Modeling inflation and Italian inflation-linked bonds

Thesis in Quantitative finance and derivatives

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Abstract

How has Italy's public debt evolved over time and what impact has inflation had on it? What is the actual composition of the debt and what are the distinctions between the mechanisms of Italy's two inflation-indexed bonds, the BTP€i and the BTP Italia?

Following the exploration of these inquiries, we conducted simulations using two distinct price index models: one grounded in historical data and the other assuming an initial inflationary shock through a mean-reverting dynamic. These models were applied to the indexation mechanisms of BTP€i and BTP Italia, and the results were examined in terms of their distribution across different scenarios.

Surprisingly, after more than 50,000 simulations, the BTP€i and BTP Italia indexation mechanisms showed the same internal rate of return in a "normal" price-increasing scenario. The situation was slightly different in the case of a price shock. In fact, in that case, the BTP Italia indexation mechanism appeared to be marginally better in hedging inflation.

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Introduction

The thesis is divided into four chapters and two appendices.
The first chapter gives an overview of public debt: its history, magnitudes, and evolution. As for the sources, in addition to those listed in the bibliography, the dataset used for the graphical representations, precisely the debt-to-GDP ratio and the absolute value of public debt, comes from the elaboration conducted by the *Osservatorio sui Conti Pubblici Italiani*. As they specified, data on debt, expressed in millions of euros and at current prices, are taken from the Bank of Italy while data on gross domestic product are instead taken from ISTAT. This chapter highlights the contribution made by inflation to both public finance indicators. The issuance of the first indexed bonds is also placed temporally and contextually.
In the second chapter, the documentation provided by the Ministry of the Treasury enables the reconstruction of not only the historical series of the composition of the public debt but also the understanding of the functioning of non-traditional instruments such as BTP€i and BTP Italia, alongside BTP Valore and BTP Futura.
In the third chapter there are two simulations of price index paths. Both are based on the Castellani, De Felice and Moriconi literature. The first price index model is based on simple historical data, while the second assumes an inflation shock that is neutralized (i.e. the inflation path returns to "normal" growth) after a certain point. The calibrations are provided and all simulations are run in MATLAB.
In the fourth chapter we will compare the behavior of BTP€i and BTP Italia. Specifically, we simulate over 50,000 different inflation scenarios using the two price models from Chapter 3 and apply some of the major bond indices to these securities. The primary objective is to compare the different approaches employed by the two bonds in managing inflation.
As for the appendices, Appendix A refers to general information on inflation-linked bonds, including a step-by-step real example of coupons and capital appreciation for BTP€i and BTP Italia. Appendix B is a mathematical appendix referring to the Ito's lemma application and integration for the second price index model.

Chapter 1

The Italian public debt: an historical overview

1.1 Historical evolution

Much has been written in the literature on the evolution of the Italian public debt. Thanks to the data provided by ISTAT and the Bank of Italy, but mainly the elaboration[1] provided by the Osservatorio sui Conti Pubblici Italiani [1], we can show the evolution over 150 years, of the public debt (on a semi-logarithmic scale due to its measures in Figure 1.1) and its relevance in relation to the GDP (Figure 1.2).
In the next subsections we will dive, with the help of various sources, into the details of each historic period. We will see not only the debt historical source details, but also the insights and impact of inflation or the effects of internal and external policies. While all the graphs are made by the data elaboration provided by the Osservatorio sui Conti Pubblici Italiani, the subsection 1.1.6 is based on a reconstruction following the same guideline as used by the Osservatorio.
The account starts from 1861, the year of Italy's unification.

[1]The absolute level of public debt is provided directly by the databases of Bank of Italy. The GDP is provided by ISTAT until the 1994, and then reconstructed by the Osservatorio dei Conti Pubblici Italiani using various forms. Debt and GDP are expressed in nominal value (i.e. current prices).

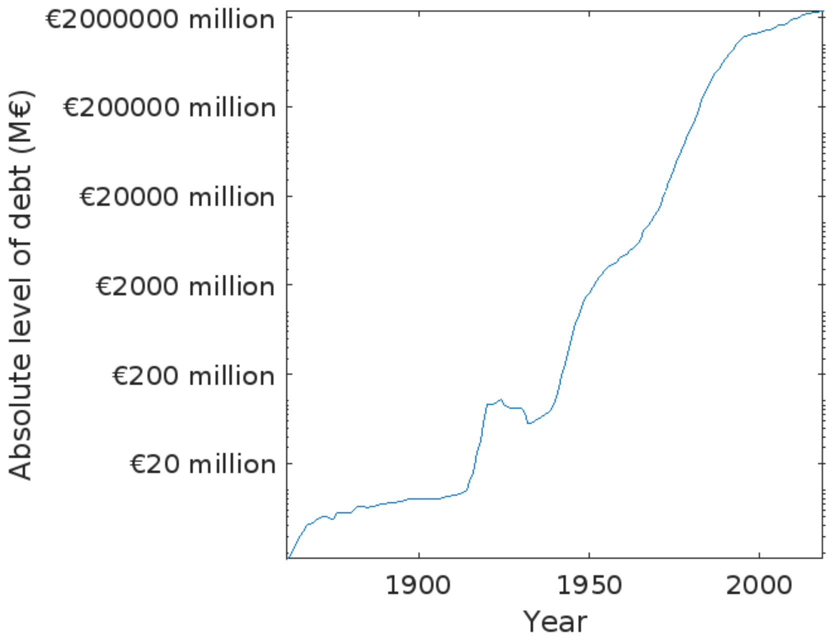

Figure 1.1: Absolute level of debt over 1861 to 2019. Semi-logarithmic scale.

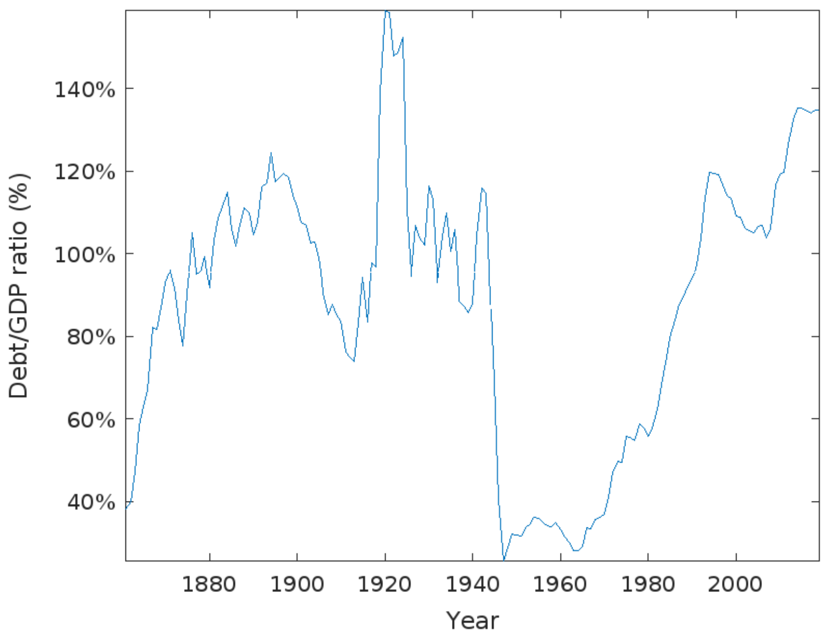

Figure 1.2: Debt-to-GDP ratio over 1861 to 2019

1.1.1 From the Unification of Italy to World War I (1861-1913)

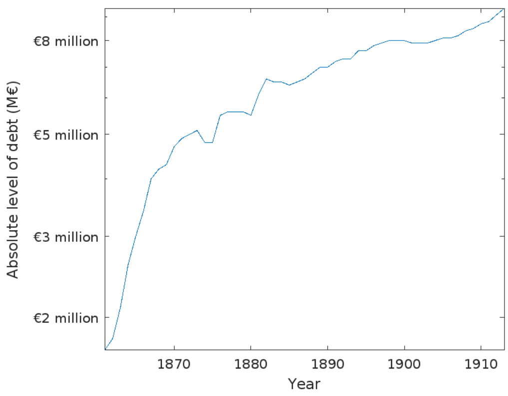

Figure 1.3: Absolute level of debt over 1861 to 1913. Semi-logarithmic scale.

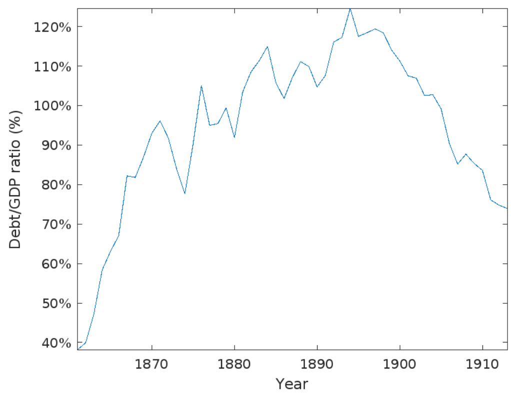

Figure 1.4: Debt-to-GDP ratio over 1861 to 1913

The Kingdom of Italy was not born debt-free. In order to ensure the respectability of the newborn State and gain the confidence of the international financial markets [2], it was necessary to assume all the debts of the pre-unitary states[2]. As a result (Figure 1.4), the country began its history with a public debt-to-GDP ratio of 39%, a figure not high by today's European fiscal standards, but certainly relevant in the historical context considered.

Over the next 40 years after unification, the debt-to-GDP ratio rose rapidly, from 39% in 1861 to 111% in the early 20th century, with a peak of 124% in 1894. The reason of this growth is to address to a sharp increase in the public spending in order to support a vast program of public works. The areas of government intervention range from military expenses (due to the military unification of the state) to war payments and public investments [4]. It could not have been otherwise: the unification took place in the midst of the Second Industrial Revolution, an economic process in which Italy participated as a heavily agricultural country with weak industry and an unfinished transportation network [5]. In addition, the unified Italy was still without some territories, which led to the Third War of Independence (1866), the first fought by the newly formed Kingdom, in which Veneto, the province of Mantua and part of Friuli were annexed, and the seizure of Rome (1870). In both cases, all the debts of the conquered territories were assumed by the Italian government [4]. These high expenditures were only gradually supported by ample "own" revenues. The report of the General Directorate of the Public Debt referred [6] to the first years after unification as those of "emergency financing": a period characterized by frequent borrowing, hasty public disposals and a tax system implemented over time[3]. The steep trend (Figure 1.3) in government debt in absolute terms also reflects this. The peak in the growth of the debt-to-GDP ratio

[2]The *"Grand Book of the Public Debt"*, the register where all state debts were recorded, was created, following the example of what was done in France with the *"Grand-livre de la dette publique"*. See [3].

[3]It is important to highlight the contribution to the first balanced current budget, excluding interest and investment expenditures in 1876, an important milestone achieved through unpopular measures. See [7].

occurred nevertheless at the turn of the century, due to the trade crisis with France, the abolition of metal non-exchangeability, and heavy spending on public works and railroads [8].

From the beginning of the 20th century, the Italian economy started to succeed, thanks to a positive global growth. This was the period of industrial takeoff: the time when the country benefited from past infrastructure investments coupled with a positive global trend. This period is commonly known and referred to as the "*Belle Epoque*" [5]. For our purposes, it is important to mention how this growth benefited the Kingdom's debt: indeed, from 1900, when the debt level was around 105% of GDP, the debt-to-GDP ratio began to slowly decrease until it reached 73% in 1913. Given the positive outlook and the economic growth of the Kingdom, it is also worth to mention [6], emphasizing the international trust in the Italian Treasury, the Annuity conversion operation in 1906, one of the greatest financial operation conducted by the Italian government in which it was offered to holders of securities bearing 5% gross and 4% net annuities the choice between converting to a newly created security, bearing interest at 3.75 percent until December 31, 1911 and 3.50 percent from January 1, 1912 onward, and repaying the debt at par. A successful financial operation in a context in which other countries also did the same, not always with the same results, such as France in 1902, as described by De Cecco [9].

Thus, in the first 50 years of its existence, the current expenses of the Italian state were financed by different methods: taxes and the sale of public assets, direct monetization[4], loans and the issue of various forms of fixed-interest securities. Regarding the composition of the public debt, thanks to Zamagni [10] it is possible to state that the main composition of the debt was long-term maturity fixed-income instruments (if not, perpetual) and postal financial collection.

1.1.2 From World War I to the aftermath of World War II (1914-1947)

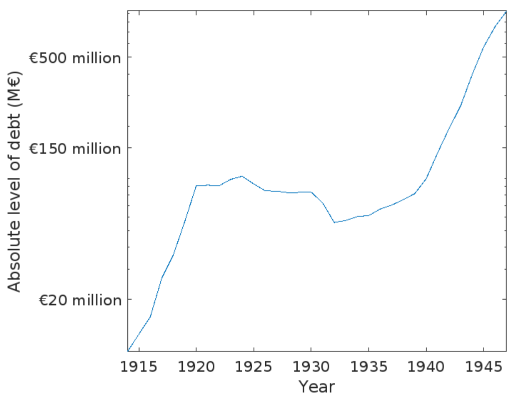

Figure 1.5: Absolute level of debt over 1914 to 1947. Semi-logarithmic scale.

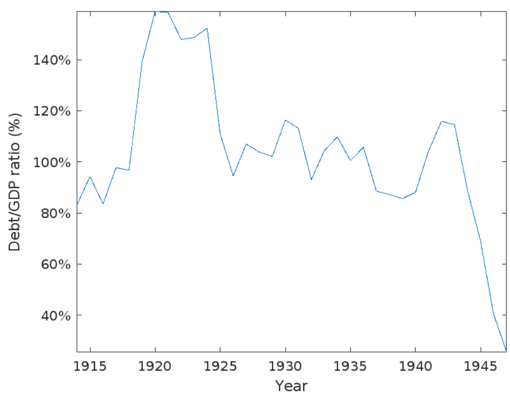

Figure 1.6: Debt-to-GDP ratio over 1914 to 1947

The outbreak of the World War I slowed down the positive prospects of global development. As far as Italy's finances were concerned, the war and its economic

[4]The path of direct debt monetization, including its price consequences, is well described in [6].

and financial consequences brought the debt-to-GDP ratio to an all-time high of 159% in 1920 as shown in Figure 1.6. From the Italian point of view, the World War I was mainly financed by debt, as pointed out by Carnazza [4] and Zamagni [10]. The war effort caused a rapid and exponential increase (Figure 1.5) in public debt: in absolute terms, the Osservatorio sui Conti Pubblici Italiani estimates [1] this increase at more than 800%. This fact has several consequences worth considering. First, most of the debt was financed abroad[5], as the domestic market was unable to meet the growing needs of the state. This was a new fact compared to the previous 50 years, when the state budget was supported on the one hand by domestic and international loans, but mainly by alienations and taxes. Figure 1.5, which shows the absolute increase of the debt, provides direct evidence of this. Additionally, as Répaci [11] pointed out, the administrative management of the budget exacerbated public finance indicators. Disconcerted control, extra-budgetary expenditures, and confusing, sometimes excessively distorting tax legislation greatly contributed to the increase in debt during the war. Besides, post-war inflationary expectations (but also in-war inflation) contributed for the first time to the preference of investors for short-term securities, as pointed out by Francese and Pace [12].

The war ended and the Italian public finances recovered until the Great Depression, thanks to rampant inflation, the cancellation[6] and rescheduling[7] of part of the debt owed to the United Kingdom and the United States, and a general economic recovery. The decrease in public spending during the early years of the regime cannot be underestimated either, contributing to the decline in the ratio [8]. Before the great depression, in 1928 the debt-to-GDP ratio was around 103%.

The debt-to-GDP ratio started to deteriorate again in the 1930s, given the crisis of 1929: while in the U.S. the *New Deal* program began, Italy was also going down the path of deficit spending with heavy investments, with the only difference that were aimed at autarchy and self-sufficiency [5]. The main source of financing for this road was the issuance of long-term bonds, the rescheduling of fixed-income securities yield[8] and the strengthening of the postal collection. The latter will be central to financing the war effort in World War II.
Moreover, while the Ethiopian War of 1935 was financed by forced loans and extraordinary taxes [6], it is also worth mentioning the exit from the gold exchange standard in 1936, which paved the way for direct monetization[9] of debt and its consequences [4]. Not surprisingly, what improved the debt-to-GDP ratio was the impact of an inflationary component in the late 1930s. A component that will be unleashed during the war contributing to much improvement in public finance indicators.

The World War II debt financing, in contrast to the World War I, was based mainly on national savings, direct monetization and domestic debt [10], fueled by

[5]And more specifically, with inter-allied loans. See [8].

[6]The war payments were sizable until 1925 and 1926, the years in which Washington and London agreements were signed. As noted by Astore and Fratianni [13] "with the two debt agreements Italy obtained an average haircut of 84% percent." The haircut must be considered not in the nominal value but in real value.

[7]There was a modification made by the regime in 1926 with the "Prestito del Littorio," which forced the conversion of some previously issued loans [14]. An operation substantially similar to the one of the 1906 but methodologically different [4].

[8]In 1934 another debt conversion happened: consolidated debt bearing 5 percent was converted to 3.5 percent [15]. An unsuccessful operation in light of investors behavior [6].

[9]As pointed out by Maier and Catanzaro [16], in the period 1938-1944 the prices increased by a factor of 16, whereas money supply increased by a factor of 14.

patriotic sentiments and propaganda systems, as well as, of course, political motivations. Again, extra-budgetary management took place. Several bodies were created to organize the state at war, resulting in the fragmentation of the budget and breaking its unity [11]. The public debt-to-GDP ratio rose to a maximum of 114% in 1943 and then fell sharply, thanks to the effects of high rampant inflation. In fact in the immediate aftermath of the war, between 1945 and 1947, very high inflation brought the national debt to 25% of GDP (See Figure 1.6), since all debt issued has always been at fixed rates [6]. An optimal starting point for the revival of the next period: that of the economic boom.

1.1.3 The economic boom and the 1970s (1948-1970)

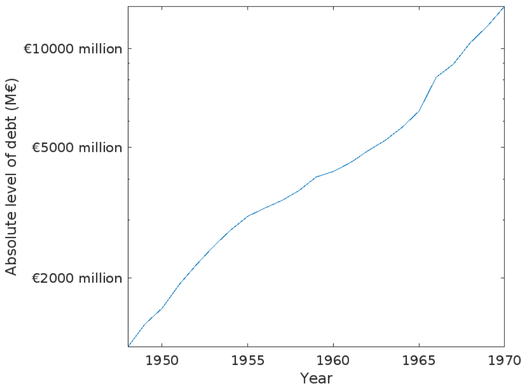

Figure 1.7: Absolute level of debt over 1948 to 1970. Semi-logarithmic scale.

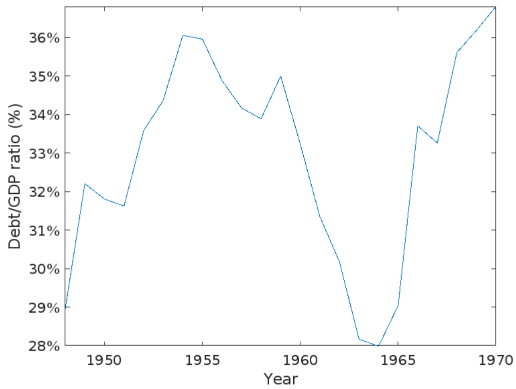

Figure 1.8: Debt-to-GDP ratio over 1948 to 1970

The years after the World War II marked significant economic and political changes. Although the regime was blamed for the financial collapse, it is crucial to note that Italy chose not to repudiate its debt and instead honored all the obligations of the prior administration[10] [17]. Therefore, in 1948 the debt-to-GDP ratio is 28% (Figure 1.8). The adherence of the newborn Republic to international systems, specially the Bretton Woods agreements, facilitated a rapid increase in production. A system of fixed exchange rates, international cooperation[11], the availability of technology and capital, as well as cheap labor and determined government intervention were just some of the key elements that allowed the economy to exponentially grow after World War II [5]. This is also evident from the evolution (Figure 1.7) of public debt over the 15-year period after the war[12]: in light of the consistent and constant rise of public debt in absolute terms, the same increase cannot be asserted about the growth in debt-to-GDP ratio. In fact, this ratio was still 28% in 1964. At this first stage, in the immediate post-war period, as highlighted by the report of the General Directorate of Public Debt [6] and Francese and Pace [12] and given inflationary

[10]Exactly as done with the unification of Italy.

[11]International cooperation included not only reconstruction aid, such as the Marshall Plan (starting in 1948) and the Mutual Security Program (from 1951) or food aid from Unrra in 1947, but also international trade and the reduction of the autarchic initiatives that were born during the regime and also flourished in other countries after the crisis of 1929 [18].

[12]Public debt has historically been focused on funding capital expenditures rather than current ones. Domestic resources have largely financed the latter during the economic boom [6].

fears, a significant part of the public debt was represented by BOTs, short-term fixed-income securities. The use of BOTs was appreciated by the market and useful to reduce the use of direct monetization, a major cause of inflationary flare-ups and problems in the exchange rate. The focus of investors shifted in the 1950s when securities maturing after one year and postal deposits gained prominence [6].

