Assignment

Homework HW4

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Musical Acoustics

POLITECNICO
MILANO 1863

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Design of a Recorder

The recorder design will consist of the dimensioning of the bore and the positioning of the two finger holes.

Bore dimensioning

The conical semiangle of the bore is $\alpha = 0.75°$ and its length is $l = 0.45\,[m]$ (*treble recorder*). The goal is for the instrument to play a E4 note ($f_{E4} = 329.63\,[Hz]$) when all the finger holes are closed, and this implies an accurate choice of the radiuses (r_1, r_2) of the truncated conical bore (refer to Figure 1).

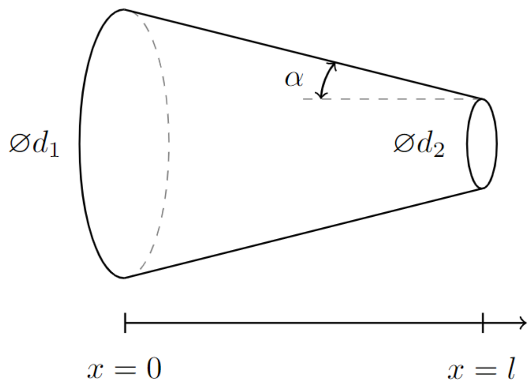


Figure 1: Truncated cone analytical description.

The fundamental frequency f_{E4} needs to correspond to a maximum of the input admittance for the truncated cone - including the mouth correction. The acoustic length of the cone is determined by employing the correction for flanged radiation, since we cannot neglect the thickness of the recorder wall:

$$L = l + 0.85 \cdot r_2 \tag{1}$$

For a conical horn with a throat of area S_1, a mouth of area S_2 at position x_2, and a bore length l, we can express the input impedance as:

$$Z_{\mathrm{IN}} = \frac{\rho c}{S_1}\left\{\frac{jZ_L\left[\sin\left(kL - \theta_2\right)/sin\theta_2\right] + \left(\rho c/S_2\right)\sin kL}{Z_L\left[\sin\left(kL + \theta_1 - \theta_2\right)/\sin\theta_1\sin\theta_2\right] - \left(j\rho c/S_2\right)\left[\sin\left(kl + \theta_1\right)/\sin\theta_1\right]}\right\} \tag{2}$$

where $\theta_1 = \arctan\left(kx_1\right)$ and $\theta_2 = \arctan\left(kx_2\right)$. The distance x_1 represents the length of the truncated part of the cone, while x_2 is the length of the complete cone.

Since we already included the radiation correction in the computation, we can neglect the load impedance Z_L, hence reducing the expression to:

$$Z_{\mathrm{IN}} = \frac{j\rho c}{S_1} \cdot \frac{\sin\left(kL\right)\sin\left(k\theta_1\right)}{\sin\left[k\left(L + \theta_1\right)\right]} \tag{3}$$

where S_1 is the input cross section and L is the corrected bore length. Recall that:

$$\theta_1 = \arctan(kx_1) \tag{4}$$

$$x_1 = \frac{r_1}{\tan\alpha} - l \tag{5}$$

We must now take into account the mouth impedance, since the resonance condition for the whole instrument is that the impedance of the air column in series with the impedance of the mouth must be a minimum. Such an impedance Z_m is essentially an inertance $j\omega M$, and by equating it to the theoretical input impedance expression for a cylindrical bore we obtain:

$$j\omega M = jZ_0 \tan(k\Delta L) \xrightarrow{k\Delta L \ll \pi/2} \Delta L = \frac{MS}{\rho} = 0.04\ [m] \Longrightarrow M = \frac{\rho \Delta L}{S_1} \tag{6}$$

where $0.04\ [m]$ is a typical value of ΔL for an alto recorder. Thus:

$$Z_{\text{tot}}(r_1) = \frac{j\rho c}{S_1} \cdot \frac{\sin(kL)\sin(k\theta_1)}{\sin[k(L+\theta_1)]} + j\omega \frac{\rho \Delta L}{S_1} \tag{7}$$

