Short Homework 1

Andrés Lizcano Rodriguez

$a, b, c, d, e, f, g, h, i, j, k, l$ generic complex numbers

$$p(x;y) := ax + bxy^2 + cx^2y + dx^2y^3 + ex^3y + fx^3y^2 = 0$$
$$q(x;y) := gxy^3 + hxy^4 + ixy^5 + jx^2y^2 + kx^2y^3 + lx^3y^5 = 0$$

a) find # of isolated solutions $(x;y)$, with $(x;y) \in (\mathbb{C}\setminus\{0\})^2$

By Bernstein Theorem, the system $p(x;y) = q(x;y) = 0$ has $2!\,\mathrm{Vol}(\mathrm{New}(p), \mathrm{New}(q))$ isolated solution in $(\mathbb{C}\setminus\{0\})^2$

We know that $2!\,\mathrm{Vol}(\mathrm{New}(p), \mathrm{New}(q)) = [\mathrm{Vol}(\mathrm{New}(p) + \mathrm{New}(q)) - \mathrm{Vol}(\mathrm{New}(p)) - \mathrm{Vol}(\mathrm{New}(q))]$

Since $\text{support}(p) = \{x; xy^2; x^2y; x^2y^3; x^3y; x^3y^2\}$

$\text{support}(q) = \{xy^3; xy^4; xy^5; x^2y^2; x^2y^3; x^3y^5\}$

Thus $P := \text{New}(p) = \text{conv}\{(1;0), (1;2), (2;1), (2;3), (3;1), (3;2)\}$

$Q := \text{New}(q) = \text{conv}\{(1;3), (1;4), (1;5), (2;2); (2;3); (3;5)\}$

Since the polytopes P and Q are 2-dimensional their volumes and that of $P+Q$ are just the areas of those polygons.

That is: $\text{Vol}(P+Q) = 19$ ("Just triangulate")

$\text{Vol}(P) = 4$

$\text{Vol}(Q) = 4$

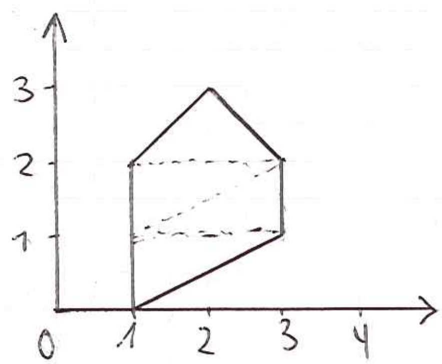
And $p(x;y) = q(x;y) = 0$ has 11 isolated solutions in $(\mathbb{C} \setminus \{0\})^2$

b) • The answer changes if we drop both adjectives: nonzero ~~and~~ isolated because there wold be infinite solutions: all elements of the form $(x;0)$, $(0;y)$ $x;y \in \mathbb{C}$ woul be solutions.

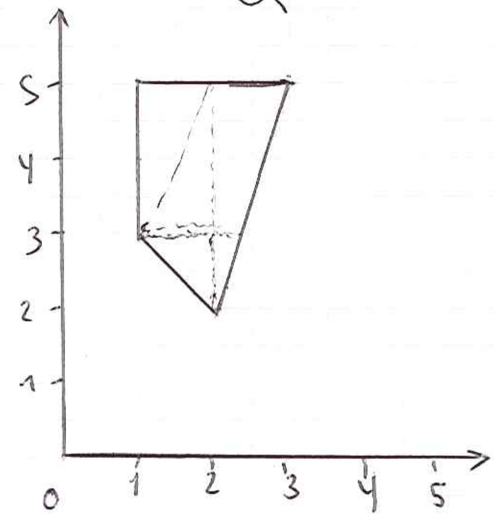
• If we drop nonzero, the answer doesn't change, since all solutions with $x=0$ or $y=0$ are not isolated

• If we drop isolated, the answer doesn't change because non-zero solutions which are not isolated would generate a continuous relation between the coeficients, thus they wouldn't be generic.

