lecture 33
4/18/07

(some of the many) <u>Applications of Tutte polynomials</u>

Recall: If f is a gen. T-G invariant:

loop, coloop

$$f(M) = a f(M\backslash e) + b f(M/e) \qquad e \neq L, C$$

$$f(M) = f(e) f(M\backslash e) \qquad e = L, C$$

then

$$f(M) = a^{\#M - r(M)} b^{r(M)} T_M\left(\frac{f(C)}{b}, \frac{f(L)}{a}\right)$$

<u>Examples:</u>

① $b(M) = \#$ bases of M

$b(M) = b(M\backslash e) + b(M/e)$
$b(C) = 1 \quad b(L) = 1$
$\longrightarrow \boxed{b(M) = T_M(1,1)}$

② $i(M) = \#$ indep sets of M

$i(M) = i(M\backslash e) + i(M/e)$
$i(C) = 2 \quad i(L) = 1$
$\longrightarrow \boxed{i(M) = T_M(2,1)}$

③ $s(M) = \#$ spanning sets of M

$s(M) = s(M\backslash e) + s(M/e)$
$s(C) = 1 \quad s(L) = 2$
$\longrightarrow \boxed{s(M) = T_M(1,2)}$

④ $2^{\#\text{elts of } M}$

$2^{\#M} = 2^{\#M\backslash e} + 2^{\#M/e}$
$2^{\#C} = 2 \quad 2^{\#L} = 2$
$\longrightarrow 2^{\#M} = T_M(2,2)$

$$\boxed{\#M = \log_2 T_M(2,2)}$$

⑤ $\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M/e}(q)$

$\chi_C(q) = q-1$ C (-1), ∅ (1)

$\chi_L(q) = 0$ L (0)

$\longrightarrow \boxed{\chi_M(q) = (-1)^{r(M)} T_M(1-q, 0)}$

Also,

$$T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}$$

$\boxed{r(M) = \text{highest power of } x \text{ in } T_M(x,y)}$

All these invariants can be easily read off from the Tutte polynomial!

◦ <u>Network reliability</u>

A connected graph G represents a network. Suppose each edge fails with probability p. (independent of the others) What is the probability that the resulting G' is connected? Call it $R_G(p)$ – "reliability polynomial"

Let e be an edge.

◦ If it fails: (prob. p)

G' is connected $\Longleftrightarrow (G\setminus e)'$ connected (prob. $R_{G\setminus e}(p)$)

◦ If it works: (prob 1-p)

G' is connected $\Longleftrightarrow (G/e)'$ connected (prob. $R_{G/e}(p)$)

So

$$R_G(p) = p\, R_{G\setminus e}(p) + (1-p)\, R_{G/e}(p) \qquad e \neq L, C$$

Similarly $R_G(p) = R_e(p)\, R_{G\setminus e}(p) \qquad e = L, C$

So we just need

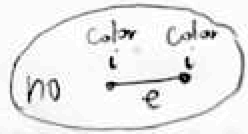
$$R_C(p) = 1-p \qquad R_L(p) = 1$$

$$\Rightarrow \boxed{R_G(p) = p^{E-V+1}(1-p)^{V-1}\, T_G(1, 1/p)}$$

∘ Graph colorings

$G = (V,E)$

$\chi_G(q) = \#$ proper q-colorings of V



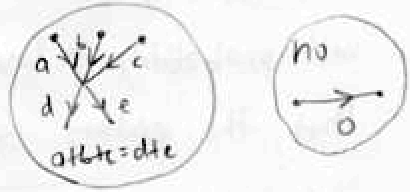
$$\boxed{\chi_G(q) = (-1)^{V-C} q^C\, T_G(1-q, 0)}$$

∘ Graph flows

$G = (V,E)$

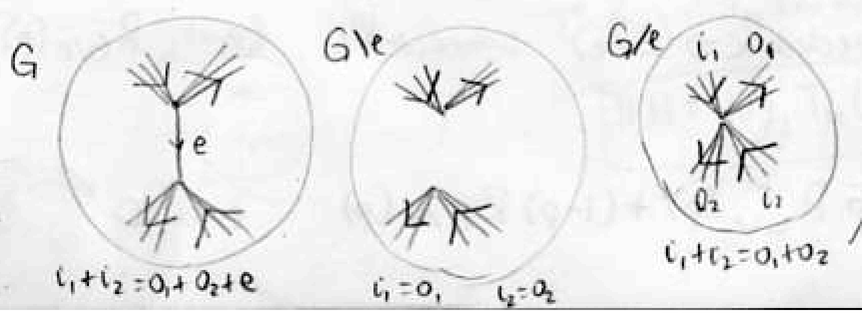
Orient edges of G arbitrarily

$F_G(q) = \#$ nowhere zero flows of G over $\mathbb{Z}_q$



$$\boxed{F_G(q) = (-1)^{E-V+C}\, T_G(0, 1-q)}$$

Proof: $F_G(q) = F_{G/e}(q) - F_{G\setminus e}(q) \qquad e \neq L, C$, etc



If $i_1 = o_1$
→ flow on $G\setminus e$

If $i_1 \neq o_1$
→ unique flow on G