# Recent results

## 6.1 Norm of localization operators: results from Nicola-Tilli

In Section 4.2 we obtained some basic results for the norm of Daubechies' localization operators, independently of the choice of the window $\phi$ for the STFT. It is reasonable to think that, for specific windows those estimates can be improved and, hopefully, find some sharp bounds. Thanks to 5.7 Nicola and Tilli accomplished this task in the case the window of the STFT is a normalized Gaussian. As done previously, since the windows is fixed one for all we will drop the pedex $\phi$.

As for the results in Section 4.2, assumptions on the weight function $F$ are related to its integrability and boundedness. The problem we are consider is, in fact, finding an optimal estimate of the type

$$\|L_F\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C \tag{6.1}$$

where $F$ satisfies the following constraints:

$$\|F\|_\infty \leq A \quad \text{and} \quad \|F\|_p \leq B. \tag{6.2}$$

Clearly the constant $C$ will depend on $p$, $A$ and $B$. In [16] this problem is completely solved: the constant $C$ is computed (explicitly in some cases), weight functions $F$ which achieve this bound are explicitly found and also function $f$ and $g$ such that $|\langle L_f f, g\rangle| = \|L_F\| = C$ are found. Before reporting the main Theorem in [16], we define the following number which will appear many times in the following

$$\kappa_p := \frac{p-1}{p}. \tag{6.3}$$

**Theorem 6.1.** *Assume $p \in [1,\infty)$, $A \in (0,\infty]$ and $B \in (0,\infty)$ with the additional condition that $A < \infty$ when $p = 1$. Let $F$ satisfy the constraints in (6.2).*

*(i) If $p = 1$, then*

$$\|L_F\| \leq A\, G(B/A) \tag{6.4}$$

*and equality occurs if and only if, for some* $\theta \in \mathbb{R}$ *and some* $z_0 \in \mathbb{R}^{2d}$

$$F(z) = Ae^{i\theta}\chi_{\mathcal{B}}(z - z_0) \quad \forall z \in \mathbb{R}^{2d} \tag{6.5}$$

*where* $\mathcal{B} \subset \mathbb{R}^{2d}$ *is the ball of measure* $B/A$ *centred at the origin.*

*(ii) If* $p > 1$ *and* $\frac{B}{A} \le \kappa_p^{d/p}$*, then*

$$\|L_F\| \le \kappa_p^{d\kappa_p} B, \tag{6.6}$$

*with equality if and only if, for some* $\theta \in \mathbb{R}$ *and some* $z_0 \in \mathbb{R}^{2d}$*,*

$$F(z) = e^{i\theta}\lambda e^{\frac{\pi}{p-1}|z-z_0|^2} \quad \forall z \in \mathbb{R}^{2d} \tag{6.7}$$

*where* $\lambda = \kappa_p^{-d/p} B$*.*

*(iii) If* $p > 1$ *and* $\frac{B}{A} > \kappa_p^{d/p}$*, then*

$$\|L_F\| \le \int_0^A G(u_\lambda(t))dt, \tag{6.8}$$

*where* $u_\lambda(t) = \left[-\log\left((\frac{t}{\lambda})^{p-1}\right)\right]^d$ *and* $\lambda > A$ *is uniquely determined by the condition* $p\int_0^A t^{p-1}u_\lambda(t)dt = B^p$*. Equality in (6.8) if and only if, for some* $\theta \in \mathbb{R}$ *and some* $z_0 \in \mathbb{R}^{2d}$*,*

$$F(z) = e^{i\theta}\min\{\lambda e^{-\frac{\pi}{p-1}|z-z_0|^2}, A\} \tag{6.9}$$

*Finally, in all the cases, condition* $|\langle L_F f, g\rangle| = \|L_F\|$ *holds for some,* $f, g \in L^2(\mathbb{R}^d)$ *such that* $\|f\|_2 = \|g\|_2 = 1$*, if and only if both* $f$ *and* $g$ *are of the kind (5.12), possibly with different* $c$*'s, but with the same* $(x_0, \omega_0) \in \mathbb{R}^{2d}$ *which coincides with the centre of* $F$*.*

We will not give the proof of these results since some of its parts are similar to the one we will see in the following Section. Moreover, we point out that the case $A = \infty$ means we are dropping the $L^\infty$ constraint.

## 6.2 Generic case

In this Section we will deal with a generalized version of the problem considered in [16]. We want to find the optimal constant $C$ such that

$$\|L_F\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C$$

under the following constraints on $F$:

$$\|F\|_p \le A \quad \text{and} \quad \|F\|_q \le B. \tag{6.10}$$

where $p, q \in (1, \infty)$ and $A, B \in (0, \infty)$. In this setting it is no more possible to find an explicit expression for $C$ and $F$, although they can be easily computed numerically.

