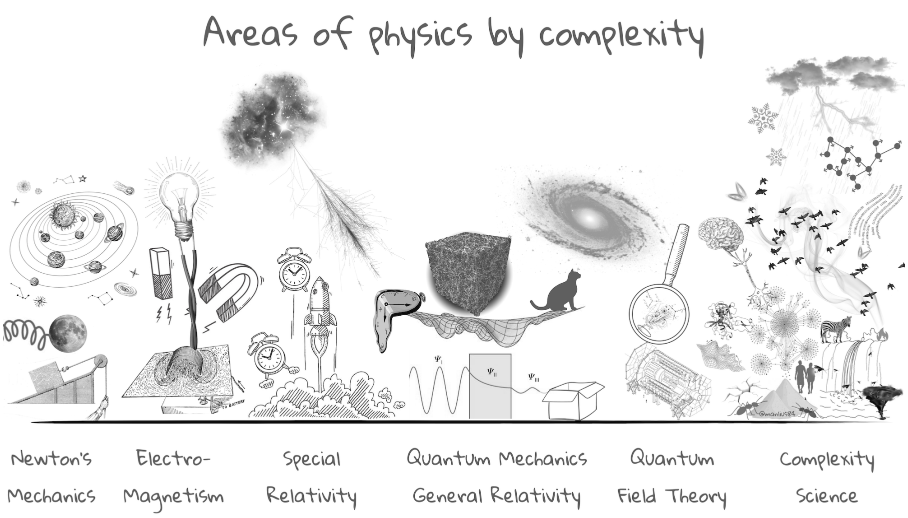
Final Report

Physics of Complex Networks: Structure and Dynamics

UNIVERSITÀ DEGLI STUDI DI PADOVA



Project # 11: Growth-Shrink Models

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1 | Theoretical Approach

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We study networks as idealizations of real world complex systems. Therefore it is essential to count for as many real features as possible in our models. A noticeable characteristic of these complex systems is that many times they evolve both creating and loosing connections and links. In this brief analysis we study the so called Growth-Shrink Models to describe these phenomena. Fist let us introduce the theoretical description needed. We will follow two different approaches to describe these underlying features.

1.1 | Artime's approach

In this section we will follow the article [1]. In this view we introduce an undirected unweighted simple network of N nodes. This parameter will be kept fixed. We then consider an evolution of the network done by fixed time steps. For each time step we will consider two phenomena:

- Addition of $\alpha_a N/2$ edges between two nodes picked uniformly at random.
- Resetting of $r_a N$ nodes picked uniformly at random. This means the deletion of all edges of the chosen nodes.

In general we could apply this model to any kind of network but for simplicity we start from a graph with no edges. We can set the degree distribution as $p_k(t=0) = \delta_{k,0}$ at time zero. This distribution is associated with the following master equation (1.1).

$$\frac{dp_k}{dt} = \alpha_a p_{k-1} - \alpha_a p_k - r p_k + r_a(k+1)p_{k+1} - r_a k p_k + r_a \delta_{k,0} \tag{1.1}$$

In equation (1.1) the first term describe the addition of an edge in nodes that had degree k-1; the second and third describe the loss of nodes with degree k due to the addition of an edge to nodes that already had degree k and due to the resetting of other nodes with degree k; the fourth is the gain of nodes with degree k due to the loss of an edge suffered by nodes with degree k+1; the fifth, again, a loss due to the removal of a neighbor of nodes with degree k; and lastly the delta encodes for the nodes left with no links.

From equation (1.1), using the generating function $g(z,t) = \Sigma_{k=0}^{\infty} z^k p_k(t)$, we obtain the PDE (1.2).

$$\frac{\partial p_k}{\partial t} = [\alpha_a z - (\alpha_a + r_a)]g + r_a(1-z)\frac{\partial g}{\partial z} + r_a \tag{1.2}$$

Fixing the conditions $g(1,t) = 1$ and $g(z,0) = 1$ and using the method of the characteristics, we can solve this PDE and, from it, derive the degree distribution (1.3).

$$p_k(t) = \frac{r_a}{\alpha_a}[1 - Q(k+1, c(t))] + \frac{c(t)^k}{k!}e^{-c(t)-r_a t} \tag{1.3}$$

$$Q(a,b) = \frac{1}{\Gamma(a)}\int_b^{\infty} x^{a-1}e^{-x}dx$$

$$c(t) = \frac{\alpha_a}{r_a}(1 - e^{-r_a t})$$

We can also compute the steady mean degree, which reads $< k >= \alpha_a/(2r_a)$.

