Assignment

Homework HW2

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Plate characterization and string-plate interaction

Quasi-longitudinal and longitudinal waves propagation speed

The considered thin, square plate has a thickness h of 1 mm, and it is made of aluminum ($E = 69\ GPa$, $\rho = 2700\ {}^{kg}/_{m^3}$, $\nu = 0.334$). For a thin plate, the velocity of propagation of *quasi-longitudinal waves* c_L can be expressed as a function of the characteristic parameters of the constituent material:

$$c_L = \sqrt{\frac{E}{\rho\,(1-\nu^2)}} = 5363.25\ {}^{m}/_{s}$$

A similar relation describes the propagation speed for *purely longitudinal waves* in thin plates:

$$c_L' = \sqrt{\frac{E\,(1-\nu)}{\rho\,(1+\nu)\,(1-2\nu)}} = 6199.16\ {}^{m}/_{s}$$

Bending waves propagation speed

Bending waves in thin plates are dispersive, and the relationship between the frequency and the velocity of propagation of such a wave is described by the following expression:

$$v\,(f) = \sqrt{1.8 f h c_L}$$

which, as expected, represents a simple square root function (Figure 1).

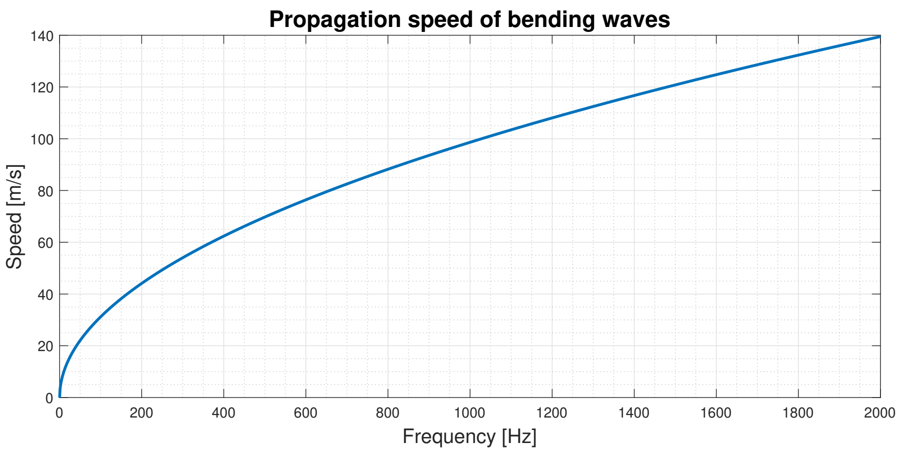


Figure 1

Plate's bending modes

The modal frequencies associated to the bending modes of a square thin plate can be expressed as function of its fundamental frequency, as illustrated in the following figure (Figure 2):

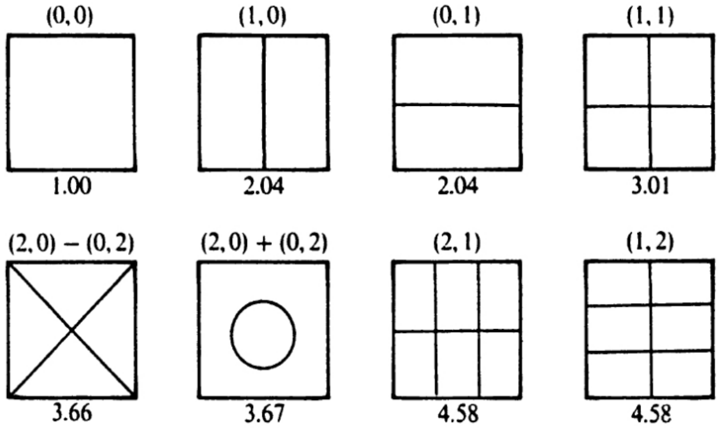


Figure 2

Obviously, the $(1, 0)$ mode and the $(0, 1)$ mode vibrate at the same frequency, just as the $(2, 1)$ and the $(1, 2)$ modes. Thus, the first six modal frequencies of the plate for bending vibrations can be determined as follows:

$$\underline{k} = \left\{1.00 \quad 2.04 \quad 3.01 \quad 3.66 \quad 3.67 \quad 4.58\right\}^T$$

$$\Rightarrow \underline{f} = f_{00} \cdot \underline{k} = \left\{394.26 \quad 804.29 \quad 1186.72 \quad 1442.99 \quad 1446.93 \quad 1805.70\right\}^T \; Hz$$

