Remember:

$$M_1 = (E_1, C_1), \quad M_2 = (E_2, C_2) \longrightarrow M_1 \oplus M_2 = (E_1 \cup E_2, C_1 \cup C_2)$$

no circuits involving both E_1 and E_2

So given a matroid M, how do we "factor" it as a direct sum $M = M_1 \oplus \cdots \oplus M_c$

<u>Def</u> Say $i \sim j$ if i, j are in a circuit of M

(Idea: They are then in the same M_i)

<u>Exercise</u> This is an equivalence relation:

$$\begin{matrix} i \sim j & i,j \in C_1 \\ j \sim k & j,k \in C_2 \end{matrix} \longrightarrow D \subset C_1 \cup C_2 - j$$

need a strong elimination axiom to get $i, k \in D$

If the equivalence classes are $T_1, \dots, T_k$ then

$$M = (M|T_1) \oplus \cdots \oplus (M|T_k)$$

This is the unique decomposition of M into <u>connected components</u>.

Ex: $M(\ ) = M(\ ) \oplus M(-) \oplus M(\ )$

<u>Def</u> If $P = \mathrm{conv}(p_1, \dots, p_n) \in \mathbb{R}^d$
$Q = \mathrm{conv}(q_1, \dots, q_m) \in \mathbb{R}^e$ then

$$P \times Q = \mathrm{conv}(\underline{p_i}\ \underline{q_j} \mid 1 \le i \le n, 1 \le j \le m) \in \mathbb{R}^{d+e}$$

<u>Ex</u>. $P =$ $\quad Q =$ $\quad P \times Q =$

$P_{M(\ )}$ $\quad P_{M(\ )}$ $\quad P_{M(\ )}$

<u>Prop</u> $M = M_1 \oplus M_2 \Rightarrow P_M = P_{M_1} \times P_{M_2}$

(use lecture 4.2)

easy: • $\dim P\times Q = \dim P + \dim Q$

Prop. $\dim P_M = |E|-1$ for M connected (only eqn is $\sum x_i = r(M)$)

Proof. Any $i\neq j$ are in a circuit C. Complete $C-j$ to a basis B
$\Rightarrow C$ is the basic circuit of B with respect to $i \Rightarrow B\cup i - j \in \mathcal{B}$
So any $e_i - e_j$ is an edge of P_M and their span is $(|E|-1)$-dim. ■

Theorem $\dim P_M = |E| - (\#\text{conn. compr. of } M)$

Pf $M = \bigoplus N_i \longrightarrow \dim P_M = \sum_i \dim N_i = \sum_i (|N_i|-1) = |E| - c(M)$. ■

In other words, the only equalities are $\left\{\begin{array}{l} x_2+x_3+x_4+x_5=2 \\ x_1=1 \\ x_6=0 \end{array}\right\}$ one per conn. comp.

This brings us to: Which inequalities are facets?

$$P_M = \{x\in\mathbb{R}^E \mid x_i \geq 0,\ \sum_{i\in S} x_i \leq r(S),\ \sum_{i\in E} x_i = r\}$$

Note: $x_i\geq 0$ redundant: $x_i = r - \sum_{j\in E-i} x_i \geq r - r(E-i) \geq 0$.

Def A set $F\subset E$ is a "facet" of M if $\sum_{i\in F} x_i \leq r(F)$ is a facet of P_M.

Prop M connected
F facet $\Leftrightarrow M|F, M/F$ connected

Proof: The bases maximizing weight $\underbrace{111}_{F}000$ are the bases of the matroid

$$(M|F)\oplus(M/F) \rightarrow \dim = |E| - c(M|F) - c(M/F).$$

$\uparrow$ $\dim = |F| - c(M|F)$

$\uparrow$ $\dim = |E-F| - c(M/F)$

$\uparrow$ 2 iff $M|F$, M/F connected ■

Def A cyclic flat is a flat which is a union of circuits.

Prop. F is a cyclic flat of M $\Leftrightarrow$ F flat of M, $E-F$ flat of M^*

Prop Facets are (size 1, n-1, or) cyclic flats.

Pf. M/F conn $\rightarrow$ M/F loopless $\rightarrow F$ flat

Dual: $(M|F)^* = M^*\backslash(E-F)$ conn. $\rightarrow E-F$ flat in M^* ■