Auxiliary Normal Inverse Gamma Similarity Function

We consider an "Auxiliary Normal-Inverse Gamma" similarity function:

$$\xi_h = (\mu; \sigma_2)$$
$$\mu \sim \mathcal{N}\left(\mu_c, \frac{\sigma^2}{\lambda_c}\right)$$
$$\sigma^2 \sim IG(a_c, b_c)$$
$$x \mid \xi_h \sim \mathcal{N}(\mu, \sigma^2)$$

We compute the similarity function, i.e., $\int_{\mathbb{R}\times\mathbb{R}_+} \prod_{i=1}^n q(x_i \mid \xi_h) q(\xi_h) d\xi_h$.

$$\int_{\mathbb{R}\times\mathbb{R}_+} \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\} \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda_c}{\sigma^2}} \exp\left\{-\frac{\lambda_c}{2\sigma^2}(\mu-\mu_c)^2\right\} \frac{b_c^{a_c}}{\Gamma(a_c)}(\sigma^2)^{-a_c-1} \exp\left\{-\frac{b_c}{\sigma^2}\right\} d\mu d\sigma^2$$

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}\times\mathbb{R}_+} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right\} \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda_c}{\sigma^2}} \exp\left\{-\frac{\lambda_c}{2\sigma^2}(\mu-\mu_c)^2\right\} \frac{b_c^{a_c}}{\Gamma(a_c)}(\sigma^2)^{-a_c-\frac{n}{2}-1} \exp\left\{-\frac{b_c}{\sigma^2}\right\} d\mu d\sigma^2$$

We can rewrite the first exponent:

$$\begin{aligned}
\sum_{i=1}^n (x_i-\mu)^2 &= \sum_{i=1}^n ((x_i-\bar{x})-(\mu-\bar{x}))^2 = \sum_{i=1}^n (x_i-\bar{x})^2 + \sum_{i=1}^n (\mu-\bar{x})^2 - 2\sum_{i=1}^n (x_i-\bar{x})(\mu-\bar{x}) \\
&= \sum_{i=1}^n (x_i-\bar{x})^2 + n(\mu-\bar{x})^2 - 2\sum_{i=1}^n (x_i\mu - x_i\bar{x} - \bar{x}\mu + \bar{x}^2) \\
&= \sum_{i=1}^n (x_i-\bar{x})^2 + n(\mu-\bar{x})^2 - 2(n\bar{x}\mu - n\bar{x}^2 - n\bar{x}\mu + n\bar{x}^2) = \sum_{i=1}^n (x_i-\bar{x})^2 + n(\mu-\bar{x})^2
\end{aligned}$$

The similarity function becomes:

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}\times\mathbb{R}_+} \sqrt{\frac{\lambda_c}{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(n(\mu-\bar{x})^2 + \lambda_c(\mu-\mu_c)^2)\right\} \\
&\quad \times \frac{b_c^{a_c}}{\Gamma(a_c)}(\sigma^2)^{-a_c-\frac{n}{2}-1} \exp\left\{-\frac{1}{\sigma^2}\left(b_c + \frac{1}{2}\sum_{i=1}^n (x_i-\bar{x})^2\right)\right\} d\mu d\sigma^2
\end{aligned}$$

We can rewrite the first exponent:

$$\begin{aligned}
n(\mu-\bar{x})^2 + \lambda_c(\mu-\mu_c)^2 &= n\mu^2 + n\bar{x}^2 - 2n\mu\bar{x} + \lambda_c\mu^2 + \lambda_c\mu_c^2 - 2\lambda_c\mu\mu_c \\
&= (n+\lambda_c)\mu - 2\mu(n\bar{x}+\lambda_c\mu_c) + n\bar{x}^2 + \lambda_c\mu_c^2 = (n+\lambda_c)\left(\mu_c^2 - 2\mu\frac{n\bar{x}+\lambda_c\mu_c}{n+\lambda_c} + \frac{n\bar{x}^2+\lambda_c\mu_c^2}{n+\lambda_c}\right) \\
&= (n+\lambda_c)\left(\mu^2 - 2\mu\frac{n\bar{x}+\lambda_c\mu_c}{n+\lambda_c} + \frac{(n\bar{x}+\lambda_c\mu_c)^2}{(n+\lambda_c)^2} + \frac{n\bar{x}^2+\lambda_c\mu_c^2}{n+\lambda_c} - \frac{(n\bar{x}+\lambda_c\mu_c)^2}{(n+\lambda_c)^2}\right) \\
&= (n+\lambda_c)\left(\mu - \frac{n\bar{x}+\lambda_c\mu_c}{n+\lambda_c}\right)^2 + \frac{n^2\bar{x}^2 + n\lambda_c\bar{x}^2 + \lambda_c^2\mu_c^2 + n\lambda_c\mu_c^2 - n^2\bar{x}^2 - \lambda_c^2\mu_c^2 - 2n\lambda_c\bar{x}\mu_c}{n+\lambda_c} \\
&= (n+\lambda_c)\left(\mu - \frac{n\bar{x}+\lambda_c\mu_c}{n+\lambda_c}\right)^2 + \frac{n\lambda_c}{n+\lambda_c}(\bar{x}-\mu_c)^2
\end{aligned}$$