Starting from the 1960s, the country experienced excess demand inflation during a period of economic growth. To tackle this problem, the National Bank decided to implement a temporary restrictive monetary policy, leaving the support of economic growth[13] exclusively to fiscal policy [4]. Similar to other European countries, there was a rise, which effects will be especially visible in the following decades, in spending and debt tied to the implementation of diverse welfare state systems during this period [20]. Regarding the composition of public debt in the 1960s, after the slowdown of short-term securities in favor of medium- to long-dated securities and especially postal funding at the end of the 1950s, the share of monetary financing returns to increase [6], in a sort of forced coexistence between the Banca d'Italia and the Treasury, as indicated by Spaventa [19]. By all means, what is expressed above is the reason why at the early 1970s, the debt-to-GDP ratio stood at 36%.

1.1.4 The 1970s and 1980s (1971-1990)

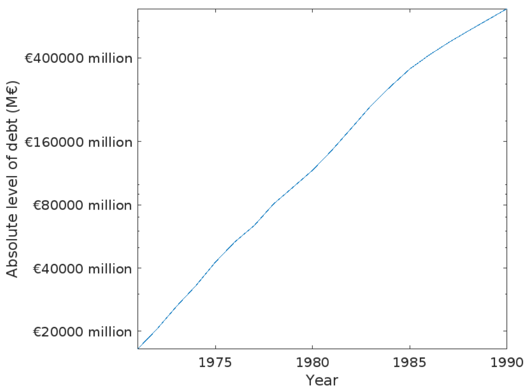

Figure 1.9: Absolute level of debt over 1971 to 1990. Semi-logarithmic scale.

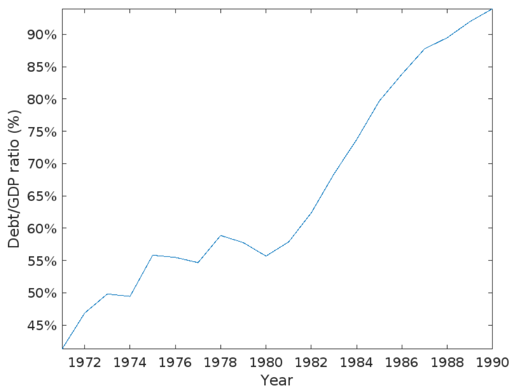

Figure 1.10: Debt-to-GDP ratio over 1971 to 1990

The 1970s and 1980s were quite a difficult time economically and financially. In 1971, the United States unilaterally withdrew from the Bretton Woods Agreements: the world moved from a system of fixed exchange rates to a system of floating exchange rates, and currency markets experienced a period of severe turbulence [21], with serious consequences for export-oriented countries. In 1973 and 1979, the situation in the Middle East led OPEC to raise the price of oil significantly, with enormous repercussions for importing countries, including Italy [22] [23]. The main consequence of this was an high rate of inflation [5]. However, in contrast to the periods of the World Wars, the debt-to-GDP ratio did not fall, but rose, for many reasons. In fact the inflation was incorporated into investors' expectations [19] and the continuing large deficits to support the massive welfare spending plan that began

[13] Until that time there was a peaceful coexistence between fiscal and monetary policy as indicated by Spaventa [19].

in the late 1960s were only partially backed by increased revenue[14]. Added to this was the support provided by the government to help households and businesses cope with the inflationary crisis [5]. Therefore, what happens is that until the 1980s, the debt-to-GDP ratio rises discontinuously (Figure 1.10): welfare spending has become standard practice (sometimes for even simple electoral purposes [25]), and only phenomena such as fiscal drag and increased tax revenues (and of course, inflation) have helped to keep this ratio from growing excessively [4]. It is also important to consider the Bank of Italy's contribution to keeping interest rates under control, given that a significant portion of the public debt was directly monetized and then in its hands [26]. Regarding the composition of public debt, although the 1960s saw a readjustment of debt in favor of longer-term securities, from the 1970s onward, due to oil shocks, currency market turbulence and prices level, investors began to prefer short-term securities. Suffice it to say that in 1981 the average maturity of government debt on the market was 9 months! [6]

While in 1970 the debt-to-GDP ratio amounted to 36%, in 1980 it reached 55%. Public finance indicators worsen during the 1980-90 decade.
Fighting inflation, now rampant[15] and a cause of social problems, was a key issue at the time. This is why Italy joined the European Monetary System in 1979, accepting a semi-fixed exchange rate system[16], issued the first medium-term bonds indexed to the BOT yield[17] (also noteworthy is the unsuccessful first attempt to issue fixed-income securities in real term, with the CTR bond[18] [28]) and finally led to the 'separation' of the Treasury and the Bank of Italy in 1981, effectively limiting the direct monetization of debt and expanding and enhancing the autonomy of the national institute. These were not the only disinflationary measures. They were complemented by wage moderation, partly induced by unemployment and partly by the depowering of the sliding wage scale[19] with the result that effectively inflation decreases to a minimum of 4,7% in 1987. However, the reduction in inflation and its impact on debt was not followed by an adequate spending review [26]. On the contrary, the high level of public spending and the independence of the Bank of Italy in the financing of the public debt led to a sharp rise in interest payments, with negative consequences for the public budget, which was burdened by enormous interest payments that reached almost one-fifth of total public spending in the early 1990s [4].

As a result, the debt-to-GDP ratio rose steadily to 93% in 1990.

1.1.5 The 1990s and the new millennium (1991-2019)

The 1990s was a period of great political and economic transformation. The upward trend in the debt-to-GDP ratio of the previous decade continued, despite attempts

[14]In the 70's there was a huge tax reform that established IRPEF and IVA taxes (and more) [24].

[15]Between 1973 and 1984 it never fell below 10 percent [27], with peaks of 18 percent in 1978 and 21 percent in 1980 [18].

[16]The exchange rate could fluctuate within a well-defined band.

[17]CCTs were issued with a maturity of three years [6]. These were bonds whose coupons were indexed to the short-term interest rate, useful to reduce the risk of capital loss for subscribers [18].

[18]CTR bond was a fixed-income security in real term which was issue for the first and last time in 1983. It was unsuccessful due to the complexity and slowness of the indexation, based on the GDP implicit deflator.

[19]Which was the indexation of full or partial wages to inflation.

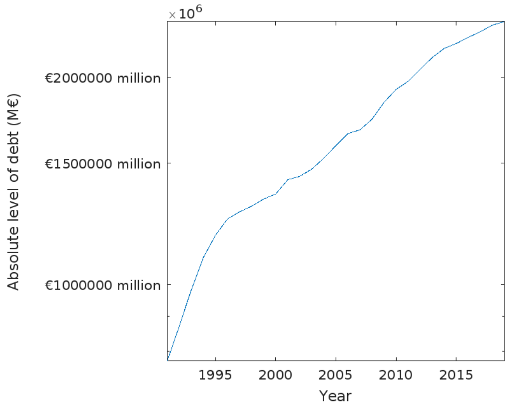

Figure 1.11: Absolute level of debt over 1991 to 2019. Semi-logarithmic scale.

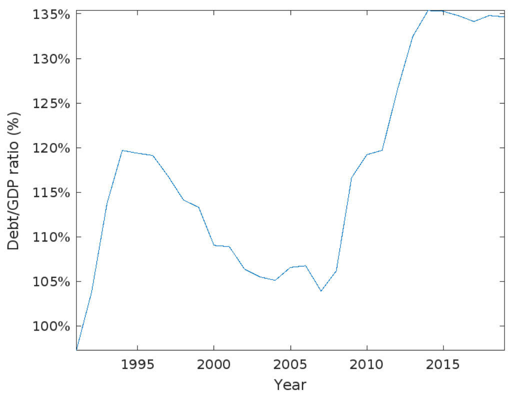

Figure 1.12: Debt-to-GDP ratio over 1991 to 2019

in the early 1990s [29] and due to high debt burdens, until 1995, when the debt-to-GDP ratio reached almost 120% (Figure 1.12). After 1995, the ratio of public debt to GDP began to decline: this result was mainly due to the process of European integration, identified in 1992 with the Maastricht Treaty (whose basic concepts were also reaffirmed in 1997 with the Amsterdam Treaty, i.e. the Stability and Growth Pact), which provides for convergence mainly on issues related to public finances. Precisely to allow convergence with the strict European parameters, the period of "privatizations" begins [30], during which the state puts more than 30 companies on the market, raising more than 100 billion euros [31] and contributing decisively to the reduction of the level of public debt to 106.4% in 2005. Of course, this was not the only instrument used in those years: Carnazza [4] highlights the increase in taxes or the reduction in public spending, while Battilani and Fauri [18] also focus on the important savings in terms of the cost of debt, stimulated by a situation of low interest rates and a correct fiscal "posture". The issuance of Italy's first inflation-indexed bond in 2003 [32], the BTP€i, indexed to European inflation (through the HICP Ex-tobacco index), is worth mentioning among the recent developments in public debt.

The downward trend in the debt-to-GDP ratio is interrupted in 2008 with the Great Recession. One of the first effects of the 2008 crisis is a sharp fall in GDP. In fact, as reported by ISTAT [33] in 2011, the main cause of the increase in the debt-to-GDP ratio is the so-called "snowball effect": that is, the combined effect of a low GDP growth rate and high interest rates[20]. Added to this, of course, were the counter-cyclical measures taken by the government that have affected the deficit. In 2010, the debt-to-GDP ratio reached 119 percent, up from 103 percent before the crisis. Worsening the public finance indicators is the ensuing sovereign debt crisis, which began with the public revelation of the manipulation of Greece's public accounts in late 2010. Italy, a heavily indebted country with weak growth in previous years, was one of the countries most affected by debtor distrust. This led to a sharp rise in Italian bond yields: in 2011, at the peak of the crisis, the spread over German government bonds even exceeded 500 basis points, according to the IMF's reconstruction [34] (Figure 1.13). In this complex situation, the Treasury

[20] *"According to the snowball-effect, the debt-to-GDP ratio tends to increase if the GDP growth rate is lower than the interest rate paid on public debt"* (Semik, Zimmermann 2022).

initiated the inaugural issuance of BTP Italia in 2012, aiming to engage investors and provide reassurance, particularly in terms of real return [35]. As far as gross

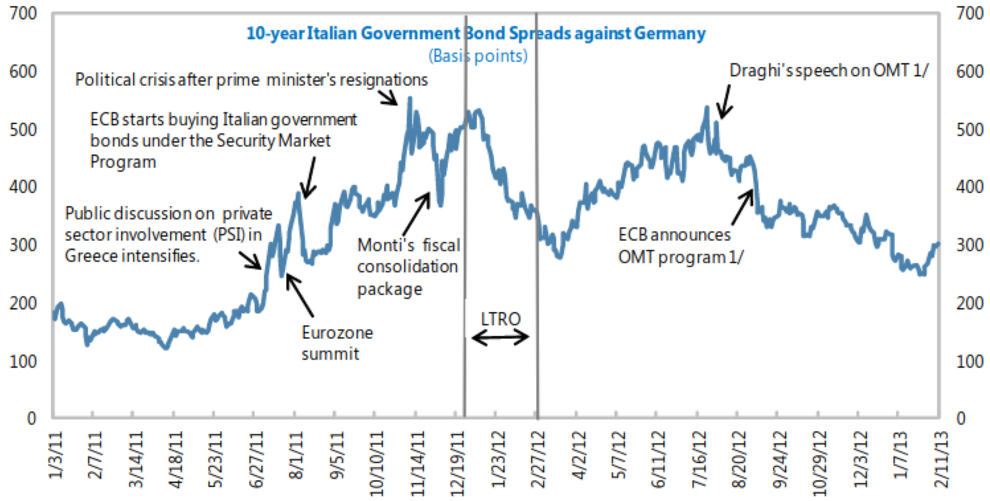

Figure 1.13: Italy's spread reconstruction. Source IMF

domestic product is concerned, suffice it to say that in 2014 Italian GDP was still below its pre-crisis peak[21]. This has an obvious impact on the debt-to-GDP ratio, which has risen to 135% in 2014. From the sovereign debt crisis to 2019, one step away from the COVID-19 pandemic, the situation hardly improves. One broad explanation to hold for this phenomenon, is that given by Bini Smaghi [36]: the snowball effect has always been around the corner, and seeing the data[22] (Figure 1.14) it is quite plausible. In fact, the country has suffered from low growth and

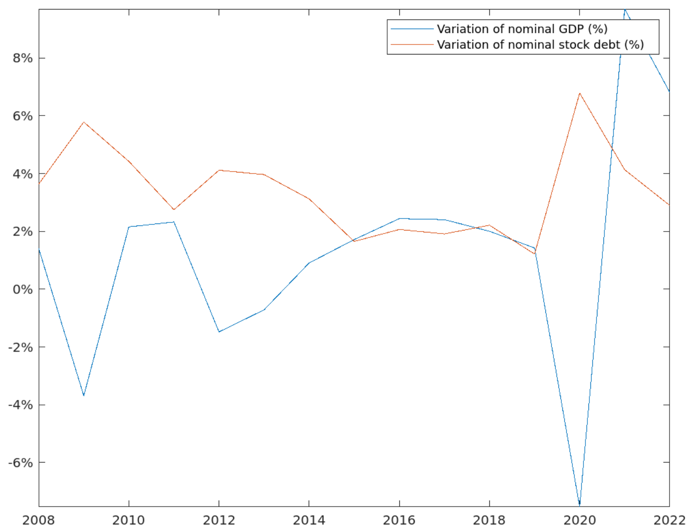

Figure 1.14: The snowball effect? Source: elaboration on Bank of Italy and ISTAT data

remarkable interest rates, in addition to a high share of GDP absorbed by debt

[21]Source: ISTAT

[22]The source of the nominal variation of GDP is a computation based on ISTAT data. The percentage variation of nominal stock of debt is based on CPI elaboration on Bank of Italy data.

service, even though quantitative easing has helped keep interest rates low. For this reason, the public debt-to-GDP ratio remained stable at 134% in 2019.

1.1.6 Today's situation (2020-)

According to the Bank of Italy and ISTAT, using the same way of measuring the quantities used in the previous subsections, we can summarize the public debt indicators in the three-year term as indicated in the Table 1.1.

Year	Variation nominal GDP	Variation abs. level of debt	Debt-to-GDP ratio
2020	-7.54%	6.79%	154.92%
2021	9.70%	4.13%	147.06
2022	6.81%	2.90%	141.67%

Table 1.1: Public finance indicators over 2020 to 2022. Source: ISTAT and Bank of Italy.

As described in the Table above, the COVID-19 effects on the public finance was terrible. The slowdown of the economy, brought an huge decrease in the GDP of 2020, which has been only partially recovered in real term in 2021. In conjunction with this, the absolute level of public debt between 2019 and 2020 increased by 6.79%: indeed, it must be considered the decrease in tax revenue and the *"extensive funding of social safety nets"* [37] during the whole pandemic period which contributed to this debt increase. Although the debt has increased, interest rates have been kept lower than they would have been without the ECB's monetary policy measures, in particular the Pandemic Emergency Purchase Program (PEPP). As described in the 2021 Public Debt Report [38], provided by MEF, the 2021 was the year of the "rebound", in which the economy started to the recovery, thanks to vaccination, and a still accomodating monetary policy. With regard to the contribution to the public finance indicators, however, it is important to note that in both 2021 and especially 2022, albeit in the context of an economic recovery, a non-negligible part of the contribution to the increase in nominal GDP comes from inflation. Inflation driven by energy costs and bottlenecks in the global supply chain. In March 2021, as indicated by the Bank of Italy [39], inflation reaches the highest level since the 1990s, while in 2022 it worsens by reaching almost 9% on average.

At the end of 2022, the debt-to-GDP ratio was addressed at 141.67%.

Chapter 2

The public debt structure

Public debt is a general term that refers to the entire structure of government liabilities. For our purposes, we will focus on the financial instruments in the regulated markets. The source of the information below is the breakdown of government debt provided by the Department of the Treasury [40].

2.1 Current government securities

2.1.1 Treasury bills (BOTs)

BOTs are short-term bonds with a maturity up to one year (3, 6 and 12 months). Remuneration is based on the difference between the issue price and the redemption price, which classifies them as zero-coupon bonds. The redemption is at par, with a single payment at maturity.

2.1.2 Treasury Bonds (BTPs)

BTPs are treasury bonds with medium/long term maturities. The minimum is 18 months and the maximum is 50 years. The remuneration is a semi-annual coupons in arrears and possible discount at issuance. The redemption is at par, with a single payment at maturity.

2.1.3 Treasury Bonds Green (BTPs Green)

BTPs Green are normal BTPs with the details that any income it is bound within the ecological transition. Thus, the remuneration is based on semi-annual coupons in arrears and possible discount at issue. Maturities are usually over 10 years. The first BTP Green was issued by the Italian Treasury on 2021. Another peculiarity of this security is that each investment paid for by the proceeds of the bonds is written in the "Allocation Impact Report". See [41] for more information.

2.1.4 Treasury Certificates (CCTs-eu)

CCTs-eu are floating-rate securities, with a maturity between 3 and 7 years. As described by the Treasury "interest is paid with semi-annual coupons in arrears indexed to 6-month Euribor[1]. The difference between the nominal value and the issue price accounts for the yield.”. As we outlined in the subsection 1.1.4, CCTs were first issued in the 1970s to attract capital in a context of high inflation. It is important to note that the CCT has not always been the same: in 2010, the Italian Treasury launched the current CCT-eu, gradually replacing the existing (and original) CCT. The difference is that while the original CCT was indexed to short-term BOT, the CCT-eu is indexed to Euribor. The reason is well described by the press statement [42] issued by the Treasury: there was the need “enlarge to the investors base in floater instruments issued by the MEF as well as to improve the efficiency and liquidity of the secondary market of such instruments”. In addition, it is a better hedge against other indexed liabilities to have a floating-rate security indexed to Euribor. De Felice, Moriconi and Salvemini [43] refer to this instrument, citing Spaventa, as a “centaur”: a short-term security with regard the yield but a long a term security with regard the maturity. Not surprisingly, an instrument used to relieve pressure on the Treasury during the years of roaring inflation.

To obtain the indexed coupon, it is necessary to calculate the gross annual simple yield by adding to the six-month Euribor rate[2] the spread defined when the bond was issued (spread that remains constant throughout the life of the security). Thus:

$$Gross\ annual\ simple\ yield \ = \ Euribor\ six\ months \ + \ spread \tag{2.1}$$

Let r_t be the the six-month coupon rate and dd the actual days of the reference semester. Then:

$$r_t \ = \ Gross\ annual\ simple\ yield \ \cdot \ \frac{dd}{360} \tag{2.2}$$

Ultimately, the coupon amount is calculated multiplying the 6 month coupon rate by the bond’s face value (in hundreds).

2.1.5 Treasury Bonds Linked To Euro-Zone Inflation (BTPs€i)

BTPs€i are medium/long term coupon bonds whose coupons are adjusted for inflation in the euro zone and are paid semi-annually. In addition, the principal to be repaid at maturity is also adjusted for inflation. The inflation index is the Harmonized Index of Consumer Prices (HICP) excluding tobacco. In case there is a price fall, the principal is still redeemed at its nominal value. The maturities are between 5 and 30 years. BTPs€i were issued the first time in 2003, to protect investors from European inflation. Cannata [32] states the reasons for those new kind of products: it was necessary to broaden the audience of investors in Italian public debt and “reduce the volatility of the cost of debt in real terms”. Although at first glance an indexed bond may appear to be detrimental to government finances, especially after the historical insights of the previous subsections[3], there are actually positive

[1]As defined by the European Money Market Institute, the Euribor is “the rate at which wholesale funds in euro could be obtained by credit institutions in current and former European Union and European Free Trade Association countries in the unsecured money market”.

[2]To be more specific, “the six-month Euribor is observed on the second business day before the first accrual day of the coupon (according to what is published on the page of the Reuters circuit EURIBOR01, at 15:00 a.m. CET, or from another source of equal rank in case this is not available”.

[3]Inflation has historically benefited, especially in the period immediately after both World Wars, the government budget.

effects. Such instruments can extend the reach of the market, allowing financing even during periods of inflation and at potentially lower cost. As far as the latter is concerned, it is enough to think back to the end of 1980s. Expectations of future and persistent inflation were built into investors' behavior, with the result that they opted for short-term or, at best, medium-term securities with high interest rates. In such a situation, the issuance of indexed bonds can help public finances through the availability of securities that, after the period of inflation has passed, return to a yield that is not excessively high and costly for government budget.

