

Remember:

lecture 43
5/11/07

$$M_1 = (E_1, C_1) \rightarrow M_1 \oplus M_2 = (E_1 \cup E_2, C_1 \cup C_2)$$

no circuits involving both E_1 and E_2

So given a matroid M , how do we "factor" it as a direct sum $M = M_1 \oplus \dots \oplus M_k$

Def Say $i \sim j$ if ij are in a circuit of M
(Ideas: They are then in the same M_i)

Exercise This is an equivalence relation:

$$\begin{matrix} i \sim j & i, j \in C_1 \\ j \sim k & j, k \in C_2 \end{matrix} \rightarrow D \subset C_1 \cup C_2 - j \quad \text{need a strong elimination axiom to get } i, k \in D$$

If the equivalence classes are T_1, \dots, T_k then

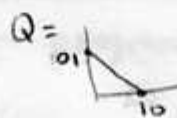
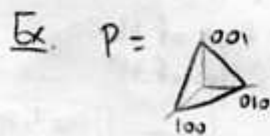
$$M = (M|T_1) \oplus \dots \oplus (M|T_k)$$

This is the unique decomposition of M into connected components.

Ex: $M(\text{---}) = M(\bigcirc) \oplus M(\text{---}) \oplus M(\bigcirc)$

Def If $P = \text{conv}(p_1, \dots, p_n) \in \mathbb{R}^d$ then
 $Q = \text{conv}(q_1, \dots, q_m) \in \mathbb{R}^e$

$$P \times Q = \text{conv}(\underline{p_i}, \underline{q_j} \mid 1 \leq i \leq n, 1 \leq j \leq m) \in \mathbb{R}^{d+e}$$



\uparrow
 $P_M(\bigcirc)$

$P_M(\bigcirc)$

$P_M(\bigcirc)$

Prop $M = M_1 \oplus M_2 \Rightarrow P_M = P_{M_1} \times P_{M_2}$

(use lecture 42)

easy: $\dim P \times Q = \dim P + \dim Q$

Prop. $\dim P_M = |E| - 1$ for M connected (only exn is $\sum X_i = r(M)$)

Proof. Any $i \neq j$ are in a circuit C . Complete $C - j$ to a basis B

$\Rightarrow C$ is the basic circuit of B with respect to $i \Rightarrow B \cup \{j\} \in B$

So any $e_i - e_j$ is an edge of P_M , and their span is $(|E| - 1) - \dim$. ■

Theorem $\dim P_M = |E| - (\# \text{ conn. comps. of } M)$

Pf $M = \bigoplus N_i \rightarrow \dim P_M = \sum_i \dim P_{N_i} = \sum_i (|N_i| - 1) = |E| - c(M)$. ■

In other words, the only equalities are $\left\{ \begin{array}{l} x_2 + x_3 + x_4 + x_5 = 2 \\ x_1 = 1 \\ x_6 = 0 \end{array} \right\}$ one per conn. comp.

This brings us to: Which inequalities are facets?

$$P_M = \{x \in \mathbb{R}^E \mid x_i \geq 0, \sum_{i \in S} x_i \leq r(S), \sum_{i \in E} x_i = r\}$$

Note: $x_i \geq 0$ redundant: $x_i = r - \sum_{j \in E-i} x_j \geq r - r(E-i) \geq 0$.

Def A set $F \subseteq E$ is a "facet" of M if $\sum_{i \in F} x_i \leq r(F)$ is a facet of P_M .

Prop M connected
 F facet $\Leftrightarrow M/F$, $M \setminus F$ connected

Proof: The bases maximizing weight $\overline{1111000}$ are the bases of the matroid F

$$(M/F) \oplus (M \setminus F) \rightarrow \dim = |E| - c(M/F) - c(M \setminus F)$$

$$\uparrow \\ \dim = |F| - c(M/F)$$

$$\uparrow \\ \dim = |E - F| - c(M \setminus F)$$

$$\uparrow \\ 2 \text{ iff } M/F, \\ M \setminus F \text{ connected.} \blacksquare$$

Def A cyclic flat is a flat which is a union of circuits.

Prop F is a cyclic flat of M
 $\Leftrightarrow F$ flat of M
 $E - F$ flat of M^*

Prop Facets are cyclic flats. Size 1, $n-1$, or

Pf. M/F conn $\rightarrow M/F$ loopless
 $\rightarrow F$ flat

Real: $(M/F)^* = M^* \setminus (E - F)$ conn.
 $\rightarrow E - F$ flat in M^* ■