MATH 420/720 . Assigned readings and homework

All numbered readings and exercises are from Miklos Bóna's *A walk through combinatorics*, Third Edition.

Reading

Jan 26, 31: Chapter 1
Feb 2, 7: Chapter 2
Feb 9, 14: Chapter 3
Feb 16, 21, 23: Chapter 4
Feb 28, Mar 2: Chapters 5 and 6
Mar 7, 9: Chapter 6 and 7
Mar 14, 16: No reading.
Mar 28, 30, Apr 4: Chapter 8
Apr 11, 13: Chapters 9 and 10
Apr 18, 20: No reading.
Apr 25: Chapter 10.
Apr 27, May 2, May 4: Chapters 11, 12.

A. Mathematical Homework

Instructions.

I very much encourage you to collaborate with others on the homework, as long as you state who you worked with. If you are stuck after working on a problem for a while, I am always happy to give you a hint during office ours. If you are still stuck, you are welcome to use the internet (or in some cases the solutions in the book) as long as you state precisely your sources; for example, which internet page did you use?.

You will write your solutions on your own, and in your own words. The grader and I have no interest in reading or giving you credit for work that is not yours. We would much rather read your (possibly incomplete) understanding of a problem, so the homework will serve as an open and honest communication channel. This will allow us to address the challenges that arise and adapt the course accordingly.

Keep in mind that not everything your classmates say is correct, and not everything the internet says is correct. I'm sure occasionally I'll say things that are incorrect, too; I'll try not to! :) Make sure that you think critically about what you are learning in the homework, and that you are confident about what you are writing. I strongly recommend that you put your notes and your internet away before you start writing your solutions.

Writing Guidelines.

Please remember to make a special effort to write your solutions clearly, not only to show the grader that you understand how to solve the problems, but also so that your reader can follow your proof easily and enjoyably. LaTeXsolutions are welcome, but only if you include the same pictures that you would have included in a handwritten proof. (It's fine to hand-draw a nice picture, take a photo of it with your phone, and attach it to the LaTeXfile.)

Additional Grading Criteria.

• (Book or Internet Solutions) If you use the book solutions or if you use the internet to help you solve a problem, please indicate this clearly. That problem will be graded out of 8/10 (a high B). I interpret 8/10 as "Good work".

• (Incomplete Solutions) If you explain clearly why your incomplete solution is incomplete, and what might help you complete it, you will earn an additional 2/10.

MATH 420 vs MATH 720 Homework.

Each homework askes you to solve a problems and turn in your best b, where $b \leq a$. You are invited to write up all a problems, but please tell the grader which b problems you would like graded. If you don't do this, the grader will grade the first a problems you wrote.

Some problems are labelled "MATH 420 only". MATH 720 students may count those towards their a solved problems, but not towards their b problems turned in.

Homework 1 (due Thursday, Feb. 2)

Solve at least 5 problems (tell me which ones), and turn in your best 3.

1. (420 only) 1.16
2. 1.20
3. 1.23
4. 1.26
5. 1.31
6. 1.33
7. Prove that there is a power of 2 whose first 2017 digits are $100\ldots0$.

Homework 2 (due Thursday, Feb. 9)

PART A. (In groups - worth 20 points) Let $f(n)$ be the smallest number for which the following statement is true:

> "*Given any deck of n cards in any order, we can sort it using at most f(n) pairwise comparisons.*"

What can you prove about $f(13)$? $f(52)$? $f(n)$?

PART B. (Individual) Solve at least 3 problems (tell me which ones) and turn in your best one.

1. (420 only) 2.20
2. 2.21
3. 2.28
4. 2.33
5. 2.35

Homework 3 (due Thursday, Feb. 16)

PART A. (Individual - worth 10 points) Solve 5 of these numerical problems (tell me which ones) and turn in your best 4.

1. 3.3
2. 3.5
3. 3.15
4. 3.18
5. 3.31
6. 3.32

PART B. (Individual - worth 20 points) Solve at least 3 of these problems (tell me which ones) and turn in your best 2.

1. (420 only) 3.35
2. 3.41
3. (420 only) 3.44
4. 3.47
5. Guess and prove a formula for $F_{n-1}^2 + F_n^2$. Can you give a combinatorial proof using tilings?
6. Prove that
$$4^n \prod_{k=1}^{n-1} \left(\frac{1}{4} + \cos^2 \frac{k\pi}{2n-1} \right) = F_{2n}.$$

Here $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots$ denotes the Fibonacci sequence.