An indexed bond can be considered as a fixed-income security in real term. In fact, as directly defined by the Treasury BTPs€i "provide constant rates of interest in real terms, that is in terms of purchasing power, fixed at their date of issue (known as the real annual coupon rate)". To understand and apply indexation, we need to link the coupon and the principal to the rate of inflation. Let's start with the indexation of the principal and introduce the *Indexation Coefficient*, which allows the calculation of adjusted values of the nominal principal amount, for a day d of a month m, on the basis of price inflation. The formula is:

$$IC_{d,m} = \frac{Reference\ Inflation_{d,m}}{Base\ Inflation} \tag{2.3}$$

where:

- Reference Inflation is the inflation for a day d of a month m and it is obtained from Eurostat Indices for the second and third month prior to the month for which the calculation is being made. The formula is as follows:

  $$RI_{d,m} = EI_{m-3} + \frac{d-1}{gg}(EI_{m-2} - EI_{m-3}) \tag{2.4}$$

  $RI_{d,m}$ is the reference inflation of a day d of a month m;
  EI_{m-3} is the Eurostat Index value for the month three months prior to that being calculated;
  EI_{m-2} is the Eurostat Index value for the month two months prior to that being calculated;
  d is the day of the month being calculated;
  gg is the actual number of days in month m.

  As can be seen from the structure of (2.4), this is nothing more than a linear interpolation. The motivation is simple: since the publication of statistical indices is not immediate and it is time-consuming, the reference inflation index for the month in question is estimated by linear interpolation[4] using data from three and two months earlier. This is also the reason why the daily indexing coefficients for the weeks following current days are posted on the Treasury website.

- Base Inflation is the value of Reference Inflation on the date of enjoyment of the security. In the case of BTP€i, this number is fixed and does not change during the life of the security.

[4]We will indicate on the x-axis the months and on the y-axis the value of the time index y

The Indexation Coefficient and the base inflation index are both published by the Treasury. The principal amount to be repaid at maturity is calculated by multiplying the nominal amount subscribed by the Indexation Coefficient calculated for the maturity date[5]. Therefore:

$$Principal\ repaid\ at\ maturity = Nominal\ principal\ value \cdot Indexation\ coefficient_{d,m} \tag{2.5}$$

If the Indexation Coefficient is less than one (i.e. deflation), the nominal value of the principal is repaid. This means that the above equation can be written as follows:

$$Principal\ repaid\ at\ maturity = Nominal\ principal\ value \cdot MAX(IC_{d,m}, 1) \tag{2.6}$$

We can also only account for capital appreciation, which can be broken down as follows:

$$Principal\ appreciation\ at\ maturity = Nominal\ principal\ value \cdot MAX(IC_{d,m} - 1, 0) \tag{2.7}$$

Indexation is also applied when selling on the secondary market: the negotiation price is determined by applying the Indexation Coefficient (of the day of the transaction) to the quoted price.
With regard to the indexation of the coupon, it must be taken into account that the BTP€i pays a constant interest in real terms. To calculate the indexed coupon, we have to multiply the real annual coupon rate (divided by two as the coupon is semi-annual) on the revalued capital by the Indexation Coefficient. Thus:

$$Coupon_t = \frac{Annual\ real\ coupon\ rate}{2} \cdot Nominal\ principal\ value \cdot IC_{d,m} \tag{2.8}$$

It is important to note that while there is a "floor" to the principal repayment (i.e. the nominal principal is repaid at maturity in the event of deflation), there is no similar statement by the Treasury for the semi-annual coupons.

Since trades are done at 'clean price' (i.e. *corso secco*) if the BTP€i is bought or sold on any day between coupon payment dates, the buyer must pay the seller the amount of interest accrued from the last coupon payment date to the settlement day of the transaction (day d of month m). Thus, as written by the Treasury, the Coupon Amount is calculated in two stages:

1. The percentage of the coupon that has accrued to the settlement date of the transaction (AC%) is calculated:

   $$AC\% = Coupon\% \cdot \frac{Relevant\ days}{Days\ between\ payment\ of\ two\ coupons} \tag{2.9}$$

   where "relevant days" is the number of days between the payment date of the previous coupon and the settlement date (day d of month m).

2. The figure obtained is then multiplied by the subscribed amount of principal recalculated as at the settlement date (equal to the nominal amount subscribed multiplied by the Indexation Coefficient):

   $$AC_{d,m} = AC\% \cdot Nominal\ principal\ value \cdot IC_{d,m} \tag{2.10}$$

[5] In addition, in order to disclose the health of the public finances, the revalued principal amount for existing BTP€is is published each month.

For more information on inflation-linked bonds, including real-world examples, see Appendix A.

2.1.6 Treasury Bonds Linked To Italian Inflation (BTPs Italia)

BTP Italia is a newer floating-rate security whose principal and coupons are indexed to the Italian inflation rate as measured by the FOI National Index (with the exclusion of tobacco products), provided by ISTAT. Maturity is between 4 and 8 years. Coupons are paid each semester. It was issued for the first time in 2012 to create an instrument with a specific hedge on Italian inflation[6].

The indexation of this security is similar to the indexation of the previous one. There is an Indexation Coefficient, which formula is as is follows:

$$IC_{d,m} = \frac{Index\ number_{d,m}}{Index\ number_{\bar{d},\bar{m}}} \tag{2.11}$$

where:

- $Index\ number_{d,m}$ indicates the price index number at day d of month m corresponding to the coupon payment date. This index number is computed by the following formula, which has the same structure of one used in BTP€i:

  $$Index\ number_{d,m} = FoiIN_{m-3} + \frac{d-1}{gg}(FoiIN_{m-2} - FoiIN_{m-3})$$

  $Index\ number_{d,m}$ indicates the index number of a day d of a month m;
  $FoiIN_{m-3}$ is the FOI Index value, excluding tobacco products, for the month three months prior to that being calculated;
  $FoiIN_{m-2}$ is the FOI Index value, excluding tobacco products, for the month two months prior to that being calculated;
  d is the day of the month being calculated;
  gg is the actual number of days in month m;
  Also in this case, the numerator of the Indexation Coefficient, the $Index_{d,m}$ is nothing but the result of a linear interpolation.

- $Index\ number_{\bar{d},\bar{m}}$ indicates the price index number as of the previous coupon payment date (6 months earlier). In the case of the payment of the first coupon, when the coupon payment date coincides with the security's enjoyment date, then the base index number for the IC is taken as that on the security's enjoyment date.
  This is a huge difference if compared to the base index used in BTP€i. This means that the inflation between the two semesters is taken into account for the calculation of the coupon (and the capital appreciation) and not the inflation from a general and sole starting point as it is instead in BTP€i.

Furthermore, while in BTP€i there is only the semi-annual coupon (except for the principal appreciation paid at maturity), in the BTP Italia the semi-annual total

[6]Until then, only BTP€i based on European inflation was issued.

return is given by two components: a semi-annual principal revaluation and the coupon. The formula can be formalized as follows:

$$Semester\ total\ return_t = \ Coupon_t\ +\ Principal\ appreciation_t \tag{2.12}$$

We know from the BTP€i that the principal revaluation follows this formula:

$$Principal\ revaluation = \ Nominal\ principal\ value\ \cdot Indexation\ coefficient_{d,m} \tag{2.13}$$

or more correctly considering the floor protection

$$Principal\ revaluation = \ Nominal\ principal\ value \cdot MAX(IC_{d,m}, 1) \tag{2.14}$$

But if we consider the part to be added, which indeed is the appreciation, then we can write it like this:

$$Principal\ appreciation_t = \ Nominal\ principal\ value \cdot MAX(IC_{d,m} - 1, 0) \tag{2.15}$$

An important difference with the BTP€i is the semi-annual appreciation of the capital: whereas in the case of the BTP€i, the increase in the value of the capital was at the time of redemption, in the case of the BTP Italia, the capital appreciation is expected semi-annually (in the case of positive inflation, of course). Additionally, there is a floor if the Indexation Coefficient is negative (i.e. there was deflation in the previous half-year), which means that there is no capital depreciation in the event of negative half-year inflation. It is important to state that, in the following period, if the Indexation Coefficient on a semi-annual basis goes again above 1, the highest price index reported in the previous semesters is used as a base[7].

As an inflation-linked bond, BTP Italia provides a fixed-income in real term. Thus, with regard to the coupon, it is well know at the issue the real coupon rate. The variable semi-annual coupons are calculated by multiplying half the real annual coupon rate, by the nominal principal revalued at the coupon payment date (that corresponds to the nominal value of principal subscribed multiplied by the adjusted Indexation Coefficient at the coupon payment date). Therefore:

$$Coupon_t \ = \ \frac{Annual\ real\ coupon\ rate}{2} \cdot Nominal\ principal\ value \cdot MAX(IC_{d,m}, 1) \tag{2.16}$$

In case of deflation the nominal semi-annual coupon rate equals the real semi-annual coupon rate, which represents the guaranteed minimum return. Again, there is a "floor" to protect the individual in the event of deflation. As in the case of semi-annual appreciation, in the following period, if the Indexation Coefficient on a semi-annual basis is greater than one, the maximum value of the price index number recorded in the previous semi-annual periods will be used as a base index. See A.3 for a detailed example. Usually BTP Italia also has a loyalty premium for investors who hold the bond until maturity.

Since, inflation-linked bond exchanges are done through the "clean real price" (i.e. *corso secco aggiustato per l'inflazione*) it is necessary to compute the accrued interest of this particular security. As the Treasury states "the countervalue at which

[7]This clause can be seen as the price of protection against deflation.

the trade is settled is obtained by multiplying the quoted/trading price by the Indexation Coefficient as of the settlement date of the transaction, and then adding the accrued interest also revalued by the Indexation Coefficient (without adjustments for taking into account potential deflation during the semi-annual period)". The calculation of accrued coupon interest $RC_{d,m}$ and the accrual of the principal revaluation $RRC_{d,m}$ can be done in several steps.

1. The percentage of the coupon that has accrued to the settlement date of the transaction (AC%) is calculated

$$AC\% = \frac{Annual\ real\ coupon\ rate}{2} \frac{Relevant\ days}{Days\ between\ payment\ of\ two\ coupons} \tag{2.17}$$

   where the "relevant days" is the number of days between the payment date of the previous coupon and the settlement date of the transaction (day d of month m). The value obtained above is then multiplied by the nominal value of principal subscribed revalued at the settlement date (equal to nominal value of principal subscribed multiplied by the Indexation Coefficient):

$$RC_{d,m} = AC\% \cdot Nominal\ principal\ value \cdot IC_{d,m} \tag{2.18}$$

2. The accrual of the principal revaluation (RRCd,m) is calculated as follows:

$$RRC_{d,m} = Nominal\ principal\ value \cdot \frac{Pr}{100} \cdot (IC_{d,m} - 1) \tag{2.19}$$

   where Pr indicates the "real" price quotation on the market at the date of trade (day d of month m).

For more information on inflation-linked bonds, including real-world examples, see Appendix A.

2.1.7 Treasury Bonds For Retail Investors (BTPs Valore)

BTPs Valore are floating-rate securities with a maturity between 4 and 5 years. It is a recent security with regular coupons with a step-up mechanism in terms of rates and extra bonus payment at maturity. There are only two issues that are exclusively for individual investors and others similarly situated. The first issue of BTP Valore addresses a semi-annual coupon, while the second issue coupons are paid every quarter. The series of guaranteed minimum coupon rates is communicated prior to issuance and may be confirmed or revised upward only at closing. It is important to note that each issue of BTPs Valore has different technicalities in terms of financial structure and maturity. In the Table 2.1 there is a breakdown of annual coupon rates.

There is also an additional loyalty premium for those who hold it until maturity.

2.1.8 Treasury Bonds Step-Up For Retail Investors (BTPs Futura)

BTPs Futura are floating-rate securities with maturities between 8 and 16 years. It is a special type of security that has step-up coupons. They are designed to be

Year	First emission BTP Valore (May, 2023)	Second emission BTP Valore (October, 2023)
1	3.25%	4.10%
2	3.25%	4.10%
3	4.00%	4.10%
4	4.00%	4.50%
5		4.50%

Table 2.1: Nominal annual coupon rates for BTPs Valore. Source: Department of Treasury

purchased only by individual persons, and as the BTP Valore, the step-up path is disclosed at the beginning of the issue and can only be revised upward at the end of the issue. There is a loyalty premium paid only to investors who purchase the bond at issuance and hold it until its final maturity: it is a final bonus equal to the average of nominal GDP annual growth rate[8] over the bond life. To date, there have been 4 issues of BTP Futura, summarized in the Table[9] 2.2.

Year	First emission BTP Futura (July, 2020)	Second emission BTP Futura (November, 2020)	Third emission BTP Futura (April, 2021)	Fourth issue BTP Futura (November, 2021)
1	1.15%	0.35%	0.75%	0.75%
2	1.15%	0.35%	0.75%	0.75%
3	1.15%	0.35%	0.75%	0.75%
4	1.15%	0.60%	0.75%	0.75%
5	1.30%	0.60%	1.20%	1.25%
6	1.30%	0.60%	1.20%	1.25%
7	1.30%	1.00%	1.20%	1.25%
8	1.45%	1.00%	1.20%	1.25%
9	1.45%		1.65%	1.70%
10	1.45%		1.65%	1.70%
11			1.65%	1.70%
12			1.65%	1.70%
13			2.00%	
14			2.00%	
15			2.00%	
16			2.00%	

Table 2.2: Nominal annual coupon rates for BTPs Valore. Source: Department of Treasury

Regarding the fidelity premium, it arises from a specific mechanism that inter-

[8]Through data collected by ISTAT.

[9]The nominal growth paths are lower than those of the BTP Valore. The reason is simple: in 2020 and 2021, the period in which BTPs Futura were issued to help the country recover from the pandemic, interest rates were generally low.

twines prolonged ownership with an indexation method tethered to GDP expansion. These financial instruments are denoted as "GDP-linked." The four fidelity premiums of BTP Futura vary notably in their approach to calculating the GDP growth bonus. Notably, the latest issuances target individual investors by offering substantial incentives to retain these securities until their maturity. Our focus will be on elucidating the indexation mechanism employed in the most recent BTP Futura issuance, which exhibits a more intricate structure compared to its predecessors. So for a 12-year bond, assuming an individual holds the bond to maturity, we have, quoting the Treasury Department [44]:

1. A first intermediate fidelity premium after 8 years which consist of a minimum of 0.4% and a maximum of 1.2% of the invested capital, based on the average annual variation of the Italian nominal GDP over the first eight years of the bond's life;

2. A second final fidelity premium in the last 4 years, which consist of two components:

   (a) the first one is calculated on the basis of the average annual variation of the Italian nominal GDP over the first eight years of the bond's life, with a minimum of 0.6% and a maximum of 1.8% of the invested capital;

   (b) the second one is calculated on the basis of the average annual variation of the Italian nominal GDP from the ninth year until the twelfth year of the bond's life, with a minimum of 1% and a maximum of 3% of the invested capital.

In a more formal way, let GDP_t be the Italian nominal GDP in year t and X_t the GDP growth rate in time t with respect to time *t-1*. Then X_t is the result of

$$X_t = \frac{(GDP_t \ - \ GDP_{t-1})}{GDP_{t-1}} \tag{2.20}$$

Moreover, let m be the issuance year and n the year before the maturity of the first eight years, the average annual variation[10] of the nominal GDP would be:

$$\bar{X}_t = \frac{1}{n-m} \sum_{t=m+1}^{n} X_t \tag{2.21}$$

Thus, the intermediate bonus will be

$$\begin{cases} 0.4\% \quad if\ 0.4\bar{X}_1 \leq 0.4\% \\ 0.4\bar{X}_1 \quad if\ 0.4\% < 0.4\bar{X}_1 < 1.2\% \\ 1.2\% \quad if\ 0.4\bar{X}_1 \geq 1.2\% \end{cases} \tag{2.22}$$

As for the final loyalty reward, the two different components, that ultimately add up to each other, need to be calculated. The first component is equal to the

[10] The nominal annual growth rate and the average annual variation of nominal GDP are rounded to the second decimal

60% of the average annual variation of the Italian nominal GDP ($\bar{X}_1$) of the same period used in the intermediate premium. Thus:

$$\begin{cases} 0.6\% & if\ 0.6\bar{X}_1 \leq 0.6\% \\ 0.6\bar{X}_1 & if\ 0.6\% < 0.6\bar{X}_1 < 1.8\% \\ 1.8\% & if\ 0.6\bar{X}_1 \geq 1.8\% \end{cases} \tag{2.23}$$

The second component will be calculated on the basis of the average annual variation of the Italian nominal GDP during the following four years of the bond's life. Let's keep n as the year before the maturity of the first eight years and define p as the year before the final bond's maturity. Then the average annual variation of the Italian nominal GDP $\bar{X}_2$ will be:

$$\bar{X}_2 = \frac{1}{p-n} \sum_{t=n+1}^{p} X_t \tag{2.24}$$

and the second component of the final premium is represented as follows:

$$\begin{cases} 1\% & if\ \bar{X}_2 \leq 1\% \\ \bar{X}_2 & if\ 1\% < \bar{X}_2 < 3\% \\ 2\% & if\ \bar{X}_2 \geq 3\% \end{cases} \tag{2.25}$$

2.2 Debt structure over the past 40 years to date

Starting from 1982 and thanks to the data provided by the Treasury, we can reconstruct the historical series of the composition of the public debt and its weighted average life. In particular, we can show the structure of the public debt by categorizing it according to the type of remuneration offered. The categories and their components are defined as follows:

- *Zero-coupon bonds*: BOTs, CTZs[11], BTEs[12];
- *Fixed-income bonds*: BTPs, BTPs Green, fixed rate CCTs[13], CTEs[14], CTOs[15];
- *Floating-rate bonds (non indexed)*: variable rate CCTs, CCTs-eu;
- *Inflation-linked bonds*: CTR, BTP€i, BTP Italia;
- *Step-up bonds*: BTPs Futura, BTPs Valore.

The Figure 2.1 is an elaboration based on Treasury data. The y-axis shows the share of the bond category in the total debt composition. What Treasury calls "foreign

[11] Government zero-coupon bonds with a maturity of 24 months.
[12] BOTs traded in ECU.
[13] In the 1970s, CCTs were issued with the option of being converted into fixed-rate securities at a later date [6]. Although the initial issuance of such convertible securities cannot be seen due to data limitations, the share of CCTs that were converted into fixed-rate securities remains.
[14] Government fixed-income bonds traded in ECU. They were similar to fixed rate CCTs with the difference that were traded in ECU.
[15] CTOs were a type of "putable bond", a fixed-income security that gives investors the right to redeem the bond early only after half its life.

debt"[16] and "other" have been omitted. It is worth mentioning that the share of both in the total debt had a negligible average weight of about 3.9 percent in the total debt composition over a 40-year period. The debt composition shown in Figure 2.1 is consistent with what we described in the subsection 1.1.4: in the 1970s and 1980s, high inflation drove investors to short-term securities, which were in fact the main securities of public debt. The situation was tamed thanks to the issuance of floating-rate securities, which gained traction in the second half of the 1980s. Since the turn of the millennium, the new inflation-linked bonds have become attractive to private and institutional investors. Over the last 20 years, however, their share has remained almost constant, partly due to low inflation in the European area. The huge effort of the Treasury in the emission of alternative appealing securities, it is also clear from the historical series of the weighted average life of debt shown in Figure 2.2. In 2022, the inflation-linked bonds accounts for 10% of total debt (7.38% for BTP€i and 3.24% for BTP Italia).
Thanks to Treasury data, it is also possible to show the weighted average life of government debt, expressed in years and fractions of years. As shown in the figure 2.2, the trend is positive.

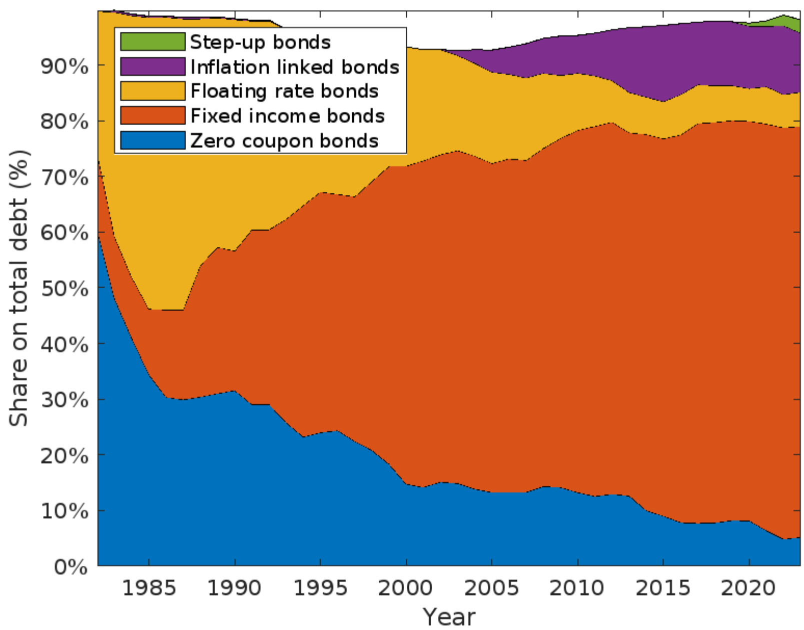

Figure 2.1: Debt composition over 40 years. Source: elaboration on Treasury data.

Everything written above, including historical series of debt composition, will be our starting point for simulating scenarios for public debt.

[16]Foreign debt includes issues in international markets with foreign currencies. For example, there are Italian government bonds that are traded in JPY, USD and AUD. See [45].

