Primal and mixed formulation for the linear elasticity problem

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1 Theoretical results

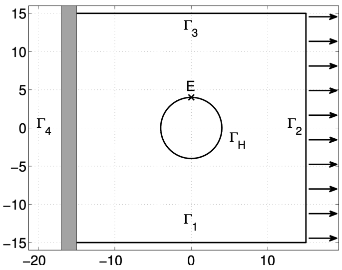


Figure 1: Plate with a hole

We consider as our domain Ω a plate of side length $L = 30$ with a central hole of radius $R = 4$. The plate is assumed elastic and weightless, its left end is clamped to a rigid wall, its right end is pulled by a constant force $\mathbf{f}$ and no force is present on the rest of the boundary. Our goal is to compute the displacement of the plate from its rest position; this problem can be reduced to solving the well known Lamé equation

$$\begin{cases} -\text{div}\,\sigma(\mathbf{u}) = 0 \quad \text{in } \Omega \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1,\, \Gamma_3,\, \Gamma_H \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{f} \quad \text{on } \Gamma_2 \\ \mathbf{u} = 0 \quad \text{on } \Gamma_4 \end{cases} \tag{1}$$

where the Cauchy tensor $\sigma(\mathbf{u})$ is defined as

$$\sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda Tr(\epsilon(\mathbf{u}))I, \quad \epsilon(\mathbf{u}) = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2}. \tag{2}$$

Weak formulation

Before stating the weak formulation associated to the problem, we aim at finding suitable Hilbert spaces and norms. It is a standard Sobolev Spaces theory result (see for example [2] or [3]) that if we require a function to be vanishing only on a positive measure subset of $\partial\Omega$ a Poincaré type inequality still holds. This implies in particular that for any $\mathbf{u} \in [H^1_{\Gamma_4}(\Omega)]^d = \{\mathbf{u} \in [H^1(\Omega)]^d : \tau_{\Gamma_4}\mathbf{u} = 0\}$ the norms $||\cdot||_{[H^1(\Omega)]^d}$ norm and $|\cdot|_{[H^1(\Omega)]^d}$ are indeed equivalent. However we decide for our problem to choose the former as the norm in $[H^1_{\Gamma_4}(\Omega)]^2$, which we denote as V for the rest of the work.

The weak formulation associated to problem (1) is constructed as follows: since V is clearly a Hilbert space as it is a closed subspace of $[H^1(\Omega)]^2$, the relation

$$-\int_\Omega \text{div}(\sigma(\mathbf{u})) \cdot \mathbf{v} = \int_\Omega \sigma(\mathbf{u}) : \nabla\mathbf{v} - \int_{\partial\Omega} (\sigma(\mathbf{u}) \cdot \mathbf{n}) \cdot \mathbf{v} \quad \forall \mathbf{v} \in V$$

directly leads to the following variational problem:

$$\text{find } \mathbf{u} \in V \text{ s.t.} \quad a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{3}$$

with

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \underline{\underline{\epsilon}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\mathbf{v}) + \lambda \int_\Omega \text{div}\mathbf{u}\,\text{div}\mathbf{v},$$

$$F(\mathbf{v}) = \int_{\Gamma_2} \mathbf{f} \cdot \mathbf{v}.$$

Using the Lax-Milgram Lemma and exploiting the Korn and Poincarè inequalities we easily conclude well posedness of this problem since

$$
\begin{aligned}
|a(\mathbf{u},\mathbf{v})| &\leq \max\{2\mu,\lambda\}||\mathbf{u}||_V||\mathbf{v}||_V, \\
a(\mathbf{u},\mathbf{u}) &\geq 2\mu||\underline{\underline{\epsilon}}(\mathbf{u})||^2_{L^2(\Omega)} \geq \frac{2\mu\kappa}{1+C_\Omega^2}||\mathbf{u}||^2_V, \\
|F(\mathbf{v})| &\leq ||\mathbf{f}||_{[L^2(\partial\Omega)]^2}C_\tau||\mathbf{v}||_V,
\end{aligned}
$$

where κ is the Korn's inequality constant and for the continuity of $F(\mathbf{v})$ we used the continuity of the trace operator $\tau : [H^1(\Omega)]^2 \mapsto [L^2(\partial\Omega)]^2$. We will call α the coercivity constant of $a(\cdot,\cdot)$ for the rest of the work. We thus conclude that there exists a unique solution in V to (3) and always by Lax-Milgram we also obtain continuous dependence (independently of λ) on the boundary stress $\mathbf{f}$ by means of

$$||\mathbf{u}||_V \leq \frac{C_\tau}{\alpha}||\mathbf{f}||_{[L^2(\partial\Omega)]^2}. \tag{4}$$

Note that the coercivity constant depends on μ so for $\mu \to \infty$ $\mathbf{u}$ is vanishing.

Convergence estimates

In order to formulate the discrete problem, let us introduce $V_h \subset V$ that is a closed subspace of V. The discrete problem associated to (3) reads:

$$\text{find } \mathbf{u_h} \in V_h \text{ s.t. } a(\mathbf{u_h},\mathbf{v_h}) = F(\mathbf{v_h}) \qquad \forall \mathbf{v_h} \in V_h,$$

with

$$
\begin{aligned}
a(\mathbf{u_h},\mathbf{v_h}) &= 2\mu\int_\Omega \underline{\underline{\epsilon}}(\mathbf{u_h}) : \underline{\underline{\epsilon}}(\mathbf{v_h}) + \lambda\int_\Omega \mathrm{div}\mathbf{u_h}\,\mathrm{div}\mathbf{v_h}, \\
F(\mathbf{v_h}) &= \int_{\Gamma_2} \mathbf{f}\cdot\mathbf{v_h}.
\end{aligned}
$$

Since V_h is a closed subspace of V, the bilinear form $a(\cdot,\cdot)$ remains continuous and coercive on V_h with the same continuity and coercivity constants obtained in the continuous case. Thus, the problem is well-posed: let $\mathbf{u_h}$ be its unique solution. Note first that $\mathbf{u}-\mathbf{u_h}$ satisfies the *Galerkin orthogonality*

$$a(\mathbf{u}-\mathbf{u_h},\mathbf{v_h}) = 0 \quad \forall \mathbf{v_h} \in V_h. \tag{5}$$

Then, using (5), we derive easily that $\forall \mathbf{v_h} \in V_h$

$$\alpha||\mathbf{u_h}-\mathbf{v_h}||^2_V \leq a(\mathbf{u_h}-\mathbf{v_h},\mathbf{u_h}-\mathbf{v_h}) \underset{(5)}{=} a(\mathbf{u}-\mathbf{v_h},\mathbf{u_h}-\mathbf{v_h}) \leq M||\mathbf{u}-\mathbf{v_h}||_V||\mathbf{u_h}-\mathbf{v_h}||_V, \tag{6}$$

where $M = \max\{2\mu,\lambda\}$ is the continuity constant of $a(\cdot,\cdot)$. Since $\mathbf{v_h}$ was not characterized during the previous computation, we can consider without loss of generality the infimum over all $\mathbf{v_h} \in V_h$, and a simple triangular inequality yields

$$||\mathbf{u}-\mathbf{u_h}||_V \leq \left(1+\frac{M}{\alpha}\right)\inf_{\mathbf{v_h}\in V_h}||\mathbf{u}-\mathbf{v_h}||_V. \tag{7}$$

The bound (7) relates the error on the discrete solution to the best interpolation fit in V_h. Note that for $\lambda \to \infty$ the error on the solution may be large even if $\inf_{\mathbf{v_h}\in V_h}||\mathbf{u}-\mathbf{v_h}||_V$ is small.

