Assignment

Homework HW1

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Musical Acoustics

POLITECNICO
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Characterization of a Resonator

Resonance Frequency

The resonator consists of a *mass* - $m = 0.1\ kg$ - and a *spring* whose constant is equal to $K = 2.53 \cdot 10^4\ N/m$. Its structure is represented in Figure 1.

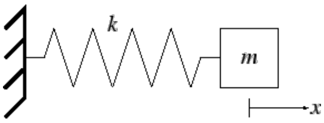

Figure 1

In order to compute its resonant frequency, we need to take into account Newton's second law:

$$F_{in} = m\ddot{x} = -Kx = F_{el}$$

By rearranging the equation and dividing both the terms by m, we obtain

$$\ddot{x} + \frac{K}{m}x = 0 \quad \longrightarrow \quad \ddot{x} + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{K/m}$ is the resonant frequency:

$$\omega_0 = \sqrt{\frac{2.53 \cdot 10^4}{0.1}} \approx 5.03 \cdot 10^2\ \ rad/s \quad \Longrightarrow \quad \boxed{f_0 = 80.05\ \ Hz}$$

Decay Time

Through experiments, it is observed that the motion of this system decays by $\Delta\mathcal{A} = -5\,dB$ in a time $t_{-5} = 0.576\,s$ (where $\mathcal{A}$ is a symbol employed to indicate a generic amplitude). In order to compute the decay time τ of the system, we express $\Delta\mathcal{A}$ according to the decibel definition:

$$10 \cdot \log_{10}\left(\frac{\mathcal{A}}{\mathcal{A}_0}\right) = -5 \quad \longrightarrow \quad \frac{\mathcal{A}}{\mathcal{A}_0} = 10^{-1/2} = \frac{1}{\sqrt{10}}$$

In accordance with the definition of the time constant τ, we can write $\mathcal{A} = \mathcal{A}_0 e^{-t/\tau}$. Thus, we are able to express the amplitude of the motion after t_{-5} seconds in two different ways and consequently solve the resulting system for τ:

$$\begin{cases} \mathcal{A} = \mathcal{A}_0 \cdot 10^{-1/2} \\ \mathcal{A} = \mathcal{A}_0 e^{-t_{-5}/\tau} \end{cases} \xrightarrow{t_{-5} = 0.576\,s} \boxed{\tau = 0.500\,s}$$

Quality Factor

Being τ equal to $1/\alpha$ and Q defined as the ratio between ω_0 and 2α (i.e. between the *central frequency* and the *bandwidth*), we can compute the quality factor as follows:

$$Q = \frac{\omega_0}{2\alpha} = \frac{\omega_0 \tau}{2} = 125.8$$

Mechanical Resistance

There are different ways to compute the system's resistance R (i.e. the *real part of its impedance*). One of the possibilities is to use the relation that links it to the resonator's mass m via the time constant τ:

$$R = \frac{2m}{\tau} = 2m\alpha = 0.3998 \ ^{kg}/_{s}$$

-3 dB Bandwidth

Let's consider the receptance of the system in exam, defined as the ratio between the displacement of the system and the corresponding harmonic forcing, for which the following expression stands true:

$$\Theta(\omega) \propto \frac{1}{\omega^2 - 2j\alpha\omega - \omega_0^2}$$

We know that the receptance decreases by $3\,dB$ for $\omega = \omega_d \pm \alpha$. Since the system is lightly damped, we can approximate the value of ω_d with ω_0. Under this assumption, the $3\,dB$-bandwidth, that corresponds to 2α, is the one depicted in the figure 2:

$$BW_{[3\,dB]} = 2\alpha = 3.998\,Hz$$

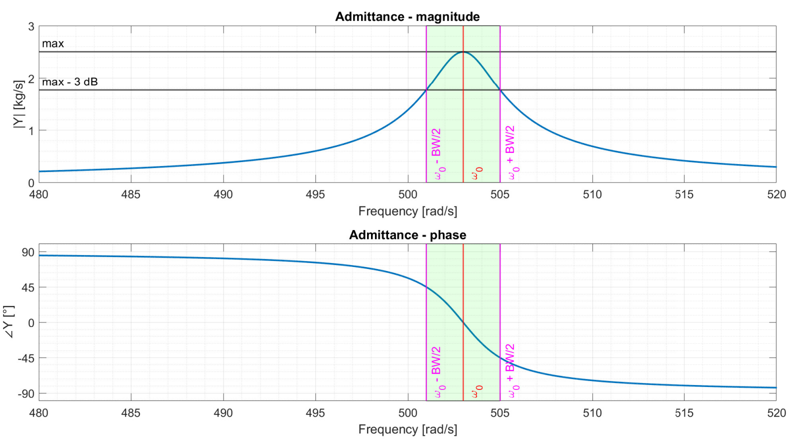

Figure 2: Admittance curves.

Time Response

Let's consider both the transient and the steady state components that contribute to the overall response of the system to an harmonic input force as the following one:

$$\underline{F}(t) = F_0 \sin\left(2\pi \underline{f}_I t\right)$$
$$\underline{f}_I = \begin{bmatrix} 60 & 80 & 100 & 120 & 140 & 160 \end{bmatrix}^T Hz = \{f_i\}_{i=1,2,\ldots,6}$$

where $F_0 = 0.1\,N$. We can express the displacement corresponding to the complete response of the system as follows, i.e. as the sum of the two aforementioned components:

$$x_i\,(t) = Ae^{-\alpha t}\cos\left(\omega_d t + \phi\right) + \frac{F_0}{\Omega_i Z\left(\Omega_i\right)}\sin\left(\Omega_i t + \angle Z\left(\Omega_i\right)\right) \quad i = 1, 2, \ldots, 6 \tag{1}$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ and $\Omega_i = 2\pi f_i$. The constants A and ϕ are to be determined via the assigned initial conditions:

$$\text{IC}: \begin{cases} \underline{x}(0) = 0 \;\; [m] \\ \underline{\dot{x}}(0) = 0 \;\; [m/s] \end{cases}$$

$$\longrightarrow \begin{cases} x_i(0) = 0 = A\cos\left(\phi\right) + F_0\left(\Omega_i Z\left(\Omega_i\right)\right)^{-1}\sin\left(\angle Z\left(\Omega_i\right)\right) \\ \dot{x}_i(0) = 0 = -\alpha A\cos\left(\phi\right) - \omega_0 A\sin\left(\phi\right) + F_0 Z^{-1}\left(\Omega_i\right)\cos\left(\angle Z\left(\Omega_i\right)\right) \end{cases} \qquad i = 1, 2, \ldots, 6$$

$$\{\phi\}_i = \arctan\left[\frac{F_0\left(Z\left(\Omega_i\right)\right)^{-1}\cos\left(\angle Z\left(\Omega_i\right)\right) - \alpha\left[x_i(0) - F_0\left(\Omega_i Z\left(\Omega_i\right)\right)^{-1}\sin\left(\angle Z\left(\Omega_i\right)\right)\right] - \dot{x}_i(0)}{\omega_d\left[x_i(0) - F_0\left(\Omega_i Z\left(\Omega_i\right)\right)^{-1}\sin\left(\angle Z\left(\Omega_i\right)\right)\right]}\right]$$

$$= \begin{Bmatrix} 0.0061 & 1.11 & -0.026 & -0.018 & -0.016 & -0.015 \end{Bmatrix}^T rad$$

$$\{A\}_i = \frac{x_i(0) - F_0\left(\Omega_i Z\left(\Omega_i\right)\right)^{-1}\sin\left(\angle Z\left(\Omega_i\right)\right)}{\cos\phi}$$

$$= \begin{Bmatrix} 8.93\cdot 10^{-6} & 4.45\cdot 10^{-4} & -7.13\cdot 10^{-6} & -3.18\cdot 10^{-6} & -1.92\cdot 10^{-6} & -1.32\cdot 10^{-6} \end{Bmatrix}^T m$$

Now, by simply substituting the two constants into equation 1, the overall response of the system can be obtained.

In the following figures, we proceed to plot and represent the impedance graph and the total response of the system for each element of the given excitation frequency array.