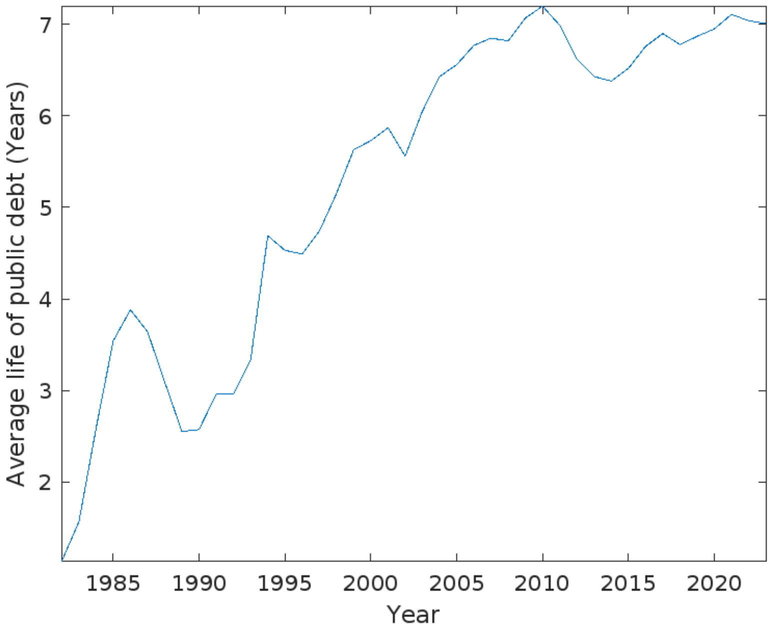

Figure 2.2: Weighted average life of Govt. debt over 40 years. Source: elaboration on Treasury data.

Chapter 3

Models for price index

In this chapter we will create and simulate two different models to reproduce the evolution of a price index over a period of N years.

3.1 A *simple* single-factor model for price index

Model structure. To simulate the behaviour of Italian inflation-linked bonds, we first need to simulate inflation. Castellani, De Felice and Moriconi, in their iconic book series (and to be more specific *Manuale di finanza III* [46]), created a simple model for simulating a price index such as the CPI index. In this first model, we suppose the following assumptions:

1. A stochastic model for the general price level $p(t)$ is chosen, which is estimated on an observed time-series of the CPI which best explains general inflation;

2. Expected future inflation rates are constant (i.e. there is neither a stochastic component in the expected future inflation nor a deterministic function).

Since tomorrow's CPI price index is a random value, the collection of daily CPI indexes is a collection of random values, which can be represented through a stochastic process. For this reason, we will describe the inflation diffusion process using a stochastic differential equation (SDE).
The general case of a SDE [46] has this form:

$$dY = a(Y,t)dt + b(Y,t)dZ \tag{3.1}$$

where the a is the drift function while b^2 is the diffusion function. Both are generally a function of time t and the value $Y(t)$ of the process in t. Since the evolution of a phenomenon is due to a random and a deterministic part, the components that multiply dt usually address the deterministic evolution, while the components that multiply dZ address the random contribution.
Therefore, defining $p(t)$ as the price index at time t, the evolution of inflation can be expressed as

$$dp(t) = f_p(p(t),t)dt + g_p(p(t),t)dW_p(t) \tag{3.2}$$

where:

- $dW_p(t)$ is an infinitesimal increment of the Wiener process. A Wiener process is particular stochastic process which can be summarized as the continuous version of the concept of *random walk*[1] [47].
  It is easy to demonstrate [47] that the infinitesimal increment of a Wiener process can be represented using a standard normal distribution. In fact $dW_p(t) = \sqrt{dt}Z_t$ where $Z \sim N(0,1)$;

- $g_p(p(t),t))$ is the *diffusion* function of $p(t)$ process. We can use the simple specification:

$$g_p(p(t),t)) = \sigma_p p(t) \tag{3.3}$$

  where σ_p is the volatility parameter (a positive constant).

- $f_p(p(t),t)$ is the *drift* function of $p(t)$ process. We can use the simple specification:

$$f_p(p(t),t) = y(t)p(t) \tag{3.4}$$

Let's rewrite the equation 3.2 according to the above specifications. Thus, the index price follows a diffusion process described by the following SDE:

$$dp(t) = y(t)p(t)dt + \sigma_p p(t)dW_p(t) \tag{3.5}$$

but what is $y(t)$? Since the expected value of a Wiener process is zero[2] (i.e. $E[dW_p(t)] = 0$) then,

$$\begin{aligned}
dp(t) &= y(t)p(t)dt + \sigma_p p(t)dW_p(t) \\
\frac{dp(t)}{p(t)dt} &= y(t) + \frac{\sigma_p p(t)dW_p(t)}{p(t)dt} \\
E\left[\frac{dp(t)}{p(t)dt}\right] &= E[y(t)] + E\left[\frac{\sigma_p p(t)dW_p(t)}{p(t)dt}\right] \\
E\left[\frac{dp(t)}{p(t)dt}\right] &= y(t) \qquad (3.6)
\end{aligned}$$

$y(t)$ is the *expected instantaneous inflation rate*. In fact, it can be observed that the equation 3.6 is the ratio of a rate[3] to the length of the reference period in infinitesimal terms. The expected instantaneous inflation rate can be written in other form. As a matter of fact:

$$\frac{dp(t)}{p(t)dt} = \frac{dp(t)}{dt}\frac{1}{p(t)} = \frac{p'(t)}{p(t)} \tag{3.7}$$

The notation $\frac{p'(t)}{p(t)}$, but more generally, every notation $\frac{y'}{y}$ [48], is nothing but the expected temporal semi-elasticity[4] of p. Thus, $y(t)$ represents the expected sensitivity

[1] A random walk is the mathematical formalization of step increments in random directions.

[2] See [47] for a demonstration.

[3] Since $dp(t) = p(t+dt) - p(t)$ then $\frac{\frac{p(t+dt)-p(t)}{p(t)}}{dt} = \frac{dp(t)}{p(t)dt}$ which is indeed our instantaneous inflation rate in differential form.

[4] Generally speaking, semi-elasticity measures the percentage change in X to an absolute (not percentage) change in Y, while elasticity measures relative percentage changes.

to the temporal variation of the price function, expressed in percentage change for time unit.
Since we assume that expected instantaneous inflation rates are constant (i.e. $y(t) = j_p$), we can reformulate the SDE 3.5 as follows:

$$dp(t) = j_p p(t)dt + \sigma_p p(t)dW_p(t) \tag{3.8}$$

The stochastic process which solves the SDE above is a *Geometric Brownian Motion*, with j_p as drift parameter and σ_p as volatility parameter. At this point it is necessary to define the parameters and execute numerically the model.

Model Calibration. Calibrating the drift and volatility parameters in this case is fairly straightforward. Castellani, De Felice and Moriconi [46] suggest looking at the historical volatility of price index time series and defining the expected instantaneous inflation rate exogenously (e.g. with surveys or other institutional features) or through panel estimation. In our case, we will even define j_p using the real time series. By downloading the HICP Ex-tobacco [49] and FOI Ex-tobacco [50] [51] indices, we can calculate the volatility and the mean of these time series over a period of more than 25 years.
First, however, a small digression is necessary. In fact, while the HICP Ex-tobacco index is given without any necessary manipulation[5], we need to reconstruct a single comprehensive FOI index time series[6]. In order to appreciate the differences between the European and the Italian price index and compute correct calculation on Italian price index volatility, we need to rebase the whole latter time series (which starts from 1996) to 2015.
Thanks to the reconciliation coefficient provided by ISTAT [52] we obtain the Figure 3.1.
Once we have the correct time series, it is time to calculate the parameters. According to the same Castellani, De Felice and Moriconi [53], we can use the tools normally used in time series analysis and portfolio optimization.
Before continuing, note that while the price index trajectories are described by a geometric Brownian motion, the infinitesimal relative return can be outlined by a *Arithmetic Brownian Motion* (Eq. 3.9). In fact, it is easy to demonstrate [47] the logical relationship that allows us to calculate the parameters by manipulating our time series into a tolerably stationary one (Figures 3.4, 3.5).

$$\begin{aligned} \frac{p(t+dt)-p(t)}{p(t)} &= j_p dt + \sigma_p dW_p(t) \\ \frac{dp(t)}{p(t)} &= j_p dt + \sigma_p dW_p(t) \\ dp(t) &= j_p p(t)dt + \sigma_p p(t)dW_p(t) \end{aligned} \tag{3.9}$$

Therefore, we are going to compute the *monthly conjunctural inflation rate* for both price indexes in order to obtain parameters for the volatility and the mean. Given a price index time series, we compute the inflation rates as follows

$$R_t = \frac{p_t - p_{t-1}}{p_{t-1}} \tag{3.10}$$

[5]The ECB provides a single complete price index time series since 1996. The year 2015 is set as base 100.

[6]Between 1996, 2010 and 2015, ISTAT provides three base-different price index time series.

that is, on monthly bases, right the monthly conjunctural inflation rate.
If we compute an empirical probability density function for both inflation rates, we obtain the Figures 3.2 and 3.3.

Thus, defining T as the number of observations and the general formulas for

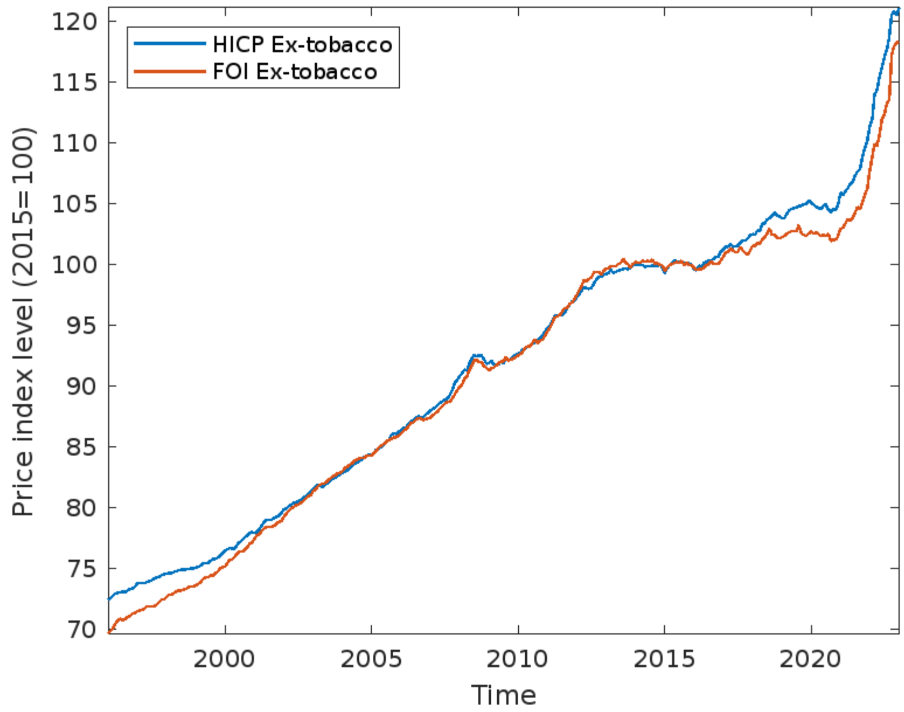

Figure 3.1: HICP and FOI Ex-tobacco monthly time series (1996-2023). Source: elaboration on ECB and ISTAT data.

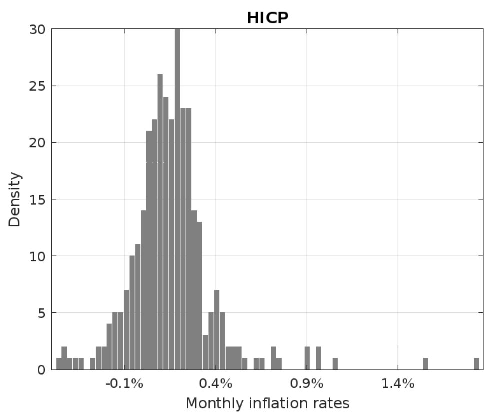

Figure 3.2: Empirical PDF for HICP monthly conj. inflation rates. Elaboration on ECB data

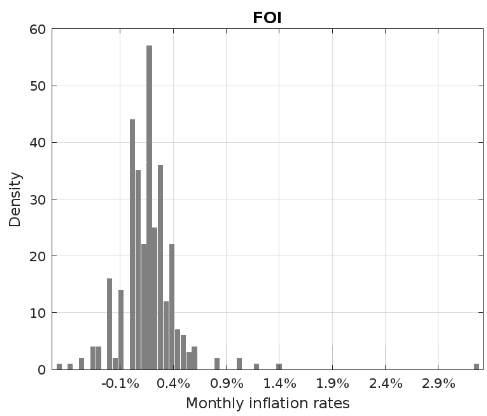

Figure 3.3: Empirical PDF for FOI monthly conj. inflation rates. Elaboration on ISTAT data

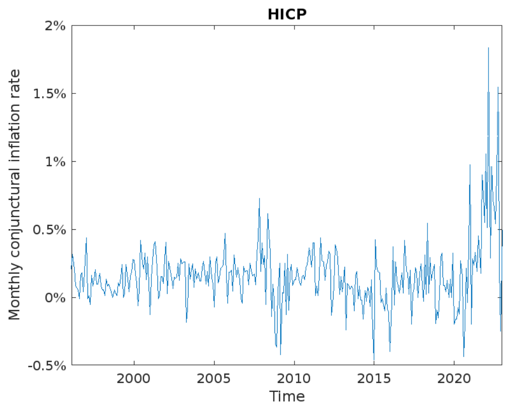

Figure 3.4: Monthly HICP inflation rate over time. Elaboration on ECB data

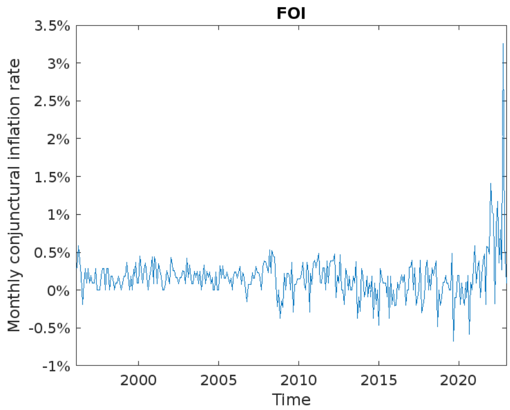

Figure 3.5: Monthly FOI inflation rate over time. Elaboration on ISTAT data

sample mean and volatility as

$$\text{sample average} = \hat{j}_p = \frac{\sum_{t=1}^{T} R_t}{T} \tag{3.11}$$

$$\text{volatility} = \hat{\sigma_p} = \sqrt{\frac{\sum_{t=1}^{T}[R_t - \hat{j}_p]^2}{T-1}} \tag{3.12}$$

we obtain, according to our calculations (see code 3.1),

```
%% Parameters j and sigma
DATA = xlsread('inflation_data.xlsx', 'data', 'B2:D336');
HICP = DATA(1:325,1);
FOI = DATA(1:325,3);

HICP_infl = diff(HICP)./HICP(1:end-1,1);
j_HICP = mean(HICP_infl);
vol_HICP = sqrt(var(HICP_infl));

FOI_infl = diff(FOI)./FOI(1:end-1,1);
j_FOI = mean(FOI_infl);
vol_FOI = sqrt(var(FOI_infl));
```

MATLAB code 3.1: Parameters for the simple single factor price index model

the parameters in Table 3.1. An optimal starting point for the simulation of inflation dynamics.

	HICP	FOI
sample average	0.0016	0.0016
volatility	0.0024	0.0029

Table 3.1: Empirical parameters of $\hat{j}_p$ and $\hat{\sigma_p}$ for price indexes

Model execution. Thanks to Ito's Lemma [46] [47] by solving the SDE 3.8 over the interval [0,T] we obtain at time T the following simulated price index:

$$p(T) = p(0)e^{(\hat{j}_p - \frac{1}{2}\hat{\sigma}_p^2)T + \hat{\sigma_p} dW(T)}$$

which is:

$$p(T) = p(0)e^{(\hat{j_p} - \frac{1}{2}\hat{\sigma_p^2})T + \hat{\sigma_p}\sqrt{T}Z_T} \tag{3.13}$$

Therefore, using the parameters of Table 3.1 and defining $\Delta t = 1$ (which implies a monthly addition after each step), we are now able to plot our simulated price index path[7] based on historical data. Aided by literacy [47], through MATLAB, we get:

```
%% Simulations of price indexes based on historical data
%number of simulated paths
n_simul = 20;

%simulation steps (m)
years = 27;
m = years*12;

%drift coefficients (in months)
j_HICP = 0.0016;
j_FOI = 0.0016;

%volatility coefficient (in months)
sigma_HICP = 0.0024;
sigma_FOI = 0.0029;

%time increment (in months)
Dt = 1;

%price index starting point
P0_HICP = 74;
P0_FOI = 70;

%time vector for each simulated paths
time = 0:Dt:Dt*(m-1);
time = repmat(time,[n_simul,1]);

%Random walk component
W = sqrt(Dt).*cumsum([zeros(n_simul,1), randn(n_simul,m-1)],2);

%Price index paths
P_tk_HICP = P0_HICP * exp((j_HICP - 0.5*sigma_HICP^2)*time +
    sigma_HICP*W);
P_tk_FOI = P0_FOI * exp((j_FOI - 0.5*sigma_FOI^2)*time + sigma_FOI*
    W);

fig1 = figure(1)
plot(P_tk_HICP')
title('HICP')
xlabel("Months after the start of the simulation")
ylabel('Simulated price index level')
set(gca,'FontName','cmr12')
axis tight
saveas(fig1, 'sim1_HICP.png')

fig2 = figure(2)
plot(P_tk_FOI')
title('FOI')
```

[7] Simulating a path is equivalent to applying a time vector to the equation 3.13.

```
xlabel("Months after the start of the simulation")
ylabel('Simulated price index level')
set(gca,'FontName','cmr12')
axis tight
saveas(fig2, 'sim1_FOI.png')
```

MATLAB code 3.2: A simple single-factor model for price index

To compare the simulated paths with the real ones, we chose the same starting point and the same time period (27 years). The results shown in Figures 3.6 and 3.7 are quite convincing with respect the real time series.

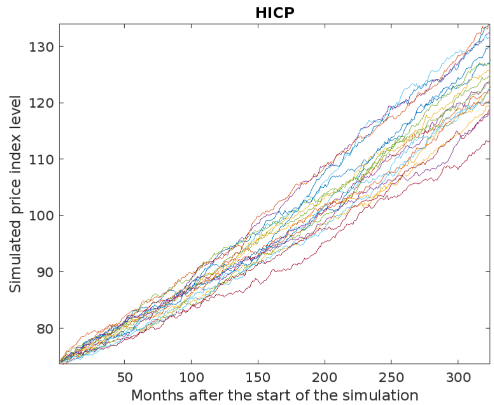

Figure 3.6: Simulated paths of the HICP index.

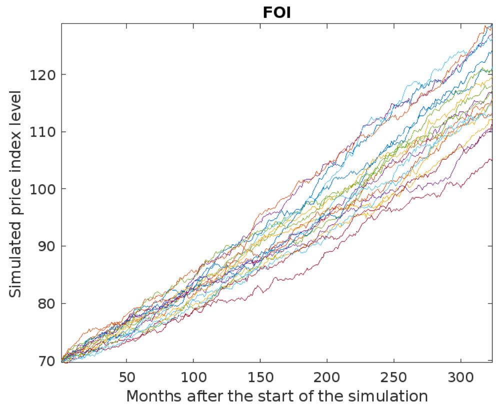

Figure 3.7: Simulated paths of the FOI index.

3.2 A single-factor model for price index

Model structure. Differently from the model proposed in the former section, the second price index model, even mentioned by the same De Felice and Moriconi in a recent paper [54], differs from the previous one just in the evolution of $y(t)$. While before we stated that the expected instantaneous inflation rate is constant, now we are supposing that it evolves accordingly to a deterministic law. Therefore, while the price index has still the same general dynamics,

$$dp(t) = y(t)p(t)dt + \sigma_p p(t)dW_p(t) \tag{3.14}$$

the expected instantaneous inflation rate $y(t)$ (i.e. the expected monthly semi-elasticity of p), evolves accordingly to a *mean-reverting dynamics* expressed as follows:

$$\begin{gathered} dy(t) = \alpha_y\,[\gamma_y - y(t)]dt \\ \alpha_y, \gamma_y \ \textit{costant and}\ \alpha_y > 0 \end{gathered} \tag{3.15}$$

The drift component described by the above differential equation has the peculiarity of being "mean-reverting", which means that it tends, with a given strength α_y, to a long-term value indicated by γ_y [46]. As far as we are concerned, this kind of drift model is useful when we are modeling the price index after an inflation shock: indeed, as we will see later, we can assume that the expected price semi-elasticity

would tend to the ECB's target in the long run from a given starting point (which may be a starting point of an inflation shock).
The assumption of a mean-reverting drift, although smoother than the former, is still a strong assumption. As a matter of fact, as even stated by the authors, the choice of a deterministic expected semi-elasticity for the price index "involves an underestimate of the inflation uncertainty". That's the reason why Moriconi provides a more complex two-factor price index model in which y(t) is described by a stochastic differential equation[8]. But that's another story.

