The VIX and related derivatives

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Documentation

History of the market for variance derivatives [1], [2]

The concept of volatility derivatives was born in the 1970s, with the development of the Black-Scholes-Merton model. This model revealed that the value of an option is highly sensitive to volatility. Traders used strategies like straddles and strangles to manage risk, but those are influenced by both the underlying asset's price and its volatility. This led to demand for direct exposure to volatility.

The first volatility derivatives appeared in OTC markets in the 1990s. These included variance swaps and volatility swaps. Variance swaps pay out based on the difference between realized and implied variance, while volatility swaps are based on standard deviation. These instruments enable investors to:

- Speculate on volatility: If an investor believes that current or future volatility is mispriced, they can bet on volatility using variance derivatives.
- Hedge against market downturns: Since volatility often spikes during market crashes, institutions use variance derivatives to hedge portfolios, benefiting from their negative correlation with equity markets.

A big advancement came in 1993 with the introduction of the CBOE Volatility Index (VIX), which become a benchmark for expected 30-day volatility. Later, VIX futures were launched in 2004 and VIX options in 2006. The 2008 global financial crisis was an important moment. As markets became more volatile, demand for hedging tools rose, and interest in volatility products increased. In the 2010s, exchange-traded products such as VXX became popular among retail and institutional investors. Today, volatility derivatives include a wide choice of instruments: listed VIX products, variance and volatility swaps, corridor variance swaps, and exotic options.

VIX volatility index [3]

The VIX is a real-time index that measures the market's expectations of volatility over the next 30 days for the S&P 500. It is often called the "fear index" because it tends to rise when markets are uncertain or falling. The VIX uses implied volatility, which reflects the market's expectations of future price movements based on option prices. Unlike historical volatility, implied volatility estimates how much the market thinks the S&P 500 will move going forward. While the VIX itself is not directly tradable, various financial instruments, such as VIX futures, options, and exchange-traded products, have been developed to allow investors to hedge against or speculate on changes in market volatility.

Main variance derivatives

VIX futures and VIX options: standardized derivatives traded on the CBOE that allow investors to trade expected market volatility based on the VIX index. In particular, VIX futures are contract that enable traders to speculate on or hedge against future values of the VIX. The price of a VIX futures contract reflects the market's view of the value of the VIX on their expiration date [4]. Instead, VIX options give holders the right, but not the obligation, to buy or sell the VIX at a specified strike price at maturity (being european options [5]). These are often used by asset managers to hedge equity positions or speculate on volatility spikes, especially in times of expected market stress.

Variance and volatility swap [2]: over-the-counter (OTC) derivative that allows investors to trade future realized variance of an underlying asset without exposure to its price movements. It provides a pure play on volatility, making it useful for hedging or speculative purposes. The payoff can be calculated as Notional $\times (\sigma^2_{\text{realized}} - \kappa_{\text{var}})$, where $\sigma^2_{\text{realized}}$ is the realized variance over the life of the contract, and κ_{var} is the variance strike set. The realized volatility as: $\sigma^2_{\text{realized}} = \frac{AF}{n-1} \sum_{i=0}^{n-1} \left(\log\left(\frac{S_{i+1}}{S_i}\right)\right)^2$, where S_i is the asset price at time i, and AF is the annualization factor based on the number of trading days in a year. AF, defined by $\frac{n}{T}$, would be 252 if the maturity of the swap were 1 year with daily sampling (T =1, n=252). Closely related are volatility swaps, which offer a payoff based on the difference between realized volatility and a predetermined volatility strike. Unlike variance swaps, volatility swaps deal directly with the standard deviation of returns, making their payoff function nonlinear and more complex to hedge. What distinguishes the VIX from a variance swap is that the VIX represents the market's expectation of future implied volatility based on option prices. In contrast, a variance swap provides a direct exposure to the actual realized variance of the underlying asset over a specified period. Thus, the VIX reflects anticipated volatility, while variance swaps settle on observed historical volatility.

Variance futures: standardized contracts traded on exchanges that let investors bet on the future variance of an asset. They have daily settlements, which means profits and losses are realized every day.The key difference between the variance future and the variance swap is that the swap is an over-the-counter contract settled at maturity based on realized variance, while the future is exchange-traded and settled daily. This makes variance futures more liquid and transparent.

Volatility-linked ETFs and ETNs: exchange-traded products designed to provide investors with exposure to volatility, typically through VIX futures. These instruments are often used by retail investors to hedge or speculate on market volatility, but they can carry significant risks due to their complex structure and daily rebalancing.

1 The Carr-Madan formula

1.1

To prove the result it is sufficient to apply integration by parts (IP) to $H''(k)(x-k)$ and the twice Fundamental Theorem of Calculus (FT) to $H''(k)$

$$
\begin{aligned}
\int_{x_0}^{x} H''(k)(x-k)\,dk &\overset{\text{IP+FT}}{=} H'(k)(x-k)\Big|_{x_0}^{x} - \int_{x_0}^{x} H'(k)\cdot(-1)\,dk \\
&= H'(k)(x-k)\Big|_{x_0}^{x} + \int_{x_0}^{x} H'(k)\,dk \\
&\overset{\text{FT}}{=} -H'(x_0)(x-x_0) + H(x) - H(x_0)
\end{aligned}
$$

from which the result follows immediately.

