Numerical Methods for Pricing Options with Transaction Costs

Vega Federico Gaspar[♭], Gonzalez Sosa Renzo[†] and Aguilar Juan David [*]

[♭]*Facultad de Ingeniería, Universidad Nacional de La Plata, Buenos Aires, Argentina.* federicogaspar@gmail.com
[†]*Facultad de Ciencias Económicas, Universidad Nacional de Cuyo, Mendoza, Argentina.* rensogonsales@gmail.com
[*]*Keysoftware Ltda, Bogotá, Colombia.* jd.aguilar.varela@gmail.com

Abstract: Black-Scholes option pricing theory rests on an arbitrage argument where it should be possible to adjust the portfolio continuously. In presence of transaction costs this assumption leads to a problematic situation. Indeed, diffusion process have infinite variation and a continuous trading would be ruinously expensive. Formally, the arbitrage argument will no longer be valid and a discrete revision of Black-Scholes will be needed. In this work we examine Leland and Amster models where the costs behave as a non-increasing linear function through the implementation of numerical scheme using implicit and explicit finite differences methods.

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1 Introduction

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Fischer Black and Myron Scholes in [1] in 1973 and previously by Robert Merton in [2]. The approach used to calculate the fair market value of a derivative asset price was based on the formation of risk-free portfolios. Infinitesimal adjustments in portfolio weights and changes in the option price $C_t(S_t)$, are used to replicate unexpected movements in the underlying asset, S_t[1]. This eliminates all the risk so the price of the portfolio becomes deterministic and therefore, its appreciation must equal the earnings of a risk-free investment with rate r during an interval dt in order to avoid arbitrage. The dynamics of the call option is given in the famous Black-Scholes equation:

$$-rC + r\frac{\partial C}{\partial S}S_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} = 0\,. \tag{1}$$

The solution of this linear equation provides a closed-form formula for C_t as a function of the underlying and a hedging portfolio that replicates the option. This model works under several assumptions, however, transaction costs, large investor preferences and incomplete markets makes the classical model strongly or fully nonlinear, where both the volatility and the drift can depend on the time, the stock price or the derivatives of the option price itself.

In presence of transaction costs Black-Scholes theory can no longer be used. Replicating the option by a dynamic strategy would be infinitely costly and no effective option price bounds emerge. Additionally, transactions costs themselves are random and path-dependent: they depend not only on the initial and final stock prices, but also on the entire sequence of stock prices in between.

2 Options with transaction costs

The Black–Scholes model requires a continuous rebalancing of the portfolio and the ratio of the holdings in the asset and the derivative products, so it is risk free only instantaneously. If the cost associated with the rehedging are independent of the time scale of rehedging then the infinite number of transactions needed to maintain a hedged position until expiration leads to infinite total transaction costs. Thus, the hedger needs to find the balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. As a result to this "imperfect" hedging, the option might be over or under priced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that

[1]Here it is assumed that prices S_t are Log-normally distributed with median $\mu - \sigma_t^2/2$ and deviation/volatility of σ_t.

there is no single equilibrium price but a range of feasible prices. Additionally different market agents have different levels of transaction costs; as a general rule there are economies of scale, so that the larger traders book are, the less significant are their cost. Thus, the contrary to Black-Scholes model, we may expect that there is no unique option value. Instead, the value of the option depend on the investor.

2.1 LELAND MODEL

In 1985 Leland [3] has proposed a very simple modification to Black-Scholes model for vanilla calls and puts where the portfolio and transaction costs are revised every finite sized time-step δt. In that sense, the hedging process is no longer continuous but from now on occurs in discrete intervals of size δt. As a consequence of this assumption, it was necessary to switch from the continuous Geometric Brownian Motion for the underlying assets price to their discrete counterpart. Leland model was based on the assumption that the transaction cost are proportional to the monetary value of the assets bought or sold: $\frac{k}{2}|\nu|S$. Here, k denotes the round trip transaction cost per unit dollar of the transaction and the number of assets bought ($\nu > 0$) or sold ($\nu < 0$) at price S. Finally it was assumed that the hedge of the portfolio has an expected return equal to that from a riskless zero coupon bond.

Following Black-Scholes model but introducing this new insights it is possible to prove that the dynamics of the option is given by a modified version of the BS equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \,, \quad \text{where} \quad \frac{\widetilde{\sigma}^2}{\sigma^2} = 1 - \frac{2k}{\sigma}\sqrt{\frac{2}{\pi\,\delta t}}\; sign\left(\frac{\partial^2 C}{\partial S^2}\right). \tag{2}$$

Here, as in BS model, r stands for the risk free rate and σ_t stands for the constant volatility of the stock.

