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PERRON METHOD FOR THE LAPLACE EQUATION

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Alla mia cara nonna Maria

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Contents

1 Introduction

The main aim of this thesis is to discuss existence and uniqueness of classical solutions to the Dirichlet Problem for the Laplace Equation on an arbitrary set.

While uniqueness is easily demonstrated for an arbitrary bounded domain and follows from the properties of harmonic functions, i.e. the solutions to the Laplace Equation, the existence discussion will require some additional work, culminating precisely in the presentation of *Perron Method*.
The Dirichlet Problem for the Laplace Equation goes back to George Green who studied it on general domains with general boundary conditions in his Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, published in 1828. His methods were not rigorous by today's standards, but the ideas were highly influential in the subsequent developments. The English mathematician reduced the problem of the existence of the solution on a domain to the existence of a Green's Function associated to that same domain, providing simultaneously an intuitive integral formula for the solution depending only on the Green's Function itself and on the boundary condition. He also showed that such a Green's Function exists for any domain with sufficiently smooth boundary.

The problem of defining what "sufficiently smooth boundary" means in the sentence above was part of the work of the German mathematician Oskar Perron, which consists in a very elegant and powerful method for deriving the solution when it exists; moreover the Perron's Theorem gives a sufficient and necessary condition, concerning the smoothness of the boundary of the domain, for the solution to exist in the case of continuous boundary data.
One can therefore say, to make the long story short, that Perron's Theorem (which dates back to 1923) concluded almost a century of study about the classical Dirichlet Problem for the Laplace Equation; such a result turns out to be very important well beyond the treatment of the Laplace Equation since it can be easily extended to more general classes of second order elliptic equations.

In the last decades, the theory of partial differential equations has further developed and it is nowadays strongly conditioned by Sobolev spaces theory and by the concept of weak solution to a partial differential equation. Nevertheless, as it is explained in the dedicated chapter, the result of Perron is something unique and very powerful which cannot be recovered easily by variational methods, and therefore it can be seen as a complementary theorem which still maintains its theoretical and practical importance and not as something which has been replaced by the modern approach of Sobolev Spaces.

The above remarks motivate my choice of carrying out a thesis on the work of both Perron and Wiener that culminated in the above mentioned theorem. I've found this final work extremely challenging and exciting, as it was literally a "full immersion" in the topic of harmonic and subharmonic functions; all the theory behind these classes of functions turned out to be really elegant, interesting and deep, and one might say that this work is just scratching the surface by this point of view.

While the heart and core of the thesis is obviously the presentation of the Perron's work concerning the Laplace Equation during his days as an ordinary professor between the universities of Heidelberg and Munich, the elaborate also presents an important theoretical introduction to connected topics.

As it is customary in the partial differential equations framework, the initial idea is to go deep in the analytical study of the Laplace Equation itself, trying to already establish some general properties about the solutions, before even starting the discussion about existence and uniqueness. To this purpose, the elaborate starts by showing the properties of harmonic functions, a class of extremely smoothly behaving functions that satisfy the Laplace Equation and whose analytical study is necessary in order to go further.
In view of getting a nice representation formula for the solution on an arbitrary domain, we introduce the fundamental solution of the Laplace Equation in $\mathbb{R}^n$ which enables us to present also the Green's Functions, which are a particularly smart and convenient modification of the fundamental solution that generalizes the latter to the case of bounded domains. The definition of Green's Functions carries a very deep mathematical meaning, also by a functional analysis point of view, which is just briefly presented; nevertheless, their effectiveness in providing very simple representation formulas is widely discussed, together with some of the most famous examples, i.e. the half plain and the ball, studied firstly by George Green himself.
On those domains, the existence of a proper Green's Function is exploited to prove the existence of a classical solution for the Laplace Equation. After these particular (but very important and instructive) cases, wondering for which domains a Green's Function actually exists, the elaborate goes on, coherently with what happened historically, with the presentation of the Perron Method.

The idea of the Perron Method is pretty intuitive, as it is usually in mathematics, if one thinks about it in one dimension, and heavily relies on the properties of the so called subfunctions of the boundary datum. Indeed the *Perron solution* is derived as the supremum of all subfunctions corresponding to a continuous function on the boundary of the domain.
As it will be seen, the discussions on the interior of the domain and on the boundary are not necessarily related: the existence of a solution to the Laplace Equation in the interior of the domain, without considering the boundary condition, follows by constructing a harmonic function as the uniform limit of a sequence of appropriately modified and locally harmonic subfunctions. On the other hand, the main result is that, given some mild condition of regularity on the boundary of the domain, one is able to continuously extend this solution from the interior of the domain up to its closure, making it coincide with the boundary datum and hence obtaining a solution in the classical sense to the

Dirichlet Problem for the Laplace Equation.
Perron's work ended with a necessary and sufficient condition for the existence of a solution over a bounded region of space, and hence gave also a sufficient condition for the existence of a Green's Function on that same region. The surprising fact, once again, is that this condition concerns only the regularity of the boundary of the domain and is satisfied by a large class of domains.

The topological and geometrical study of the domains satisfying the Perron's Theorem condition was part of the Wiener's study in the work carried out by the two mathematicians at the beginning of the twentieth century; since it relies on the concept of capacity of a set and on the study of the Wiener coefficients, these topics overcome the scope of the elaborate, and they will be only briefly introduced for the sake of completeness; on the other hand, in order to stick to the core of the elaborate, some sufficient conditions on the boundary of the domains such that the Perron's Theorem applies will be presented in Section 7.4.

The second to last chapter is then dedicated to an immediate application of the Perron's Theorem in the study of another very important boundary value problem in dynamics of fluids and electromagnetism known as the Dirichlet Problem for the Poisson Equation; here one sees that indeed, using the linearity of all involved expressions, one can easily conclude existence and uniqueness of a classical solution with weak assumptions on the data. In this respect it can be noticed that Perron's Theorem represents a fundamental basis for showing well-posedness of other well known boundary value problems in the applications.

In the last chapter, the relation of Perron Method with the modern approach of Partial Differential Equations is tackled, concerning again the Dirichlet Problem for the Laplace Equation since the generalization follows easily. In particular, a comparison is made with Variational Methods, which are widely used since the introduction of Sobolev Spaces in the 1950s and represent nowadays a valid alternative to a classical environment with the drawback of having a less smooth and well behaving solution.
The proposed comparison concerns with the practical usefulness of the two approaches, but mostly focuses on the regularity properties of the solution obtained in the two cases. While, on one hand, concluding well-posedness using weak formulations is much faster and easier, on the other hand it is not easy to come back aiming to derive smoothness properties on the obtained solution. In this context Perron Method fits perfectly as an alternative which combines weaker assumptions on the datum together with a much more powerful result. Perron's Theorem therefore is not an ancient result that has been replaced by the theory of Sobolev Spaces, but still today represents a meaningful characterization of a classical solution to the Dirichlet Problem for the Laplace Equation with continuous boundary datum.

2 Definitions and notation

Definition 2.1. A domain Ω is an open, connected subset of $\mathbb{R}^n$.

Given a smooth function $u = u(x) : \overline{\Omega} \to \mathbb{R}$ where $x = (x_1, ..., x_n) \in \overline{\Omega}$, the following notation is used:

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$
$$\partial_\nu u(y) = \nabla u(y) \cdot \nu(y)$$
$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

Definition 2.2 (Laplace Equation). Given a function $u : \overline{\Omega} \to \mathbb{R}$ the Laplace equation is given by $-\Delta u = 0$, holding $\forall x \in \Omega$.

To determine, when it is possible, a unique solution one has to provide some conditions on the boundary $\partial\Omega$. In the mostly studied problems, this boundary conditions can be basically of the following types:

$$\text{DIRICHLET PROBLEM} \quad -\Delta u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$
$$\text{NEUMANN PROBLEM} \quad -\Delta u = 0 \quad \text{in } \Omega, \quad \partial_\nu u = h \quad \text{on } \partial\Omega$$
$$\text{ROBIN PROBLEM} \quad -\Delta u = 0 \quad \text{in } \Omega, \quad \partial_\nu u + \alpha u = h \quad \text{on } \partial\Omega$$

Recall that this elaborate treats in particular the Dirichlet Problem for the Laplace Equation.

Definition 2.3. One defines in $\mathbb{R}^n$ the following quantities:

$$\omega_n = |\partial B_1(0)| = \frac{2\sqrt{\pi}^n}{\Gamma(2^{-n})}$$
$$\alpha_n = |B_1(0)|$$

where

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$$

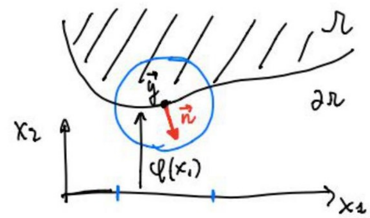
More in general, one can derive via a change of variables

$$|\partial B_r(x)| = r^{n-1}\omega_n$$
$$|B_r(x)| = r^n \alpha_n$$

with the relation

$$\alpha_n = \int_{B_1(0)} dx = \int_0^1 |\partial B_r(0)| dr = \omega_n \int_0^1 r^{n-1} dr = \frac{\omega_n}{n}$$

Definition 2.4. Let Ω a bounded domain. Then Ω is said to be a C^k domain (and it is denoted as $\partial\Omega \in C^k$) if $\forall x_0 \in \partial\Omega\ \exists R > 0$ and $\Psi : R^{n-1} \to R \in C^k(R^{n-1})$ such that, via reorientation of the axes, $\Omega \cap B_R(x_0) = \{x \in B_R(x_0) : x_n \geq \Psi(x_1, ..., x_{n-1})\}$. If Ψ is simply Lipschitz continuous, Ω is said to be Lipschitz continuous.



3 Properties of harmonic functions

Definition 3.1. A function $u \in C^2(\Omega)$ is said to be
• **harmonic** if $\Delta u(x) = 0 \ \forall x \in \Omega$.
• **subharmonic** if $\Delta u(x) \geq 0 \ \forall x \in \Omega$.
• **superharmonic** if $\Delta u(x) \leq 0 \ \forall x \in \Omega$.

In particular a harmonic function is both superharmonic and subharmonic.

It turns out that harmonic functions have many properties which allow to conclude powerful results about their regularity and stability.

An important first result and a well known characterization for harmonic functions is the following:

Theorem 3.2 (Mean value property).
Let $u \in C^2(\Omega)$ be an harmonic function on a domain Ω. Then for any $\overline{B_r(x)} \subset \Omega$, it holds that

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)dS(y) \tag{1}$$

Proof. Define the function

$$r \mapsto \phi(r) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)dS(y)$$

Via a suitable change of variables ($z = \frac{y-x}{r}$) one finds

$$\phi(r) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_1(0)} u(x+rz)r^{n-1}dS(z) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x+rz)dS(z)$$

Then since $u \in C^2(\Omega)$, it is uniformly continuous and one can pass the limit inside so that

$$\phi(0) = \lim_{r\to 0} \phi(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} \lim_{r\to 0} u(x+rz)dS(z) = u(x)$$

On the other hand, one can also exchange integration and derivation obtaining

$$\begin{aligned}
\phi'(r) = & \ \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{d}{dr} u(x+rz)dS(z) \\
= & \ \frac{1}{\omega_n} \int_{\partial B_1(0)} \nabla u(x+rz) \cdot z dS(z) \qquad \text{(z is the normal unit vector)} \\
= & \ \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(x)} \partial_\nu u(y)dS(y) \qquad \text{(change of variables } y = x+rz\text{)} \\
= & \ \frac{1}{r^{n-1}\omega_n} \int_{B_r(x)} \Delta u(y)dy = 0 \qquad \text{(since } u \text{ is harmonic in } \Omega\text{)}
\end{aligned}$$

Therefore

$$\phi(r) = \phi(0) = u(x) \quad \forall r > 0 \quad \text{such that } \overline{B_r(x)} \subset \Omega.$$

From this equation one also easily finds that

$$\int_{B_r(x)} u(y)dy = \int_0^r \left(\int_{\partial B_t(x)} u(y)dS(y)\right)dt = \int_0^r u(x)|\partial B_t(x)|dt = u(x)|B_r(x)|$$

$\square$

It turns out that the converse is also true, namely:

Theorem 3.3.
If $u \in C^2(\Omega)$ satisfies the mean value property, then u is harmonic in Ω.

Proof. Suppose by contradiction that there exists $x \in \Omega$ such that $\Delta u(x) \neq 0$, and assume $\Delta u(x) > 0$ without loss of generality.
Then by continuity there exists $r > 0 : \Delta u(y) > 0 \; \forall y \in B_r(x)$ with $\overline{B_r(x)} \subset \Omega$.
Defining now

$$r \mapsto \phi(r) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(x)} u(y)dS(y)$$

as above, one would have by the same calculations that

$$\begin{aligned} \phi(0) &= u(x) \\ \phi'(r) &= \frac{1}{r^{n-1}\omega_n} \int_{B_r(x)} \Delta u(y)dy > 0 \end{aligned}$$

a contradiction with the fact that $\phi'(r) = 0$ due to the mean value property.

$\square$

Therefore formula (1) characterizes any harmonic function in Ω.
The surprising fact is that, in Theorem 3.3, one can drop the hypothesis $u \in C^2(\Omega)$ and the theorem remains true for any $u \in C^0(\Omega)$, i.e. if a function $u \in C^0(\Omega)$ satisfies the mean value formula (1) for any closed ball contained in Ω, then it is automatically $C^2(\Omega)$ and harmonic. More than that, it is of class $C^\infty(\Omega)$.

To prove this one first needs to recall some mollification and convolution theory in $\mathbb{R}^n$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\phi(r) = \begin{cases} Ce^{-\frac{1}{r^2-1}} & \text{for } r \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

and let then $\eta : \mathbb{R}^n \to \mathbb{R}$, $\eta(x) = \phi(||x||)$ where C is chosen such that

$$\int_{\mathbb{R}^n} \eta(x)dx = 1$$

Finally, let $\eta_\epsilon : \mathbb{R}^n \to \mathbb{R}$ such that

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{||x||}{\epsilon})$$

with $\int_{B_1(0)} \eta_\epsilon(x)dx = 1$ and $\mathrm{supp}(\eta_\epsilon) \subset \overline{B\epsilon(0)}$; it is easy to check that $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$.

Definition 3.5 (Mollification of a locally integrable function).
Let $\epsilon > 0$ and define first $\Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$. Let f locally integrable on a domain Ω (integrable on every $\Omega' \subset\subset \Omega$).
The ϵ-mollification of f is a function $f_\epsilon : \Omega_\epsilon \to \mathbb{R}$ defined as:

$$f_\epsilon(x) = (\eta_\epsilon * f)(x) = \int_\Omega \eta_\epsilon(x-y)f(y)dy \tag{2}$$

The following results are holding:

Theorem 3.6. Let f_ϵ as in (2). Then

a) Let $f \in C^0(\mathbb{R}^n)$. Then f_ϵ converges to f as $\epsilon \to 0^+$ uniformly on compact subsets of $\mathbb{R}^n$.

b) If $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$ then $f_\epsilon \to f$ as $\epsilon \to 0^+$ in $L^p(\mathbb{R}^n)$.

c) Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f \in L^1_{loc}(\Omega)$, for any compact set $V \subset\subset \Omega$ one has that $||f_\epsilon - f||_{L^1(V)} \to 0$ as $\epsilon \to 0^+$.

Theorem 3.7. Any function $u \in C^0(\Omega)$ satisfying the mean value property is harmonic and is $C^\infty(\Omega)$.

Proof. Let $x \in \Omega$ and $u_\epsilon = \eta_\epsilon * u : \Omega_\epsilon \to \mathbb{R}$ the ϵ-mollification of u given by (2). Then

$$\begin{aligned}
u_\epsilon(x) = & \int_\Omega \eta_\epsilon(x-y)u(y)dy \\
= & \int_{B_\epsilon(x)} \eta_\epsilon(x-y)u(y)dy && \text{(since supp}(\eta_\epsilon(x-y)) \subset \overline{B_\epsilon(x)}) \\
= & \int_{B_\epsilon(x)} \frac{1}{\epsilon^n}\phi(\frac{||x-y||}{\epsilon})u(y)dy && \text{(by definition)} \\
= & \int_0^\epsilon (\int_{\partial B_t(x)} \frac{1}{\epsilon^n}\phi(\frac{t}{\epsilon})u(y)dS(y))dt && \text{(by radial symmetry of } \eta_\epsilon) \\
= & \int_0^\epsilon \frac{1}{\epsilon^n}\phi(\frac{t}{\epsilon})u(x)|\partial B_t(x)|u(x)dt && \text{(by the mean value property)} \\
= & \; u(x)\int_{B_\epsilon(x)} \eta_\epsilon(x-y)dy = u(x)
\end{aligned}$$

Therefore $u(x) = u_\epsilon(x)\ \forall x \in \Omega_\epsilon$ and therefore $u \in C^\infty(\Omega_\epsilon)$. But since $\epsilon > 0$ is arbitrary indeed $u \in C^\infty(\Omega)$ and since it is in particular $C^2(\Omega)$, it is harmonic by Theorem 3.3.

□

3.1 Maximum principle

A very important property for subharmonic/superharmonic (and hence also for harmonic) functions is now proved. Roughly speaking, it states that a non constant subharmonic (resp. superharmonic) function in $\Omega \subset \mathbb{R}^n$ cannot assume its maximum (resp. minimum) value inside Ω.

Theorem 3.8 (Strong maximum principle).
Let $u \in C^2(\Omega)$ a subharmonic (resp. superharmonic) function. If $\exists y \in \Omega$ such that $u(y) = \max_{x\in\overline{\Omega}} u(x)$ (resp. $u(y) = \min_{x\in\overline{\Omega}} u(x)$), then u is constant.

Proof. Let u subharmonic and M $= \max_{x\in\overline{\Omega}} u(x)$; define then $\Omega_M = \{y \in \Omega : u(y) = M\}$. Assume now that $\exists y \in \Omega$ such that u(y) = M, then $\Omega_M \neq \emptyset$ and Ω_M is also relatively closed in Ω by continuity of u. Let now $y \in \Omega_M$ and consider $r > 0 : \overline{B_r(x)} \subset \Omega$; then

$$M = u(y) \leq \frac{1}{|B_r(y)|}\int_{B_r(y)} u(z)dz \leq M$$

which can be true only if $u(x) = M\ \forall x \in B_r(y)$ and therefore $B_r(y) \subset \Omega_M$ $\forall y \in \Omega_M$. Then Ω_M is also relatively open and coincides with Ω, which is connected by definition.