Now, let τ_h be a triangulation of Ω and let us introduce the subspace $V_h = \{\mathbf{v_h} \in V : \mathbf{v_h}|_K \in \mathbb{P}_2(K), \forall K \in \tau_h\}$, namely the space of piecewise quadratic finite elements. Assuming for the moment that the solution $\mathbf{u}$ is smooth enough (and in particular that $\mathbf{u} \in (C^0(\overline{\Omega}))^2$), we can take $\mathbf{v_h}$ in equation (7) to be the standard interpolation operator $I_h(\mathbf{u})$ such that $I_h(\mathbf{u}(a_j)) = \mathbf{u}(a_j)$ at each node a_j of the mesh τ_h. Then $\exists C > 0$ independent of the meshsize h such that

$$
\begin{aligned}
||\mathbf{u}-I_h(\mathbf{u})||_{[L^2(\Omega)]^2} &\leq Ch^3|\mathbf{u}|_{[H^3(\Omega)]^2}, \\
||\nabla(\mathbf{u}-I_h(\mathbf{u}))||_{[L^2(\Omega)]^2} &\leq Ch^2|\mathbf{u}|_{[H^3(\Omega)]^2},
\end{aligned} \tag{8}
$$

with C independent of h. Inserting this result in (7), we obtain

$$||\mathbf{u}-\mathbf{u_h}||_V \leq \left(1+\frac{M}{\alpha}\right)Ch^2|\mathbf{u}|_{[H^3(\Omega)]^2}. \tag{9}$$

We aim now at deriving also an error estimate in $[L^2(\Omega)]^2$. To do this we will take ispiration from the Aubin-Nitsche trick. Let $\mathbf{e_h} = \mathbf{u} - \mathbf{u_h}$, and consider the *dual problem*

$$\text{find } \phi \in V \text{ such that} \quad a(\mathbf{v}, \phi) = \int_\Omega \mathbf{e_h} \cdot \mathbf{v} \quad \forall \mathbf{v} \in V, \tag{10}$$

which corresponds to a slight modification of the initial problem (1) where $\mathbf{f} = 0$ and we impose $-\mathrm{div}\, \sigma(\mathbf{u}) = \mathbf{e_h}$. (10) can be easily proved to be well-posed (since $\mathbf{e_h} \in [L^2(\Omega)]^2$) with $||\phi||_V \leq \frac{1}{\alpha}||\mathbf{e_h}||_{L^2}$. Since $\mathbf{e_h} \in [L^2(\Omega)]^2$ and $\mathbf{v} \in [L^2(\Omega)]^2$, we then have

$$||\mathbf{e_h}||^2_{[L^2(\Omega)]^2} = a(\mathbf{e_h}, \phi) = \underbrace{a(\mathbf{e_h}, \phi - \mathbf{w_h})}_{\text{Galerkin orthogonality}} \leq M||\mathbf{e_h}||_V \inf_{\mathbf{w_h} \in V_h} ||\phi - \mathbf{w_h}||_V,$$

where M is still the continuity constant of $a(\cdot, \cdot)$. If we assume for the moment that $\phi \in [H^2(\Omega)]^2$ with $||\phi||_{[H^2(\Omega)]^2} \leq C||\mathbf{e_h}||_{[L^2(\Omega)]^2}$, then we have

$$\inf_{\mathbf{w_h} \in V_h} ||\phi - \mathbf{w_h}||_V \leq Ch|\phi|_{[H^2(\Omega)]^2}.$$

Thus

$$||\mathbf{e_h}||^2_{[L^2(\Omega)]^2} \leq M||\mathbf{e_h}||_V Ch|\phi|_{[H^2(\Omega)]^2} \leq \tilde{C}h||\mathbf{e_h}||_V||\mathbf{e_h}||_{[L^2(\Omega)]^2},$$

and plugging in (9) we obtain

$$||\mathbf{u} - \mathbf{u}_h||_{[L^2(\Omega)]^2} \leq \hat{C}h^3|u|_{[H^3(\Omega)]^2}, \tag{11}$$

for some constant $\hat{C} = \hat{C}(M, \alpha)$. We see that in both the estimates (9) and (11), if the compressibility of the material doesn't hold true ($\lambda \to \infty$), we may have very poor accuracies.

Error estimates on functionals of the solution

We aim now at estimating the errors committed in the numerical computation of the two quantities

$$Q_1(\mathbf{u}) = \frac{1}{|\Gamma_2|}\int_{\Gamma_2} u_1, \quad Q_2(\mathbf{u}) = u(0, R). \tag{12}$$

We follow the same trick as before, imposing the RHS of (10) to be our functional Q. That is, we introduce the dual problem

$$\text{find } \phi \in V \text{ such that} \quad a(\mathbf{v}, \phi) = Q(\mathbf{v}) \quad \forall \mathbf{v} \in V. \tag{13}$$

Then it holds that

$$Q(\mathbf{u}) - Q(\mathbf{u_h}) = Q(\mathbf{u} - \mathbf{u_h}) = a(\mathbf{u} - \mathbf{u_h}, \phi) = \underbrace{a(\mathbf{u} - \mathbf{u_h}, \phi - \mathbf{w_h})}_{\text{Galerkin orthogonality}} \leq M||\mathbf{u} - \mathbf{u_h}||_V \inf_{\mathbf{w_h} \in V_h} ||\phi - \mathbf{w_h}||_V,$$

where again one is using (5). Depending on the smoothness of the solution ϕ we obtain the desired estimates. We present here the case of Q_1, then a similar analysis will apply to the case of Q_2 (for which we need at least $\mathbf{u} \in [H^2(\Omega)]^2$ otherwise $Q_2(u)$ may not even be well defined). The well-posedness is again straightforward to verify: indeed, (13) with $Q = Q_1$ corresponds to our initial problem (3) with $\mathbf{f} = (\frac{1}{|\Gamma_2|}, 0)^T$. The same analysis that led to (4) yields immediately that $||\phi_1||_V \leq \frac{C_\tau}{\alpha\sqrt{|\Gamma_2|}}$. Given the observation above, it is now reasonable to assume the same degree of regularity on $\mathbf{u}$ and ϕ_1. Hence, if we have that $\mathbf{u} \in [H^3(\Omega)]^2$ and $\phi_1 \in [H^3(\Omega)]^2$, it holds that

$$|Q(\mathbf{u}) - Q(\mathbf{u_h})|_V \leq Ch^4|\mathbf{u}|_{[H^3(\Omega)]^2}|\phi_1|_{[H^3(\Omega)]^2},$$

Note that this corresponds to a doubled convergence rate with respect to the error estimate $||\mathbf{u} - \mathbf{u_h}||_V$ independently of the regularity of the solution. However, also in this case, the dependence of C on λ may compromise the quality of our approximation in the incompressible limit. The regularity of the solution to (13) in the case $Q = Q_2$ is more delicate as (RHS) corresponds to the case in which we apply a boundary stress equal to δ_E which is clearly not smooth. We refer to the last section for the numerical results concerning this part.

Mixed formulation

As we saw in the previous section, multiple times for $\lambda \to \infty$ the error estimates may be compromised even if the interpolation fit is good enough. To control this incompressible limit in principle one should impose $\text{div}(\mathbf{u}) = 0$ for the second term of $a(\cdot,\cdot)$ to be bounded. In practice, λ is not $+\infty$ but just really large. Still, to control this nearly incompressible case, we take inspiration from the incompressible one and we rewrite the weak formulation (3) in the form of a saddle point problem by introducing a *pressure* $p = -\lambda \text{div}(\mathbf{u}) \in L^2(\Omega)$. We denote the space $L^2(\Omega)$ as Q for the rest of the work. Then the mixed formulation reads

$$\text{find } (\mathbf{u}, p) \in V \text{ x } Q \text{ such that} \qquad \begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = F(\mathbf{v}) & \forall \mathbf{v} \in V \\ b(\mathbf{u},q) - c(p,q) = 0 & \forall q \in Q \end{cases} \tag{14}$$

where

$$a(\mathbf{u},\mathbf{v}) = 2\mu \int_\Omega \underline{\underline{\epsilon}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\mathbf{v}), \quad b(\mathbf{v},p) = -\int_\Omega p \, \text{div}\mathbf{v},$$
$$c(p,q) = \frac{1}{\lambda}\int_\Omega p\, q, \quad F(\mathbf{v}) = \int_{\Gamma_2} \mathbf{f}\cdot\mathbf{v}.$$

It is not difficult to interpret this mixed compressible formulation as an $\epsilon = \frac{1}{\lambda}$ perturbation of the incompressible formulation for which we impose $b(\mathbf{u},q) = 0 \; \forall q \in Q$. If we introduce now the coupled space $W = V$ x Q endowed with the norm $||(\mathbf{u},p)||_W = (||\mathbf{u}||_V^2 + ||p||_Q^2)^{1/2}$, then we can write a global weak formulation: we aim at finding $(\mathbf{u},p) \in W$ such that

$$A((\mathbf{u},p),(\mathbf{v},q)) = F(\mathbf{v}), \tag{15}$$

where $A((\mathbf{u},p),(\mathbf{v},q)) = a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + b(\mathbf{u},q) - c(p,q)$.

