Theorem (Gelfand, Goresky, MacPherson, Serganova, 1987)

Let $\mathcal{B}$ be a collection of k-sets of E.

Let $P_{\mathcal{B}} = \mathrm{conv}(V_B : B \in \mathcal{B})$

$(E,\mathcal{B})$ is a matroid $\iff$ every edge of P_M is of the form $e_i - e_j$.

$\Rightarrow$ Let V_A, V_B form an edge. Then A, B are the (only) w-max bases for some weight vector.

Let $a \in A-B$. By symmetric exchange, find $b \in B-A$ with

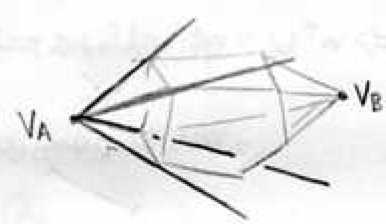
$$A-a\cup b,\ B-b\cup a \in \mathcal{B}$$

Since $w(A-a\cup b) + w(B-b\cup a) = \underbrace{w(A)}_{\text{maximum}} + \underbrace{w(B)}$, the bases

$A-a\cup b$, $B-b\cup a$ must also be maximum. $\Rightarrow A-a\cup b = B$

$$V_A - V_B = V_A - V_{A-a\cup b} = e_a - e_b. \quad \blacksquare$$

$\Leftarrow$ Let V_A, V_B be vertices of Poly_M.



$$V_B - V_A = \sum_i \alpha_i E_i$$

$\alpha_i \geq 0$, E_i: edges coming out of V_A.

Assume $V_A = 111000110000$ WLOG

$V_B = 000111110000$

$$V_B - V_A = \underbrace{-1-1-1}_{W}\ \underbrace{111}_{X}\ \underbrace{00}_{Y}\ \underbrace{0000}_{Z}$$

Suppose $E_i = e_r - e_s$ occurs. $r \notin A \to r \in X \cup Z$

$s \in A \to s \in W \cup Y$

 If $r \in Z$, $(V_B - V_A)_Z > 0 \to r \in X$ Similarly $s \in W$

Now let's prove the basis exchange axiom.

Let $a \in A - B = W$. Since $\underbrace{V_B - V_A}_{\text{"a" coord. is } -1} = \sum \alpha_i E_i$, some E_i has "a coord = -

say $E_i = e_b - e_a$

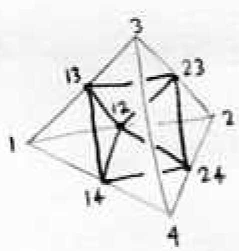
$\Rightarrow V_A + E_i = V_{A \cup b - a}$ is a vertex of $P_{\mathcal{B}}$

$\Rightarrow A - a \cup b \in \mathcal{B}$ $\blacksquare$

<u>Note</u>. A way to build $P_{\mathcal{B}}$ is to consider the standard simplex with vertices $e_1, e_2, \ldots, e_E$, and put a vertex on the baricenter of face B, which is $\frac{1}{r}\left(\sum_{b \in B} e_b\right) = \frac{1}{r} V_B$.

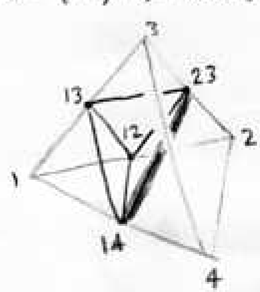
($\mathcal{B}$ is a matroid) $\Longleftrightarrow$ (edges of $P_{\mathcal{B}}$) $\parallel$ (edges of simplex)

$\mathcal{B} = \{12, 13, 14, 23, 24\}$



a matroid

$\mathcal{B} = \{12, 13, 14, 23\}$



not a matroid