# math 850 . representation theory

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## homework one . due thursday feb 4

Note. You are encouraged to work together on the homework, but please state who you worked with **in each problem**. Write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (Group actions as homomorphisms.) Write precise statements and detailed proofs of the following two facts:

   (a) A group action of a group $G$ on a set $A$ is the same thing as a homomorphism from $G$ to the symmetric group $S_A$.

   (b) A (linear) group action of a group $G$ on a vector space $V$ is the same thing as a homomorphism from $G$ to the general linear group $GL(V)$.

2. (An action of the symmetric group $S_3$ on $\mathbb{R}^2$.) Consider an equilateral triangle $V_1V_2V_3$ with center at $(0,0)$, vertex $V_1 = (1,0)$, and vertices labeled $V_1, V_2, V_3$ in countersclockwise order. Consider the action of the symmetric group $S_3$ on $\{V_1, V_2, V_3\}$ where $\pi \in S_3$ takes each vertex $V_i$ to $V_{\pi(i)}$. This extends to a unique (linear) action of $S_3$ on $\mathbb{R}^2$, say $X : S_3 \to GL_2(\mathbb{R})$. Compute the six matrices $\{X(\pi) : \pi \in S_3\}$ and show they faithfully represent $S_3$.

3. (A representation of an infinite group.) Let $SO_2(\mathbb{R})$ be the group of rotations of the circle under the operation of composition.

   (a) Prove that, considering $\mathbb{R}$ as an additive group, we have
   $$SO_2(\mathbb{R}) \cong \mathbb{R}/2\pi\mathbb{R}.$$

   (b) Prove that
   $$SO_2(\mathbb{R}) \cong \left\{A \in GL_2(\mathbb{R}) : A^tA = I, \det A = 1\right\}.$$

   (c) Consider the map $\varphi : SO_2(\mathbb{R}) \to GL_2(\mathbb{C})$ which sends $\theta \in \mathbb{R}/2\pi\mathbb{R}$ to
   $$\varphi(\theta) = \begin{bmatrix} \alpha & \alpha^2 - \alpha \\ 0 & \alpha^2 \end{bmatrix}$$
   where $\alpha = e^{i\theta}$. Prove that $\varphi$ is a group representation of $SO_2(\mathbb{R})$.

4. (The sign of a permutation.) An *inversion* in a permutation $\pi = \pi_1 \dots \pi_n$ is a pair of indices $i < j$ such that $\pi_i > \pi_j$. Let $\mathrm{inv}(\pi)$ be the number of inversions of $\pi$.

   (a) If $\pi$ is a product of $k$ transpositions, prove that $k \equiv \mathrm{inv}(\pi) \pmod 2$.

   (b) Conclude that the sign of a permutation is well defined.

   (c) Conclude that the sign representation of $S_n$ is indeed a representation.

5. (Representations of $p$-groups in characteristic $p$.) Let $G$ be a group with $|G| = p^n$ for a prime number $p$ and a positive integer $n$, and let $\mathbb{K}$ be a field of characteristic $p$. Prove that the every representation of $G$ over $\mathbb{K}$ is trivial.