□

An immediate application of the strong maximum principle to the Dirichlet Problem for the Laplace Equation is the following: if the boundary term is non negative and positive somewhere on $\partial\Omega$ the solution is positive everywhere in Ω.

Requiring Ω to be bounded a weaker version of the maximum principle can be proved: it says that a subharmonic (resp. superharmonic) function takes its maximum (resp. minimum) on the boundary of the domain.

Theorem 3.9 (Weak maximum principle).
Let Ω be a bounded domain and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ subharmonic (resp. superharmonic). Then

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} \quad (\text{resp.} \min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x))$$

Proof. Assume first that $-\Delta u < 0$ and that $\exists y \in \Omega : u(y) = M = \max_{x\in\overline{\Omega}} u(x)$. Then since $u \in C^2(\Omega)$ $\Delta u(y) \leq 0$ which is a contradiction. Suppose now in the general case that $-\Delta u \leq 0$ and consider a function $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\Delta v > 0$; one can take for example $v(x) = |x|^2 = \sum_{i=1}^n x_i^2$.

Let then $u_\epsilon = u + \epsilon\, v$, $\epsilon > 0$, then clearly $\Delta u_\epsilon > 0$ and by the discussion above $\max_{\overline{\Omega}} u_\epsilon = \max_{\partial\Omega} u_\epsilon$. Then

$$\max_{x\in\overline{\Omega}} u(x) + \epsilon \min_{x\in\overline{\Omega}} v(x) \le \max_{x\in\partial\Omega} u(x) + \epsilon \max_{x\in\partial\Omega} v(x) = \max_{x\in\partial\Omega} u_\epsilon(x)$$

The claim follows by taking the limit $\epsilon \to 0$.

□

The weaker version of the maximum principle is still valid if one relaxes the hypothesis of Ω being bounded, but with u vanishing at infinity.

Corollary 3.10 (Comparison Principle).
Let Ω a bounded domain and $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $-\Delta(u - v) \le 0$ and $u - v|_{\partial\Omega} \le 0$. Then $u - v \le 0$ in Ω.

Proof. Apply the weak maximum principle to $u - v$.

□

Some direct consequences of both versions of the maximum principle are now discussed.

Uniqueness results and stability estimates

Let Ω a bounded domain and consider the Dirichlet Problem for the Laplace Equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Suppose $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ two different solutions.
Let $w = u_1 - u_2$, then w satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and therefore by weak maximum principle

$$\max_{x\in\overline{\Omega}} w = \max_{x\in\partial\Omega} w = 0$$
$$\min_{x\in\overline{\Omega}} w = \min_{x\in\partial\Omega} w = 0$$

Hence $w(x) = 0\ \forall x \in \overline{\Omega}$ and the solution is unique.
For such a unique solution, by the weak maximum principle

$$\min_{x\in\overline{\Omega}} g(x) = \min_{x\in\partial\Omega} u(x) = \min_{x\in\overline{\Omega}} u(x) \le \max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) = \max_{x\in\partial\Omega} g(x)$$

Hence $\forall y \in \overline{\Omega}$

$$\min_{x\in\partial\Omega} g(x) \le u(y) \le \max_{x\in\partial\Omega} g(x)$$

or equivalently

$$||u||_{C^0(\overline{\Omega})} \leq ||g||_{C^0(\partial\Omega)}$$

On the same vein, if u_{g_1} and u_{g_2} are respectively solutions to the problems

$$\begin{cases} -\Delta u_{g_1} = 0 & \text{in } \Omega \\ u_{g_1} = g_1 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_{g_2} = 0 & \text{in } \Omega \\ u_{g_2} = g_2 & \text{on } \partial\Omega \end{cases}$$

with $g_1, g_2 \in C^0(\partial\Omega)$ and $||g_1 - g_2||_{C^0(\partial\Omega)} \leq \epsilon$, then

$$||u_{g_1} - u_{g_2}||_{C^0(\overline{\Omega})} \leq ||g_1 - g_2||_{C^0(\partial\Omega)} \leq \epsilon$$

i.e. small perturbations on the data imply small perturbations on the solution.

The last discussion has shown that the Dirichlet Problem for the Laplace Equation, guaranteed existence of a solution, is well-posed: the latter will also be unique and depend continuously on the data. However, as it was explained in the first chapter, one can also give other boundary conditions, the most common ones being the Neumann condition and the Robin condition.

Recall that the Neumann Problem for the Laplace Equation is

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = h & \text{on } \partial\Omega \end{cases}$$

meanwhile the Robin Problem for the Laplace Equation is defined as

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \partial_\nu u + \alpha u = h & \text{on } \partial\Omega \end{cases}$$

The well-posedness of these problems is now briefly discussed for the sake of completeness.

Consider first the Neumann Problem, and suppose for the moment that a solution exists. If u_1 and u_2 are solutions, then the function $w = u_1 - u_2$ satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ \partial_\nu w = 0 & \text{on } \partial\Omega \end{cases}$$

Then by integration by parts

$$0 = \int_\Omega -\Delta w \cdot w = \int_{\partial\Omega} \partial_\nu w \cdot w + \int_\Omega |\nabla w|^2 = \int_\Omega |\nabla w|^2$$

from which $|\nabla w|^2 = 0$: in principle one can only say that the solution is unique up to an arbitrary constant. The problem is therefore well-posed, as long as a solution exists, only in a proper quotient space where two solutions which differ by a constant are identified. Observe, on the other hand, that the boundary datum h cannot be chosen arbitrarily for the solution to exist and has to satisfy a compatibility condition

$$\int_{\partial\Omega} h = \int_{\partial\Omega} \partial_\nu u = \int_\Omega \Delta u = 0$$

Concerning instead the Robin Problem, $w = u_1 - u_2$ satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega \end{cases}$$

In this case

$$\int_\Omega |\nabla w|^2 + \int_{\partial\Omega} \alpha w^2 = 0$$

This implies $|\nabla w| = 0$ in Ω and $w = 0$ on $\partial\Omega$, hence $w = 0$ in $\overline{\Omega}$ and the solution is unique.

3.2 Analytical results

The properties presented in the previous subsection are the most important about harmonic functions in order to have a proper framework when discussing Perron Method, as the latter mostly relies on the weak maximum principle for subharmonic functions and uniqueness of the constructed solution will follow by the discussion above.
Nevertheless, harmonic functions present many other powerful properties which are now presented, starting from:

Lemma 3.11. Let v_n a sequence of harmonic functions on a domain Ω converging uniformly to v in Ω. Then v is also harmonic.

Proof. Being a uniformly-convergent limit of a sequence of continuous functions, v is continuous in Ω. Thus, in view of the mean value property (1) it suffices to show that

$$v(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} v \qquad \text{for any } \overline{B_r(x)} \subset \Omega$$

Since v_n converges uniformly to v, for any $\epsilon > 0$ one can find n sufficiently large such that

$$|\frac{1}{|B_r(x)|} \int_{B_r(x)} v_n - \frac{1}{|B_r(x)|} \int_{B_r(x)} v| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |v_n - v| \leq \epsilon$$

Thus one has $v_n(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} v_n \to \frac{1}{|B_r(x)|} v = v(x)$ and the claim follows since $\overline{B_r(x)} \subset \Omega$ is arbitrary.

$\square$

Another important and well known result about uniform convergence of a sequence of harmonic functions is the **Harnack's Theorem**; a particular case of this theorem is very useful and central in the presentation of Perron Method, for this reason both the results are presented later on in the dedicated chapter (refer to Theorems 7.9 and 7.10), meanwhile for the moment other regularity and stability properties are addressed.

Lemma 3.12. The zeros of an harmonic function cannot be isolated.

Proof. Let Ω arbitrary and $x \in \Omega$ be a zero of u. The idea is to show that any open ball centered at x contains a zero other than x. Let $r > 0$ be arbitrary, with $\overline{B_r(x)} \subset \Omega$. The mean value property implies that

$$\frac{1}{|\partial B_{r/2}(x)|}\int_{\partial B_{r/2}(x)} u = u(x) = 0$$

Then there must exist $y \in \partial B_{r/2}(x)$ such that $u(y) = 0$, since otherwise the connectedness of $\partial B_{r/2}(x)$ and the continuity of u would imply

$$\frac{1}{|\partial B_{r/2}(x)|}\int_{\partial B_{r/2}(x)} u \neq 0$$

Hence, $B_r(x)$ contains a zero of u other than x.

□

Theorem 3.13 (Harnack's inequality). Let u a non negative harmonic function in $\Omega \subset \mathbb{R}^n$. Then for any $\Omega' \subset\subset \Omega$ there exists a constant $C = C(\Omega')$ depending only on Ω', such that

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

Proof. Let $r = \frac{1}{4} dist(\Omega', \partial\Omega)$. Choose now $x, y \in \Omega'$ with $|x - y| \leq r$. Then

$$\begin{aligned} u(x) = & \frac{1}{|B_{2r}(x)|}\int_{B_{2r}(x)} udz \\ \geq & \frac{1}{\alpha_n 2^n r^n}\int_{B_r(y)} udz = \frac{1}{2^n}u(y) \qquad \text{(since } u \text{ is non-negative in } \Omega) \end{aligned}$$

and hence

$$2^n u(y) \geq u(x) \geq \frac{1}{2^n}u(y)$$

whenever $|x - y| \leq r$.
Since Ω' is connected and $\overline{\Omega'}$ is compact one can cover $\overline{\Omega'}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and with $B_i \cap B_{i-1} \neq 0$ for $i = 2, ..., N$. Then for all $x, y \in \Omega'$

$$u(x) \geq \frac{1}{2^{n(N+1)}}u(y)$$

□

These inequality asserts that the values of a non-negative harmonic function within Ω' must all be comparable: u cannot attain very large (or small) values somewhere in Ω' unless this holds everywhere in Ω'. The intuitive idea is that, since Ω' is at a positive distance from $\partial\Omega$, "there is room for the averaging effects of the Laplace's Equation to occur".

Theorem 3.14 (Liouville Theorem). Let $n \geq 2$. Let u be a harmonic and bounded function in $\mathbb{R}^n$. Then u is constant.

Proof. One needs the following preliminary result:

Lemma 3.15. Let u be harmonic in $\Omega \subset \mathbb{R}^n$. Given an arbitrary $\overline{B_r(x)} \subset \Omega$, it holds that

$$|u(x)| \leq \frac{1}{\alpha_n r^n}||u||_{L^1(B_r(x))}, \quad |\frac{\partial u}{\partial x_j}(x)| \leq \frac{2^{n+1}n}{\alpha_n r^{n+1}}||u||_{L^1(B_r(x))}, \quad j = 1, ..., n \tag{3}$$

Proof. By the mean value property

$$|u(x)| = |\frac{1}{\alpha_n r^n}\int_{B_r(x)} u| \leq \frac{1}{\alpha_n r^n}\int_{B_r(x)} |u| = \frac{1}{\alpha_n r^n}||u||_{L^1(B_r(x))} \tag{4}$$

To show the second inequality in (3) note that $\frac{\partial u}{\partial x_j}$ is harmonic.
Thus one gets that

$$\begin{aligned}
|\frac{\partial u}{\partial x_j}(x)| &= \frac{2^n}{\alpha_n r^n}|\int_{B_{r/2}(x)} \frac{\partial u}{\partial x_j}| \\
&= \frac{2^n}{\alpha_n r^n}|\int_{\partial B_{r/2}(x)} u \cdot \nu_j| && \text{(by integration by parts)} \\
&\leq \frac{2^n}{\alpha_n r^n}\int_{\partial B_{r/2}(x)} |u(y)|dS(y) \\
&\leq \frac{2^n}{\alpha_n r^n}\int_{\partial B_{r/2}(x)} \frac{2^n}{\alpha_n r^n}||u||_{L^1(B_{r/2}(x))} && \text{(applying (4) over } B_{r/2}(x)) \\
&\leq \frac{2^{2n}}{\alpha_n^2 r^{2n}}\int_{\partial B_{r/2}(x)} ||u||_{L^1(B_r(x))} && (||u||_{L^1(B_{r/2}(x))} \leq ||u||_{L^1(B_r(x))}) \\
&= \frac{2^{2n}}{\alpha_n^2 r^{2n}}||u||_{L^1(B_r(x))}\int_{\partial B_{r/2}(x)} 1
\end{aligned}$$

and the claim follows by observing that

$$\int_{\partial B_{r/2}(x)} 1 = \omega_n(\frac{r}{2})^{n-1} = n\alpha_n(\frac{r}{2})^{n-1}$$

□

Given the two results from equation (3), for any $x \in \mathbb{R}^n$ and $r > 0$

$$\begin{aligned}
|\frac{\partial u}{\partial x_j}(x)| \leq & \quad \frac{2^{n+1}n^2}{\omega_n r^{n+1}}||u||_{L^1(B_r(x))} \\
= & \quad \frac{2^{n+1}n^2}{\omega_n r^{n+1}}\int_{B_r(x)}|u| && \text{(by definition)} \\
\leq & \quad \frac{2^{n+1}n^2}{\omega_n r^{n+1}}||u||_{L^\infty(B_r(x))}\alpha_n r^n && \text{(by Holder's inequality)} \\
\leq & \quad \frac{2^{n+1}n}{r}||u||_{L^\infty(\mathbb{R}^n)} && (||u||_{L^\infty(B_r(x))} \leq ||u||_{L^\infty(R^n)})
\end{aligned}$$

Thus $\frac{\partial u}{\partial x_j} = 0 \ \forall j = 1, ..., n$, from which the statement follows.

$\square$

Indeed the Liouville Theorem holds trivially also for $n = 1$: a function $u \in C^0(\mathbb{R})$ is harmonic if and only if it is affine, and affine functions are unbounded at $\pm\infty$ unless they are constant.

4 The fundamental solution

4.1 A brief introduction to distributions

Definition 4.1. Let Ω a domain. One defines $C_0^\infty(\Omega)$ as the space of test functions in Ω endowed with the following notion of convergence: a sequence $C_0^\infty(\Omega) \ni \{\phi_k\} \to \phi \in C_0^\infty(\Omega)$ if $\forall k \in N$ $\text{supp}(\phi_k) \subset K \subset\subset \Omega$ and $\forall \alpha \in N^n$ $D^\alpha(\phi_k) \to D^a(\phi)$.

Definition 4.2. A distribution f in Ω is a linear and continuous functional $f : C_0^\infty(\Omega) \to \mathbb{R}$ with the following notion of sequential continuity:

$$< f, \phi_k > \to < f, \phi > \text{ whenever } \phi_k \to \phi \text{ as } k \to +\infty$$

It turns out that every locally integrable function f defines a distribution $f \mapsto T_f$ where T_f acts on a generic element $\phi \in C_0^\infty(\Omega)$ as follows:

$$< T_f, \phi > = \int_\Omega f(x)\phi(x)dx$$

The distribution T_f is clearly a linear functional. It is also sequentially continuous in $C_0^\infty(\Omega)$. Indeed, let $\phi_k \to \phi$ as $k \to \infty$ in $C_0^\infty(\Omega)$. In particular there must exists $K \subset\subset \Omega$ such that $\text{supp}(\phi_k) \subset K$ and $\phi_k \to \phi$ uniformly in K. It follows that

$$\begin{aligned} | < T_f, \phi_k > - < T_f, \phi > | &= | < T_f, \phi_k - \phi > | && \text{(by linearity of } T_f\text{)} \\ &= | \int_\Omega f(\phi_k - \phi)| && \text{(by definition)} \\ &\leq \int_K |f||\phi_k - \phi| && \text{(since supp}(\phi_k - \phi) \subset K\text{)} \\ &\leq ||f||_{L^1(K)}||\phi_k - \phi||_{C^0(K)} \to 0 \end{aligned}$$

With some abuse of notation, one denotes by f both the function in $L^1_{loc}(\Omega)$ and the corresponding distribution T_f.

A more interesting example of distribution is given by

Definition 4.3 (Dirac mass). Let $y \in \Omega$. The Dirac mass at y, denoted δ_y, is the distribution that satisfies $< \delta_y, \phi > = \phi(y)$ $\forall \phi \in C_0^\infty(\Omega)$.

The drawback is that the Dirac mass is not a regular distribution associated to a locally integrable function.

Proposition 4.4. There doesn't exist $f \in L^1_{loc}(\mathbb{R}^n)$ such that $T_f = \delta$, namely

$$\int_{\mathbb{R}^n} f(x)\phi(x)dx = \phi(0) \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n)$$

Proof. This is a consequence of the well known

Lemma 4.5 (Fundamental Lemma of Calculus of Variations).
Let $f \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f(x)\phi(x)dx = 0 \qquad \forall \phi \in C_0^\infty(\Omega)$$

Then $f = 0$ a.e. in Ω.

Proof. Let $\phi(x) = \eta_\epsilon(x-y)$, so that by hypotheses $f_\epsilon(y) = 0$ $\forall y \in \Omega$. Then for all $K \subset\subset \Omega$:

$$||f||_{L^1(K)} \leq ||f - f_\epsilon||_{L^1(K)} + \underbrace{||f_\epsilon||_{L^1(K)}}_{=0} = ||f - f_\epsilon||_{L^1(K)} \to 0 \quad \text{as } \epsilon \to 0$$

where the claim follows by Theorem 3.6, part c).

□

The result remains true for $\Omega = \mathbb{R}^n$.
Suppose now by contradiction that there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that $T_f = \delta$. Fixed $x_0 \in \mathbb{R}^n$ one would have

$$< \delta_{x_0}, \phi >= \int_\Omega f\phi$$

Considering $\tilde{\phi}$ with $\text{supp}(\tilde{\phi}) \subset K \subset\subset \mathbb{R}^n \backslash \{x_0\}$, then

$$\tilde{\phi}(x_0) = 0 = \int_{\mathbb{R}^n} f\tilde{\phi} \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n \backslash \{x_0\})$$

Then by the Fundamental Lemma 4.5 $f = 0$ a.e. in $\mathbb{R}^n \backslash \{x_0\}$ and actually $f = 0$ a.e. in $\mathbb{R}^n$. But this is a contradiction since δ is not the trivial distribution.

