math 850 . representation theory

federico ardila san francisco state university

homework three . due thursday mar 3

Note. You are encouraged to work together on the homework, but please state who you worked with **in each problem**. Write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (Signs of permutations and determinants.) Let $X : S_n \to GL_n(\mathbb{R})$ be the defining representation of S_n, prove that $\det X(\pi) = \text{sgn}(\pi)$ for all $\pi \in S_n$.
2. (What to do if your representation is not faithful.) Sagan, Problem 1.13.5.
3. (First isomorphism theorem for G-modules?) Let $\theta : V \to W$ be a homomorphism of G-modules.
   (a) Prove that Ker θ is a G-submodule of V.
   (b) Prove that Im θ is a G-submodule of W.
   (c) Is there a first isomorphism theorem for G-modules? (If your answer is "yes", state and prove all the necessary background definitions and results. If it is "no", explain why.)
4. (The character table of the symmetric group S_4.) Compute it. [1]

BONUS. (The character table of the dihedral group D_n.) Compute it.

[1] HINT. How many irreducible representations are there? How many of them do you know? Write down their characters. You are probably still missing some. Now construct enough (probably not irreducible) representations of S_4, so that you are able to use them to recover the missing irreducible ones.