GTDMO - 1st assignment

Group 4

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Excersise Description (i)

A company is producing three different types of glue, type A, type B,and type C. For each Kg of production of glue A, the company is spending 2euro in workers salary, 5euro in electricity, and 1euro in packaging. For each Kg of production of glue B, the company is spending 5euro in workers salary, 2euro in electricity, and 3euro in packaging. For each Kg of production of glue C,the company is spending 8euro in workers salary, 3euro in electricity, and 2euro in packaging.

In one day of production of glues A, B, C in total the company would like to spend exactly 335euro in workers salary, 170euro in electricity and 115euro in packaging. Is it possible? And if yes how many Kg of glues A, B and C should they produce per day?

Formulate the problem in mathematical terms and explain how do you find the solution.

Part A: Problem Formulation

The first step in order to formulate a problem is to identify which are the variables. It is observed that the Glue making company has 3 different lines of products and that they are produced independently, [**glue A, glue B, glue C**]. Each product uses the same 3 resources in their production, [**workers, electricity, packaging**]. In addition, it is asked to find a combination of the three lines of glue that satisfy a given cost and usage of each resource, namely [**335 workers, 170 electricity, 115 packaging**].

As the production of the products are independent, meaning that there is no interaction between them, a linear system can be built.

Therefore we can set three equations, one for each resource, as follows:
$+ WorkersCost = 2x_1 + 5x_2 + 1x_3$ $+ ElectricityCost = 5x_1 + 2x_2 + 3x_3$ $+ PackagingCost = 8x_1 + 3x_2 + 2x_3$

Being:

i) **Glue A** $= x_1$
ii) **Glue B** $= x_2$
iii) **Glue C** $= x_3$

Moreover, it is asked to spend *exactly*:

- $WorkersCost = 335$
- $ElectricityCost = 170$
- $PackagingCost = 115$

Therefore the Linear System of Equations can be written as:

- $2x_1 + 5x_2 + 1x_3 = 335$
- $5x_1 + 2x_2 + 3x_3 = 170$
- $8x_1 + 3x_2 + 2x_3 = 115$

Then, it can be seen that:

A) The cost from the total usage of each resource is equivalent to the **known numbers = vector b**

B) The cost on each resource derived from the production of 1Kg of glue type are the **coefficients = matrix A**

C) The Kgs of each glue type are the **unknowns = vector x**

As a result, the problem was formulated in the matrix form:

$Ax = b$

R code: formulation

```
## Define the terms
products <- c("glue A", "glue B", "glue C")
resources <- c("workers", "electricity", "packaging")

## Create matrix A of coefficients
A <- matrix(data = c(2, 5, 1,
                     5, 2, 3,
                     8, 3, 2),
            nrow = 3,
            ncol = 3,
            byrow = FALSE
)

## Set rownames = resources and colnames = glue type
rownames(A) <- resources
colnames(A) <- products

print(A)
```

```
##             glue A glue B glue C
## workers          2      5      8
## electricity      5      2      3
## packaging        1      3      2
```

```
## Create Vector b of resources cost
b <- c(335, 170, 115)

print(b)
```

```
## [1] 335 170 115
```

Part B: Problem Solution

Since the problem is linear and in matrix form, one path to find the solution is to create the **augmented matrix** `A_a` by adding `b` as the last column of `A` .

```
## Compute the Augmented Matrix
A_a <- cbind(A, b)
colnames(A_a) <- c(products, "cost")

print(A_a)
```

```
##             glue A glue B glue C cost
## workers          2      5      8  335
## electricity      5      2      3  170
## packaging        1      3      2  115
```

Once the augmented matrix `A_a` is defined, the next step is to, by using *elementary row* operations find a triangular matrix, more specifically, to reduce the matrix to its **"Echelon form"**.

