math 850 . representation theory

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homework two . due thursday feb 18

Note. You are encouraged to work together on the homework, but please state who you worked with **in each problem**. Write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (Some matrix representations of the symmetric group S_3.)
   (a) Let X be the left-regular representation of S_3. Compute the matrices $X(123), X(132), X(213), X(231), X(312), X(321)$ in the standard basis $\{123, 132, 213, 231, 312, 321\}$.
   (b) Consider the left coset representation of S_3 with respect to the subgroup $H = \{e, (23)\}$ of S_3. Compute the matrices of this representation in the standard basis. (See Example 1.3.5 in the book.)
   (c) Consider the left coset representation of S_3 with respect to the subgroup $K = \{e, (123), (132)\}$ of S_3. Compute the matrices of this representation in the standard basis.
2. (The one-dimensional complex representations of the cyclic groups.)
   (a) Describe all the one-dimensional complex representations of the cyclic group C_n. Which ones are inequivalent?
   (b) Describe all the one-dimensional complex representations of a finite abelian group G. (Recall that every finite abelian group is of the form $G = C_{n_1} \times \cdots \times C_{n_k}$ for some prime powers $n_1, \ldots, n_k$.) Deduce that the number of inequivalent degree 1 complex representations of G is equal to $|G|$.
3. (Counting the one-dimensional complex representations of finite groups.) Let G be a finite group and $G' = [G, G]$ be its *commutator subgroup*, which is defined to be the subgroup generated by the elements $[g, h] = g^{-1}h^{-1}gh$ for all $g, h \in G$.
   (a) Prove that G' is a normal subgroup of G.
   (b) Prove that G/G' is commutative.
   (c) Prove that for any field $\mathbb{K}$, the degree 1 representations of G over $\mathbb{K}$ are in bijection with the degree 1 representations of G/G' over $\mathbb{K}$.
   (d) Deduce that the number of degree 1 complex representations of any finite group G is equal to $|G : G'|$.
4. (Things get more complicated in characteristic p.) Let G be a group with $|G| = p^n$ for a prime number p and a positive integer n, and let $\mathbb{K}$ be a field of characteristic p. Prove that the every irreducible representation of G over $\mathbb{K}$ is trivial. (If you would like a hint, see Dummit-Foote, Exercise 18.1.22.)