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homework four . due thursday mar 17

Note. You are encouraged to work together on the homework, but please state who you worked with **in each problem**. Write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (The defining and standard representations of S_n and the fixed points of permutations.) Let V_{def}, V_{std}, and V_{triv} be the defining, standard, and trivial representations of the symmetric group S_n. Recall that $V_{def} \cong V_{std} \oplus V_{triv}$.
   (a) Prove that the standard representation $V_{std} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$ has no non-trivial S_n-invariant subspaces. Conclude that V_{std} is irreducible.
   (b) Prove that the characters of V_{std} and V_{def} are given by
   $$\chi_{def}(\pi) = f(\pi), \qquad \chi_{std}(\pi) = f(\pi) - 1$$
   where $f(\pi)$ is the number of *fixed points* of π.[1]
   (c) Compute
   $$\sum_{\pi \in S_n} f(\pi) \qquad \text{and} \sum_{\pi \in S_n} f(\pi)^2.$$
   (d) BONUS. Give combinatorial proofs for these formulas you obtained in part (c).

2. (Constructions of representations.) In class we discussed how, given finite-dimensional representations V and W of a finite group G, we can construct the following representations:
   (a) The dual V^*.
   (b) The direct sum $V \oplus W$.
   (c) The vector space $\mathrm{Hom}(V, W)$ of linear maps from V to W.
   (d) The tensor product $V \otimes W$.
   (e) The quotient V/W (iIf W is a subrepresentation of V).

   For each one of these, prove that it is indeed a representation of G, and prove the formulas for its dimension and its character.

3. (The character table of the symmetric group S_5.) Compute it.

BONUS. (The character table of the dihedral group D_n.) Compute it.

[1] A fixed point of π is an element i such that $\pi(i) = i$.

4. (Restriction and induction are transitive)

   (a) If H is a subgroup of G and χ is a character of G, we define the *restriction of the character* χ *to* H to be
   $$\chi{\downarrow}_H^G (h) = \chi(h)$$
   for all $h \in H$. Prove that restriction is transitive; that is, if $K \leq H \leq G$ we have
   $$\chi{\downarrow}_K^G = (\chi{\downarrow}_H^G){\downarrow}_K^H .$$

   (b) If H is a subgroup of G and χ is a character of H, we define the *character of* G *induced by* χ to be
   $$\chi{\uparrow}_H^G (g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi(x^{-1}gx)$$
   for all $g \in G$. Prove that induction is transitive; that is, if $K \leq H \leq G$ we have
   $$\chi{\uparrow}_K^G = (\chi{\uparrow}_K^H){\uparrow}_H^G .$$

5. (The number of irreducible representations.) Let G be a group. Let $\mathcal{F}$ be the vector space of functions $f : G \to \mathbb{C}$, and let $\mathcal{C}$ be the subspace of functions f which are constant on conjugacy classes.[2]

   (a) Prove that $f \in \mathcal{C}$ if and only if for any complex representation V, the linear map $\phi_{f,V} : V \to V$ given by
   $$\phi_{f,V}(v) = \sum_{g \in G} f(g) g \cdot v$$
   is a homomorphism of representations.

   (b) Show that the trace of $\phi_{f,V}$ is $\langle f, \chi_{V^*} \rangle$ for all $f \in \mathcal{C}$ and all representations V.

   (c) Show that if $\langle f, \chi_{V^*} \rangle = 0$ for some irreducible representation V, then $\phi_{f,V} = 0$.

   (d) Show that if $f \in \mathcal{C}$ is non-zero then $\phi_{f,R}$ is non-zero for the regular representation R.

   (e) Conclude that the characters of the irreducible representations span $\mathcal{C}$.

   (f) Conclude that the number of irreducible representations of G equals the number of conjugacy classes of G.

[2] Such functions are called *class functions*.