Assignment

Homework HL3

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Musical Acoustics

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Modeling Techniques - Exercise 1

Piano String FD Model

In the first part of the following report the finite difference method will be employed in order to model the string-hammer interaction within a piano. Specifically, the model will concern a C2 string, for which $f_1 = 65.4$ $[Hz]$. The whole implementation employed *MatLab*.

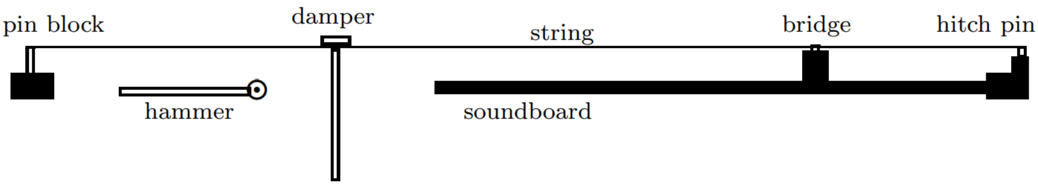


Figure 1: Simplified hammer-string mechanism illustration.

Numerical Model Introduction

The sampling of the string happens across both the spatial and temporal domain. The sampling frequency in time is $F_s = {}^1/_{T_s} = 4 \times 44.1$ $[kHz]$, while the total signal length amounts to 8 $[s]$ (L_t). Thus, the time step can be expressed in terms of the sampling time and of the simulation length: $N = {}^{L_t}/_{T_s}$. Concerning the spatial domain, the length of each string segment can be expressed as $X = {}^{L_s}/_{M}$, where L_s is the length of the string and M is the spatial step. Now, let us consider the transverse displacement *difference equation* for the stiff and lossy string in discrete time (including the string-hammer interaction):

$$y_m^{n+1} = a_1\left(y_{m+2}^n + y_{m-2}^n\right) + a_2\left(y_{m+1}^n + y_{m-1}^n\right) + a_3 y_m^n + a_4 y_m^{n-1} + a_5\left(y_{m+1}^{n-1} + y_{m-1}^{n-1}\right) + a_F F_m^n \quad (1)$$

In order to compute the coefficients a_i $(i = 1, 2, \ldots, 5, F)$ we need to set or determine the string parameters, that are reported in the following table:

L_s $[m]$	M_s $[kg]$	$\rho = {}^{M_s}/_{L_s}$ $[{}^{kg}/_{m}]$	$T = 4L_s^2\rho f_1^2$ $[N]$
1.92	$3.5 \cdot 10^{-2}$	$1.82 \cdot 10^{-2}$	$1.15 \cdot 10^{3}$
b_1 $\left[s^{-1}\right]$	b_2 $[s]$	ε	$c = \sqrt{{}^T/_\rho}$ $[{}^m/_s]$
0.5	$6.25 \cdot 10^{-9}$	$7.5 \cdot 10^{-6}$	251

Table 1: String parameters.

where b_1 and b_2 are respectively the *air damping coefficient* and the string *internal friction coefficient*, ε represents the string *stiffness coefficient*, T is the *tension* of the string and ρ is the linear density of the string. We can now compute the difference equation coefficients, in accordance with the analytical expressions reported in the following table.

a_1	a_2	a_3	a_4	a_5	a_F
$\frac{-\lambda\mu}{1+b_1T_s}$	$\frac{\lambda^2+4\lambda\mu+\nu}{1+b_1T_s}$	$\frac{2-2\lambda^2-6\lambda^2\mu-2\nu}{1+b_1T_s}$	$\frac{-1+b_1T_s+2\nu}{1+b_1T_s}$	$\frac{-\nu}{1+b_1T_s}$	$\frac{T_s^2/\rho}{1+b_1T_s}$

Table 2: Difference equation coefficients.

The auxiliary parameters μ and ν can be expressed as follows:

$$\mu = \frac{\varepsilon^2}{c^2X^2} \qquad \nu = \frac{2b_2T_s}{X^2} \tag{2}$$

while λ is the **Courant number**, which is a parameter of fundamental importance, because it allows us to *verify the validity of the discrete model* (further details will be given later on).