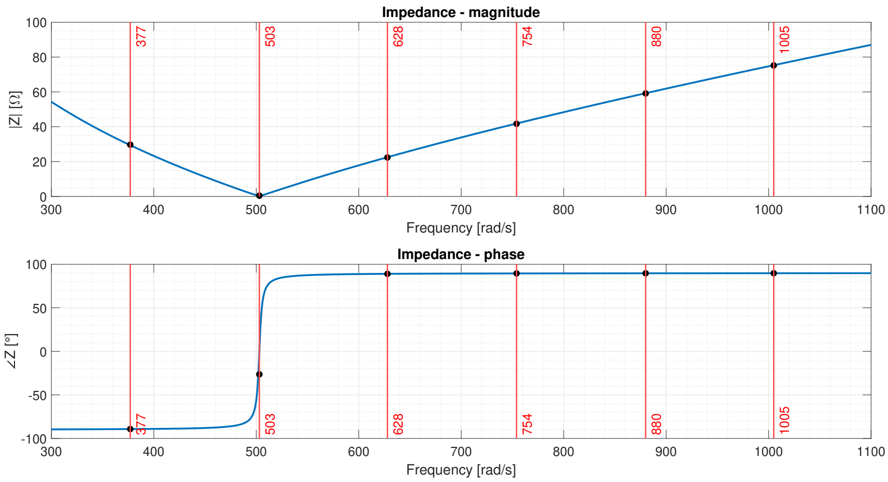

Figure 3: Impedance

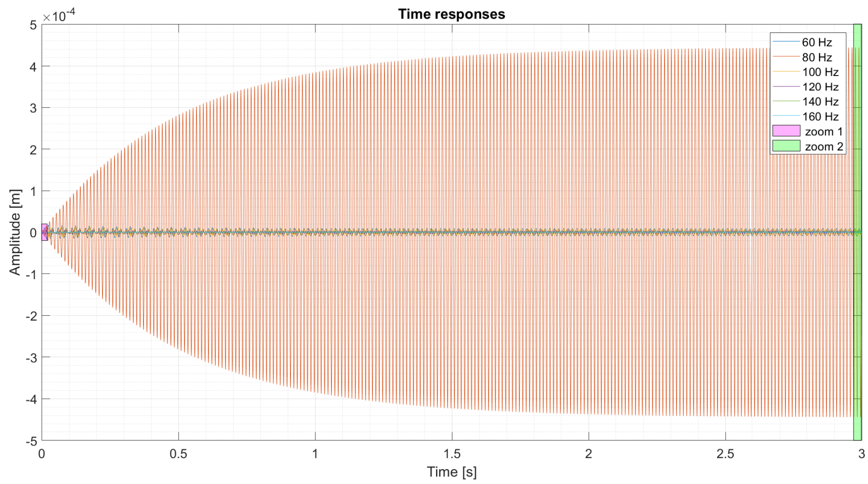

Figure 4: Time Responses

In order to increase the intelligibility of the oscillation amplitude two portions of the overall graph have been highlighted in the subsequent figures. Specifically, one magnified zone was selected as temporally distant from the transient (figure 6) and the other one was chosen just at the beginning of the motion (figure 5), where the effect of the initial conditions can be more easily noticed.