Homework 4 (due Thursday, Feb. 23)

PART A. (In groups - worth 10 points) Give at least one combinatorial proof of the combinatorial identity assigned to your group.

PART B. (Individual - worth 20 points) Solve at least 3 of these problems (tell me which ones) and turn in your best 2.

1. (420 only) 4.4
2. 4.6
3. 4.38
4. 4.47
5. 4.53
6. 4.54

Homework 5 (due Thursday, Mar. 2)

PART A. (In the groups formed in class - worth 10 points)

1. Find the probability $P(n)$ that, among n randomly chosen people, there are at least two who have the same birthday. Compute $P(23)$ explicitly. Assume that there are no leap years and that all birthdays are equally likely.

(NOTE. Don't stop there! See Bonus Problem 6.)

PART B. (Individual - worth 20 points) Solve at least 3 of these problems (tell me which ones) and turn in your best 2.

Recall that a *composition* of n is a way of expressing n as an ordered sum of positive integers, called the *parts* of the composition.

1. (420 only) How many pairs of subsets $A, B \subseteq [n]$ are there such that $A \subseteq B$?

2. How many k-tuples of subsets $S_1, \ldots, S_k$ of $[n]$ are there such that:

   (a) the S_is are pairwise disjoint?

   (b) $S_1 \cap \cdots \cap S_k = \emptyset$?

3. Find the number of compositions of n into odd parts.

4. Find the number of compositions of n into parts greater than 1.

5. Fix k and n. How many k-tuples of subsets $A_1, \ldots, A_k \subseteq [n]$ are there such that
$$A_1 \subseteq A_2 \supseteq A_3 \subseteq A_4 \supseteq \cdots ?$$

(NOTE. Don't stop there! See Bonus Problem 5.)

Homework 6 (due Thursday, Mar. 9)

PART A. (In the groups formed in class - worth 10 points) Solve one of the following two problems.

1. (a) Discover and prove a recurrence relation for the number $S(n,k)$ of set partitions of the set $\{1,2,\ldots,n\}$ into k pairwise disjoint non-empty subsets.

   (b) Prove that

$$x^n = \sum_{k=1}^{n} S(n,k)\, x(x-1)(x-2)\cdots(x-k+1)$$

2. (a) Discover and prove a recurrence relation for the number $c(n,k)$ of permutations of the set $\{1,2,\ldots,n\}$ with k cycles,

   (b) Prove that

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=1}^{n}(-1)^{n-k}c(n,k)x^k$$

PART B. (Individual - worth 20 points) Solve at least 3 of these problems (**tell me which ones**) and turn in your best 2.

1. Let $1 \le k \le n$. If you write down all the compositions of n, how many times will you see the number k as a part?
2. (420 only) 5.19 and 5.20 and 5.24
3. 5.27 and 5.28
4. 6.47
5. 6.53

Homework 7 (due Thursday, Mar. 16)

PART A. (In the groups formed in class - worth 10 points) Solve one of the following two problems, using the outline proposed in the worksheet you received in class.

1. A **derangement** is a permutation $\pi : [n] \to [n]$ with the property that $\pi(i) \neq i$ for all $1 \leq i \leq n$. Find the number of derangements of $[n]$. Approximately how large is the probability that a permutation of $[n]$ is a derangement?

2. Euler's **totient function** φ counts the number $\varphi(n)$ of integers k with $1 \leq k \leq n$ such that k and n are relatively prime. Find a simple formula for $\varphi(n)$.

PART B. (Individual - worth 20 points) Solve at least 3 of these problems (**tell me which ones**) and turn in your best 2.

1. (420 only) 7.16 and 7.17 and 7.19

2. 7.31 (a "zero row" is a row where all entries are zero)

3. 7.35

4. Suppose we have a finite set of objects and a list P of properties that each object may or may not satisfy. For each subset $S \subseteq P$, let

   b_S = # of objects satisfying the properties in S, **and no others**,
   c_S = # of objects in S satisfying **at most** the properties in S.

   (a) Express the numbers c_S in terms of the numbers b_S.
   (b) Express the numbers b_S in terms of the numbers c_S.