Theorem (ii) includes the case when $F$ satisfies just an $L^p$ constraint by taking $A$ ($L^\infty$ constraint). In the current setting we have an $L^p$ and an $L^q$ bound, hence, thanks to 6.6, it is straightforward to see that

$$\|L_F\|_{L_2\to L_2} \leq \min\{\kappa_p^{d\kappa_p}A,\, \kappa_q^{d\kappa_q}B\}$$

Suppose that the first term is less than the second, which means:

$$\kappa_p^{d\kappa_p}A \leq \kappa_q^{d\kappa_q}B \iff \frac{B}{A} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d \tag{6.11}$$

Clearly, for $B$ sufficiently large we expect that the solution of current problem is the same as the one with just an $L^p$ constraint, namely the one given by (6.7). Therefore, we want to compare its $L^q$ norm with the bound given by $B$:

$$\begin{aligned}\|F\|_q^q &= \int_{\mathbb{R}^{2d}} |F(z)|^q dz = \lambda^q \int_{\mathbb{R}^{2d}} e^{-\frac{q\pi}{p-1}|z-z_0|^2} dz \overset{z'=\left(\frac{q\pi}{p-1}\right)^{1/2}(z-z_0)}{=} \\ &= \lambda^q \left(\frac{p-1}{q\pi}\right)^d \int_{\mathbb{R}^{2d}} e^{-|z'|^2} dz' = \lambda^q \left(\frac{p-1}{q\pi}\right)^d \pi^d = \lambda^q \left(\frac{p-1}{q}\right)^d.\end{aligned}$$

Since we want $F$ to satisfy the $L^q$ constraint we should have

$$\lambda \left(\frac{p-1}{q}\right)^{d/q} \leq B \overset{\lambda=\kappa_p^{-d/p}A}{\Longrightarrow} \left(\frac{p}{p-1}\right)^{d/p} \left(\frac{p-1}{q}\right)^{d/q} A \leq B$$

which is equivalent to

$$\frac{B}{A} \geq \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{\frac{d}{q}}$$

If this condition were less restrictive than the one given by (6.11) we would have solved the problem. Unfortunately, this is not the case. Indeed it is always true, regardless of $p$ and $q$, that

$$\kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{\frac{d}{q}} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d$$

Recalling that $\kappa_p = \frac{1}{p'}$, where $p'$ is the conjugate exponent of $p$, this inequality can be stated in a different way:

$$\left(\frac{1}{p'}\right)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \geq \left(\frac{1}{p'}\right)^{\frac{1}{p'}} \left(\frac{1}{q'}\right)^{-\frac{1}{q'}} \iff \left(\frac{1}{p'}\right)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{p'}} \left(\frac{1}{q'}\right)^{\frac{1}{q'}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \geq 1$$

but, since $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}-1=\frac{1}{q'}$, in conclusion we have:

$$\left(\frac{p'}{q'}\right)^{\frac{1}{q'}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \geq 1$$

It is quite easy that this inequality holds for every pair of $p, q > 1$. Letting $x = \frac{1}{p}$ and $y = \frac{1}{q}$ we can take the log of the right-hand side and study it as a function of $x$ and $y$:

$$f(x,y) = (1-y)\left[\log(1-x) - \log(1-y)\right] + y\left[\log(x) - \log(y)\right]$$

Taking the partial derivative with respect to $x$ we have

$$\frac{\partial f}{\partial x}(x,y) = -\frac{1-y}{1-x} + \frac{y}{x} = \frac{y-x}{x(1-x)}$$

Since $x \in (0,1)$, $\frac{\partial f}{\partial x}(x,y)$ is positive for $x < y$. SCRIVERE IN APPENDICE?

Despite this fact, at least we can say that, if $B/A \geq \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{\frac{d}{q}}$ or $B/A \leq \kappa_q^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{\frac{d}{p}}$, the problem is already solved and the solution is given by Theorem 6.1. Therefore, from now on, we will suppose to be in the intermediate case, that is

$$\kappa_q^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{\frac{d}{p}} < \frac{B}{A} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{\frac{d}{q}} \tag{6.12}$$

We notice that the condition is well-posed, since it is actually true that

$$\kappa_q^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{\frac{d}{p}} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{\frac{d}{q}}$$

whenever $p \neq q$. Aggiungere dimostrazione in appendice?

Before tackling the problem in this intermediate regime, we prove a Theorem from [16] which gives a bound for $\|L_F\|$ in terms of the distribution function of $|F|$.

**Theorem 6.2.** *Assume $F \in L^p(\mathbb{R}^{2d})$ for some $p \in [1, +\infty)$ and let $\mu(t) = |\{|F| > t\}|$ be the distribution function of $|F|$. Then*

$$\|L_F\| \leq \int_0^\infty G(\mu(t))dt \tag{6.13}$$

*Equality occurs if and only if $F(z) = e^{i\theta}\rho(|z - z_0|)$ for some $\theta \in \mathbb{R}$, $z_0 \in \mathbb{R}^{2d}$ and some nonincreasing function $\rho : [0, +\infty) \to [0, +\infty)$*

*Proof.* For the sake of brevity we denote the variable $(x, \omega) \in R^{2d}$ as $z$ and therefore $dxd\omega$ as $dz$. Let $f, g \in L^2(\mathbb{R}^d)$ such that $\|f\|_2 = \|g\|_2 = 1$. Since we are in a Hilbert space $\|L_F\|$ can be computed as the supremum of $|\langle L_F f, g\rangle|$ over all normalized $f$ and $g$. Therefore we are interested in estimating the previous scalar product

$$\begin{aligned}|\langle L_F f, g\rangle| = |\mathscr{L}_F(f,g)| &\leq \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)| \cdot |\mathcal{V}g(z)| dz \overset{\text{C-S}}{\leq} \\ &\leq \left(\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}g(z)|^2 dz\right)^{1/2}\end{aligned} \tag{6.14}$$

