# POLITECNICO DI TORINO

**Corso di Laurea Magistrale**
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# Some recent results on the norm of localization operators


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Theorem from [**nicolatilli_fk**]

**Theorem 3.1.** *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2 dx d\omega \leq G(|\Omega|)$$

*where $G(s)$ is given by*

$$G(s) \coloneqq \int_0^s e^{(-d!\tau)^{1/d}} d\tau \tag{3.1}$$

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Let's now consider the case where both $p$ and $q$ are neither 1 or $+\infty$. The result presented in [**nicolatilli_norm**] include the case ...

$$\|L_F\|_{L_2 \to L_2} \leq \min\{\kappa_p^{d\kappa_p} A,\, \kappa_q^{d\kappa_q} B\}$$

Suppose that the minimum is given by $\kappa_p^{d\kappa_p} A$, therefore

$$\kappa_p^{d\kappa_p} A \leq \kappa_q^{d\kappa_q} B \iff \frac{B}{A} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d$$

We can check if the solution of the problem with just the $L^p$ bound solves also the problem with both bounds, that is $F\|_{L^q} \leq B$, where $F$ is given by ...

$$\|F\|_{L^q}^q = \int_{\mathbb{R}^{2d}} |F(z)|^q dz = ... = \lambda^q \left(\frac{p-1}{q}\right)^d$$

Since we want $F$ to satisfy the $L^q$ constraint we should have

$$\frac{B}{A} \geq \kappa_p^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \left(\frac{p}{q}\right)^{\frac{d}{q}}$$

It would be nice if this bound was less restrictive than the first one. Unfortunately that's not the case, in fact it's always true that

$$\left(\frac{p'}{q'}\right)^{\frac{1}{q'}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \geq 1$$

Following the path in [**nicolatilli_norm**] we obtain ...

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \implies u(t) = \frac{1}{d!}\left[-\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d, \ t \in (0, M)$$

Our main goal now is to show that multipliers $\lambda_1, \lambda_2$ are unique and both positive.

The easiest fact to prove is that both multipliers are not 0. In fact if one, say $\lambda_2$, was 0, we would obtain that the solution of our problem is the same as the one with just the $L^p$ bound. But we already know that this function does not satisfy the $L^q$ constraint hence it is impossible that $\lambda_2 = 0$.

Suppose now that one of the multipliers, say always $\lambda_2$, is negative. Consider an interval $[a, b] \subset (0, M)$ and a variation $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Thanks to the Gram-Schmidt process we can construct a variation orthogonal to $t^{p-1}$. Since $\eta$ is arbitrary we can suppose that it is not orthogonal to $t^{q-1}$, in particular we can suppose that $\int_a^b t^{q-1}\eta(t)dt < 0$. Therefore the directional derivative of $G$ along $\eta$ is:

$$\begin{aligned}\int_a^b G'(u(t))\eta(t)dt &= \int_a^b (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \\ &= \lambda_2 \int_a^b t^{q-1}\eta(t)dt > 0\end{aligned}$$

which contradicts the fact that $u$ is a maximizer.

Now that we now that both multipliers are positive we can prove that $u$ is continuous, which is equivalent to say that $M = T$, where $T$ is the unique positive number such that $\lambda_1 T^{p-1} + \lambda_2 T^{q-1} = 1$ (uniqueness of $T$ follows from the positivity of multipliers).
We start supposing that $M < T$ which means that $\lim_{t \to M^-} u(t) > 0$. Consider the following variation

$$\eta(t) = \begin{cases} -1 + \alpha\dfrac{t}{M} + \beta, & t \in (M - M\delta, M) \\ 1, & t \in (M, M + M\delta) \\ 0, & \text{otherwise} \end{cases}$$

where $\delta > 0$ is small enough so that $M - M\delta > 0$ and $M + M\delta < T$ while $\alpha$ and $\beta$ are constants, depending on $\delta$, to be found. Since we want this to be an admissible variation we need to impose that $\eta$ is orthogonal to $t^{p-1}$ and $t^{q-1}$. For example, the first condition is:

$$\begin{aligned}0 &= \int_{M-M\delta}^{M+M\delta} t^{p-1}\eta(t)dt = -\int_{M-M\delta}^{M} t^{p-1}dt + \int_{M-M\delta}^{M} t^{p-1}\left(\alpha\frac{t}{M} + \beta\right)dt + \int_{M}^{M+M\delta} t^{p-1}dt \overset{\tau = t/M}{=} \\ &= M^p \int_{1-\delta}^{1} \tau^{p-1}(\alpha\tau + \beta)d\tau - M^p \int_{1-\delta}^{1} \tau^{p-1}d\tau + M^p \int_{1}^{1+\delta} \tau^{p-1}d\tau \overset{1/\delta}{\Longrightarrow} \\ &\Longrightarrow \fint_{1-\delta}^{1} \tau^{p-1}(\alpha\tau + \beta)d\tau = \alpha \fint_{1-\delta}^{1} \tau^{p}d\tau + \beta \fint_{1-\delta}^{1} \tau^{p-1}d\tau = \fint_{1-\delta}^{1} \tau^{p-1}d\tau - \fint_{1}^{1+\delta} \tau^{p-1}d\tau\end{aligned}$$

The equation stemming from the orthogonality with $t^{q-1}$ is analogous. Therefore we obtained a nonhomogeneous linear system for $\alpha$ and $\beta$ **VA RESO MEGLIO**

$$\begin{pmatrix} \fint_{1-\delta}^{1} \tau^p d\tau & \fint_{1-\delta}^{1} \tau^{p-1} d\tau \\ \fint_{1-\delta}^{1} \tau^q d\tau & \fint_{1-\delta}^{1} \tau^{q-1} d\tau \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \fint_{1-\delta}^{1} \tau^{p-1} d\tau - \fint_{1}^{1+\delta} \tau^{p-1} d\tau \\ \fint_{1-\delta}^{1} \tau^{q-1} d\tau - \fint_{1}^{1+\delta} \tau^{q-1} d\tau \end{pmatrix} \tag{5.1}$$

This system has a unique solution if and only if the determinant of the matrix is not 0. If we see it as a function of $\delta$ we can expand the terms in a Taylor series:

$$\begin{aligned}
&\fint_{1-\delta}^{1} \tau^p d\tau \fint_{1-\delta}^{1} \tau^{q-1} d\tau - \fint_{1-\delta}^{1} \tau^q d\tau \fint_{1-\delta}^{1} \tau^{p-1} d\tau = \\
&= \left(1 - \frac{p}{2}\delta + \frac{p(p-1)}{6}\delta^2 + o(\delta^2)\right)\left(1 - \frac{q-1}{2}\delta + \frac{(q-1)(q-2)}{6}\delta^2 + o(\delta^2)\right) + \\
&- \left(1 - \frac{q}{2}\delta + \frac{q(q-1)}{6}\delta^2 + o(\delta^2)\right)\left(1 - \frac{p-1}{2}\delta + \frac{(p-1)(p-2)}{6}\delta^2 + o(\delta^2)\right) = \\
&= (1-1) + \delta\left(-\frac{q-1}{2} - \frac{p}{2} + \frac{p-1}{2} + \frac{q}{2}\right)\dots \\
&= \frac{p-q}{12}\delta^2 + o(\delta^2)
\end{aligned}$$

therefore the determinant is always non 0.
The derivative of $G$ along $\eta$ is nonpositive because $u$ is supposed to be a maximizer, therefore

$$\begin{aligned}
0 \geq &\int_{M-M\delta}^{M+M\delta} G'(u(t))\eta(t) dt = -\int_{M-M\delta}^{M} \left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right) dt + \\
&+ \int_{M-M\delta}^{M} \left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\left(\alpha \frac{t}{M} + \beta\right) dt + \int_{M}^{M+M\delta} dt = \\
&= -\int_{M-M\delta}^{M} \left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right) dt + \lambda_1 M^p \int_{1-\delta}^{1} t^{p-1}(\alpha t + \beta) dt + \\
&+ \lambda_2 M^q \int_{1-\delta}^{1} t^{q-1}(\alpha t + \beta) dt + M\delta
\end{aligned}$$