Model calibration. We need to calibrate the model by defining the parameters γ_y *and* α_y and a starting point $y(0)$.
Regarding the long-term monthly[9] semi-elasticity, which is indicated by γ_y, we can assume that since the annual European inflation target is set at $r_y^{TARGET} = 2\%$ by the ECB, the expected $y(t)$ of long-term is determined by an iterative procedure. As a matter of fact, fixed a price index starting point P_0, we can easily calculate the theoretical target indexes for the next N years. Indeed:

$$\begin{aligned} P_1 &= P_0 \cdot (1 + r_y^{TARGET}) \\ P_2 &= P_1 \cdot (1 + r_y^{TARGET}) \\ &\dots \\ P_k &= P_{k-1} \cdot (1 + r_y^{TARGET}) \\ &\dots \\ P_N &= P_{N-1} \cdot (1 + r_y^{TARGET}) \end{aligned} \tag{3.16}$$

If we assume a *constant* monthly semi-elasticity to get to the ECB's theoretical price index targets, which are now known and marked with an asterisk (*), the solution can be easily found using the following closed-form formula:

$$P_k^* = P_{k-1}^* \cdot (1 + \gamma_y)^{12} \tag{3.17}$$

$$| \sqrt[12]{\frac{P_k^*}{P_{k-1}^*}} | - 1 = \gamma_y \tag{3.18}$$

By doing this, we get a result of $\gamma_y = 0.001651$. A numerical proof is provided:

```
% theorical (according to ECB) price index series
format short
P0 = 100;
r = 0.02;
N = 10;
P = P0.*((1+r).^(1:N))

% gamma identification
format long
for j = 1:N-1
    r(j) = ((P(j+1)/P(j))^(1/12))-1
end
```

MATLAB code 3.3: Gamma parameter detection

[8]See [55].
[9]Remember that we are simulating monthly variations of the price index.

Nicely, the result is similar to the historical one calculated in section 3.1: in that case the parameter $\hat{j}_p$ was 0.0016.

The strength parameter α_y lies between [0,1], where *0* means that there is no adjustment towards γ_y, while *1* means an *instantaneous*[10] adjustment towards the long-term parameter. In an interesting IMF working paper analyzing more than 100 inflation shocks [56], one of the main evidence obtained is that during the energy crises of 1973-1982, the disinflation process[11] (in the resolved[12] cases) took on average 3.5 years for a list of strictly selected countries[13] (Figure 3.8).
Assuming a similar situation[14], we can write a simple algorithm to generally cal-

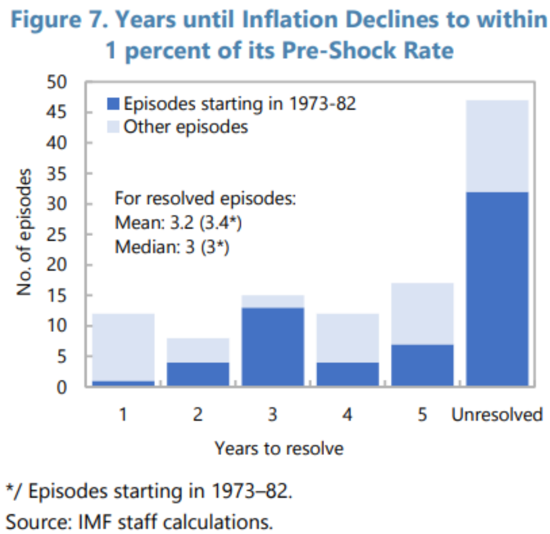

Figure 3.8: Source: One Hundred Inflation Shocks: Seven Stylized Facts [56].

culate α_y. Supposing that the monthly semi-elasticity of the price would return to a sufficiently close neighborhood of the long-run value γ_y in 3.5 years (that is, 42 months), is equivalent to deciding that, given a starting point $y(0) = y_0$, approximately 42 iterations are necessary to get *near* the long-run value. As we know, the design of a mean-reverting drift, except for the case where $\alpha_y = 1$, is asymptotic, which means that the difference $|\gamma_y - y(t)|$ will never be exactly zero! Thus, the simplest way to generally identified the parameter of α_y which admits approximately t iterations is through brute force method. Then:

```
function [alpha] = alpha_coeff(y0, gamma_y, t)
% A function that calculates the alpha coefficient of a mean-
    reverting
% process necessary to obtain a number of iterations equal to 't'
    while
```

[10] In our case, a monthly adjustment.

[11] Defined as bringing CPI inflation back to within 1 percentage point of its pre-shock rate.

[12] By citing the authors, the unresolved cases are those in which there was a temporary disinflation process. In facts, *"In about 90 percent of unresolved episodes (28 out of 32 during the 1973–79 oil crises), inflation declined materially within the first three years after the initial shock, but then either plateaued at an elevated level or re-accelerated"*

[13] Developed and emerging market economies with a diversified market, not low-income economies that were not involved in armed conflict at the time of the shock.

[14] And thus comparing the oil crises of 1973-1982 with the energy crisis following the Russian invasion of Ukraine.

```
% approaching gamma (with an accuracy of 10^-4) from a given
    starting point.
%
% [alpha] = alpha_coeff(y0, gamma_y, t)
%
%   y0 = starting point of the mean reverting non stochastic
    process;
%   gamma = long term value;
%   t = iteration necessary to get close to gamma.

j = 0;
k = 0;
alpha_temp = linspace(0.999,0.001,100000);

while abs(k-t) >= 1
  j = j+1;
  k = 1;
  y(k) = y0;
  dt = 1;

  while abs(gamma_y - y(k)) > 10^-4
    dy = alpha_temp(j)*(gamma_y-y(k))*dt;
    y(k+1) = y(k)+dy;
    k = k+1;
  end
end

alpha = alpha_temp(j);

end
```

MATLAB code 3.4: Alpha computation

The above function identifies a parameter α_y that, given a starting point and a parameter γ_y, allows a number of iterations in the neighborhood $[t-1, t+1]$, such that $|\gamma_y - y(k)| < 10^{-4}$.
Although the α_y calibration with $t = 42$ may be strong, the stylized facts that emerge from the paper are consistent with the European framework: there is a strong (and credible) central bank that has no problem tightening monetary policy. Thus, if we are not convinced about the ability of the ECB and the European economies to contain inflation (and then fall in the case of the "resolved case"), using the algorithm 3.4, we can calculate α_y referring an extended period.

Model execution.
Note: The complete list of calculations is available in Appendix B.

By integrating[15] [54] the equation 3.15 we obtain:

$$y(t) = \gamma_y - [\gamma_y - y(0)]e^{-\alpha_y t} \tag{3.19}$$

which can be replaced in our original SDE (Eq. 3.5):

$$\begin{aligned} dp(t) &= y(t)p(t)dt + \sigma_p p(t)dW_p(t) \\ dp(t) &= [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]p(t)dt + \sigma_p p(t)dW_p(t) \end{aligned} \tag{3.20}$$

[15] With $y(0) = y_0$

If we apply the Ito's lemma and integrate it, after some manipulation, we will obtain:

$$p(T) = p(0)e^{\left(\gamma_y - \frac{1}{2}\sigma_p^2\right)T + \frac{\gamma_y - y(0)}{\alpha_y}\left(e^{-\alpha_y T} - 1\right) + \sigma_p \sqrt{T} Z_T} \tag{3.21}$$

The equation 3.21 can be represented through MATLAB

```
function [P_tk] = SinglePriceModel(n_simul, Years, alpha, gamma_y,
    y0, sigma, P0)
% A function that simulates simple single factor
% monthly price index paths, according to drift and diffusion
    parameters.
%
%   n_simul = number of simulations;
%   Years = Years to be simulated;
%   alpha = strength coefficient
%   gamma = long-term semi-elasticity value
%   y0 = semi-elasticity starting point
%   P0 = price index starting point
%   sigma = volatility coefficient;

%simulation steps (m)
m = Years*12;

%time increment (in months)
Dt = 1;

%time vector for each simulated paths
time = 0:Dt:Dt*(m-1);
time = repmat(time,[n_simul,1]);

%Random walk component
W = sqrt(Dt).*cumsum([zeros(n_simul,1), randn(n_simul,m-1)],2);

%Price index paths
P_tk = P0 .* exp( (gamma_y-(1/2).*sigma^2).*time + ((gamma_y-y0)/
    alpha).*(exp(-alpha.*time) -1) + sigma.*W);

fig1 = figure(1)
plot(P_tk')
title('Price index paths')
xlabel("Months after the start of the simulation")
ylabel('Simulated price index level')
set(gca,'FontName','cmr12')
axis tight

end
```

MATLAB code 3.5: p(t) with deterministic y(t) trajectories

By simulating a 10-year price index trajectories[16], we can better appreciate the differences with the former model.

```
%% Price index paths
n_simul = 10; %number of simulations
Years = 10; %10 years simulation

sigma = 0.0029; %price index volatility parameter
```

[16]We will use the same price index volatility extrapolated from the historical series

```
P0 = 100; %price index starting point

%a function which create price index paths
gamma_y = 0.001615;
y0 = 9.48*(10^-3); %inflation shock
t = 42; %in months

[alpha] = alpha_coeff(y0, gamma_y, t)

[P_tk] = SinglePriceModel(n_simul, Years, alpha, gamma_y, y0, sigma
    , P0);
```

MATLAB code 3.6: Single factor price index model

Figure 3.9: Single-factor price index model trajectories

As we can see in Figure 3.9, this type of model is useful for simulating the evolution of the price index after an inflation shock[17]. Unfortunately, in the final model, due to the specific shape of the expressions (Eq. 3.14 and 3.21), the time t does not directly affect the time after which it curves perfectly.
For a better understanding of this feature, we will ignore the stochastic component[18] and plot the graphs by varying only the number of t (Figure 3.10). The more t is small, the more the price index quickly converges to the monthly inflation rate γ_y. Something we expected, with the only difference that t does not directly suggest graphically the time needed to reach the expected monthly inflation rate of γ_y. But this is something that is possible to work on further research. The code remains for

[17]The inflationary shock used is based on real data [57], and more specifically on a 12% annual inflation rate. Indeed, since $112 = 100 \cdot (1 + r)^{12}$ we obtain $r = 9.48 \cdot 10^{-3}$

[18]By making $sigma_p$ a really small number.

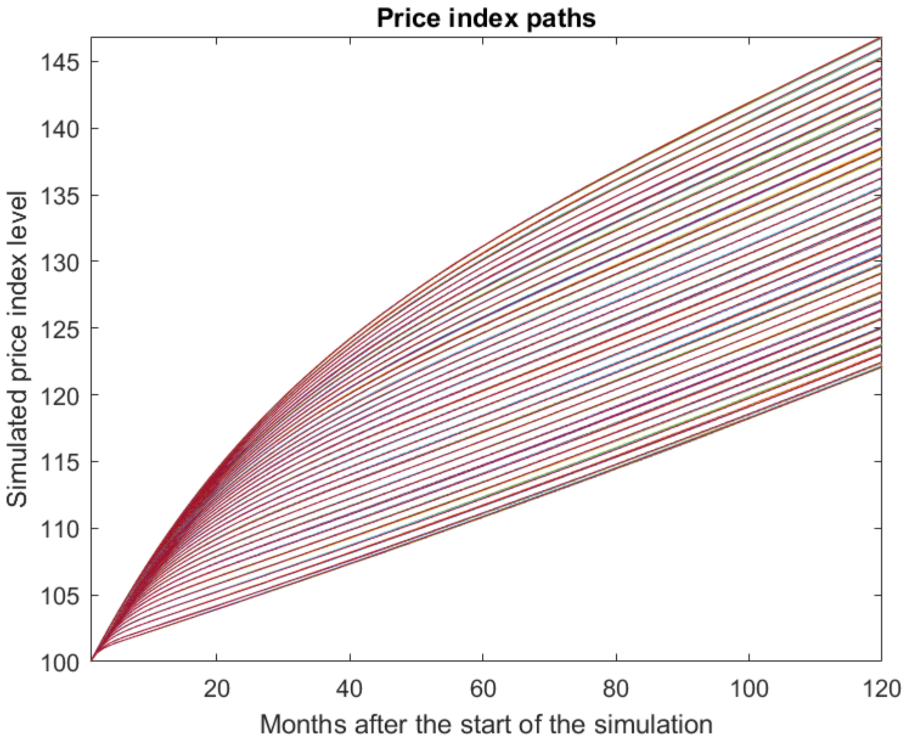

Figure 3.10: The effects of alpha on the price index evolution

further simulations (MATLAB code 3.7).

```
n_simul = 7; %number of simulations
Years = 10; %years of simulation

sigma = 0.000029; %price index volatility parameter
P0 = 100; %price index starting point

%a function which create price index paths
gamma_y = 0.001615;
y0 = 9.48*(10^-3); %inflation shock
t = 2:3:108; %in months

[alpha] = alpha_coeff(y0, gamma_y, t)

for k = 1:length(t)
    [alpha(k)] = alpha_coeff(y0, gamma_y, t(k))
    [P_tk] = SinglePriceModel(n_simul, Years, alpha(k), gamma_y, y0, sigma, P0);
    hold on
end
```

MATLAB code 3.7: The effects of alpha on the price index evolution

Let's now take a look at how these two models affect the behavior of Italian inflation-linked bonds.

Chapter 4

Simulated scenarios for Italian inflation-linked securities

4.1 Simulated coupons and capital appreciation for BTP€i and BTP Italia

In the previous chapter we analyzed and structured two models to simulate the evolution of a price index. Let's apply these models to BTP€i and BTP Italia, simulating different scenarios for coupons and redemptions.

BTP€i. Before proceeding, the following simplifying assumptions are made:

1. The price index simulation starts at the emission of the security;

2. The base index in the denominator of the Indexation Coefficient is fixed on the issue date and it is not computed through interpolation.

Assumption 1 helps us to keep our code simple. In fact, if we establish that the issue date and the starting point of our simulation are the same, we can easily individualise the functional simulated 'Eurostat indices' for the whole life of the security. Instead, with reference to Assumption 2, in the real world the base index for BTP€i is fixed at the date of enjoyment of the securities, which usually, *but not necessarily*, coincides with the issue date. This assumption is a direct consequence of the previous one. In fact, it allows us to consider the starting point of the calculations as the first element of our stochastic simulation, without admitting past indexes.
Before diving into the MATLAB code, let's analyze its theoretical structure according to a 5-year indexed bond with a semi-annual coupon (and a real rate of 0.10%), issued on November 2021 and expiring on November 2026. Following our assumptions and defining `EI` as the vector of simulated Eurostat price indices, we would have the type of recurrences indicated in Figure 4.1.
In a context with semi-annual coupon, we need to calculate a number of Indexation Coefficient, which can be synthesized by the following expressions

$$Number\ of\ IC = Maturity\ in\ years\ \cdot 2 \tag{4.1}$$
$$= \frac{Maturity\ in\ months}{6} \tag{4.2}$$

	A	B	C	D	E	F	G	H	I	J	K	L
1	BTP €I Indexation coefficient structure											
2	DATES	‘Nov 2021	May 2022	‘Nov 2022	May 2023	‘Nov 2023	May 2024	‘Nov 2024	May 2025	‘Nov 2025	May 2026	‘Nov 2026
3	Time series location	EI(1)	EI(7)	EI(13)	EI(19)	EI(25)	EI(31)	EI(37)	EI(43)	EI(49)	EI(55)	EI(61)
4	Base index	\\	EI(1)	EI(1)	EI(1)	EI(1)	EI(1)	EI(1)	EI(1)	EI(1)	EI(1)	EI(1)
5	EI (m-2)	\\	EI(5)	EI(11)	EI(17)	EI(23)	EI(29)	EI(35)	EI(41)	EI(47)	EI(53)	EI(59)
6	EI (m-3)	\\	EI(4)	EI(10)	EI(16)	EI(22)	EI(28)	EI(34)	EI(40)	EI(46)	EI(52)	EI(58)
7	Reference index (RI)	\\	Interpolation using EI(5), EI(4)	Interpolation using EI(11), EI(10)	Interpolation using EI(17), EI(16)	Interpolation using EI(23), EI(22)	Interpolation using EI(29), EI(28)	Interpolation using EI(35), EI(34)	Interpolation using EI(41), EI(40)	Interpolation using EI(47), EI(46)	Interpolation using EI(53), EI(52)	Interpolation using EI(59), EI(58)
8	Indexation coefficient	\\	RI(May 2022) / EI(1)	RI(Nov 2022) / EI(1)	RI(May 2023) / EI(1)	RI(Nov 2023) / EI(1)	RI(May 2024) / EI(1)	RI(Nov 2024) / EI(1)	RI(May 2025) / EI(1)	RI(Nov 2025) / EI(1)	RI(May 2026) / EI(1)	RI(Nov 2026) / EI(1)
9	Meaning	Emission date	Coupon #1	Coupon #2	Coupon #3	Coupon #4	Coupon #5	Coupon #6	Coupon #7	Coupon #8	Coupon #9	Coupon #10 + principal repayment

Figure 4.1: Price index location example for a 5-years semi-annual BTP€i

which imply the same number of reference inflation indices (interpolated by specific Eurostat indices EI_{m-3} and EI_{m-2}).
Therefore, according to what has been said above, it is easy to generalize the example. We can then define a function to calculate the vector of indexed coupons and the repayment of the appreciated capital, according to the functioning of the BTP€i.

```
function [coupons,principal_appreciations] = BTPei(Nom_val, EI, Real_rate, d, m, t)
% A function that calculates semi-annual coupons and
% principal indexed repayments for BTPei
% according to simulated price index trajectories.
%
% BTPei(Nom_val, EI, Real_rate, d, m, t)
%
%   Nom_val = Nominal principal amount underwritten;
%   EI = simulated time series of the price index;
%   Real_rate = real rate of the indexed bond;
%   d = emission day of the bond. A number between 1 and 31;
%   m = emission month of the bond. A number between 1 and 12;
%   t = maturity (in years).

% The effect of assumption 1
% the base index is and will always be the emission date of the bond,
% which is also the first element of the price index simulation
BaseIndex = EI(1);

% The following array is used to calculate the correct interest accrual
days_month = [31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31];
days_month = repmat(days_month,[1,2]);

% Let's transform years in months
maturity_in_months = t*12;

% The effect of assumption 2 -- We are computing the Eurostat indices
% to find the Indexation Coefficients through interpolation
EIm2 = EI(7-2:6:maturity_in_months);
EIm3 = EI(7-3:6:maturity_in_months);

% The number of Reference Indexes to be computed is the maturity in months
% divided by the coupon frequency (in this case is semi-annual)
RI = NaN(1,maturity_in_months/6);

% Reference index calculation
k = 0;
```

```
while k < length(RI)
  k = k+1;
  RI(k) = EIm3(k)-((d-1)/days_month(m))*(EIm2(k)-EIm3(k));
  k = k+1;
  RI(k) = EIm3(k)-((d-1)/days_month(m+6))*(EIm2(k)-EIm3(k));
end

IC = RI./BaseIndex;

coupons = Nom_val.*(Real_rate/2).*IC;
principal_appreciations = Nom_val*max(IC(1,end)-1,0);

end
```

MATLAB code 4.1: A function that simulates coupon and appreciation scenarios for BTP€i

Therefore, we can now simulate future coupons and principal appreciation according to the simulated price index paths. The general structure of the script to simulate coupons and principal appreciation scenarios, using one of the previous price index models, for a 5-year semi-annual BTP€i with a real rate of 0.10% and an issue date of November 15, is as follows:

```
%% Simulation of coupons capital repayment for BTPei
Real_rate = 1; %expressed in percentage format
Real_rate = Real_rate/100; %expressed in numeric format
EI = P_tk; %P_tk is the output of one of the price index model
    function
day = 15;
month = 11;
time = 5;
Nom_val = 1000;

for j = 1:n_simul
    [BTPei_coupons(j,:),BTPei_principal_appreciations(j,:)] = ...
    BTPei(Nom_val, EI(j,:), Real_rate, day, month, time)
end
```

MATLAB code 4.2: Implementation of the BTP€i function

The output will be a matrix for the coupons[1] and a vector for the simulated principal appreciation component.

BTP Italia. We can do a similar operation regarding the BTP Italia. As we already know from the subsection 2.1.6, the functioning of the BTP Italia is slightly different. First, the base index changes every semester, while in BTP€i it remains always fixed. In addition, there is a floor protection that triggers a special mechanism for selecting the base index in case of deflation between two semesters. Last but not least, the capital appreciation is paid semi-annually to the holder instead of a single lump sum at maturity.
We will make similar assumptions as in the case of BTP€i.

1. The price index simulation starts at the emission of the security;

[1] Each row of the matrix is a simulated scenario for coupons.

2. The *first* base index at the denominator of the Indexation Coefficient is fixed at the date of issue and is not calculated by interpolation. The subsequent base indexes are interpolated using the simulated price paths.