1.2

Appliying the identity

$$(x-y) = (x-y)^+ - (y-x)^+$$

to what we proved in Section 1.1 we have that

$$\begin{aligned} H(x) &= H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)(x-k)dk \\ &= H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)[(x-k)^+ - (k-x)^+]dk = (\star) \end{aligned}$$

Now consider the final integral. For its first part we have that

$$\begin{aligned} \int_{x_0}^{x} H''(k)(x-k)^+dk &= \int_{x_0}^{x} H''(k)(x-k)^+dk + \int_{x}^{+\infty} H''(k)(x-k)^+dk \\ &= \int_{x_0}^{+\infty} H''(k)(x-k)^+dk \end{aligned}$$

as $(x-k)^+ = 0 \ \forall \ k \geq x$, hence the integral for $k \geq x_0$ is zero and can be added to the first one. For the second part of the integral, instead, we have that

$$\begin{aligned} -\int_{x_0}^{x} H''(k)(k-x)^+dk &= \int_{x}^{x_0} H''(k)(k-x)^+dk \\ &= \int_{x}^{x_0} H''(k)(k-x)^+dk + \int_{0}^{x} H''(k)(k-x)^+dk \\ &= \int_{0}^{x_0} H''(k)(k-x)^+dk \end{aligned}$$

where we added the 0 integral from 0 to x as $(k-x)^+ = 0 \ \forall \ k \leq x$.
It follows that

$$\begin{aligned} (\star) = H(x) &= H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)[(x-k)^+ - (k-x)^+]dk \\ &= H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)(x-k)^+dk - \int_{x_0}^{x} H''(k)(k-x)^+dk \\ &= H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{+\infty} H''(k)(x-k)^+dk + \int_{0}^{x_0} H''(k)(k-x)^+dk \end{aligned}$$

1.3

To find a replicating strategy for the European derivative with payoff $H(S_T)$ we can use the result of the previous section (here $x = S_T$ and $x_0 = S_0$):

$$H(S_T) = H(S_0) + H'(S_0)(S_T - S_0) + \int_{0}^{S_0} H''(k)(k-S_T)^+dk + \int_{S_0}^{+\infty} H''(k)(S_T-k)^+dk \tag{1}$$

It follows by the law of one price that the derivative can be replicated with a portfolio made of

- $n_0 = H'(S_0)$ units of the underlying asset
- $a_0 = \Big(H(S_0) - S_0 H'(S_0)\Big)e^{-rT}$ units of the risk free
- $w(k) = H''(k)$ units of a put with strike k for all $k \leq x_0$
- $w(k) = H''(k)$ units of a call with strike k for all $k \geq x_0$

The price of the derivative at time zero is therefore given by

$$e^{-rT}\Big(H(S_0) - S_0 H'(S_0)\Big) + H'(S_0)S_0 + \int_0^{S_0} H''(k)p_0(T,k,S)dk + \int_{S_0}^{+\infty} H''(k)c_0(T,k,S)dk$$

If S_0 is equal to the forward price, that is if $S_0 = F_0(T) = \mathbb{E}^{\mathbb{Q}}[S_T]$, then when computing expectations in (3) under the risk neutral measure $\mathbb{Q}$ we have that

$$e^{-rT}\Big[H(S_0) + H'(S_0)\mathbb{E}^{\mathbb{Q}}[S_T - x_0]\Big] = e^{-rT}\Big[H(x_0) + H'(x_0)\mathbb{E}^{\mathbb{Q}}[S_T] - x_0\Big] = e^{-rT}H(x_0)$$

In other words our position in the underlying becomes zero (i.e., $n_0 = 0$) and our position in the risk free asset becomes $a_0 = H(x_0)e^{-rT}$.

1.4

If the derivative has payoff $H(x) = x^p$ it follows from Section 1.3 that the static replicating portfolio relative to an arbitrary reference point x_0 is given by

- $n_0 = p x_0^{p-1}$ unites of the underlying asset
- $a_0 = x_0^p(1-p)e^{-rT}$ units of the riskless asset
- $w(k) = p(p-1)k^{p-2}$ units of a put with strike k for all $k \leq x_0$
- $w(k) = p(p-1)k^{p-2}$ units of a call with strike k for all $k \geq x_0$

1.5

In practice the limits of the Carr-Madan results are the assumption that the payoff of the derivative is piecewise twice continuously differentiable and that the replicating strategy includes holding a position $w(k)$ in European put and call options with strike price k for *all* $k \leq x_0$ in the first case and $k \geq x_0$ in the second case. The assumption that k is continuous is clearly irrealistic, and in practice, a discrete version of the result could lead to an imperfect replicating strategy.

2 The VIX index

Assuming that the market is complete and that the SPX evolves according to the following dynamics:

$$\frac{dS_t}{S_t} = (r - \delta)\,dt + \sqrt{V_t}\,dB_t^{\mathbb{Q}}$$

where, r denotes the constant interest rate, δ denotes the constant dividend yield on the index, $B_t^{\mathbb{Q}}$) is a Brownian motion under the risk neutral measure, and $V_t \geq 0$ is a process that models the squared volatility of the index and which may depend on other sources of risk than just the brownian motion. We define:

$$\overline{V}_{t,T} = \int_t^T V_s \, ds$$

2.1

To prove that $x\overline{V}_{t,T} = \int_t^T \frac{dS_s}{S_s} - \log\left(\frac{S_T}{S_t}\right)$ we proceed as follows. From Itō's Lemma we have

$$d\left(\log S_t\right) = \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right) d\langle S_t \rangle = \frac{dS_t}{S_t} - \frac{1}{2} V_t \, dt$$

Integrating from t to T gives

$$\log\left(\frac{S_T}{S_t}\right) = \int_t^T \frac{dS_s}{S_s} - \frac{1}{2}\overline{V}_{t,T} \iff \frac{1}{2}\overline{V}_{t,T} = \int_t^T \frac{dS_s}{S_s} - \log\left(\frac{S_T}{S_t}\right)$$

hence we obtain $x = \frac{1}{2}$.