Theorem 1.1. *The mixed formulation (14) admits a unique solution if the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are continuous and moreover*

$$\exists\alpha > 0 : a(\mathbf{u},\mathbf{u}) \geq \alpha||\mathbf{u}||_V^2 \quad \forall \mathbf{u} \in Ker\{b\} = \{\mathbf{u} \in V : b(\mathbf{u},q) = 0 \quad \forall q \in Q\} \tag{16}$$

$$\exists\beta > 0 : \sup_{\mathbf{v}\in V} \frac{b(\mathbf{v},q)}{||\mathbf{v}||_V} \geq \beta||q||_Q \quad \forall q \in Q \tag{17}$$

The continuity of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ (we denote for the rest of the work M and γ as the respective continuity constant) and $c(\cdot,\cdot)$ is trivially satisfied. Moreover, we also need coercivity of $a(\cdot,\cdot)$ on the kernel of b and the *LBB condition* to be satisfied. While the coercivity holds globally and (16) is thus trivially satisfied, we now show the validity of a global inf-sup condition.

Lemma 1.2. *There exists $\beta > 0$ such that*

$$\sup_{v\in V} \frac{b(\mathbf{v},q)}{||\mathbf{v}||_V} \geq \beta||q||_Q \quad \forall q \in Q. \tag{18}$$

Proof. Given $q \in L^2(\Omega)$, consider the elliptic problem

$$\begin{cases} -\Delta\psi = q & \text{in } \Omega \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_4 \\ \psi = 0 & \text{on } \partial\Omega\backslash\Gamma_4 \end{cases} \tag{19}$$

Problem (19) can be easily shown to admit a unique solution ψ in $S = H^1_{\partial\Omega\backslash\Gamma_4}(\Omega)$. Indeed the weak formulation reads

$$\int_\Omega \nabla\psi\cdot\nabla v = \int_\Omega qv \quad \forall v \in S,$$

so that $a(\psi,v) = \langle\psi,v\rangle_V$ is clearly continuous and coercive with coercivity constant $\alpha = 1$. Hence we have $||\psi||_{H^1(\Omega)} \leq C_\Omega||q||_{L^2(\Omega)}$, C_Ω being the Poincaré constant. Due to elliptic regularity it is a well known result (see for example [3]) that $\psi \in H^2(\Omega)$ with

$$||\psi||_{H^2(\Omega)} \leq \hat{C}||q||_{L^2(\Omega)} \tag{20}$$

for some constant $\hat{C} > 0$. If we set $\tilde{\mathbf{v}} = \nabla\psi$, by (20) we directly obtain that $\tilde{\mathbf{v}} \in [H^1(\Omega)]^2$ and $\tilde{\mathbf{v}}\cdot n = 0$ on Γ_4. To take care of the tangential component, we invoke the trace theorem in $H^2(\Omega)$

(see for example [1] or [4]) to conclude the existence of $\phi \in H^2(\Omega)$ such that $\nabla\psi = -\text{curl}\,\phi$ on $\partial\Omega$ with $||\phi||_{H^2(\Omega)} \lesssim ||\tilde{\mathbf{v}}||_V$. Let now $\hat{\mathbf{v}} = \text{curl}\,\phi$. Then the function $\mathbf{v_q}$ defined as $\mathbf{v_q} = \tilde{\mathbf{v}} + \hat{\mathbf{v}}$ in $\Omega \cup \Gamma_4$, $\mathbf{v_q} = \tilde{\mathbf{v}}$ on $\partial\Omega \backslash \Gamma_4$ is an element of V with $-\text{div}\,\mathbf{v_q} = q$ (since $\hat{\mathbf{v}}$ is divergence free). Therefore we can conclude that

$$\sup_{v \in V} \frac{b(\mathbf{v}, q)}{||\mathbf{v}||_V} \underset{\mathbf{v}=\mathbf{v_q}}{\geq} \frac{b(\mathbf{v_q}, q)}{||\mathbf{v_q}||_V} \underset{-\text{div}\,\mathbf{v_q}=q}{\geq} \frac{||q||_Q^2}{C||q||_Q} = \frac{1}{C}||q||_Q$$

for some constant $C > 0$. The claim then follows by taking $\beta = \frac{1}{C}$ and (18) is proved. □

Another way to prove (18) is the following: given $q \in L^2(\Omega)$, consider the elliptic problem

$$\begin{cases} -\Delta\psi = q & \text{in } \Omega \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_4 \\ \frac{\partial\psi}{\partial\tau} = 0 & \text{on } \Gamma_4 \\ \psi = 0 & \text{on } \partial\Omega\backslash\Gamma_4 \end{cases} \tag{21}$$

Note that since $\psi = 0$ on $\partial\Omega\backslash\Gamma_4$, we have coherently that

$$0 = \int_{\Gamma_4} \frac{\partial\psi}{\partial\tau} = \psi(B) - \psi(A),$$

where A and B denote the two vertices of Γ_4. Hence, (21) is well posed in S. Now, the estimate (20) holds also in this case and hence the element we seek for in V is simply given by $\tilde{\mathbf{v}} = \nabla\psi$.

As a direct corollary, we conclude well-posedness of the mixed formulation (14) in V x Q. Let us now derive an estimate on the norm of the solution $(\mathbf{u}, p)$ to (15). Note first of all that due to the inf-sup condition

$$\beta||p||_Q \leq \sup_{v\in V} \frac{b(\mathbf{v}, p)}{||\mathbf{v}||_V} = \sup_{v\in V} \frac{F(\mathbf{v}) - a(\mathbf{u}, \mathbf{v})}{||\mathbf{v}||_V},$$

so it directly follows that

$$\beta||p||_Q \leq ||F||_{V^*} + M||\mathbf{u}||_V, \tag{22}$$

where V^* denotes the dual space of V. Let now $\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^\perp$ with $\mathbf{u}^0 \in V_0$ and $\mathbf{u}^\perp \in V_0^\perp$, where $V_0 = \{\mathbf{u} \in V : b(\mathbf{u}, q) = 0 \ \forall q \in Q\} = Ker\{b\}$. We have that

$$\alpha||\mathbf{u}^0||_V^2 \leq a(\mathbf{u}^0, \mathbf{u}^0) = F(\mathbf{u}^0) - a(\mathbf{u}^\perp, \mathbf{u}^0) \implies ||\mathbf{u}^0||_V \leq \frac{1}{\alpha}(||F||_V^* + M||\mathbf{u}^\perp||_V).$$

For $\mathbf{u}^\perp$ we find, in the same fashion, that since $\mathbf{u}^0 \in V_0$

$$b(\mathbf{u}, q) = b(\mathbf{u}^\perp, q) = \frac{1}{\lambda}\langle p, q\rangle_Q \leq \frac{1}{\lambda}||p||_Q||q||_Q.$$

The reversed inf-sup condition on $b(\cdot,\cdot)$ then yields

$$\beta||\mathbf{u}^\perp||_V \leq \sup_{q\in Q} \frac{b(\mathbf{u}^\perp, q)}{||q||_Q} \leq \frac{1}{\lambda}||p||_Q \implies ||\mathbf{u}^\perp||_V \leq \frac{1}{\beta\lambda}||p||_Q.$$

We can now combine the estimates obtained above to conclude that

$$\beta||p||_Q \leq (1 + \frac{M}{\alpha})||F||_V^* + (\frac{M^2}{\alpha\beta\lambda} + \frac{M}{\beta\lambda})||p||_Q, \tag{23}$$

and thus

$$(\beta - \frac{M^2}{\alpha\beta\lambda} - \frac{M}{\beta\lambda})||p||_Q \leq (1 + \frac{M}{\alpha})||F||_{V^*}.$$

Note that for λ large enough and since $\beta > 0$ we can say that $(\beta - \frac{M^2}{\alpha\beta\lambda} - \frac{M}{\beta\lambda}) \geq \delta > 0$ to conclude that

$$||p||_Q \leq \frac{1 + \frac{M}{\alpha}}{\delta}||F||_{V^*}. \tag{24}$$

Now by using a simple triangular inequality

$$||\mathbf{u}||_V \leq ||\mathbf{u}^0||_V + ||\mathbf{u}^\perp||_V \leq \frac{1}{\alpha}||F||_{V^*} + (1 + \frac{M}{\alpha})\frac{1}{\beta\lambda}||p||_Q,$$

and plugging in (24) yields

$$||\mathbf{u}||_V \leq \frac{1}{\alpha}||F||_{V^*} + (1+\frac{M}{\alpha})^2\frac{1}{\beta\lambda\delta}||F||_{V^*}. \tag{25}$$

We can thus conclude that

$$||(\mathbf{u},p)||_W \leq ||\mathbf{u}||_V + ||p||_Q \leq \frac{1}{\alpha}||F||_{V^*} + (1+\frac{M}{\alpha})^2\frac{1}{\beta\lambda\delta}||F||_{V^*} + \frac{1+\frac{M}{\alpha}}{\delta}||F||_{V^*}. \tag{26}$$

Using the fact that $||F||_V^* \leq C_\tau ||\mathbf{f}||_{[L^2(\partial\Omega)]^2}$, where C_τ is the usual continuity constant of the trace operator τ, we obtain again continuous dependence on the boundary traction. In the incompressible limit $\lambda \to \infty$ the solution $\mathbf{u}$ to (14) belongs to Ker $\{b\}$, so $\mathbf{u}^\perp = 0$ and equation (26) reads

$$||(\mathbf{u},p)||_W \leq ||\mathbf{u}||_V + ||p||_Q \leq \frac{1}{\alpha}||F||_{V^*} + \frac{1}{\beta}(1+\frac{M}{\alpha})||F||_{V^*}, \tag{27}$$

so continuous dependence on $\mathbf{f}$ is preserved, as for the primal formulation (3). Note in particular that introducing the dual formulation we lose dependence of M on λ, and this is crucial for this last claim to hold.

