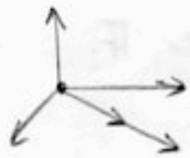
Nice facts:

- Algebraic matroids are matroids. (1937)

- Representable over $\mathbb{F}$ $\Rightarrow$ algebraic over some extension $\mathbb{K}/\mathbb{F}$

Pf:



100		x
010	$\longrightarrow$	y
001		z
$0\,\tfrac12\,\tfrac12$		$\tfrac12 y + \tfrac12 z$
011		$y+z$
000		0
in $\mathbb{F}$		in $\mathbb{F}(x,y,z)/\mathbb{F}$
(lin.)	$\longmapsto$	(alg.)

- $\left(\begin{array}{l}\text{Algebraic over } \mathbb{K}/\mathbb{F} \\ \text{Char } \mathbb{F}=0\end{array}\right) \Rightarrow$ linear over $\overline{\mathbb{F}}$, the algebraic closure of $\mathbb{F}$.

- Non-Pappus is algebraic, but not linear. (1983)
- Are all matroids algebraic? (1971)

- No. (1975, you in 2007)

- Matroids closed under deletion (obvious) and contraction (1989)

- $\boxed{\text{Matroids closed under duals? Open! Project?}}$

Many open questions!

Enumeration

So far:

① Nice families of matroids (graphical, linear, transversal, cotransversal, algebraic, ...)

② Nice characterizations of matroids (different axiom systems, geom lattices, greedy, ...)

③ Nice constructions (deletion, contraction, duals, direct sum)

Next few weeks:

④ ∘ Counting in matroids

∘ Using matroids to count things.

First: Counting in posets

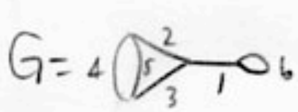
Let P be a (finite) poset. Define $\mu: P \to \mathbb{Z}$:

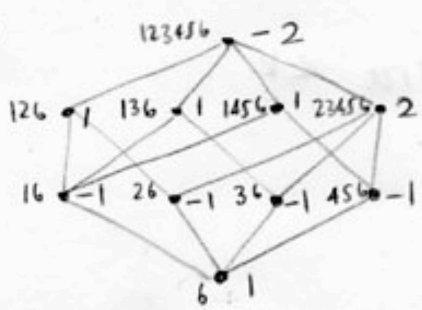
$$\mu(x) = \begin{cases} 1 & \text{if } x \text{ is minimal} \\ -\sum_{y<x} \mu(y) & \text{otherwise.} \end{cases}$$

If P is graded,

$$\chi_P(q) = \sum_{x \in P} \mu(x)\, q^{r(P)-r(x)}$$

is its characteristic polynomial

Ex. $G =$ 

$L_{M(G)}$: 

$\to -2$

$\to 5$

$\to -4$

$\to 1$

$$\chi(q) = q^3 - 4q^2 + 5q - 2$$
$$= (q-1)^2(q-2).$$

loopless

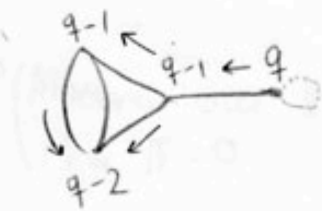
Theorem (Birkhoff 1912)

Let G be a graph with c connected components and let L be the lattice of flats of $M(G)$.
Then for $q \in \mathbb{Z}^+$,

$$q^c \chi_L(q) = \# \text{ of } \underbrace{\text{proper colorings}}_{} \text{ of } G$$

"chromatic polynomial of G"

. Colorings of the vertices with q colors such that no two neighbors have the same color.



$$q\chi_L(q) = q(q-1)^2(q-2)$$

Theorem: (Appel, Haken, 1976)
If M is a planar graphical matroid, then $\chi_{L_M}(4) \neq 0$.

A good project: Investigate the roots of characteristic polynomials.