2.2

We want to prove that: $\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = \alpha(T-t)$

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (r-\delta)\, ds + \int_t^T \sqrt{V_s}\, dB_s^{\mathbb{Q}}\right] = (r-\delta)(T-t) + 0$$

hence we obtain $\alpha = (r-\delta)$

2.3

We use the identity from point 2.2 to prove:

$$\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_t}\right)\right] = \log\left(\frac{K_0}{S_t}\right) - \left(1 - \frac{F_t(T)}{K_0}\right) - P_t(T, K_0)$$

where

$$P_t(T, K_0) := \int_0^{K_0} e^{r(T-t)} \frac{\mathrm{Put}_t(T,k)}{k^2}\, dk + \int_{K_0}^{\infty} e^{r(T-t)} \frac{\mathrm{Call}_t(T,k)}{k^2}\, dk$$

and $\mathrm{Put}_t(T,k)$, $\mathrm{Call}_t(T,k)$ denote the prices at time t of European put and call options with maturity T and strike k, respectively.

We know that $\log(x) \in C^{\infty}((x > 0))$, hence using Equation (1)

$$\log(x) = 0 + 1 \cdot (x-1) + \int_0^1 \left(-\frac{1}{\kappa^2}\right)(\kappa - x)^+ d\kappa + \int_1^{\infty}\left(-\frac{1}{\kappa^2}\right)(x-\kappa)^+ d\kappa$$

it follows that

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\log(S_T)\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\log(k_0)\right] + \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{k_0}(S_T - k_0)^+\right] - \int_0^{k_0} \frac{1}{\kappa^2}(\kappa - S_T)^+ d\kappa - \int_{k_0}^{\infty} \frac{1}{\kappa^2}(S_T - \kappa)^+ d\kappa \\ &= \log(k_0) + \frac{F_t(T)}{k_0} - 1 - P_t(T, k_0)\end{aligned}$$

from which the following results proved:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_t}\right)\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\log(S_T)\right] - \log(S_t) = \log\left(\frac{k_0}{S_t}\right) + \frac{F_t(T)}{k_0} - 1 - P_t(T, k_0) \\ &= \log\left(\frac{k_0}{S_t}\right) - \left(1 - \frac{F_t(T)}{k_0}\right) - P_t(T, k_0)\end{aligned}$$

2.4

To prove the result we start from what we showed in Section 2.1. Taking expectations on both sides yields:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{2}\overline{V}_{t,T}\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u} - \log\left(\frac{S_T}{S_t}\right)\right] \\ &= (r - \delta)(T - t) - \log\left(\frac{k_0}{S_t}\right) + \left(1 - \frac{F_t(T)}{k_0}\right) + P_t(T, k_0) = (\spadesuit)\end{aligned}$$

Now recall that from the definition of forward price we have $(r - \delta)(T - t) = \log\left(\frac{F_t(T)}{S_t}\right)$ From which it follows that

$$(\spadesuit) = \log\left(\frac{F_t(T)}{S_t} \cdot \frac{S_t}{k_0}\right) + P_t(T, k_0) + \left(1 - \frac{F_t(T)}{k_0}\right)$$

Using Taylor's formula $\log(x) \approx (x - 1) - \frac{(x-1)^2}{2}$ we get the result:

$$\begin{aligned}\frac{1}{2}\mathbb{E}_t^{\mathbb{Q}}\left[\overline{V}_{t,T}\right] &\approx \left(\frac{F_t(T)}{k_0} - 1\right) - \frac{1}{2}\left(\frac{F_t(T)}{k_0} - 1\right)^2 + P_t(T, k_0) + \left(1 - \frac{F_t(T)}{k_0}\right) \\ &= P_t(T, k_0) - \frac{1}{2}\left(1 - \frac{F_t(T)}{k_0}\right)^2\end{aligned}$$

To explain CBOE's claim regarding the VIX we connect our latest expression to its definition. We start by noticing that:

$$\int_0^{\infty} \frac{e^{r(T-t)}\, O_t(T, k)}{k^2}\,\mathrm{d}k = \int_0^{K_0} \frac{e^{r(T-t)}\ \mathrm{Put}_t(T, k)}{k^2}\,\mathrm{d}k + \int_{K_0}^{\infty} \frac{e^{r(T-t)}\ \mathrm{Call}_t(T, k)}{k^2}\,\mathrm{d}k \;=\; P_t(T, K_0).$$

With this expression we are assuming the existance of a continuum of strike prices, that is unreasonable in real life situation. We substitute the Lebesgue inegral with a Riemann sum over all the tradable strikes K_i.