Most of the coding intuitions used in the case of BTP€i are reimplemented. However, the most important change that we must apply, is the capital and coupon floor protection mechanism. As stated by the Treasury, in case of a semi-annual deflation, *"in the following period, if the Indexation Coefficient on a semi-annual basis goes again above 1, the highest price index reported in the previous semesters is used as a base"*.
As we have seen in the previous case, the theoretical structure in the case of BTP Italia is as is shown in Figure 4.2.
By adding the floor protection mechanism, we can also obtain a useful function for

	A	B	C	D	E	F	G	H	I	J	K	L
1	BTP Italia Indexation coefficient structure											
2	DATES	'Nov 2021	May 2022	'Nov 2022	May 2023	'Nov 2023	May 2024	'Nov 2024	May 2025	'Nov 2025	May 2026	'Nov 2026
3	Time series location	EI(1)	EI(7)	EI(13)	EI(19)	EI(25)	EI(31)	EI(37)	EI(43)	EI(49)	EI(55)	EI(61)
4	Base index	\\	EI(1)	EI(7)	EI(13)	EI(19)	EI(25)	EI(31)	EI(37)	EI(43)	EI(49)	EI(55)
5	EI (m-2)	\\	EI(5)	EI(11)	EI(17)	EI(23)	EI(29)	EI(35)	EI(41)	EI(47)	EI(53)	EI(59)
6	EI (m-3)	\\	EI(4)	EI(10)	EI(16)	EI(22)	EI(28)	EI(34)	EI(40)	EI(46)	EI(52)	EI(58)
7	Reference index	\\	Interpolation using EI(5), EI(4)	Interpolation using EI(11), EI(10)	Interpolation using EI(17), EI(16)	Interpolation using EI(23), EI(22)	Interpolation using EI(29), EI(28)	Interpolation using EI(35), EI(34)	Interpolation using EI(41), EI(40)	Interpolation using EI(47), EI(46)	Interpolation using EI(53), EI(52)	Interpolation using EI(59), EI(58)
8	Indexation coefficient	\\	RI(May 2022) / EI(1)	RI(Nov 2022) / RI(May 2022)	RI(May 2023) / RI(Nov 2022)	RI(Nov 2023) / RI(May 2023)	RI(May 2024) / RI(Nov 2023)	RI(Nov 2024) / RI(May 2024)	RI(May 2025) / RI(Nov 2024)	RI(Nov 2025) / RI(May 2025)	RI(May 2026) / RI(Nov 2025)	RI(Nov 2026) / RI(May 2026)
9	Meaning	Emission date	Coupon #1 + principal repayment	Coupon #2 + principal repayment	Coupon #3 + principal repayment	Coupon #4 + principal repayment	Coupon #5 + principal repayment	Coupon #6 + principal repayment	Coupon #7 + principal repayment	Coupon #8 + principal repayment	Coupon #9 + principal repayment	Coupon #10 + principal repayment

Figure 4.2: Price index location example for a 5-years semi-annual BTP Italia

BTP Italia coupons and capital appreciation scenarios.

```
function [coupons,principal_appreciations] = BTPitalia(Nom_val, EI,
    Real_rate, d, m, t)
% A function that calculates semi-annual coupons and
% principal indexed repayments for BTP Italia
% according to simulated price index trajectories.
%
% BTPitalia(Nom_val, EI, Real_rate, d, m, t)
%
%   Nom_val = Nominal principal amount underwritten;
%   EI = simulated time series of the price index;
%   Real_rate = real rate of the indexed bond;
%   d = emission day of the bond. A number between 1 and 31;
%   m = emission month of the bond. A number between 1 and 12;
%   t = maturity (in years).

% According to assumption 1
% the base index is and will always be the emission date of the
  bond,
% which is also the first element of the price index simulation
BaseIndex = EI(1);

% The following array is used to calculate the correct interest
  accrual
days_month = [31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31];
days_month = repmat(days_month,[1,2]);

% Let's transform years in months
maturity_in_months = t*12;

% The effect of assumption 2 -- We are computing the Eurostat
  indices
% to find the Indexation Coefficients through interpolation
```

```
EIm2 = EI(7-2:6:maturity_in_months);
EIm3 = EI(7-3:6:maturity_in_months);

% The number of Reference Indexes to be computed is the maturity in
    months
% divided by the coupon frequency (in this case is semi-annual)
RI = NaN(1,maturity_in_months/6);

% Reference index calculation
k = 0;
while k < length(RI)
  k = k+1;
  RI(k) = EIm3(k)-((d-1)/days_month(m))*(EIm2(k)-EIm3(k));
  k = k+1;
  RI(k) = EIm3(k)-((d-1)/days_month(m+6))*(EIm2(k)-EIm3(k));
end

% Base index vector since it changes each six-months
BaseIndex = [EI(1), RI(1:end-1)];

% Capital and coupon floor protection mechanism
for k = 1:maturity_in_months/6
    IC(k) = RI(k)/BaseIndex(k);
    if IC(k) < 1;
        IC(k) = 1;
        Max_baseindex = max(BaseIndex(1:k));
        BaseIndex(k+1) = Max_baseindex;
    end
end

coupons = Nom_val.*(Real_rate/2).*max(IC,1);
principal_appreciations = Nom_val.*max(IC-1,0);

end
```

MATLAB code 4.3: A function that simulates coupon and appreciation scenarios for BTP Italia

As before, the general structure of the script to simulate coupons and appreciation scenarios, using one of the previous price index models, for a 5-year semi-annual BTP Italia with a real rate of 0.10% and an issue date of November 15, is as follows:

```
%% Simulation of coupons capital repayment for BTP italia
Real_rate = 0.10; %expressed in percentage format
Real_rate = Real_rate/100; %expressed in numeric format
EI = P_tk; %P_tk is the output of one of the price index model
   function
day = 15;
month = 11;
time = 5;
Nom_val = 1000;
for j = 1:n_simul
    [BTPitalia_coupons(j,:),BTPitalia_principal_appreciations(j,:)]
    = BTPitalia(Nom_val, EI(j,:), Real_rate, day, month, time);
end
```

MATLAB code 4.4: Implementation of the BTP Italia function

Two matrices are the output: one for the coupons and one for the principal appreciation payment. Each row represents a different scenario.

It is important to state that both functions are designed to work only with semi-annual BTPs Italia and BTPs€i.

4.2 Comparing BTP Italia and BTP€i scenarios

In the previous section, we presented practical tools for creating different scenarios in terms of coupons and capital appreciation, for both BTP Italia and BTP€i. Now, it is time to compare these instruments. Which method of accounting for inflation in indexed bonds is more costly for the Italian Treasury? And simmetrically, *ceteris paribus*, which security seems better for an investor to hedge against inflation? Specifically, we will compare two 10-years indexed bonds with a real rate of 1% and semi-annual coupons. Why such a small real coupon rate? Because only in this way we can better understand the revaluation mechanism for BTP€i and BTP Italia. Both bonds have a minimum lot size of €1000 and an issue date of January, 15. The comparisons are provided for both price index models in Chapter 3.
Before proceeding, it is important to refine the results given by the BTP€i and BTP Italia functions.
By defining the semi-annual return as the sum of coupons and capital appreciations

$$Semester\ total\ return_t = \ Coupon_t \ + \ Principal\ appreciation_t \tag{4.3}$$

we can obtain unique matrices for both securities that contain the sum of both cash flows. Therefore,

```
% Creating semi-annual return matrices

[r,c] = size(BTPei_coupons);

BTPei_semiannual_return = BTPei_coupons + [zeros(r,c-1),
    BTPei_principal_appreciations]

BTPitalia_semiannual_return = BTPitalia_coupons +
    BTPitalia_principal_appreciations
```

MATLAB code 4.5: Refinement script for BTP€i and BTP Italia semi-annual return matrices

According to our price index models, by applying the same stochastic scheme to both securities, we can obtain two comparable cash flows after the refinement in MATLAB Code 4.5. If we apply the "historical" price index model, the code will be:

```
%% Price index paths
n_simul = 50000; %number of simulaations
Years = 15; %years simulated
j = 0.0016; sigma = 0.0029; %parameters
P0 = 100; %price index starting point

fig1 = figure(1)
[P_tk] = SimpleSinglePriceModel(n_simul, Years, j, sigma, P0);

% % input for inflation price model #2
% gamma_y = 0.001615;
% y0 = 9.48*(10^-3);
```

```
% t = 42;
% [alpha] = alpha_coeff(y0, gamma_y, t)
% [P_tk] = SinglePriceModel(n_simul, Years, alpha, gamma_y, y0,
    sigma, P0);

%% Features of BTPei and BTP Italia
Real_rate = 1; %expressed in percentage format
Real_rate = Real_rate/100; %expressed in numeric format
EI = P_tk; %P_tk_is the matrix containing trajectories of price
    index.
day = 15;
month = 1;
time = 10; %years to maturity of both securities
Nom_val = 1000; %capital subscribed

%% Simulation of coupons and capital appreciation repayment for
    BTPei and BTP Italia
for j = 1:n_simul
     [BTPei_coupons(j,:),BTPei_principal_appreciations(j,:)] = BTPei
    (Nom_val, EI(j,:), Real_rate, day, month, time);
end

for j = 1:n_simul
     [BTPitalia_coupons(j,:),BTPitalia_principal_appreciations(j,:)]
     = BTPitalia(Nom_val, EI(j,:), Real_rate, day, month, time);
end

%% Creating semi-annual return matrices
[r,c] = size(BTPei_coupons);
BTPei_semiannual_return = BTPei_coupons + [zeros(r,c-1),
    BTPei_principal_appreciations];
BTPitalia_semiannual_return = BTPitalia_coupons +
    BTPitalia_principal_appreciations;
```

MATLAB code 4.6: Comprehensive code to compare BTP€i and BTP Italia

For a better understanding, here are the first 10 semi-annual return scenarios of our simulation, for both securities. These are the first 10 rows of matrices `BTP€i_semiannual_return` and `BTPitalia_semiannual_return`.

```
>> BTPei_semiannual_return(1:10,:)

ans =
```

4.9807	5.0463	5.1105	5.2095	5.2833	5.3705	5.3607	5.3955	5.3754	5.3591	5.3601	5.4041	5.4744	5.4678	5.4910	5.5463	5.6279	5.6463	5.7245	161.5874
5.0366	5.0840	5.1676	5.1589	5.2356	5.3106	5.3496	5.3604	5.4031	5.4783	5.5747	5.6324	5.6733	5.8945	6.0573	6.1202	6.2571	6.3157	6.4179	310.2131
5.0094	5.0984	5.1789	5.3256	5.3703	5.3966	5.4180	5.4127	5.5304	5.5664	5.5815	5.6895	5.7648	5.7501	5.8608	5.9300	5.9753	6.0141	6.1435	233.6211
5.0207	5.0218	5.0236	5.0842	5.1344	5.1445	5.2135	5.3130	5.2573	5.2648	5.3557	5.3780	5.4357	5.5493	5.5832	5.6224	5.7321	5.8244	5.8710	175.9510
5.0589	5.0910	5.1436	5.2022	5.2457	5.2378	5.2937	5.3411	5.3530	5.3814	5.4146	5.4571	5.5810	5.6487	5.6739	5.7547	5.7897	5.9086	5.9317	200.0706
4.9998	5.0916	5.1738	5.2627	5.2644	5.2620	5.3305	5.4489	5.4971	5.5363	5.5627	5.6815	5.6245	5.6776	5.6844	5.8200	5.9496	6.0136	6.1541	262.5849
5.0394	5.0647	5.0888	5.1538	5.1625	5.2174	5.3403	5.4104	5.5227	5.5620	5.6881	5.7939	5.7643	5.7844	5.8737	5.8985	5.9145	5.9368	5.9396	199.9679
5.0626	5.1119	5.1627	5.3036	5.3869	5.3830	5.3567	5.3957	5.5168	5.5207	5.5463	5.6067	5.6966	5.7312	5.8277	5.9007	6.0235	6.1413	6.2059	252.9027
5.0299	5.0653	5.1553	5.2658	5.3011	5.3638	5.3995	5.4321	5.4806	5.5370	5.5936	5.6447	5.6848	5.7925	5.8341	5.9274	6.0135	6.1960	6.2599	264.3679
5.0432	4.9900	5.0354	5.0573	5.0931	5.1363	5.2098	5.2903	5.3194	5.3370	5.3431	5.3661	5.4509	5.4698	5.5056	5.5951	5.5670	5.6354	5.7702	148.0364

Figure 4.3: First 10 simulated rows of the semi-annual return matrix for BTP€i

```
>> BTPitalia_semiannual_return(1:10,:)

ans =
```

5.0000	14.3162	17.7661	24.4834	19.2316	21.5858	5.0000	9.6794	5.0000	5.0000	5.0000	6.6000	18.0729	5.0000	8.0479	15.1218	19.7806	8.2954	18.9245	14.5690
12.3522	14.4670	21.5214	5.0000	18.2250	19.4026	12.3726	7.0331	13.0066	18.9759	22.6862	15.4101	12.3021	44.1708	32.7644	15.4309	27.4872	14.4075	21.2598	20.7559
6.8849	22.8672	20.8653	33.4634	13.4419	9.9171	8.9780	5.0000	25.8549	11.5387	7.7359	24.4359	18.3047	5.0000	21.7347	16.8656	12.6819	11.5276	26.6277	5.0000
9.1595	5.2188	5.3591	17.1256	14.9307	6.9745	18.4811	24.1712	5.0000	5.0000	13.0816	9.1926	15.7759	25.9936	11.1541	12.0456	24.6199	21.1667	13.0541	5.0000
16.8379	11.3708	15.3949	16.4540	13.4012	5.0000	14.1850	13.9997	7.2489	10.3274	11.1958	12.9011	27.8021	17.2046	9.4781	19.3054	11.1191	25.6299	8.9431	11.5669
5.0000	23.4119	21.2304	22.2722	5.3194	5.0000	17.6176	27.3139	13.8969	12.1672	9.7989	26.4635	5.0000	5.0000	5.5021	28.9817	27.3777	15.8080	28.4789	25.8095
12.9275	10.0280	9.7820	17.8560	6.6871	15.6820	28.6804	18.1960	25.8511	12.1571	27.7758	23.6997	5.0000	5.0000	18.8445	9.2486	7.7208	8.7917	5.4730	10.1397
17.5891	14.7839	14.9830	32.4348	20.7716	5.0000	5.0000	6.6442	27.5563	5.7201	9.6566	15.9416	21.1250	11.0903	21.9188	17.5943	25.9147	24.6651	15.5690	9.4403
11.0198	12.0566	22.8710	26.5295	11.7383	16.8857	11.6871	11.0733	13.9680	15.3525	15.2667	14.1865	12.1433	24.0264	12.2165	21.0804	19.5904	35.5007	15.3671	9.8987
13.6878	5.0000	5.0000	7.8014	12.1228	13.5199	19.3819	20.5278	10.5212	8.3295	6.1547	9.3271	20.8711	8.4985	11.5652	21.3447	5.0000	12.2355	29.0435	5.0000

Figure 4.4: First 10 simulated rows of the semi-annual return matrix for BTP Italia

The same can be done using the second price index model of subchapter 3.2.

Before proceeding, a small clarification is needed. Since the main point of the comparison is the different mechanism by which the bond deals with the inflation link, this means that we will directly analyze the differences between the cash flows of BTP€i and BTP Italia. Thus, our comparisons will be ex-post with all the trimmings.

4.2.1 Yield to Maturity (YTM)

Normally, the real yield of an inflation-linked bond is "calculated by taking the real coupons and principal repayments as cash flows without inflation compensation" [58]. But here we will do the opposite, and more specifically, we will compare the appreciation mechanism after simulating inflation. We will use the YTM or more commonly referred the Internal Rate of Return (IRR) of a bond [59].
The Yield to Maturity is "the interest rate that makes the present value of a bond's payments equal to its price" [59], or in other words, "the rate at which the financial transaction is fair" [48]. The YTM is often used as "yieldstick" while comparing different bonds. In fact, this rate is often interpreted as a measure of the average return that will be earned on a bond if it is purchased and held to maturity [59]. It considers interest-on-interest and it assumes that the coupon payments can be reinvested at an interest rate equal to the Yield to Maturity [60]. The former statement should not be underestimated: although the YTM has the advantage of being a single measures for comparing similar bonds, it does not account for the *reinvestment risk*. This can lead to an overestimation of the effective return.
By the way, by setting the price equal to subscribed capital at the issue, the YTM is the discounted rate r^* which solves the following equation:

$$capital \;=\; \sum_{t=1}^{m} \frac{coupons}{(1+r)^t} + \frac{capital}{(1+r)^m} \tag{4.4}$$

Therefore, by considering a cash flows $\{x_0, x_1, ..., x_k, ..., x_m\}$ over a schedule $\{t_0, t_1, ..., t_k, ..., t_m\}$, the IRR would also be the r^* such that:

$$\sum_{k=0}^{m} x_k(1+r)^{-(t_k - t_0)} = 0 \tag{4.5}$$

$$NPV = 0 \tag{4.6}$$

In our case, since the sign changes once, we know there is only one solution [48].
As for the calculation, there is no closed formula, which means that the result goes through an iterative procedure. Luckily, MATLAB directly provides the built-in function `irr` which computes the IRR of a given cash flow.
We just need to add a negative disbursement at t_0 (i.e. the underwritten capital or nominal value) and the principal repayment itself at t_m, where m is the maturity date. We do not need to worry about the capital appreciation or the floor mechanisms, since they are already taken care of by the functions we saw earlier (MATLAB codes 4.1, 4.3) and the refinement script (MATLAB code 4.5). Thus, the MATLAB code 4.7 provides the PDFs shows in Figures 4.5 and 4.6.

```
%% YTMs / IRRs
format short g
% creating a cash flow containing the capital outflow and inflow
capital = Nom_val*ones(r,1);
BTPei_CF = [-capital, BTPei_semiannual_return(:,1:end-1), BTPei_semiannual_return(:,end) + capital];
BTPitalia_CF = [-capital, BTPitalia_semiannual_return(:,1:end-1), BTPitalia_semiannual_return(:,end) + capital];

% calculation and annualization of IRRs
IRR_BTPei = (irr(BTPei_CF').*2);
IRR_BTPitalia = (irr(BTPitalia_CF').*2);

%% Synthetic indices YTM - IRR
%BTP ei
BTPei_irr_mean = mean(IRR_BTPei)
BTPei_irr_var = var(IRR_BTPei)
BTPei_irr_skew = skewness(IRR_BTPei)
BTPei_irr_kurt = kurtosis(IRR_BTPei)

%BTP Italia
BTPitalia_irr_mean = mean(IRR_BTPitalia)
BTPitalia_irr_var = var(IRR_BTPitalia)
BTPitalia_irr_skew = skewness(IRR_BTPitalia)
BTPitalia_irr_kurt = kurtosis(IRR_BTPitalia)

%% Confrontation between IRRs
nbins = 2500;
IRR_BTPei = IRR_BTPei.*100; %multiply by 100 for better visualisation
IRR_BTPitalia = IRR_BTPitalia.*100; %multiply by 100 for better visualisation
fig2 = figure(2);
[xx1,N_BTPei] = F_Empirical_pdf(IRR_BTPei,nbins)
title("PDF of BTPei YTMs")
ylabel("frequencies")
xlabel('Annualized YTM')
xtickformat('percentage')
set(gca,'FontName','cmr12')

fig3 = figure(3);
[xx2,N_BTPitalia] = F_Empirical_pdf(IRR_BTPitalia,nbins)
title("PDF of BTP Italia YTMs")
xlabel('Annualized YTM')
ylabel("frequencies")
xtickformat('percentage')
set(gca,'FontName','cmr12')
```

MATLAB code 4.7: BTP€i and BTP Italia IRRs comparisons

If we study both probability distribution function by calculating the four most characteristic synthetic indices of a distribution we obtain the results in Table 4.1. We are not concerned with the absolute value of the two averages. Indeed, this depends on the characteristics of the bond and the evolution of the price index model. What is important is that the two probability distribution functions are very similar, if not identical (Figures 4.5 and 4.6), even though there are different rules for accounting coupons and capital appreciation. In other words, with over 50,000 simulations, we can expect a very similar normal distribution for both YTMs, since they actually

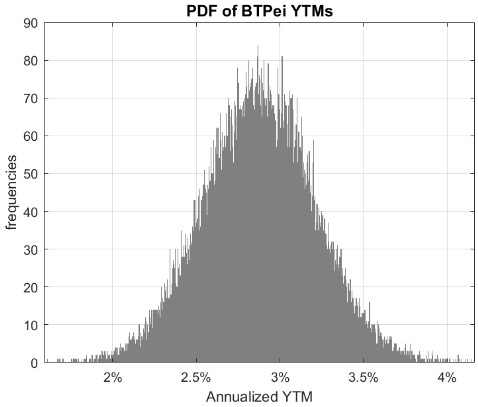

Figure 4.5: Probability distribution function of YTMs associated to BTP€i. Price model n.1

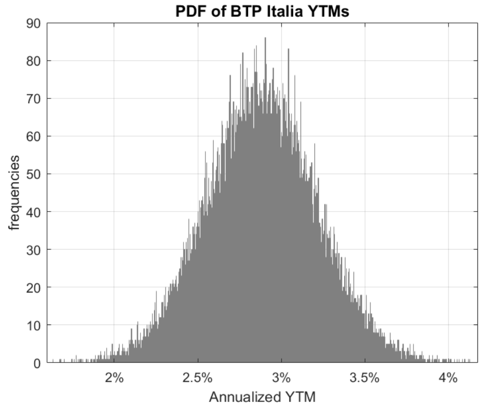

Figure 4.6: Probability distribution function of YTMs associated to BTP Italia. Price model n.1

	BTP€i	BTP Italia
Mean	0.0288	0.0288
Variance	1.0149e-05	9.9557e-06
Skewness	0.0070	0.0163
Kurtosis	2.9886	2.9906

Table 4.1: Synthetic indices for BTP€i and BTP Italia YTM distributions. Price model n.1

differ for very small amounts, probably due to the presence of the floor protection mechanism of BTP Italia and the semi-annual appreciation of the principal.
The same computations can be done using the second price index model[2] with the results evidenced in Table 4.2.