Ring and X modes

Let us now consider the plate to be made by Sitka spruce: we assume it to be realized using the quarter-cut scheme. The main dimensions of the plate are aligned with the longitudinal and radial directions of the wood. The length of the side along the radial direction is $a = 0.15\,m$. We want to predict the length b of the side parallel to the fibers so that the ring and X modes can be observed.

In order to obtain the modes that can be observed in square plates of isotropic materials even in a wood plate, the ratio between the radial and the fiber-oriented dimensions must verify the following relation:

$$\frac{b}{a} = \left(\frac{E_L}{E_R}\right)^{1/4} \xrightarrow{\textit{Sitka spruce}} b = 1.9 \cdot a = 0.29\,m$$

Tension of the coupled string

Now let us consider that a string is attached to the aluminum square plate. The string is made of iron ($\rho = 5000\,{}^{kg}/{}_{m^3}$), its cross section is circular with a radius of $r = 0.0011\,m$, and its length is $L = 0.45\,m$. Due to internal losses and sound radiation, the plate at the frequency of the considered mode dissipates energy, and the merit factor is $Q = 25$. We aim at computing the tension of the string so that its fundamental mode is tuned to the frequency of the first mode of the plate. In order to do that, we start with the expression of the propagation velocity for a wave along a string:

$$c = \lambda \cdot f_0 = 2L \cdot f_0 = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\pi r^2 \rho}} \implies T = (2L \cdot f_0)^2 \cdot \pi r^2 \rho = 2393\,N$$

Coupled system modes

The coupling of a string and a soundboard can either be *strong* or *weak*. In order to discriminate between these two cases, we refer to the following relation:

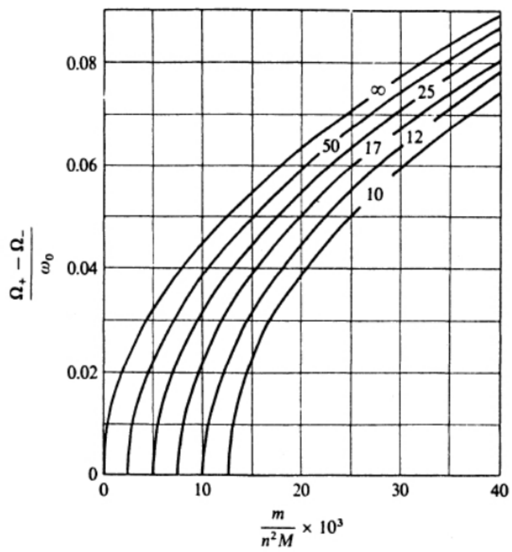
$$\begin{cases} \frac{m}{n^2 M} < \frac{\pi^2}{4Q^2} \longrightarrow \textit{Weak coupling} \\ \frac{m}{n^2 M} > \frac{\pi^2}{4Q^2} \longrightarrow \textit{Strong coupling} \end{cases}$$

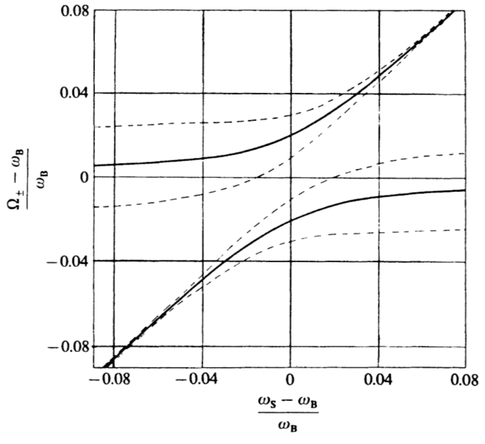
where $m = \rho_s \pi r^2 L$ is the mass of the string, $M = \rho_b abh$ is the mass of the board and n is the considered mode index. For $Q = 25$, $\pi^2/\left(4Q^2\right) = 0.0039$. The values of the ratio $m/\left(n^2 M\right)$ for $i = 1, 2, \ldots, 6$ in the examined case are:

Mode Index	**1**	**2**	**3**	**4**	**5**	**6**
$m/(n^2 M)$	0.1408	0.0352	0.0156	0.0088	0.0056	0.0039
Coupling	Strong	Strong	Strong	Strong	Strong	Weak

In order to compute the first modal frequency of the strong coupled system it is possible to refer to the graph represented in Figure 3. The relation represented is valid, in fact, only if, for the considered mode, the modal frequency of the soundboard and the one of the string are equal, and we tuned the string so that its first modal frequency coincides with the fundamental one of the board.

Figure 3





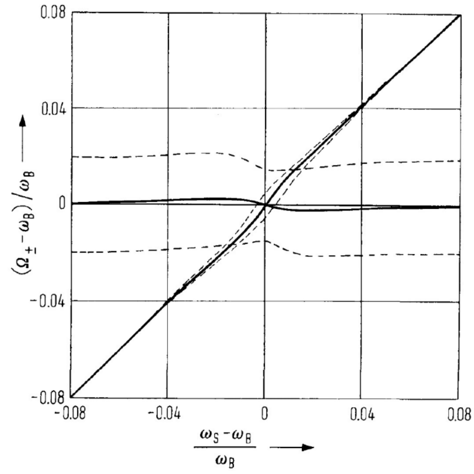


Figure 4

Since $m/M \cdot 10^3 = 140.79$, we extended the curve corresponding to $Q = 25$ by extrapolating a linear relation, which led to the following result:

$$\Delta f = f_1^+ - f_1^- = 91.72\,Hz$$
$$f_1^- = 348.40\,Hz \quad f_1^+ = 440.12\,Hz$$

Concerning the other five frequencies, we have to refer to the graphs reported in Figure 4 (in particular, to the right one for weak coupling and to the left one for strong coupling). Obviously, the modes of the soundboard that are simply a rigid rotation of other modes, are considered together. In addition, the ring mode and the X mode frequencies relative to the soundboard are both compared to the fourth natural frequency of the string. This is due to the fact that they vibrate at basically the same frequency, and thus associating the second one (i.e. the ring mode) to a higher order string mode would end up in a result incompatible with the coupling condition and in the loss of a possible natural frequency of the coupled system.

String Mode	**Board Mode**	$(f_s - f_b)/f_b$	$\Delta f^-_{\%}$	$\Delta f^+_{\%}$	$f_-\ [Hz]$	$f_+\ [Hz]$
2	$(1,0)$ or $(0,1)$	-0.020	-3	1.5	780.16	816.35
3	$(1,1)$	-0.003	-4	1	1139.2	1198.6
4	$(2,0)-(0,2)$	0.093	$\ll 1$	$\approx (f_s - f_b)/f_b$	1443.0	1577.0
4	$(2,0)+(0,2)$	0.090	$\ll 1$	$\approx (f_s - f_b)/f_b$	1446.9	1577.0
5	$(2,1)$ or $(1,2)$	0.092	$\ll 1$	$\approx (f_s - f_b)/f_b$	1805.7	1971.3

Table 1

The graph returns a percentage result in terms of the soundboard frequency. Thus, f_i^+ and f_i^- for $i = 1, 2, \ldots, 6$ are computed by applying $\Delta f^{\pm}_{\%}$ to f_{i_B}.

Conclusions

This study provides an interesting example of the effects of plate and string interactions in a coupled system. Starting from the physical characteristics of the plate and the string, it was possible to study the vibration modes and frequencies of the two objects both separately and as a complex coupled system.