□

The theory of distributions enables us to generalize in a weak sense the classical meaning of derivative:

Definition 4.6. For $f \in L^1_{loc}(\Omega)$, the distributional partial derivative f_{x_i} is defined as

$$< f_{x_i}, \phi >= -\int_\Omega f\phi_{x_i} \qquad \forall \phi \in C_0^\infty(\Omega)$$

More generally, for $\alpha \in N^n$, the α-distributional partial derivative $D^\alpha f$ is defined as

$$< D^\alpha f, \phi >= (-1)^{|\alpha|} \int_\Omega f D^\alpha \phi \qquad \forall \phi \in C_0^\infty(\Omega)$$

Therefore one can say that any locally integrable function admits distributional derivative of any order. In particular, the “distributional Laplacian” will be defined as

$$< \Delta f, \phi >= \int_\Omega f \Delta \phi \qquad \forall \phi \in C_0^\infty(\Omega)$$

4.2 The fundamental solution

The fundamental solution of the Laplace equation is a particular solution, having radial symmetry, which is of great importance in the theory of elliptic equations and serves to construct other solutions.
Consider the Laplace equation in $\Omega = \mathbb{R}^n$, the idea is to construct explicitly a radially symmetrical solution by letting $u(x) = v(|x|)$, with $v : \mathbb{R}_+ \to \mathbb{R}$. Let then $r = |x| = \sqrt{\sum_{i=1}^n x_i^2}$, via a proper change of variables one finds

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \qquad \frac{\partial^2 r}{\partial x_i^2} = \frac{1}{r} - \frac{x_i^2}{r^3}, \qquad i = 1, ..., n$$

$$\frac{\partial v}{\partial x_i} = v'(r)\frac{x_i}{r}, \qquad \frac{\partial^2 v}{\partial x_i^2} = v''(r)\frac{x_i^2}{r^2} + v'(r)(\frac{1}{r} - \frac{x_i^2}{r^3}) \qquad i = 1, ..., n$$

and the equation becomes

$$\Delta v = \sum_{i=1}^n v_{x_i x_i} = v''(r) + v'(r)\frac{n-1}{r} = 0 \qquad \forall r > 0 \tag{5}$$

The general solution to equation (5) has the form

$$v(r) = \begin{cases} C_1 \log(r) + C_2 & \text{if n = 2} \\ C_1 r^{2-n} + C_2 & \text{if n > 2} \end{cases}$$

and the fundamental solution corresponds to a particular choice of the two constants C_1 and C_2.

Definition 4.6. The function $\Gamma : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ defined as

$$\Gamma(x) = \Gamma(|x|) = \begin{cases} -\frac{1}{2\pi} \log(|x|) & \text{if n = 2} \\ \\ \frac{1}{(n-2)\omega_n} |x|^{2-n} & \text{if n > 2} \end{cases} \tag{6}$$

is called the **fundamental solution of the Laplace Equation**.

Namely, $C_2 = 0$ is chosen such that, at least in the case $n > 2$, the function $\Gamma(x) \to 0$ as $|x| \to +\infty$, while C_1 is made to fulfill the condition

$$-\int_{\partial B_r(0)} \partial_\nu \Gamma(x) dS(x) = 1, \quad \forall r > 0 \tag{7}$$

Note that if one were to apply the divergence theorem to the left hand side of (7) (which is not possible in principle since Γ is not differentiable in $\{y\}$), one would get

$$\int_{B_r(0)} -\Delta_x \Gamma(x) dx = 1, \quad \forall r > 0$$

which is absurd since -$\Delta_x \Gamma(x) = 0$. This contradiction is solved by formally saying that the fundamental solution solves in a distributional sense

$$-\Delta_x \Gamma(x) = \delta_0, \quad x \in \mathbb{R}^n$$

where δ_0 indicates the Dirac point mass in 0. The meaning of the previous affirmation is going to be cleared and formally justified once the Green's Representation Formula will be presented.

4.3 Regularity results

In order to prove the latter, it is important to check the local integrability properties of the fundamental solution derived above around its singularity.

1. $\Gamma(x-y)$ is locally integrable around y as

$$\int_{B_r(y)} \frac{1}{2\pi} \log|x-y| = \int_0^r \Big(\int_{\partial B_t(x)} \frac{1}{2\pi} \log(t) dS(y)\Big)dt$$
$$= \int_0^r t \log t dt = \frac{r^2}{4}(2\log(r) - 1) < +\infty \quad \text{if n} = 2$$

$$\int_{B_r(y)} \frac{1}{(n-2)\omega_n} \frac{1}{|x-y|^{n-2}} dx = \int_0^r \Big(\int_{\partial B_s(y)} \frac{1}{(n-2)\omega_n} \frac{1}{s^{n-2}} dS(x-y)\Big)dt$$
$$= \int_0^r \frac{s}{n-2} ds = \frac{r^2}{2(n-2)} < +\infty \quad \text{if n} > 2$$

2. $\partial_{x_i}\Gamma(x-y)$ is locally integrable around y as

$$\partial_{x_i}\Gamma(x-y) = -\frac{1}{\omega_n} \frac{x_i - yi}{|x-y|^n}$$
$$\int_{B_r(y)} |\partial_{x_i}\Gamma(x-y)| \leq \int_{B_r(y)} \frac{1}{\omega_n r^{n-1}} = \int_0^r ds = r < +\infty$$

3. $\partial_{x_i x_j}\Gamma(x-y)$ is not locally integrable around y as

$$\partial_{x_i x_j}\Gamma(x-y) = -\frac{1}{\omega_n}\Big(\frac{|x-y|^2\delta_{ij} - n(x_i - yi)(x_j - y_j)}{|x-y|^{n+2}}\Big)$$
so that $\displaystyle\int_{B_r(y)} |\partial_{x_i x_j}\Gamma(x-y)| = +\infty$

Hence $\partial_{x_i x_j}\Gamma(x-y)$ is not locally integrable around the singularity of $\Gamma(x-y)$; however, it is "barely not integrable" and it suffices to multiply the second derivative by $|x-y|^\alpha$ for any $\alpha > 0$ to make the function locally integrable, i.e. $|\cdot - y|^\alpha \partial_{x_i x_j}\Gamma(\cdot - y) \in L^1_{loc}(\Omega)$ for any $\alpha > 0$ and any $i, j = 1, ..., n$.

5 Green's Representation Formula

Proposition 5.1 (First and Second Green's identities).
Let Ω a domain and $u, w \in C^2(\Omega)$. Then the following two identities are holding:

$$\int_\Omega u\Delta w = -\int_\Omega \nabla u \cdot \nabla w + \int_{\partial\Omega} u\partial_\nu w \tag{8}$$

$$\int_\Omega (u\Delta w - w\Delta u) = \int_{\partial\Omega} (u\partial_\nu w - w\partial_\nu u) \tag{9}$$

Proof. First one needs to prove the identity

$$div(u\nabla w) = \nabla u \cdot \nabla w + u\Delta w \tag{10}$$

Proof of equation (10). Consider the classical product rule of derivation, namely

$$\frac{\partial}{\partial x_i}(uw_{x_i}) = u_{x_i}w_{x_i} + uw_{x_ix_i}$$

Summing over all indices $i = 1, ..., n$ yields the result.

Integrating then both sides on Ω gives

$$\int_\Omega div(u\nabla w) = \int_\Omega \nabla u \cdot \nabla w + \int_\Omega u\Delta w$$

and applying the Divergence Theorem to the left hand side term, namely

$$\int_\Omega div(u\nabla w) = \int_{\partial\Omega} u\partial_\nu w$$

the claim follows and equation (8) is proved.

Consider now equation (10) for u, w as above and then switching the two functions, namely

$$div(u\nabla w) = \nabla u \nabla w + u\Delta w$$
$$div(w\nabla u) = \nabla w \nabla u + w\Delta u$$

By linearity, the difference between the two equations gives

$$div(u\nabla w - w\nabla u) = u\Delta w - w\Delta u$$

from which the claim follows: integrating on Ω

$$\int_\Omega div(u\nabla w - w\nabla u) = \int_\Omega u\Delta w - \int_\Omega w\Delta u$$

and then applying the divergence theorem to the left hand side term, that is

$$\int_\Omega div(u\nabla w - w\nabla u) = \int_{\partial\Omega} u\partial_\nu w - w\partial_\nu u$$

also equation (9) is proved.

$\square$

Moreover, given the properties of the fundamental solution referred to in the preceding chapter, one is now set to state and prove

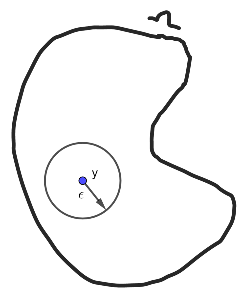
Theorem 5.2 (Green's Representation Formula).
Let Ω a bounded domain with C^1 boundary and $u \in C^2(\Omega)$. For any $y \in \Omega$ the following formula holds:

$$u(y) = -\int_{\Omega} \Gamma(x-y)\Delta u(x)dx - \int_{\partial\Omega} \partial_\nu \Gamma(x-y)u(x)dS(x) + \int_{\partial\Omega} \Gamma(x-y)\partial_\nu u(x)dS(x) \tag{11}$$

Proof. Recall the Green's second identity: for any $u, v \in C^2(\Omega)$

$$\int_\Omega \Delta u \cdot v - \int_\Omega \Delta v \cdot u = \int_{\partial\Omega} \partial_\nu u \cdot v - \int_{\partial\Omega} \partial_\nu v \cdot u$$

The idea is now to choose $v = \Gamma(x-y)$. However, the Laplacian of $x \mapsto \Gamma(x-y)$ is not defined in y, so one first writes the identity on $\Omega_\epsilon = \Omega \backslash \overline{B_\epsilon(y)}$ for a fixed $\epsilon > 0$ such that $\overline{B_\epsilon(y)} \subset \Omega$ and then takes the limit as $\epsilon \to 0$.



Then one has that

$$\begin{aligned}
&\int_{\Omega_\epsilon} \Delta u(x)\Gamma(x-y)dx - \int_{\Omega_\epsilon} \Delta_x \Gamma(x-y)u(x)dx \\
&= \int_{\partial\Omega_\epsilon} \partial_\nu u(x)\Gamma(x-y)dS(x) - \int_{\partial\Omega_\epsilon} \partial_\nu \Gamma(x-y)u(x)dS(x)
\end{aligned} \tag{12}$$

which immediately becomes

$$\int_{\Omega_\epsilon} \Delta u(x)\Gamma(x-y)dx = \int_{\partial\Omega_\epsilon} \partial_\nu u(x)\Gamma(x-y)dS(x) - \int_{\partial\Omega_\epsilon} \partial_\nu \Gamma(x-y)u(x)dS(x)$$

since $\Delta_x\Gamma(x-y)) = 0$ in Ω_ϵ. For the left hand side term one finds that

$$\begin{aligned}
&|\int_\Omega \Delta u(x)\Gamma(x-y)dx - \int_{\Omega_\epsilon} \Delta u(x)\Gamma(x-y)dx| \\
&\leq \int_{B_\epsilon(y)} |\Delta u(x)\Gamma(x-y)|dx \leq \max_{B_\epsilon(y)} |\Delta u| \int_{B_\epsilon(y)} |\Gamma(x-y)|dx \quad \to 0 \quad \text{as } \epsilon \to 0
\end{aligned}$$

Here one has used the fact that $u \in C^2(\Omega)$ and that

$$\int_{B_\epsilon(y)} |\Gamma(x-y)|dx = \frac{\epsilon^2}{4}(2\log(\epsilon)-1) \quad \text{if n = 2}$$
$$\int_{B_\epsilon(y)} |\Gamma(x-y)|dx = \frac{\epsilon^2}{2(n-2)} \qquad \text{if n} \geq 3$$

and therefore

$$\int_{\Omega_\epsilon} \Delta u(x)\Gamma(x-y)dx \to \int_\Omega \Delta u(x)\Gamma(x-y)dx \quad \text{as } \epsilon \to 0$$

On the right hand side, since $\partial\Omega_\epsilon = (\partial\Omega \cup \partial B_\epsilon(y))$, for the contributes of $\partial B_\epsilon(y)$ one finds that for the first term

$$|\int_{\partial B_\epsilon(y)} \partial_\nu u(x)\Gamma(x-y)dS(x)| \leq \int_{\partial B_\epsilon(y)} |\partial_\nu u(x)\Gamma(x-y)|dS(x)$$
$$\leq \max_{B_\epsilon(y)} |\partial_\nu u| \int_{\partial B_\epsilon(y)} \Gamma(x-y)dS(x) \to 0 \quad \text{as } \epsilon \to 0$$

and for the second term, considering the Taylor expansion of u in $y \in \Omega$ gives

$$\int_{\partial B_\epsilon(y)} \partial_\nu \Gamma(x-y)u(x)dS(x) = \int_{\partial B_\epsilon(y)} \partial_\nu \Gamma(x-y)$$
$$[u(y) + Du(y+\Theta(x-y))\underbrace{(x-y)}_{\sim\,\epsilon}]dS(x) = u(y)\int_{\partial B_\epsilon(y)} \partial_\nu\Gamma(x-y) = u(y)$$

where in the last equality equation (7) is used with the outwards normal vector pointing towards the interior of Ω, and higher order terms are vanishing in the limit as $\epsilon \to 0$.

Therefore in the end, equation (12) becomes

$$\int_\Omega \Delta u(x)\Gamma(x-y)dx = \int_{\partial\Omega} \partial_\nu u(x)\Gamma(x-y)dS(x)$$
$$- \int_{\partial\Omega} \partial_\nu\Gamma(x-y)u(x)dS(x) - u(y) \tag{13}$$

which yields the claim.

$\square$

Remark

If one considers the Green's Representation Formula (11) applied to a function $\phi \in C_0^\infty(\Omega)$, the boundary terms are vanishing and hence

$$< \delta_y, \phi >= \phi(y) = -\int_\Omega \Gamma(x-y)\Delta_x\phi(x)dx = - < \Delta\Gamma(\cdot - y), \phi >$$

which justifies saying that the fundamental solution satisfies in a distributional sense

$$-\Delta\Gamma(x-y) = \delta_y$$

Once again this is coherent with the contradiction at the end of the presentation of the fundamental solution and with the fact that $\partial_{x_i x_j}\Gamma(x-y)$ is not locally integrable around y, since the Dirac delta is not a regular distribution and hence cannot be associated to a locally integrable function on every compact set containing y.

6 Green's Functions

6.1 The idea behind constructing a Green's Function

One wants to find an explicit representation formula for the solution to the considered boundary value problems. Recall Green's Representation Formula (11) and consider Green's second identity (9) applied to u and to an harmonic function $\xi \in C^2(\Omega)$, that is

$$\int_\Omega \Delta u \cdot \xi - \int_\Omega \Delta \xi \cdot u = \int_{\partial\Omega} \partial_\nu u \cdot \xi - \int_{\partial\Omega} \partial_\nu \xi \cdot u$$

which becomes

$$0 = -\int_\Omega \Delta u \cdot \xi + \int_{\partial\Omega} \partial_\nu u \cdot \xi - \int_{\partial\Omega} \partial_\nu \xi \cdot u \tag{14}$$

Then summing equations (11) and (14) gives

$$\begin{aligned} u(y) = &-\int_\Omega \Delta u(x)[\Gamma(x-y)+\xi(x)]dx \\ -\int_{\partial\Omega} \partial_\nu[\Gamma(x-y)+\xi(x)]u(x)dS(x) &+ \int_{\partial\Omega} \partial_\nu u(x)[\Gamma(x-y)+\xi(x)]dS(x) \end{aligned} \tag{15}$$

The idea would be now to construct a particularly convenient ξ_y satisfying

$$\begin{cases} \Delta \xi_y = 0 \quad \text{in } \Omega \\ \Gamma(x-y) + \xi_y(x) = 0 \quad \text{on } \partial\Omega \end{cases}$$

to further simplify the expression above (15). Note that there is a strong dependence on Ω.

Consider for example the Dirichlet Problem for the Poisson Equation

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega \end{cases}$$

where $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$ and Ω is a bounded domain with a sufficiently smooth boundary.

Supposing for the moment that one is able to find the **corrector function** ξ_y for each y $\in \Omega$, then the expression above simplifies to

$$u(y) = \int_\Omega f(x)G(x,y)dx - \int_{\partial\Omega} \partial_\nu G(x,y)g(x)dS(x) \tag{16}$$

where $G(x,y) = \Gamma(x-y) + \xi_y(x)$ is the Green's Function of the first kind for Ω, also called simply the **Green's Function** for the domain Ω.

In the Laplace equation, $f = 0$ so equation (16) further simplifies to

$$u(y) = -\int_{\partial\Omega} \partial_\nu G(x,y) g(x) dS(x) \tag{17}$$

Note that at this point the existence of the solution to the Dirichlet Problem for the Laplace Equation for continuous boundary datum on a domain has been reduced to the existence of a Green's Function for that domain.

6.2 Properties of Green's Functions

Given a bounded domain Ω, the Green's Function $G(x,y)$ satisfies in a distributional sense

$$\begin{cases} -\Delta_x G(x,y) = \delta_y & \text{in } \Omega \\ G(x,y) = 0 & \text{on } \partial\Omega \end{cases}$$

This condition is trivial to check using the linearity of all expressions involved.