Dividing by $M\delta$ and rearranging we obtain:

$$\fint_{M-M\delta}^{M} \left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right) dt \geq 1 + \lambda_1 M^{p-1} \fint_{1-\delta}^{1} t^{p-1}(\alpha t + \beta) dt + \lambda_2 M^{q-1} \fint_{1-\delta}^{1} t^{q-1}(\alpha t + \beta) dt \tag{5.2}$$

We notice that the last two terms are exactly the one that appear in the orthogonality condition, therefore to understand their behavior as $\delta$ approaches 0 we need to study the right-hand side of the system (5.1). If we expand the first term in its Taylor series with respect to $\delta$ we have:

$$\left(1 - \frac{p-1}{2}\delta + o(\delta)\right) - \left(1 + \frac{p-1}{2}\delta + o(\delta)\right) = -(p-1)\delta + o(\delta)$$

and the same is for the other term. Since they both are of order $\delta$ if we let $\delta \to 0^+$ in (5.2) we obtain

$$\lambda_1 M^{p-1} + \lambda_2 M^{q-1} \geq 1$$

Since the function $\lambda_1 t^{p-1} + \lambda_2 t^{q-1}$ is strictly increasing (because $\lambda_1$ and $\lambda_2$ are both positive) $M \geq T$ which is absurd because we supposed that $M < T$.

Lastly we shall prove that multipliers $\lambda_1, \lambda_2$, and hence maximizer, are unique. For this proof it is convenient to express $u$ in a slightly different way:

$$u(t) = \frac{1}{d!}\left[\mathrm{Log}_-\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^d$$

To emphasize that $u$ is parametrized by $c_1, c_2$ we may write $u(t; c_1, c_2)$. Now we define

$$f(c_1, c_2) = p\int_0^T t^{p-1} u(t; c1, c2)dt, \quad g(c_1, c_2) = q\int_0^T t^{q-1} u(t; c1, c2)dt$$

We want to highlight that, even if not explicit, also $T$ depends on $c_1$ and $c_2$. Nevertheless these functions are differentiable since both $T$ and $u$ are differentiable with respect to $(c_1, c_2)$, functions $t^{p-1}u$ and $t^{q-1}u$ and their derivatives are bounded in $(0, T)$. Our maximizer $u$ satisfies the constraints only if $f(c_1, c_2) = A^p$, $g(c_1, c_2) = B^q$. Therefore to prove uniqueness of the maximizer we need to show that level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.
First of all we are studying endpoints. For example, if $c_2 = 0$:

$$\begin{aligned} f(c_1,0) &= p\int_0^{1/c_1} t^{p-1}\frac{1}{d!}\left[-\log(c_1 t)^{p-1}\right]^d dt \overset{\tau = c_1 t}{=} \\ &= \frac{p(p-1)^d}{c_1^p d!}\int_0^1 \tau^{p-1}\left[-\log(\tau)\right]^d d\tau = \frac{\kappa_p^d}{c_1^p} = A^p \implies c_{1,f} = \frac{\kappa_p^{d/p}}{A} \end{aligned}$$

The same can be done for $g$ and setting $c_1 = 0$ thus we obtain four points

$$c_{1,f} = \frac{\kappa_p^{d/p}}{A}, \ c_{1,g} = \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B}, \ c_{2,f} = \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A}, \ c_{2,g} = \frac{\kappa_q^{d/q}}{B}$$