P



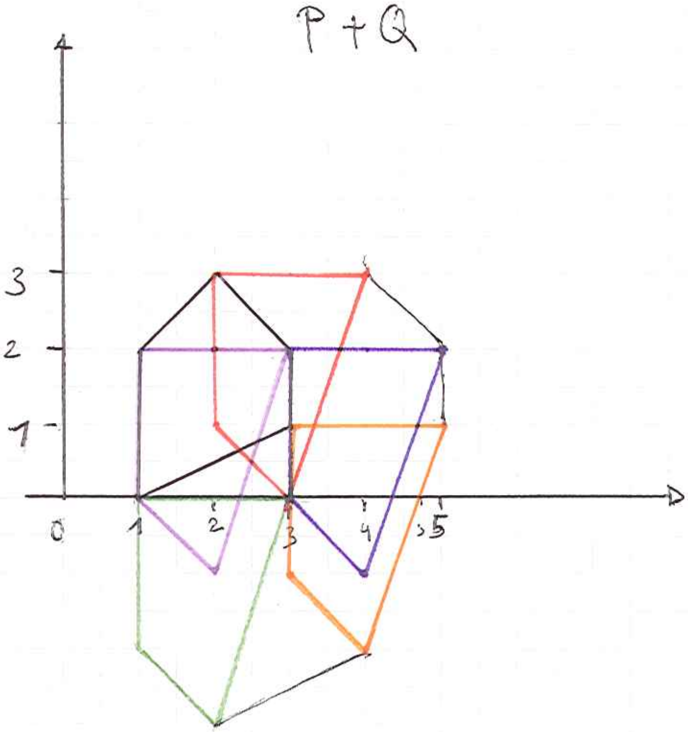
Q

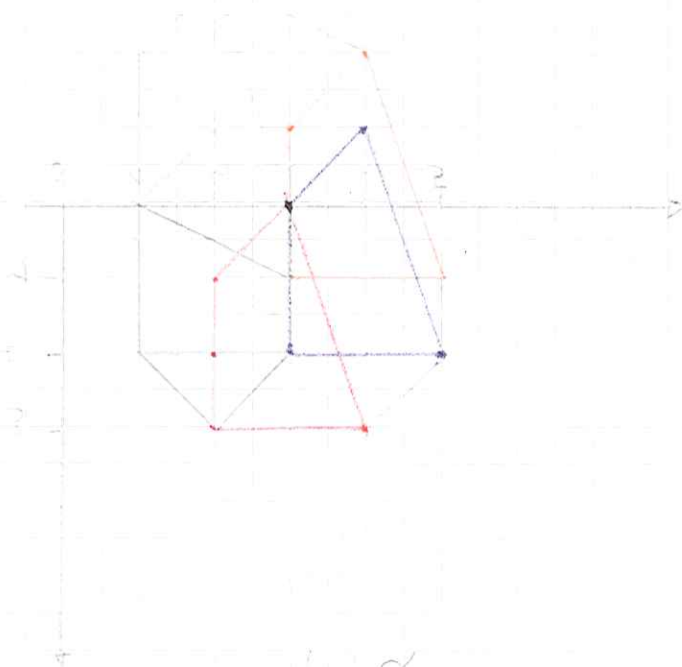


$\mathrm{Vol}(P) = 4$

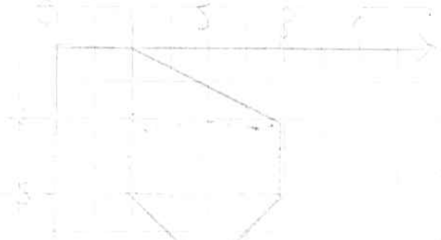
$\mathrm{Vol}(Q) = 4$

P + Q





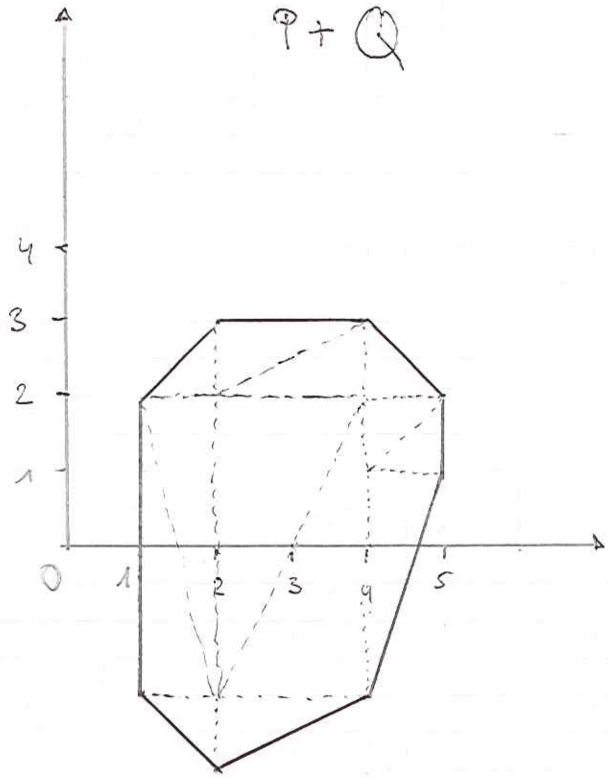
b + a



b

a

$P + Q$



$\mathrm{Vol}(P+Q) = 19$