5. Let $M_3(n)$ be the number of 3×3 matrices whose entries are non-negative integers such that every row sum and every column sum is equal to n. Prove that

$$M_3(n) = \binom{n+5}{5} - \binom{n+2}{5}$$

B. Bonus Problems / Projects

You may turn these in at any time during the semester. You may do them individually or in groups of at most 5. Some of these may become your final projects.

Some of these questions are stated imprecisely on purpose. Part of your task (though I am happy to help) will be to formulate them precisely in a way that is mathematically interesting.

1. (a) Prove that, no matter where the 29 of us are in the classroom (Thornton Hall 211), there are two of us at a distance of at most x from each other. (The smaller your x, the better.)

   (b) Find a way of distributing the 29 of us in the classroom so that the minimum distance between two of us is y. (The larger your y, the more impressive your construction will be.)

2. Count the domino tilings of the United Farm Workers logo.

3. (Courtesy of Jonathan, thank you!) Mark has n songs in his iPod, and he wants to create a playlist of p songs such that

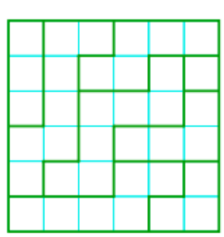
   - each song is played at least once, and
   - a song can be played again only if at least k other songs have been played in between.

   In how many ways can he create such a playlist?

4. Problem 4.48

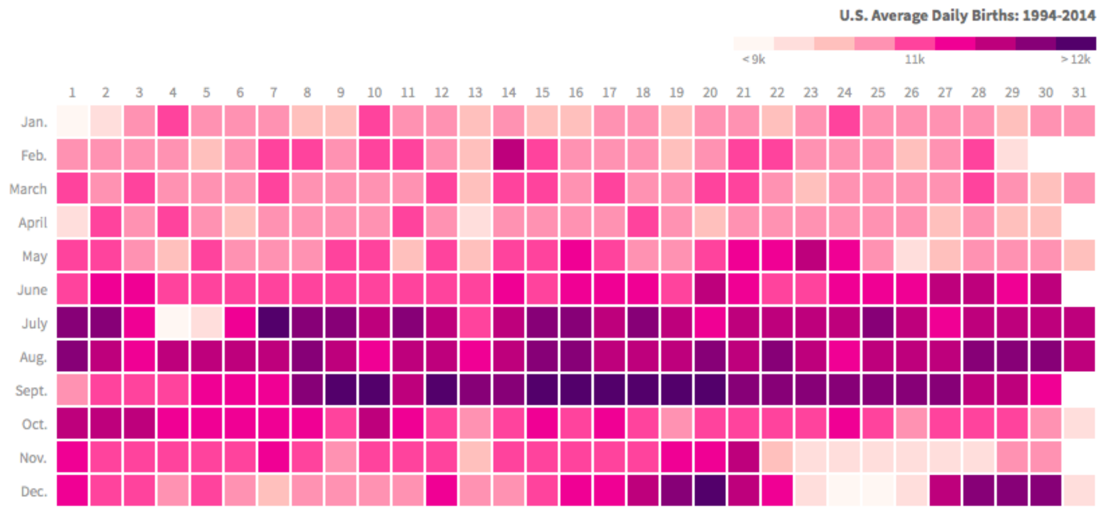
5. Consider an $n \times n$ square grid. An *NE-snake* is a sequence of one or more unit squares such that each unit square is either directly above or directly to the right of the previous unit square. Find the number of ways of tiling the $n \times n$ square grid with NE-snakes.

   A tiling of a 6×6 grid with NE-snakes is shown below.



6. Let $P(n)$ be the probability that, if we assume that all birthdays are equally likely, then among n people chosen randomly there are at least two with the same birthday. (See HW 5.)

   Let $P_S(n)$ be the probability that, among n people chosen randomly in the location S, there are at least two with the same birthday.



   It is **not** true that all birthdays are equally likely. For example, the figure above shows the birthday distribution in the United States.

   (a) There are many interesting observations to be made about the birthday distribution above. State some of them, and for each one, explain its likely causes.

   (b) Which one is larger, $P_{USA}(n)$ or $P(n)$?

   (c) Which one is larger, $P_{CA}(n)$ or $P_{NY}(n)$?