For every strike j we approximate the expression by its midpoint value on the panel $[K_{j-1}, K_{j+1}]$. This produces the quadrature width $\Delta K_j = \frac{K_{j+1} - K_{j-1}}{2}$, Using the local midpoint rule we replaced the continuous integral by the Riemann sum

$$2\sum_j \frac{\Delta K_j}{K_j^2}\, e^{r(T-t)}\, O_t\Big(T, K_j\Big),$$

which is exactly the first term in the CBOE's VIX definition once the quadratic cash–forward correction $-\Big(1 - F_t(T)/K_0\Big)^2$ is added unchanged. Obiviously, as it is an approximation from an integral to a finite sum we will have some approximation but that can be ignored, or recoduct to just bid-ask noise, if the strike prices-grid is thin enough. Hence we can conclude that:

$$\left(\frac{\text{VIX}}{100}\right)^2 \approx \frac{1}{\eta}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{2}\overline{V}_{t,T}\right]$$

So we proved that, under the no–arbitrage assumption, $(\text{VIX}_t/100)^2$ is nothing less than the market–implied forecast of the variance that will be *realised* over the next thirty days, as the CBOE claims. Unlike the backward-looking realised variance of the last month's returns, it is forward-looking, can be traded or hedged immediately via VIX futures and options, and condenses the information contained in *all* out-of-the-money SPX options, so it is far less exposed to one-day price shocks. Relative to a Black–Scholes implied volatility extracted from a single option, the VIX is model-free (it requires no constant-volatility assumption), captures the full volatility smile—including skew and kurtosis—and sidesteps any mispricing or illiquidity that may affect an individual strike. All the previous results come from a particular framework.

- **Risk–neutral diffusion.** Under the no–arbitrage measure Q the index satisfies the dynamics with constant interest rate r and dividend yield δ, a standard Brownian motion B_t^Q and a non-negative volatility process V_t.
- **Continuity (no jumps).** The sample paths of S_t are continuous; hence Itô's formula for $\log S_t$ holds without jump terms and gives the identity that ties realised variance to the log contract.
- **Integrability of variance.** The process V_t is square–integrable so that $\int_t^T V_u \, du$ is finite; no further distributional assumptions are imposed, making the result model-free with respect to volatility.
- **Market completeness.** The absence of arbitrage together with a complete market for European options ensures the existence of the unique measure Q above and allows the static Carr–Madan replication used in the derivation.

3 Futures pricing

3.1

In this section we aim to prove that $\left(\frac{\mathrm{VIX}_T}{100}\right)^2 = \frac{1}{\eta}(a + bV_T)$ under the assumption that

$$\left(\frac{\mathrm{VIX}_T}{100}\right)^2 = \frac{1}{\eta}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_T^{T+\eta} V_u\,du\right] \tag{2}$$

Recall the dynamics of V_t

$$dV_t = \lambda(\theta - V_t)dt + \xi\rho\,\sqrt{V_t}dB_t^{\mathbb{Q}} + \xi\sqrt{|1-\rho^2|V_t}\,dZ_t^{\mathbb{Q}} \tag{3}$$

By linearity of expectation and Fubini's theorem we have that

$$\mathbb{E}^{\mathbb{Q}}[dV_T] = d\mathbb{E}^{\mathbb{Q}}[V_T] = \mathbb{E}^{\mathbb{Q}}\left[\lambda(\theta - V_T)dt + \xi\rho\,\sqrt{V_t}dB_t^{\mathbb{Q}} + \xi\sqrt{|1-\rho^2|V_t}\,dZ_t^{\mathbb{Q}}\right] = \lambda(\theta - \mathbb{E}^{\mathbb{Q}}[V_t])dt$$

that is

$$d\mathbb{E}^{\mathbb{Q}}[V_t] = \lambda(\theta - \mathbb{E}^{\mathbb{Q}}[V_t])dt$$

For the purpose of this exercise, define $s := T + \eta$ and $\mu(s) := \mathbb{E}_T^{Q}[V_s]$, hence getting

$$\frac{d\mu(s)}{ds} = \lambda\theta - \lambda\mu(s)$$

The solution to the ODE is given by

$$\mu(s) = \theta + Ce^{-\lambda s}$$

To get the value of the constant C we use the fact that $\mu(T) = \mathbb{E}_T^{Q}[V_T] = V_T$:

$$\mu(T) = \theta + Ce^{-\lambda T} = V_T \iff C = (V_T - \theta)e^{\lambda T}$$

The final expression is therefore given by

$$\mathbb{E}_T^{Q}[V_{T+\eta}] = \theta + (V_T - \theta)e^{-\lambda\eta} \tag{4}$$

Now we explicitly compute the expectation. From Equation (3) we can write

$$\lambda V_t\,dt = \lambda\theta\,dt + \xi\rho\,\sqrt{V_t}\,dB_t^{\mathbb{Q}} + \xi\sqrt{|1-\rho^2|V_t}\,dZ_t^{\mathbb{Q}} - dV_t$$

