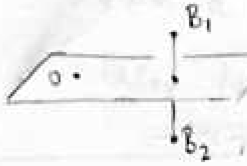
lecture 45
5/15/07

Why are these $e_i - e_j$ edges exciting?

Place a mirror $\perp$ to each edge of P_M.



Note: $OB_1 = \sqrt{1^2+0^2+1^2+1^2+0^2} = \sqrt{r}$

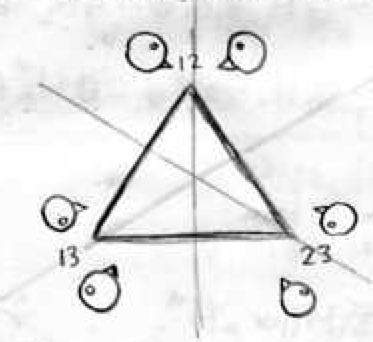
$OB_2 = \sqrt{r}$

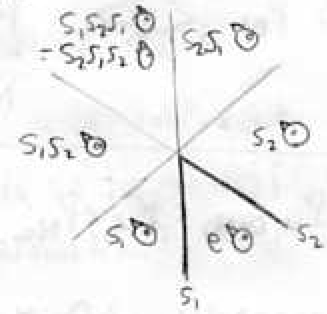
So the origin is on the mirror.

Let ρ be the reflection across the mirror.

Example:

$P_{U_{2,3}}$





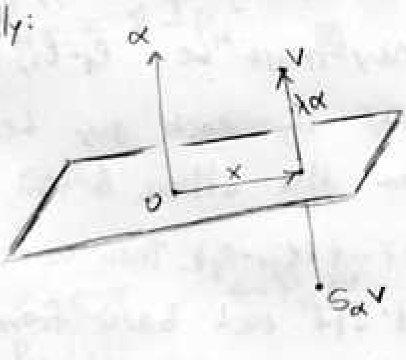
Finite # of images!

Group generated by the reflections: $\{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} = S_3$

Generators: s_1, s_2

Relations: $s_1^2 = 1 \quad s_2^2 = 1 \quad (s_1s_2)^3 = 1$

More generally:



$v = x + \lambda\alpha \qquad s_\alpha v = x - \lambda\alpha = v - 2\lambda\alpha$

$(\alpha, v) = 0 + \lambda(\alpha,\alpha)$

$$s_\alpha v = v - 2\frac{(\alpha, v)}{(\alpha,\alpha)}\alpha$$

$$\alpha = e_i - e_j \longrightarrow s_\alpha v = v - \frac{2(e_i - e_j, v)}{(e_i-e_j, e_i-e_j)}(e_i - e_j) = v - (v_i - v_j)(e_i - e_j)$$

$$= v_1 \dots \underset{\text{swap}}{v_j \dots v_i} \dots v_n$$

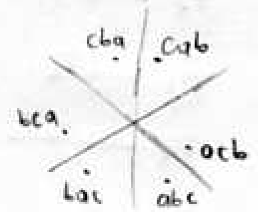
So reflecting across a hyperplane transposes coordinates i and j.

Reflecting across several hyperplanes permutes the coordinates.

If M connected $\rightarrow$ every $e_i - e_j$ appears

$\rightarrow$ Can swap coordinates i and j

$\rightarrow$ Can attain any permutation



Prop. The "reflection group" generated by reflecting on the mirrors of a connected matroid polytope on $[n]$ is the symmetric group S_n.

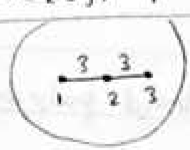
$\rightarrow$ If you build this system of mirrors you get 3 basic reflections s_1, s_2, s_3 which generate S_4 (24 images of yourself!) (Vain!)

- $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$ tetra
- $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ cube octa
- $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}$ dodec icosa

gens: s_1, s_2, s_3

rels: $s_1^2 = s_2^2 = s_3^2 = 1$

$(s_1 s_2)^3 = (s_2 s_3)^3 = 1 \qquad (s_1 s_3)^2 = 1$



In general,

$S_n =$ 

gens: $s_1, \ldots, s_{n-1}$

rels: $s_i^2 = 1$

$(s_i s_{i+1})^3 = 1$

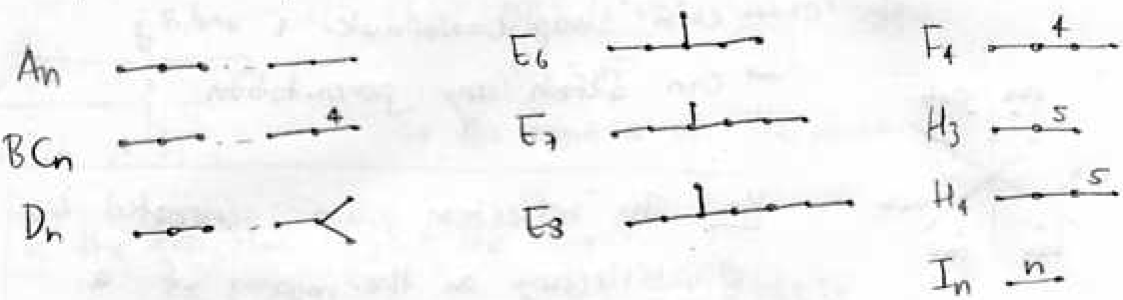
$(s_i s_j)^2 = 1 \qquad |j-i| \geq 2$

- omit edges 

- omit labels 

How rare is it for a reflection group to be finite?
<u>Very rare</u>.

The only (irreducible) ones:



Platonic solids:

simplex → A_n
hypercube/crosspolytope → BC_n
dodeca/icosahedron → H_3
24-cell → F_4
120-cell / 600-cell → H_4

(These Coxeter groups/Weyl groups appear in <u>many</u> places in mathematics.)

<u>Def</u> A <u>Coxeter matroid polytope</u> is a polytope whose edge-mirror reflections generate a finite group.

(Gelfand-Serganova, 1987)

So what is a Coxeter matroid, then?

See Borovik-Gelfand-White, "Coxeter matroids".

This is a recent field full of open problems.