Finding the Echelon Rectangular Matrix of augmented A

The first element `E[1,1]` has to be non-zero. As it already is it is not needed to switch rows. Then we divide the first row `E[1,]` by 2 to turn the pivot into a 1:

```
## Make the first column a pivot = 1
E <- A_a
E[1,] <- E[1,] * 1/2

print(E)
```

```
##             glue A glue B glue C  cost
## workers          1    2.5      4 167.5
## electricity      5    2.0      3 170.0
## packaging        1    3.0      2 115.0
```

Then we subtract in the second row `E[2,]` 5 times the first row `E[1,]` to eliminate the first element and we subtract in the third `E[3,]` row 1 time the first row `E[1,]`:

```
## Eliminate the first element of the second row
E[2,] <- E[2,] - (E[1,] * 5)

## Eliminate the first element of the third row
E[3,] <- E[3,] - E[1,]

print(E)
```

```
##             glue A glue B glue C   cost
## workers          1    2.5      4  167.5
## electricity      0  -10.5    -17 -667.5
## packaging        0    0.5     -2  -52.5
```

We convert the second pivot to 1 by diving by -10.5 and eliminate the number beneath it `E[3,2]` by subtracting to the third row `E[3,]` a half of the second row `E[2,]`:

```
## Make the second pivot = 1
E[2,] <- E[2,] / -10.5

## Eliminate the second element of the third row
E[3,] <- E[3,] - (E[2,] * 0.5)

print(E)
```

```
##             glue A glue B    glue C      cost
## workers          1    2.5  4.000000 167.50000
## electricity      0    1.0  1.619048  63.57143
## packaging        0    0.0 -2.809524 -84.28571
```

The last pivot is founded by multiplying the last row `E[3,]` by -(1/`E[3,]`):

```
## Make the third pivot = 1
e_1 <- 1 / E[3,3]

E[3,3] <- E[3,3] * e_1
E[3,4] <- E[3,4] * e_1

print(E)
```

```
##             glue A glue B   glue C      cost
## workers          1    2.5 4.000000 167.50000
## electricity      0    1.0 1.619048  63.57143
## packaging        0    0.0 1.000000  30.00000
```

We can now say as we have at one pivot by column there are no **free variables**. To proceed, we can write a system of equations with the values of the Echelon matrix:

$x_1 + 2.5x_2 + 4x_3 = 167.50$
$0x_1 + x_2 + 1.6x_3 = 63.57$
$0x_1 + 0x_2 + x_3 = 30.00$

Finally we can find the values of $\mathbf{x}$:

```
## X3
x3 <- E[3,4]

## X2
x2 <- E[2,4] - (E[2,3] * x3)

## X1
x1 <- E[1,4] - (E[1,2] * x2) - (E[1,3] * x3)

solution <- matrix(data = c(x1, x2, x3),
                   nrow = 1,
                   ncol = 3)
colnames(solution) <- products
```

```r
rownames(solution) <- "Kg"

print(solution)
```

```
##    glue A glue B glue C
## Kg     10     15     30
```

Part C: Explanation

It was proved that the problem had a solution, which is unique, represented in the vector `solution` = [10, 15, 30]. The properties of the Reduced Echelon form anticipated this result as the rightmost column was not a pivot (the prior column was `E[3,3]`) and the last row `E[3,]` was not a row of *0*, thus the **Existence and Uniqueness Theorem** was satisfied.

Excercise Description (ii)

Which problem should you solve instead to know how many Kg of A,B, C should they produce, if the company wants to spend in total 620euro per day, whatever the splitting into costs for workers salary, electricity and packaging? Do you think that the problem has a unique solution in this case?

Part D: Total Cost Problem

The problem now has only 1 row, as the products are still 3 but the only issue is the total cost, which has to be equal to 620. Therefore the system will be written as:

- $TotalCost = (2+5+1)x_1 + (5+2+3)x_2 + (8+3+1)x_3$

Where the coefficients of x_1, x_2, x_3 are the column sums of matrix `A` and vector `b` is equal to 620.

Then it is evident that the problem has infinite solutions as there are 3 variables and 1 equation, or 1 basic variables and 2 free variables.