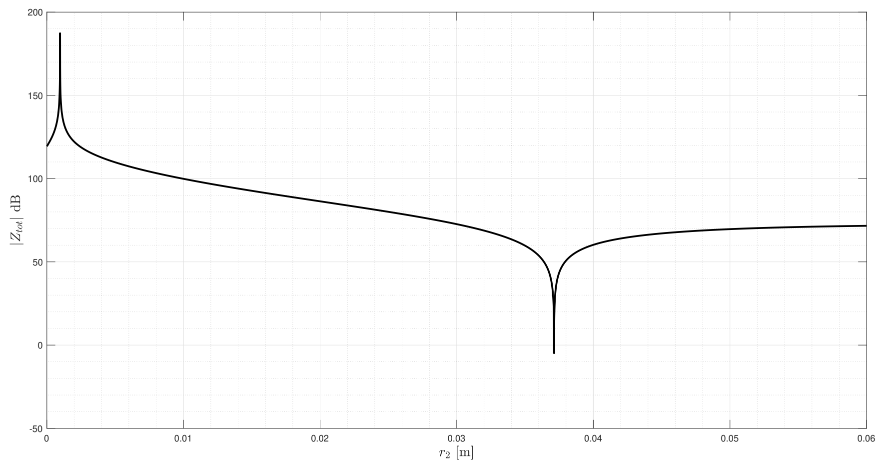


Figure 2

The input radius r_1 we are looking for is the one that minimizes the total impedance:

$$r_{1_0} = Z_{\text{tot}}^{-1}\left(\min_{r_1}\{Z_{\text{tot}}\}\right) = 0.0430\ [m] \tag{8}$$

Thus the radius r_2 of the open end can be determined as follows:

$$r_2 = r_1 - l \cdot \tan\alpha = 0.0371\ [m] \tag{9}$$

$$\Longrightarrow \begin{cases} d_1 = 0.0860\ [m] \\ d_2 = 0.0742\ [m] \end{cases} \tag{10}$$

First finger hole

Finger holes obviously alter the acoustic properties of the resonator. In order to design an instrument that plays a F4 ($f' = 349.23\ [Hz]$) note when the first hole is closed, the following approximations are assumed valid:

- $S_h = S_2 \Longleftrightarrow r_h = r_2$, *i.e.* the radius of the hole is assumed to be the same as the mouth radius of the recorder;
- $L_h = \Delta L = 0.85 \cdot r_2$, *i.e.* the acoustic length of the hole is assumed to be the same as the virtual elongation of the resonator's foot.

Let us consider the input impedance expression for a cone:

$$Z_{\mathrm{IN}} = \frac{\rho c}{S_1} \left\{ \frac{jZ_L \left[\sin\left(kL - \theta_2\right) / sin\theta_2\right] + \left(\rho c / S_2\right) \sin kL}{Z_L \left[\sin\left(kL + \theta_1 - \theta_2\right) / \sin\theta_1 \sin\theta_2\right] - \left(j\rho c/S_2\right)\left[\sin\left(kl + \theta_1\right) / \sin\theta_1\right]} \right\} \tag{11}$$

In the examined case, we refer to the distance between the throat of the recorder and the finger hole as L_{1h} and to the one between the hole and the mouth as $L_{h2} = D' + \Delta L = D' + 0.85 \cdot r_2$, while $S_h = \pi r_h^2$ is the cross-section of the cone in correspondence with the hole, just as before:

$$Z_{\mathrm{IN}_h} = \frac{\rho c}{S_1} \left\{ \frac{jZ_L \left[\sin\left(k'L_{1h} - \theta_1\right) / sin\theta_1\right] + \left(\rho c / S_h\right) \sin k'L_{1h}}{Z_L \left[\sin\left(k'L_{1h} + \theta_h - \theta_1\right) / \sin\theta_1 \sin\theta_h\right] - \left(j\rho c/S_h\right)\left[\sin\left(k'L_{1h} + \theta_h\right) / \sin\theta_h\right]} \right\} \tag{12}$$

where $\theta_h = \arctan\left(k'x_h\right)$ and x_h is the distance between the finger hole position along the cone axis and the truncated apex of the cone. The same relations hold for θ_1, x_1. In this case, the load impedance Z_L is the equivalent impedance $Z_L = Z_h \parallel Z'$, where Z_h is the impedance of the hole and Z' is the impedance of the end portion of the bore:

$$\begin{cases} Z_h = \left[-jS_h(\rho c)^{-1} \cot\left(k'L_h\right)\right]^{-1} \\ Z' = j\rho c S_h^{-1} \sin\left(k'L_{h2}\right) \sin\left(k'\theta_h\right) \left\{\sin\left[k'\left(L_{h2} + \theta_h\right)\right]\right\}^{-1} \\ L_h = r_h + 0.85 \cdot r_2 \end{cases} \tag{13}$$