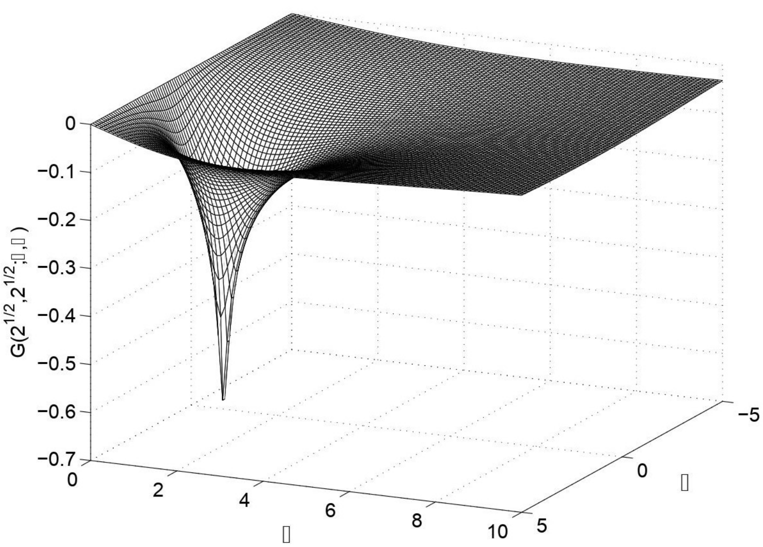


Figure 1: Plot of the Green's Function $-G(x,y)$ for the Laplacian operator in the upper half plane in $\mathbb{R}^2$ and for $y = (\sqrt{2}, \sqrt{2})$

The Green's Function of a boundary value problem for a linear differential equation on a certain domain Ω is the fundamental solution of this equation satisfying homogeneous boundary conditions. This means that if $-\Delta$ is the Laplacian operator, the Green's Function $G(x,y)$ is the solution of the equation $-\Delta G(x,y) = \delta_y$, where δ is Dirac's delta function, and the solution of the initial value problem $-\Delta u = f$ is the convolution $G*f$. By this point of view and given also the vanishing condition on $\partial\Omega$, the Green's Function can be considered as a generalization of the fundamental solution $\Gamma(\cdot - y)$ to a bounded domain in distribution theory.

It is moreover clear that, if a Green's Function exists for a domain Ω, then it is unique. Indeed suppose that $G_1(x, y)$ and $G_2(x, y)$ are both Green's Functions for a domain Ω; then $G^*(x, y) = G_1(x, y) - G_2(x, y)$ satisfies by linearity

$$\begin{cases} -\Delta G^*(x, y) = 0 & \text{in } \Omega \\ G^*(x, y) = 0 & \text{on } \partial\Omega \end{cases}$$

and hence by weak maximum principle $G^*(x, y) = 0\ \forall x \in \overline{\Omega}$, so $G_1(x, y) = G_2(x, y)\ \forall x \in \Omega$ and the Green's Function for Ω is unique.

Therefore the only thing left to discuss is whether, given a domain Ω, a Green's Function exists. This question, which is of interest because of the representation formula (17), will be given an answer in the following chapter, meanwhile for the moment other important properties of the Green's Functions are presented.

Proposition 6.1 (Symmetry of Green's Function).
Let $\Omega \subset \mathbb{R}^n$ a bounded domain with a smooth boundary and G be the Green's Function for the domain Ω. Then $\forall x, y \in \Omega\ G(x, y) = G(y, x)$.

Proof. One needs to use

Corollary 6.2. Let $x, y \in \Omega$ and $v(z) = G(z, x)$, $w(z) = G(z, y)$ for $z \in \Omega$. Then for $0 < \epsilon < \frac{|x-y|}{2}$

$$\int_{\partial B_\epsilon(x)} (v\partial_\nu w - w\partial_\nu v) = \int_{\partial B_\epsilon(y)} (w\partial_\nu v - v\partial_\nu w) \tag{18}$$

where ν denotes the *inward* unit normal vector on $B_\epsilon(x) \cup B_\epsilon(y)$.

Proof. For sufficiently small ϵ one can apply the Green's second identity to the functions v and w on $\Omega_\epsilon = \Omega \backslash (B_\epsilon(x) \cup B_\epsilon(y))$ to obtain

$$\int_{\Omega_\epsilon} \Delta w \cdot v - \int_{\Omega_\epsilon} \Delta v \cdot w = \int_{\partial\Omega_\epsilon} \partial_\nu w \cdot v - \int_{\partial\Omega_\epsilon} \partial_\nu v \cdot w$$

Then noting that $\Delta v(z) = 0$ for $z \neq x$, $\Delta w(z) = 0$ for $z \neq y$, and $v|_{\partial\Omega} = w|_{\partial\Omega} = 0$, one gets that

$$0 = \int_{\partial\Omega_\epsilon} \partial_\nu w \cdot v - \int_{\partial\Omega_\epsilon} \partial_\nu v \cdot w$$

from which equation (18) follows by construction of $\partial\Omega_\epsilon$.

□

One now shows that $v(y) = w(x)$ by computing the limits of the two terms on both sides of (18) as $\epsilon \to 0^+$. Since w is smooth near x, $|\nabla w| \le M$ on $B_\epsilon(x)$ provided ϵ is small enough, and thus one has that

$$\begin{aligned}
|\int_{\partial B_\epsilon(x)} v\frac{\partial w}{\partial \nu}| \le & \quad M\epsilon^{n-1} \sup_{\partial B_\epsilon(x)} |v| \\
\le & \quad M\epsilon^{n-1}(M' + \sup_{z\in\partial B_\epsilon(x)} |\Gamma(x-z)|) \\
\le & \quad M\epsilon^{n-1}(M' + |\ln(\epsilon) + 1|\epsilon^{2-n}) \quad \to 0 \quad \text{as} \quad \epsilon \to 0
\end{aligned}$$

where one has used the smoothness of the corrector function $|\xi_x| \le M'$ on $\partial B_\epsilon(x)$. Moreover, since $|\xi_x|$ is smooth in Ω, the same argument as in the proof of the Green's representation formula gives

$$\lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(x)} w\partial_\nu v = w(x)$$

Hence, as $\epsilon \to 0^+$, $RHS\ of$ (18) $\to -w(x)$, meanwhile $LHS\ of$ (18) $\to -v(y)$. Then $G(x,y) = G(y,x)\ \forall x,y \in \Omega$.

□

By maximum principle one can also get an estimate on the sign of $G(x,y)$:

Proposition 6.3.
Let $G(x,y)$ the Green's Function for a bounded domain Ω. Then $G(x,y) > 0$.

Proof. Fix $y \in \Omega$. Then the function $h(x) = G(x,y)$ is harmonic in $\Omega\backslash\{y\}$ and $G(x,y) \to +\infty$ as $x \to y$. Therefore there must exist $r > 0$ such that $h(x) > 0$ when $x \in \overline{B_r(y)}$.
Consider now $\Omega_{y,r} = \Omega\backslash\overline{B_r(y)}$; since $h \ge 0$ on $\partial\Omega_{y,r}$ and is non constant as it is strictly positive on $\partial B_r(y)$, by strong maximum principle $h(x) = G(x,y) > 0$ also in $\Omega_{y,r}$, and hence in all Ω.

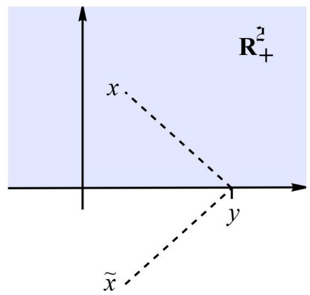
□

6.3 Explicit examples of Green's Functions

It turns out that in some well known cases which were studied by George Green himself everything works properly, i.e. there are simple domains with particular symmetries for which one is able to explicitly find the equation for the Green's Function. Recall that the main task is finding the corrector function for each internal point of the domain.

Green's Function for the half plain

The half plain is defined as $\mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) : x_n \ge 0\}$. Given a point $y = (y_1, ..., y_n) \in \mathbb{R}^n_+$, consider the reflection point with respect to to $\partial\mathbb{R}^n_+$ $y^* = (y_1, ..., -y_n) \notin \overline{\mathbb{R}^n_+}$ as in the following picture.



Then a good choice for the auxiliary function ξ_y will be given by $\xi_y(x) = -\Gamma(x - y^*)$, so that $G(x, y) = \Gamma(x - y) - \Gamma(x - y^*)$.
The Green's function constructed above is harmonic in $\mathbb{R}^n_+ \backslash \{y\}$; moreover whenever $x \in \partial\mathbb{R}^n_+$ $|x - y| = |x - y^*|$, hence it is also vanishing on $\partial\mathbb{R}^n_+$ by construction. In view of expressing the solution to the Dirichlet Problem for the Laplace Equation using equation (17), the calculation of $\partial_\nu G(x, y)$ gives

$$\partial_\nu G(x, y) = -\frac{\partial G(x, y)}{\partial x_n} = \frac{1}{\omega_n}[\frac{x_n - y_n}{|x - y|^n} - \frac{x_n - y_n^*}{|x - y^*|^n}]$$

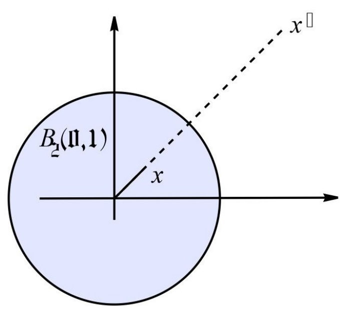
Whenever $x \in \partial\Omega$ $|x - y| = |x - y^*|$ and hence the expression above simply becomes

$$K(x, y) = \frac{2y_n}{\omega_n|x - y|^n} \tag{19}$$

where $K(x, y) = -\partial_\nu G(x, y)$ is called the **Poisson Kernel** associated to $\mathbb{R}^n_+$.

Green's Function for the ball $B_R(0)$

In this case the idea is a little bit less immediate: given $y \in B_R(0)$, instead of considering simply the reflection point of y with respect to the boundary, let $\tilde{y} = \frac{R^2}{y^2}y$ the **dual point** to y as in the following picture.



Consider then $G(x, y) = \Gamma(x - y) - \Gamma(\frac{|y|}{R}(x - \tilde{y}))$. Clearly the auxiliary function $\xi_y = -\Gamma(\frac{|y|}{R}(\cdot - \widetilde{y}))$ is harmonic up to the boundary since $\widetilde{y} \notin B_R(0)$. Moreover notice that

$$\frac{|y|^2}{R^2}|x - \widetilde{y}|^2 = \frac{|y|^2}{R^2}|x|^2 + y^2 - 2x \cdot y$$

and whenever $x \in \partial B_R(0)$

$$\frac{|y|^2}{R^2}|x - \widetilde{y}|^2 = |x - y|^2$$

so that $G(x, y) = 0$ on $\partial B_R(0)$.
Then one can construct the Poisson Kernel for $B_R(0)$

$$\frac{\partial G(x, y)}{\partial x_i} = \frac{1}{\omega_n}(-\frac{x_i - y_i}{|x - y|^n} + \frac{R^{n-2}}{|y|^{n-2}} \frac{x_i - \widetilde{y}_i}{|x - \widetilde{y}|^n})$$

If $x \in \partial B_R(0)$, since in that case $|x - y|^2 = \frac{|y|^2}{R^2}|x - \widetilde{y}|^2$, the expression above further simplifies giving

$$\partial_\nu G(x, y) = -\frac{1 - \frac{|y|^2}{R^2}}{\omega_n |x - y|^n} x_i \cdot \nu$$

In the end, being $x \cdot \nu = R$, one gets that

$$K(x, y) = \frac{R^2 - |y|^2}{\omega_n R |x - y|^n} \tag{20}$$

which is the **Poisson Kernel** associated to $B_R(0)$.

Note that, in accordance with the maximum principle, the Poisson Kernel is always strictly positive on $\partial\Omega$ for any Ω which admits a Green's Function. This follows easily from the representation formula

$$u(y) = \int_{\partial\Omega} K(x, y) g(x) dS(x)$$

If $g \geq 0$ on $\partial\Omega$ then necessarily by weak maximum principle $u \geq 0$ in $\overline{\Omega}$ which implies that $K(x, y) > 0$ for all $x \in \partial\Omega$.

7 Dirichlet Problem for the Laplace Equation

Recall from equation (17) that, given the boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

one can give a very simple and powerful representation formula of the solution given by

$$u(y) = \int_{\partial\Omega} K(x,y)g(x)dS(x) \tag{21}$$

being $K(x,y)$ the Poisson Kernel associated to the domain Ω. The idea is now to use the calculations made in the previous chapter to conclude that the constructed function is a classical solution, i.e. it is continuous up to the boundary of the domain and harmonic.

This is achieved rather easily for $\mathbb{R}^n_+$ and $B_R(0)$.

7.1 Solution on the ball $B_R(0)$

Theorem 7.1. Given any $g \in C^0(\partial B_R(0))$ the function

$$u(y) = \begin{cases} \frac{R^2-|y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(x)}{|x-y|^n} dS(x) & \text{y} \in B_R(0) \\ \\ g(y) & \text{y} \in \partial B_R(0) \end{cases}$$

is $C^2(B_R(0)) \cap C^0(\overline{B_R(0)})$ and is the unique solution of the Dirichlet Problem

$$\begin{cases} -\Delta u = 0 & \text{in } B_R(0) \\ u = g & \text{on } \partial B_R(0) \end{cases}$$

Proof. The harmonicity of u is clear as one can notice that $G(x,y) = G(y,x)$ with $G(y,x)$ harmonic. Moreover if $G(x,y)$ is harmonic then $\partial_\nu G(x,y)$ is also harmonic, so that

$$\Delta_y u(y) = \Delta_y \int_{\partial B_R(0)} K(x,y)g(x)dS(x) = \int_{\partial B_R(0)} \Delta_y K(x,y)g(x)dS(x) = 0$$

Let now $y_0 \in \partial\Omega$, one wants to show that $\lim_{B_R(0)\ni y\to y_0} u(y) = g(y_0)$, i.e. $\forall \epsilon > 0\ \exists \delta_\epsilon > 0 : |y-y_0| < \delta_\epsilon \to |u(y) - g(y_0)| < \epsilon$. First note that choosing $u = 1$ in equation (15) gives

$$1 = \int_{\partial B_R(0)} K(x,y)dS(x)$$

and therefore

$$\begin{aligned}
|u(y)-g(y_0)| = &\; |\int_{\partial B_R(0)} K(x,y)(g(x)-g(y_0)dS(x)| \\
\leq &\; \int_{\partial B_R(0)} K(x,y)|g(x)-g(y_0)|dS(x) \\
= &\; \int_{\partial B_R(0)\cap B_{\delta_\epsilon(y_0)}} K(x,y)|g(x)-g(y_0)|dS(x) \\
+ &\; \int_{\partial B_R(0)\setminus B_{\delta_\epsilon(y_0)}} K(x,y)|g(x)-g(y_0)|dS(x) = A+B
\end{aligned}$$

where one finds for the first integral that

$$\begin{aligned}
A = &\; \int_{\partial B_R(0)\cap B_{\delta_\epsilon(y_0)}} K(x,y)|g(x)-g(y_0)|dS(x) \\
\leq &\; \epsilon \int_{\partial B_R(0)\cap B_{\delta_\epsilon(y_0)}} K(x,y)dS(x) \leq \epsilon \int_{\partial B_R(0)} K(x,y)dS(x) = \epsilon
\end{aligned}$$

On $\partial B_R(0)\setminus B_{\delta_\epsilon(y_0)}$ one has that $|x-y_0| \geq \delta_\epsilon$. Letting now $|y-y_0| \leq \delta^{'} \leq \frac{\delta_\epsilon}{2}$ gives

$$|x-y_0| \leq |x-y|+|y-y_0| \underbrace{\Longrightarrow}_{x\in\partial B_R(0)\setminus B_{\delta_\epsilon(y_0)}} |x-y| \geq \frac{\delta_\epsilon}{2}$$

and the denominator of $K(x,y)$ is well defined; moreover

$$K(x,y) = \frac{(R-|y|)(R+|y|)}{\omega_n R|x-y|^n} \leq \frac{2^{n+1}\delta^{'}}{\omega_n \delta_\epsilon^n}$$

Therefore one has that

$$B = \int_{\partial B_R(0)\setminus B_{\delta_\epsilon(y_0)}} K(x,y)|g(x)-g(y_0)|dS(x) \leq 2\max_{\partial\Omega}|g|\frac{2^{n+1}\delta^{'}}{\omega_n \delta_\epsilon^n}$$

By choosing now

$$\delta^{'} \leq \min(\frac{\delta_\epsilon}{2}, \frac{\epsilon\omega_n\delta_\epsilon^n}{2^{n+1}2\max_{\partial\Omega}|g|})$$

one can conclude that $|u(y)-g(y_0)| \leq 2\epsilon$. Therefore the constructed function is continuous up to $\partial B_R(0)$ and uniqueness follows by applying the maximum principle.