Since the result is symmetric in $f$ and $g$ we can study just one of the terms. Letting $m = \operatorname{ess\,sup} |F(z)|$ and assuming $m > 0$ (otherwise every result is trivial) we can use the "layer cake" representation [13, Theorem 1.13]:

$$|F(z)| = \int_0^m \chi_{\{|F|>t\}}(z)dt$$

in order to find

$$\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz = \int_{\mathbb{R}^{2d}} \left( \int_0^m \chi_{\{|F|>t\}}(z)dt \right) |\mathcal{V}f(z)|^2 dz \overset{\text{Tonelli}}{=}$$
$$= \int_0^m \left( \int_{\mathbb{R}^{2d}} \chi_{\{|F|>t\}}(z) |\mathcal{V}f(z)|^2 dz \right) dt = \int_0^m \left( \int_{\{|F|>t\}} |\mathcal{V}f(z)|^2 dz \right) dt$$

We notice that the quantity in the inner integral is exactly the one in the theorem 5.7, hence

$$\int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)|^2 dz \leq \int_0^m G\left(|\{|F| > t\}|\right) dt = \int_0^m G(\mu(t))dt \tag{6.15}$$

We point out that since $\mu(t) = 0$ for $t > m$ and that $G(0) = 0$, the previous expression is equivalent to (6.13).

Because $p < \infty$, $L_F$ is a compact operator, there exist normalized $f$ and $g$ which achieve equality in the supremum of the norm, namely $\langle L_F f, g\rangle = \|L_F\|$. Therefore equality in (6.13) occurs if and only if all the previous inequalities become equalities. Equality in (6.15) occurs if and only if

$$\int_{\{|F|>t\}} |\mathcal{V}f(z)|^2 dz = G(\mu(t)) \tag{6.16}$$

for a.e. $t \in (0, m)$. Thanks to Theorem 5.7, for just one $t_0 \in (0, m)$ we can infer that $\{|F| > t_0\}$ is (equivalent to) a ball centred in $z_0 = (x_0, \omega_0)$ and that $f$ is a Gaussian of the kind (5.12) with the same centre $z_0$. Then, still by theorem 5.7, since (6.16) holds a.e in $(0, m)$ and that $f$ is always the same we have that also the other levels sets $\{|F| > t\}$ are equivalent to balls centred at the same $z_0$. Finally, we can extend the result to every $t \in (0, m)$ because $\{|F| > t\} = \bigcup_{s>t}\{|F| > s\}$. Since Theorem 5.7 is a "if and only if", these conditions on $F$ and $f$ are also sufficient to guarantee equality in (6.15). Clearly the same result holds for $g$ which has to be a Gaussian, possibly with different coefficient $c$ but the same centre.

In the end it turns that $|F|$ is spherically symmetric and radially decreasing as claimed in theorem's statement.

Conditions for $f$ and $g$ imply that $\mathcal{V}g = e^{i\alpha}\mathcal{V}f$ for some $\alpha \in \mathbb{R}$. This provides equality in (6.14) when using Cauchy-Schwarz inequality. Lastly we shall prove that also the first inequality in (6.14), that is

$$\left| \int_{\mathbb{R}^{2d}} F(z)\mathcal{V}f(z)\overline{\mathcal{V}g(z)}dz \right| \leq \int_{\mathbb{R}^{2d}} |F(z)| \cdot |\mathcal{V}f(z)| \cdot |\mathcal{V}g(z)|dz$$

becomes an equality, which is true if and only if

$$e^{-i\theta}\int_{\mathbb{R}^{2d}} F(z)\cdot|\mathcal{V}f(z)|^2dz = \int_{\mathbb{R}^{2d}} |F(z)|\cdot|\mathcal{V}f(z)|^2dz$$

for some $\theta \in \mathbb{R}$. This, in turn, is equivalent to the condition

$$e^{-i\theta}F(z)\cdot|\mathcal{V}f(z)|^2 = |F(z)|\cdot|\mathcal{V}f(z)|^2 \quad \text{for a.e. } z\in\mathbb{R}^{2d}$$

but since $|\mathcal{V}f(z)|^2 > 0$, equality in 6.14 with $f$ and $g$ as (5.12) occurs if and only if $F(z) = e^{i\theta}|F(z)|$. □

In light of the previous Theorem it is natural to seek for sharp upper bound for the right-hand side of (6.13). Since this involves the distribution function $|F|$ we shall search this bound between all the possible distribution functions. In order to do so we need to rephrase constraints (6.10) in terms of $\mu$. This can be easily done one more time thanks to a more general version of the "layer cake" representation (see [13, Theorem 1.13] or [7, Proposition 1.1.4]):

$$\|F\|_p^p = p\int_0^\infty t^{p-1}|\{|F|>t\}|dt.$$

Hence, constraints (6.10) become

$$p\int_0^\infty t^{p-1}u(t)dt \le A^p \quad \text{and} \quad q\int_0^\infty t^{q-1}u(t)dt \le B^q \tag{6.17}$$

and we can define the proper space of possible distribution functions

$$\mathcal{C} = \{u : (0,+\infty) \to [0,+\infty) \text{ such that } u \text{ is decreasing and satisfies (6.17)}\}. \tag{6.18}$$

Up to now we have rephrased our original question in terms of the following variational problem:

$$\sup_{v\in\mathcal{C}} I(v) \quad \text{where} \quad I(v) := \int_0^{+\infty} G(v(t))dt \tag{6.19}$$

Firstly, we shall prove existence of maximizers.