At this point, we add the mouth impedance Z_m just as we did before:

$$Z_{\mathrm{tot}} = Z_{\mathrm{IN}_h} + Z_m = Z_{\mathrm{IN}_h} + j\omega' \frac{\rho \Delta L}{S_2} \tag{14}$$

The hole must be positioned in correspondence with the distance D' that minimizes Z_{tot}:

$$D'_0 = Z_{\mathrm{tot}}^{-1} \left(\min_{D'} \left\{ Z_{\mathrm{tot}} \right\} \right) = 0.0329\ [m] \tag{15}$$

The graph of the total impedance as a function of the distance D' is reported in Figure 3.

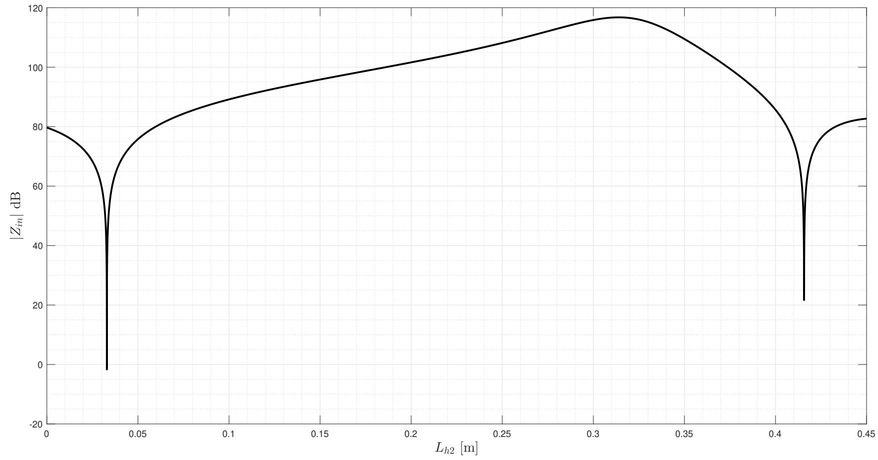


Figure 3: Total input impedance as a function of the distance between the last finger hole and the mouth of the resonator.

Second finger hole

The positioning process for the second finger hole is totally analogous to the one described in the previous section. Specifically, the second hole should be positioned so that the corresponding resonant frequency is $f'' = 392\,[Hz]$ (G4) (i.e. when the two finger holes are open). The adopted assumptions are to the ones assumed valid for the first hole. For the sake of brevity, we will not show all the calculations and analytical expressions, but in Figure 4 is reported the graph of the total impedance as a function of the distance D'' between the two holes.

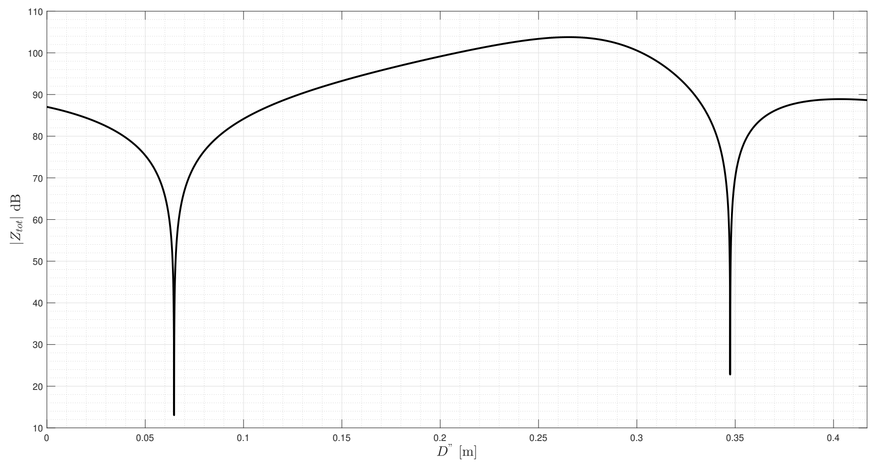


Figure 4: Total input impedance as a function of the distance between the two holes.

$$D''_0 = Z_{\text{tot}}^{-1}\left(\min_{D''}\{Z_{\text{tot}}\}\right) = 0.0647\;[m] \tag{16}$$

Prediction of the input impedance

By employing the model of the instrument as a transmission line we will now attempt to predict the input impedance for the following three configurations of the resonator:

- both finger holes closed;
- just the last finger hole open;
- both finger holes open.