Taylor-Hood Finite Elements

In order to construct a finite element approximation of the solution to (15), let us introduce the finite element spaces $X_h^2 = \{\mathbf{v_h} \in C^0(\overline{\Omega}) : \mathbf{v_h}|_K \in \mathbb{P}_2(K)\, \forall K \in \tau_h\}$ and $X_h^1 = \{q_h \in C^0(\overline{\Omega}) : q_h|_K \in \mathbb{P}_1(K)\, \forall K \in \tau_h\}$. Then the Taylor-Hood elements correspond to the choice of $V_h = X_h^2 \cap V$ for the displacement $\mathbf{u}$ and $Q_h = X_h^1 \cap Q$ for the pressure p. We aim at obtaining error estimates. Let us assume that the solution $(\mathbf{u_h}, p_h) \in V_h$ x Q_h exists, namely the conditions of Theorem 1.1 are holding also at the discrete level and $\exists \alpha^*, \beta^* > 0$ such that $\alpha_h \geq \alpha^*$ and $\beta_h \geq \beta^*$. Then for any $\mathbf{u_I} \in V_h, p_I \in Q_h$ the couple $(\mathbf{u_h} - \mathbf{u_I}, p_h - p_I)$ solves

$$\begin{cases} a(\mathbf{u_h}-\mathbf{u_I},\mathbf{v_h}) + b(\mathbf{v_h}, p_h - p_I) = a(\mathbf{u}-\mathbf{u_I},\mathbf{v_h}) + b(\mathbf{v_h}, p-p_I) \\ b(\mathbf{u_h}-\mathbf{u_I}, q_h) - c(p_h-p_I,q_h) = b(\mathbf{u}-\mathbf{u_I},q_h) - c(p-p_I,q_h) \end{cases} \tag{28}$$

We denote for simplicity $\mathbf{w_h} = \mathbf{u_h} - \mathbf{u_I}$. As in the continuous case, we write $\mathbf{w_h} = \mathbf{w_h}^0 + \mathbf{w_h}^\perp$ with $\mathbf{w_h}^0 \in Ker(B_h)$ and $\mathbf{w_h}^\perp \in (Ker(B_h))^\perp$. Then

$$\alpha_h||\mathbf{w_h}^0||_V^2 \leq a(\mathbf{w_h}^0,\mathbf{w_h}^0) = a(\mathbf{u}-\mathbf{u_I},\mathbf{w_h^0}) + b(\mathbf{w_h^0}, p-p_I) - a(\mathbf{w_h}^\perp,\mathbf{w_h}^0)$$

$$\implies ||\mathbf{w_h}^0||_V \leq \frac{M}{\alpha_h}||\mathbf{u}-\mathbf{u_I}||_V + \frac{\gamma}{\alpha_h}||p-p_I||_Q + \frac{M}{\alpha_h}||\mathbf{w_h}^\perp||_V. \tag{29}$$

Thus we have

$$||\mathbf{w_h}||_V \leq ||\mathbf{w_h}^0||_V + ||\mathbf{w_h}^\perp||_V \leq \frac{M}{\alpha_h}||\mathbf{u}-\mathbf{u_I}||_V + \frac{\gamma}{\alpha_h}||p-p_I||_Q + (1+\frac{M}{\alpha_h})||\mathbf{w_h}^\perp||_V, \tag{30}$$

where by the inversed inf-sup condition

$$\beta_h||\mathbf{w_h}^\perp||_V \leq \sup_{q_h\in Q_h}\frac{b(\mathbf{w_h}^\perp,q_h)}{||q_h||_Q} \leq \gamma||\mathbf{u}-\mathbf{u_I}||_V + \frac{1}{\lambda}||p-p_I||_Q + \frac{1}{\lambda}||p_h-p_I||_Q, \tag{31}$$

We can now use the first equation in (28) to derive a bound on $p_h - p_I$:

$$\beta_h||p_h-p_I||_Q \leq \sup_{\mathbf{v_h}\in V_h}\frac{b(\mathbf{v_h},p_h-p_I)}{||\mathbf{v_h}||_V} \leq M||\mathbf{u}-\mathbf{u_I}||_V + \gamma||p-p_I||_Q + M||\mathbf{w_h}||_V, \tag{32}$$

and plugging in (30) into (32) we conclude that

$$\beta_h||p_h-p_I||_Q \leq M||\mathbf{u}-\mathbf{u_I}||_V + \gamma||p-p_I||_Q + M||\mathbf{w_h}||_V \tag{33}$$

$$\leq (M+\frac{M^2}{\alpha_h})||\mathbf{u}-\mathbf{u_I}||_V + \gamma(1+\frac{M}{\alpha_h})||p-p_I||_Q + (M+\frac{M^2}{\alpha_h})||\mathbf{w_h}^\perp||_V \tag{34}$$

$$\leq c_{1,h,\lambda}||\mathbf{u}-\mathbf{u_I}||_V + c_{2,h,\lambda}||p-p_I||_Q + (M+\frac{M^2}{\alpha_h})\frac{1}{\lambda\beta_h}||p_h-p_I||_Q \tag{35}$$

$$\implies (\beta_h - \frac{M}{\lambda\beta_h} - \frac{M^2}{\alpha_h\lambda\beta_h})||p_h-p_I||_Q \leq c_{1,h,\lambda}||\mathbf{u}-\mathbf{u_I}||_V + c_{2,h,\lambda}||p-p_I||_Q \tag{36}$$

with $c_{1,h,\lambda} = (M + \frac{M^2}{\alpha_h})(1 + \frac{\gamma}{\beta_h})$ and $c_{2,h,\lambda} = \frac{M\gamma}{\alpha_h} + (M + \frac{M^2}{\alpha_h})\frac{1}{\lambda\beta_h}$. If we take now λ large enough $(\beta_h - \frac{M}{\lambda\beta_h} - \frac{M^2}{\alpha_h\lambda\beta_h}) \geq \delta > 0$ so that in the end

$$||p_h - p_I||_Q \leq \frac{c_{1,h,\lambda}}{\delta}||\mathbf{u} - \mathbf{u_I}||_V + \frac{c_{2,h,\lambda}}{\delta}||p - p_I||_Q \tag{37}$$

$$||\mathbf{u_h} - \mathbf{u_I}||_V \leq c_{3,h,\lambda}||\mathbf{u} - \mathbf{u_I}||_V + c_{4,h,\lambda}||p - p_I||_Q, \tag{38}$$

where $c_{3,h,\lambda} = \frac{M}{\alpha_h} + \frac{1+\frac{M}{\alpha_h}}{\beta_h}(\gamma + \frac{c_{1,h,\lambda}}{\lambda\delta})$ and $c_{4,h,\lambda} = \frac{\gamma}{\alpha_h} + \frac{1+\frac{M}{\alpha_h}}{\lambda\beta_h}(1 + \frac{c_{2,h,\lambda}}{\delta})$. By using a standard triangular inequality we obtain from (37) and (38) the bounds on the error estimates

$$||p - p_h||_Q \leq \frac{c_{1,h,\lambda}}{\delta} \inf_{\mathbf{u_I}\in V_h} ||\mathbf{u} - \mathbf{u_I}||_V + (1 + \frac{c_{2,h,\lambda}}{\delta}) \inf_{p_I \in Q_h} ||p - p_I||_Q \tag{39}$$

$$||\mathbf{u} - \mathbf{u_h}||_V \leq (1 + c_{3,h,\lambda}) \inf_{\mathbf{u_I}\in V_h} ||\mathbf{u} - \mathbf{u_I}||_V + c_{4,h,\lambda} \inf_{p_I \in Q_h} ||p - p_I||_Q, \tag{40}$$

Note that as we expected, the errors on both p and $\mathbf{u}$ are depending on the best interpolation fits in both finite element spaces. In particular, for the Taylor Hood elements, assuming $u \in [H^3(\Omega)]^2$ and $p \in H^2(\Omega)$ we then expect a second order convergence rate in both $||u - u_h||_V$ and $||p - p_h||_Q$. We thus have a balance between the rates in the two spaces of the dual formulation which is often desirable, in particular $\exists C > 0$ such that

$$||(\mathbf{u} - \mathbf{u_h}, p - p_h)||_W \leq Ch^2(|u|_{[H^3(\Omega)]^2} + |p|_{H^2(\Omega)}). \tag{41}$$

Concerning the behaviour of the estimate (41) when $\lambda \to \infty$, note that none of the constants involved in (39) and (40) is exploding, so we expect the accuracy of our approximation not to depend on λ contrarily to what happens in the primal formulation. Once again, as in the continuous case, it is crucial that M is not depending on λ anymore.

