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Some recent results on the norm of localization operators

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3.1 STFT

3.1.1 Properties of STFT

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3.3 Faber-Krahn Inequality for the STFT

Theorem from [**nicolatilli_fk**]

Theorem 3.1. *For every $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$ and every measurable subset $\Omega \subset \mathbb{R}^{2d}$ with finite measure we have*

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2 dx d\omega \leq G(|\Omega|)$$

where $G(s)$ is given by

$$G(s) := \int_0^s e^{(-d!\tau)^{1/d}} d\tau \tag{3.1}$$

Capitolo 4

Localization Operators

4.1 Definition and properties

4.2 Eigenvalues and eigenfunctions

Capitolo 5

Recent results from Nicola-Tilli

5.1 Case $q = +\infty$

5.2 Generic case

Let's now consider the case where both p and q are neither 1 or $+\infty$. The result presented in [**nicolatilli_norm**] include the case ...

$$\|L_F\|_{L_2 \to L_2} \leq \min\{\kappa_p^{d\kappa_p} A,\ \kappa_q^{d\kappa_q} B\}$$

Suppose that the minimum is given by $\kappa_p^{d\kappa_p} A$, therefore

$$\kappa_p^{d\kappa_p} A \leq \kappa_q^{d\kappa_q} B \iff \frac{B}{A} \geq \left(\frac{\kappa_p^{\kappa_p}}{\kappa_q^{\kappa_q}}\right)^d$$

We can check if the solution of the problem with just the L^p bound solves also the problem with both bounds, that is $F\|_{L^q} \leq B$, where F is given by ...

$$\|F\|_{L^q}^q = \int_{\mathbb{R}^{2d}} |F(z)|^q dz = ... = \lambda^q \left(\frac{p-1}{q}\right)^d$$

Since we want F to satisfy the L^q constraint we should have

$$\frac{B}{A} \geq \kappa_p^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \left(\frac{p}{q}\right)^{\frac{d}{q}}$$

It would be nice if this bound was less restrictive than the first one. Unfortunately that's not the case, in fact it's always true that

$$\left(\frac{p'}{q'}\right)^{\frac{1}{q'}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \geq 1$$

Following the path in [**nicolatilli_norm**] we obtain ...

$$G'(u(t)) = \lambda_1 t^{p-1} + \lambda_2 t^{q-1} \implies u(t) = \frac{1}{d!}\left[-\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right)\right]^d, \ t \in (0, M)$$

Our main goal now is to show that multipliers λ_1, λ_2 are unique and both positive.
The easiest fact to prove is that both multipliers are not 0. In fact if one, say λ_2, was 0, we would obtain that the solution of our problem is the same as the one with just the L^p bound. But we already know that this function does not satisfy the L^q constraint hence it is impossible that $\lambda_2 = 0$.
Suppose now that one of the multipliers, say always λ_2, is negative. Consider an interval $[a, b] \subset (0, M)$ and a variation $\eta \in L^\infty(0, M)$ supported in $[a, b]$. Thanks to the Gram-Schmidt process we can construct a variation orthogonal to t^{p-1}. Since η is arbitrary we can suppose that it is not orthogonal to t^{q-1}, in particular we can suppose that $\int_a^b t^{q-1}\eta(t)dt < 0$. Therefore the directional derivative of G along η is:

$$\begin{aligned}\int_a^b G'(u(t))\eta(t)dt &= \int_a^b (\lambda_1 t^{p-1} + \lambda_2 t^{q-1})\eta(t)dt = \\ &= \lambda_2 \int_a^b t^{q-1}\eta(t)dt > 0\end{aligned}$$

which contradicts the fact that u is a maximizer.

PROOF OF CONTINUITY

Lastly we shall prove that multipliers λ_1, λ_2, and hence maximizer, are unique. For this proof it is convenient to express u in a slightly different way

$$u(t) = \frac{1}{d!}\left[\mathrm{Log}_-\left((c_1 t)^{p-1} + (c_2 t)^{q-1}\right)\right]^d$$

To emphasize that u is parametrized by c_1, c_2 we may write $u(t; c_1, c_2)$. Now we define

$$f(c_1, c_2) = p\int_0^T t^{p-1}u(t; c1, c2)dt, \quad g(c_1, c_2) = q\int_0^T t^{q-1}u(t; c1, c2)dt$$

We want to highlight that, even if not explicit, also T depends on c_1 and c_2. Our maximizer u satisfies the constraints only if $f(c_1, c_2) = A^p$, $g(c_1, c_2) = B^q$. Therefore to prove uniqueness of the maximizer we need to show that level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.
First of all we are studying endpoints. For example, if $c_2 = 0$:

$$\begin{aligned}f(c_1,0) &= p\int_0^{1/c_1} t^{p-1}\frac{1}{d!}\left[-\log(c_1 t)^{p-1}\right]^d dt \overset{\tau = c_1 t}{=} \\ &= \frac{p(p-1)^d}{c_1^p d!}\int_0^1 \tau^{p-1}\left[-\log(\tau)\right]^d d\tau = \frac{\kappa_p^d}{c_1^p} = A^p \implies c_{1,f} = \frac{\kappa_p^{d/p}}{A}\end{aligned}$$

The same can be done for g and setting $c_1 = 0$ thus we obtain four points

$$c_{1,f} = \frac{\kappa_p^{d/p}}{A}, \ c_{1,g} = \left(\frac{p-1}{q}\right)^{d/q}\frac{1}{B}, \ c_{2,f} = \left(\frac{q-1}{p}\right)^{d/p}\frac{1}{A}, \ c_{2,g} = \frac{\kappa_q^{d/q}}{B}$$

In the regime we are considering one has that $c_{1,f} < c_{1,g}$ and $c_{2,f} > c_{2,g}$, indeed

$$c_{1,f} < c_{1,g} \iff \frac{\kappa_p^{d/p}}{A} < \left(\frac{p-1}{q}\right)^{d/q} \frac{1}{B} \iff \frac{B}{A} < \kappa_p^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{p}{q}\right)^{d/q}$$

$$c_{2,f} > c_{2,g} \iff \left(\frac{q-1}{p}\right)^{d/p} \frac{1}{A} > \frac{\kappa_q^{d/q}}{B} \iff \frac{B}{A} > \kappa_q^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{q}{p}\right)^{d/p}$$