As we have seen in section 3.2, the underlying second price index model implies a

	BTP€i	BTP Italia
Mean	0.0370	0.0379
Variance	1.0102e-05	1.0622e-05
Skewness	0.0171	0.0278
Kurtosis	2.9869	2.9750

Table 4.2: Synthetic indices for BTP€i and BTP Italia YTM distributions. Price model n.2

period of rapid price increases and later "normalization" to monthly growth in line with the ECB target. With the same number of simulations, the expected YTM of the BTP Italia is slightly higher than that of the BTP€i. This could be due to higher capital appreciation (due to rampant inflation) over the life of the bond (i.e. a lower number of events triggering the floor protection mechanism[3]). The

[2]The second price index model used, suppose the following parameters: $\gamma_y = 0.001615, t = 42, y_0 = 9.48 \cdot 10^{-3}$

[3]The situation in which the semi-annual Indexation Coefficient is less than one is less likely when there is rampant inflation.

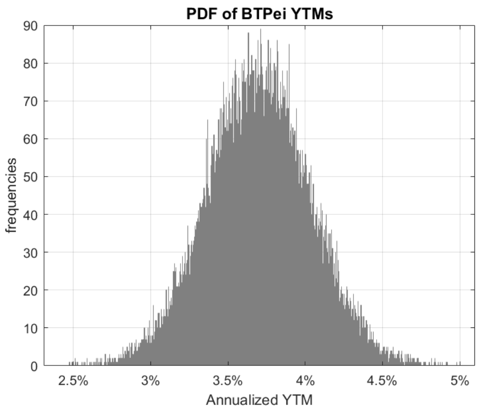

Figure 4.7: Probability distribution function of YTMs associated to BTP€i. Price model n.2

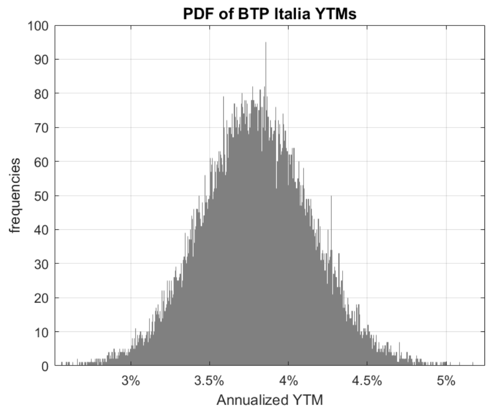

Figure 4.8: Probability distribution function of YTMs associated to BTP Italia. Price model n.2

representation of both PDFs is in Figures 4.7 and 4.8.

4.2.2 Temporal and variability indices

When we start comparing time and variability indices, things get a bit more complicated.
Macaulay duration from a "statistical viewpoint" can be interpreted as the "weighted average of the payment dates of the cash flows" [47] (or, from a physical point of view, "the temporal barycenter" of the investment [48]). Castellani, De Felice and Moriconi refer to the Macaulay duration as a temporal index [48].
Defined a cash flows $x = \{x_0, x_1, ..., x_k, ..., x_m\}$ over a schedule $t = \{t_0, t_1, ..., t_k, ..., t_m\}$ the general Macaulay duration formula is as follows:

$$D(t_0, x) = \frac{\sum_{t=1}^{m} (t_k - t_0) x_k (1 + i_k)^{-(t_k - t_0)}}{\sum_{t=1}^{m} x_k (1 + i_k)^{-(t_k - t_0)}} \tag{4.7}$$

But since we do not have a yield term structure, we can calculate the *flat yield curve duration* using the IRR obtained in the previous subsection. In fact, as stated by Castellani, De Felice and Moriconi [48] it provides a satisfactory approximation. Therefore, given r^* as the YTM, the formula would be the following:

$$D(t_0, x) = \frac{\sum_{t=1}^{m} (t_k - t_0) x_k (1 + r^*)^{-(t_k - t_0)}}{\sum_{t=1}^{m} x_k (1 + r^*)^{-(t_k - t_0)}} \tag{4.8}$$

What is more, it is strictly linked to the price sensitivity of an investment. Defined the present value of a cash flow as:

$$PV(t_0, x) = \sum_{t=1}^{m} x_k (1 + r^*)^{-(t_k - t_0)} \tag{4.9}$$

where $PV'(t_0, x)$ is the first derivative w.r.t. r, the element:

$$\frac{PV'(t_0, x)}{PV(t_0, x)} = -\frac{1}{(1 + r^*)} D(t_0, x) \tag{4.10}$$

is a semi-elasticity or indeed, "the price sensitivity of the investment" [47]. This is also known as the *modified duration* [48], one of the most important tools used in the nominal interest rate sensitivity of traditional bonds; in fact, duration quantifies the estimated impact of interest rate fluctuations on a bond's value [60]. The modified duration is considered as a variability index [48].
The real duration of an inflation-linked bond, as stated by Unicredit [61], is the "duration calculated using the real coupon and the non-revaluated principal". It is clear that ILB prices rise when real yields fall and fall when real yields rise.[4].
In our case, however, we are comparing the different indexing mechanisms. We are tracking and making ex-post comparisons. Let's try to apply the above formulas to the realized cash flows.
Both the Macaulay Duration and the Modified duration can be define through a MATLAB function:

```
function [Mac_Dur,Mod_Dur] = Flat_Dur(x,t,t0, i)
% Dur function: a function which compute the Macaulay duration and
% Modified duration with flat yield structure for semi-annual cash flow
%   x = vector of cash flows (row vector);
%   t = vector of the time schedule (row vector);
%   t0 = starting point of the investment (scalar);
%   i = annual flat interest rate curve (scalar);

% Discount factor
v = ((1+(i/2)).^-(t-t0));
% Present value
PV = x*v';

% Macaulay duration
for k = 1:length(t)
    vv(k) = ((1+(i/2)).^-(t(k)-t0));
    Mac_Dur(k) = ((t(k)-t0)*x(k)*vv(k));
end
Mac_Dur = sum(Mac_Dur)/PV;

% Modified duration adjusted for semi-annual coupons
Mod_Dur = -(1/(1+(i/2)))*Mac_Dur;
```

MATLAB code 4.8: Flat yield curve duration

By applying the above function to the simulated path of BTP€i and BTP Italia (MATLAB code 4.9)

```
%% temporal and volatility indexes
t0 = 0;
t = 0.5:0.5:time;
IRR_BTPei = IRR_BTPei./100;
IRR_BTPitalia = IRR_BTPitalia./100;
for j = 1:n_simul
    [BTPei_Mac_Dur(j), BTPei_Mod_Dur(j)] = Flat_Dur(BTPei_semiannual_return(j,:),t,t0, IRR_BTPei(j));
end

for j = 1:n_simul
```

[4] History shows that real interest rates have a lower volatility than nominal interest rates. As a result, ILBs tend to be less volatile in relation to their nominal counterparts [58] [62].

```
    [BTPitalia_Mac_Dur(j), BTPitalia_Mod_Dur(j)] = Flat_Dur(
   BTPitalia_semiannual_return(j,:),t,t0, IRR_BTPitalia(j));
end
```

MATLAB code 4.9: Computation of BTP€i and BTP Italia Macaulay Durations and Modified

we would obtain a probability distribution for BTP€i and BTP Italia Macaulay Duration and Modified Duration.

```
%PDFs for BTPei Macaulay duration and Modified duration
fig4 = figure(4)
subplot(2,1,1)
F_Empirical_pdf(BTPei_Mac_Dur,nbins)
title('PDF of BTPei Macaulay duration')
xlabel('Macaulay duration (in years)')
ylabel("frequencies")

subplot(2,1,2)
F_Empirical_pdf(BTPei_Mod_Dur,nbins)
title('PDF of BTPei Modified duration')
xlabel('Modified duration (in percentage)')
xtickformat('percentage')
ylabel("frequencies")
set(gca,'FontName','cmr12')

% PDFs for BTPei Macaulay duration and Modified duration
fig5 = figure(5)
subplot(2,1,1)
F_Empirical_pdf(BTPitalia_Mac_Dur,nbins)
title('PDF of BTP Italia Macaulay duration')
xlabel('Macaulay duration (in years)')
ylabel("frequencies")
set(gca,'FontName','cmr12')

subplot(2,1,2)
F_Empirical_pdf(BTPitalia_Mod_Dur,nbins)
title('PDF of BTP Italia Modified duration')
xlabel('Modified duration (in percentage)')
xtickformat('percentage')
ylabel("frequencies")
set(gca,'FontName','cmr12')
```

MATLAB code 4.10: PDFs for BTP€i Macaulay duration and Modified duration

The synthetic indices of the PDFs in Figures 4.9 and 4.10, using the price model n.1 (the "historical" one) are expressed in Table 4.3.
The BTP€i has a higher average Macaulay duration than the BTP Italia. The reason is obvious. It is due to the design of the BTP€i, which does not provide for capital appreciation until maturity. That is also the reason why Macaulay duration outliers are more likely to occur in the BTP€i than in the BTPItalia.
Let us now turn to the modified duration comparison. The only thing that is different between the two securities is the way inflation is incorporated into the coupons and capital appreciation. By doing a little forcing, considering how the YTMs of these two securities depend on the real interest rate and the inflation that has occurred, we can consider modified duration as a proxy for the relative sensitivity to these two components.

BTP€i seems on average more sensitive than BTP Italia, with less outliers and more variance.

It is important to note that in the first price index models, the semi-annual inflation in BTP Italia triggers the floor protection mechanism more often than in the second. In absence of positive inflation shock (or in "normal situation"), the price index model is more "subject to the stochastic component", which can bring often a semi-annual Indexation Coefficient less than 1.
This may be the reason why the results with the second price index models, where inflation follows a stepped shape, differ in magnitude from those in the first price index model. The inflation shock implies a lower probability of triggering the floor mechanism, as shown in the results presented in the Table 4.4. The PDFs are shows in Figures 4.11 and 4.12.

	BTP€i		BTP Italia	
	Macaulay duration	Modified duration	Macaulay duration	Modified duration
Mean	8.2583	-8.1410	5.2183	-5.1442
Variance	0.0367	0.0311	0.0968	0.0945
Skewness	-0.7121	0.7764	0.0302	-0.0458
Kurtosis	4.0058	4.1690	3.0581	3.0635

Table 4.3: Time and volatility synthetic indices with price index model n.1

	BTP€i		BTP Italia	
	Macaulay duration	Modified duration	Macaulay duration	Modified duration
Mean	8.5800	-8.4240	4.3152	-4.2348
Variance	0.0139	0.0106	0.0607	0.0580
Skewness	-0.5208	0.6004	-0.0683	0.0560
Kurtosis	3.5763	3.7347	3.1011	3.1037

Table 4.4: Time and volatility synthetic indices with price index model n.2

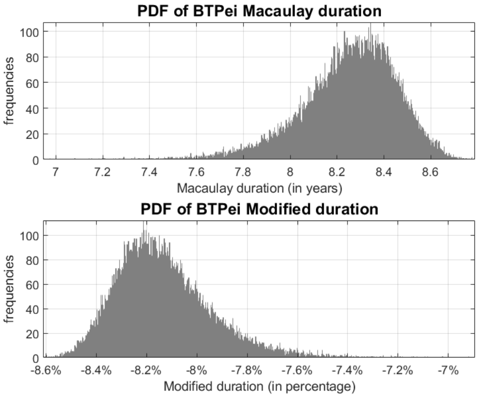

Figure 4.9: PDFs for BTP€i Macaulay duration and Modified duration. Price index model n.1

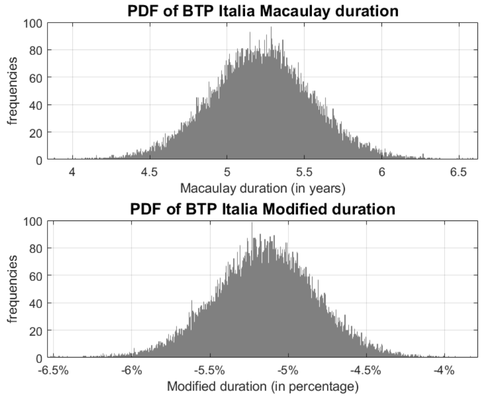

Figure 4.10: PDFs for BTP Italia Macaulay duration and Modified duration. Price index model n.1

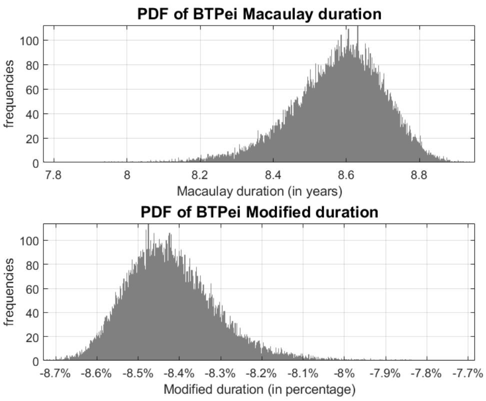

Figure 4.11: PDFs for BTP€i Macaulay duration and Modified duration. Price index model n.2

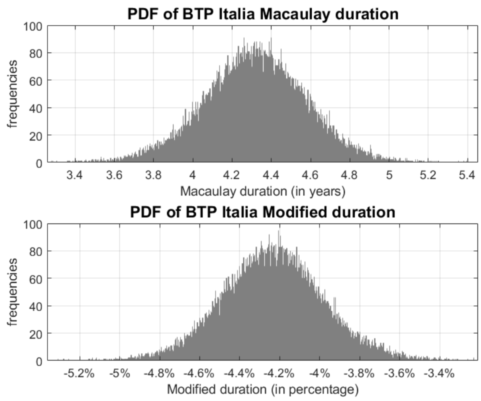

Figure 4.12: PDFs for BTP Italia Macaulay duration and Modified duration. Price index model n.2

Conclusions

In the historical overview of public debt, we saw many times the effect of inflation. As also stated by Cottarelli and Gottardo [63], inflation can affect the public debt in two ways: on one hand, it can increase the nominal GDP and then decrease the total debt-to-GDP ratio. On the other hand, investors may demand a higher yield to account for future inflation expectations, with obvious consequences for the interest portion of public spending. According to the Italian debt history, both effects happened. If we exclude the effect of fiscal policy on the ratio, we observe that after the World Wars, a significant positive contribution to the public finance indicator was made primarily due to inflation, which eroded national savings and the real value of fixed-income securities.
However, the contribution of inflation has not always been positive. In the 1980s, the short-term orientation of investors, together with a high rate of inflation and the abandonment of direct monetization, led to an exponential increase in debt, with still actual consequences. Although Italy emerged from that situation with the issuance of floating-rate securities, such as CCT-eu, glossing over the unsuccessful attempt of the CTR, only starting at the beginning of the 21st century, Italy started to issue inflation-linked bonds. In fact, BTP€i and BTP Italia were issued, covering both European and Italian inflation. At the end of 2022, the inflation-linked securities issued by the Italian Treasury accounted for 10% of the total debt composition. Overlooking the technicalities highlighted in Chapter 2, the differences between BTP€i and BTP Italia can be summarized as in Table 4.5.

Before proceeding with the practical simulation of BTP€i and BTP Italia indexation, in accordance with the literature by Castellani, De Felice, and Moriconi, we needed to simulate the evolution of the price index. We explored two models: the first model is simply based on historical data from the evolution of the real HICP ex-tobacco and FOI index, while the second one assumes a drift evolution coefficient that evolves with mean-reverting dynamics. It turned out that this second price index model is useful for simulating price shock scenarios.
Thanks to our simulated price index models in Chapter 3, we were able to numerically assess the differences between them. Surprisingly, after more than 50,000 simulations, they showed the same internal rate of return in a "normal" price-increasing scenario. The situation was slightly different in the case of a price shock. In fact, in that case the BTP Italia seemed to be marginally better in hedging inflation.
For both, the different design features fell back on the temporal and variability indices. In fact, the BTP€i showed, on average, a higher Macaulay duration in both price evolution scenarios, which is the direct result of the temporal design of the revaluation payments. Moreover, in the second scenario, we observed how BTP Italia is better equipped to cope with the inflationary shock: in fact, the barycenter of payments in the second price scenario was anticipated with respect to the same

	BTP€i	BTP Italia
Indexation Coefficient formation	The base index used for the Indexation Coefficient (the index at the denominator) is fixed at the issuance. Thus, the inflation is considered from a sole and unique starting point.	The base index used for the Indexation Coefficient (the index at the denominator) change each semester. Thus, the inflation is considered between semesters.
Semi-annual return	The semi-annual return consist in the revalued coupon. The appreciated principal is repaid at maturity.	The semi-annual return consist in the revalued coupon + the capital appreciation. The principal is repaid at maturity.
Floor protection mechanism	Only the capital repaid at maturity has a floor protection. In case of deflation (from the initial starting point) the coupons are not protected.	Both the coupons and the capital appreciation have a floor protection mechanism. However, the floor protection mechanism implies a downside for the next semester (the highest price index registered in the past is used as the base index).

Table 4.5: BTP€i and BTP Italia differences

temporal index of the first scenario.
Those results are also reflected in the modified duration, with all the trimmings.

Appendix A

Inflation-linked bonds

A.1 Deepening indexed securities

A.1.1 Introduction to indexed bonds

Inflation-linked bonds (ILBs) generally refer to fixed-income securities whose coupon and principal are linked to a specified price index. While ordinary fixed-income securities take inflation into account through pressure on nominal interest rates[1], inflation is taken into account by design in inflation-linked bonds, as these types of securities yield a real rate of return, meaning that the nominal return varies with the rate of inflation realized over the life of the bond.[64]. Even if from a theoretical point of view the indexation can be applied to any kind of indexes (wages, commodities and currencies), in practice price indexes are mainly used (if we exclude floating interest rate securities linked to institutional rates) [65]. There are several reasons for the existence of this type of security, and indeed, inflation-linked bonds are less innovative securities than we think. The first example of inflation-linked bonds can be found in Massachusetts in the 1780s. Here was issued the first bond whose coupon and principal were linked to the price of a basket of goods [66]. If we jump to the XX century, we can divide the issuance of indexed government bonds in two groups: the issuance by necessity and the deliberate policy choice. With regard to the first group, we can consider the emission of indexed bonds of whose countries have always had some problems in containing the inflation, such as Chile (1956), Brazil (1964), Colombia (1967), Argentina (1973) and also Italy (1983). The second group is populated by United Kingdom (in 1981), Australia (in 1985), Sweden (in 1994) and New Zealand (in 1995) [66]. The aim of these issues is to reduce the inflationary expectations of the investors by strengthening the government policies in the fight against inflation.

However, the reasons for issuing index-linked bonds are not only related to cost savings or credibility (for the issuer). In fact, inflation-linked bonds also have important consequences for investors. There is a positive example from a welfare point of view: suffice it to consider contributions as a hedge against inflation for pension fund management[2]. More specifically, unanticipated inflation (or deflation) implies an unintended transfer of wealth from lenders to borrowers, a situation that

[1]Think of the sovereign debt issuance in Italy in the 80s.

[2]Campbell and Viceira (2002) state that the real risk-free asset is the inflation-linked Treasury bill, since it provides a stable and constant stream of consumption over time.

is hedged by inflation-linked securities. It must also be considered the favorable externalities related to a greater comprehensiveness of the financial markets, with positive outcome in the savings rate and the public debt financing. For the reasons mentioned above, countries such as United States (1997), France (1998), Japan (2004), Germany (2006) issue inflation-linked bonds [66]. Additionally, if we refer on the issuer side, it is important to note that since government revenues (i.e. taxes) are somehow linked to inflation, the issuance of inflation-linked securities allows for a more accurate matching of government assets and liabilities [66]. An important element that should not be underestimated, as also stated in the World Bank's public debt guidelines [67].