2 Numerical results

We aim now at comparing the results we obtained numerically with the theoretical results obtained before. The domain Ω is a section with side length which seems in principle prohibitive for our computational power available: to work with mesh size h small enough we thus integrate (1) on a rescaled section of length $L = 7.5$ with a central hole of radius $R = 1$. On the other hand, we don't rescale the boundary traction on Γ_2 as it has already the unit measure of a pressure. Moreover, since no explicit analytical solution is given, the errors will be computed with respect to a reference solution calculated with Taylor-Hood elements on a sufficiently refined mesh (also here, the maximum possible). In particular, we take a reference mesh refined up to $I + 2$ truncations of an initial coarse mesh, while for the iterations which produce the plots we truncate at most I times. Before entering into details, let us briefly recall that the relation between the Lamé parameters μ and λ is $\frac{2\mu}{\lambda} = \frac{1-2\nu}{\nu}$, where ν is the so called Poisson ratio. When $\lambda >> \mu$ such a ratio tends to 0.5 and the compressibility hypothesis is not holding anymore. In the numerical tests below the boundary stress on Γ_2 is $f = (100, 0)^T$, μ is always fixed at $8 \cdot 10^5$ and we increase λ iteratively.

Convergence estimates

Taylor-Hood Elements

At $\lambda = 10^5$, the simple and the mixed form are equivalent. The optimal rates of convergence derived in the theoretical part are however not reached, which suggests that the solution doesn't match the assumed regularity. Nevertheless, the orders of convergence "scale" well between themselves: the L^2 error converges approximately with one additional power of h than the H^1. Increasing λ to 10^7, the mixed and the simple form become decoupled and the former is behaving better. The more we increase λ, the higher the gap we observe between the two forms. However, surprisingly the simple form maintains a good behaviour and it keeps on having the same convergence rate of the mixed formulation, even if the incompressible limit is more than approached at $\lambda = 10^9$ or $\lambda = 10^{11}$. In order to see the simple form collapse λ has to be increased until 10^{17}. Moreover, even at this point, we don't see a plateau appearing but the error in the simple form is just exploding, hence we can assume that we don't recover the convergence rates for h small enough. The results we obtain seem to suggest that $\mathbb{P}_2$ elements are particularly robust in this case also in the incompressible limit. We see on the other hand that the L^2 norm of the pressure is converging with the same rate as the H^1 norm of the displacement, as we fully expected for Taylor-Hood elements.

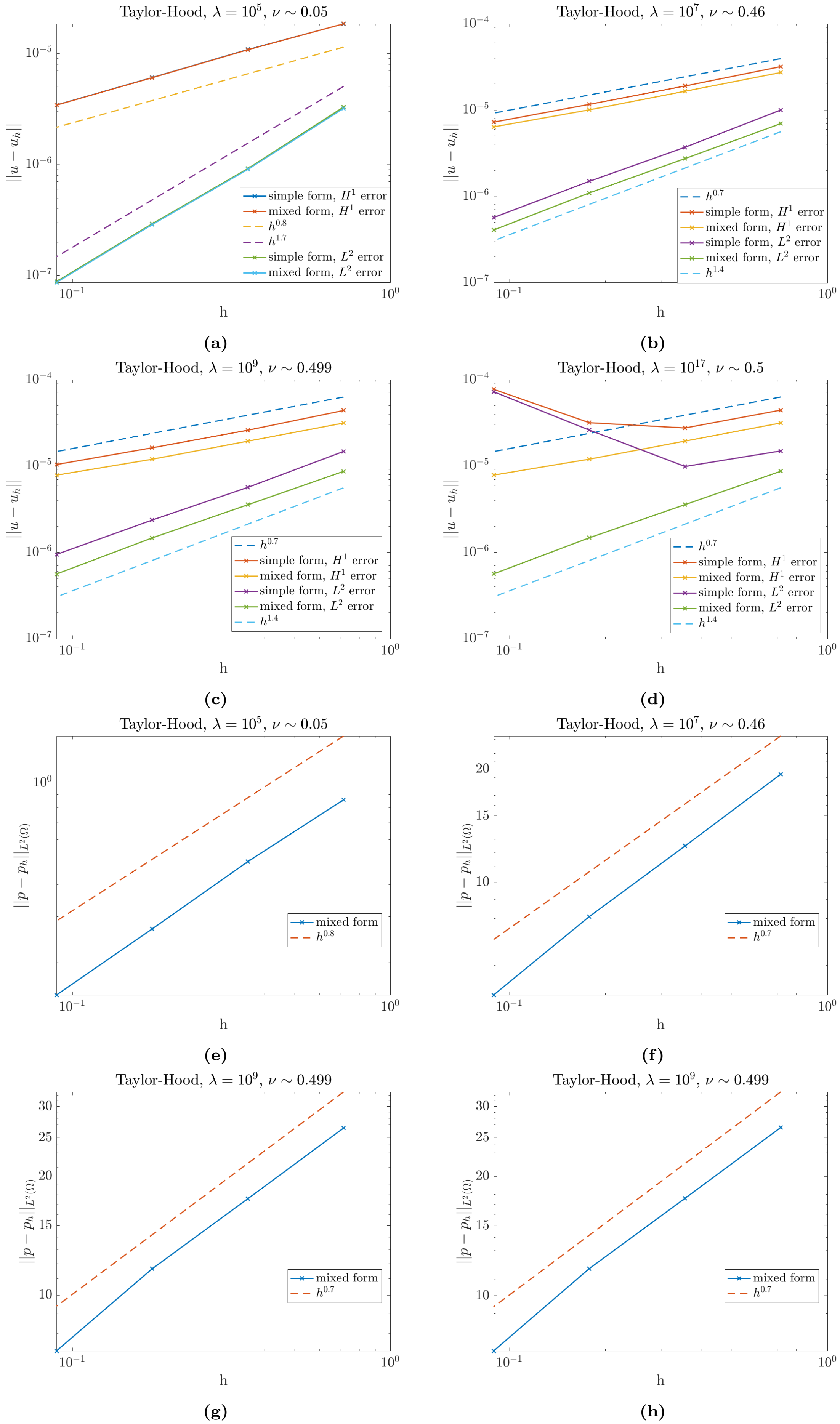


Figure 2: Convergence results in L^2 and H^1 norms of the displacement for Taylor-Hood elements (mixed form) and P_2^C elements (simple form): a) $\lambda = 10^5$, b) $\lambda = 10^7$, c) $\lambda = 10^9$ and d) $\lambda = 10^{17}$. Convergence results in the L^2 norm of the pressure for Taylor- Hood elements: e) $\lambda = 10^5$, f) $\lambda = 10^7$, g) $\lambda = 10^9$ and h) $\lambda = 10^{17}$.

MINI-Element

Given the results presented in the previous subsection, it was of interest to conduce similar tests using the MINI-Element for the mixed formulation and $\mathbb{P}_1^C$ elements for the simple one. In particular, we show that using a less robust choice for the simple form the incompressible limit gives rise to problems well earlier with respect than what was observed before. Indeed, the simple and the mixed form are equivalent at $\lambda = 10^5$ (fig. 3a), but as in the previous case we don't reach the theoretical orders of convergence (we expect at most linear convergence in the H^1 norm and quadratic in the L^2 norm). Approaching the incompressible limit ($\lambda = 10^9$, $\nu \approx 0.499$), the simple and the mixed form are clearly decoupled (fig. 3c): the convergence orders of the mixed form are only slightly modified, while the error of the simple form is much bigger, and a plateau is observed for the latter. For $\lambda = 10^{11}$, the mixed form remains stable, while the simple form is stuck at high errors and is not converging anymore: it should recover a good order of convergence only for very small (maybe computationally unreachable) mesh size h. As we added the plots for the MINI-Element mostly to show the difference between $\mathbb{P}_2^c$ and $\mathbb{P}_1^c$ elements for the simple formulation, the convergence results for the L^2 norm of the pressure are not presented.