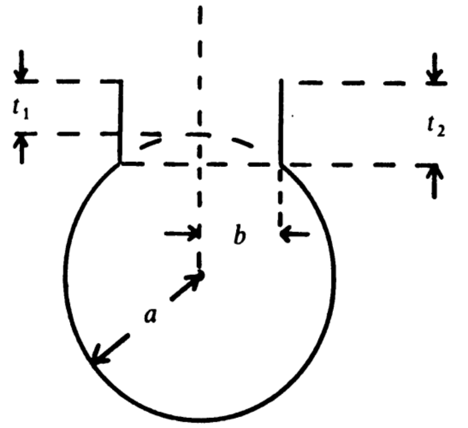
The idea behind the method is to divide the bore of the instrument into a large number of small lengths, in each of which the geometry is adequately represented by a short length of cone. The acoustic impedance coefficients Z_{ij} representing the relationship between the pressures and acoustic volume flows p_1, U_1 and p_2, U_2 at the upper and lower ends of such a section are:

$$\begin{cases} p_1 = Z_{11}U_1 + Z_{12}U_2 \\ p_2 = Z_{21}U_1 + Z_{22}U_2 \end{cases} \tag{17}$$

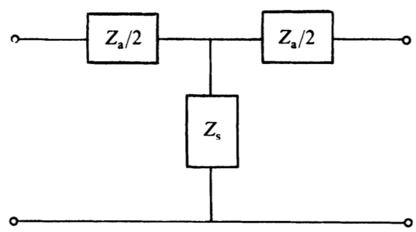
Thus, if $Z_2 = p_2/U_2$ is the acoustic load attached to the end 2, the impedance $Z_1 = p_1/U_1$ seen at the entrance to end 1 is:

$$Z_1 = Z_{11} + \frac{Z_{12}^2}{Z_2 - Z_{22}} \tag{18}$$

since $Z_{12} = Z_{21}$.



(a) Dimension definitions for a side hole in a cylindrical tube.



(b) T-section network representing the acoustic effect of a side hole in a tube.

Figure 5

For a circular finger hole of radius $b = r_2$ and chimney height t in a tube of radius $a = r_h$, as shown in Figure 5a, the equivalent network has the form shown in Figure 5b, and the imaginary parts of the series and shunt elements have the following forms:

$$Z_s = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \cdot \begin{cases} -j\cot kt & \text{(closed)} \\ jkt_e & \text{(open)} \end{cases} \tag{19}$$

$$Z_a = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \cdot (-jkt_a) \qquad \text{(closed or open)} \tag{20}$$

Equation 19 reflects the fact that an open hole behaves like a shunt inertance, as we have already discussed, while a closed hole behaves like a shunt compliance because of its enclosed volume. Consequently, for a partially open tonehole, the shunt impedance may be described as an inertance in parallel with a compliance. The T-section model formulation implies that the tonehole has null geometric length along the bore axis. However, a real finger hole results in a reduction of the effective acoustic length of the bore. This is why length corrections must be applied to the main bore in presence of a hole. The **tonehole effective geometrical height** t, the **open hole effective (*acoustical*) wavelength** t_e and the **series equivalent length** for an open or closed tonehole $t_a^{(o)}, t_a^{(c)}$ can be estimated as follows:

$$t = t_w + 0.125 \cdot \frac{b^2}{a}\left[1 + 0.172\left(\frac{b}{a}\right)^2\right] \tag{21}$$

$$t_e = \frac{(1/k)\tan(kt) + b\left[1.40 - 0.58\,(b/a)^2\right]}{1 - 0.61kb\tan(kt)} \tag{22}$$

$$t_a^{(o)} = \frac{0.47b(b/a)^4}{\tanh(1.84t/b) + 0.62(b/a)^2 + 0.64(b/a)} \tag{23}$$

$$t_a^{(c)} = \frac{0.47b(b/a)^4}{\coth(1.84t/b) + 0.62(b/a)^2 + 0.64(b/a)} \tag{24}$$

The instrument input impedance was predicted in the requested three cases basing the computation on this model (**Keefe's T-section model**). The graphic results are reported in the following figures.