**Proposition 6.3.** *The supremum in (6.19) is finite and it is attained by at least one function $u \in \mathcal{C}$. Moreover, every extremal function $u$ achieves equality in at least one of the constraints (6.17).*

*Proof.* Considering, for example, the first constraint in (6.17), we see that

$$t^pu(t) = p\int_0^t \tau^{p-1}u(t)d\tau \overset{u \text{ decreasing}}{\le} p\int_0^t \tau^{p-1}u(\tau)d\tau \le A^p$$

hence functions in $\mathcal{C}$ are pointwise bounded by $A^p/t^p$. It is straightforward to verify that $G$ in (5.11) is increasing, that $G(s) \le s$ and that $G(s) \le 1$. Using these properties we have:

$$I(u) = \int_0^{+\infty} G(u(t))dt = \int_0^1 G(u(t))dt + \int_1^{+\infty} G(u(t))dt \overset{G(s)\le 1}{\le} 1 + \int_1^{+\infty} G(u(t))dt \overset{G \text{ increasing}}{\le}$$

$$\le 1 + \int_1^{+\infty} G(A^p/t^p)dt \overset{G(s)\le s}{\le} 1 + \int_1^{+\infty}\frac{A^p}{t^p}dt < \infty$$

therefore the supremum in (6.19) is finite.

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ be a maximizing sequence. Since every $u_n$ is pointwise bounded by $A^p/t^p$, thanks to Helly's selection theorem we can say that, up to a subsequence, $u_n$ converges pointwise to a decreasing function $u$, which is still in $\mathcal{C}$, indeed:

$$\int_0^{+\infty} t^{p-1}u(t) = \int_0^{+\infty} \lim_{n\to\infty} t^{p-1}u_n(t)dt \overset{\text{Fatou's lemma}}{\leq} \liminf_{n\to\infty} \int_0^{+\infty} t^{p-1}u_n(t)dt \leq \frac{A^p}{p}$$

and the same holds for $q$ instead of $p$.

Now we have to prove that $u$ is actually achieving the supremum. We already saw that the following holds:

$$|G(u_n(t))| \leq \chi_{(0,1)}(t) + \frac{A^p}{t^p}\chi_{(1,+\infty)}$$

and that the left-hand side is a function in $L^1(0,+\infty)$. This allows us to use dominated convergence theorem to conclude that

$$I(u) = \int_0^{+\infty} G(u(t)) = \lim_{n\to\infty} \int_0^{\infty} G(u_n(t))dt = \lim_{n\to\infty} I(u_n) = \sup_{v\in\mathcal{C}} I(v)$$

Lastly we need to show that $u$ achieves equality at least in one of the constraints (6.17). Suppose that this is not true. If we let $u_\varepsilon(t) = (1+\varepsilon)u(t)$, then for $\varepsilon > 0$ sufficiently small constraints are still satisfied and since $G$ is strictly increasing $I(u_\varepsilon) > I(u)$, which contradicts the hypothesis that $u$ is a maximizer. □

In order to do some "meaningful" calculus of variations over $I$ we need to enlarge $\mathcal{C}$, because the monotonicity assumption is quite strict. We will show that removing this hypothesis leaves the supremum unchanged and that maximizers are indeed monotonic.

**Proposition 6.4.** *Let $\mathcal{C}' = \{u : (0,+\infty) \to [0,+\infty)$ such that $u$ is measurable and satisfies (6.17)$\}$. Then*

$$\sup_{v\in\mathcal{C}} I(v) = \sup_{v\in\mathcal{C}'} I(v). \tag{6.20}$$

*In particular, any function $u \in \mathcal{C}$ achieving the supremum on the left-hand side also achieves it on the right-hand side.*

*Proof.* Let $u \in \mathcal{C}'$. We define its *decreasing rearrangement* as

$$u^*(s) = \sup\{t \geq 0 : |\{u > t\}| > s\} \tag{6.21}$$

AGGIUNGERE DIMOSTRAZIONE IN APPENDICE?? □

Before stating and proving the theorem that gives the only maximal function of (6.19), we introduce the following notation:

$$\mathrm{Log}_{-}(x) = \max\{-\log(x),\ 0\}, \quad x > 0.$$

**Theorem 6.5.** *There exist a unique function $u \in \mathcal{C}$ achieving the supremum in (6.19) that is:*

$$u(t) = \frac{1}{d!}\left[\mathrm{Log}_-\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d, \quad t > 0 \tag{6.22}$$

*where $\lambda_1, \lambda_2$ are both positive and uniquely determined by*

$$p\int_0^{+\infty} t^{p-1}u(t)dt = A^p, \quad q\int_0^{+\infty} t^{q-1}u(t)dt = B^q$$

*Proof.* We will split the proof in several parts. Firstly we will show that maximizers are given by (6.22). Then we will show that multipliers $\lambda_1$ and $\lambda_2$ are both strictly positive and unique.