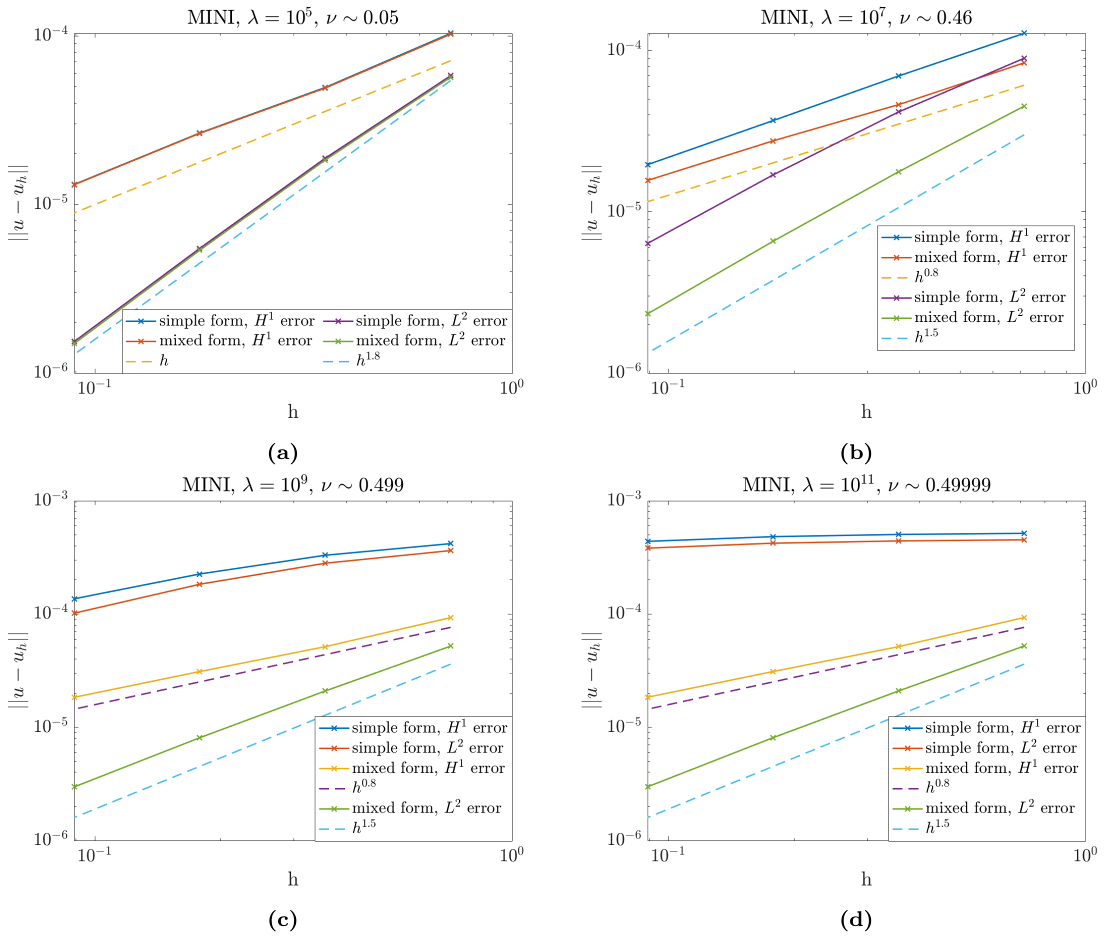


Figure 3: Convergence results in L^2 and H^1 norms of the displacement for MINI-Element (mixed form) and P_1^C elements (simple form) : a) $\lambda = 10^5$, b) $\lambda = 10^7$, c) $\lambda = 10^9$ and d) $\lambda = 10^{11}$.

Error estimates on functionals of the solution

Taylor-Hood Elements

Concerning the convergence of the functionals (12), as we expected from the theoretical analysis, the convergence of Q_1 is approximately twice as fast as the one in the H^1 norm. On the other hand, Q_2 converges slower, probably because the solution of the corresponding dual problem is not smooth enough. While the gap between the two forms increases accordingly to λ and the mixed form remains always the most accurate, to spoil the order of convergence of the simple form once again λ needs to be increased until 10^{17}. Contrarily to what we expected from the theory, the convergence rate in the L^2 norm of the displacement and the one in the functionals are approximately the same.

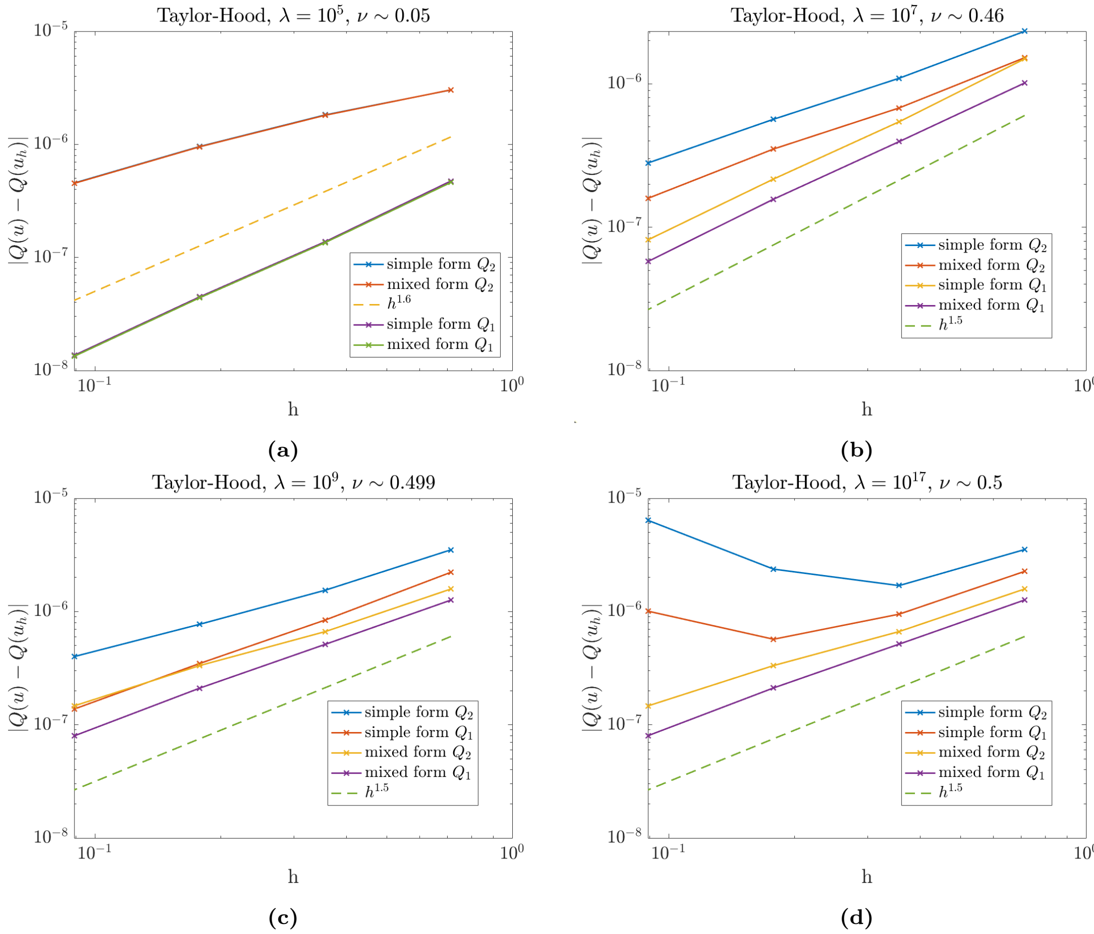
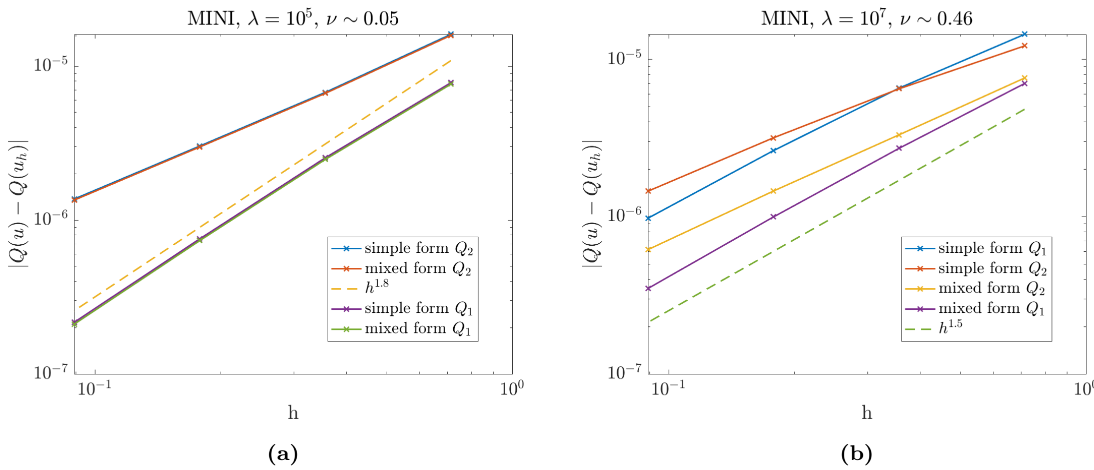
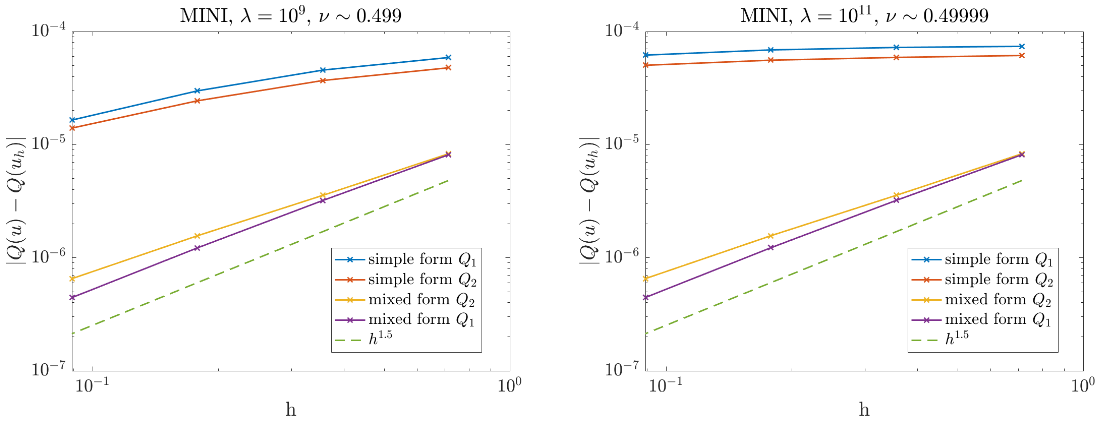