Boundary Conditions

The ***boundary conditions*** at both ends of the string need to be taken into account, resulting in different expressions of the wave difference equation for $m = 0, 1, M-1, M$:

- Recurrence equation for $\boxed{m = 0}$ (left hinged end):

$$y_m^{n+1} = b_{L1}y_m^n + b_{L2}y_{m+1}^n + b_{L3}y_{m+2}^n + b_{L4}y_m^{n-1} + b_{LF}F_m^n \tag{3}$$

b_{L1}	b_{L2}	b_{L3}	b_{L4}	b_{LF}
$\frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T_s+\zeta_l\lambda}$	$\frac{4\lambda^2\mu+2\lambda^2}{1+b_1T_s+\zeta_l\lambda}$	$\frac{-2\lambda^2\mu}{1+b_1T_s+\zeta_l\lambda}$	$\frac{-1b_1T_s+\zeta_l\lambda}{1+b_1T_s+\zeta_l\lambda}$	$\frac{T_s^2/\rho}{1+b_1T_s+\zeta_l\lambda}$

Table 3: Difference equation coefficients (left end).

The $\zeta_l = 10^{20}$ $[\Omega/kg{\cdot}m^{-2}{\cdot}s^{-1}]$ parameter is the normalized left end impedance.

- Recurrence equation for $\boxed{m = M}$ (bridge):

$$y_m^{n+1} = b_{R1}y_m^n + b_{R2}y_{m-1}^n + b_{R3}y_{m-2}^n + b_{R4}y_m^{n-1} + b_{RF}F_m^n \tag{4}$$

b_{R1}	b_{R2}	b_{R3}	b_{R4}	b_{RF}
$\frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T_s+\zeta_b\lambda}$	$\frac{4\lambda^2\mu+2\lambda^2}{1+b_1T_s+\zeta_b\lambda}$	$\frac{-2\lambda^2\mu}{1+b_1T_s+\zeta_b\lambda}$	$\frac{-1b_1T_s+\zeta_b\lambda}{1+b_1T_s+\zeta_b\lambda}$	$\frac{T_s^2/\rho}{1+b_1T_s+\zeta_b\lambda}$

Table 4: Difference equation coefficients (bridge).

The $\zeta_b = 10^3$ $[\Omega/kg{\cdot}m^{-2}{\cdot}s^{-1}]$ parameter is the normalized bridge impedance.

– Recurrence equation for $\boxed{m = 1}$:

$$
\begin{aligned}
y_m^{n+1} = a_1 \left(y_{m+2}^n - y_m^2 + 2y_{m-1}^n \right) + a_2 \left(y_{m+1}^n + y_{m-1}^n \right) + \dots \\
\dots + a_3 y_m^n + a_4 y_m^{n-1} + a_5 \left(y_{m+1}^{n-1} + y_{m-1}^{n-1} \right) + a_F F_m^n
\end{aligned}
\tag{5}
$$

– Recurrence equation for $\boxed{m = M - 1}$:

$$
\begin{aligned}
y_m^{n+1} = a_1 \left(2y_{m+1}^n + y_{m-2}^n \right) + a_2 \left(y_{m+1}^n + y_{m-1}^n \right) + \dots \\
\dots + a_3 y_m^n + a_4 y_m^{n-1} + a_5 \left(y_{m+1}^{n-1} + y_{m-1}^{n-1} \right) + a_F F_m^n
\end{aligned}
\tag{6}
$$

Initial Conditions

The string is assumed to be at rest at time $t = 0\ [s]$, hence:

$$y_m^0 = 0 \tag{7}$$

At time $t = \Delta t$ (i.e. $n = 1$) the hammer displacement η is given by:

$$\eta(1) = v_{h0} \Delta t \tag{8}$$

where $v_{h0} = 2.5\ [m/s]$ is the initial hammer velocity. At this time we cannot use Equation 1 to compute the string displacement because the recurrence involves four time steps, but it is possible to approximate y_m^1 via the Taylor series:

$$y_m^1 = \frac{1}{2} \left[y_{m+1}^0 + y_{m-1}^0 \right] \tag{9}$$

We can define the force density term F_m^n of Equation 1 as:

$$
\begin{cases}
F_m^n = F_H(n) \cdot g\,(m, m_0) \\
F_H(n) = K \left| \eta^n - y_{m_0}^n \right|^p
\end{cases}
\tag{10}
$$