In the regime we are considering one has that $c_{1,f} < c_{1,g}$ and $c_{2,f} > c_{2,g}$, indeed

$$\begin{aligned} c_{1,f} < c_{1,g} &\iff \frac{\kappa_p^{d/p}}{A} < \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B} \iff \frac{B}{A} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{p}{q}\right)^{d/q} \\ c_{2,f} > c_{2,g} &\iff \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A} > \frac{\kappa_q^{d/q}}{B} \iff \frac{B}{A} > \kappa_q^{d\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{q}{p}\right)^{d/p} \end{aligned}$$

Since there is this dispositions of these points we expect there is an intersection between the level sets. Firstly we notice that, for every $c_1 \in (0, c_{1,f})$ there exist a unique value of $c_2$ for which $f(c_1, c_2) = A^p$. Indeed, from previous computations we notice that $f(c_1,0)$ is a decreasing function hence $f(c_1,0) > A^p$, while $\lim_{c_2\to+\infty} f(c_1,c_2) = 0$. The uniqueness of this value follows from strict monotonicity of $f(c_1,\cdot)$, in fact:

$$\frac{\partial f}{\partial c_1}(c_1,c_2) = -\frac{p(p-1)}{(d-1)!}c_1^{p-2}\int_0^T \frac{t^{2(p-1)}}{(c_1t)^{p-1}+(c_2t)^{q-1}}\left[-\log\left((c_1t)^{p-1}+(c_2t)^{q-1}\right)\right]^{d-1}dt \tag{5.3}$$

is always strictly negative. We point out that the term $\frac{\partial T}{\partial c_1}(c_1,c_2)u(T;c_1,c_2)$ is zero because $u$ is 0 in $T$. The same is true for $g$, therefore on the interval $(0,c_{1,f})$ the level sets of $f$ and $g$ can be seen as the graph of two functions $\varphi,\gamma$. Since $\frac{\partial f}{\partial c_2},\frac{\partial g}{\partial c_2} < 0$ for every $(c_1,c_2)$ from the implicit function theorem we have that $\varphi$ and $\gamma$ are differentiable with respect to $c_1$.

## Possibile disegno??

After defining $\varphi$ and $\gamma$ we want to prove that $(\varphi-\gamma)' < 0$. Still from the implicit function theorem we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c_1}(\varphi-\gamma)(c_1) = -\frac{\frac{\partial f}{\partial c_1}(c_1,\varphi(c_1))}{\frac{\partial f}{\partial c_2}(c_1,\varphi(c_1))} + \frac{\frac{\partial g}{\partial c_1}(c_1,\gamma(c_1))}{\frac{\partial g}{\partial c_2}(c_1,\gamma(c_1))} < 0 \iff \\ \frac{\partial f}{\partial c_1}(c_1,\varphi(c_1))\frac{\partial g}{\partial c_2}(c_1,\gamma(c_1)) - \frac{\partial f}{\partial c_2}(c_1,\varphi(c_1))\frac{\partial g}{\partial c_1}(c_1,\gamma(c_1)) > 0\end{aligned}$$

As for (5.3) the other derivatives are computed. To simplify the notation we define $h(t;c_1,c_2) = \frac{1}{(d-1)!}\frac{1}{(c_1t)^{p-1}+(c_2t)^{q-1}}\left[-\log\left((c_1t)^{p-1}+(c_2t)^{q-1}\right)\right]^{d-1}$. From Fubini's theorem we can write the product of the integrals as a double integral

$$\begin{aligned}&p(p-1)q(q-1)c_1^{p-2}\gamma(c_1)^{q-2}\iint_{[0,T]^2} h(t;c_1,\varphi(c_2))h(s;c_1,\gamma(c_2))t^{2(p-1)}s^{2(q-1)}dtds- \\ &p(q-1)q(p-1)c_1^{p-2}\varphi(c_1)^{q-2}\iint_{[0,T]^2} h(t;c_1,\varphi(c_2))h(s;c_1,\gamma(c_2))t^{p+q-2}s^{p+q-2}dtds\end{aligned}$$

