Tutte polynomials and duality:

Theorem

$$T_{M^*}(x,y) = T_M(y,x)$$

Pf $T_{M^*}(x,y) = \sum_{A \subseteq E} (x-1)^{r(M^*)-r^*(A)} (y-1)^{|A|-r^*(A)}$

But $r^*(A) = |A| - r(M) + r(E-A)$

- $|A| - r^*(A) = r(M) - r(E-A)$
- $r(M^*) - r^*(A) = |E| - r(M) - r^*(A) = |E| - |A| - r(E-A)$

So
$$T_{M^*}(x,y) = \sum_{A \subseteq E} (x-1)^{|E-A|-r(E-A)} (y-1)^{r(M)-r(E-A)} = T_M(y,x) \quad \blacksquare$$

A finite field method for computing Tutte polynomials (Ardila 02) (Greene 76)

$\mathcal{A}$ - arrangement with $\mathbb{Z}$-coeffs. in $\mathbb{k}^n \longrightarrow M$-matroid

$\mathcal{A}_q$ - induced arrangement in $\mathbb{F}_q^n$

Theorem For q large enough,

$$\sum_{p \in \mathbb{F}_q^n} t^{h(p)} = q^{n-r} \overline{\chi}_M(q,t)$$

$h(p) = \#$ of hyperplanes of $\mathcal{A}_q$ containing p

"coboundary polynomial" - basically the Tutte poly:

$$\overline{\chi}_{\mathcal{A}}(q,t) = (t-1)^r \, T_{\mathcal{A}}\left(\frac{q+t-1}{t-1}, t\right)$$

$$T_{\mathcal{A}}(x,y) = \frac{1}{(y-1)^r} \overline{\chi}_{\mathcal{A}}\left((x-1)(y-1), y\right)$$

Pf Analogous to finite field method:

$$\begin{pmatrix} \# \text{ of } p \in \mathbb{F}_q^n \\ \text{on no hyps} \end{pmatrix} = q^{n-r} \chi_M(q) \qquad (t=0)$$

Ex $\mathcal{H}_n$: $x_i = 0 \quad 1 \le i \le n$

$$\sum_{p \in \mathbb{F}_q^n} t^{h(p)} = \sum_{p \in \mathbb{F}_q^n} t^{(\#\text{ of zero coords})}$$

$$(\text{coeff of } t^k) = \underbrace{\binom{n}{k}}_{\substack{\text{which } k \\ \text{coords are} \\ \text{equal to } 0}} \underbrace{(q-1)^{n-k}}_{\substack{\text{what are} \\ \text{the other} \\ \text{coords.}}}$$

So we get

$$\overline{\chi}_{\mathcal{H}_n}(q,t) = \sum_{k\ge 0}^{n} \binom{n}{k}(q-1)^{n-k} t^k = (q+t-1)^n$$

$$T_{\mathcal{H}_n}(x,y) = \frac{1}{(y-1)^n}\,\overline{\chi}_{\mathcal{H}_n}((x-1)(y-1), y) = \frac{1}{(y-1)^n}\left((x-1)(y-1)+y-1\right)^n = x^n$$

Ex $\mathcal{B}_n$: $x_i = x_j \quad 1 \le i < j \le n \qquad M = M(K_n)$

$T_{\mathcal{B}_n}$ and $\overline{\chi}_{\mathcal{B}_n}$ are messy, but the generating function is simple:

$$\sum_{n \ge 0} \overline{\chi}_{\mathcal{B}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\ge 0} t^{\binom{n}{2}} \frac{x^n}{n!}\right)^q$$

Pf: F. Ardila "Computing the Tutte poly of a hyp. arr."
(Pacific J of Math or my website) ∎