   (d) Which one is larger, $P_{NY}(n)$ or $P_{VT}(n)$?

   (e) Which one is larger, $P_{USA}(n)$ or $P_{COL}(n)$?

7. In class on Feb. 21 we analyzed the *Towers (not) of Hanoi* problem from the point of view of the largest disk. What other interesting points of view are we missing? What new things do those points of view tell us about the problem that we didn't already know?

8. Let $p_{\leq a, \leq b}(n)$ be the number of partitions of n into at most a parts, all of which are less than or equal to b. Let q be a prime number and let $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ be the field of q elements.

   (a) Find a recurrence relation for $p_{\leq a, \leq b}(n)$.

   (b) Prove that the number of a-dimensional subspaces of the vector space $\mathbb{F}_q^{a+b}$ is
   $$\sum_{n \geq 0} p_{\leq a, \leq b}(n)\, q^n.$$

   (c) Why is $p_{\leq a, \leq b}(n) = p_{\leq b, \leq a}(n)$? Give a combinatorial proof and an algebraic proof.

9. Usually when we see a Venn diagram in a textbook, it only involves 2 or 3 sets. Why is this?

   (a) Is it possible to draw a Venn diagram for 4 sets using circles?

   (b) Is it possible to draw a Venn diagram for 4 sets using shapes other than circles?

   (c) Is it possible to draw a Venn diagram for $n > 4$ sets using circles?

   (d) Is it possible to draw a Venn diagram for $n > 4$ sets using shapes other than circles?

10. In class on March 14 and 16, we asked you to begin organizing:

   - the domino tilings of a $2 \times n$ rectangle
   - the configurations of points on a line

   in a way that is useful and beautiful. The goal of that exercise is to explore the internal structure of the "space of all possibilities" for each one of these problems. Answer some or all of these questions:

   (a) How would you do this? **Pictures, please!** (You may use the same strategy you used in class or a different one.)

   (b) When you look at your answer to part (a), what do you wonder? What questions and/or answers arise when you look at these mathematical objects organized in this way?

   (c) What similarities and differences do you notice between these two "spaces of possibilities"?

(d) Are there ways in which your "design decisions" lead to mathematical consequences? Are there ways in which your mathematical knowledge of these objects influenced your design decisions?

11. Give a combinatorial proof that the probability that a path from $(0,0)$ to $(2n,0)$ using steps NE $(1,1)$ and SE $(1,-1)$ does not go below the x axis is $1/(n+1)$.

12. Prove that for any permutation w of length 3, the number of permutations of $[n]$ avoiding w is the Catalan number C_n.

13. Let $C_n(a,b)$ be the number of Dyck paths from $(0,0)$ to $(2n,0)$ such that the initial number of NE steps is a and the number of returns to the x axis is b. Prove that $C_n(a,b) = C_n(b,a)$.

C. Short Homework

Short Homework 1 – in groups (due Tuesday, Apr. 11)

Solve one or (for extra credit) both of these problems:

1. Prove that the number of binary trees with $n+1$ leaves is the Catalan number C_n.
2. Draw all the binary trees with 5 leaves, connecting two trees if one can be obtained from the other by regrafting a branch without changing its parent node. Organize the trees to obtain the most beautiful and useful picture that you can get. Looking at your picture, what do you wonder? What mathematical questions arise? Can you answer them?

Short Homework 2 (due Thursday, Apr. 20)

1. (Individual) Come up with an example of a graph (or family of graphs) that is interesting to you. This might be a graph that is interesting mathematically, has real-life applications, is really very beautiful, etc.

   Describe in words or pictures the graph's vertices, edges, paths, cycles, Eulerian paths, Eulerian cycles. (Bonus: Guess, conjecture, prove interesting facts about your graph and the notions above.)

   Of course, the extent to which these questions can be answered depends on the graph that you chose.

2. (In pairs/trios)

   (a) Draw a tree on [9], compute its Prüfer code, and give it to your partner. Don't show them your tree.

   (b) Use your partner's code to recover the tree they drew.

   (c) Compare your answers. If they don't match, do it again.

Short Homework 3 (due Thursday, May 4) The goal of this exercise is to complete the proof of the formula for the number of memory wheels for an alphabet of size n and windows of length k. Let $B(n,k)$ be the de Bruijn graph for this problem for $k \geq 2$, as defined in class. We begin with some definitions.