□

7.2 Solution on the half plain $\mathbb{R}^n_+$

Theorem 7.2. Given any $g \in C^0(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n)$ the function

$$u(y) = \begin{cases} \int_{\partial\mathbb{R}^n_+} \frac{2y_n}{\omega_n|x-y|^n} g(x)dS(x) & \text{y} \in \mathbb{R}^n_+ \\ \\ g(y) & \text{y} \in \partial\mathbb{R}^n_+ \end{cases}$$

is $C^2(\mathbb{R}^n_+) \cap C^0(\overline{\mathbb{R}^n_+})$ and is the unique solution of the Dirichlet Problem

$$\begin{cases} -\Delta u = 0 \quad \text{in } \mathbb{R}^n_+ \\ u = g \quad \text{on } \partial\mathbb{R}^n_+ \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases}$$

Proof. The harmonicity of u is clear; in fact $\Delta_y G(x,y) = 0$ for $x \neq y$. Moreover since G is symmetric $\Delta_y G(x,y) = \Delta_y G(y,x) = 0$ for $x \neq y$. Then one has that

$$\begin{aligned} \Delta_y u(y) &= -\Delta_y \int_{\partial\mathbb{R}^n_+} \partial_\nu G(x,y) g(x) dS(x) \\ &= -\int_{\partial\mathbb{R}^n_+} \Delta_y \nabla_x G(x,y) \cdot \nu(x) g(x) dS(x) \\ &= -\int_{\partial\mathbb{R}^n_+} \nabla_x \Delta_y G(x,y) \cdot \nu(x) g(x) dS(x) = 0 \end{aligned}$$

Let now $y_0 \in \partial\mathbb{R}^n_+$, one wants to show that $\lim_{\mathbb{R}^n_+ \ni y \to y_0} u(y) = g(y_0)$, i.e. $\forall \epsilon > 0$ $\exists \delta_\epsilon > 0 : |y - y_0| < \delta_\epsilon \to |u(y) - g(y_0)| < \epsilon$. As in the previous case

$$1 = \int_{\partial\mathbb{R}^n_+} K(x,y) dS(x)$$

and therefore

$$\begin{aligned} |u(y) - g(y_0)| = & \quad |\int_{\partial\mathbb{R}^n_+} K(x,y) g(x) dS(x) - \int_{\partial\mathbb{R}^n_+} K(x,y) g(y_0)| \\ \leq & \quad \int_{\partial\mathbb{R}^n_+} K(x,y) |g(x) - g(y_0)| dS(x) \\ = & \quad \int_{\partial\mathbb{R}^n_+ \cap B_{\delta_\epsilon(y_0)}} K(x,y) |g(x) - g(y_0)| dS(x) \\ + & \quad \int_{\partial\mathbb{R}^n_+ \setminus B_{\delta_\epsilon(y_0)}} K(x,y) |g(x) - g(y_0)| dS(x) = I + J \end{aligned}$$

where one finds on the first integral that

$$I = \int_{\partial\mathbb{R}^n_+ \cap B_{\delta_\epsilon(y_0)}} K(x,y) |g(x) - g(y_0)| dS(x) \leq \epsilon \int_{\partial\mathbb{R}^n_+} K(x,y) dS(x) \leq \epsilon$$

On J, $|x - y_0| > \delta_\epsilon$. Let now $|y - y_0| \leq \frac{\delta_\epsilon}{2}$. Then

$$|x - y_0| \leq |x - y| + |y - y_0| \leq |x - y| + \frac{\delta_\epsilon}{2} \leq |x - y| + \frac{|x - y_0|}{2}$$

so that in $\partial\mathbb{R}^n_+ \backslash B_{\delta_\epsilon(y_0)}$, $|x-y| \geq \frac{|x-y_0|}{2}$. Therefore

$$\begin{aligned}
J = & \int_{\partial\mathbb{R}^n_+ \backslash B_{\delta_\epsilon(y_0)}} K(x,y)|g(x)-g(y_0)|dS(x) \\
\leq & \ 2\max_{\partial\mathbb{R}^n_+} |g| \int_{\partial\mathbb{R}^n_+ \backslash B_{\delta_\epsilon(y_0)}} \frac{2y_n}{\omega_n |x-y|^n} dS(x) \\
\leq & \ 2\max_{\partial\mathbb{R}^n_+} |g| \int_{\partial\mathbb{R}^n_+ \backslash B_{\delta_\epsilon(y_0)}} \frac{2^{n+1}y_n}{\omega_n |x-y_0|^n} \\
= & \ \frac{2^{n+2} y_n \max_{\partial\mathbb{R}^n_+} |g|}{\omega_n} \int_{\partial\mathbb{R}^n_+ \backslash B_{\delta_\epsilon(y_0)}} \frac{1}{|x-y_0|^n} dS(x)
\end{aligned}$$

and also J can be made arbitrarily small taking y sufficiently close to y_0. Therefore the constructed function is continuous up to $\partial\mathbb{R}^n_+$ and uniqueness follows once again by maximum principle.

□

7.3 Perron Method

The discussion above has shown that for two particularly convenient domains one can explicitly prove the continuity of the solution to the Laplace Equation by means of a relatively easy to construct Green's Function, concluding also the existence and uniqueness of a classical solution on those domains. For general domains, in contrast, there is no hope for an explicit representation of the Green's Function, and not even its existence is clear. Moreover, even if existence of $G(x,y)$ were at hand, in order to obtain solutions via the Green's Function representation we would also need $\partial_\nu G(x,y)$ to exist and be well defined on $\partial\Omega$ and to have similar properties to those of the Poisson Kernels constructed before. These are complicated matters which cannot be approached without a more elaborate theory. Thus, the Green's Function approach is not well-suited for the existence theory on general domains Ω, and here we indeed give up on that approach. However, by a different strategy, i.e. by the Perron Method which is presented in the current section, we will indeed establish the solvability of the Dirichlet Problem for the Laplace Equation on quite general domains. Remark that, once this is achieved, as side benefit one also obtains the existence of the Green's Function $G(x,y)$ and the availability of the representation formula (17).

Indeed Perron's Theorem gives precisely a necessary and sufficient condition on the regularity of the boundary for the existence of the solution to the Dirichlet Problem for the Laplace equation on a bounded domain Ω, together with a very elegant method to derive such a solution. The Perron Method works by finding the largest subharmonic function with boundary values below the desired values: the *Perron solution* then coincides with the actual solution of the Dirichlet Problem if the problem is solvable.

It is then convenient to further characterize subharmonic functions before entering the discussion of Perron Method.

Definition 7.3 (Subharmonic and superharmonic functions).
A function $u \in C^0(\Omega)$ is said to be subharmonic (superharmonic) in Ω if for any $\overline{B_r(x)} \subset \Omega$ it satisfies

$$u(y) \leq (\geq) \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) dS(y), u(y) \leq (\geq) \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy \quad (22)$$

Therefore in principle a subharmonic function is just a continuous function which satisfies equation (22); recall that for harmonic functions equality holds as in equation (1).
However, for subharmonic/superharmonic functions, one cannot hope to recover the same regularity as in the case of u harmonic (recall Theorem 3.4), i.e. if a function $u \in C^0(\Omega)$ satisfies (22) for any $\overline{B_r(x)} \subset \Omega$, it is not true, in general, that $u \in C^\infty(\Omega)$, neither that $-\Delta u \leq 0$ (which might not even exist).

Proposition 7.4 (Characterization of subharmonic functions).
Let $u \in C^0(\overline{\Omega})$. Then the following conditions are equivalent:

a) For every $y \in \Omega$ and any ball $\overline{B_R(y)} \subset \Omega$ one has

$$u(y) \leq \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x)$$

b) For every $y \in \Omega$ there exists $\delta = \delta(y)$ such that $\forall r : 0 < r \leq \delta$ one has

$$u(y) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} u(x) dS(x)$$

c) For every $\overline{B_R(y)} \subset \Omega$ and every harmonic function $h \in C^0(\overline{B_R(y)})$ in $B_R(y)$ satisfying $h \geq u$ on $\partial B_R(y)$, it holds that $h \geq u$ in $B_R(y)$.

Proof. $\{a \to b\}$. a) clearly implies b).
$\{b \to c\}$. Suppose that b) holds. Then the function $u - h$ is subharmonic and $u - h|_{\partial B_R(y)} \leq 0$ by assumption. Therefore by weak maximum principle $u - h \leq 0$ over all $B_R(y)$.
$\{c \to a\}$. Let h the solution to

$$\begin{cases} -\Delta h = 0 & \text{in } B_R(y) \\ h = u & \text{on } \partial B_R(y) \end{cases}$$

which exists since $u \in C^0(\partial B_R(y))$. Then the claim follows since

$$\frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(y)} h(x) dS(x) = h(y) \geq u(y)$$

□

In particular, the characterization given by c) will be largely used in the Perron Method and somehow justifies the name of this class of functions: a subharmonic function is a function which is not larger than any harmonic function on an arbitrary ball of the domain, given that it is not larger on the boundary of that ball. This is coherent with what the one dimensional intuition suggests: the harmonic functions are the affine ones and the subharmonic functions are the convex ones which stay below them if they are not larger on the boundary of any connected interval.

Clearly if one requires $u \in C^2(\Omega)$, then the characterization given by (22) becomes equivalent to the usual one with the Laplacian of u, namely $-\Delta u \leq 0$. The proof can be easily produced by following the very same footsteps of the proofs of Theorem 3.2 and 3.3, only the value of $-\Delta u$ in the last passage changes.

Given this characterization of subharmonic functions, one is now ready to enter the discussion of Perron Method.

The idea of the Perron Method can be better illustrated via the one dimensional example: consider the boundary value problem

$$\begin{cases} u'' = 0 \quad \text{in } (0,\, 1) \\ u(0) = u_0 \\ u(1) = u_1 \end{cases}$$

The solution is a straight line connecting $(0,\, u_0)$ and $(1,\, u_1)$. Now let U_0 be any function connecting these two points (or more in general such that $U_0(0) \leq u_0$ and $U_0(1) \leq u_1$), and modify U_0 as follows: take any subinterval $(a, b) \subset (0, 1)$ and replace $U_0|_{(a,b)}$ by the solution of

$$\begin{cases} u'' = 0 \quad \text{in (a,b)} \\ u(a) = U_0(a) \\ u(b) = U_0(b) \end{cases}$$

Doing this again and again, the sequence approaches the straight line, which is the needed solution.

In general it is not easy to make this argument rigorous for general U_0, but restricting to U_0 subharmonic then a crucial observation is that every modification makes the function "larger".
On the other hand all these approximations of the solution are bounded from above by $\max\{u_0, u_1\}$; therefore there must be an upper bound which coincides with the solution to the initial problem by uniqueness.

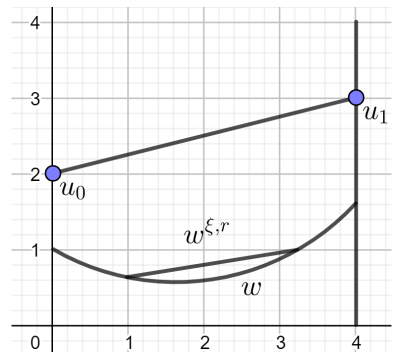


Figure 2: The real solution is approached by modifying locally the subharmonic functions

This argument can be generalized to higher dimensional problems and made rigorous, the general ideas being
• Modifying any subharmonic function by patching it with harmonic functions;
• The modified function is still subharmonic;
• Subharmonic functions are bounded by above by their boundary values

One starts with some definitions needed by the framework:

Definition 7.5. A function $w \in C^0(\overline{\Omega})$ is called
1. **subfunction** (relative to g) if it is subharmonic and $w \leq g$ on $\partial\Omega$.
2. **superfunction** (relative to g) if it is superharmonic and $w \geq g$ on $\partial\Omega$.

If u is a solution to a Dirichlet problem and $w \in C^0(\overline{\Omega})$ is a subfunction relative to u, then $w - u$ is subharmonic and $(w-u)|_{\partial\Omega} \leq 0$. By the maximum principle this implies that $w \leq u$ in $\overline{\Omega}$, that is, if u is a solution, u is not smaller than any subfunction.
On the other hand if $v \in C^0(\overline{\Omega})$ is a superfunction relative to u, then $v - u$ is superharmonic with $(v-u)|_{\partial\Omega} \geq 0$, and therefore once again by maximum principle $v \geq u$ in $\overline{\Omega}$, i.e. u is also not greater than any superfunction.

Let S_g be the set of all subfunctions, which is non empty since $w(x) = \inf_{\partial\Omega} g$, $\forall x \in \overline{\Omega}$ belongs to S_g. The idea is to construct the solution as

$$u(x) = \sup_{w \in S_g} w(x), \quad \mathrm{x} \in \overline{\Omega} \tag{23}$$

Therefore one needs to show that such a function is harmonic in Ω and satisfies the boundary condition $u|_{\partial\Omega} = g$.

The completion of the proof of this result will require the introduction of some technical lemmas and constructions, which are necessary in order to prove rigorously that the function defined in (23) is indeed a valid solution, but that also give the intuition behind all the construction of Perron Method.

Lemma 7.6. Let $w_1, w_2, ..., w_n \in S_g$. Then $\max\{w_1, w_2, ..., w_n\} \in S_g$.

Proof. Let $w = \max\{w_1, w_2, ..., w_n\}$. It is immediate to conclude that $w \leq g$ on $\partial\Omega$, as all w_i satisfy this condition. Therefore one just needs to show that w is subharmonic in Ω, but this follows immediately. Indeed for any w_i

$$w_i(x) \leq \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} w_i(y)dS(y) \leq \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} w(y)dS(y)$$

and therefore

$$w(x) = \max\{w_1(x), w_2(x), ..., w_n(x)\} \leq \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} w(y)dS(y)$$

which is precisely the characterization given by (22). Hence w is subharmonic and is a subfunction.

□

As it was explained previously, the idea is now to modify locally any given subfunction by means of a harmonic function such that this modification is still continuous in $\overline{\Omega}$; the generalization to higher dimensions of the "harmonic and local straightening" applied in one dimension as in Figure 2 is obtained by introducing the following function:

Definition 7.7 (Harmonic lift). Let $w \in S_g$, $\xi \in \Omega$ and $r > 0$ such that $\overline{B_r(\xi)} \subset \Omega$. The **harmonic lift** of w in $B_r(\xi)$ is defined as

$$w^{\xi,r} : \overline{\Omega} \to R, \;\; w^{\xi,r}(x) = \begin{cases} w(x) & \text{if x} \in \overline{\Omega}\backslash B_r(\xi) \\ \Delta w^{\xi,r}(x) = 0 & \text{if x} \in B_r(\xi) \end{cases} \tag{24}$$

Such a function in (24) exists and is well defined since the Dirichlet Problem for the Laplace Equation on a ball with continuous boundary datum admits a unique solution. By maximum principle it follows immediately that

$$w^{\xi,r} \geq w \quad \text{in } \overline{\Omega} \tag{25}$$

Equation (25) explains precisely the meaning of this operation and the name of this modification, as the subfunction w is literally lifted up locally, making it harmonic on $B_r(\xi)$. Again this idea is clearer in the one dimensional example of Figure 2: any subfunction w is just a convex function which in $(0,1)$ is not larger than its **affine lift** in $(\xi - r, \xi + r)$, i.e. the function satisfying

$$\begin{cases} -\Delta v = 0 & \text{in } (\xi - r, \xi + r) \\ v = w & \text{in } [0,1]\backslash(\xi - r, \xi + r) \end{cases}$$

Moreover the set S_g is closed under this operation of lifting on an arbitrary ball, i.e. the harmonic lift of any subfunction is still a subfunction:

Lemma 7.8. For any $w \in S_g$ and $\overline{B_r(\xi)} \subset \Omega$, $w^{\xi,r} \in S_g$.

Proof. Consider $x \in \Omega$ and $R > 0$ arbitrary such that $B_R(x) \subset \Omega$ and let h the solution to

$$\begin{cases} -\Delta h = 0 & \text{in } B_R(x) \\ h = w^{\xi,r} & \text{on } \partial B_R(x) \end{cases}$$

where such a function h exists since $w^{\xi,r} \in C^0(\partial B_R(x))$.
Since $w \leq w^{\xi,r} = h$ on $\partial B_R(x)$ and w is subharmonic, by maximum principle $w \leq h$ in $\overline{B_R(x)}$ and in particular $w^{\xi,r} = w \leq h$ in $B_R(x) \backslash B_r(\xi)$.
On the other hand, if $B = B_R(x) \cap B_r(\xi) \neq \emptyset$, one has that $w^{\xi,r}$ and h are harmonic in B and $w^{\xi,r} \leq h$ on ∂B (because they coincide on $\partial B_R(x)$ and on $(\partial B \cap \partial B_r(\xi)) \subset \overline{B_R(x)}$ one has $w^{\xi,r} = w \leq h$ by the previous point), so once again by maximum principle $w^{\xi,r} \leq h$ in B. Hence $w^{\xi,r} \leq h$ in $B_R(x)$ and finally

$$\begin{aligned} 0 = & \quad \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} (h - w^{\xi,r}) dS & & (\text{since } h = w^{\xi,r} \text{ on } \partial B_R(x)) \\ = & \quad h(x) - \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} w^{\xi,r} dS & & (\text{since } h \text{ is harmonic in } B_R(x)) \\ \geq & \quad w^{\xi,r}(x) - \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} w^{\xi,r} dS & & (\text{since } h \geq w^{\xi,r} \text{ in } B_R(x)) \end{aligned}$$

and hence $w^{\xi,r}$ is also subharmonic and a subfunction.

□

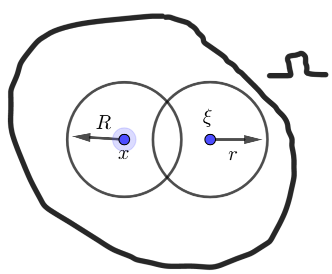


Figure 3: $w^{\xi,r}$ is subharmonic in $B_R(x) \cap B_r(\xi)$ and in $B_R(x) \backslash B_r(\xi)$

The previous results are clearly going in the direction of approaching the solution defined in (23) by modifying any subfunction locally, making it harmonic. A step further is made by establishing an important result of uniform convergence of sequences of harmonic functions.

Theorem 7.9. Let $\{v_k\}_{k\in\mathbb{N}}$ be a monotone non-decreasing sequence of harmonic functions in $B_R(x)$ and let $\{v_k(x)\}_{k\in N}$ be bounded. Then $\{v_k\}$ converges uniformly on any closed ball $B_r(x)$, $r \leq R$ to a harmonic function v in $B_R(x)$.

Proof. Since $\{v_k(x)\}_{k\in N}$ is bounded and non-decreasing, the limit $\lim_{k\to\infty} v_k(x) = v(x)$ exists. Then, $\forall \epsilon > 0$, $\exists N_\epsilon$ s.t. $0 \leq v_l(x) - v_k(x) < \epsilon$ for any $l \geq k > N_\epsilon$. Let now $0 < r < R$, $y \in \overline{B_r(x)}$ and $\delta < R - r$. Then since $v_l - v_k$ is non negative and harmonic in $B_R(x)$

$$\begin{aligned}
0 \leq \quad & v_l(y) - v_k(y) \\
= \quad & \frac{1}{|B_\delta(y)|}\int_{B_\delta(y)} (v_l(z) - v_k(z))dz && \text{(by mean value property)} \\
\leq \quad & \frac{1}{\delta^n \alpha_n}\int_{B_{r+\delta}(x)} (v_l - v_k)(z)dz && \text{(positivity of measure)} \\
= \quad & \frac{(r+\delta)^n}{\delta^n}(v_l(x) - v_k(x)) \leq (1 + \frac{r}{\delta})^n \epsilon && (\overline{B_{r+\delta}(x)} \subset B_R(x) \text{ as } \delta < R - r)
\end{aligned}$$

Hence one has that

$$\sup_{y\in\overline{B_r(x)}} |v_l(y) - v_k(y)| \leq (1 + \frac{r}{\delta})^n \epsilon \quad \forall l \geq k > N_\epsilon$$

which implies that $\{v_k\}_k$ converges uniformly in $\overline{B_r(x)}$ to a continuous function v. Moreover since all v_k are harmonic, Lemma 3.11 implies that v is also harmonic in $B_R(x)$.