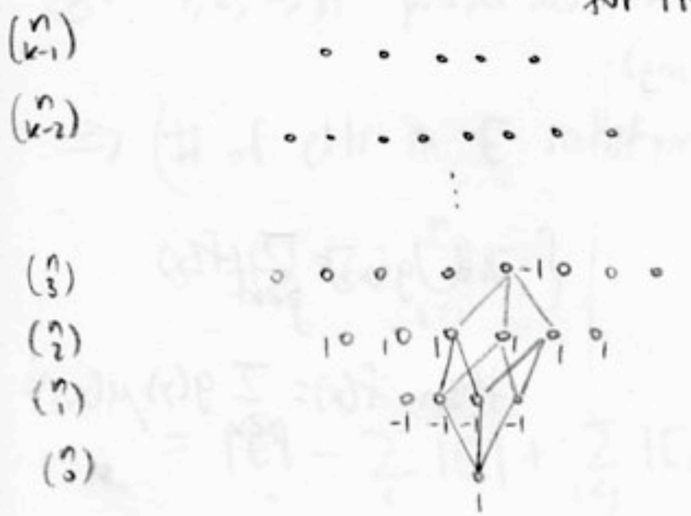
Example

$M = U_{k,n}$ $(n > k)$

Note: $r(A) = \begin{cases} |A| & \text{for } |A| \leq k \\ k & \text{for } |A| > k \end{cases}$

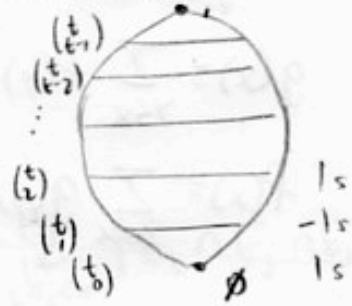
So the flats are: ∘ all sets of size $< k$
∘ $[n]$.

$L(M)$ for $M = U_{k,n}$.



Claim. $\mu(A) = (-1)^{|A|}$

Pf. Induct.



$\mu(A) = a$ such that

$$\binom{t}{0} - \binom{t}{1} + \binom{t}{2} - \dots + (-1)^{t-1}\binom{t}{t-1} + a = 0$$

$$(1-1)^t - (-1)^t \binom{t}{t} + a = 0$$

$$a = (-1)^t$$

$$\boxed{\chi_L(q) = \sum_{t=0}^{k-1} \binom{n}{t}(-1)^t q^{n-k-t} \underbrace{- \sum_{t=0}^{k-1}\binom{n}{t}(-1)^t}_{\mu(\hat{1})}}$$

<u>Def</u>. The two-variable Möbius function of P is

$$\mu : \{(x,y) \mid x \le y \text{ in } P\} \longrightarrow \mathbb{Z}$$

defined by

$$\mu(x,y) = \begin{cases} 1 & \text{if } y = x \\ -\sum_{x \le z < y} \mu(x,z) & \text{otherwise.} \end{cases}$$

In other words, $\boxed{\mu(x,y) = \mu_{P_{\ge x}}(y)}$

where $P_{\ge x}$ is the poset of elts $\ge x$ in P

Möbius Inversion Formula.

P poset

$f, g : P \to \mathbb{Z}$ (or any other ring)

If $\quad g(x) = \sum_{y \le x} f(y)$

then $\quad f(x) = \sum_{y \le x} g(y)\, \mu(y,x).$

If $g(x) = \sum_{y \ge x} f(y)$

then $f(x) = \sum_{y \ge x} g(y)\, \mu(x,y)$

Ex. D_n = poset of divisors of n

$g(d) = \#$ of $1 \le k \le n$ such that $d \mid k \quad = n/d$

$f(d) = \#$ of $1 \le k \le n$ such that $(n,k) = d$

$$\Rightarrow \quad g(d) = \sum_{d|m|n} f(m)$$

$$\Rightarrow \quad f(d) = \sum_{d|m|n} \mu(d,m)\, g(m) = \sum_{d|m|n} \mu(m/d)\, \frac{n}{m}$$

Möbius fn. of number theory (HW 5 #3)

$f(1) = \#$ of $1 \le k \le n$ with $(n,k) = 1$

$\quad = \varphi(n) = n\left(1 - 1/p_1\right) \cdots \left(1 - 1/p_k\right) \quad$ (p_is = primes dividing n)

Ex. 1, 2, ..., n properties that elts of a set E might have

$\Rightarrow$ (# of elts in E satisfying none of the properties)

$$= \#\left\{E - \bigcup_{i=1}^{n} E_i\right\}$$

$$= |E| - \sum_i |E_i| + \sum_{i<j} |E_i \cap E_j| - \sum_{i<j<k} |E_i \cap E_j \cap E_k| + \cdots$$

$$+ (-1)^n |E_1 \cap \cdots \cap E_n|$$

This is Möbius inversion applied to $P = 2^{[n]}$.

