Another application: Knot Theory

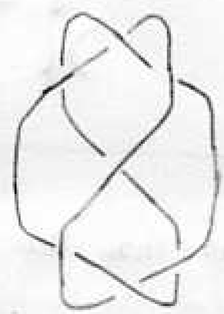
A <u>knot</u> is an embedding of a circle in $\mathbb{R}^3$.

(Take a string, wrap it around itself, and join its ends to form a loop)

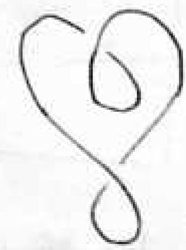
A <u>link</u> is an embedding of k circles in $\mathbb{R}^3$.

Knot diagrams:

trefoil



trefoil?
unknot?



unknot

Two knots K, L are <u>ambient isotopic</u> if K can be deformed smoothly (without cutting or crossing segments) to obtain L.

Fox (1949): There are some very complicated knots. For example, a "<u>wild</u>" knot has no piecewise linear representation.

A "<u>tame</u>" knot is one ambient isotopic to a <u>simple</u> polygon in $\mathbb{R}^3$.
(All knots we consider are tame) $\quad$ simple $\downarrow$ no crossings

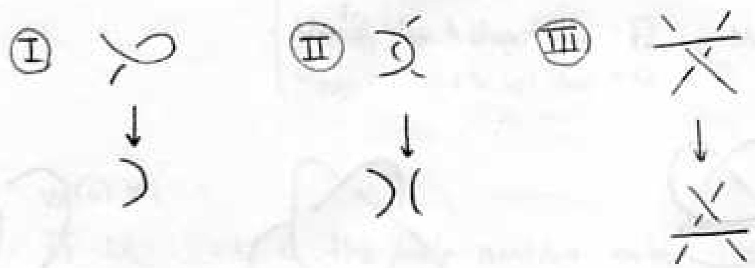
Recognition problem: Given two knot diagrams, are they equivalent?

Unknotting problem: Given a knot diagram, is it the unknot?

<u>Open:</u> Is there a polynomial time algorithm?

Theorem (Reidemeister 1935)

Two knot diagrams represent the same knot if and only if they can be obtained from each other by Reidemeister moves:



This tells us how to show that two knots are equivalent.

Q. How do we show that two knots are not equivalent?

Def A knot invariant is a "quantity" that is the same for equivalent knots.

A powerful example:

Theorem (Jones 1985)

There is a polynomial (in $\sqrt{t}, \frac{1}{\sqrt{t}}$) knot invariant V of oriented knots such that:

- $V(\text{unknot}) = 1$
- $\frac{1}{t}V($$) - t\,V($$) = (\sqrt{t} - \frac{1}{\sqrt{t}})\,V($$)$

Ex. $\frac{1}{t}V($$) - t\,V($$) = (\sqrt{t} - \frac{1}{\sqrt{t}})\,V($$)$

Prop $V(K)$ does not depend on the orientation of K. "Jones polynomial" of K

$V($$) = \dfrac{\frac{1}{t} - t}{\sqrt{t} - \frac{1}{\sqrt{t}}} = -(\sqrt{t} + \frac{1}{\sqrt{t}})$

<u>Exercise</u> Compute $V($$)$ to show that the trefoil knot is not the unknot. Is $V($$) = V($$)$?

<u>Note</u> The faces of a knot diagram can be "checkerboard-colored"

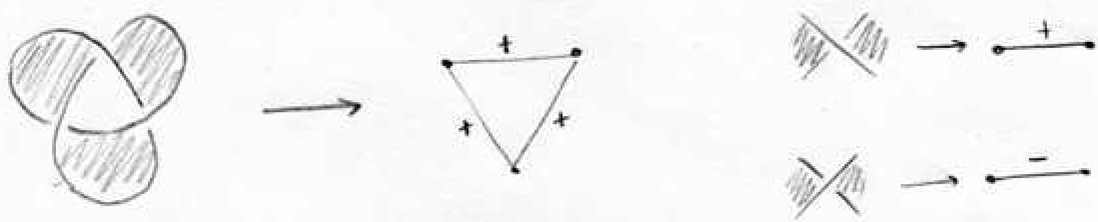


diagram D

signed graph S(D)

- This is a bijection (link diagrams) $\longleftrightarrow$ (signed plane graphs)

(alternating link diagrams) $\longleftrightarrow$ (plane graphs)

$\uparrow$

crossings: above, below, above, below, ...

All small links (≤ 7 crossings) have alternating link diagrams.

> <u>Theorem.</u>
> K alternating link $\longrightarrow$ $G=(V,E)$ positive signed graph
>
> $$V(K) = (-1)^{w(K)}\, t^{\frac{1}{2}(3w(K)-2V+E+2)}\, T_G(-t, -1/t)$$

(choose coloring to make graph pos.)

$w(K)$ = "writhe" of the diagram

$= \#($$) - \#($$)$

<u>Exercise</u>
Compute $V($$)$
and $V($$)$ again
using this.

Ex: $w($$) = 3 - 0 = 3$