Let $M = \sup\{t \in (0, +\infty) \ : u(t) > 0\}$. From Proposition 6.3 we know that $u$ has to achieve at least one of the constraints, therefore $M > 0$. Consider now a closed interval $[a, b] \subset (0, M)$ and a function $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Without loss of generality we can suppose that $\eta$ is orthogonal, in the $L^2$ sense, to $t^{p-1}$ and $t^{q-1}$, explicitly

$$\int_a^b t^{p-1}\eta(t)dt = 0, \quad \int_a^b t^{q-1}\eta(t)dt = 0. \tag{6.23}$$

On $[a, b]$ we have that $u(t) \geq u(b) > 0$, hence, for $|\varepsilon|$ sufficiently small, $u + \varepsilon\eta$ is still a nonnegative function which satisfies (6.17), therefore $u + \varepsilon\eta \in \mathcal{C}'$. Since we are supposing that $u$ is a maximizer, the function $\varepsilon \mapsto I(u + \varepsilon\eta)$ has a maximum for $\varepsilon = 0$. Since $\eta$ is supported in a compact interval we can differentiate under the integral sign and obtain

$$0 = \frac{d}{d\varepsilon}I(u + \varepsilon\eta)|_{\varepsilon=0} = \int_a^b G'(u(t))\eta(t)dt.$$

We would like to extend this result to every $\eta$ in $L^2(a, b)$ satisfying (6.23). Since $L^\infty(a, b)$ is dense in $L^2(a, b)$, there exist a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset L^\infty(a, b)$ such that $\eta_k \to \eta$ in $L^2(a, b)$. We can consider the projection operator $P$ such that, given $\psi \in L^2(a, b)$, $P\psi$ is the orthogonal projection of $\psi$ onto $X = \mathrm{span}\{t^{p-1}, t^{q-1}\}^\perp \subset L^2(a, b)$. Since $P$ is continuous we have that $P\eta_k \to P\eta = \eta$, hence

$$0 = \int_a^b G'(u(t))P\eta_k(t)dt = \langle G'(u), P\eta_n\rangle_{L^2(a,b)} \to \langle G'(u), \eta\rangle_{L^2(a,b)} = \int_a^b G'(u(t))\eta(t)dt \implies$$
$$\implies \int_a^b G'(u(t))\eta(t)dt = 0. \tag{6.24}$$

Since (6.24) holds for every $\eta \in X$ it must be that

$$G'(u) \in X^\perp = \left(\mathrm{span}\{t^{p-1}, t^{q-1}\}^\perp\right)^\perp = \mathrm{span}\{t^{p-1}, t^{q-1}\} \quad \text{in } (a, b).$$

By letting $a \to 0^+$ and $b \to M^-$ we then obtain

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \quad \text{for a.e. } t \in (0, M) \tag{6.25}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Since $u$ is decreasing actually (6.25) holds for every $t \in (0, M)$. We point out that this argument enables us to say that maximizers have to achieve equality in both constraints in (6.17). Indeed, if, for example, we had that $q \int_0^\infty t^{q-1}u(t)dt < B^q$, the second condition of orthogonality in (6.23) could be removed (since, for sufficiently small $\varepsilon$, a variation non-orthogonal to $t^{q-1}$ would be admissible), providing us with the solution of the same variational problem but without the $L^q$ constraint. Since we already know that actually this solution does not satisfy the $L^q$ constraint we conclude that $u$ has to achieve equality in both constraints. With the very same thinking we can say that neither $\lambda_1$ nor $\lambda_2$ can be 0.

Recalling the expression of (5.11) we see that $G'(s) = e^{-(d!s)^{1/d}}$ and easily invert (6.25), thus obtaining

$$u(t) = \begin{cases} \frac{1}{d!}\left[-\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d & t \in (0, M) \\ 0 & t \in (M, +\infty) \end{cases} \tag{6.26}$$

We remark that a priori it was possible that $M = +\infty$, but from the explicit expression of maximizers we see that this is not possible since $u$ has to be nonnegative.

Our main goal now is to show that multipliers $\lambda_1, \lambda_2$ are both positive and unique since this will give us uniqueness for the maximizer.

We start by proving that both multipliers are positive. Suppose that one of them, for example $\lambda_2$, is negative. Consider an interval $[a, b] \subset (0, M)$ and a variation $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Thanks to the Gram-Schmidt process we can construct a variation orthogonal to $t^{p-1}$. Since $\eta$ is arbitrary we can also suppose that it is not orthogonal to $t^{q-1}$, in particular we can ask that $\int_a^b t^{q-1}\eta(t)dt < 0$. Therefore, the directional derivative of $G$ along $\eta$ is:

$$\int_a^b G'(u(t))\eta(t)dt = \int_a^b (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \lambda_2 \int_a^b t^{q-1}\eta(t)dt > 0$$

which contradicts the fact that $u$ is a maximizer.

Now that we now that both multipliers are positive we can prove that $u$ is continuous, which is equivalent to say that $M = T$, where $T$ is the unique positive number such that $\lambda_1 T^{p-1} + \lambda_2 T^{q-1} = 1$ (uniqueness of $T$ follows from the positivity of multipliers).

We start supposing that $M < T$, which means that $\lim_{t\to M^-} u(t) > 0$. Consider the following variation

$$\eta(t) = \begin{cases} -1 + \alpha\frac{t}{M} + \beta, & t \in (M - M\delta, M) \\ 1, & t \in (M, M + M\delta) \\ 0, & \text{otherwise} \end{cases}$$

where $\delta > 0$ is small enough so that $M - M\delta > 0$ and $M + M\delta < T$, while $\alpha$ and $\beta$ are constants, depending on $\delta$, to be determined. Since we want this to be an admissible variation we need to impose that $\eta$ is orthogonal to $t^{p-1}$ and $t^{q-1}$. For example, the first condition is:

$$0 = \int_{M-M\delta}^{M+M\delta} t^{p-1}\eta(t)dt = -\int_{M-M\delta}^{M} t^{p-1}dt + \int_{M-M\delta}^{M} t^{p-1}\left(\alpha\frac{t}{M}+\beta\right)dt + \int_{M}^{M+M\delta} t^{p-1}dt \overset{\tau=t/M}{=}$$