Figure 4: Convergence results on the functionals Q_1 and Q_2 for Taylor-Hood elements (mixed form) and P_2^C elements (simple form) : a) $\lambda = 10^5$, b) $\lambda = 10^7$, c) $\lambda = 10^9$ and d) $\lambda = 10^{17}$.

MINI-Element

We present to conclude also the results for the estimates on the functionals also for Q_1 and Q_2 for the MINI-Element, as they are perfectly in line with the ones observed in Figure 3. As for the Taylor Hood elements, the convergence of Q_1 has doubled rate with respect to the H^1 norm while the error in Q_2 converges slower. For $\lambda = 10^9$ and $\lambda = 10^{11}$ the simple form is suffering again from the incompressibility of the material, so that a plateau is observed for the former and convergence is completely spoiled for the latter. Also in this case, the mixed form is the most accurate independently of λ and the errors remain always of the same magnitude, another symptom of the stability of the dual formulation.





(c) **(d)**

Figure 5: Convergence results on the functionals Q_1 and Q_2 for the MINI-Element (mixed form) and $\mathbb{P}_1^C$ elements (simple form) : a) $\lambda = 10^5$, b) $\lambda = 10^7$, c) $\lambda = 10^9$ and d) $\lambda = 10^{11}$.

Inf-sup stability

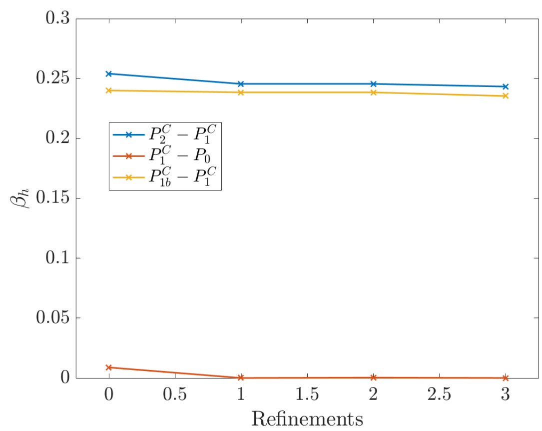


Figure 6: Inf-sup stability tests

We see that, coherently with what we expect from the theory, the $\mathbb{P}_2^c - \mathbb{P}_1^c$ Taylor-Hood Elements and the MINI-Element are inf-sup stable, as the constant β_h remains bounded from below by some constant $\beta > 0$ as the mesh is refined. While these two choices give the desired result, we recall that for example, the $\mathbb{P}_1^c - \mathbb{P}_0$ finite elements are not satisfying this condition and are therefore unstable: as we can see β_h is vanishing when performing the refinements. To conclude, note that the following upper bound on the inf-sup constant is always satisfied:

$$\beta_h \leq \inf_{v_h \in V_h} \sup_{q_h \in Q_h} \frac{b(v_h, p_h)}{||v_h||_V ||p_h||_Q} \leq 1. \tag{42}$$

3 Discussion and final remarks

For the Taylor Hood elements, while we expected to see results which were more in line with the ones discussed during the course (see e.g. exercise session 7), we couldn't obtain results significantly closer to those. While many tests were carried out in order to spoil the reasons for this, perhaps we observed (e.g. plotting the solutions and checking their physical coherence) that the accuracy of the simple form was hard to break in some sense. On the other hand we always observed coherently the correct scaling between the convergence in the H^1 norm and the one of the functional Q_1, while for the functional Q_2 it was not so trivial to find a theoretical expected behaviour. The MINI-Element results were completely as expected. While it is hard to consider our numerical results fully satisfactory, we find evidence of the difference between the two forms in both cases, the mixed form being always at least as accurate as the simple one.

4 Implementation aspects

We attach in the following pages the code used for running all the numerical tests in *Free Fem++*, referring then to the *.zip* in the submission for the code for the inf-sup tests which produced the last plot and for all the post-processing files in *Matlab*.
Concerning the code below, we refined the mesh during the iterations using the command *trunc* and this helped us stabilizing the convergence results, especially for the functionals and for the pressure.

```

// PROJECT: PRIMAL AND MIXED FORMULATION FOR LINEAR ELASTICITY

load "UMFPACK64"

// PHYSICAL PARAMETERS

func f1 = 100;
func f2 = 0;
real mu = 8e5;
real lambda = 1e5;

// MESH PARAMETERS

int g = 4; // divisions of the coarsest mesh
int I = 4; // iterations for the convergence study -> 3 refinements
int ref = 5; // refinements for the reference solution

// BORDERS : RESCALED DOMAIN
// f1 is not rescaled because it is a pressure and not a force.

real radius = 1;
real size = 7.5;

border down(t = -size/2.,size/2.){x = t; y = -size/2.; label=1;};
border right(t = -size/2.,size/2.){x = size/2.; y = t; label=2;};
border up(t=size/2.,-size/2.){x = t;y = size/2.; label=3;};
border left(t=size/2.,-size/2.){x = -size/2.; y = t; label=4;};
border hole(t = 0,2*pi){x = radius*cos(t); y = radius*sin(t);
       label=99;};

// COARSEST MESH

mesh Th = buildmesh(down(2^g) + right(2^g) + up(2^g) +
          left(2^g) + hole(-2^(g)));
//plot(Th,wait = 1);

// MOST REFINED MESH FOR THE REFERENCE SOLUTION :
// TRUNCATION OF THE COARSEST MESH

mesh Thr = trunc(Th, 1, split=2^ref);


// DEFINE FINITE ELEMENT SPACES : we calculate the reference solution
// using the mixed form.

fespace Vhr(Thr,[P2,P2]); // Taylor-Hood
// fespace Vhr(Thr,[P1b,P1b]); // MINI element
fespace Qhr(Thr,P1);

Vhr [u1r,u2r];
Vhr [v1r,v2r];
Qhr pr,qr;

// COMPUTE THE REFERENCE SOLUTION

// problem
solve lamer([u1r,u2r,pr],[v1r,v2r,qr]) =
                    int2d(Thr)(-qr*(dx(u1r) + dy(u2r))
                    - pr*(dx(v1r) + dy(v2r)) - (1.0/lambda)*pr*qr
                    + 2.*mu*(dx(u1r)*dx(v1r) + dy(u2r)*dy(v2r)
                    + 0.5*(dy(u1r)+dx(u2r))*(dy(v1r)+ dx(v2r))))
                    - int1d(Thr,2)(f1*v1r + f2*v2r) // boundary stress
                    + on(4,u1r=0,u2r=0); // boundary condition