Integrating on both sides, taking expectations and using the fact that stochastic integrals are martingale we obtain

$$\mathbb{E}_T^{\mathbb{Q}}\left[\int_T^{T+\eta} V_u\,du\right] = \theta\eta - \frac{1}{\lambda}\mathbb{E}_T^{\mathbb{Q}}\left[\int_T^{T+\eta} dV_u\right] = \theta\eta - \frac{1}{\lambda}\mathbb{E}_t^{\mathbb{Q}}[V_{T+\eta}] + \frac{V_T}{\lambda}$$

and combining with Equation (4) we finally get

$$\mathbb{E}_T^{\mathbb{Q}}\left[\int_T^{T+\eta} V_u\,du\right] = \theta\eta - \frac{1}{\lambda}\mathbb{E}_t^{\mathbb{Q}}[V_{T+\eta}] + \frac{V_T}{\lambda} = \theta\eta + (V_T-\theta)\left[\frac{1-e^{-\lambda\eta}}{\lambda}\right]$$

which allows to express the final result as

$$\begin{aligned}\left(\frac{\mathrm{VIX}_T}{100}\right)^2 &= \frac{1}{\eta}\mathbb{E}_t^{Q}\left[\int_T^{T+\eta} V_u\,du\right] = \frac{1}{\eta}\left[\theta\eta + (V_T-\theta)\frac{1-e^{-\lambda\eta}}{\lambda}\right]\\ &= \frac{1}{\eta}\left[\underbrace{\theta\eta\left(1-\frac{1-e^{-\lambda\eta}}{\lambda\eta}\right)}_{=:a} + \underbrace{\frac{1-e^{-\lambda\eta}}{\lambda}}_{=:b}V_T\right]\end{aligned}$$

3.2

To tackle this problem we can either apply Feynman-Kac theorem in its converse form or use the martingale property of $f(t,V_t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-sV_T}\right]$. We choose the second way.
Let's first apply Itō's Lemma to f:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial V_t}\mathrm{d}V_t + \frac{1}{2}\frac{\partial^2 f}{\partial V_t^2}\mathrm{d}\langle V\rangle_t$$

We can now substitute the dynamics of V_t and, since f is a martingale, setting its drift to 0. Doing so we arrive to the following PDE:

$$\frac{\partial f}{\partial t} + \lambda(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2 f}{\partial v^2} = 0 \tag{5}$$

Inspired by what we want to prove, we conjecture that the function has the form:

$$f(t,v) = e^{-c(T-t,s)-d(T-t,s)v}$$

Let's first impose the final conditions:

$$f(T,v) = e^{-c(0,s)-d(0,s)v} = e^{-sv}$$

from which it follows immediately that we must have $c(0,s)=0$ and $d(0,s)=s$.
Now we can plug our guess function in [PDE]. Before doing that note:

$\frac{\partial f}{\partial t} = (c'+d')f$; $\frac{\partial f}{\partial v} = -df$; $\frac{\partial^2}{\partial v^2}(f) = d^2 f$

Hence (5) becomes

$$(c'+d'v)f + \lambda(\theta-v)(-df) + \frac{1}{2}\xi^2 v d^2 f = 0$$

which, dividing by f (which is never 0), becomes

$$c' + d'v - \lambda(\theta-v)d + \frac{1}{2}\xi^2 v d^2 = 0$$

Now with separation of variable we can group coefficients of v and constant terms, hence getting

$$d' = -\lambda d - \frac{1}{2}\xi^2 d^2, \qquad d(0) = s$$

$$c' = \lambda \theta d, \qquad c(0) = 0$$

Using Wolphram Alpha we found that

$$d = d(T-t, s) = \frac{2\lambda s}{e^{\lambda(T-t)}(2\lambda + s\xi^2) - s\xi^2}$$

and as a consequence we have that

$$c(T-t) = \frac{2\lambda\theta}{\xi^2} \log\left(\frac{(2\lambda + s\xi^2) - s\xi^2 e^{-\lambda(T-t)}}{2\lambda}\right)$$

3.3

Variance futures price Using risk-neutral pricing and the variance future definition:

$$f_t^{\mathrm{VA}}(T) = \mathbb{E}_t^{\mathbb{Q}}\left[f_T^{\mathrm{VA}}(T)\right] = \frac{10{,}000}{T-t_0}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_{t_0}^{T} V_u\,du\right]$$

Splitting the integral:

$$f_t^{\mathrm{VA}}(T) = \frac{10{,}000}{T-t_0}\left(\int_{t_0}^{t} V_u\,du + \mathbb{E}_t^{\mathbb{Q}}\left[\int_{t}^{T} V_u\,du\right]\right)$$

From subsection 3.1 , we know:

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_{t}^{T} V_u\,du\right] = \theta(T-t) - \theta\cdot\frac{1-e^{-\lambda(T-t)}}{\lambda} + \frac{1-e^{-\lambda(T-t)}}{\lambda}V_t$$

Then the future variance price becomes

$$f_t^{\mathrm{VA}}(T) = \frac{10{,}000}{T-t_0}\left(\int_{t_0}^{t} V_u\,du + a^*(T-t) + b^*(T-t)V_t\right)$$

where: $a^* := \theta\left(1 - \frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)}\right)$ and $b^* := \frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)}$

VIX futures price

From the definition of the VIX in subsection 3.1:

$$\left(\frac{\mathrm{VIX}_T}{100}\right)^2 = \frac{1}{\eta}\left(a + bV_T\right) \Rightarrow \mathrm{VIX}_T = \frac{100}{\sqrt{\eta}}\sqrt{a + bV_T}$$

Then the price of a VIX futures at time t is:

$$f_t^{\mathrm{VIX}}(T) = \mathbb{E}_t^{\mathbb{Q}}\left[f_T^{\mathrm{VIX}}(T)\right] = \frac{100}{\sqrt{\eta}}\,\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{a' + b'V_T}\right] \tag{6}$$

where: $a' = a, \quad b' = b$

3.4

We aim to prove that:

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - e^{-sx}\right)\frac{ds}{s^{3/2}}, \quad x > 0 \tag{7}$$

It is known, from the Laplace transform tables, that the Laplace transform of $\sqrt{x}$ is $\mathcal{L}\{\sqrt{x}\} = \int_0^\infty e^{-sx}\sqrt{x}\,dx = \frac{\sqrt{\pi}}{2s^{3/2}}$.