1.2 Moore's approach

The second approach taken in consideration is the one described by the article [3]. We will force this method to be as much coherent to the previous one as possible.

First, let us take an undirected unweighted simple network of N nodes. In this case we do not fix immediately the number of nodes, that could in general change in time. We then consider again an evolution of the network done by fixed time steps. For each time step we will consider two phenomena:

- Addition of 1 node to the network. This nodes form immediately α_m links with α_m different existing nodes, picked uniformly at random.
- Removal of r_m nodes, picked uniformly at random, from the network.

Notice that the first point could include the case of a so called preferential attachment in which the probability of a new node to form a link with a given existing node can be modulated by, for example, its degree. For simplicity, and to be more coherent with the previous case, we consider uniform attachment. To conform to the Artime's approach, we want N to be fixed in time. To do so we simply set r_m to 1.

Following Moore's study, we can then write the update equation (1.4).

$$Np'_k = Np_k + \alpha_m p_{k-1} - \alpha_m p_k - p_k + (k+1)p_{k+1} - kp_k + \delta_{k,\alpha_m} \tag{1.4}$$

In equation (1.4) we study the number of nodes with degree k at the next time step Np'_k. In it, the first term takes into account the number of nodes with degree k at the previous time; the middle terms can be compared to the first five terms in the Artime's study and, lastly, the delta encodes for the new nodes, entering with degree α_m.

Again, using the generating function $g(z,t) = \Sigma_{k=0}^{\infty} z^k p_k(t)$, and studying directly the asymptotic behaviour ($p'_k = p_k$), we can obtain the final, time independent degree distribution (1.5), (1.6).

$$p_k = \frac{e^{\alpha_m}}{\alpha_m^{\alpha_m+1}}[\Gamma(\alpha_m+1) - \Gamma(\alpha_m+1,\alpha_m)]\frac{\Gamma(k+1,\alpha_m)}{\Gamma(k+1)}, \quad \text{if} \quad k < \alpha_m \tag{1.5}$$

$$p_k = \frac{e^{\alpha_m}}{\alpha_m^{\alpha_m+1}}[\Gamma(k+1) - \Gamma(k+1,\alpha_m)]\frac{\Gamma(\alpha_m+1,\alpha_m)}{\Gamma(k+1)}, \quad \text{if} \quad k \geq \alpha_m \tag{1.6}$$

To uniform the two strategies we will also take a network with no edges at time zero. Notice that this model differs from the prior one mainly for the method of adding new links. On one hand, we randomly add edges while resetting some nodes. On the other hand, we could imagine to reset a node and then to give it the new edges. In other words, only the reset nodes receive all α_m new links. To be coherent, and add the same amount of edges, we will set $\alpha_m = \alpha_a N/2$. Also, since r_m is fixed to one we will use $r_a = 1/N$.

Even for this model we can compute the steady mean degree, which reads $< k >= \alpha_m$.

2 | Simulations

Task leader(s): *Merlo Federico*

We now aim to verify numerically the Theoretical approaches studied in the first task. We also take the opportunity to confront the outcome of the two different methods, to which we will refer as Artime's and Moore's models. To do that we implement a code written in R [2] that takes the same undirected unweighted simple network, initially without any edge, and implements the two different strategies described before to study its evolution. In This simulations the number of iterations is kept insufficiently small due to the luck of computational power.

2.1 | Degree Distribution

First of all, we have studied the degree distribution of the network after its evolution following either Artime's or Moore's method.