Let D be a directed graph with vertex set V..

- The **adjacency matrix** $A = A(D)$ is the $V \times V$ matrix whose entries are
$$a_{uv} = \#\text{ of edges } u \to v.$$
and the **directed Laplacian matrix** $\overrightarrow{L} = \overrightarrow{L}(D)$ is the $V \times V$ matrix whose entries are
$$\overrightarrow{L}_{uv} = \begin{cases} -(\#\text{ of edges } u \to v) & \text{if } u \neq v, \\ \text{outdeg}(u) - (\#\text{ of edges } u \to u) & \text{if } u = v. \end{cases}$$

- If D is a directed graph, its *line graph* D' is defined as follows:
o D' has a vertex called uv for each edge $u \to v$ of D.
o D' has an edge $uv \to vw$ for each path $u \to v \to w$ of length 2 in D.

- A graph is *k-regular* if $\text{outdeg}(u) = k$ for every vertex u.

Now, here are the questions. You may use any results proved in class.

1. For a k-regular graph D, prove that $\overrightarrow{L}(D) = kI - A(D)$.
2. (Bonus) Prove that if D is a k-regular digraph and $A(D)$ has eigenvalues $\alpha_1, \ldots, \alpha_v$, then $\overrightarrow{L}(D)$ has eigenvalues $k - \alpha_1, \ldots, k - \alpha_v$
3. (Bonus.) If $A(D)$ has eigenvalues $\alpha_1, \ldots, \alpha_v$, prove that $A(D')$ has eigenvalues $\alpha_1, \ldots, \alpha_v, 0, \ldots, 0$.
4. Prove that $B(n, k)$ is equal to the line graph of $B(n, k-1)$ for $k \geq 3$. Draw $B(2, 4)$ as the line graph of $B(2, 3)$ to illustrate this.
5. Draw the graph $B(4, 2)$. More generally, describe the graph $B(n, 2)$.
6. Find the eigenvalues of $\overrightarrow{L}(B(n, 2))$ and $A(B(n, 2))$.
7. Find the eigenvalues of $A(B(n, k))$ and $\overrightarrow{L}(B(n, k))$ for $k \geq 2$.
8. Prove that the number of memory wheels for an alphabet of size n and windows of length k is $(n!)^{n^{k-1}}/n^k$.

D. Other Homework

Instructions. Mathematics is a human endeavor and, when you enter mathematics, you do not leave yourself at the door. The goal of these assignments is for us to think explicitly about some personal, communital, and cultural aspects of mathematics.

Assignment 0 (daily, in aphabetical order).

I want to continue playing some music before class and in between things, to bring some more light into that classroom. On your designated day, please pick a song to share that makes you feel comfortable / joyful / at home. If you'd like to, you can tell us a bit about the song or why it's meaningful to you. You can send me a link to the song the day before, or bring a record and let me know to bring the record player that day.

Assignment 1 (due Thursday, Jan. 26)

Reread the Agreement in the syllabus, and the list of words that we used to describe Carlos Embales' guaguancó, and which I think should (mostly) apply to our class this semester:

> community . joy . polyrhythm . family . crescendo . playful
> encouraging . unexpected . churchlike . inviting . dancing
> conversation . courage . motivation . cheerful . Spanish .
> learning . rhythm . celebration . style . culture . festive

In the groups formed in class, please write down 2-3 concrete practices we can follow that will help us create this atmosphere and achieve the goals of the agreement.

Assignment 2 (due Tuesday, Jan. 31)

Write a 2 page mathematical autobiography. Some guiding questions: How did you become interested in mathematics? What are some of your most memorable (positive and negative) mathematical experiences? What do you like and dislike the most about mathematics? What is your favorite way of learning mathematics?

Of course there is no wrong answer here. The goal of this assignment is for you to reflect on your own mathematical experience, and to help me understand your mathematical practices and plan the best possible course for you.

Assignment 3 (due Tuesday, Apr 4)

Read the articles "*The importance of stupidity in scientific research*" by Martin A. Schwartz and "*The Secret to Raising Smart Kids*" by Carol Dweck, which were emailed to you. Think about them in relation to your own personal experiences in mathematics in general and in this class in particular. Write down your thoughts in a one-page essay.