where K is the hammer felt stiffness, p is the stiffness exponent, m_0 is the striking point along the string and $g\,(m, m_0)$ is the spatial window (more details on the spatial window will be given further on). Hence, for $n = 1, 2$:

$$F_H(1) = K \left| \eta(1) - y_{m_0}^1 \right|^p \tag{11}$$

$$F_H(2) = K \left| \eta(2) - y_{m_0}^2 \right|^p \tag{12}$$

while the hammer displacement is given by:

$$\eta(n+1) = d_1 \eta(n) + d_2 \eta(n-1) + d_F F_H(n) \tag{13}$$

The analytical expressions of the coefficients d_i $(i = 1, 2, F)$ are reported in Table 5 ($M_H = 4.9 \cdot 10^{-3}\ [kg]$ is the hammer mass, $b_H = 10^{-4}\ \left[s^{-1}\right]$ is the fluid damping coefficient).

d_1	d_2	d_F
$\frac{2}{1+b_H T_s/2M_H}$	$\frac{-1+b_H T_s/2M_H}{1+b_H T_s/2M_H}$	$\frac{-T_s^2/M_H}{1+b_H T_s/2M_H}$

Table 5

This yields:

$$\eta(2) = 2\eta(1) - \eta(0) - T_s^2 F_H(1) \cdot M_H^{-1} \tag{14}$$

When the first three time steps are known, it is possible to employ Equation 1 to compute the future displacement y_m^{n+1} (assuming the present force $F_H(n)$ is known). Obviously, the hammer loses contact with the string when:

$$\eta(n+1) < y_{m_0}^{n+1} \tag{15}$$

After this condition is met, the string if left to free vibrations.

Hammer Contact Window

Let us consider the aforementioned spatial window $g(m, m_0)$ (see Equation 10). We dubbed M the number of spatial samples, and the window is centered on m_0 - i.e. on the spatial sample corresponding to the striking position - thus we can define the *relative striking position* as:

$$a = 0.12 = {}^{m_0}/_{M} \iff m_0 = \lceil aM \rfloor \tag{16}$$

The operator $\lceil \square \rfloor$ represents the operation of rounding to the nearest integer. Hence, we can determine the number of spatial samples that make contact with the hammer during the striking m_w as follows:

$$m_w = w/X \tag{17}$$

where $w = 0.2$ is the width of the window and X is the spatial sampling interval. At this point, we can define a Hanning window centered at m_0 with a length m_w.

Spatial Sampling

In order to ensure numerical stability, the spatial step should be limited. Two different stability conditions can be found in literature (*bibliographic references reported in the next page*):

$$X_{\max_1} = \sqrt{\frac{1}{2}\left(c^2T_s^2 + 4b_2T_s + \sqrt{\left(c^2T_s^2 + 4b_2T_s\right)^2 + 16\varepsilon^2T_s^2}\right)} \qquad [1] \tag{18}$$

$$\Rightarrow M_{\max_1} = \left\lfloor \frac{L_s}{X_{\max_1}} \right\rfloor = 1348 \tag{19}$$

$$M_{\max_2} = \left\lfloor \sqrt{\left(-1 + \sqrt{1 + 16\varepsilon\gamma^2}\right) \cdot 8\varepsilon} \right\rfloor = 479 \qquad [2] \tag{20}$$

The γ parameter is given by:

$$\gamma = \frac{f_{\mathrm{Nyq}}}{f_1} \tag{21}$$

where $f_{\mathrm{Nyq}} = F_s/2$ is the Nyquist frequency. Hence, γ is related to the anti-aliasing condition.

Given the significant difference between the two limit values of M, we decided to work in favour of security, thus choosing the minimum of the two as the maximum number of spatial steps:

$$M = \min\left\{M_{\mathrm{max}_1} \quad M_{\mathrm{max}_2}\right\} = 479 \tag{22}$$

Courant-Friedrichs-Lewy Condition

We already mentioned the Courant number when computing the coefficients of the string displacement difference equation. The Courant-Friedrichs-Lewy (CFL) condition is a sufficient condition to satisfy the Von Neumann stability requirements:

$$\lambda \leq 1 \tag{23}$$

In the examined case, the Courant number amounts to:

$$\lambda = \frac{cT_s}{X} = 0.355 \leq 1 \tag{24}$$

and thus satisfies the CFL condition.