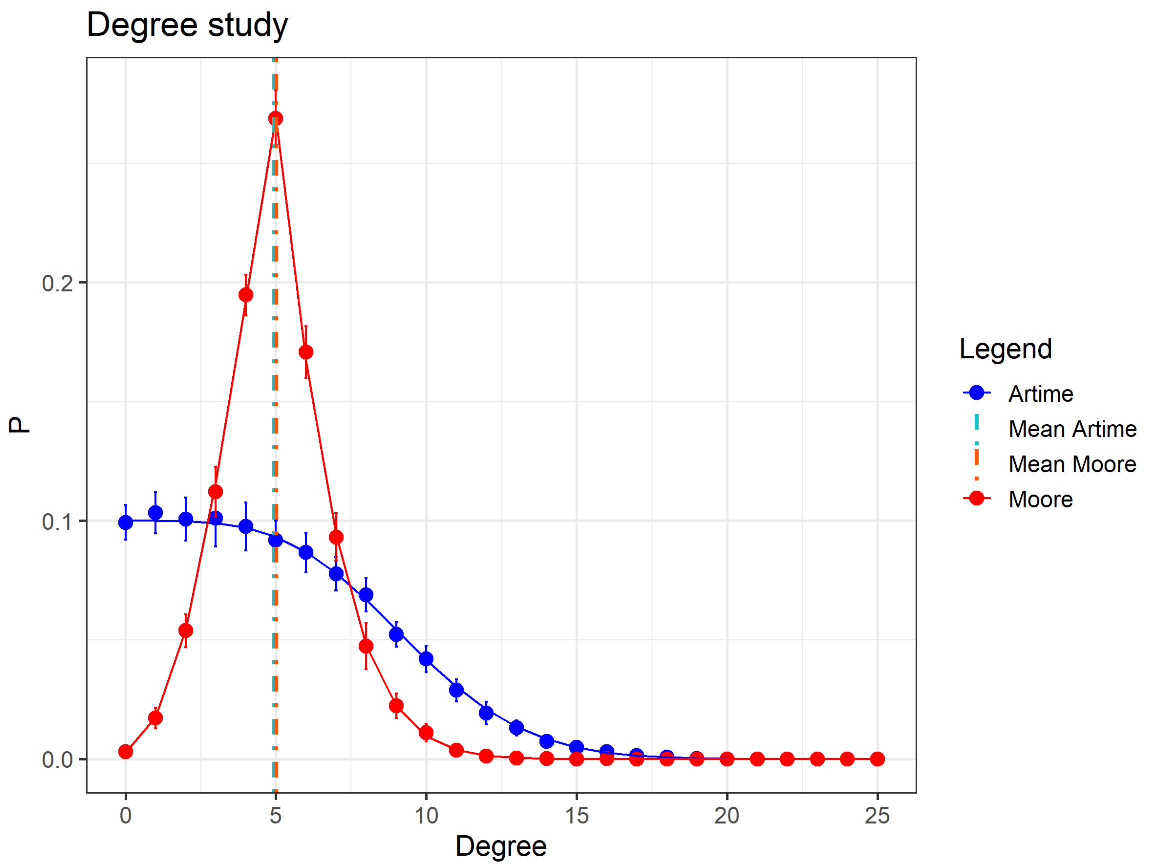

Figure 2.1: Network size: 1000; Time-steps: 5000; Iterations: 20

In figure 2.1 we can see the final degree distribution for the two methods. For both of them it was used an initial network with 1000 nodes and no edges. Then the evolution was brought on for 5000 time-steps with values: $\alpha_a = 10/1000$ and $r_a = 1/1000$ for Artime's method and $\alpha_m = 5$ and $r_m = 1$ for Moore's method. In this figure we see that, for both models, the data coming from the simulations align with the theoretical prediction from task 1. Confronting the two models, we can see that both have mean degree tending to 5, as expected. Moore's method, however, has a sharper degree in which, obviously, the maximum probability is associated with 5, the number of new edges given to each reset node. Artime's distribution is instead smoother, with 0 being one of the most likely degree in the network, since the reset nodes are left with no edges.

2.2 Size of the LCC

Secondly we have analysed the appearance of a Largest Connected Component (LCC), looking at its size.