$$= M^p\int_{1-\delta}^{1}\tau^{p-1}(\alpha\tau+\beta)d\tau - M^p\int_{1-\delta}^{1}\tau^{p-1}d\tau + M^p\int_{1}^{1+\delta}\tau^{p-1}d\tau \xrightarrow{1/\delta}$$

$$\Longrightarrow \fint_{1-\delta}^{1}\tau^{p-1}(\alpha\tau+\beta)d\tau = \alpha\fint_{1-\delta}^{1}\tau^{p}d\tau + \beta\fint_{1-\delta}^{1}\tau^{p-1}d\tau = \fint_{1-\delta}^{1}\tau^{p-1}d\tau - \fint_{1}^{1+\delta}\tau^{p-1}d\tau$$

The equation stemming from the orthogonality with $t^{q-1}$ is analogous. Therefore we obtained a nonhomogeneous linear system for $\alpha$ and $\beta$ **VA RESO MEGLIO**

$$\begin{pmatrix}\fint_{1-\delta}^{1}\tau^{p}d\tau & \fint_{1-\delta}^{1}\tau^{p-1}d\tau\\ \fint_{1-\delta}^{1}\tau^{q}d\tau & \fint_{1-\delta}^{1}\tau^{q-1}d\tau\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\fint_{1-\delta}^{1}\tau^{p-1}d\tau - \fint_{1}^{1+\delta}\tau^{p-1}d\tau\\ \fint_{1-\delta}^{1}\tau^{q-1}d\tau - \fint_{1}^{1+\delta}\tau^{q-1}d\tau\end{pmatrix} \tag{6.27}$$

This system has a unique solution if and only if the determinant of the matrix is not 0. We can show this directly:

$$\fint_{1-\delta}^{1}\tau^{p}d\tau\fint_{1-\delta}^{1}\tau^{q-1}d\tau - \fint_{1-\delta}^{1}\tau^{q}d\tau\fint_{1-\delta}^{1}\tau^{p-1}d\tau =$$

$$= \frac{1}{\delta^2}\int_{(1-\delta,1)^2}\left(\tau^p\sigma^{q-1} - \tau^{p-1}\sigma^q\right)d\tau d\sigma = \frac{1}{\delta^2}\int_{(1-\delta,1)^2}\tau^{p-1}\sigma^{q-1}(\tau-\sigma)\,d\tau d\sigma =$$

$$= \frac{1}{\delta^2}\left(\int_{Q_1}\tau^{p-1}\sigma^{q-1}(\tau-\sigma)\,d\tau d\sigma + \int_{Q_2}\tau^{p-1}\sigma^{q-1}(\tau-\sigma)\,d\tau d\sigma\right) =$$

where $Q_1 = (1-\delta,1)^2\cap\{\tau>\sigma\}$ and $Q_2 = (1-\delta,1)^2\cap\{\tau<\sigma\}$. In the second integral we can consider the change of variable that swaps $\tau$ and $\sigma$. In this case the new domain is $Q_1$, hence:

$$= \frac{1}{\delta^2}\int_{Q_1}\left(\tau^{p-1}\sigma^{q-1} - \tau^{q-1}\sigma^{p-1}\right)(\tau-\sigma)\,d\tau d\sigma$$

In $Q_1$ $\tau-\sigma>0$ and the sign of $\tau^{p-1}\sigma^{q-1} - \tau^{q-1}\sigma^{p-1}$ is constant, in fact:

$$\tau^{p-1}\sigma^{q-1} - \tau^{q-1}\sigma^{p-1} > 0 \iff \left(\frac{\tau}{\sigma}\right)^{p-q} > 1 \overset{\tau>\sigma}{\iff} p>q$$

Therefore the determinant of the matrix is always not 0.

The derivative of $G$ along $\eta$ is nonpositive because $u$ is supposed to be a maximizer, therefore

$$0 \geq \int_{M-M\delta}^{M+M\delta} G'(u(t))\eta(t)dt = -\int_{M-M\delta}^{M}\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)dt +$$

$$+ \int_{M-M\delta}^{M}\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\left(\alpha\frac{t}{M}+\beta\right)dt + \int_{M}^{M+M\delta} dt =$$

$$= -\int_{M-M\delta}^{M}\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)dt + \lambda_1 M^p\int_{1-\delta}^{1} t^{p-1}(\alpha t+\beta)dt +$$

$$+ \lambda_2 M^q\int_{1-\delta}^{1} t^{q-1}(\alpha t+\beta)dt + M\delta$$

Dividing by $M\delta$ and rearranging we obtain:

$$\fint_{M-M\delta}^{M} \left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right) dt \geq 1 + \lambda_1 M^{p-1} \fint_{1-\delta}^{1} t^{p-1}(\alpha t + \beta) dt + \lambda_2 M^{q-1} \fint_{1-\delta}^{1} t^{q-1}(\alpha t + \beta) dt \tag{6.28}$$

We notice that the last two terms are exactly the one that appear in the orthogonality condition, therefore, to understand their behavior as $\delta$ approaches 0, we need to study the right-hand side of the system (6.27). If we expand the first term in its Taylor series with respect to $\delta$ we have:

$$\left(1 - \frac{p-1}{2}\delta + o(\delta)\right) - \left(1 + \frac{p-1}{2}\delta + o(\delta)\right) = -(p-1)\delta + o(\delta)$$

and the same is for the other term. Since they both are of order $\delta$, if we let $\delta \to 0^+$ in (6.28) we obtain

$$\lambda_1 M^{p-1} + \lambda_2 M^{q-1} \geq 1$$

Since the function $\lambda_1 t^{p-1} + \lambda_2 t^{q-1}$ is strictly increasing (because $\lambda_1$ and $\lambda_2$ are both positive) $M \geq T$ which is absurd because we supposed that $M < T$. This allows us to write $u$ as in (6.22).