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lecture 28
3/28/07

Proof of Möbius Inversion: **Lemma.** $\displaystyle\sum_{\substack{y \text{ s.t.} \\ z \le y \le x}} \mu(y,x) = \begin{cases} 1 & x = z \\ 0 & x > z \end{cases}$

Pf. "Poset induction"

- True for $x = z$
- Sup true for all $a < x$ $(z \le a)$ $\Rightarrow \displaystyle\sum_{y:\, z \le y \le x} \mu(y,x) = \sum_{y:\, z \le y < x} \left(-\sum_{a:\, y \le a < x} \mu(y,a)\right) + \mu(x,x)$

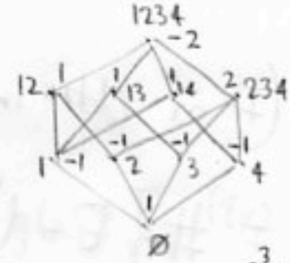
$$= \sum_{a:\, z \le a < x} \underbrace{\left(\sum_{y:\, z \le y \le a} -\mu(y,a)\right)}_{\substack{-1 \text{ for } a=z \\ 0 \text{ otherwise}}} + 1 = -1 + 1 = 0 \quad \blacksquare$$

Ok, now: $\displaystyle\sum_{y \le x} g(y)\mu(y,x) = \sum_{y \le x} \left(\sum_{z \le y} f(z)\right) \mu(y,x) = \sum_{z \le y \le x} f(z)\mu(y,x)$

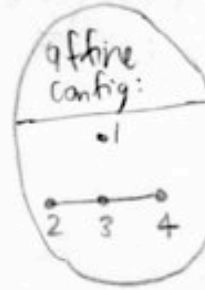
$$= \sum_{z \le x} f(z) \underbrace{\left(\sum_{z \le y \le x} \mu(y,x)\right)}_{\substack{0 \text{ if } z < x \\ 1 \text{ if } z = x}} = f(x) \quad \blacksquare$$

Recall the characteristic polynomial of M is

$$\chi_M(q) = \sum_{F \in L_M} \mu(F)\, q^{r-r(F)}$$



q^3-4q^2+5q-2



Recall:

A-hyperplane arrangement

$\longrightarrow M_A$-matroid

$\{H_1, \ldots, H_k\}$ "indep" $\iff$ normal vectors $\{v_1, \ldots, v_k\}$ lin. indep.

$\iff \operatorname{codim}(H_1 \cap \cdots \cap H_k) = k$

$\longrightarrow L_A$ = poset of intersections of A

= lattice of flats of M_A.

Theorem (Zaslavsky, 1975)

A - arrangement in $\mathbb{R}^n$

L_A - intersection poset

$r(A) = \#$ of regions in $\mathbb{R}^n - A$ $\longrightarrow$ $r(A) = |\chi_A(-1)|$

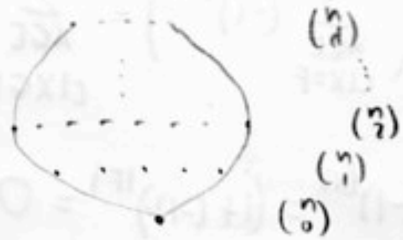
$b(A) = \#$ of bounded regions $\qquad\qquad b(A) = |\chi_A(1)|$

<u>Ex.</u> For the arrangement shown, $r(A) = |-1-4-5-2| = 12$

$b(A) = |1-4+5-2| = 0$

<u>Ex</u> For n hyperplanes in $\mathbb{R}^d$ in general position,

$L_{A_{n,d}} =$



$$\chi_{A_{n,d}}(q) = \binom{n}{0} q^d - \binom{n}{1} q^{d-1} + \binom{n}{2} q^{d-2} - \dots + (-1)^d \binom{n}{d} q^d$$

$$\rightarrow r(A_{n,d}) = \binom{n}{d} + \binom{n}{d-1} + \dots + \binom{n}{1} + \binom{n}{0}$$

$$b(A_{n,d}) = \binom{n}{d} - \binom{n}{d-1} + \dots \pm \binom{n}{1} \mp \binom{n}{0}$$

10 cuts on an apple: $\leq \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 176$ pieces

$\leq \binom{10}{3} - \binom{10}{2} + \binom{10}{1} - \binom{10}{0} = 84$ pieces without peel

To prove this, we need to know more about μ and χ for geometric lattices.