Implementation of the Finite Difference Scheme

The implementation of the model described in the previous section was achieved via a loop. In order to setup the recursive procedure, the computation starts from the evaluation of the initial conditions (*first three time steps*, i.e. $n = 0, 1, 2$ - see Equations 7 to 14) and proceeds by determining the recursive equation coefficients (Table 2). From $n = 3$ on, the program computes in parallel the hammer force term $F_H(n)$, the string displacement y_m^n and the hammer displacement $\eta(n)$. If the condition expressed in Equation 15 is met, the force term is excluded from the recursive scheme before proceeding further.

We treated the string as if only a twelve samples wide portion (positioned specularly with respect to the hammer impact position) of it was generating sound (from $m = M - m_0 - 6$ to $m = M + m_0 + 6$).

The string displacement at a fixed time instant, the generated piano signal and its spectral content are reported in the following figures.

[1]Saitis, C. (2008). *Physical modeling of the piano: an investigation into the effect of string stiffness on the hammer-string interaction.* Belfast: Sonic Arts Research Centre.

[2]Chaigne, A. & Askenfelt, A. (1994). *Numerical simulations of piano strings. I. A physical model for a struck string using finite difference methods.* The Journal of the Acoustical Society of America, 95 (2), 1112-1118.

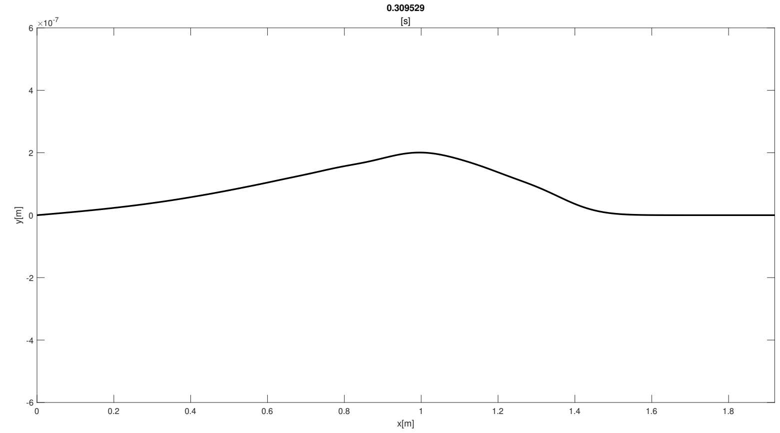


Figure 2: String displacement at $t \approx 0.31$ $[s]$.

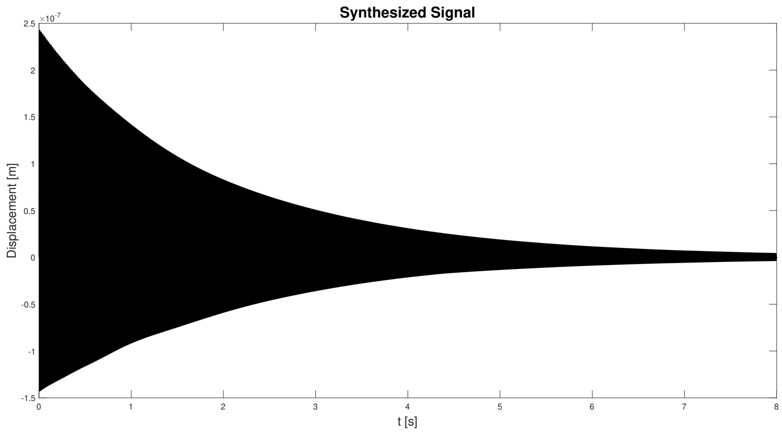


Figure 3: Generated C2 signal.