The similarity function becomes:

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}\times\mathbb{R}_+} \sqrt{\frac{\lambda_c}{2\pi\sigma^2}} \exp\left\{-\frac{n+\lambda_c}{2\sigma^2}\left(\mu - \frac{n\bar{x}+\lambda_c\mu_c}{n\bar{x}+\lambda_c}\right)^2\right\} \\
&\quad \times \frac{b_c^{a_c}}{\Gamma(a_c)}(\sigma^2)^{-a_c-\frac{n}{2}-1} \exp\left\{-\frac{1}{\sigma^2}\left(b_c + \frac{1}{2}\sum_{i=1}^n (x_i-\bar{x})^2 + \frac{1}{2}\frac{n\lambda_c}{n+\lambda_c}(\bar{x}-\mu_c)^2\right)\right\} d\mu d\sigma^2
\end{aligned}$$

We can rewrite the second exponent:

$$\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{n\lambda_c}{n+\lambda_c}(\bar{x}-\mu_c)^2=\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\bar{x}+n\bar{x}^2+\frac{n\lambda_c}{n+\lambda_c}(\bar{x}^2+\mu_c^2-2\bar{x}\mu_c)$$
$$=\sum_{i=1}^{n}x_i^2-n\bar{x}^2+\frac{n\lambda_c}{n+\lambda_c}(\bar{x}^2+\mu_c^2-2\bar{x}\mu_c)=\sum_{i=1}^{n}x_i^2+n\bar{x}^2\left(\frac{\lambda_c}{n+\lambda_c}-1\right)+\frac{n\lambda_c}{n+\lambda_c}(\mu_c^2-2\bar{x}\mu_c)$$
$$=\sum_{i=1}^{n}x_i^2+\frac{1}{n+\lambda_c}(-n^2\bar{x}^2+n\lambda_c\mu_c^2-2n\lambda_c\bar{x}\mu_c)$$
$$=\sum_{i=1}^{n}x_i^2+\frac{1}{n+\lambda_c}(-n^2\bar{x}^2-2n\lambda_c\bar{x}\mu_c-\lambda_c^2\mu_c^2+n\lambda_c\mu_c^2+\lambda_c^2\mu_c^2)=$$
$$\sum_{i=1}^{n}x_i^2-\frac{(n\bar{x}+\lambda_c\mu_c)^2}{n+\lambda_c}+\frac{n\lambda_c\mu_c^2+\lambda_c^2\mu_c^2}{n+\lambda_c}=\sum_{i=1}^{n}x_i^2-\frac{(n\bar{x}+\lambda_c\mu_c)^2}{n+\lambda_c}+\lambda_c\mu_c^2$$

Returning to the similarity function:

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}\times\mathbb{R}_+}\sqrt{\frac{\lambda_c}{2\pi\sigma^2}}\exp\left\{-\frac{n+\lambda_c}{2\sigma^2}\left(\mu-\frac{n\bar{x}+\lambda_c\mu_c}{n\bar{x}+\lambda_c}\right)^2\right\}$$
$$\times\frac{b_c^{a_c}}{\Gamma(a_c)}(\sigma^2)^{-a_c-\frac{n}{2}-1}\exp\left\{-\frac{1}{\sigma^2}\left(b_c+\frac{1}{2}\left(\sum_{i=1}^{n}x_i^2-\frac{(n\bar{x}+\lambda_c\mu_c)^2}{n+\lambda_c}+\lambda_c\mu_c^2\right)\right)\right\}d\mu d\sigma^2$$

The part inside the integral is the pdf of a Normal-Inverse Gamma distribution (in particular, it's the posterior distribution of the conjugate Normal-Normal-Inverse Gamma model), therefore we just have to adjust the normalizing constants and the integral is equal to 1.

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\sqrt{\frac{\lambda_c}{\lambda_c+n}}\frac{\Gamma(a_c+\frac{n}{2})}{\Gamma(a_c)}b_c^{a_c}\left(b_c+\frac{1}{2}\left(\sum_{i=1}^{n}x_i^2-\frac{(n\bar{x}+\lambda_c\mu_c)^2}{n+\lambda_c}+\lambda_c\mu_c^2\right)\right)^{-a_c-\frac{n}{2}}$$

The same result (with different notation) can be found here.