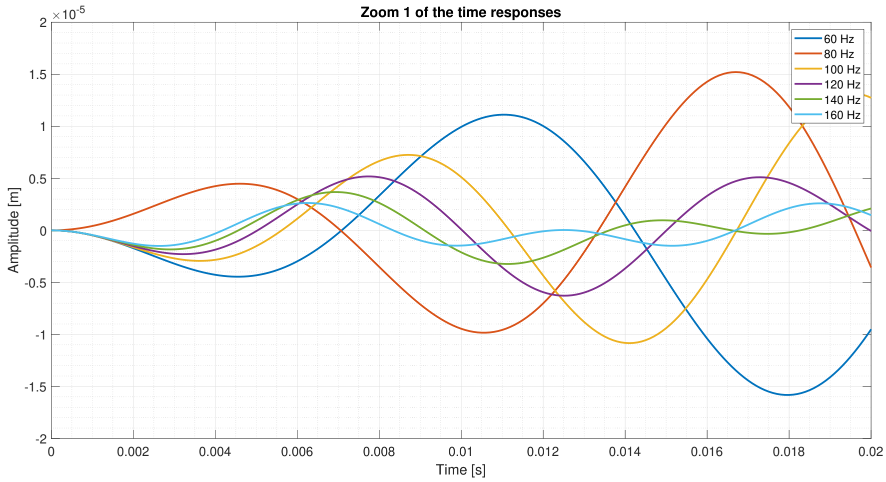

Figure 5: Zoom 1

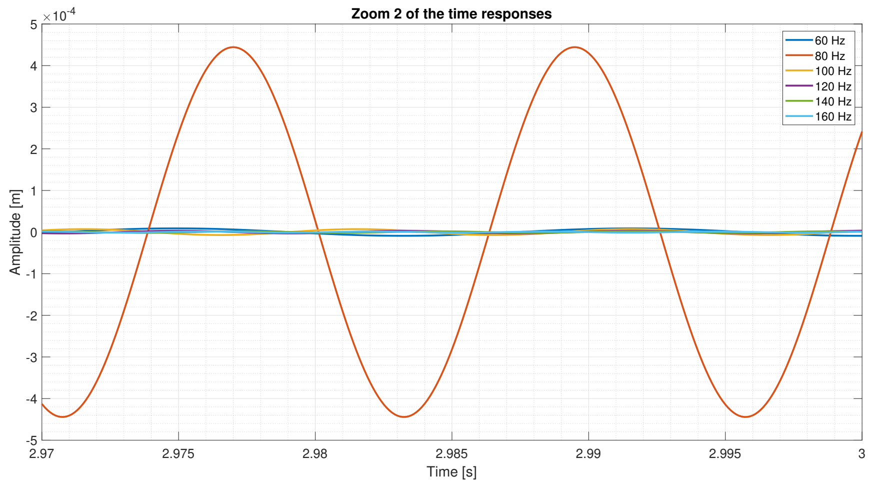

Figure 6: Zoom 2

Conclusions

As a consequence of the characteristics of the system and the chosen initial conditions, numerous oscillations are required for the system to stabilise on the steady state response (i.e for the transient to be extinguish itself) as shown in figure 4. As expected, after the extinction of the transient, the oscillation whose magnitude appears decisively greater than the others is the one that almost coincides with the admittance peak (see figure 2 for reference) corresponding to the *resonance frequency*, i.e. the one at 80 Hz. The behaviour is obviously compatible with the information given by the impedance graph in figure 3, and shows the importance of the resonance phenomenon when dealing with vibrating structures, given the amplitude gap of the oscillations when approaching resonance conditions. In particular, it is quite clear from the comparison in figure 7 that, with the only exception of the 80 Hz response, all the graphs show a sudden increase after which the amplitude decreases to reach the steady-state regime. The 80 Hz graph is the only one in which the curve trend simply increases progressively from zero to the steady-state values. These different behaviours are due to the superposition of the free motion of the system, which is governed by the natural frequency, and the response to the harmonic forcing (zero state response).

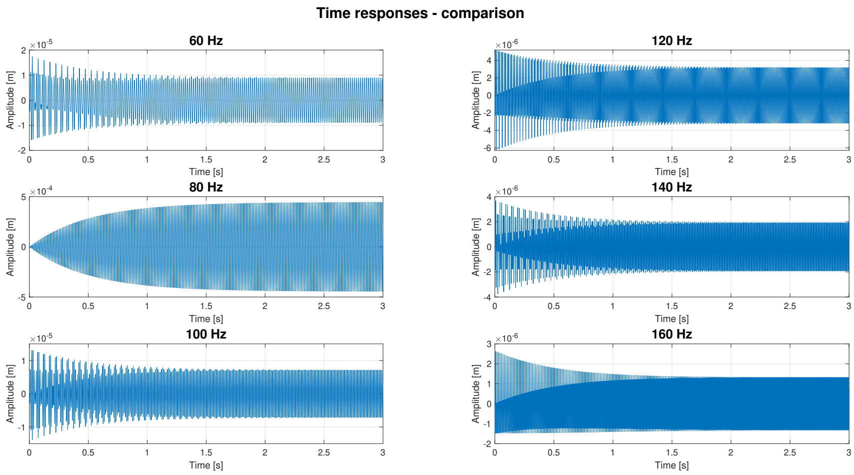

Figure 7: Time responses - comparison