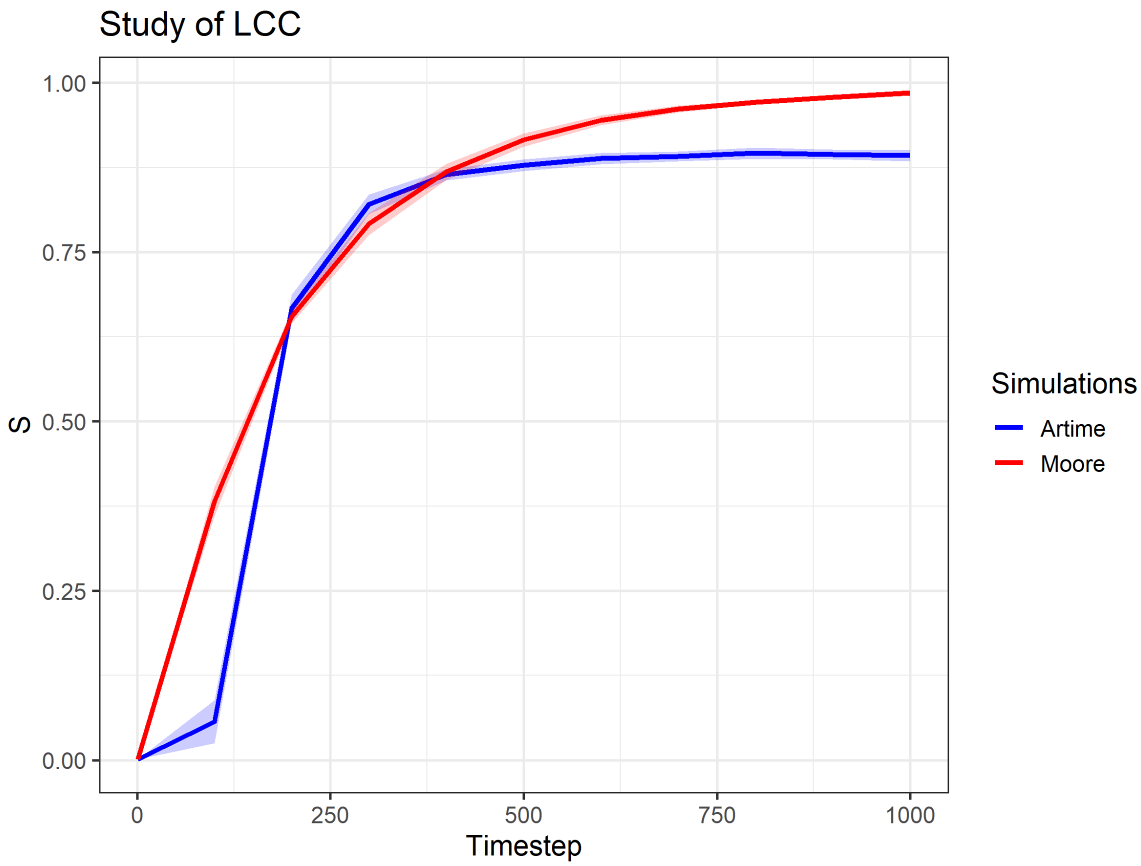

Figure 2.2: Network size: 1000; Time-steps: 1000; Iterations: 20

In figure 2.2 we report the outcome of a simulation run for 1000 time-steps and with the same initial network and parameters set before. We can see the emergence of an LCC in both models. However, while Moore's network seems to start immediately to form a well connected cluster, Artime's one starts with a flatter curve. We can also notice that, on the contrary, Artime's curve has a sharper transition. Finally, Moore's model builds a more connected network in which almost all node belong to the LCC, although, even for Artime's network we reach a very well connected structure.

In conclusion, Both models are well described theoretically and show similarity in their final outcome. However, their different approach leads also to significant diversities that could be compared with the real world complex systems, in order to apply to them the more suitable model.

3 | Bibliography

[1] Oriol Artime. Stochastic resetting in a networked multiparticle system with correlated transitions. *Journal of Physics A: Mathematical and Theoretical*, 55(48): 484004, 2022.

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[3] Cristopher Moore, Gourab Ghoshal, and Mark EJ Newman. Exact solutions for models of evolving networks with addition and deletion of nodes. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, 74(3):036121, 2006.