Lastly we shall prove that multipliers $\lambda_1, \lambda_2$, and hence maximizer, are unique. For this proof it is convenient to express $u$ in a slightly different way:

$$u(t) = \frac{1}{d!}\left[\mathrm{Log}_-\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^d$$

To emphasize that $u$ is parametrized by $c_1, c_2$ we may write $u(t; c_1, c_2)$. Now we define

$$f(c_1, c_2) = p\int_0^T t^{p-1} u(t; c1, c2) dt, \quad g(c_1, c_2) = q\int_0^T t^{q-1} u(t; c1, c2) dt$$

We want to highlight that, even if it is not explicit, also $T$ depends on $c_1$ and $c_2$. Nevertheless these functions are differentiable since both $T$ and $u$ are differentiable with respect to $(c_1, c_2)$, functions $t^{p-1}u$ and $t^{q-1}u$ and their derivatives are bounded in $(0, T)$. Our maximizer $u$ satisfies the constraints only if $f(c_1, c_2) = A^p$, $g(c_1, c_2) = B^q$. Therefore to prove uniqueness of the maximizer we need to show that level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.
First of all we are studying endpoints. For example, if $c_2 = 0$:

$$\begin{aligned} f(c_1, 0) &= p\int_0^{1/c_1} t^{p-1}\frac{1}{d!}\left[-\log(c_1 t)^{p-1}\right]^d dt \overset{\tau = c_1 t}{=} \\ &= \frac{p(p-1)^d}{c_1^p d!}\int_0^1 \tau^{p-1}\left[-\log(\tau)\right]^d d\tau = \frac{\kappa_p^d}{c_1^p} = A^p \implies c_{1,f} = \frac{\kappa_p^{d/p}}{A} \end{aligned}$$

The same can be done for $g$ and setting $c_1 = 0$ thus we obtain four points

$$c_{1,f} = \frac{\kappa_p^{d/p}}{A}, \; c_{1,g} = \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B}, \; c_{2,f} = \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A}, \; c_{2,g} = \frac{\kappa_q^{d/q}}{B}$$

In the regime we are considering one has that $c_{1,f} < c_{1,g}$ and $c_{2,f} > c_{2,g}$, indeed

$$c_{1,f} < c_{1,g} \iff \frac{\kappa_p^{d/p}}{A} < \left(\frac{p-1}{q}\right)^{d/q} \frac{1}{B} \iff \frac{B}{A} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{d/q}$$
$$c_{2,f} > c_{2,g} \iff \left(\frac{q-1}{p}\right)^{d/p} \frac{1}{A} > \frac{\kappa_q^{d/q}}{B} \iff \frac{B}{A} > \kappa_q^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{d/p}$$

which are exactly conditions in (6.12). Since there is this dispositions of these points we expect there is an intersection between level sets. Firstly we notice that, for every $c_1 \in (0, c_{1,f})$, there exist a unique value of $c_2$ for which $f(c_1, c_2) = A^p$. Indeed, from previous computations we notice that $f(c_1,0)$ is a decreasing function hence $f(c_1,0) > A^p$, while $\lim_{c_2 \to +\infty} f(c_1, c_2) = 0$. The uniqueness of this value follows from strict monotonicity of $f(c_1, \cdot)$, in fact:

$$\frac{\partial f}{\partial c_1}(c_1, c_2) = -\frac{p(p-1)}{(d-1)!} c_1^{p-2} \int_0^T \frac{t^{2(p-1)}}{(c_1 t)^{p-1} + (c_2 t)^{q-1}} \left[-\log\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^{d-1} dt \tag{6.29}$$

is always strictly negative. We point out that the term $\frac{\partial T}{\partial c_1}(c_1, c_2) u(T; c_1, c_2)$, that should appear since $T$ depends on $c_1$, is zero because $u$ is 0 in $T$. The same is true for $g$, therefore on the interval $(0, c_{1,f})$ the level sets of $f$ and $g$ can be seen as the graph of two functions $\varphi, \gamma$. Since $\frac{\partial f}{\partial c_2}, \frac{\partial g}{\partial c_2} < 0$ for every $(c_1, c_2)$, from the implicit function theorem we have that $\varphi$ and $\gamma$ are differentiable with respect to $c_1$.
After defining $\varphi$ and $\gamma$ we want to prove that $(\varphi - \gamma)' < 0$. Still from the implicit function theorem we have

$$\frac{d}{dc_1}(\varphi - \gamma)(c_1) = -\frac{\frac{\partial f}{\partial c_1}(c_1, \varphi(c_1))}{\frac{\partial f}{\partial c_2}(c_1, \varphi(c_1))} + \frac{\frac{\partial g}{\partial c_1}(c_1, \gamma(c_1))}{\frac{\partial g}{\partial c_2}(c_1, \gamma(c_1))} < 0 \iff$$
$$\mathcal{I}(c_1) = \frac{\partial f}{\partial c_1}(c_1, \varphi(c_1)) \frac{\partial g}{\partial c_2}(c_1, \gamma(c_1)) - \frac{\partial f}{\partial c_2}(c_1, \varphi(c_1)) \frac{\partial g}{\partial c_1}(c_1, \gamma(c_1)) > 0$$