A.1.2 Indexation lag

Indexed bonds track a benchmark price index. Generally, most of the indexes used are CPI-type, which means that are consumer price indexes linked to a bundle purchased by the average household. As we have seen in subsections 2.1.5 and 2.1.6, the indexation is done according to an Indexation Coefficient computed using CPI index measures. Since there is not enough time between the effective monthly measure of prices and the coupon payment date, the indexation is done through linear interpolation. If we delve deeper, this approach, which has the benefit of provide a day-by-day Indexation Coefficient, has some downturns. In fact, the indexation-lag does not provide hedging at the end of the life of bond[3], effectively making the bond nominal at maturity [65]. That's the reason why the indexation-lag must not be underestimated since in certain situation as hyperinflation, it can be hedge-inefficient [65]. The indexing lag also applies when the bond is traded in the secondary market, as accrued interest (which is the real rate plus inflation[4]) is taken into account. The longer the indexing lag[5] and the shorter the time to maturity, the more relevant the effect.

A.1.3 The break-even inflation

Can we understand when it is worthwhile to invest in an indexed security compared with its nominal alternative? Or specularly, can we identify the effective cost savings by issuing an inflation-linked bonds? The answer is Yes. Let's take the Fisher's identity:

$$(1+i) = (1+r)(1+\pi) \tag{A.1}$$

where i is the nominal rate, r is the real rate and π is the inflation rate.
The equation A.1 can be approximated as

$$i \approx r + \pi \tag{A.2}$$

The Fisher's identity (as it is clear in its approximated form) states that the nominal interest rate i is equal to the sum of the real rate r plus the expected inflation rate π. The break-even inflation (BEI) is then the yield difference between a nominal

[3] If an inflation-linked principal expires on June 15, 2023, if we suppose that the indexation interpolation is given through the price index of three and two months before (as in BTP€i) the inflation of May and half of June will not considered.

[4] This is calculated based on the inflation of the previous months.

[5] In the U.K., prior to 2005, for some gilts the indexation lag was eight months ! [68].

bond and an inflation-linked bond with the same maturity and credit risk. The break-even inflation is computed then as the spread between the nominal and the real rate.

$$BEI \approx i - r \tag{A.3}$$

Therefore, the break-even inflation is the inflation rate at which the expected return of the two bonds would be the same, or in other words, it is the level of inflation that equalizes the return of fixed and inflation-linked bonds [61]. Since the nominal bonds are issued with nominal coupon rate and the ILBs are issued with the real one, it can be calculated each trading day and can be useful to understand how much the market estimates the future inflation. Although the above statement is theoretically true, we have to take into account some small problems when considering the spreads between bond yields as market expectations for future inflation. In fact, we must also take into account a kind of inflation premium that is embedded in the uncertainty of long term the nominal bond and a liquidity premium, since the market share of ILBs is smaller than the nominal one [66]. Additionally, the inflation index considered by the monetary authorities (and upon which monetary policies are based) is often different from the the price index used for bonds indexation[6]. This means that the market expectation that we can extract can not necessarily be a mark for future monetary policies and the above statement is partially true only in a risk-neutral environment [61].

A.2 A real-world example of indexation for BTP€i

To give a practical example, let us take as a reference the BTP€i issued on November 15, 2021 and maturing on May 15, 2033[7], and let's simulate the coupon calculation with a detachment date of November 15, 2023. Even if the date is in the past, we will assume and simulate the calculation as it should be done.

First, it is necessary to calculate the Indexation Coefficient. Although this index is calculated for all days of the year,[8] we will calculate it specifically for November 15, 2023, the ex-dividend date of the coupon. From a simple algebraic point of view, the Indexation Coefficient is a fraction composed of a reference inflation index and a base inflation index:

$$IC_{d,m} = \frac{Reference\ Inflation\ Index_{d,m}}{Base\ Inflation} \tag{A.4}$$

[6]The bonds which are indexed to European inflation usually refer to the HICP Ex-tobacco index while the price index considered by the ECB imply also the tobacco. The same can be stated for the United States autorities: the CPI is used for indexation while the Federal Reserve prefers to evaluate the core inflation, "a measure of inflation that strips out volatile food and energy prices" [69]

[7]Isin code: IT0005482994.

[8]As we know from subsection 2.1.5, capital appreciation also occurs in the case of secondary market trading: the price at which BTP€i are bought and/or sold is obtained by multiplying the quoted price by the "Indexation Coefficient" relative to the settlement date of the transaction. This is useful to calculate the accrued interest to be paid at the time of the secondary market exchange, as we will see later. It is therefore clear how necessary it is to calculate the Indexation Coefficient on a daily basis.

Where the reference inflation index $RI_{d,m}$ is the result of the following interpolation:

$$RI_{d,m} = EI_{m-3} + \frac{d-1}{gg}(EI_{m-2} - EI_{m-3}) \tag{A.5}$$

For the day November 15, 2023, we must then take the Eurostat index values of August 2023 (EI_{m-3}) and September 2023 (EI_{m-2}). Therefore by downloading the monthly HICP Ex-tobacco indices from the ECB database [49] we get:

$$RI_{15,Nov} = 123.66 + \frac{15-1}{30}(124.06 - 123.66) \tag{A.6}$$

$$= 123.8466667 \tag{A.7}$$

The index should be truncated to the sixth decimal place (123.846666) and rounded to the fifth digit (123.846667).
We must now calculate the base index, which is computed on the date of enjoyment of the bond (this date is also indicated on *Borsa Italiana*). In this case it is November 15, 2021. We therefore apply the formula A.5 to obtain the base inflation on the security's enjoyment date:

$$BI = 107.54 + \frac{15-1}{30} \times (108.06 - 107.54) \tag{A.8}$$

$$= 107,78266666 \tag{A.9}$$

Truncated to the sixth decimal place (107.782666) and rounded to the fifth digit (107.78267). In this case EI_{m-3} and EI_{m-2} will indicate the Eurostat ex-tobacco HICP indices for August 2021 and September 2021.

Now that we have computed both $Reference\ Inflation\ Index_{d,m}$ and the $Base\ Inflation$ we can calculate the Indexation coefficient:

$$IC_{15,Nov} = \frac{Reference\ Inflation\ Index_{15,Nov}}{Base\ Inflation} \tag{A.10}$$

$$= \frac{123.846667}{107,78267}$$

$$= 1.14904 \tag{A.11}$$

The result is consistent with what is published on the Treasury Department website [70]. Now that we have calculated the coefficient indexation we can revalue the coupon and principal.

The variable amount of the semi-annual coupons is calculated by multiplying the real annual coupon rate divided by two by the revalued principal amount on the coupon payment date. Assuming that the bond in question has a real annual coupon rate of 0.10%, we have that for a minimum principal amount of €1,000, the gross semi-annual coupon is:

$$Coupon_{15,Nov} = \frac{Annual\ real\ coupon\ rate}{2} \cdot Nominal\ principal\ value \cdot IC_{15,Nov} \tag{A.12}$$

$$= \frac{0.0010}{2} \cdot 1000 \cdot IC_{15,Nov}$$

$$= 0.57452 \tag{A.13}$$

This result is consistent with what is published on the Bank of Italy website [71]. To calculate the revalued principal at maturity, we will simply have to multiply the nominal principal amount subscribed by the Indexation Coefficient of November 15, 2033. On the off chance that this is less than unity, the nominal principal will be repaid.

If we wanted to calculate the clean price of the bond instead, we need to calculate the accrued interest. Assume that we want to buy the above stock on November 2, 2023. On this day the Indexation Coefficient equals 1.14743. As a first step, we will need to calculate the percentage of coupon being compensated (1) and then the accrual of the principal revaluation (2):

1. The percentage of the coupon that has accrued to the settlement date of the transaction (AC%) is calculated as follows:

$$AC\% = Coupon\% \cdot \frac{Relevant\ days}{Days\ between\ payment\ of\ two\ coupons} \tag{A.14}$$

   where "relevant days" is the number of days between the payment date of the previous coupon and the settlement date (day d of month m). In this case, since the last coupon has been paid on May, 15 of the same year, the relevant days are 171 days, while the number of the days between the two coupons are 184. Thus:

$$\begin{aligned} AC\% &= \frac{0.0010}{2} \cdot \frac{171}{184} && \text{(A.15)} \\ &= \frac{171}{368000} && \text{(A.16)} \\ & && \text{(A.17)} \end{aligned}$$

2. The figure obtained is then multiplied by the subscribed amount of principal recalculated as at the settlement date (equal to the nominal amount subscribed - let's suppose 1000€- multiplied by the Indexation Coefficient):

$$\begin{aligned} RC_{2,Nov} &= AC\% \cdot Nominal\ principal\ value \cdot IC_{2,Nov} && \text{(A.18)} \\ &= \frac{171}{368000} \cdot 1000 \cdot 1,14743 \\ &= 0.53318 && \text{(A.19)} \end{aligned}$$

   which seems consistent with the coupon result stated in (A.13).

Finally, the trading price to which the accrued coupon must be added must be calculated. The trading price is calculated by multiplying the quoted price by the Indexation Coefficient of the transaction date.

A.3 A real-world example of indexation for BTP Italia

To give a practical example, let us take as a reference the BTP Italia issued on June 28, 2022 and maturing on June 28, 2030[9]. In order to a complete understanding

[9] Isin code: IT0005497000.

the clause and the floor mechanism of BTP Italia, we are going to simulate the semi-annual return (coupon + principal appreciation) with detachment dates of December 28, 2022 and June 28, 2023. We will assume and simulate the calculation as it should be done.

As we already know it is necessary to compute the Indexation Coefficient for December 28, 2022.

$$IC_{28,Dec} = \frac{Index\ number_{28,Dec}}{Index\ number_{\bar{28},J\bar{u}n}} \tag{A.20}$$

$$Index\ number_{28,Dec} = FoilN_{Dec-3} + \frac{d-1}{gg}(FoilN_{Dec-2} - FoilN_{Dec-3}) \tag{A.21}$$

$$Index\ number_{\bar{28},J\bar{u}n} = FoilN_{Jun-3} + \frac{d-1}{gg}(FoilN_{Jun-2} - FoilN_{Jun-3}) \tag{A.22}$$

Let's apply the downloaded data from ISTAT [50] [51].

Since

$$\begin{aligned} Index\ number_{28,Dec} &= 113.5 + \frac{28-1}{31}(117.2 - 113.5) && \text{(A.23)} \\ &= 116.72258 && \text{(A.24)} \\ Index\ number_{\bar{28},J\bar{u}n} &= 109.9 + \frac{28-1}{30}(109.7 - 109.9) && \text{(A.25)} \\ &= 109.72 && \text{(A.26)} \end{aligned}$$

the Indexation Coefficient would be:

$$\begin{aligned} IC_{28,Dec} &= \frac{116.72258}{109.72} && \text{(A.27)} \\ IC_{28,Dec} &= 1.06382 && \text{(A.28)} \end{aligned}$$

The result is consistent with what is published by the Treasury [72].

Now we are able to compute the total return of the semester, composed by the appreciation of capital and the coupon. Supposed we are referring to a nominal of €1000, the coupon appreciation at December 28, 2022, would be:

$$\begin{aligned} Principal\ appreciation_{28,Dec} &= Nominal\ value \cdot MAX(IC_{28,Dec} - 1, 0) && \text{(A.29)} \\ &= 1000 \cdot MAX(1.06382 - 1, 0) \\ &= 63.82 && \text{(A.30)} \end{aligned}$$

while the coupon would be:

$$\begin{aligned} Coupon_t &= \frac{0.016}{2} \cdot 1000 \cdot MAX(1.06382, 1) && \text{(A.31)} \\ &= 8.51056 && \text{(A.32)} \end{aligned}$$

The results are consistent with what is published by the Bank of Italy [73].

If the intensity of inflation decreases in the following six months but remains positive, the result will not change in substance. At the next coupon date, which will be June 28, 2023, the base index used for the Indexation Coefficient of the coupon of June 28, 2023 will be the reference inflation index of December 28, 2022 (i.e. in this specific case 116.72258). If the price level has risen, the index for the

next six months will be higher than the index for the previous six months, and the ratio used to calculate the Indexation Coefficient will also be positive.
Therefore, using the numbers of our example, if we assume that in the next semester the semi-annual inflation would be about 2% (instead of 6.3% of the previous period), we will have an Indexation Coefficient given by a reference inflation index on June 28, 2023, which will be 119.05703 and a base index of 116.72258. The Indexation Coefficient for the coupon and capital appreciation on June 28, 2023 would then be:

$$IC_{28,Dec} = \frac{119,05703}{116.72258} \tag{A.33}$$
$$= 1.02000 \tag{A.34}$$

which is a number greater than 1. Therefore, what we have already seen above applies to the calculation of the coupon and the principal appreciation.

The situation becomes more complicated if there is deflation in the period between the 2 coupons. Let's suppose, by ignoring the real data, that we are in a scenario in which 6 months later the inflation is negative, which means that the price level computed by interpolation (i.e. the reference inflation index, the numerator of the Indexation Coefficient) is less than its base index. Let's suppose 114.00000. Then we have an Indexation Coefficient which is less than 1.

$$IC_{28,Jun} = \frac{Index\ number_{28,Jun}}{Index\ number_{\bar{28},\bar{Dec}}} \tag{A.35}$$
$$= \frac{114}{116.72258} \tag{A.36}$$
$$= 0.97667 \tag{A.37}$$

In this case, the floor mechanism is activated by protecting the coupon and the capital appreciation. With respect to the coupon, the minimum real coupon rate is given, while the capital appreciation would be zero.
But there are no free lunches. This hedging imply that the for the next semester, which would be December 28, 2023, if the index coefficient on a six-month basis returns above unity, the maximum value of the price index number recorded in the previous semesters is taken as the base index. In fact, the most important implication is that, considering how the Indexation Coefficient is structured, in order to get a return above the minimum, we need the numerator to exceed the denominator, which, within this clause, is set at its all-time high value. This clause, together with the fact that the inflation taken into account is that which has occurred every six months (and not that which has occurred since a certain point in time like BTP€i), makes the BTP Italia a more complex instrument than the BTP€i, in particular with regard to retail investors (for whom it is, moreover, intended, as is also shown by the information provided by the Ministry of the Treasury [74]).

If we want to calculate the clean real price of the bond instead, we need to compute the accrued interest and the accrued capital appreciation. With regard the accrued interest the computation is the same as the one seen in BTP€i.
Let's suppose we buy the least amount of BTP Italia of the example above on December 15, 2022. The Indexation Coefficient for that day is 1,04968. The accrued

interest to add at the quoted price would be:

$$
\begin{aligned}
AC\% = Coupon\% \cdot \frac{Relevant\ days}{Days\ between\ payment\ of\ two\ coupons} \qquad \text{(A.38)} \\
= \frac{0.016}{2} \cdot \frac{170}{183} \\
= \frac{34}{4575} \qquad \text{(A.39)}
\end{aligned}
$$

and therefore the interest share to be added:

$$
\begin{aligned}
RC_{15,Dec} &= AC\% \cdot Nominal\ principal\ value \cdot IC_{15,Dec} \qquad \text{(A.40)} \\
&= \frac{34}{4575} \cdot 1000 \cdot 1,04968 \\
&= 7.8009 \qquad \text{(A.41)}
\end{aligned}
$$

While the principal appreciation to correspond would have been:

$$
\begin{aligned}
RRC_{15,Dec} &= Nominal\ principal\ value \cdot \frac{Pr}{100} \cdot (IC_{15,Dec} - 1) \qquad \text{(A.42)} \\
&= 1000 \cdot \frac{98.23}{100} \cdot 1,04968 \\
&= 48.80 \qquad \text{(A.43)}
\end{aligned}
$$

where Pr indicates the price quotation on the market at the date of trade (day *"d"* of month *"m"*). As stated on *Borsa Italiana* on that day day price would have been of €98.23.

Appendix B

Calculations for the single-factor price index model

B.1 Ito's Lemma application to dp(t) with deterministic y(t)

By replacing $y(t)$ into $dp(t)$ we obtain:

$$
\begin{aligned}
dp(t) &= y(t)p(t)dt + \sigma_p p(t)dW_p(t) \\
dp(t) &= [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]p(t)dt + \sigma_p p(t)dW_p(t)
\end{aligned}
\tag{B.1}
$$

Given a function $F(p(t), t)$ of time t and the stochastic process *p(t)*[1] we can obtain its differential $(dF(p(t), t) = F(p(t) + dp(t)), t + dt) - F(p(t), t)$ through the Taylor approximation:

$$
d\ F(p(t), t) \simeq \frac{\partial F}{\partial p(t)}dp(t) + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial^2 p(t)}dp(t)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial^2 t}dt^2 + \frac{\partial^2 F}{\partial p(t)\partial t}dtdp(t)
\tag{B.2}
$$

As evidenced by Cesarone [47], the *"rationale behind the informal justification of Ito's lemma is to examine each addend of the Taylor approximation, to neglect all terms of order greater than dt"*. Therefore, we can eliminate also $\frac{\partial^2 F}{\partial^2 t}dt^2$.
By defining:

$$
\begin{aligned}
F(p(t), t) &= ln(p(t)) \\
d\ F(p(t), t) &= d\ ln(p(t))
\end{aligned}
$$

which means:

$$
\begin{aligned}
\frac{\partial F}{\partial p(t)} &= \frac{1}{p(t)} \\
\frac{\partial^2 F}{\partial^2 p(t)} &= -\frac{1}{p(t)^2} \\
\frac{\partial F}{\partial t} &= 0
\end{aligned}
$$

[1] The function is twice differentiable in $p(t)$ and once in t.

The equation B.2 can be written as:

$$d\ ln(p(t)) = \frac{1}{p(t)}dp(t) - \frac{1}{2}\frac{1}{p(t)^2}dp(t)^2 + \frac{\partial^2 F}{\partial p(t)\partial t}dtdp(t) \tag{B.3}$$

Thus,

$$d\ ln(p(t)) = \frac{1}{p(t)}dp(t) - \frac{1}{2}\frac{1}{p(t)^2}dp(t)^2 \tag{B.4}$$

By analyzing the term $dp(t)^2$ we notice other negligible components. In fact:

$$\begin{aligned} dp(t)^2 &= \{[\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]p(t)dt + \sigma_p p(t)dW(t)\}^2 \\ dp(t)^2 &= [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]^2 p(t)^2 dt^2 + \\ &+ 2\{[\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]p(t)dt\}\{\sigma_p p(t)dW(t)\} \\ &+ (\sigma_p p(t)dW(t))^2 \end{aligned} \tag{B.5}$$

Therefore:

$$dp(t)^2 = \sigma_p^2 p(t)^2 dW(t)^2 \tag{B.6}$$

Let's go back to the equation B.4. Since the quadratic variation of a Wiener process can be considered as "non-random"[2], which implies $dW(t)^2 = dt$, now we have:

$$\begin{aligned} d\ ln(p(t)) &= [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]dt + \sigma_p dW_p(t) - \frac{1}{2}\sigma_p^2 dW(t)^2 \\ d\ ln(p(t)) &= [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t}]dt + \sigma_p dW_p(t) - \frac{1}{2}\sigma_p^2 dt \end{aligned} \tag{B.7}$$

We are ready to integrate it.

B.2 Integration of *d ln(p(t))*

By collecting the equation B.7 for dt we have:

$$d\ ln(p(t)) = [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t} - \frac{1}{2}\sigma^2]dt + \sigma_p dW_p(t) \tag{B.8}$$

which can be integrated from 0 to T:

$$\int_0^T d\ ln(p(t)) = \int_0^T [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t} - \frac{1}{2}\sigma^2]dt + \int_0^T \sigma_p dW_p(t) \tag{B.9}$$

Specifically:

$$\int_0^T d\ ln(p(t)) = ln(p(T)) - ln(p(0)) \tag{B.10}$$

$$\int_0^T \sigma_p dW_p(t) = \sigma_p W(T) = \sigma_p \sqrt{T} Z_T \tag{B.11}$$

[2]See [47].

while the largest component:

$$\int_0^T [\gamma_y - (\gamma_y - y(0))e^{-\alpha_y t} - \frac{1}{2}\sigma^2]dt$$

is broken down into:

$$\int_0^T \gamma_y dt = \gamma_y T \tag{B.12}$$

$$\int_0^T -(\gamma_y - y(0))e^{-\alpha_y t}dt = -(\gamma_y - y(0))\int_0^T e^{-\alpha_y t}dt \tag{B.13}$$

$$= \frac{\gamma_y - y(0)}{\alpha_y}(e^{-\alpha_y T} - 1) \tag{B.14}$$

$$\int_0^T -\frac{1}{2}\sigma_p^2 dt = -\frac{1}{2}\sigma_p^2 T \tag{B.15}$$

Therefore by manipulating we have:

$$ln(p(T)) = ln(p(0)) + (\gamma_y - \frac{1}{2}\sigma_p)T + \frac{\gamma_y - y(0)}{\alpha_y}(e^{-\alpha_y T} - 1) + \sigma_p\sqrt{T}Z_T \tag{B.16}$$

$$\tag{B.17}$$

And ultimately:

$$p(T) = p(0)e^{(\gamma_y - \frac{1}{2}\sigma_p)T + \frac{\gamma_y - y(0)}{\alpha_y}(e^{-\alpha_y T} - 1) + \sigma_p\sqrt{T}Z_T} \tag{B.18}$$

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