□

This is indeed a particular case of a more general theorem:

Theorem 7.10 (Harnack's Theorem). Let $\{v_k\}_{k\in N}$ be a monotone non-decreasing sequence of harmonic functions in a domain Ω and suppose that the sequence $\{v_k(x)\}_{k\in N}$ is bounded for some point $x \in \Omega$. Then $\{v_k\}$ converges uniformly on any bounded subdomain $\Omega' \subset\subset \Omega$ to a harmonic function.

Given all the results presented above, one is now ready to state the first main result, which in literature is also called the **Wiener's Theorem**:

Lemma 7.11. The function $u = \sup_{w\in S_g} w$ is well defined and harmonic in Ω.

Proof. First notice that by maximum principle any $w \in S_g$ satisfies

$$w(x) \leq \max_{\partial\Omega} g \quad \forall x \in \Omega$$

so that the function u is well defined in $\overline{\Omega}$.

To prove now that u is an harmonic function, the idea is to show that it is point-wise equal, on any arbitrary ball contained in Ω, to a harmonic function obtained as the limit of a properly defined sequence of harmonic lifts on that ball.

To this aim, let $\xi \in \Omega$ and $\{w_k \in S_g\}_{k\in N}$ a sequence of subfunctions such that $\lim_{k\to\infty} w_k(\xi) = u(\xi)$. One can define

$$\{\tilde{w}_k = \max_{j=1,\dots,k} w_j\}_{k\in N} \tag{26}$$

which is now a non-decreasing sequence of subfunctions (by Lemma 7.6), such that $\lim_{k\to\infty} \tilde{w}_k(\xi) = u(\xi)$.
Now take a ball $\overline{B_r(\xi)} \subset \Omega$ and define the sequence

$$\{\tilde{w}_k^{\xi,r}\}_{k\in N} \tag{27}$$

where $\tilde{w}_k^{\xi,r}$ is the harmonic lift of $\tilde{w}_k$ in $B_r(\xi)$. By Lemma 7.8, each $\tilde{w}_k^{\xi,r} \in S_g$.

To recap the preceding paragraph, one first considers a sequence of subfunctions converging to u in ξ; the idea is to approach u in $B_r(\xi)$ by constructing the sequence in (26) inductively (the k-th term of $\{\tilde{w}_k\}$ is either w_k or $\tilde{w}_{k-1}$) and then to make every term of such a non-decreasing sequence harmonic locally in $B_r(\xi)$, while maintaining the subfunction property, by taking the harmonic lift of $\{\tilde{w}_k\}_{k\in N}$ given by $\{\tilde{w}_k^{\xi,r}\}_{k\in N}$.

Moreover by maximum principle $\tilde{w}_k^{\xi,r} \le \tilde{w}_l^{\xi,r}$ in Ω if $k < l$, so that the sequence in (27) is non-decreasing and bounded from above by the boundary term with $\tilde{w}_k^{\xi,r}(\xi) \to u(\xi)$ as $k \to \infty$. But then due to Theorem 7.9 necessarily $\tilde{w}_k^{\xi,r}$ converges to a harmonic function w on $B_r(\xi)$, uniformly on any $\overline{B_\rho(\xi)}$, $\rho < r$. Clearly since $\lim_{k\to\infty} \tilde{w}_k^{\xi,r}(\xi) = w(\xi)$ by uniqueness of the limit $w(\xi) = u(\xi)$ and $w(y) \le u(y)\ \forall y \in B_r(\xi)$.

Claim. It holds that $w = u$ on $B_r(\xi)$.

Suppose by contradiction that this is not the case and let $\eta \in B_r(\xi)$, $\eta \ne \xi$ such that $w(\eta) < u(\eta)$. Then, there exists $\overline{u} \in S_g$ such that $w(\eta) < \overline{u}(\eta) \le u(\eta)$.
Define now

$$v_k = \max\{\overline{u}, \tilde{w}_k^{\xi,r}\} \tag{28}$$

where the $\tilde{w}_k^{\xi,r}$ terms are those of the sequence defined in (27) and v_k by construction is non-decreasing. If $v_k^{\xi,r}$ is the harmonic lift of v_k in $B_r(\xi)$, it holds that $v_k^{\xi,r} \ge v_k \ge \tilde{w}_k^{\xi,r}$ in $B_r(\xi)$ so that necessarily $\lim_{k\to\infty} v_k^{\xi,r}(\xi) = u(\xi)$.
Note that this sequence is modified with respect to v_k only in the terms where the maximum is $\overline{u}$, as the harmonic lift of an already lifted subfunction is not modifying it.
Once again $\{v_k^{\xi,r}\}$ is non-decreasing in k and bounded, hence it converges to a harmonic function v in $B_r(\xi)$ such that $w \le v \le u$ in $B_r(\xi)$, $w(\xi) = v(\xi) = u(\xi)$, and $w(\eta) < \overline{u}(\eta) = v(\eta)$.
But then $v - w$ is a non negative, non constant (because $(v - w)(\eta) > 0$), harmonic function in $B_r(\xi)$ which takes its minimum $(v-w)\ (\xi) = 0$ at the interior

point ξ, which cannot happen by the strong maximum principle.

Therefore u is harmonic in $B_r(\xi)$ and also in Ω since ξ, r were arbitrary.

$\square$

It is better to give the intuition of what is happening in the previous proof by the one dimensional construction presented before.
The solution one wants to approach is the straight line connecting points u_0 and u_1 in Figure 4; consider once again a sequence $\{w_k\}_{k \in N}$ of subfunctions such that $w_k(\xi) \to u(\xi)$ for a fixed point $\xi \in (0,1)$.

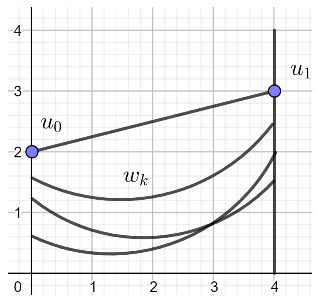


Figure 4: $\{w_k\}$ is any sequence of subfunctions in (0,1)

Consider now $r > 0$ such that $[\xi - r, \xi + r] \subset (0,1)$ and construct first the sequence $\{\tilde{w}_k\}_{k \in N}$ which is non-decreasing by construction. On such an interval for all $k \in N$ consider the harmonic lift as in Figure 2, that is substitute any $\tilde{w}_k$ in $(\xi - r, \xi + r)$ by an affine function so that the modified function is still continuous in [0,1].
This sequence of subfunctions $\{\tilde{w}_k^{\xi,r}\}_{k \in N}$ is still non-decreasing, bounded by $\max\{u_0, u_1\}$ and harmonic in $(\xi - r, \xi + r)$, so it converges to a harmonic function w in $(\xi - r, \xi + r)$ with $u(\xi) = w(\xi)$ and surely $w \leq u$ over the whole $(\xi - r, \xi + r)$.

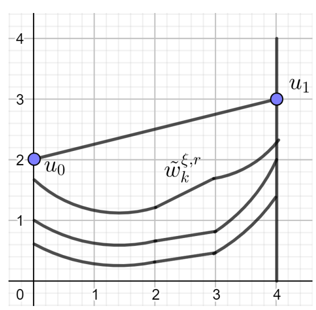


Figure 5: $\{\tilde{w}_k^{\xi,r}\}_{k \in N}$ converges to an harmonic function in $(\xi - r, \xi + r)$

In order to prove that in fact $w = u$ in $(\xi - r, \xi + r)$, the idea is to show that there is no space for another subfunction to be strictly in between w and u anywhere in $(\xi - r, \xi + r)$. Suppose by contradiction that this is the case and let $\overline{u}$ such a subfunction, with $w(\eta) < \overline{u}(\eta) \leq u(\eta)$ for some point $\eta \neq \xi \in (\xi - r, \xi + r)$.

Then the sequence given point-wise by the harmonic lift of $\max\{\overline{u}, \tilde{w}_k^{\xi,r}\}$ again satisfies the hypotheses of Theorem 7.9 and hence converges to a harmonic function $v \geq w$ in $(\xi - r, \xi + r)$ with $v(\xi) = u(\xi)$. This yields the contradiction of the strong maximum principle applied to $v - w$.
Then u is harmonic in $(\xi - r, \xi + r)$ and also in $(0, 1)$ as r, ξ were arbitrary.

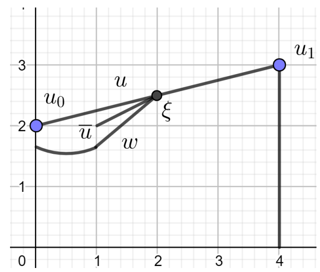


Figure 6: w and u must coincide over all $(\xi - r, \xi + r)$

In the picture above, the intuition is that w necessarily "reaches" the values attained by u all over the subinterval, otherwise there is space for another subfunction to stay in the middle and yield a contradiction as explained in the previous paragraph.
Another important observation to be made with reference to the last picture is the following: one can assume without loss of generality that the solution $u = \sup_{w \in S_g} w$ corresponds to the harmonic lift of a subfunction in $(\xi - r, \xi + r)$. This is coherent with the characterization of subharmonic functions given previously in Lemma 7.4, since $w^{\xi,r} \geq w$ by equation (25) for any subfunction: given any subfunction $v \in S_g$ one can replace it with $v^{\xi,r} \in S_g$ and assume the latter as the candidate to be the Perron solution.

One has now proved that the function defined in (23) is harmonic in Ω.

However, as it was explained in the introduction, in the Perron Method the study of the boundary behaviour of the solution is essentially separate from the existence discussion in the interior of the domain.

Therefore in order to show that $u|_{\partial\Omega} = g$ it is necessary to further characterize all the points of $\partial\Omega$. The continuous assumption of boundary values is connected to the geometric properties of the boundary through the concept of regular point.

Definition 7.12. A boundary point $\xi \in \partial\Omega$ is said to be **regular** if there exists a superharmonic function p such that $p > 0$ in $\overline{\Omega} \backslash \{\xi\}$ and $p(\xi) = 0$.

Such a function p is also called a **barrier function** at $\xi \in \partial\Omega$. This definition comes from complex analysis: a barrier function in the theory of complex functions is a function the existence of which for all boundary points of the domain implies that Ω is a domain of holomorphy, i.e. each point $x_0 \in \partial\Omega$ has a holomorphic function in Ω which cannot be holomorphically extended to x_0. In particular, a barrier exists at a boundary point ξ of a domain Ω if there is an analytic function defined in Ω that is unbounded at ξ.

Theorem 7.13. Let $u = \sup_{w \in S_g} w$ and ξ a regular boundary point. Then u is continuous at ξ and $u(\xi) = g(\xi)$.

Proof. Since ξ is regular, there exists by definition a superharmonic function p such that $p > 0$ in $\overline{\Omega}\backslash\{\xi\}$ and $p(\xi) = 0$. Moreover being g continuous at ξ, $\forall \epsilon > 0\ \exists \delta_\epsilon > 0$ such that $|g(x) - g(\xi)| < \epsilon$ for all $x \in \partial\Omega$ with $|x - \xi| < \delta_\epsilon$. On the other hand, since $p > 0$ in $\overline{\Omega}\backslash\{\xi\}$ one finds K $= \min_{\overline{\Omega}\backslash B_{\delta_\epsilon}(\xi)} p > 0$, so that if $M = ||g||_{C^0(\partial\Omega)}$ the following functions are well defined:

$$\underline{w}(x) = g(\xi) - \epsilon - \frac{2M}{K}p(x), \qquad \overline{w}(x) = g(\xi) + \epsilon + \frac{2M}{K}p(x)$$

Clearly $\underline{w}$ is subharmonic and for any $x \in \partial\Omega$

$$|x - \xi| < \delta_\epsilon \to \underline{w}(x) = \underbrace{g(\xi) - g(x) - \epsilon}_{\leq 0 \text{ since } g(\xi) - g(x) \leq \epsilon} - \frac{2M}{K}p(x) + g(x) \leq g(x)$$

$$|x - \xi| \geq \delta_\epsilon \to \underline{w}(x) = g(x) - \epsilon + \underbrace{g(\xi) - g(x) - \frac{2M}{K}p(x)}_{\leq 0 \text{ as } p(x) \geq K \text{ in } \overline{\Omega}\backslash B_{\delta_\epsilon}(\xi)} \leq g(x)$$

Hence, $\underline{w}$ is a subfunction. By a similar argument $\overline{w}$ is superharmonic and for any $x \in \partial\Omega$

$$|x - \xi| < \delta_\epsilon \to \overline{w}(x) = \underbrace{g(\xi) - g(x) + \epsilon}_{\geq 0 \text{ since } g(\xi) - g(x) \geq -\epsilon} + \frac{2M}{K}p(x) + g(x) \geq g(x)$$

$$|x - \xi| \geq \delta_\epsilon \to \overline{w}(x) = g(x) + \epsilon + \underbrace{g(\xi) - g(x) + \frac{2M}{K}p(x)}_{\geq 0 \text{ since } p(x) \geq K \text{ in } \overline{\Omega}\backslash B_{\delta_\epsilon}(\xi)} \geq g(x)$$

so that $\overline{w}$ is a superfunction.
Since if u is a solution it is not smaller than any subfunction and not greater than any superfunction, then $\underline{w}(x) \leq u(x) \leq \overline{w}(x)\ \forall x \in \overline{\Omega}$, which explicitly means that

$$g(\xi) - \epsilon - \frac{2M}{K}p(x) \leq u(x) \leq g(\xi) + \epsilon + \frac{2M}{K}p(x)$$

or equivalently

$$|u(x) - g(\xi)| \leq \epsilon + \frac{2M}{K}p(x)$$

But now since $p(x) \to 0$ as $x \to \xi$, one can find $\tilde{\delta} > 0$: $p(x) < \frac{K}{2M}\epsilon$ for all $|x - \xi| < \tilde{\delta}$, so that $|u(x) - g(\xi)| \leq 2\epsilon$.

$\square$

Given all the previous discussion, one is now ready to state

Theorem 7.14 (Perron's Theorem).
The Dirichlet Problem for the Laplace Equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is solvable for any $g \in C^0(\partial\Omega)$ if and only if all boundary points are regular.

Proof. If all the boundary points are regular, then by the two preceding theorems 7.11 and 7.13 the Dirichlet Problem admits a solution.
Suppose instead that the Dirichlet problem is solvable for any continuous boundary datum. Consider, in particular, the boundary datum $g(x) = |x - \xi|$, with $\xi \in \partial\Omega$, which is obviously continuous on $\partial\Omega$. The solution u of the corresponding Dirichlet Problem is harmonic, strictly positive in $\overline{\Omega}\backslash\{\xi\}$ by the strong maximum principle and $u(\xi) = g(\xi) = 0$, hence ξ is regular.

□

The preceding theorem has also the following natural consequence, which is actually very deep and was already anticipated at the beginning of the section: Perron's Theorem gives a sufficient condition for the existence of a Green's Function on a bounded domain.

Corollary 7.15. Let Ω a bounded domain with regular boundary points. Then there exists a unique Green's Function for Ω.

Proof. If Ω has regular boundary points, the Dirichlet Problem for the Laplace Equation admits a solution for any continuous boundary datum. Then a Green's Function for Ω exists, is well defined and is unique by maximum principle.

□

This also implies that the Perron solution must have the Green's Function representation in the form of (17).

Some sufficient conditions for a domain to fulfill the Perron's condition on $\partial\Omega$ are now presented as they don't require much effort and are actually easier to deal with with respect to the characterizations of domains with regular boundary points given by the theory of functions through the concept of barrier function. The conditions associated to the latter are in fact not easy to check in most of the cases and require entering another whole field of mathematics, which is out of scope here.

Definition 7.16. A domain Ω is said to be satisfying an **exterior sphere condition** if $\forall \xi \in \partial\Omega$ there exists $y \in \overline{\Omega}^c$ and r > 0 such that $\overline{B_r(y)} \cap \overline{\Omega} = \{\xi\}$.

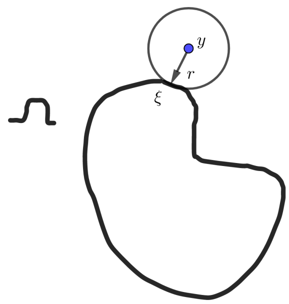


Figure 7: Exterior sphere condition

Theorem 7.17. All domains with C^2 boundary satisfy the exterior sphere condition.

Proof. Given a point $\xi \in \partial\Omega$, one can always choose an orthogonal coordinate system $x_1, ..., x_n$ for $\mathbb{R}^n$ so that the point ξ lies at $(0, 0, ..., 0)$ and the exterior normal to Ω in 0 is $e_n = (0, 0, ..., 1)$. Let $r > 0$ and $f \in C^2(R^{n-1})$ such that $\Omega \cap B_r(0) = \{x_n < f(x_1, ..., x_{n-1})\}$. Note that $f(0) = 0$ and $Df(0) = 0$. By the Taylor's Theorem, $|f(x)| \le M(x_1^2 + ... + x_{n-1}^2)$ for some M and x in a neighbourhood of 0. Let now $y = \delta e_n$, $\delta > 0$ so that $y \in \mathbb{R}^n \backslash \Omega$. One can choose $\delta > 0$ sufficiently small so that $B_\delta(y) \subset \mathbb{R}^n \backslash \overline{\Omega}$, $\overline{B_r(y)} \cap \overline{\Omega} = \{0\}$. If $x \in \overline{B_\delta(y)}$, $x \neq 0$, then

$$|x - y|^2 = |x|^2 - 2\delta x_n + \delta^2 \le \delta^2$$

so that $|x|^2 \le 2\delta x_n$. Then

$$f(x_1, ..., x_{n-1}) \le M|x|^2 \le 2M\delta x_n$$

and one can choose $\delta < \frac{1}{2M}$ so that $f(x_1, ..., x_{n-1}) < x_n$, i.e. $x \in \mathbb{R}^n \backslash \Omega$.

□

Contrarily to what one may think, requiring $\partial\Omega \in C^1$ is not sufficient: to show this, consider as a counterexample $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > x_1^2 \log |x_1|\}$. Then $\partial\Omega$ is of class C^1 but in $(0, 0)$ the exterior sphere condition is not satisfied. To prove this, let $f(x) = x^2 \log |x|$ and $g(x) = \sqrt{r^2 - x^2} - r$, for $r > 0$ to be fixed. Note that $f(0) = g(0) = 0$. If the exterior sphere condition was satisfied, then there would exist $r > 0$ such that $f(x) > g(x)$ for every $x \in (-r, r)$, $x \neq 0$. But this is impossible since $f'(x) < g'(x)$ as $x \to 0^+$ and $f'(x) > g'(x)$ as $x \to 0^-$.