As for (6.29) the other derivatives are computed. To simplify the notation we define $h(t; c_1, c_2) = \frac{1}{(d-1)!} \frac{1}{(c_1 t)^{p-1} + (c_2 t)^{q-1}} \left[-\log\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^{d-1}$. From Fubini's theorem we can write the product of the integrals as a double integral

$$\mathcal{I}(c_1) = p(p-1)q(q-1) c_1^{p-2} \gamma(c_1)^{q-2} \int_{[0,T]^2} h(t; c_1, \varphi(c_2)) h(s; c_1, \gamma(c_2)) t^{2(p-1)} s^{2(q-1)} dt ds +$$
$$- p(q-1)q(p-1) c_1^{p-2} \varphi(c_1)^{q-2} \int_{[0,T]^2} h(t; c_1, \varphi(c_2)) h(s; c_1, \gamma(c_2)) t^{p+q-2} s^{p+q-2} dt ds$$

When level sets intersect we have $\varphi(c_1) = \gamma(c_1)$. In this situation we can factorize the terms outside the integral and notice that the sign of $\mathcal{I}$ depends on the sign of:

$$\int_{[0,T]^2} h(t; c_1, \varphi(c_1)) h(s; c_1, \gamma(c_1)) \left(t^{2(p-1)} s^{2(q-1)} - t^{p+q-2} s^{p+q-2}\right) dt ds =$$
$$= \int_{[0,T]^2} h(t; c_1, \varphi(c_1)) h(s; c_1, \gamma(c_1)) t^{p-2} s^{q-2} \left(t^p s^q - t^q s^p\right) dt ds$$

In order to simplify the notation once again we set $H(t, s; c_1) = h(t; c_1, \varphi(c_1))h(s; c_1, \gamma(c_1))$. Let $T_1 = [0, T]^2 \cap \{t > s\}$ and $T_2 = [0, T]^2 \cap \{t < s\}$. We can split the above integral in two parts:

$$\int_{T_1} H(t, s; c_1)t^{p-2}s^{q-2} (t^p s^q - t^q s^p) \, dtds + \int_{T_2} H(t, s; c_1)t^{p-2}s^{q-2} (t^p s^q - t^q s^p) \, dtds$$

Then, considering the change of variables that swaps $t$ and $s$, the domain of integration becomes $T_1$ and since $H$ is symmetric in $t$ and $s$ we have that the previous quantity is equal to

$$\begin{aligned} &\int_{T_1} H(t, s; c_1) \left(t^{p-2}s^{q-2} - t^{q-2}s^{p-2}\right) (t^p s^q - t^q s^p) \, dtds = \\ =&\int_{T_1} H(t, s; c_1) \frac{1}{t^2 s^2} (t^p s^q - t^q s^p)^2 \, dtds \end{aligned}$$

which is strictly positive.

Now we are able to prove the uniqueness of multipliers.

First of all, since $(\varphi - \gamma)' < 0$ whenever $\varphi(c_1) = \gamma(c_1)$, for every point of intersection there exist $\delta > 0$ such that $\varphi(t) > \gamma(t)$ for $t \in (c_1 - \delta, c_1)$ while $\varphi(t) < \gamma(t)$ for $t \in (c_1, c_1 + \delta)$. Define $c_1^* := \sup\{c_1 \in [0, c_{1,f}] : \forall t \in [0, c_1] \ \varphi(t) \geq \gamma(t)\}$. This is an intersection point between $\varphi$ and $\gamma$ (if $\varphi(c_1^*) > \gamma(c_1^*)$ due to continuity there would be $\varepsilon > 0$ such that $\varphi(c_1^* + \varepsilon) > \gamma(c_1^* + \varepsilon)$ which contradicts the definition of $c_1^*$) and it is the first one, because we saw that after every intersection point there is an interval where $\varphi < \gamma$. Lastly, since $\varphi(0) > \gamma(0)$ and $\varphi(c_{1,f}) = 0 < \gamma(c_1, f)$ we have that $0 < c_1^* < c_{1,f}$.

Suppose now that there is a second point of intersection $\tilde{c}_1$ after the first one. Since immediately after $c_1^*$ we have that $\varphi$ becomes smaller than $\gamma$, this second point of intersection is given by $\tilde{c}_1 = \sup\{c_1 \in [c_1^*, c_{1,f}] : \forall t \in [c_1^*, c_1] \ \varphi(t) \leq \gamma(t)\}$. Considering that this is an intersection point, there exist an interval before $\tilde{c}_1$ where $\varphi$ is strictly greater than $\gamma$ which is absurd, hence $c_1^*$ is the only intersection point between $\varphi$ and $\gamma$.

Therefore $(c_1^*, \varphi(c_1^*) = c_2^*)$ is the unique pair of multipliers for which

$$p \int_0^T t^{p-1} u(t; c_1^*, c_2^*) dt = A^p, \quad q \int_0^T t^{q-1} u(t; c_1^*, c_2^*) dt = B^q$$

and, in the end, $u(t; c_1^*, c_2^*)$ is the unique maximizer for (6.19). □

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