Prop. M simple matroid, L_M lattice of flats ← non-recursive description of Möbius function

$$\mu(F) = \sum_{\substack{X \subseteq E \\ \mathrm{cl}\, X = F}} (-1)^{|X|}$$

Proof:

Need to check:

- $\mu(\emptyset) = 1$ $\qquad \sum_{\substack{X \subseteq E \\ \mathrm{cl}X = \emptyset}} (-1)^{|X|} = (-1)^0 = 1$

- $\sum_{G \leq F} \mu(G) = 0$ for $F \neq 0$ flat

$$\sum_{G \leq F} \left(\sum_{\substack{X \subseteq E \\ \mathrm{cl}X = F}} (-1)^{|X|} \right) = \sum_{\substack{X \subseteq E \\ \mathrm{cl}X \subseteq F}} (-1)^{|X|}$$

F flat: $\mathrm{cl}X \subseteq F \Leftrightarrow X \subseteq F$

$$= \sum_{X \subseteq F} (-1)^{|X|} = (1 + (-1))^{|F|} = 0 \quad \blacksquare$$

Whitney's Theorem.

$$\chi_M(q) = \sum_{X \subseteq E} (-1)^{|X|} q^{r(M) - r(X)}$$

Proof.

$$\chi_M(q) = \sum_{F \text{ flat}} \mu(F) q^{r - r(F)} = \sum_{F \text{ flat}} \sum_{\substack{X \subseteq E \\ \mathrm{cl}X = F}} (-1)^{|X|} q^{r - r(X)}$$

$$= \sum_{X \subseteq E} (-1)^{|X|} q^{r - r(\mathrm{cl}\, X)} = \sum_{X \subseteq E} (-1)^{|X|} q^{r - r(X)} \quad \blacksquare$$

Prop (Deletion-contraction)

M matroid, e element $\longrightarrow \chi_M(q) = \begin{cases} \chi_{M\setminus e}(q) - \chi_{M/e}(q) & e \neq \text{loop, coloop} \\ (q-1)\,\chi_{M\setminus e}(q) & e = \text{coloop} \quad M\setminus e \cong M/e \\ 0 & e = \text{loop} \end{cases}$

$$\chi_M(q) = \underbrace{\sum_{X \subseteq E-e} (-1)^{|X|} q^{r(M) - r_M(X)}}_{①} + \underbrace{\sum_{X \subseteq E-e} (-1)^{|X \cup e|} q^{r(M) - r_M(X \cup e)}}_{②}$$

$e \neq$ coloop $\longrightarrow r(M\setminus e) = r(M)$ $\longrightarrow$ ① $= \chi_{M\setminus e}(q)$ $\qquad$ ($q\chi_{M\setminus e}(q)$ if coloop)

$r_{M\setminus e}(X) = r_M(X)$

$e \neq$ loop $\longrightarrow r(M/e) = r(M) - 1$ $\longrightarrow$ ② $= -\chi_{M/e}(q)$ $\qquad$ ($-\chi_{M/e}(q)$ if loop)

$r_{M/e}(X) = r_M(X \cup e) - 1$

loop, coloop $\rightarrow$ similar

Deletion, Contraction for arrangements

A arrangement in $\mathbb{F}^n$, $H \in A$

deletion: $A \setminus H = \{H' \in A : H' \neq H\}$ $\qquad$ arrangement in $\mathbb{F}^n$

contraction: $A/H = \{H' \cap H : H' \in A, H' \neq H\}$ $\qquad$ arrangement in $H \cong \mathbb{F}^{n-1}$

Note. If A has matroid M, $\qquad \longleftarrow$ (This is for A central. If the hyperplanes don't go through the origin, you have to do something a bit different. See Stanley's notes)

$A \setminus H$ has matroid $M \setminus H$

A/H has matroid M/H

Prop

A arrangement, $H \in A$.

$$r(A) = r(A \setminus H) + r(A/H)$$

$$b(A) = \begin{cases} b(A\setminus H) + b(A/H) & H \neq \text{coloop} \\ 0 & H = \text{coloop} \end{cases}$$

