The coeffs of the Tutte polynomial

Note: From
$$\begin{cases} T_M(x,y) = T_{M\setminus e}(x,y) + T_{M/e}(x,y) & e \neq C, L \\ T_M(x,y) = x\,T_{M\setminus e}(x,y) & e = C \\ T_M(x,y) = y\,T_{M/e}(x,y) & e = L \end{cases}$$

we know the coeffs of T_M are in $\mathbb{N}$. What do they count?

Ex: T  $(x,y) = x^3y + x^2y^2 + x^2y + xy^2 + xy^3$

Since $T_M(1,1) = \#$ bases

maybe coeff of $x^iy^j = \#$ bases such that...

Let M be a matroid and order E in any way.

Let $B \in \mathcal{B}(M)$.

Def. An element $e \notin B$ is externally active if it is the smallest element of the basic circuit $C(B,e)$.

Def An element $i \in B$ is internally active if it is the smallest element of the basic circuit $C(B^*,i)$ of M^*

Ex $M =$ 

B	12	13	14	23	24
$E(B)$	–	–	3	1	13
$I(B)$	12	1	1	–	–
B^*	34	24	23	14	13

$M^* =$ 

$\downarrow$

$x^2 + x + xy + y + y^2$ (compare with T )

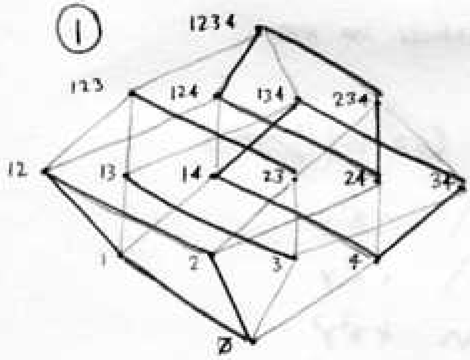
$\boxed{\text{Theorem (Crapo 1969)} \quad T_M(x,y) = \sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}}$

$i(B) = |I(B)|$ = internal activity
$e(B) = |E(B)|$ = external activity

So (coeff of $x^i y^e$) = # basis with int. act. i, ext. act. e.

<u>Note</u>. The RHS is then independent of the ordering on E! This is not at all clear!

<u>Sketch of proof.</u>

1. 

- The Boolean lattice $2^{[n]}$ is partitioned into intervals $[B - I(B), B \cup E(B)]$ i.e.,
- Every set $S \subseteq E$ is uniquely $S = B - I \cup E \qquad I \subseteq I(B),\ E \subseteq E(B)$

$[\emptyset, 12], [3, 13], [4, 134], [23, 123], [24, 1234]$

2. If $S = B - I \cup E$, then $r(S) = r - |I|$.

<u>Then</u>:

$$\sum_{S \subseteq [n]} (x-1)^{r - r(S)} (y-1)^{|S| - r(S)} = \sum_{\substack{S = B - I \cup E \\ B \in \mathcal{B} \\ I \subseteq I(B) \\ E \subseteq E(B)}} (x-1)^{|I|} (y-1)^{|E|}$$

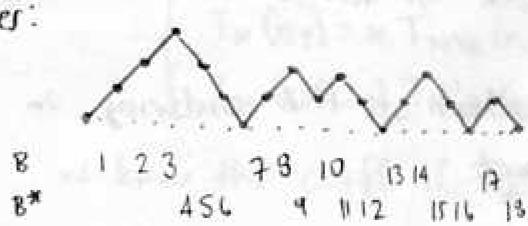
$$= \sum_{B \in \mathcal{B}} \left(\sum_{I \subseteq I(B)} (x-1)^{|I|} \right) \left(\sum_{E \subseteq E(B)} (y-1)^{|E|} \right) = \sum_{B \in \mathcal{B}} \left(\sum_{k=0}^{i(B)} \binom{i(B)}{k} (x-1)^k \right) \left(\sum_{k=0}^{e(B)} \binom{e(B)}{k} (y-1)^k \right)$$

$$= \sum_{B \in \mathcal{B}} (1 + (x-1))^{i(B)} (1 + (y-1))^{e(B)} . \quad \blacksquare$$

An example:

C_n = Catalan matroid.

Bases:



The ground set has a "natural" numbering.

Exercise: $I(B) = 123$ = initial string of upsteps

$E(B) = 6\ 12\ 16\ 18$ = downsteps where B bounces on x-axis.

Theorem. (Ardila 02, Bonin-deMier-Noy 02)

$$T_{C_n}(x,y) = \sum_{P \text{ Dyck}} x^{a(P)} y^{b(P)}$$

$a(P)$ = # of upsteps before first downstep
$b(P)$ = # of bounces on x-axis

Ex. $C_3 = M($  $)$

x^3y
$+ x^2y$
$+ x^2y^2$
$+ xy^2$
$+ xy^3$

Corollary.
(# of paths with $a(P) = r$, $b(P) = s$)
= (# of paths with $a(P) = s$, $b(P) = r$)

→ # of paths with r initial upsteps
= # of paths with r bounces

Proof.
Recall that $C_n^* \cong C_n$, so $T_{C_n}(x,y) = T_{C_n^*}(x,y) = T_{C_n}(y,x)$. ∎

Exercise.
Prove that there are $\binom{2n}{n}$ paths of $2n$ steps ↗ or ↘ which stay above the x-axis.

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