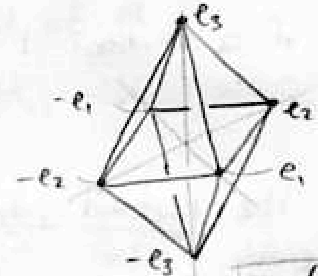
A <u>polytope</u> in $\mathbb{R}^d$ is the n-dimensional analog of a polygon.

You can describe it in two ways:

① <u>Vertices</u>

A polytope is the "convex hull" of a finite set of points $V = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^d$.



$\mathrm{conv}(\pm e_1, \pm e_2, \pm e_3)$
octahedron
(6 vertices)

$$P = \mathrm{conv}(V) = \left\{ x = \sum_{i=1}^{n} a_i v_i \mid a_i \geq 0, \sum_{i=1}^{n} a_i = 1 \right\}$$

Think: Snap a rubber band (or rubber suit?) around V.

This is the smallest convex set containing P.

② <u>Facets</u>

A polytope is a bounded intersection of halfspaces defined by linear inequalities

$$P = \{ x \in \mathbb{R}^d \mid a_1 \cdot x \leq z_1, \ldots, a_m \cdot x \leq z_m \}$$

octahedron:

$x_1 + x_2 + x_3 \leq 1$

$x_1 - x_2 + x_3 \leq 1$

$\vdots$

$\pm x_1 \pm x_2 \pm x_3 \leq 1$ (8 facets)

<u>Theorem</u>

A subset $S \subseteq \mathbb{R}^d$ is a convex hull of a finite set of points iff it is a bounded intersection of halfspaces.

This is great because we can go from the V-description to the H-description freely; for example:

- (polytope) $\cap$ (polytope) = polytope   H
- (polytope) $\cap$ (affine subspace) = polytope   H
- (projection of polytope) = polytope   V

To define a face F of P, one "points in its direction" w:

$$F_w = \{x \in P \mid w \cdot x \text{ is maximum}\}$$ (a polytope of smaller dimension)

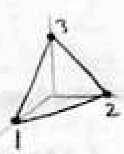
Faces of octahedron: 1 3-D, 8 2-D, 12 1-D, 6 0-D, 1 (-1)-D

<u>Def</u> The <u>matroid polytope</u> P_M of M (or matroid basis polytope) is

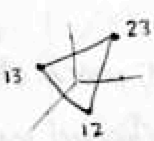
$$P_M = \text{conv}\left(\{e_{b_1} + \cdots + e_{b_r}\} \mid \{b_1, \ldots, b_r\} \text{ is a basis of } M\right) \quad \text{in } \mathbb{R}^E.$$

<u>Ex</u> $|E| = 3$

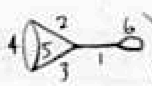
$M = U_{1,3}$



$M = U_{2,3}$



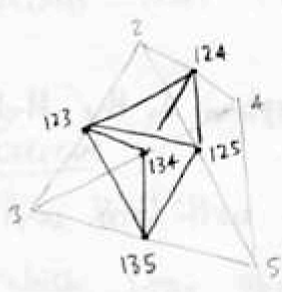
$M($  $)$

<u>Ex</u> $M($  $) = \text{conv}(111000, 110100, 110010, 101100, 101010)$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 123 & 124 & 125 & 134 & 135 \end{array}$$

This is in $\mathbb{R}^6$! But:
- $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3$
- $x_1 = 1$
- $x_6 = 0$

$\rightarrow$ So 3-D



<u>Q</u> What is $\dim(P_M)$?