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Some recent results on the norm of localization operators

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Theorem from [**nicolatilli_fk**]

Theorem 3.1. *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2 dx d\omega \leq G(|\Omega|)$$

where $G(s)$ is given by

$$G(s) \coloneqq \int_0^s e^{(-d!\tau)^{1/d}} d\tau \tag{3.1}$$

Capitolo 4

Localization Operators

4.1 Definition and properties

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Recent results from Nicola-Tilli

5.1 Case $q = +\infty$

5.2 Generic case

Let's now consider the case where both p and q are neither 1 or $+\infty$. The result presented in [**nicolatilli_norm**] include the case ...

$$\|L_F\|_{L_2 \to L_2} \leq \min\{\kappa_p^{d\kappa_p} A,\, \kappa_q^{d\kappa_q} B\}$$

Suppose that the minimum is given by $\kappa_p^{d\kappa_p} A$, therefore

$$\kappa_p^{d\kappa_p} A \leq \kappa_q^{d\kappa_q} B \iff \frac{B}{A} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}} \right)^d$$

We can check if the solution of the problem with just the L^p bound solves also the problem with both bounds, that is $F\|_{L^q} \leq B$, where F is given by ...

$$\|F\|_{L^q}^q = \int_{\mathbb{R}^{2d}} |F(z)|^q dz = ... = \lambda^q \left(\frac{p-1}{q} \right)^d$$

Since we want F to satisfy the L^q constraint we should have

$$\frac{B}{A} \geq \kappa_p^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \left(\frac{p}{q} \right)^{\frac{d}{q}}$$

It would be nice if this bound was less restrictive than the first one. Unfortunately that's not the case, in fact it's always true that

$$\left(\frac{p'}{q'} \right)^{\frac{1}{q'}} \left(\frac{p}{q} \right)^{\frac{1}{q}} \geq 1$$

Following the path in [**nicolatilli_norm**] we obtain ...

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \implies u(t) = \frac{1}{d!}\left[-\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d, \ t \in (0, M)$$

Our main goal now is to show that multipliers λ_1, λ_2 are unique and both positive.
The easiest fact to prove is that both multipliers are not 0. In fact if one, say λ_2, was 0, we would obtain that the solution of our problem is the same as the one with just the L^p bound. But we already know that this function does not satisfy the L^q constraint hence it is impossible that $\lambda_2 = 0$.
Suppose now that one of the multipliers, say always λ_2, is negative. Consider an interval $[a,b] \subset (0,M)$ and a variation $\eta \in L^\infty(0,M)$ supported in $[a,b]$. Thanks to the Gram-Schmidt process we can construct a variation orthogonal to t^{p-1}. Since η is arbitrary we can suppose that it is not orthogonal to t^{q-1}, in particular we can suppose that $\int_a^b t^{q-1}\eta(t)dt < 0$. Therefore the directional derivative of G along η is:

$$\begin{aligned}\int_a^b G'(u(t))\eta(t)dt &= \int_a^b (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \\ &= \lambda_2 \int_a^b t^{q-1}\eta(t)dt > 0\end{aligned}$$

which contradicts the fact that u is a maximizer.

PROOF OF CONTINUITY

Lastly we shall prove that multipliers λ_1, λ_2, and hence maximizer, are unique. For this proof it is convenient to express u in a slightly different way

$$u(t) = \frac{1}{d!}\left[\mathrm{Log}_-\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^d$$

To emphasize that u is parametrized by c_1, c_2 we may write $u(t; c_1, c_2)$. Now we define

$$f(c_1, c_2) = p\int_0^T t^{p-1}u(t; c1, c2)dt, \quad g(c_1, c_2) = q\int_0^T t^{q-1}u(t; c1, c2)dt$$

We want to highlight that, even if not explicit, also T depends on c_1 and c_2. Nevertheless these functions are differentiable since both T and u are differentiable with respect to (c_1, c_2), functions $t^{p-1}u$ and $t^{q-1}u$ and their derivatives are bounded in $(0, T)$. Our maximizer u satisfies the constraints only if $f(c_1, c_2) = A^p$, $g(c_1, c_2) = B^q$. Therefore to prove uniqueness of the maximizer we need to show that level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.
First of all we are studying endpoints. For example, if $c_2 = 0$:

$$\begin{aligned}f(c_1{,}0) &= p\int_0^{1/c_1} t^{p-1}\frac{1}{d!}\left[-\log(c_1 t)^{p-1}\right]^d dt \overset{\tau = c_1 t}{=} \\ &= \frac{p(p-1)^d}{c_1^p d!}\int_0^1 \tau^{p-1}\left[-\log(\tau)\right]^d d\tau = \frac{\kappa_p^d}{c_1^p} = A^p \implies c_{1,f} = \frac{\kappa_p^{d/p}}{A}\end{aligned}$$

The same can be done for g and setting $c_1 = 0$ thus we obtain four points

$$c_{1,f} = \frac{\kappa_p^{d/p}}{A}, \ c_{1,g} = \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B}, \ c_{2,f} = \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A}, \ c_{2,g} = \frac{\kappa_q^{d/q}}{B}$$