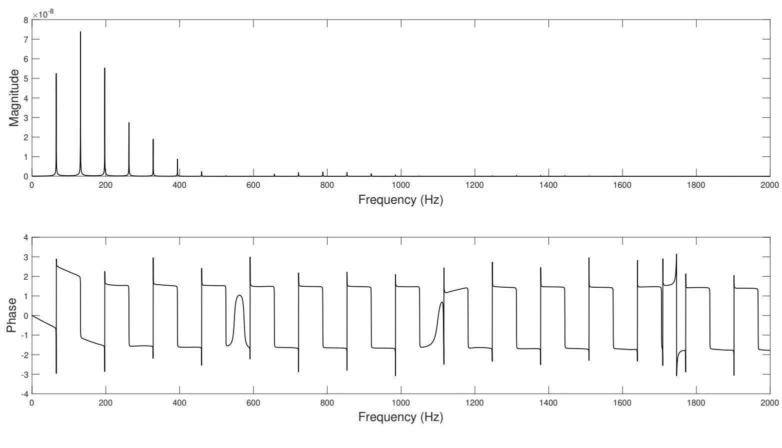


Figure 4: Spectral content of the generated signal.

Modeling Techniques - Exercise 2

Complete Guitar Model

A guitar can be modeled in the first instance as a **two-mass system** (Figure 5) whose vibrational elements are represented by the top plate (m_p, K_p) and the air volume (m_h, V) contained in the sound box, which are in turn modeled as *lumped oscillators*. The corresponding equivalent electric circuit is represented on the right side of the illustration.

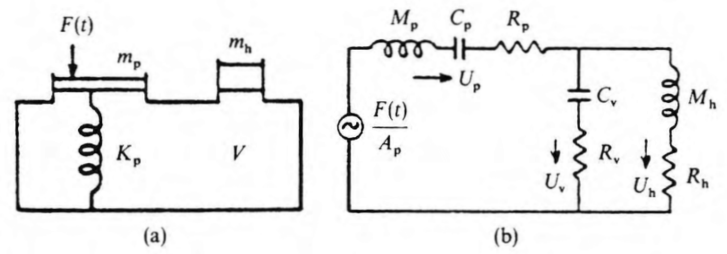


Figure 5

As we can see, resonances are modeled by inserting a **filter bank** in the circuit: each RLC branch models a particular filter with *one associated resonance*. Obviously, a real guitar plate has more than two resonances, and in the examined case a total of **twenty RLC branches**, i.e. twenty resonances, were added to the model (see the right portion of the electrical circuit represented in Figure 6). Each of the elements that make up the filters was *empirically tuned* to obtain the resonances measured from a real guitar. This kind of refined model ensures a **better yield for higher frequencies**.

We are interested in the time response of the modeled guitar to a **pluck excitation**: therefore, we added a signal generator circuit that is able to simulate a plucked string vibrational behaviour. We are considering an **ideal undamped string**, and the string whose behaviour is the closest to this approximation in an actual guitar is the **E4 string** (i.e. *the high E*): therefore, we will study the response of the model to a plucking of the first string.

In the following page, Figure 6 and 7 represents the twenty-resonances model (with the explicit parameters values) and the plucking signal generation circuit, while Figure 8 shows the complete circuit just as it was implemented in MatLab/Simulink.

Twenty Resonances Without String Model

First of all, we perform the simulation without considering the plucked string model. The resulting signal, reported in Figure 9 (page 9) and whose spectral content is shown in Figure 10, is clearly a rough approximation of the guitar sound, much less accurate than the one obtained via the complete model. By looking at the spectrogram in Figure 11, in fact, we can observe that the harmonic content of such a signal remains present for the whole length of the signal almost without decaying.

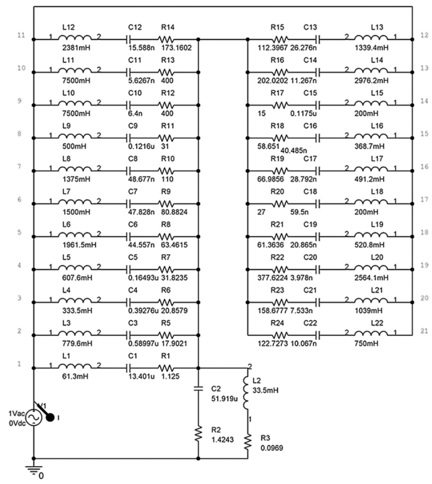


Figure 6: Guitar model - RLC branches.