Let's define: $g(x) = \frac{1}{2\sqrt{\pi}}\int_0^\infty (1 - e^{-sx})\frac{ds}{s^{3/2}}$. If $g(x)$ has the same Laplace trasform of $\sqrt{x}$, then, being both functions continuous and of exponential order, those two functions are equal. Therefore:

$$\begin{aligned}
\mathcal{L}\{g(x)\} &= \frac{1}{2\sqrt{\pi}}\int_0^\infty \left[\int_0^\infty e^{-sx}(1 - e^{-s'x})\,dx\right]\frac{ds'}{s'^{3/2}} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left[\int_0^\infty \left(e^{-sx} - e^{-(s+s')x}\right)dx\right]\frac{ds'}{s'^{3/2}} \\
&= \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(\frac{s'}{s(s+s')}\right)\frac{ds'}{s'^{3/2}} = \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{s}\int_0^\infty \frac{1}{(s+s')\sqrt{s'}}\,ds'
\end{aligned}$$

Let $s' = su \Rightarrow ds' = s\,du$. Then:

$$\int_0^\infty \frac{1}{(s+s')\sqrt{s'}}ds' = \int_0^\infty \frac{1}{s(1+u)\cdot\sqrt{su}}\cdot s\,du = \frac{1}{\sqrt{s}}\int_0^\infty \frac{1}{(1+u)\sqrt{u}}\,du$$

It is a known integral that:

$$\int_0^\infty \frac{1}{(1+u)\sqrt{u}}\,du = \pi \Rightarrow \mathcal{L}\{g(x)\} = \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{\sqrt{s}}\cdot\pi = \frac{\sqrt{\pi}}{2s^{3/2}}$$

We conclude:

$$g(x) = \sqrt{x} \Rightarrow \sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - e^{-sx}\right)\frac{ds}{s^{3/2}}$$

Substituting $x = a' + b'V_T$ and applying the identity (7) to the VIX future price equation (6),we obtain:

$$f_t^{\mathrm{VIX}}(T) = \frac{100}{\sqrt{\eta}}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - e^{-s(a'+b'V_T)}\right)\frac{ds}{s^{3/2}}\right]$$

Using Fubini's theorem:

$$f_t^{\mathrm{VIX}}(T) = \frac{100}{2\sqrt{\eta\pi}}\int_0^\infty \mathbb{E}_t^{\mathbb{Q}}\left[1 - e^{-s(a'+b'V_T)}\right]\frac{ds}{s^{3/2}}$$

We define the function:

$$\ell(s, T-t, V_t) := -\log \mathbb{E}_t^{\mathbb{Q}}\left[e^{-s(a'+b'V_T)}\right] \Rightarrow \mathbb{E}_t^{\mathbb{Q}}\left[e^{-s(a'+b'V_T)}\right] = e^{-\ell(s,T-t,V_t)}$$

So the final expression for the VIX futures price becomes:

$$f_t^{\mathrm{VIX}}(T) = \frac{50}{\sqrt{\pi\eta}} \int_0^\infty \left(1 - e^{-\ell(s,T-t,V_t)}\right) \frac{ds}{s^{3/2}}$$

From direct calculations we get that: $\ell(s, T-t, V_t) = s \cdot a + c(T-t) + d(T-t)V_t$

3.5

In an arbitrage-free market, the price of a variance futures contract maturing at time $T = t + \eta$ must equal the square of the VIX index at time t. That is, $f_t^{VA}(t+\eta) = \mathrm{VIX}_t^2$.

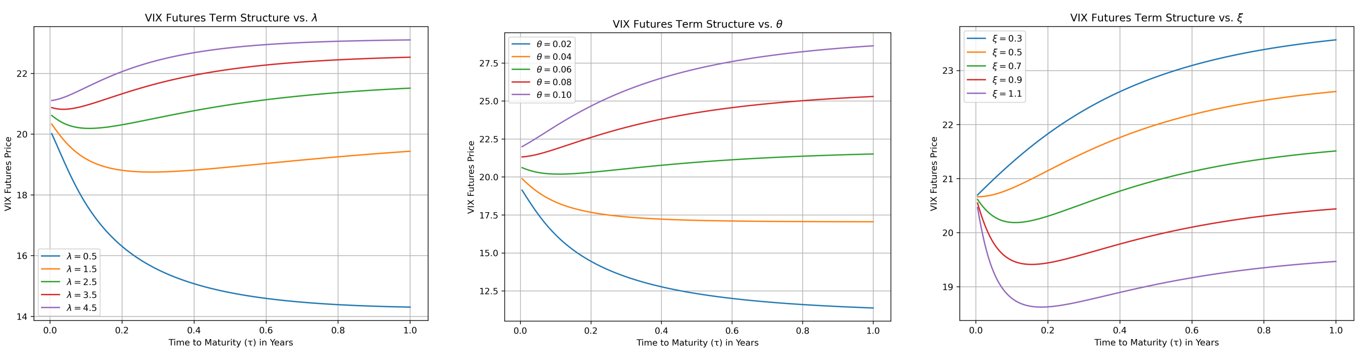
If this equality does not hold, it creates an arbitrage opportunity. The strategy depends on the sign of the mispricing:

- **If** $\mathrm{VIX}_t^2 > f_t^{VA}(t+\eta)$, sell the replicating VIX portfolio and buy the variance futures.
- **If** $\mathrm{VIX}_t^2 < f_t^{VA}(t+\eta)$, buy the replicating VIX portfolio and sell the variance futures.