Proof By picture:

$$\#(\text{total regions}) = \#\begin{pmatrix}\text{regions I had} \\ \text{without } H\end{pmatrix} + \#\begin{pmatrix}\text{regions that } H \\ \text{cut into two}\end{pmatrix}$$

$$r(A) = r(A\setminus H) + r(A/H)$$

($b(A)$ is similar.) ▨

Zaslavsky's Theorem

A arrangement in $\mathbb{R}^n$, M = matroid of A

$r(A) = (-1)^{r(M)} \chi_M(-1)$

$b(A) = (-1)^{n-r(M)} \chi_M(1)$

Proof. Induct on $|A|$. $|A| = 0$ clear. Sup true for $|B| < |A|$. Sup. $H \neq$ coloop

$$r(A) = r(A\setminus H) + r(A/H) = (-1)^{r(M\setminus H)} \chi_{M\setminus H}(-1) + (-1)^{r(M/H)} \chi_{M/H}(-1)$$

$$= (-1)^{r(M)} \chi_{M\setminus H}(-1) + (-1)^{r(M)-1} \chi_{M/H}(-1) = (-1)^{r(M)} \chi_M(-1).$$

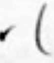
(Similar for H = coloop, or for $b(A)$. ■

A topological proof of Zaslavsky's theorem (sketch)

Δ topological space $\longrightarrow$ "Euler characteristic" $\chi(\Delta) = \text{Euler}(\Delta)$

This number carries topological information. If Δ, Δ' are "the same" (homeomorphic, or even homotopic) then $\text{Euler}(\Delta) = \text{Euler}(\Delta')$

Ex: $\text{Euler}(\ominus) = 2$

$\text{Euler}($  $) = 1$ $\longrightarrow$ different.

Def/Thm. If a "nice" space Δ is decomposed "nicely" into "nice" cells: f_d d-dim cells, f_{d-1} $(d-1)$-dim, $\ldots$, f_0 0-dim,

$$\text{Euler}(\Delta) = f_0 - f_1 + f_2 - \ldots$$

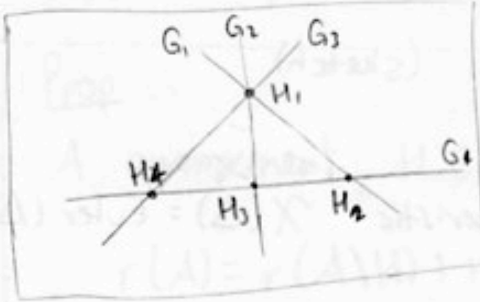
(independent of the decomposition)

Ex

$\text{Euler}(\ominus) = 6 - 12 + 8 = 2$

$\boxed{\textbf{Prop.}\ \text{Euler}(\mathbb{R}^n) = (-1)^n}$

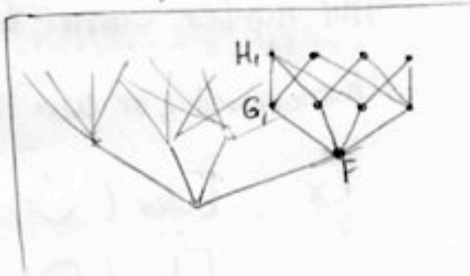
F



F flat of arrangement - already decomposed nicely.

$$\text{Euler}(F) = f_0 - f_1 + f_2$$

$$= [r(H_1) + \cdots + r(H_4)] - [r(G_1) + \cdots + r(G_4)] + r(F)$$



For each i-dim flat F

$$\text{Euler}(F) = \sum_{\substack{G \geq F \\ \text{in } L_A}} (-1)^{\dim G}\, r(G)$$

$$\downarrow$$

$$(-1)^{\dim F}\, r(F) = \sum_{\substack{G \geq F \\ \text{in } L_A}} \mu(F,G)\, \underbrace{\text{Euler}(G)}_{(-1)^{\dim G}}$$

Now put $F = \hat{0} = \mathbb{R}^n$

$$\downarrow$$

$$(-1)^n r(\mathbb{R}^n) = \sum_{G \in L_A} \mu(G)(-1)^{r - r(G)} = \chi(-1) \quad \blacksquare$$

Similar for $b(A)$, except we need to prove:

$$\text{Euler}(F_{\text{bounded}}) = b_0 - b_1 + b_2 \overset{?}{=} 1$$

$\uparrow$

union of the bounded faces.

<u>Q</u> Topology of A_{bounded}?