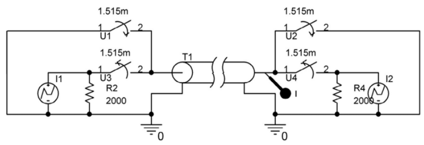


Figure 7: String model - Plucking signal generator.

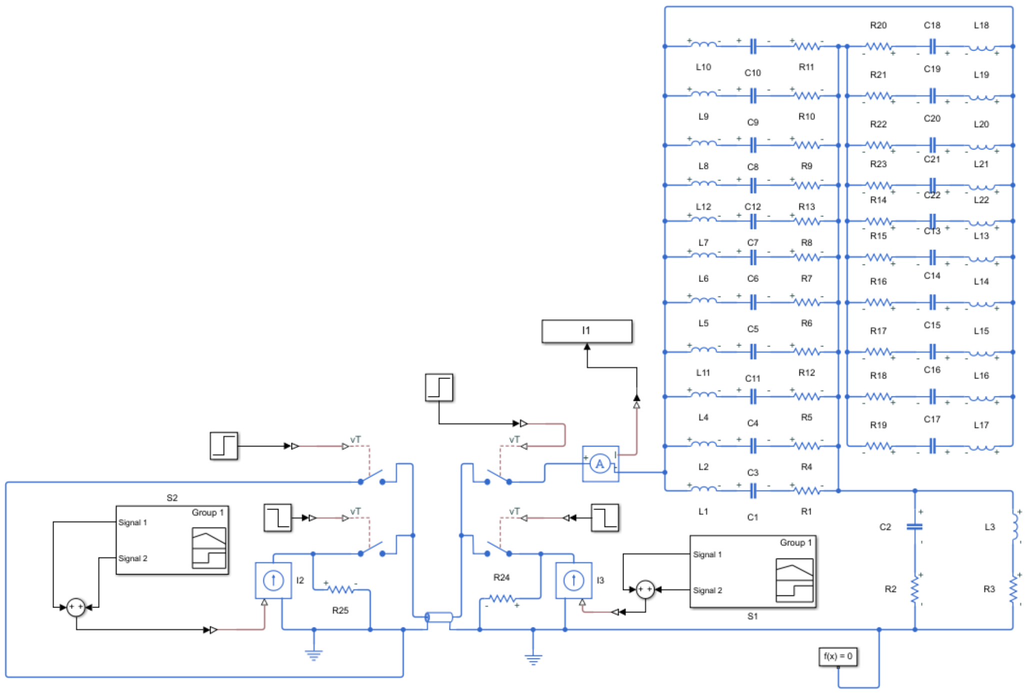


Figure 8: Complete Simulink circuit.

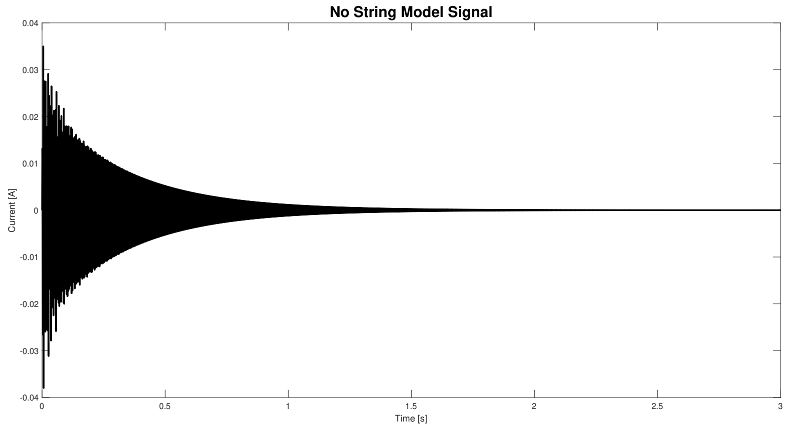


Figure 9: Generated guitar signal without string simulation.