At maturity $T = t + \eta$, both the VIX-based position and the variance futures converge to the realized variance, locking in a risk-free profit equal to $\left|\mathrm{VIX}_t^2 - f_t^{VA}(t+\eta)\right|$.

3.6

We now analyze the behavior of the VIX futures price with respect to the key model parameters λ, θ, and ξ. For each parameter, we plot the futures price against the time to maturity (τ) to observe the impact on the entire term structure. In this analysis, the base parameters are fixed at $V_t = 0.04$, with the other parameters set to $\lambda = 2.5$, $\theta = 0.06$, and $\xi = 0.7$.



(a) vs. λ (b) vs. θ (c) vs. ξ

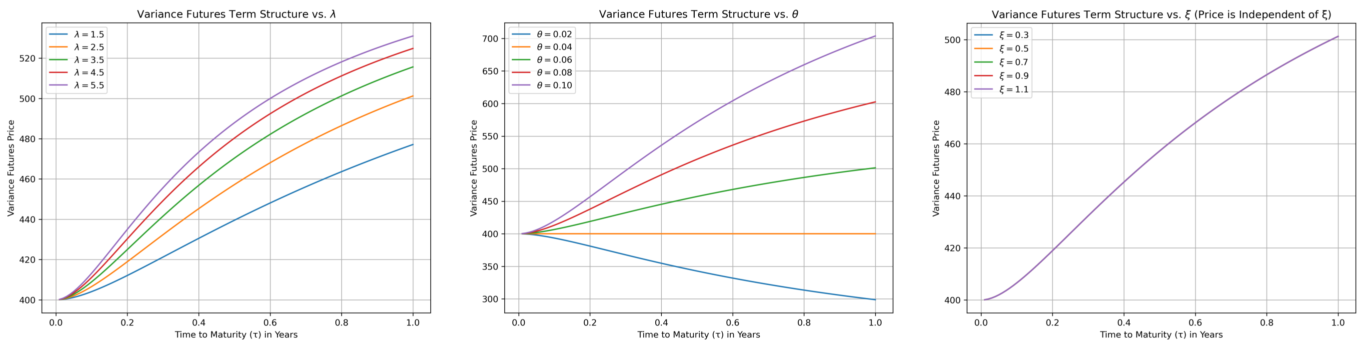
Figure 1: Sensitivity analysis of the VIX futures term structure with respect to the model parameters λ, θ, and ξ.

Figure 1 illustrates the impact of the model parameters.

- **(a) Sensitivity to λ:** A higher value of λ leads to a higher futures price for any given maturity $\tau > 0$. This is because the initial variance ($V_t = 0.04$) is lower than the long-run mean ($\theta = 0.06$), and a faster reversion speed pulls the expected future variance up towards the mean more quickly. The starting price of each curve at $\tau \approx 0$ is different, which is consistent with the fact that the theoretical spot VIX price depends on the value of λ.
- **(b) Sensitivity to θ:** The parameter θ acts as an anchor for the entire term structure. A higher value of θ uniformly shifts the futures price curve upwards. When the initial variance $V_t = 0.04$ is higher than θ (e.g., $\theta = 0.02$), the term structure is downward sloping. Conversely, when V_t is lower than θ, the term structure is upward sloping.
- **(c) Sensitivity to ξ:** All curves converge to the exact same point as τ approaches zero, confirming the theory that the spot VIX price is independent of ξ. For any maturity $\tau > 0$, a higher value of ξ results in a lower VIX futures price due to the concavity of the square-root payoff. Greater uncertainty (higher ξ) leads to a larger discount on the expected payoff.

3.7

We perform a similar sensitivity analysis for the Variance Futures prices computed in the preceding section. The results illustrate how the term structure of variance futures responds to the model parameters, revealing behavior that is notably different from that of VIX futures. The same base parameters are used for this analysis.



(a) vs. λ (b) vs. θ (c) vs. ξ

Figure 2: Sensitivity analysis of the Variance futures term structure with respect to the model parameters λ, θ, and ξ.

Figure 2 shows the impact of the model parameters on variance futures.

- **(a) Sensitivity to λ:** All curves originate from the same point at $\tau = 0$, as the price is determined only by the fixed accrued variance. For $\tau > 0$, a higher value of λ leads to a higher futures price, since our base case assumes $V_t < \theta$.
- **(b) Sensitivity to θ:** Again, all curves start at the same price. The plot clearly illustrates the role of θ as the long-term anchor. A key result is shown where $\theta = 0.04 = V_t$, in which case the term structure is flat.