- contractible (Conj. by Zaslavsky 1975, Ziegler 1988)
- homeomorphic to a ball if A is simple (Dong, 2006)
- pure (Dong, 2006)

The finite field method

A very useful way to compute characteristic polynomials:

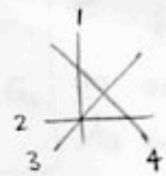
Theorem. (Crapo-Rota 1970, Athanasiadis 1996)

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$, or $\mathbb{Q}^n, \mathbb{C}^n, \ldots$ whose equations have $\mathbb{Z}$ (or $\mathbb{Q}$) coefficients. Let $\mathcal{A}_q$ be the arrangement with the same equations in $\mathbb{F}_q^n$ (for some prime power q). Then, for q large enough,

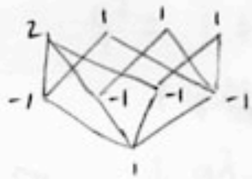
$$\chi_{\mathcal{A}}(q) = \#\text{ of points in } \mathbb{F}_q^n \text{ on no hyperplane of } \mathcal{A}_q$$

$$= \left| \mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H \right|$$

Ex: $x=0,\ y=0,\ x=y,\ x+y=1$

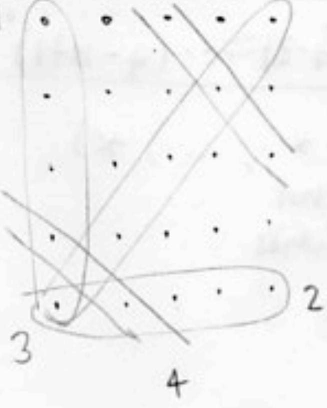


$\longrightarrow$



$\longrightarrow q^2 - 4q + 5$

In $\mathbb{F}_q^2$:



$$\rightarrow q^2 - \underbrace{q}_{\text{on } 1} - \underbrace{(q-1)}_{\substack{\text{on } 2\\ \text{not } 1}} - \underbrace{(q-1)}_{\substack{\text{on } 3\\ \text{not } 1,2}} - \underbrace{(q-3)}_{\substack{\text{on } 4\\ \text{not } 1,2,3}}$$

$$= q^2 - 4q + 5$$

Ex $\mathcal{H}_n$: $x_1=0, x_2=0, \ldots, x_n=0 \longrightarrow$ $2^{[n]}$

(Coordinate hyperplanes)

$$\chi_{\mathcal{H}_n}(q) = q^n - \binom{n}{1}q^{n-1} + \binom{n}{2}q^{n-2} - \cdots$$

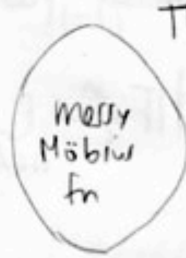
Finite field method:

$\chi_{\mathcal{H}_n}(q) =$ # of pts in $\mathbb{F}_q^n$ s.t. no $x_i = 0$

$$= \underset{\substack{(*,\\ (\neq 0)}}{(q-1)}\,\underset{\substack{*,\\ (\neq 0)}}{(q-1)} \cdots \underset{\substack{*)\\ (\neq 0)}}{(q-1)} = \boxed{(q-1)^n = \chi_{\mathcal{H}_n}(q)}$$

Ex $\mathcal{B}_n$: $x_i = x_j \quad 1 \leq i < j \leq n \longrightarrow$ Π_n

("braid arrangement")



$$\chi_{\mathcal{B}_n}(q) = \text{mess?}$$

Finite field method:

$\chi_{\mathcal{B}_n}(q) =$ # of pts in $\mathbb{F}_q^n$ s.t. no $x_i = x_j$

$$= \boxed{\underset{\substack{(*,\\ \text{any}}}{q}\;\underset{\substack{*,\\ \text{not}\\ x_1}}{(q-1)}\;\underset{\substack{*,\\ \text{not}\\ x_1,x_2}}{(q-2)}\;\underset{\substack{*,\\ \text{not}\\ x_1,x_2,x_3}}{(q-3)} \cdots \underset{*)}{(q-n+1)} = \chi_{\mathcal{B}_n}(q)}$$

Proof.

① If q is large enough, $L_A \cong L_{A_q}$

Equivalently, A and A_q have the same (semi)matroid,

i.e.,

$$\dim(\underbrace{H_a \cap \cdots \cap H_c}_{\substack{\text{sols to a system}\\ \text{of linear eqns}\\ Ax=b}}) \overset{?}{=} \dim(\underbrace{(H_a)_q \cap \cdots \cap (H_c)_q}_{\substack{\text{sols to the same}\\ \text{system (mod } q)\\ Ax=b}})$$

nonempty $\Leftrightarrow$ rank $\boxed{A|b}$ = rank $\boxed{A}$

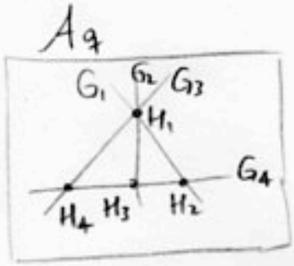
$\dim(\cap) = n - \text{rank} \boxed{A}$

Key A and A_q have the same rank for q large enough, because

rank $(A) \geq t \Leftrightarrow$ some $t \times t$ minor $\det B \neq 0$

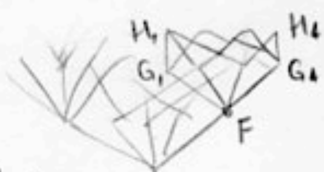
So if $q >$ these $(\det B)$s, ranks agree.