Theorem 7.18 (Poincaré Criterion). If a bounded domain Ω satisfies the exterior sphere condition then all its boundary points are regular.

Proof. Since Ω satisfies the exterior sphere condition, given $\xi \in \partial\Omega$ there exists $y \in \overline{\Omega}^c$ and $r > 0$ such that $\overline{B_r(y)} \cap \overline{\Omega} = \{\xi\}$. For $n \ge 3$, the function

$$\phi(x) = \frac{1}{r^{n-2}} - \frac{1}{|x - y|^{n-2}}, \qquad x \in \overline{\Omega}$$

is a barrier at $\xi \in \partial\Omega$. Indeed, ϕ is harmonic in Ω as it is of the same form of the fundamental solution (6) with $y \notin \Omega$; moreover clearly $\phi(\xi) = 0$ and $\phi(x) > 0$ for $x \in \overline{\Omega} \backslash \{\xi\}$.
For $n = 2$, it is again straightforward to check that

$$\phi(x) = \log \frac{1}{r} - \log \frac{1}{|x - y|}, \qquad x \in \overline{\Omega}$$

is a barrier at ξ.

$\square$

The two preceding theorems have shown that

$$\partial\Omega \in C^2 \implies \text{exterior sphere condition} \implies \text{all boundary points are regular}$$

and therefore the Dirichlet Problem for the Laplace Equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

with $g \in C^0(\partial\Omega)$ is solvable on every C^2 domain.

7.4 Capacity and Wiener's Criterion

The theory of all domains with regular boundary points (in literature they are also called simply **regular domains**) was developed through a very deep geometrical and analytical study by the American mathematician Norbert Wiener. Its connection with potential theory and electromagnetism, which gives another justification for the deep bond between physics and the boundary value problems that are discussed in the elaborate, surprisingly turns out to be very strong and is here only presented briefly for the sake of completeness.

The capacity of a set is a mathematical analogue of a set's ability to hold electrical charge. More precisely, it is the capacitance of the set: the total charge a set can hold while maintaining a given potential energy. The physical concept of capacity provides another means of characterizing regular and exceptional boundary points. Let Ω a bounded domain in $\mathbb{R}^n (n \geq 3)$ and with a smooth boundary $\partial\Omega$. Let then u be the harmonic function (sometimes also called the *Newtonian potential*) satisfying

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n \backslash \Omega \\ u = 1 & \text{on } \partial\Omega \\ \lim_{|x| \to \infty} u(x) = 0 \end{cases}$$

Then one defines the **harmonic capacity** of the set Ω as

$$C(\Omega) = \int_{R^n \backslash \Omega} |\nabla u|^2 dx$$

The existence of such a function u is easily established as the unique limit of harmonic functions u' in an expanding sequence of bounded domains having $\partial\Omega$ as a inner boundary and with outer boundaries going to infinity. Namely, the harmonic capacity can also be understood as a limit of the condenser capacity. In fact, let S_r the sphere of radius r centered in the origin. Since Ω is bounded, for sufficiently large r, S_r will enclose Ω and (Ω, S_r) will form a condenser pair. The harmonic capacity is then defined as

$$C(\Omega) = \lim_{r\to\infty} C(\Omega, S_r)$$

Capacity can also be defined for domains with non smooth boundaries and for any compact set by considering the unique limit of the capacities of a nested sequence of approximately smoothly bounded domains.
But indeed, equivalent definitions of capacity can be given directly without the use of approximate domains. In particular, the variational characterization of the capacity defines the latter as

$$C(\Omega) = \inf_{v\in K} \int_\Omega |\nabla v|^2$$

where

$$K = \{v \in C_0^1(\mathbb{R}^n) : v = 1 \quad \text{in } \Omega\}$$

The deep connection between the concept of capacity and the characterization of regular points on surfaces arises by

Proposition 7.19 (Wiener Criterion). Let Ω a bounded domain. For any $\lambda \in (0,1)$, let C_j be the capacity of the set $B_{\lambda^j}(\xi) \cap \Omega^c$; then ξ is a regular point if and only if

$$\sum_{j=0}^{\infty} C_j/\lambda^{j(n-2)}$$

diverges.

The proof of Wiener's Criterion is omitted.

Although this last condition has a very deep theoretical meaning, it is not easy to verify it and as it was explained before, it requires entering another field of mathematics, deviating from the intent of this elaborate. For this reason it is better to treat the sufficient conditions of the previous section such as the exterior sphere condition or $\partial\Omega \in C^2$: these two characterizations are the simplest and the most intuitive ones, as they were tackled also previously in the elaborate and they arise often in the study of very similar problems.

8 Dirichlet Problem for the Poisson Equation

Consider now the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

The construction of a Green's Function leads again to an explicit formula for the solution given by

$$u(y) = \int_{\Omega} G(x,y)f(x)dx - \int_{\partial\Omega} \partial_\nu G(x,y)g(x)dS(x) \tag{29}$$

and as before for simple domains one can construct explicitly the Green's Function and the Poisson Kernel. Moreover if a solution exists, it is always in the form above:

Theorem 8.1. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of the Dirichlet Problem for the Poisson equation above, with $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$, then the solution is of the form in equation (29).

8.1 Newtonian Potential

An alternative approach would be to use the linearity of the given problem as follows: define the **Newtonian Potential** as

$$w(y) = \int_{\Omega} \Gamma(x-y)f(x)dx \tag{30}$$

The idea is to prove that, under certain assumptions for f, $-\Delta w = f$.
If for example $f \in C_0^2(\mathbb{R}^n)$, it is easy to check using convolution theory that everything works properly. Indeed a more detailed analysis suggests that one can actually require less regularity on f.

Definition 8.2. A function $f : \overline{\Omega} \to R$ is said to be uniformly α-Holder continuous for some $\alpha \in (0,1]$ in Ω if

$$\sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$$

If a function is uniformly α-Holder continuous in Ω, then it is uniformly continuous in $\overline{\Omega}$. The space of uniformly α-Holder continuous functions on $\overline{\Omega}$, denoted as $C^{0,\alpha}(\overline{\Omega})$, is a Banach space with norm given by

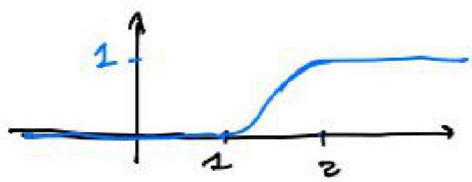
$$||f||_{C^{0,\alpha}(\overline{\Omega})} = ||f||_{C^0} + |f|_{C^{0,\alpha}}$$

with the seminorm

$$|f|_{C^{0,\alpha}} = \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

Theorem 8.3. Let Ω a bounded domain and $f \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$ and bounded. Then it holds that $-\Delta w = f$ and $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Proof. Let $\eta : \mathbb{R}_+ \to [0,1]$ such that $\eta(r) = 0$ for $0 \le r \le 1$, $\eta(r) = 1$ for $r \ge 2$ and moreover $0 \le \eta'(r) \le 2$ $\forall r$.



For instance one can consider

$$\eta(t) = \begin{cases} 0 \quad t \le 1 \\ \frac{1}{2} + \frac{1}{2}\sin(\pi(t - 3/2)) \quad 1 \le t \le 2 \\ 1 \quad t \ge 2 \end{cases}$$

Let $\eta_\epsilon : \mathbb{R} \to [0,1]$, $x \mapsto \eta(\frac{|x|}{\epsilon})$ a radial function and define the **mollified fundamental solution** as

$$\Gamma_\epsilon(x-y) = \Gamma(x-y)\eta_\epsilon(x-y)$$

such that

$$\Gamma_\epsilon(x-y) = \begin{cases} 0 \quad x \text{ in } B_\epsilon(y) \\ \Gamma(x-y) \quad x \text{ in } R^n \backslash B_{2\epsilon}(y) \end{cases}$$

In the end consider the **smoothed Newtonian Potential**

$$w_\epsilon(y) = \int_\Omega \Gamma_\epsilon(x-y) f(x) dx$$

Whereas, by the initial assumptions, it follows easily that $w_\epsilon \in C^0(\mathbb{R}^n)$ one needs to show that $w_\epsilon(y) \to w(y)$ as $\epsilon \to 0$ uniformly on any compact subset of $\mathbb{R}^n$, i.e. that

$$\begin{aligned} |w(y) - w_\epsilon(y)| \le & \int_\Omega |\Gamma(x-y) - \Gamma_\epsilon(x-y)||f(x)|dx \\ \le & \int_\Omega |(1-\eta_\epsilon(x-y))||\Gamma(x-y)||f(x)|dx \\ \le & \sup_\Omega |f| \int_{B_{2\epsilon}(y)} |(1-\eta_\epsilon(x-y))||\Gamma(x-y)|dx \\ \le & \sup_\Omega |f| \int_{B_{2\epsilon}(y)} |\Gamma(x-y)|dx \\ = & \begin{cases} \sup_\Omega |f| \;\; \epsilon^2(2\log(2\epsilon) - 1) & \text{if } n = 2 \\ \sup_\Omega |f| \;\; \frac{2\epsilon^2}{n-2} & \text{if } n > 2 \end{cases} \quad \to 0 \quad \text{as } \epsilon \to 0 \end{aligned}$$

This shows that $w \in C^0(\mathbb{R}^n)$. Now one knows that $w_\epsilon \in C^1$ since $\Gamma_\epsilon \in C^1$, then defining

$$v_i(y) = \int_\Omega \partial_{y_i}\Gamma(x-y)f(x)dx$$

one finds that

$$\begin{aligned}
|v_i(y) - \partial_{y_i}w_\epsilon(y)| \leq & \int_\Omega |\partial_{y_i}\Gamma(x-y) - \partial_{y_i}(\Gamma(x-y)\eta_\epsilon(x-y))||f(x)|dx \\
= & \int_{B_{2\epsilon}(y)} |\partial_{y_i}(1-\eta_\epsilon(x-y))\Gamma(x-y) \\
& + (1-\eta_\epsilon(x-y))\partial_{y_i}\Gamma(x-y)||f(x)|dx \\
\leq & \sup_\Omega |f| \int_{B_{2\epsilon}(y)} |\frac{2}{\epsilon}\Gamma(x-y) + \partial_{y_i}\Gamma(x-y)|dx \\
\leq & \sup_\Omega |f| \int_{B_{2\epsilon}(y)} |\frac{2}{\epsilon}\Gamma(x-y)| + \sup_\Omega |f| \int_{B_{2\epsilon}(y)} |\partial_{y_i}\Gamma(x-y)| \\
\leq & \begin{cases} \sup_\Omega |f| 2\epsilon(2\log(2\epsilon)-1) + \sup_\Omega |f|\epsilon & \text{if } n=2 \\ \\ \sup_\Omega |f| \frac{4\epsilon}{(n-2)} + \sup_\Omega |f|\epsilon & \text{if } n>2 \end{cases} \quad \to 0 \quad \text{as } \epsilon \to 0
\end{aligned}$$

Then $\partial_{y_i}w_\epsilon(y) \to v_i(y)$ uniformly on any compact subset of $\mathbb{R}^n$, so that $v_i \in C^0(\mathbb{R}^n)$ and moreover

$$\begin{aligned}
w(y+se_i) - w(y) &= \lim_{\epsilon\to 0}(w_\epsilon(s+e_i) - w_\epsilon(s)) = \lim_{\epsilon\to 0}\int_0^s \partial_{y_i}w_\epsilon(y+te_i)dt \\
&= \int_0^s \lim_{\epsilon\to 0}\partial_{y_i}w_\epsilon(y+te_i)dt = \int_0^s v_i(y+te_i)dt
\end{aligned}$$

Hence $v_i = \partial_{y_i}w$, so that $\partial_{y_i}w_\epsilon(y) \to \partial_{y_i}w(y)$ and therefore $w \in C^1(\mathbb{R}^n)$.
Define now the following two quantities, where $\widetilde{f}$ is any zero and smooth extension of f and Ω_0 is a domain with C^1 boundary containing Ω:

$$\begin{aligned}
v_{ij}(y) &= \int_{\Omega_0} \partial_{y_iy_j}\Gamma(x-y)|x-y|^\alpha(\frac{|\widetilde{f}(x)-\widetilde{f}(y)|}{|x-y|^\alpha}) + f(y)\int_{\Omega_0}\partial_{y_iy_j}\Gamma(x-y)dx \\
&= \int_{\Omega_0} \partial_{y_iy_j}\Gamma(x-y)|x-y|^\alpha(\frac{|\widetilde{f}(x)-\widetilde{f}(y)|}{|x-y|^\alpha}) + f(y)\int_{\partial\Omega_0}\partial_{y_i}\Gamma(x-y)\nu_j dS(x)
\end{aligned}$$

$$\begin{aligned}
\partial_{y_i}u_\epsilon^j(y) &= \int_{\Omega_0} \partial_{y_i}(\partial_{y_j}\Gamma(x-y)\eta_\epsilon(x-y)|x-y|^\alpha(\frac{|\widetilde{f}(x)-\widetilde{f}(y)|}{|x-y|^\alpha}) \\
&+ f(y)\int_{\partial\Omega_0}\partial_{y_i}\Gamma(x-y)\nu_j dS(x)
\end{aligned}$$

Then one has that

$$|v_{ij}(y) - \partial_{y_i} u_\epsilon^j(y)| \leq \int_{\Omega_0} \partial_{y_i y_j}(\Gamma(x-y) - \eta_\epsilon(x-y)\Gamma(x-y)|x-y|^\alpha(\frac{|\widetilde{f}(x) - \widetilde{f}(y)|}{|x-y|^\alpha})$$
$$\leq \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \int_{B_{2\epsilon}(y)} |\partial_{y_i}(1-\eta_\epsilon(x-y))\partial_{y_j}\Gamma(x-y)||x-y|^\alpha$$
$$+ \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \int_{B_{2\epsilon}(y)} |(1-\eta_\epsilon(x-y))\partial_{y_i y_j}\Gamma(x-y)||x-y|^\alpha$$
$$\leq \sup_{x,y\in B_{2\epsilon}(y)} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \int_{B_{2\epsilon}(y)} (\frac{2}{\epsilon}|\partial_{y_i}\Gamma(x-y)| + |\partial_{y_i y_j}\Gamma(x-y)|)|x-y|^\alpha \to 0 \quad \text{as } \epsilon \to 0$$

It follows that $\partial_{y_i} u_\epsilon^j \to v_{ij}$ uniformly on any compact subset of Ω. On the other hand $u_\epsilon^j \to v_j = \partial_{y_j} w$ uniformly in $\overline{\Omega}$. Then $v_{ij} = \partial_{y_i y_j} w$ and hence $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$.
Finally for any $y \in \Omega$ such that $B_\epsilon(y) \subset \Omega$

$$\begin{aligned}\Delta w(y) = \sum_i v_{ii}(y) &= \int_{\Omega_0 \setminus B_\epsilon(y)} \underbrace{\Delta\Gamma(x-y)}_{=0}(\widetilde{f}(x) - f(y)dx \\ &+ \sum_i \int_{B_\epsilon(y)} \underbrace{\partial_{ii}\Gamma(x-y)|x-y|^\alpha}_{=O(\epsilon^\alpha)}(\frac{|\widetilde{f}(x) - \widetilde{f}(y)|}{|x-y|^\alpha})dx \\ &+ f(y)\int_{\partial\Omega_0} \nabla_y \Gamma(x-y)\cdot \nu dS(x)\end{aligned}$$

Therefore one has that

$$\Delta w(y) = f(y)\int_{\partial\Omega_0} \nabla_y \Gamma(x-y)\cdot \nu dS(x)$$

and taking now $\Omega_0 = B_R(y)$ with R sufficiently large so that $\Omega \subset \Omega_0$ gives

$$\Delta w(y) = f(y)\int_{\partial B_R(y)} \partial_\nu \Gamma(x-y) dS(x) = -f(y)$$

which concludes the proof.

□

Suppose now Ω with all regular boundary points and $g \in C^0(\partial\Omega)$. Let $u_0 : \overline{\Omega} \to R$ the unique solution to

$$\begin{cases} -\Delta u_0 = 0 & \text{in } \Omega \\ u_0 = g - w & \text{on } \partial\Omega \end{cases}$$

where such a function exists by Perron's Theorem since $g - w \in C^0(\partial\Omega)$. Then if $f \in C^{0,\alpha}(\overline{\Omega})$ by linearity $w + u_0$ is the unique solution to the initial problem since

$$\begin{cases} -\Delta(w + u_0) = f & \text{in } \Omega \\ w + u_0 = g & \text{on } \partial\Omega \end{cases}$$

9 Perron Method vs Variational Methods

As it was explained in the introduction, the modern theory of PDE's is widely influenced by the new concept of weak solution. The idea is to work initially in Sobolev Spaces, i.e. "spaces of functions with integrable distributional derivatives", where it is easier to conclude existence and uniqueness of a solution to a boundary value problem, and then try to come back with regularity theory to recover a classical solution out of the weak initial one.

It is then interesting, given the work carried out previously, to discuss how the Perron's result fits together with this modern approach.

To this aim, one has to give a brief introduction to Sobolev Spaces, to construct the proper framework for introducing the weak formulation of the Dirichlet Problem for the Laplace Equation.

9.1 A brief introduction to Sobolev Spaces

Definition 9.1. For $1 \leq p \leq \infty$ the space $W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \quad \forall \alpha \in N^n, |\alpha| \leq k\}$ is called a Sobolev space. For p = 2 the spaces $W^{k,2}(\Omega)$ are also called $H^k(\Omega)$.
On the space $W^{k,p}(\Omega)$ one defines the seminorm

$$|f|_{W^{k,p}(\Omega)} = (\sum_{|\alpha|=k} \int_\Omega |D^\alpha f|^p)^{\frac{1}{p}}$$

and the norm

$$||f||_{W^{k,p}(\Omega)} = (\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha f|^p)^{\frac{1}{p}}$$

Lemma 9.2. The application $||\cdot||_{W^{k,p}(\Omega)}$ is a norm for any $1 \leq p \leq \infty$.