- **(c) Sensitivity to ξ:** This plot provides the most striking result, showing that all curves lie perfectly on top of one another. This visually confirms that the price of a variance future is entirely independent of ξ, as its price depends only on the expectation of future variance, not its distribution.

3.8

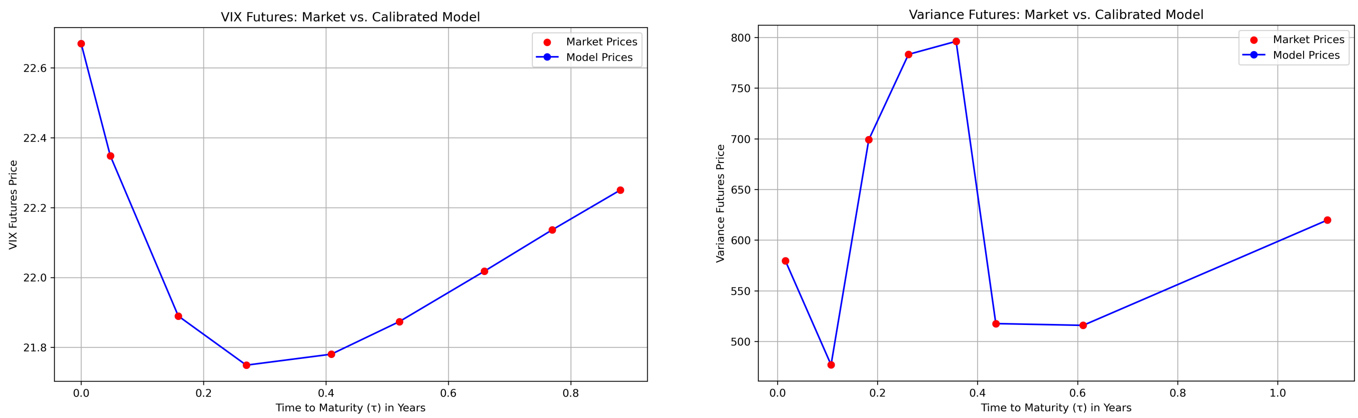
In this section, we calibrate the four parameters of the stochastic volatility model, $(V_t, \lambda, \theta, \xi)$, by jointly fitting the model to the observed prices of the VIX futures and S&P 500 Variance Futures term structures. The calibration is performed by minimizing the sum of squared errors between the model-generated prices and the market prices using an optimization algorithm.

The optimization yielded the following calibrated parameters:

- **Current Squared Volatility (V_t):** 0.0499, which implies a current VIX level of 22.34.
- **Mean-Reversion Speed (λ):** 1.2405.
- **Long-Run Mean Variance (θ):** 0.0711, implying a long-run average VIX of 26.66.
- **Volatility of Volatility (ξ):** 0.5250.

The calibrated long-run mean θ suggests a significant variance risk premium is embedded in the futures prices, as the implied VIX of 26.66 is considerably higher than the spot VIX. The mean-reversion speed λ of 1.2405 is economically reasonable and suggests a moderate rate at which volatility reverts to its long-term mean.

The calibration achieved a final minimized error of 0.00, and the corresponding model prices show a perfect fit to the market data. This fit is visually confirmed in Figure 3.



(a) VIX Futures Calibration (b) Variance Futures Calibration

Figure 3: Comparison of market futures prices against the prices generated by the calibrated model.

3.9

To estimate the correlation parameter ρ we need a derivative that depends both on the underlying and on the variance process. An example could be an OTM European option on the SPX as it depends on both the processes, for instance a put. Options, differently from forwards and futures, don't depend only on the first moment of the variance process, hence when applying Ito to find the PDE the correlation term will pop out.

3.10

We are considering a ATM call with strike price K and maturity T=1. Given that the implied volatility surface is described as follows:

$$\sigma_t(T,K) = \alpha(t,1,V_t) \; + \; \underbrace{\beta(t,1,V_t)\log\frac{K}{S_t}}_{=0} \; + \; \underbrace{\gamma(t,1,V_t)\Big(\log\frac{K}{S_t}\Big)^2}_{=0},$$

$$\sigma_0 = \alpha(t,1,V_t),$$

The Black–Scholes price of the call is

$$C_t = e^{-\delta}\, S_t\, \Phi(d_1) - e^{-r} K\, \Phi(d_2), \qquad d_1 = \tfrac{1}{2}\sigma_0, \quad d_2 = d_1 - \sigma_0,$$

Let X_t be a self-financing portfolio holding π^S SPX futures and π^{VA} variance futures. To replicate the call, we impose $\mathrm{d}X_t = \mathrm{d}C_t$ in particular we aim to match the diffusion terms.

We get that $\pi^S = \Delta_C = \partial_s C(t,s,v) = \Phi(d_1)$. To find the right number of Variance futures we use the sensitivity of the call and of the future price to V. We get that

$$\partial_V f_0^{VA}(1) = 10000\frac{1-e^{-\lambda}}{\lambda} \text{ and } \partial_V C(t,s,v) = Ke^{-\delta}\phi(d_1)\alpha_V$$

Hence $\pi^{VA} = \frac{e^{-\delta}\phi(d_1)K\alpha_V\lambda}{10000(1-e^{-\lambda})}$ To make the portfolio self financing and to match the initial price of the call we need to invest/borrow the difference in price of the call and the portfolio.

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