In the regime we are considering one has that $c_{1,f} < c_{1,g}$ and $c_{2,f} > c_{2,g}$, indeed

$$c_{1,f} < c_{1,g} \iff \frac{\kappa_p^{d/p}}{A} < \left(\frac{p-1}{q}\right)^{d/q} \frac{1}{B} \iff \frac{B}{A} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{d/q}$$
$$c_{2,f} > c_{2,g} \iff \left(\frac{q-1}{p}\right)^{d/p} \frac{1}{A} > \frac{\kappa_q^{d/q}}{B} \iff \frac{B}{A} > \kappa_q^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{q}{p}\right)^{d/p}$$

Since there is this dispositions of these points we expect there is an intersection between the level sets. Firstly we notice that, for every $c_1 \in (0, c_{1,f})$ there exist a unique value of c_2 for which $f(c_1, c_2) = A^p$. Indeed, from previous computations we notice that $f(c_1,0)$ is a decreasing function hence $f(c_1,0) > A^p$, while $\lim_{c_2 \to +\infty} f(c_1, c_2) = 0$. The uniqueness of this value follows from strict monotonicity of $f(c_1, \cdot)$, in fact:

$$\frac{\partial f}{\partial c_1}(c_1, c_2) = -\frac{p(p-1)}{(d-1)!} c_1^{p-2} \int_0^T \frac{t^{2(p-1)}}{(c_1 t)^{p-1} + (c_2 t)^{q-1}} \left[-\log\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^{d-1} dt \tag{5.1}$$

is always strictly negative. We point out that the term $\frac{\partial T}{\partial c_1}(c_1, c_2) u(T; c_1, c_2)$ is zero because u is 0 in T. The same is true for g, therefore on the interval $(0, c_{1,f})$ the level sets of f and g can be seen as the graph of two functions φ, γ. Since $\frac{\partial f}{\partial c_2}, \frac{\partial g}{\partial c_2} < 0$ for every (c_1, c_2) from the implicit function theorem we have that φ and γ are differentiable with respect to c_1.

Possibile disegno??

If we manage to prove that $(\varphi - \gamma)' < 0$ when $\{f = A^p\}$ and $\{g = B^q\}$ intersect we have uniqueness of the intersection point. Indeed, since $c_{2,f} = \varphi(0) > \gamma(0) = c_{2,g}$, if there were more intersection point for at least one of them it has to be $(\varphi - \gamma)' \geq 0$. Still from the implicit function theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}c_1}(\varphi - \gamma)(c_1) = -\frac{\dfrac{\partial f}{\partial c_1}(c_1, \varphi(c_1))}{\dfrac{\partial f}{\partial c_2}(c_1, \varphi(c_1))} + \frac{\dfrac{\partial g}{\partial c_1}(c_1, \gamma(c_1))}{\dfrac{\partial g}{\partial c_2}(c_1, \gamma(c_1))} < 0 \iff$$
$$\frac{\partial f}{\partial c_1}(c_1, \varphi(c_1)) \frac{\partial g}{\partial c_2}(c_1, \gamma(c_1)) - \frac{\partial f}{\partial c_2}(c_1, \varphi(c_1)) \frac{\partial g}{\partial c_1}(c_1, \gamma(c_1)) > 0$$

As for (5.1) the other derivatives are computed. To simplify the notation we define $h(t; c_1, c_2) = \frac{1}{(d-1)!} \frac{1}{(c_1 t)^{p-1} + (c_2 t)^{q-1}} \left[-\log\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^{d-1}$. From Fubini's theorem we can write the product of the integrals as a double integral

$$p(p-1)q(q-1)c_1^{p-2}\gamma(c_1)^{q-2} \iint_{[0,T]^2} h(t; c_1, \varphi(c_2)) h(s; c_1, \gamma(c_2)) t^{2(p-1)} s^{2(q-1)} dt ds -$$
$$p(q-1)q(p-1)c_1^{p-2}\varphi(c_1)^{q-2} \iint_{[0,T]^2} h(t; c_1, \varphi(c_2)) h(s; c_1, \gamma(c_2)) t^{p+q-2} s^{p+q-2} dt ds$$

At the intersection point $\varphi(c_1) = \gamma(c_1)$ hence the sign of the previous expression depends only on the sign of

$$\iint_{[0,T]^2} h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))\left(t^{2(p-1)}s^{2(q-1)} - t^{p+q-2}s^{p+q-2}\right) dtds =$$
$$= \iint_{[0,T]^2} h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))t^{p-2}s^{q-2}\left(t^p s^q - t^q s^p\right) dtds$$

In order to simplify the notation once again we set $H(t,s;c_1) = h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))$. Let $T_1 = [0,T]^2 \cap \{t > s\}$ and $T_2 = [0,T]^2 \cap \{t < s\}$. We can split the above integral in two parts

$$\iint_{T_1} H(t,s;c_1)t^{p-2}s^{q-2}\left(t^p s^q - t^q s^p\right) dtds + \iint_{T_2} H(t,s;c_1)t^{p-2}s^{q-2}\left(t^p s^q - t^q s^p\right) dtds$$

We can exchange t with s in the second integral. With this change of variables the domain of integration becomes T_1 and since H is symmetric in t and s we have that the previous quantity is equal to

$$\iint_{T_1} H(t,s;c_1)\left(t^{p-2}s^{q-2} - t^{q-2}s^{p-2}\right)\left(t^p s^q - t^q s^p\right) dtds =$$
$$\iint_{T_1} H(t,s;c_1)\frac{1}{t^2 s^2}\left(t^p s^q - t^q s^p\right)^2 dtds$$

which is strictly positive.