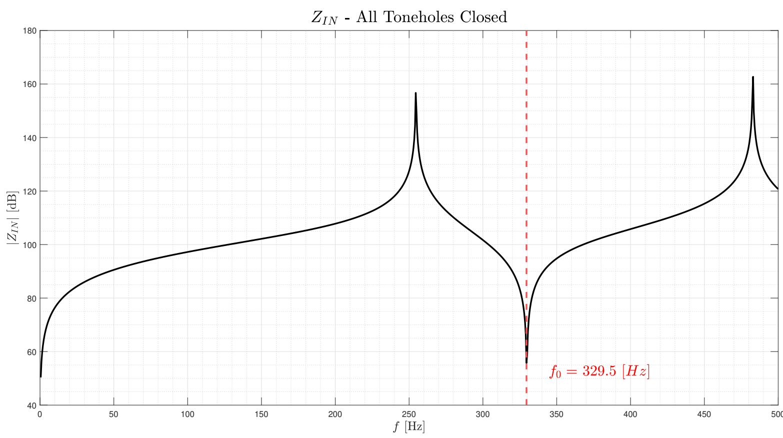


Figure 6: Predicted input impedance when all the toneholes are closed.

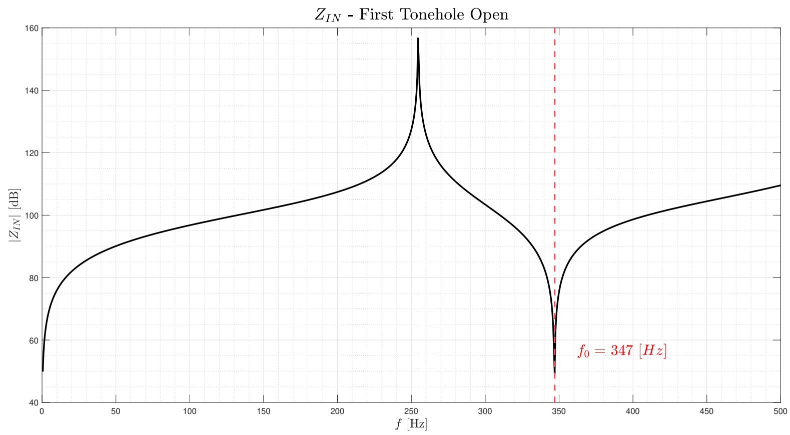


Figure 7: Predicted input impedance when the first tonehole is open.

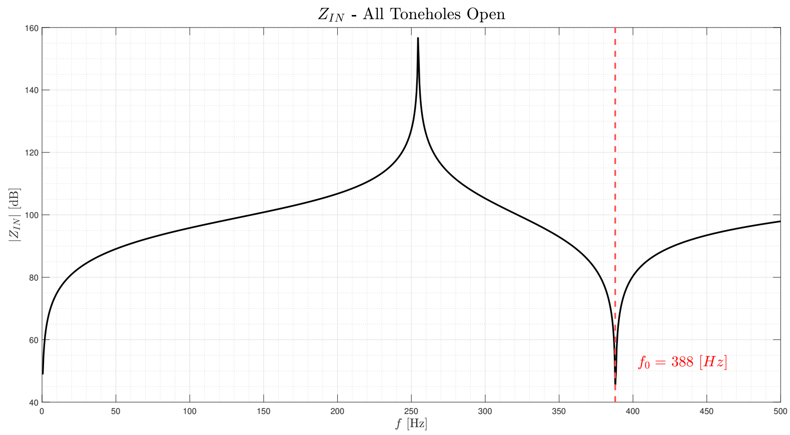


Figure 8: Predicted input impedance when all the toneholes are open.

Conclusions

The approach adopted in the dimensioning of the bore produced results that are not coherent with the actual dimensions of a treble recorder, while the positioning of the finger holes resulted in the overlapping of the two holes (since $D_0'' < 2r_h$) and in the trespassing of the first hole through the limit of the mouth of the instrument (since $D_0' > r_h$). Thus, we can state that the positioning process was not successful, being the theorized instrument impossible to build. This may be due to the large number of approximations involved both from a dimensional (*e.g.* (overestimation of the tonehole radius)) and an acoustical perspective (*e.g* acoustic length and load impedance estimation).
On the other end, the prediction of the input impedance based on Keefe's T-section model produced coherent impedance curves that show the first impedance minima in correspondence with the assigned resonance frequency, hence validating this approach in the acoustic modelling of a woodwind instrument.