2.2 AMSTER MODEL

Amster et al. in [4] extended Leland model, where the transaction costs behave as a nonincreasing linear function $\delta P_t = \delta P_t^0 - (a - b|\nu|)|\nu|S$. The price of the portfolio due to this transaction model leads to a fully non-linear differential equation:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2 S^2 \frac{\partial^2 F}{\partial S^2} + b\sigma^2 S^3 \left(\frac{\partial^2 F}{\partial S^2}\right)^2 + rS\frac{\partial F}{\partial S} - rF = 0\,. \tag{3}$$

In that article they obtained solutions for the stationary problem. Moreover, they gave conditions for the existence of solutions of the general evolution equation.

3 NUMERICAL METHODS

The models explored in the previous section are described through non-linear differential equations that can not be solved analytically. In this section we will find the numerical solution, using the finite difference method to approximate the first and second derivative through the forward-backward and central difference. The reader is referred to [5] for an extensive introduction to this topic.

3.1 EXPLICIT METHOD

3.1.1 Black Scholes and Leland model

Using backward difference for the time derivative and the central difference for first and second Stock derivative in (1) we obtain:

$$\mathbf{F}^{m-1} = \mathbf{M}\mathbf{F}^m \Rightarrow \begin{bmatrix} C_1^{m-1} \\ \vdots \\ C_{N-1}^{m-1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 1-\gamma_1 & \beta_1 & 0 \\ \ddots & \ddots & \ddots & \\ 0 & \alpha_{N-1} & 1-\gamma_{N-1} & \beta_{N-1} \end{bmatrix} \begin{bmatrix} C_0^m \\ \vdots \\ C_N^m \end{bmatrix}$$

where we have defined: $\alpha_n = \frac{1}{2}\left(\sigma^2 n^2 - rn\right) dt\,, \quad \beta_n = \frac{1}{2}\left(\sigma^2 n^2 + rn\right) dt\,, \quad \gamma_n = rdt + \alpha_n + \beta_n\,.$

Boundary conditions on the European option for large S and for $S = 0$ makes the C_N^m and C_0^m coefficients known for every time step m. Additionally, the final time condition (the payoff of the option) makes the C_n^M coefficients known for all n. Therefore for every time step $m+1$, the value of the portfolio can be obtained by the multiplication of the matrix $\mathbf{M}$ and the price of the portfolio at time step m.

3.1.2 Amster model

Using the same scheme for the derivatives in eqn. (3) we obtained:

$$\mathbf{F}^{m-1} = \mathbf{M}\mathbf{F}^m + b\sigma^2 \frac{dt}{dS} \quad \mathbf{\Gamma}^{\dagger m}\mathbf{B}\mathbf{\Gamma}^m$$

with $\mathbf{B} = i^3\delta_{ij}$ and $\mathbf{\Gamma}_n^m = F_{n-1}^m - 2F_n^m + F_{n+1}^m$

3.2 Implicit Method

3.2.1 Black Scholes and Leland model

Using forward difference for the time derivative and the central difference for first and second Stocks derivative in (2) we obtain:

$$\begin{bmatrix} 1+\gamma_1 & -\beta_1 & & 0 \\ -\alpha_2 & 1+\gamma_2 & \ddots & \\ & \ddots & \ddots & -\beta_{N-2} \\ 0 & & -\alpha_{N-1} & 1+\gamma_{N-1} \end{bmatrix} \begin{bmatrix} C_1^{m-1} \\ \vdots \\ C_{N-1}^{m-1} \end{bmatrix} = \begin{bmatrix} C_1^m \\ \vdots \\ C_{N-1}^m \end{bmatrix} + \begin{bmatrix} \alpha_1 C_0^m \\ 0 \\ \vdots \\ 0 \\ \beta_{N-1} C_N^m \end{bmatrix}$$

A couple of observations should be made here. Vectors are of length $N-1$, while $\mathbf{M}$ matrix is of size $(N-1) \times (N-1)$. Final condition is imposed when $m = M$, while the boundary condition is incorporated in the vector $\mathbf{b}^{m-1}$.

4 Conclusions

In this article we studied two models of transaction costs using finite difference method implemented through an OOP algorithm. For the Leland case, the price formation of an European option was computed using an explicit and implicit scheme. In contrast, for the Amster case, the price was computed using an explicit scheme exclusively. Indeed, implicit method was not suitable for this model because this method derives in a non-linear system of equations, which requires an iterative method based on unknown initial parameters to solve it. Both cost models had a smaller region of stability compared to the numerical solution of the Black Scholes model with the same scheme. This behaviour is inherited from the non-linear nature of the equation of evolution when transaction costs are taken into account.

