NUMERICAL METHODS FOR PRICING OPTIONS WITH TRANSACTION COSTS

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Abstract: Black-Scholes option pricing theory rests on an arbitrage argument where it should be possible to adjust the portfolio continuously. In presence of transaction costs (as it happens in real markets) this assumption leads to a problematic situation. Indeed, diffusion process have infinite variation and a continuous trading would be ruinously expensive. Formally, the arbitrage argument will no longer be valid because replicating the portfolio will be infinitely costly so a discrete revision of Black-Scholes will be needed. In this work we examine Leland model where the costs behave as a non-increasing linear function through the implementation of numerical scheme using implicit and explicit finite differences methods.

Keywords: *Option Pricing, Transaction costs, Finite Difference Methods*
2000 AMS Subject Classification: 21A54 - 91G60

1 INTRODUCTION

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Fischer Black and Myron Scholes in [1] in 1973 and previously by Robert Merton in [2]. The solution of the famous (linear) Black–Scholes equation provides both the price for an European option and a hedging portfolio that replicates the option under several assumptions. Under these assumptions the market is complete, which means that any asset can be replicated with a portfolio of other assets in the market. However, transaction costs, large investor preferences and incomplete markets makes the classical model strongly or fully nonlinear, where both the volatility and the drift can depend on the time, the stock price or the derivatives of the option price itself.

In presence of transaction costs the arbitrage argument in the Black-Scholes theory to price options can no longer be used. Replicating the option by a dynamic strategy would be infinitely costly and no effective option price bounds emerge. Additionally, transactions costs themselves are random and path-dependent: they depend not only on the initial and final stock prices, but also on the entire sequence of stock prices in between.

2 FINANCE BASICS

2.1 OPTIONS: CONTRACT DEFINITION

Definition 1 (European Options) An European option is a contract that allows a buyer the right to buy (call) or sell (put) an underlying asset or financial instrument at a specified strike price K on a specified expiration date T. The option can be purchased for a price called the premium, at time $t < T$.

Therefore, the seller has the corresponding obligation to fulfill the transaction (i.e., to sell or buy) if the buyer (owner) "exercises" the option. A call option would normally be exercised only when the strike price is below the market value of the underlying asset, while a put option would normally be exercised only when the strike price is above the market value. When an option is exercised, the cost to the buyer of the asset acquired is the strike price plus the premium, if any. The premium is income to the seller, and normally a capital loss to the buyer.

The most desirable way of pricing a call option is to find a closed-form formula for C_t that expresses the latter as a function of the underlying asset's price and the relevant parameters. At time t, the only known "formula" concerning C_t is the one that determines its value at the time of expiration. In fact, if the option is expiring out of money, the underlying can be purchased in the market for less than the strike and the option will have no value. From the other hand, if the option expires in the money, the holder will clearly exercise the option buying the underlying at the strike price K and then sell it in the market at a higher price S_T with a net profit of $S_T - K$. Market participants, being aware of this, will place a value of $S_T - K$ on the option.

2.2 Delta Hedging and Black Scholes Formula

The approach used by BS to calculate the fair market value of a derivative asset price was based on the formation of risk-free portfolios. Infinitesimal adjustments in portfolio weights and changes in the option price are going to be used to replicate unexpected movements in the underlying asset, S_t. This eliminates all the risk from the portfolio, at the same time it impose restrictions on the way $C_t(S_t)$, S_t, and the risk-free asset could jointly move over time.

The dynamics of the call options, and a closed-form formula for C_t that expresses the latter as a function of the underlying asset's price is given in the famous Black-Scholes equation:

$$-rC + r\frac{\partial C}{\partial S}S_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} = 0\,. \tag{1}$$

3 Options with transaction costs

The Black–Scholes model requires a continuous rebalancing of the portfolio and the ratio of the holdings in the asset and the derivative products, so it is risk free only instantaneously. If the cost associated with the rehedging are independent of the time scale of rehedging then the infinite number of transactions needed to maintain a hedged position until expiration leads to infinite total transaction costs. Thus, the hedger needs to find the balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. As a result to this "imperfect" hedging, the option might be over or under priced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices. Additionally different people have different levels of transaction costs; as a general rule there are economies of scale, so that the larger traders book are, the less significant are their cost. Thus,the contrary to Black-Scholes model, we may expect that there is no unique option value. Instead, the value of the option depend on the investor.