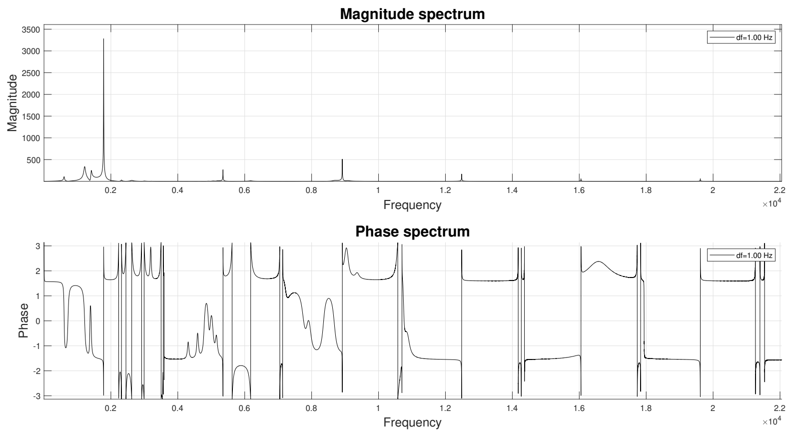


Figure 10: Generated guitar signal without string simulation - Spectral analysis.

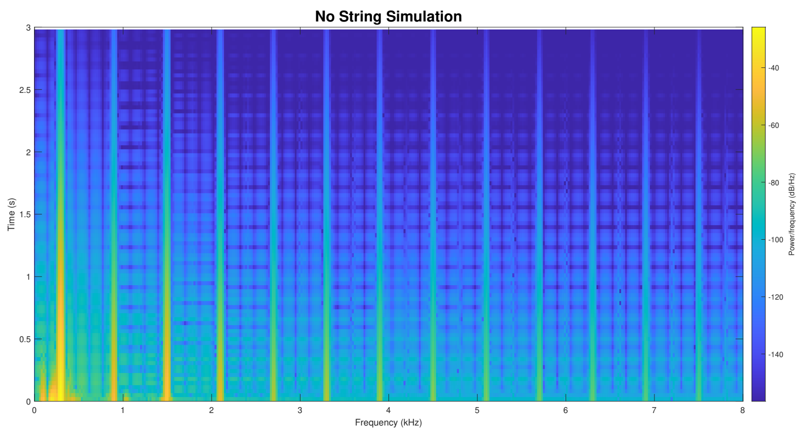


Figure 11: Generated guitar signal without string simulation - Spectrogram.

Complete Guitar Model (including Plucked String Simulation)

Now we perform the simulation with the inclusion of the plucked string simulation circuit. The resulting signal is reported in Figure 12 and its spectral content is shown in Figure 13. This is clearly a much more accurate approximation of a guitar sound: it much more rich from an harmonic point of view, its harmonics are better spread across the frequency spectrum (as integer multiples of the fundamental one, of course) and the spectral content decays in time as expected (refer to Figure 14).

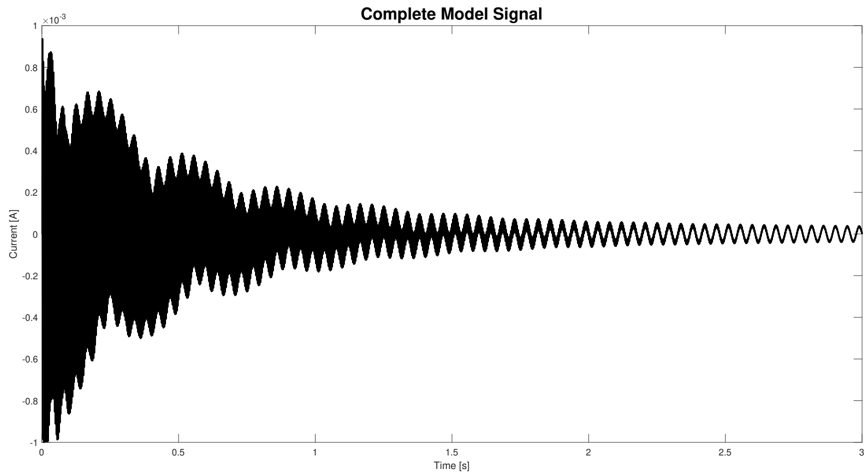


Figure 12: Generated guitar signal, including string simulation.