```

```
// PLOTTING TO CHECK SYMMETRY WITH RESPECT TO X-AXIS AND PHYSICAL
// COHERENCE OF THE SOLUTION

//plot(u1r,value = true, wait=1, fill=1);
//plot(u2r,value = true, wait=1, fill=1);
//plot(pr,value = true, wait=1, fill=1);
//plot(Thr, [u1r, u2r], value=true, wait=true);


// ITERATIONS TO CHECK THE NUMERICAL CONVERGENCE RATES


// NUMERICAL TESTS FOR THE MIXED FORMULATION

real[int] hvect(I);
real[int] errQ1(I);
real[int] errQ2(I);
real[int] errH1(I);
real[int] errL2(I);
real[int] errP(I);

for(int i = 0; i < I; ++i){

// REFINING THE MESH BY TRUNCATION

mesh Thi = trunc(Th, 1, split=2^(i));
//plot(Thi,wait = 1);

// FINITE ELEMENT SPACE

fespace Vh(Thi,[P2,P2]); // Taylor-Hood
// fespace Vh(Thi,[P1b,P1b]); // MINI-element
fespace Qh(Thi,P1);

Vh [u1i,u2i];
Vh [v1i, v2i];
Qh pi, qi;

// h // ----------------------------------------
fespace Mh(Thi,P0);
Mh h = hTriangle;
hvect[i]=h[].max; // calculating the mesh size

// COMPUTE THE ith SOLUTION

// problem
solve lame([u1i,u2i,pi],[v1i,v2i,qi]) =
            int2d(Thi)(-qi*(dx(u1i) + dy(u2i))
            - pi*(dx(v1i) + dy(v2i)) - (1.0/lambda)*pi*qi
            + 2.*mu*(dx(u1i)*dx(v1i) + dy(u2i)*dy(v2i)
            + 0.5*(dy(u1i)+dx(u2i))*(dy(v1i)+ dx(v2i))))
            - int1d(Thi,2)(f1*v1i + f2*v2i) // boundary stress
            + on(4,u1i=0,u2i=0); // boundary condition

// ERROR COMPUTATION WITH RESPECT TO THE REFERENCE SOLUTION

// L2 error for the pressure

errP[i] = int2d(Thi)((pi - pr)^2);
errP[i] = sqrt(errP[i]);
cout << "Error p at iteration : " << i << "\t" << errP[i] << endl;

// L2 error for the displacement

errL2[i] = int2d(Thi)((u1i - u1r)^2 + (u2i - u2r)^2);
errL2[i] = sqrt(errL2[i]);
cout << "Error L2 at iteration : " << i << "\t" << errL2[i] << endl;

// H1 error for the displacement

errH1[i] = int2d(Thi)((u1i - u1r)^2 + (u2i - u2r)^2)
      + int2d(Thi)((dx(u1i) - dx(u1r))^2 + (dy(u1i) - dy(u1r))^2
      + (dx(u2i) - dx(u2r))^2 + (dy(u2i) - dy(u2r))^2);
errH1[i] = sqrt(errH1[i]);
cout << "Error H1 at iteration : " << i << "\t" << errH1[i] << endl;

```

```
// errors on the functionals

errQ1[i] = 1/size*int1d(Thi,2)(u1i - u1r);
errQ1[i] = abs(errQ1[i]);
cout << "Error Q1 at iteration: " << i << "\t" << errQ1[i] << endl;

errQ2[i] = sqrt((u1i(0,radius) - u1r(0,radius))^2
         + (u2i(0,radius) - u2r(0,radius))^2);
cout << "Error Q2 at iteration: " << i << "\t" << errQ2[i] << endl;

}

// EXPORTING ERRORS FOR THE MIXED FORM

cout << "Saving data: " << endl;

ofstream filehmixed("h_vect_mixed.dat");
filehmixed << hvect << endl;

ofstream fileErrorPmixed("errorP_mixed.dat");
fileErrorPmixed << errP << endl;

ofstream fileErrorL2mixed("errorL2_mixed.dat");
fileErrorL2mixed << errL2 << endl;

ofstream fileErrorH1mixed("errorH1_mixed.dat");
fileErrorH1mixed << errH1 << endl;

ofstream fileErrorQ1mixed("errorQ1_mixed.dat");
fileErrorQ1mixed << errQ1 << endl;

ofstream fileErrorQ2mixed("errorQ2_mixed.dat");
fileErrorQ2mixed << errQ2 << endl;


// NUMERICAL TESTS FOR THE SIMPLE FORMULATION

for(int i = 0; i < I; ++i){

// REFINING THE MESH BY TRUNCATION

mesh Thi = trunc(Th, 1, split=2^(i));
//plot(Thi, wait = 1);

// FINITE ELEMENT SPACES

fespace Vh(Thi,[P2,P2]); // P2 continuous to compare with Taylor-Hood
// fespace Vh(Thi,[P1,P1]); // P1 continuous to compare with MINI
Vh [u1i,u2i];
Vh [v1i,v2i];

// h //-----------------------------------------
fespace Qh(Thi,P0);
Qh h = hTriangle;
hvect[i]=h[].max; // calculating the mesh size

// COMPUTE THE ith SOLUTION

// problem
solve lame([u1i,u2i],[v1i,v2i]) =
               int2d(Thi)(lambda*(dx(u1i) + dy(u2i))*(dx(v1i) + dy(v2i))
               + 2.*mu*(dx(u1i)*dx(v1i) + dy(u2i)*dy(v2i)
               + 0.5*(dy(u1i)+dx(u2i))*(dy(v1i)+ dx(v2i))))
               - int1d(Thi,2)(f1*v1i + f2*v2i) // boundary stress
               + on(4,u1i=0,u2i=0); // boundary condition

//plot(u1i,value=true, wait=1, fill=1);
//plot(u2i,value=true, wait=1, fill=1);
//plot(Thi, [u1i, u2i], value=true, wait=true);

// ERROR COMPUTATION WITH RESPECT TO REFERENCE SOLUTION

// L2 error for the displacement

errL2[i] = int2d(Thi)((u1i - u1r)^2 + (u2i - u2r)^2);
errL2[i] = sqrt(errL2[i]);
```

```
cout << "Error L2 at iteration : " << i << "\t" << errL2[i] << endl;

// H1 error for the displacement

errH1[i] = int2d(Thi)((u1i - u1r)^2 + (u2i - u2r)^2)
      + int2d(Thi)((dx(u1i) - dx(u1r))^2 + (dy(u1i) - dy(u1r))^2
      + (dx(u2i) - dx(u2r))^2 + (dy(u2i) - dy(u2r))^2);
errH1[i] = sqrt(errH1[i]);
cout << "Error H1 at iteration : " << i << "\t" << errH1[i] << endl;

// errors on the functionals

errQ1[i] = 1/size*int1d(Thi,2)(u1i - u1r);
errQ1[i] = abs(errQ1[i]);
cout << "Error Q1 at iteration: "<< i << "\t" << errQ1[i] << endl;

errQ2[i] = sqrt((u1i(0,radius) - u1r(0,radius))^2
         + (u2i(0,radius) - u2r(0,radius))^2);
cout << "Error Q2 at iteration: "<< i << "\t" << errQ2[i] << endl;

}

// EXPORTING ERRORS FOR THE SIMPLE FORM

cout << "Saving data: " << endl;
ofstream filehsimple("h_vect_simple.dat");
filehsimple << hvect << endl;

ofstream fileErrorL2simple("errorL2_simple.dat");
fileErrorL2simple << errL2 << endl;

ofstream fileErrorH1simple("errorH1_simple.dat");
fileErrorH1simple << errH1 << endl;

ofstream fileErrorQ1simple("errorQ1_simple.dat");
fileErrorQ1simple << errQ1 << endl;

ofstream fileErrorQ2simple("errorQ2_simple.dat");
fileErrorQ2simple << errQ2 << endl;
```

References

[1] Douglas N Arnold, L Ridgway Scott, and Michael Vogelius. "Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon". In: *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 15.2 (1988), pp. 169–192.

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