Vertices of P_M

$P_M = \operatorname{conv}(v_B : B \in \mathcal{B}(M))$ $\qquad v_B = (0011\overbrace{0101}^{B}) \in \mathbb{R}^E$

Prop. The vertices of P_M are all the v_B $(B \in \mathcal{B}(M))$

Pf The linear function $\sum_{i \in B} x_i$ is maximized at v_B.

Theorem

$$P_M = \{x \in \mathbb{R}^E \mid \underset{(\forall i \in E)}{x_i \geq 0}, \ \underset{\forall S \subseteq E}{\sum_{i \in S} x_i \leq r(S)}, \ \sum_{i \in E} x_i = r(M)\}$$

To prove this, we recall some basics of linear programming:

Linear program: Given
- a matrix $A \in \mathbb{R}^{m \times n}$
- a vector $b \in \mathbb{R}^m$
- a vector $c \in \mathbb{R}^n$,

(maximize a linear function over a polytope)

maximize $c^T x$ over all $x \in \mathbb{R}^n$
such that $Ax \leq b$, $x \geq 0$

Many practical problems can be phrased in this way, and there are good algorithms (simplex alg., ellipsoid method) for this.

Dual linear program: minimize $b^T y$ over all $y \in \mathbb{R}^m$
such that $A^T y \geq c$, $y \geq 0$

Duality Theorem

Weak: $c^T x \leq b^T y$ for any feasible x, y.

Strong: If primal has an optimal solution u, then dual has one v, and $c^T u = b^T v$.

Proof of weak duality thm:

$$c^T x \leq (A^T y)^T x = y^T A x \leq y^T b \quad \blacksquare$$

<u>Note:</u> If we find x, y feasible such that $c^T x = b^T y$, then they must both be optimal.

<u>Proof.</u>

Let Q_M be that polytope. Want: $Q_M = P_M$.

(A) $P_M \subseteq Q_M$

Each $v_B = (\overbrace{0\,0\,1\,1\,0\,1\,0\,1\,0\,0}^{B})$ has $(v_B)_i \geq 0$, $\sum (v_B)_i = r$

$$\sum_{i \in S} (v_B)_i = |B \cap S| \leq r(S)$$

So $v_B \in Q_M \rightarrow P_M = \mathrm{conv}(v_B) \subseteq Q_M$.

(B) $Q_M \subseteq P_M$

Plan: Every vertex of Q_M is a v_B (a vertex of P_M).

So $Q_M = \mathrm{conv}(\text{vertS. of } Q_M) \subseteq \mathrm{conv}(v_B) = P_M$

Each vertex p of Q_M is defined by a direction w

p maximizes $w^T x$ over Q_M

<u>Linear program:</u>

maximize $w^T x = \boxed{\quad}\,\boxed{\ }$

over

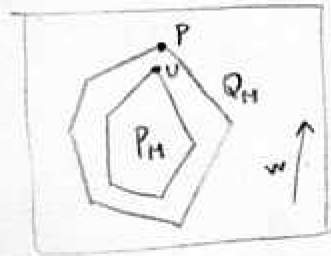
$$\begin{array}{c} 2^E \end{array}
\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}}^{E}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} E
\leq
\begin{bmatrix} r(\emptyset) \\ r(1) \\ r(2) \\ r(3) \\ r(12) \\ r(13) \\ r(23) \\ r(123) \\ -r(123) \end{bmatrix} 2^E
\qquad
\left\{ \begin{array}{l} Ax \leq r \\ x \geq 0 \end{array} \right.$$

Dual linear program

minimize $r^T y$

over

$$\begin{bmatrix} 0&1&0&0&1&1&0&1&-1 \\ 0&0&1&0&1&0&1&1&-1 \\ 0&0&0&1&0&1&1&1&-1 \end{bmatrix} \begin{bmatrix} y(\emptyset) \\ y(1) \\ y(2) \\ y(3) \\ y(12) \\ y(13) \\ y(23) \\ y(123) \\ y \end{bmatrix} \geq \begin{bmatrix} w(1) \\ w(2) \\ w(3) \end{bmatrix} \quad \longrightarrow \quad \begin{cases} A^T y \geq w \\ y \geq 0 \end{cases}$$



Suppose w: $\underbrace{W_a = W_b = W_c}_{w(1)} > \underbrace{W_d = W_e}_{w(2)} > \underbrace{W_f = W_g = W_h}_{w(3)}$

Candidate to maximize $w^T U$

$$U = \begin{array}{cccccccc} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

$r(abc)$ 1s

$r(abcde)$ 1s

$r(abcdefgh)$ 1s

$S_1 = abc$

$S_2 = abcde$

$S_3 = abcdefgh$

$$\Rightarrow w^T U = w(1) r(S_1) + w(2)(r(S_2) - r(S_1)) + w(3)(r(S_3) - r(S_2))$$

$$= r(S_1)(w(1) - w(2)) + r(S_2)(w(2) - w(3)) + r(S_3)\, w(3) = r^T v$$

where $v = \begin{bmatrix} 0 \\ 0 \\ w(1)-w(2) \\ 0 \\ 0 \\ w(2)-w(3) \\ 0 \\ 0 \\ w(3) \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow S_1 \\ \\ \\ \leftarrow S_2 \\ \\ \leftarrow S_3 \\ \\ \end{array}$

Also, v is feasible:

For each $i \in E$, $\displaystyle\sum_{S \ni i} v(S) \overset{?}{\geq} w_i$ $\qquad i \in S_a, S_{a+1}, \ldots, S_{last}$

$v(S_a) + v(S_{a+1}) + \cdots + v(S_{last}) \overset{?}{\geq} w_i$

$(w(a) - w(a+1)) + \cdots + w(last) \overset{?}{\geq} w_i \qquad w(a) = w(i)$ ✓

$w^T U = r^T v \Rightarrow U$ optimal $\Rightarrow p = U$ ← vertex of P_M.
(v optimal)