At the intersection point $\varphi(c_1) = \gamma(c_1)$ hence the sign of the previous expression depends only on the sign of

$$\begin{aligned}&\iint_{[0,T]^2} h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))\left(t^{2(p-1)}s^{2(q-1)} - t^{p+q-2}s^{p+q-2}\right)dtds = \\ &= \iint_{[0,T]^2} h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))t^{p-2}s^{q-2}\left(t^ps^q - t^qs^p\right)dtds\end{aligned}$$

In order to simplify the notation once again we set $H(t,s;c_1) = h(t;c_1,\varphi(c_1))h(s;c_1,\gamma(c_1))$. Let $T_1 = [0,T]^2 \cap \{t > s\}$ and $T_2 = [0,T]^2 \cap \{t < s\}$. We can split the above integral in two parts

$$\iint_{T_1} H(t,s;c_1)t^{p-2}s^{q-2}\left(t^ps^q - t^qs^p\right)dtds + \iint_{T_2} H(t,s;c_1)t^{p-2}s^{q-2}\left(t^ps^q - t^qs^p\right)dtds$$

We can exchange $t$ with $s$ in the second integral. With this change of variables the domain of integration becomes $T_1$ and since $H$ is symmetric in $t$ and $s$ we have that the previous quantity is equal to

$$\iint_{T_1} H(t,s;c_1)\left(t^{p-2}s^{q-2} - t^{q-2}s^{p-2}\right)(t^p s^q - t^q s^p)\,dtds =$$
$$\iint_{T_1} H(t,s;c_1)\frac{1}{t^2 s^2}\left(t^p s^q - t^q s^p\right)^2 dtds$$

which is strictly positive.

Now we are able to prove the uniqueness of multipliers.
First of all, since $(\varphi-\gamma)' < 0$ whenever $\varphi(c_1) = \gamma(c_1)$, for every point of intersection there exist $\delta > 0$ such that $\varphi(t) > \gamma(t)$ for $t \in (c_1-\delta, c_1)$ while $\varphi(t) < \gamma(t)$ for $t \in (c_1, c_1+\delta)$. Define $c_1^* \coloneqq \sup\{c_1 \in [0, c_{1,f}] : \forall t \in [0, c_1]\ \varphi(t) \geq \gamma(t)\}$. This is an intersection point between $\varphi$ and $\gamma$ (if $\varphi(c_1^*) > \gamma(c_1^*)$ due to continuity there would be $\varepsilon > 0$ such that $\varphi(c_1^*+\varepsilon) > \gamma(c_1^*+\varepsilon)$ which contradicts the definition of $c_1^*$) and it is the first one, because we saw that after every intersection point there is an interval where $\varphi < \gamma$. Lastly, since $\varphi(0) > \gamma(0)$ and $\varphi(c_{1,f}) = 0 < \gamma(c_1, f)$ we have that $0 < c_1^* < c_{1,f}$.
Suppose now that there is a second point of intersection $\tilde{c}_1$ after the first one. Since immediately after $c_1^*$ we have that $\varphi$ becomes smaller than $\gamma$ this second point of intersection is given by $\tilde{c}_1 = \sup\{c_1 \in [c_1^*, c_{1,f}] : \forall t \in [c_1^*, c_1]\ \varphi(t) \leq \gamma(t)\}$. Considering that this is an intersection point, there exist an interval before $\tilde{c}_1$ where $\varphi$ is strictly grater than $\gamma$ which is absurd, hence $c_1^*$ is the only intersection point between $\varphi$ and $\gamma$.
Therefore $(c_1^*, \varphi(c_1^*) = c_2^*)$ is the unique pair of multipliers for which $p\int_0^T t^{p-1}u(t;c_1^*,c_2^*)dt = A^p$, $q\int_0^T t^{q-1}u(t;c_1^*,c_2^*)dt = B^q$ and in the end $u(t;c_1^*,c_2^*)$ is the unique maximizer for **CITARE PROBLEMA VARIAZIONALE**