Proof. One has that

$$\begin{aligned} ||f+g||_{k,p} = \quad & (\sum_{|\alpha|=k} ||D^\alpha f + D^\alpha g||^p_{L_p})^{\frac{1}{p}} \\ \leq \quad & (\sum_{|\alpha|=k} (||D^\alpha f||^p_{L_p} + ||D^\alpha g||^p_{L_p}))^{\frac{1}{p}} \qquad (||\cdot||_{L^p} \text{ is a norm}) \\ \leq \quad & (\sum_{|\alpha|=k} ||D^\alpha f||^p_{L_p})^{\frac{1}{p}} + (\sum_{|\alpha|=k} ||D^\alpha g||^p_{L_p})^{\frac{1}{p}}. \end{aligned}$$

□

Lemma 9.3 The spaces $(W^{k,p}(\Omega), ||\cdot||_{k,p})$ are Banach spaces for any $k \in N$ and $1 \leq p \leq \infty$. In particular, the spaces $H^k(\Omega)$ are Hilbert spaces with the inner product

$$< f, g >_{H^k} = \sum_{|\alpha|\leq k} < D^\alpha f, D^\alpha g >_{L^2}$$

Proof. Let $\{f_n\}$ a Cauchy sequence in $W^{k,p}(\Omega)$, i.e. $\forall \epsilon > 0\ \exists N > 0 : ||f_n - f_m||_{k,p} < \epsilon\ \forall n, m \geq N$. Therefore for each $\alpha \in N^n$ with $|\alpha| \leq k$, the sequence $\{D^\alpha f_n\}$ is Cauchy in $L^p(\Omega)$ and hence converges to f_α in $L^p(\Omega)$. Moreover for $\alpha = (0, ..., 0)$ also $f_n \to f$. Indeed one has that

$$\int_\Omega f D^\alpha \phi = \lim_{n\to\infty} \int_\Omega f_n D^\alpha \phi = \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha f_n \phi = (-1)^{|\alpha|} \int_\Omega f_\alpha \phi$$

and therefore $f_\alpha = D^\alpha f$ for all $|\alpha| \leq k$, where above one can exchange limit and integral due to Holder's inequality since $D^\alpha \phi \in L^q(\Omega)$ with q conjugate exponent of p. The same argument applies to show that $\lim_{n\to\infty} \int_\Omega D^\alpha f_n \phi = \int_\Omega f_\alpha \phi$. Therefore $D^\alpha f_n \to D^\alpha f$ in $L^p(\Omega)$ and hence $f_n \to f$ in $W^{k,p}(\Omega)$.

□

Density results

Recall that $C_0^\infty(\Omega)$ and $C_0^\infty(\mathbb{R}^n)$ are dense in $L^p(\Omega)$ for any $p \neq \infty$. The problem intuitively with concluding the same result for $W^{k,p}(\Omega)$ is that the extension by zero of a function outside Ω creates a discontinuity point and hence the distributional derivative of the considered function may not be an $L^p(\mathbb{R}^n)$ function anymore. In fact, this is true only in the trivial case $\Omega = \mathbb{R}^n$.

For this reasons, given that the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ does not coincide with the whole space, the following definition becomes interesting:

Definition 9.4 The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is defined as the space $W_0^{k,p}(\Omega)$, which can be made a Banach space with the induced norm by $W^{k,p}(\Omega)$. In particular the spaces $W_0^{k,2}(\Omega) = H_0^k(\Omega)$ are Hilbert spaces with the induced inner product from $H^k(\Omega)$.

Some particularly important density results are now presented, and although their proof is out of the scope of the elaborate, the idea is always to consider mollifications (with proper translations when needed to avoid discontinuities on $\partial\Omega$) of the initial function.

Theorem 9.5 (Meyers-Serrin Theorem). Let $\Omega \subset \mathbb{R}^n$ be arbitrary and $f \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{f_n\} \in C_0^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $f_n \to f$ in $W^{k,p}(\Omega)$.

Assuming some smoothness on $\partial\Omega$ one can conclude more powerful results:

Definition 9.6 (Segment condition). A domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition if for every $x \in \partial\Omega$ there exists a neighbourhood U_x and a non-zero vector y_x such that $U_{x,1} = \{z + ty_x, z \in U_x \cap \overline{\Omega}, t \in (0,1)\} \subset \Omega$.

Domains with C^1 or Lipschitz continuous boundaries satisfy this condition.

Theorem 9.7 If Ω satisfies the segment condition, then $C_0^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$.

Traces in $W^{k,p}(\Omega)$

Let $\Omega \in \mathbb{R}^n$ be a domain with nonempty bounded and smooth boundary $\partial\Omega$, say of class C^1. Recall that a function $f \in L^p(\Omega)$ is defined only up to zero measure sets (i.e. it is defined only almost everywhere), therefore it is meaningless to talk about the "boundary value" of f since one can modify its value on $\partial\Omega$ without changing the equivalence class which f belongs to. On the other hand, for a function $f \in W^{k,p}(\Omega)$ with $k \geq 1$, it does make sense to talk about the value of f (also called the "Trace") on $\partial\Omega$. The way to proceed is the following: since $C_0^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ by Theorem 9.7, one can find a sequence $f_k \in C_0^\infty(\overline{\Omega})$ such that $f_k \to f$ in $W^{k,p}(\Omega)$. For each f_k, the trace $f_k|_{\partial\Omega}$ is uniquely defined, so that one defines the "trace of f on $\partial\Omega$" as $\lim_{k\to\infty} f_k|_{\partial\Omega}$ in $L^p(\partial\Omega)$. This definition is justified by

Theorem 9.8 Let $\Omega \subset \mathbb{R}^n$ be a domain with a bounded boundary $\partial\Omega$ of class C^1. Then there exists a linear and bounded operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, $1 \leq p < \infty$, s.t. $\tau f = f|_{\partial\Omega}$ for any $f \in C^0(\Omega) \cap W^{1,p}(\Omega)$.

From the previous result, one can give the following characterization of the functions in $H_0^1(\Omega)$:

Proposition 9.9 Let Ω bounded with C^1 boundary and $f \in H^1(\Omega)$. Then $f \in H_0^1(\Omega)$ if and only if $\tau f = 0$.

9.2 Weak Formulation of the Dirichlet-Laplace Problem

Recall that in a classical environment the Dirichlet Problem for the Laplace Equation is

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{31}$$

In the Sobolev Spaces framework, once given the definition of a solution to (31) in a weak sense, one is able to prove existence of solutions (with less regularity) in a much easier way than that of the Perron Method.

Suppose for the moment that there exists $u_g \in H^1(\Omega)$ such that $u_g|_{\partial\Omega} = g$ and consider the function $\overline{u} = u - u_g$.
Consider now $v \in C_0^\infty(\Omega)$, multiplying both sides of (31) by v and integrating on Ω one can write

$$-\int_\Omega \Delta\overline{u} \cdot v + \int_\Omega \Delta u_g \cdot v = 0 \qquad \forall v \in C_0^\infty(\Omega)$$

Assuming now that the right hand side of the previous equation is smooth enough one can integrate by parts to obtain

$$\int_\Omega \nabla \overline{u} \cdot \nabla v = -\int_\Omega \nabla u_g \cdot \nabla v \qquad \forall v \in C_0^\infty(\Omega) \tag{32}$$

where one can extend by density the formulation (32) to $v \in H_0^1(\Omega)$.
This is called the weak formulation to the boundary value problem (31) in $H^1(\Omega)$ and one can prove rather easily that such a variational problem admits a unique solution in $H_0^1(\Omega)$. Before entering this discussion, it is important to recall that the weak formulation associated to a boundary value problem in principle is not unique, so in general it is a little bit of an "art" to decide the most convenient one.

Definition 9.10. A 2-form $a : V \mathrm{x} V \to \mathbb{R}$ is said to be

1. **bilinear** if $a(\alpha u + \beta w, v) = \alpha a(u, v) + \beta a(w, v)$ $\forall u, v, w \in V,\ \alpha, \beta \in \mathbb{R}$
2. **bounded** if $\exists M > 0$: $|a(u, v)| \leq M||u||_V ||v||_V$
3. **coercive** if $\exists \alpha > 0$: $a(u, u) \geq \alpha ||u||_V^2$.

The following result is a standard tool when dealing with Sobolev Spaces with an Hilbert structure to conclude well-posedness of the weak formulations like the one above:

Theorem 9.11 (Lax-Milgram Theorem). Let V an Hilbert space and $a : V \mathrm{x} V \to R$ a bilinear, bounded and coercive form on V. Then for any $F \in V^*$ the abstract problem

$$\text{find } u \in V \text{ such that} \quad a(u, v) = F(v) \quad \forall v \in V \tag{33}$$

has a unique solution $u \in V$ such that $||u||_V \leq \frac{||F||_{V^*}}{\alpha}$, being α the coercivity constant of the bilinear form $a(\cdot, \cdot)$.

Given this result, one can easily conclude that the weak formulation (32) is well posed in $H_0^1(\Omega)$ when such a space, as a consequence of the well known Poincaré's Inequality, is endowed with the norm $||\, |\nabla u|\, ||_{L^2(\Omega)}$. In fact

$$|a(u, v)| = |\int_\Omega \nabla \overline{u} \cdot \nabla v| \leq ||\overline{u}||_{H_0^1(\Omega)} ||v||_{H_0^1(\Omega)}$$

$$a(u, u) = \int_\Omega |\nabla \overline{u}|^2 = ||\overline{u}||_{H_0^1(\Omega)}^2$$

$$|F(v)| = |\int_\Omega \nabla u_g \cdot \nabla v| \leq ||\nabla u_g||_{L^2(\Omega)} ||v||_{H_0^1(\Omega)}$$

One can conclude by Lax-Milgram Theorem that there exists a unique solution $\overline{u}$ to the problem

$$\text{find } \overline{u} \in H_0^1(\Omega) \text{ such that} \quad \int_\Omega \nabla \overline{u} \cdot \nabla v = -\int_\Omega \nabla u_g \cdot \nabla v \quad \forall v \in H_0^1(\Omega)$$

which satisfies

$$||\overline{u}||_{H_0^1(\Omega)} \leq ||\nabla u_g||_{L^2}$$

Hence $u = \overline{u} + u_g$ is a solution in $H^1(\Omega)$ of the boundary value problem (31) such that

$$||u||_{H^1(\Omega)} \leq ||u_g||_{H^1(\Omega)}$$

Now one must address for which g this is eventually the case: supposing that $\partial\Omega \in C^1$, in particular g must be the trace of a $H^1(\Omega)$ function. Since the trace operator τ is not surjective, one has to define the space

$$H^{1/2}(\Omega) = \tau(H^1(\Omega))$$

i.e. the image of H^1 under τ. Such a space can be made a Banach space when it is endowed with the norm

$$||g||_{H^{1/2}(\Omega)} = \inf_{w \in H^1(\Omega), \tau w = g} ||w||_{H^1(\Omega)}$$

One can indeed prove that $H^{1/2}(\Omega)$ is dense in $L^2(\partial\Omega)$, but in principle not every function in $L^2(\partial\Omega)$ is the image of a $H^1(\Omega)$ function under τ, so one has to require $\partial\Omega \in C^1$ and $g \in H^{1/2}(\Omega)$ in order for the problem to be well-posed.

9.3 Comparison between the two methods

In this section, one shows that the above assumptions on g and on $\partial\Omega$ are indeed not enough in order to recover a classical solution in any dimension.
Recall that Perron's Theorem states that given any $g \in C^0(\partial\Omega)$, a classical solution exists and is unique as long as all boundary points are regular. On the other hand, the weak formulation above proved that a solution exists in $H^1(\Omega)$ as long as $\partial\Omega \in C^1$ and $g \in H^{1/2}(\Omega)$. One is now interested in looking for a connection between the two results, if such a connection actually exists.

In trying to go back to a classical environment with the solution obtained in $H^1(\Omega)$ some results can be concluded using basic embedding theory and regularity results from L^p-theory. Some easy stability estimates can be immediately derived as particular cases of the following well known and more general theorems:

Theorem 9.12 (Rellich-Kondrachov Theorem). Let Ω bounded with $\partial\Omega \in C^1$. Then

- if $n > 2$, then $H^1(\Omega) \hookrightarrow L^q(\Omega) \qquad \forall q \in [1, \frac{2n}{n-2}]$
- if $n = 1$, then $H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$
- if $n = 2$, then $H^1(\Omega) \hookrightarrow L^q(\Omega) \qquad \forall q \in [1, \infty)$.

This implies as a first easy result that as long as $n = 1$ the weak solution is continuous up to the boundary.

Theorem 9.13 (General Sobolev Inequalities). Let Ω a domain with $\partial\Omega \in C^1$ and assume $u \in W^{k,p}(\Omega)$. Then

1. if
$$k < \frac{n}{p}$$
then $u \in L^q(\Omega)$ for
$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

2. if
$$k > \frac{n}{p}$$
then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})$ where
$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1 & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

This implies in particular that, for $p = 2$ the solution $u \in H^k(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$ if $k \geq [\frac{n}{2}] + 1$. This means that in order for the solution to be continuous up to the boundary, to an increase in the dimension of the domain must correspond an increase on k. The needed regularity is further achieved by means of

Theorem 9.14. Let $\partial\Omega \in C^1$. Then the Dirichlet Problem for the Laplace Equation admits a unique solution in $H^1(\Omega)$ and

1. **Interior regularity:** $u \in H^k_{loc}(\Omega)$ for every $k \in \mathbb{N}$.

2. **Global regularity:** If $\partial\Omega \in C^{k+2}$ and $g \in H^{k+3/2}(\Omega)$, then $u \in H^{k+2}(\Omega)$.

Since by the interior regularity result $u \in H^k_{loc}(\Omega)$, then
$$u \in C^m(\overline{\Omega}') \quad \forall \Omega' \subset\subset \Omega, \quad \text{with } k > \frac{n}{2} \text{ and } m = k - [\frac{n}{2}] - 1$$

The previous property holding for any $k \in \mathbb{N}$ and for any $\Omega' \subset\subset \Omega$, we may conclude that $u \in C^\infty(\Omega)$ without any restriction on the dimension on the domain. This is a very important result, but one cannot claim yet that the solution is a classical one, as one needs continuity of u up to the boundary to conclude.
By the global regularity estimate, requiring that $\partial\Omega \in C^{k+2}(\Omega)$ and $g \in H^{k+3/2}(\partial\Omega)$ (i.e. the trace of a $H^{k+2}(\Omega)$ function) then
$$u \in H^{k+2}(\Omega) \subset C^m(\overline{\Omega}) \quad \text{with } 2(k+2) > n \text{ and } m = k + 1 - [\frac{n}{2}]$$

If one wants $u \in C^0(\overline{\Omega})$ then $m \geq 0$ means that $k \geq [\frac{n}{2}] - 1$. For example

• for $n = 2$ and $n = 3$ one needs $k \geq 0$ so the minimum requirements are $\partial\Omega \in C^2$ and $g \in H^{3/2}(\Omega)$.

• for $n = 4$ and $n = 5$ one needs $k \geq 1$ so the minimum requirements are $\partial\Omega \in C^3$ and $g \in H^{5/2}(\Omega)$.

The conclusion is the following: by Variational Methods

$$\begin{cases} \partial\Omega \in C^{k+2} \\ g \in H^{k+3/2}(\Omega) \\ k \geq [\frac{n}{2}] - 1 \end{cases} \implies u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$$

meanwhile by the Perron's Theorem and the consequent analysis in Chapter 7

$$\begin{cases} \partial\Omega \in C^2 \\ g \in C^0(\partial\Omega) \\ \text{for every } n \in \mathbb{N} \end{cases} \implies u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$$

With the very simple analysis above, which does not presume to be completely exhaustive, it becomes clear that when one tries to gain on the dimension of Ω the hypotheses become more stringent both with respect to Perron's theorem and with respect to what is necessary to make the weak formulation (32) well posed: this is already the case for $n = 2$ or $n = 3$, where most of the time the applications are set, with the assumptions on g and on the regularity of $\partial\Omega$. An important result is that at least on the interior of the domain the solution is infinitely differentiable in any dimension, i.e. it is harmonic and smoothly behaving as it was concluded for any classical solution to (31) by Theorem 3.4.

On the other hand, Perron's Theorem requires a mild condition of regularity on the boundary of the domain (but recall that already $\partial\Omega \in C^2$ is sufficient and a large class of domains satisfies this condition) and for every $g \in C^0(\partial\Omega)$ the problem is well-posed without any restriction on the dimension. Clearly, as long as one is looking to obtain a solution in a classical sense, Perron's Theorem gives a much more powerful statement, and the idea of the method is very simple and intuitive as it was illustrated in Section 7.3, the main concern being just the high level of technicality needed by the framework of subfunctions and convergence results for sequences of harmonic functions.
The thing is that just like the Perron Method easily generalizes to other classes of boundary value problems of the second order, variational methods are much more elastic and ductile and in many applications one can conclude very important results also with functions which are just in $L^q(\Omega)$ for some $q > 2$, i.e. not necessarily classical solutions; this is why embedding results such as the Rellich-Kondrachov Theorem or the General Sobolev Inequalities are very important and the modern theory is pushing towards a weak solutions and Sobolev Spaces

type of approach.

Nevertheless, Perron's Theorem is a very important tool of the Partial Differential Equations theory that is able to return under very general (and not very limiting) assumptions a classical solution to the Dirichlet Problem for the Laplace Equation on a bounded domain in any dimension, although the formula of the solution is not an explicit one. In the end, it is important to recall that Perron's Theorem dates back to 1923, before all the theory of Sobolev Spaces was developed and studied properly. At the time, this was an astonishingly important and revolutionary result, which certainly started a more serious treatment of many other boundary value problems, with the aim of recovering all the necessary regularity on a solution perhaps with weak assumptions on the data: a very first example was presented in Section 8. This treatment is nowadays going on and sees Perron's Theorem as one of the most important results which gives, without any constraints on the dimension, a necessary and sufficient condition for the existence of a classical solution to the Dirichlet Problem for the Laplace Equation on a bounded domain.

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