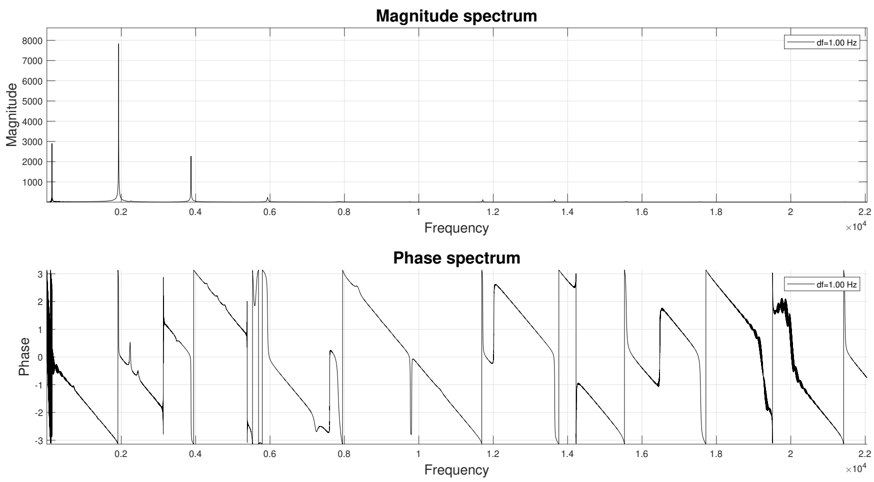


Figure 13: Generated guitar signal, including string simulation - Spectral analysis.

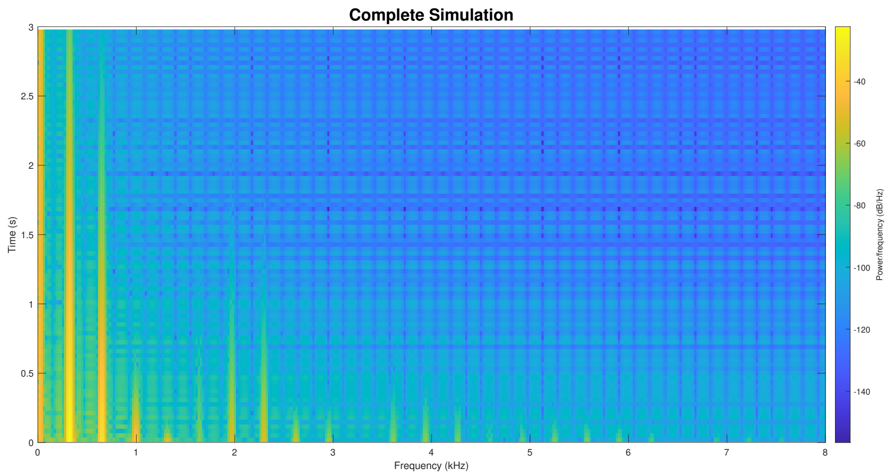
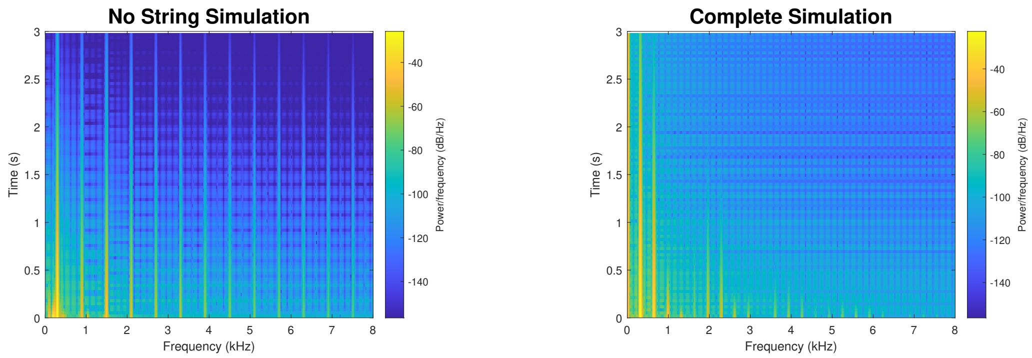


Figure 14: Generated guitar signal, including string simulation - Spectrogram.



(a) **(b)**

Figure 15

The previous figure depicts the spectrograms of the two generated signal one next to the other in order to better appreciate the differences between the two and the improvement that resulted from the addition of the string model to the RLC branches.