3.1 Leland model

In 1985 Leland [3] has proposed a very simple modification to Black-Scholes model for vanilla calls and puts where the portfolio and transaction costs are revised every finite sized time-step δt. In that sense, the hedging process is no longer continuous but from now on occurs in discrete intervals of size δt. As a consequence of this assumption, it was necessary to switch from the continuous Geometric Brownian Motion for the underlying assets price to their discrete counterpart. Leland model was based on the assumption that the transaction cost are proportional to the monetary value of the assets bought or sold: $\frac{k}{2}|\nu|S$. Here, k denotes the round trip transaction cost per unit dollar of the transaction and the number of assets bought ($\nu > 0$) or sold ($\nu < 0$) at price S. Finally it was assumed that the hedge of the portfolio has an expected return equal to that from a riskless zero coupon bond.

Following Black-Scholes model but introducing this new insights it is possible to prove that the dynamics of the option is given by a modified version of the BS equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2 S^2\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0\ , \quad \text{where} \quad \frac{\widetilde{\sigma}^2}{\sigma^2} = 1 - \frac{2k}{\sigma}\sqrt{\frac{2}{\pi\,\delta t}}\ sign\left(\frac{\partial^2 C}{\partial S^2}\right)\,. \tag{2}$$

4 Numerical Methods

Lets recall the first and second derivative approximations used in the finite difference method, that will be used in the next section:

◇ the forward difference: $\frac{\partial u}{\partial x}(x,\tau) = \frac{u(x+\delta x,\tau)-u(x,\tau)}{\delta x} + \mathcal{O}(\delta x)$

◇ the backward difference: $\frac{\partial u}{\partial x}(x,\tau) = \frac{u(x,\tau)-u(x-\delta x,\tau)}{\delta x} + \mathcal{O}(\delta x)$

◇ the central difference: $\frac{\partial u}{\partial x}(x,\tau) = \frac{u(x+\delta x,\tau)-u(x-\delta x,\tau)}{2\delta x} + \mathcal{O}(\delta x^2)$

For second partial derivatives, the symmetric central difference is given by:

$$\frac{\partial^2 u}{\partial x^2}(x,\tau) = \frac{u(x+\delta x,\tau) - 2u(x,\tau) + u(x-\delta x,\tau)}{(\delta x)^2} + \mathcal{O}(\delta x^2)\,. \tag{3}$$

4.1 Explicit Method

4.1.1 Black Scholes and Leland model

Using backward difference for the time derivative and the central difference for first and second Stock derivative in (1) we obtain:

$$\begin{bmatrix} \alpha_1 & 1-\gamma_1 & \beta_1 & 0 \\ \ddots & \ddots & \ddots & \\ 0 & \alpha_{N-1} & 1-\gamma_{N-1} & \beta_{N-1} \end{bmatrix} \begin{bmatrix} C_0^m \\ \vdots \\ C_N^m \end{bmatrix} = \begin{bmatrix} C_1^{m-1} \\ \vdots \\ C_{N-1}^{m-1} \end{bmatrix}$$

where we have defined:

$$\alpha_n = \frac{1}{2}\left(\sigma^2 n^2 - rn\right)dt\,, \quad \beta_n = \frac{1}{2}\left(\sigma^2 n^2 + rn\right)dt\,, \quad \gamma_n = rdt + \alpha_n + \beta_n\,.$$

Boundary conditions on the European option for large S and for $S = 0$ makes the C_N^m and C_0^m coefficients known for every time step m. Additionally, the final time condition (the payoff of the option) makes the C_n^M coefficients known for all n. Therefore for every time step $m+1$, the value of the portfolio can be obtained by the multiplication of the matrix $\mathbf{M}$ and the price of the portfolio at time step m.

Implicit Method

Black Scholes and Leland model

Using forward difference for the time derivative and the central difference for first and second Stocks derivative in (2) we obtain:

$$\begin{bmatrix} 1+\gamma_1 & -\beta_1 & & 0 \\ -\alpha_2 & 1+\gamma_2 & \ddots & \\ & \ddots & \ddots & -\beta_{N-2} \\ 0 & & -\alpha_{N-1} & 1+\gamma_{N-1} \end{bmatrix} \begin{bmatrix} C_1^{m-1} \\ \vdots \\ C_{N-1}^{m-1} \end{bmatrix} = \begin{bmatrix} C_1^m \\ \vdots \\ C_{N-1}^m \end{bmatrix} + \begin{bmatrix} \alpha_1 C_0^m \\ 0 \\ \vdots \\ 0 \\ \beta_{N-1} C_N^m \end{bmatrix}$$

A couple of observations should be made here. Vectors are of length $N-1$, while $\mathbf{M}$ matrix is of size $(N-1)\times(N-1)$. Final condition is imposed when $m = M$, while the boundary condition is incorporated in the vector $\mathbf{b}^{m-1}$.