②



F flat in an arrangement A_q in $\mathbb{F}_q^3$

$$q^2 = 1+1+1+1 \qquad (H_1, \ldots, H_4)$$
$$+ (q-2)+(q-2)+(q-2)+(q-3) \qquad (G_1, \ldots, G_4)$$
$$+ q^2 - 4q + 5 \qquad (F)$$



L_{A_q}

$$\Rightarrow q^{\dim F} = \sum_{G \geq F} g(G)$$

where $g(G)$ = points on G, but not on flats smaller than G

$$\Rightarrow g(F) = \sum_{G \geq F} \mu(F,G)\, q^{\dim G}$$

$$\Rightarrow g(\hat{0}) = \sum_{G} \mu(G)\, q^{r - r(G)} = \chi_A(q) \blacksquare$$

The Tutte polynomial

Recall:

- $r(A) = r(A\backslash H) + r(A/H)$ $\quad H \neq$ loop, coloop
- $b(A) = b(A\backslash H) + b(A/H)$ $\quad$ "
- $\chi_M(q) = \chi_{M\backslash e}(q) - \chi_{M/e}(q)$ $\quad e \neq L, I$
- $|\mathcal{B}(M)| = |\mathcal{B}(M\backslash e)| + |\mathcal{B}(M/e)|$
- $|\mathcal{I}(M)| = |\mathcal{I}(M\backslash e)| + |\mathcal{I}(M/e)|$

$\vdots \qquad \vdots$

Are these instances of a general phenomenon?

Def.

A Tutte-Grothendieck invariant is a function

$f: (\text{matroids}) \longrightarrow (\text{comm. ring})$ such that

- $M \cong N \Rightarrow f(M) = f(N)$ $\quad$ "invariant"
- $f(M) = f(M\backslash e) + f(M/e)$ $\quad e \neq$ loop, coloop $\quad$ "T-G"
- $f(M) = f(M\backslash e)\, f(e)$ $\quad e =$ loop, coloop

A generalized T-G invariant is one where

$$f(M) = a\, f(M\backslash e) + b\, f(M/e) \qquad a, b \in R$$

$$e \neq L, I$$

Notice, if you define $f(I), f(L)$, that determines f completely. But does it "overdefine" it?

<u>No!</u>

(Brylawski, 1972)

<u>Theorem</u>. There is a unique function $T: (\text{matroids}) \to \mathbb{Z}[x,y]$ such that

- $M \cong N \Rightarrow T_M(x,y) = T_N(x,y)$
- $T_{\text{coloop}}(x,y) = x, \quad T_{\text{loop}}(x,y) = y$
- $T_M(x,y) = T_{M\backslash e}(x,y) + T_{M/e}(x,y) \qquad e \neq \text{loop, coloop}$
- $T_M(x,y) = \underbrace{T_e(x,y)}_{\substack{x \text{ if coloop} \\ y \text{ if loop}}} T_{M\backslash e}(x,y) \qquad e = \text{coloop, loop}$

Furthermore, any T-G invariant is an evaluation of it:

$$f(M) = T_M(f(\text{coloop}), f(\text{loop}))$$

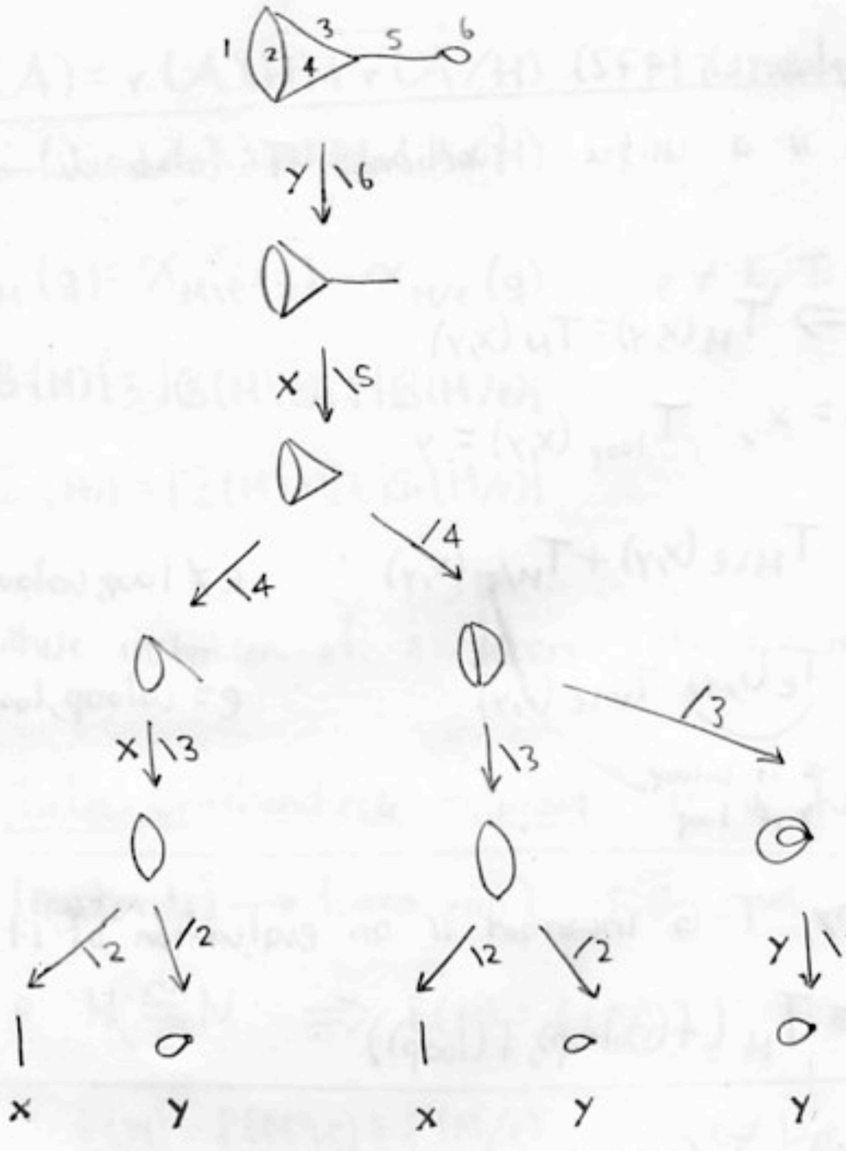
Any generalized T-G invariant is, also:

$$f(M) = a^{|E|-r(E)} b^{r(E)} T_M\left(\frac{f(\text{coloop})}{a}, \frac{f(\text{loop})}{b}\right)$$

So: The <u>Tutte polynomial</u> $T_M(x,y)$ is the mother of all (T-G) invariants. ("universal T-G invariant")

(Compare: M comm. monoid $\to$ Grothendieck gp = universal way of ($M \to$ abelian gp))

Before proving this, let's compute $T_M(x,y)$ for $M = M($$)$



$$x^3y + x^2y^2 + x^2y + xy^2 + xy^3 = T_M(x,y)$$

Why does this not depend on the order of the elements deleted and contracted?

Proof. Clearly $T_M(x,y)$ is determined by $T_{loop}(x,y)$, $T_{coloop}(x,y)$.

Why is it not overdetermined?

Key: Deletion and contraction commute, so:

$$\begin{tikzcd}[column sep=tiny]
 & & M \arrow[dl] \arrow[dr] & & \\
 & M\backslash e \arrow[dl] \arrow[d] & & M/e \arrow[d] \arrow[dr] & \\
M\backslash e\backslash f & M\backslash e/f & & M/e\backslash f & M/e/f
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=tiny]
 & & M \arrow[dl] \arrow[dr] & & \\
 & M\backslash f \arrow[dl] \arrow[d] & & M/f \arrow[d] \arrow[dr] & \\
M\backslash f\backslash e & M\backslash f/e & & M/f\backslash e & M/f/e
\end{tikzcd}$$

give the same result ∎

Theorem.

$$\boxed{T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}}$$

Proof: Need to check:

- This is an invariant ✓
- This satisfies the T-G recursion. (Exercise: see how we proved this for χ_M, and imitate that proof.) ∎

Exercise.

- Compute $T_{U_{2,4}}$ recursively () and directly from the formula.
- Compute $T_{U_{m,n}}$