In figure 1 we compared the option price and the greek gamma for Leland model (with 4 different rehedge times δt) versus the standard Black-Scholes case at the initial time of the contract. From figure 1a. two observations can be extracted. From one hand, for large rehedge times $\tilde{\sigma} \to \sigma$ and Leland model converges to BS as expected. From the other, if the portfolio is rehedged almost continuously the price of the option vanishes. In this case, transaction costs overcome the value of the portfolio and it becomes in a total loss for the owner of the option. The figure 1b. shows how the gamma tends to a Dirac delta centered in the strike price for small rehedge times. Indeed, for $\delta t \to 0$ the numerical model mimics the continuous rehedge frequency of the analytical case and the value of the option is zero unless the value of the stock is greater than the strike price. It is useful to analyze these results in the light of those already known for the Black-Scholes model. Indeed, for standard model it can be proved that $\Gamma > 0$ for one call held long. Assuming the same behaviour in the presence of transaction costs, the Leland equation (2) becomes linear with an adjusted constant variance $\widetilde{\sigma}^2 = \sigma^2 - Le$, where Le is some positive constant. Therefore a long position has a lower apparent volatility than in the classical model, and the price of the option drops, in agreement with the loss of risk. This attenuated volatility can be explained as follows: when the asset price rises the owner of the option must sell some of the asset to remain delta hedged, however the effect of the bid-offer spread on the underlying is to reduce the price at which the asset is sold and so the effective increase in the asset price is less than the actual increase.

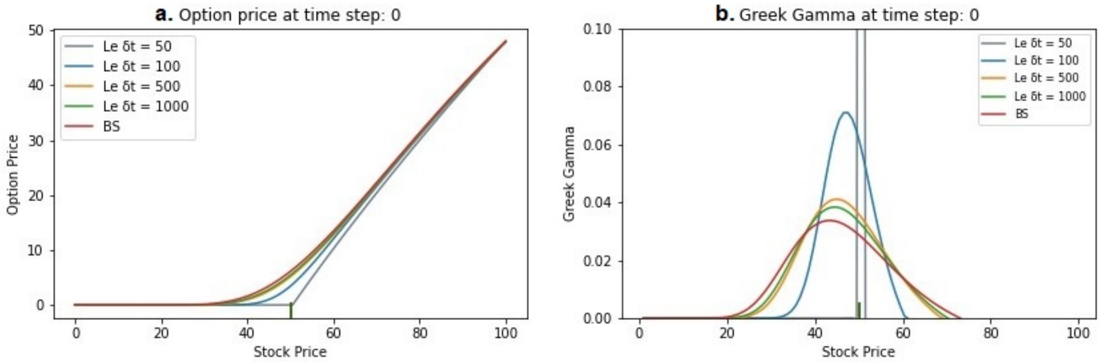

Figure 1: **a.** shows the option price and **b.** the greek gamma at time $t = 0$ for BS and Leland model with different rehedge times δt. The numerical model was introduced for a long participant with a strike price and an initial stock value of 50 monetary units. The volatility was set to 0.4, the annualized expiry date of the option was set to $5/12$ and the risk free rate to $r = 0.1$. For Leland model the round trip transaction cost was set to 2.

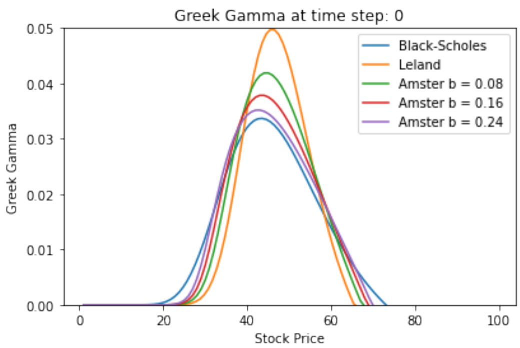

Figure 2: greek gamma at time $t = 0$ for Leeland and Amster model with same k and varying b.

A final financial interpretation can be made. Leland model switches from the continuous geometric brownian motion to their discrete counterpart. This discretization eliminates the leading order component of randomness $\mathcal{O}(dt)$ leaving a small component of order $\mathcal{O}(\sqrt{dt})$ proportional to Γ (which is a measure of the degree of mishedging of this imperfect hedged portfolio). Thus, the gamma is related to the amount of rehedging that takes place in the next time interval and hence to the cost expected.

Finally, even when the Amster model was stable for a small region of parameters ($b < 0.3$) it exhibits a behaviour qualitatively compatible with the economies of scale. Indeed, as larger the number of assets traded, the less significant are their cost. This is a consequence of the parameter b that acts like a dumping constant, it attenuates the critical point of the greek gamma in the neighborhoods of the strike K and the price under this model is closer to the usual Black Scholes predicting price. Neverless, it worth to mention that for $b = 0$ Amster converges to Leland model and for large b, Amster model is able to maintain the portfolio hedged with less number of steps than Leland model.

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