5 Conclusions

Even when implicit and explicit methods have different range of convergence area, Leland model exhibits a smaller region of stability compared to the Black Scholes one. This behaviour is inherited from the non-linear nature of the equation of evolution when transaction costs are taken into account.

Two observations can be extracted from the figure 1a. below. From one hand, for large rehedge times $\tilde{\sigma} \to \sigma$ and Leland model converges to BS as expected. From the other, if the portfolio is rehedged almost continuously the price of the option vanishes. In this case, transaction costs overcome the value of the portfolio and it becomes in a total loss for the owner of the option. The figure 1b. shows how the gamma tends to a Dirac delta centered in the strike price for small rehedge times. Indeed, for $\delta t \to 1$ the numerical model mimics the continuous rehedge frecuency of the analytical case. In this regime, the value of the option is zero unless the value of the stock is greater than the strike price.

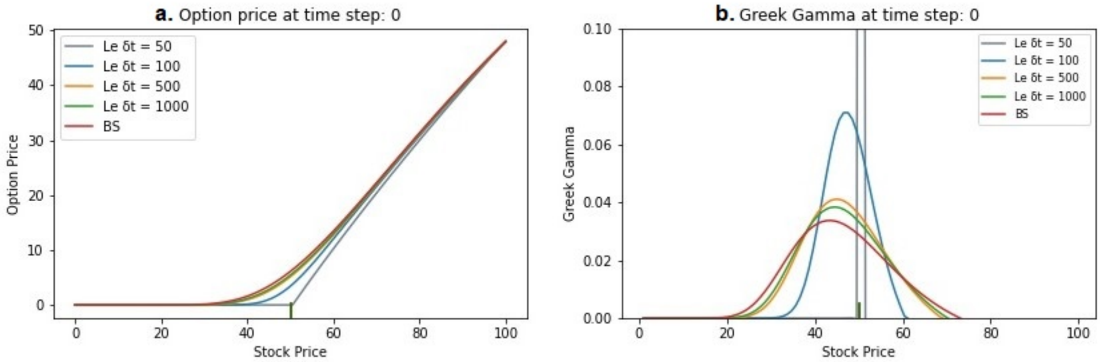

Figure 1: **a.** shows the option price and **b.** the greek gamma at time $t = 0$ for BS and Leland model with different rehedge times δt. The numerical model was introduced for a long participant with a strike price and an initial stock value of 50 monetary units. The volatility was set to 0.4, the annualized expiry date of the option was set to $5/12$ and the risk free rate to $r = 0.1$. For Leland model the round trip transaction cost was set to 2.

Even when non-linearity has a profound impact in the model, it is possible to analyze the results in term of the usual BS model. Indeed, for BS model it can be proved that $\Gamma > 0$ for one call held long. Assuming the same behaviour in the presence of transaction costs, the Leland equation (2) becomes linear with an adjusted constant variance $\widetilde{\sigma}^2 = \sigma^2 - Le$, where Le is some positive constant. Therefore a long position has a lower apparent volatility than in the classical model, and the price of the option drops, in agreement with the loss of risk. This attenuated volatility can be explained as follows: when the asset price rises the owner of the option must sell some of the asset to remain delta hedged, however the effect of the bid-offer spread on the underlying is to reduce the price at which the asset is sold and so the effective increase in the asset price is less than the actual increase.

Finally a financial interpretation can be made. Leland model switches from the continuous geometric brownian motion to their discrete counterpart. This discretization eliminates the leading order component of randomness $\mathcal{O}(dt)$ leaving a small component of order $\mathcal{O}(\sqrt{dt})$ proportional to Γ (which is a measure of the degree of mishedging of this imperfect hedged portfolio). Thus, the gamma is related to the amount of rehedging that takes place in the next time interval and hence to the cost expected.

References

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