Lecture 44
5/13/07

Facets are some of the cyclic flats of M.

Not much is known about them.

(Bonin-de Mier 2005) The cyclic flats of M form a lattice (ordered by inclusion), and every lattice is isomorphic to one of these.

Research question: Study the facets of M combinatorially.

Study the combinatorics of P_M.

For example, $P_{M(K_4)}$ is a self-dual polytope. (f-vector = (16, 54, 78, 54, 16))

Is there a good explanation?

K_4 is self-dual, but that's not it:

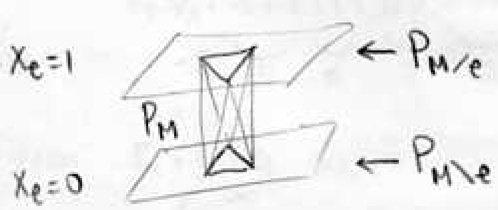
$\boxed{\underline{\text{Prop}}.\ P_{M^*} \cong P_M}$

Pf. $v_B = (101000) \longrightarrow -v_B = (-1\,0\,-1\,0\,0\,0) \longrightarrow \mathbb{1} - v_B = (010111)$

vertex of P_M — vertex of P_{M^*}

So $P_{M^*} = \mathbb{1} - P_M$. ∎

By the way, notice:



Prop
$P_{M/e}$ and $P_{M\setminus e}$ are the faces $x_i = 1$, $x_i = 0$ of P_M, respectively.

How much of M does P_M capture?

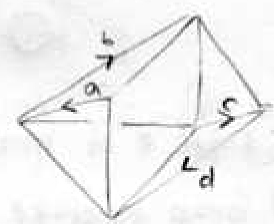
- not loops: M and $M \oplus L$ have the same polytope
- not coloops: M and $M \oplus C$ "
- no distinction between M, M^*: M and M^* "

<u>Theorem</u>

$$P_M \cong P_N \iff \begin{aligned} M &= M_1 \oplus \cdots \oplus M_a \oplus N_1 \oplus \cdots \oplus N_b \oplus \text{loops, coloops} \\ N &= M_1 \oplus \cdots \oplus M_a \oplus N_1^* \oplus \cdots \oplus N_b^* \oplus \text{loops, coloops} \end{aligned}$$

Since $P_{M_1 \oplus M_2} = P_{M_1} \times P_{M_2}$, enough to show:

For M, N connected, $P_M \cong P_N \iff N \cong M$ or M^*.



Pf. Notice:
$$\begin{aligned} (e_i - e_j,\ e_i - e_k) &= 1 && 60^\circ \\ (e_i - e_j,\ e_j - e_k) &= -1 && 120^\circ \\ (e_i - e_j,\ e_k - e_l) &= 0 && 90^\circ \end{aligned}$$

Consider P_M. Since M connected, every $e_i - e_j$ is an edge. From the edges of P_M, pick n adding up to 0, so that no sub-sum adds up to 0. They must be $e_1 - e_2, e_2 - e_3, \ldots, e_n - e_1$ with the order determined by each guy being at 120° from the previous one and 90° from the others. In turn, this determines each $e_i - e_j = (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j)$. Then we have labelled all edges of P_M, and can read off each basis from the exchanges coming out of it. Another order $a_1 a_2 \ldots a_n$ instead of $12 \ldots n$ gives an isomorphic matroid. The other cyclic order of the summands "inverts all arrows", and gives the dual matroid. ▣